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Michael Falk, Vadim Schechtman, & Alexander Varchenko

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BGG RESOLUTIONS VIA CONFIGURATION SPACES

by Michael Falk, Vadim Schechtman & Alexander Varchenko

To the memory of I.M. Gelfand, on the occasion of his centenary (1913–2013)

Abstract. — We study the blow-ups of configuration spaces. These spaces have a structure of what we call an Orlik–Solomon manifold; it allows us to compute the intersection cohomology of certain flat connections with logarithmic singularities using some Aomoto type complexes of logarithmic forms. Using this construction we realize geometrically the $\mathfrak{sl}_2$ Bernstein–Gelfand–Gelfand resolution as an Aomoto complex.

Résumé (Résolutions BGG via les espaces de configurations). — Nous étudions les éclatements d'espaces de configuration. Ces espaces ont une structure de variété que nous appelons d'Orlik-Solomon ; elle permet de calculer la cohomologie d'intersection de certaines connexions plates avec singularités logarithmiques à l'aide de complexes de formes logarithmiques du type d'Aomoto. En utilisant cette construction, nous donnons une réalisation géométrique de la résolution de Bernstein–Gelfand–Gelfand pour $\mathfrak{sl}_2$ comme un complexe d'Aomoto.

Contents

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1. Introduction

Let us discuss briefly some general perspective and motivation.

Localization of $\mathfrak{g}$-modules: two patterns

(a) *Localization on the flag space.* — Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra whence the root system $R \subset \mathfrak{h}^*$; fix a base of simple roots $\Delta \subset R$ whence a decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. The classical Bernstein–Gelfand–Gelfand resolution is the left resolution of a simple finite dimensional $\mathfrak{g}$-module L_χ of highest weight $\chi - \rho$ (where ρ is the half-sum of the positive roots) of the form

$$0 \longrightarrow C_n \longrightarrow \ldots \longrightarrow C_0 \longrightarrow L_\chi \longrightarrow 0, \tag{1.1}$$

where

$$C_i = \bigoplus_{w \in W_i} M_{w\chi},$$

cf. [BGG75]. Here M_λ denotes the Verma module of the highest weight $\lambda - \rho$, and $W_i \subset W$ is the set of elements of the Weyl group of length i.

We can pass to contragredient duals and use the isomorphism $L_\chi = L_\chi^*$ given by the Shapovalov form to get a right resolution

$$0 \longrightarrow L_\chi \longrightarrow C_0^* \longrightarrow \ldots \longrightarrow C_n^* \longrightarrow 0, \tag{1.2}$$

where

$$C_i^* = \bigoplus_{w \in W_i} M_{w\chi}^*.$$

A geometric explanation of the last complex was given by Kempf, [Kem78], who interpreted (1.2) as a Cousin complex connected with the filtration of the flag space G/B by unions of Schubert cells (G being a semisimple group with Lie algebra $\mathfrak{g}$ and $B \subset G$ the Borel subgroup with $\mathrm{Lie}(B) = \mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}_+$). Here the i-th term is interpreted as a relative cohomology space with support in the union of Schubert cells of codimension i. This geometric picture is a part of Beilinson–Bernstein theory which says that some reasonable category of $\mathfrak{g}$-modules is equivalent to a category of (twisted) $\mathscr{D}$-modules over G/B, [BB81].

(b) *Localization on configuration spaces.* — In a different direction, contragredient Verma modules and irreducible representations have been realized in [SV91] in certain spaces of logarithmic differential forms on configuration spaces. This may be upgraded to an equivalence between some category of $\mathfrak{g}$-modules and some category of $\mathscr{D}$-modules over configuration spaces, cf. [KS97, BFS98, KV06].

Blow-ups and their "Schubert" stratifications. — In this note we propose a construction which provides a geometric interpretation of the resolutions similar to the BGG resolution in (1.2). The main new idea is to use the blow-ups of hyperplane arrangements (in our case – the configuration arrangements) studied in [ESV92, STV95, BG92, Var95, DCP95]. We define some natural stratifications on such blow-ups which play the role of the Schubert stratification on G/B. On each stratum we consider the

Aomoto complex of logarithmic Orlik-Solomon forms; they are subcomplexes of the de Rham complexes of standard local systems from [SV91]. (In fact the stratification itself depends on a local system).

This way we get double complexes with one differential induced by the de Rham differential and the other one being the residue. The residue differential gives rise to BGG-like complexes. For the trivial local system we get the complexes considered in [BG92]; in our case the combinatorics of the "Schubert stratification" depends on the Cartan matrix and a finite number of dominant weights.

We illustrate this construction for $\mathfrak{g} = \mathfrak{sl}_2$. In this case we obtain the BGG resolutions of tensor products of finite dimensional $\mathfrak{g}$-modules, and the complex associated with our double complex calculates the intersection cohomology of the corresponding local system.

We expect to develop a similar picture for Kac-Moody algebras with nontrivial Serre's relations. In this program, one considers discriminantal arrangements associated with a Kac-Moody algebra $\mathfrak{g}$, see [SV91]. One resolves the singularities of such an arrangement and considers the associated double complex of Orlik-Solomon forms as in this paper. Serre's relations of $\mathfrak{g}$ correspond to certain strata of the resolution. By using these strata, one expects to define a double subcomplex of the double complex. The spaces of the double subcomplex will correspond to the subspaces of the associated BGG resolution.

In Section 2, we consider a complex analytic manifold X, a divisor $D \subset X$ with normal crossings and a holomorphic flat connection on X. We construct a complex which calculates the cohomology of X with coefficients in the local system associated with the flat connection.

In Section 3, we define an Orlik-Solomon manifold, a flat connection with logarithmic singularities on an Orlik-Solomon manifold, and the associated finite-dimensional Aomoto complex. Theorem 3.2 says that the Aomoto complex calculates the cohomology of the Orlik-Solomon manifold with coefficients in the local system associated with the connection. Theorem 3.2 is our first main result.

In Section 4, we discuss the minimal resolution of singularities of an arrangement. In Section 5, we introduce weighted Orlik-Solomon manifolds associated with weighted arrangement of hyperplanes. In Section 6, we review the definition of the BGG resolution for the Lie algebra $\mathfrak{sl}_2$. In Section 7, we realize geometrically the $\mathfrak{sl}_2$ BGG resolution as the skew-symmetric part of the Aomoto complex of a suitable weighted Orlik-Solomon manifold. Theorem 7.7 is our second main result. In Section 7.8, we discuss the relations between the BGG resolution and the complex of flag forms. In Section 7.9, we discuss the relations between the BGG resolution and intersection cohomology.

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2. Residue complex of a filtered manifold

2.1. Local system of a flat connection. — Let X be a smooth connected complex analytic manifold. Given a natural number r, let ∇ be a holomorphic flat connection on the trivial bundle $X\times\mathbb{C}^r\to X$. The sheaf $\mathscr{L}$ on X of flat sections of ∇ is a locally constant sheaf. If s is a differential form with values in $\mathbb{C}^r$, we denote $d_{\mathscr{L}}s:=\nabla s=ds+\omega\wedge s$ where ω is the connection form, a differential 1-form with values in $\operatorname{End}(\mathbb{C}^r)$. We have $d^2_{\mathscr{L}}=0$.

Let $(\Omega^\bullet_X\otimes\mathbb{C}^r,d_{\mathscr{L}})$ be the de Rham complex of sheaves of $\mathbb{C}^r$-valued holomorphic differential forms on X with differential $d_{\mathscr{L}}$. The cohomology $H^\bullet(X;\mathscr{L})$ of X with coefficients in $\mathscr{L}$ is canonically isomorphic to the hypercohomology $H^\bullet(X,\Omega^\bullet_X\otimes\mathbb{C}^r)$.

2.2. Residue complex of sheaves. — Let $D\subset X$ be a divisor with normal crossings. Namely, we assume that X is covered by charts such that in each chart D is the union of several coordinate hyperplanes or the empty set. Such charts are called *linearizing.* We define

$$\mathscr{Z}=\{X=Z_0\supset D=Z_1\supset Z_2\supset\cdots\}$$

the associated filtration of X by closed subsets as follows. A point $x\in X$ belongs to Z_i if in a linearizing chart x belongs to the intersection of i distinct coordinate hyperplanes of D. Thus $\operatorname{codim}_X Z_i=i$ if Z_i is nonempty. We denote by $C_{i,j}$, $j=1,2,\dots$, the connected components of $Z_i\smallsetminus Z_{i+1}$. Each $C_{i,j}$ is a smooth connected complex analytic submanifold of X of codimension i. We set $C_{0,1}=X\smallsetminus D$.

Let $\Omega^\ell_{C_{i,j}}$ be the sheaf of holomorphic differential ℓ-forms on $C_{i,j}$. Let $f:C_{i,j}\hookrightarrow X$ be the natural embedding and $f_*\Omega^\ell_{C_{i,j}}$ the direct image sheaf. We denote

$$\Omega^\ell_{X,\mathscr{Z}}=\bigoplus_{i,j}f_*\Omega^{\ell-2i}_{C_{i,j}}.$$

Let $d_{\mathscr{L}}:f_*\Omega^\ell_{C_{i,j}}\otimes\mathbb{C}^r\to f_*\Omega^{\ell+1}_{C_{i,j}}\otimes\mathbb{C}^r$ be the differential of the connection $\nabla|_{C_{i,j}}$ and

$$\operatorname{res}:f_*\Omega^\ell_{C_{i,j}}\otimes\mathbb{C}^r\longrightarrow f_*\Omega^{\ell-1}_{C_{i+1,j'}}\otimes\mathbb{C}^r$$

the residue map, if $C_{i+1,j'}$ lies in the closure $\overline{C_{i,j}}$, and the zero map otherwise. The map $\widetilde{d}=d_{\mathscr{L}}+\operatorname{res}$ defines the complex of sheaves on X,

$$0\longrightarrow\Omega^0_{X,\mathscr{Z}}\otimes\mathbb{C}^r\xrightarrow{\widetilde{d}}\Omega^1_{X,\mathscr{Z}}\otimes\mathbb{C}^r\xrightarrow{\widetilde{d}}\Omega^2_{X,\mathscr{Z}}\otimes\mathbb{C}^r\xrightarrow{\widetilde{d}}\cdots$$

The natural embeddings $\Omega^\ell_X\otimes\mathbb{C}^r\hookrightarrow\Omega^\ell_{C_{0,1}}\otimes\mathbb{C}^r\hookrightarrow\Omega^\ell_{X,\mathscr{Z}}\otimes\mathbb{C}^r$ define an injective homomorphism of complexes

$$(\Omega^\bullet_X\otimes\mathbb{C}^r,d_{\mathscr{L}})\longleftrightarrow(\Omega^\bullet_{X,\mathscr{Z}}\otimes\mathbb{C}^r,\widetilde{d}).\tag{2.1}$$

Theorem 2.1. — *The homomorphism* (2.1) *is a quasi-isomorphism.*

Proof. — It is enough to check this statement locally on X. In that case we may assume that $X=\{z=(z_1,\dots,z_k)\in\mathbb{C}^k\mid |z|<1\}$ and D is the union of several coordinate hyperplanes in X. For that example, the statement is checked by direct calculation. □

2.3. Residue complex of global sections. — Let $\Gamma(C_{i,j},\Omega^{\ell}_{C_{i,j}})$ be the space of global sections of $\Omega^{\ell}_{C_{i,j}}$. Denote

$$\Gamma^{\ell}(X,\mathscr{X};\mathbb{C}^r) = \bigoplus_{i,j}\Gamma(C_{i,j},\Omega^{\ell-2i}_{C_{i,j}})\otimes\mathbb{C}^r.$$

The map $\widetilde{d}=d_{\mathscr{L}}+\operatorname{res}$ defines the complex of vector spaces

$$0\longrightarrow\Gamma^0(X,\mathscr{X};\mathbb{C}^r)\xrightarrow{\ \widetilde{d}\ }\Gamma^1(X,\mathscr{X};\mathbb{C}^r)\xrightarrow{\ \widetilde{d}\ }\Gamma^2(X,\mathscr{X};\mathbb{C}^r)\xrightarrow{\ \widetilde{d}\ }\cdots$$

Theorem 2.2. — *In addition to assumptions of Sections* 2.1 *and* 2.2*, we assume that for any i,j, the manifold $C_{i,j}$ is a Stein manifold. Then there is the natural isomorphism $H^{\bullet}(X;\mathscr{L})\simeq H^{\bullet}(\Gamma^{\bullet}(X,\mathscr{X};\mathbb{C}^r),\widetilde{d})$.*

Proof. — For the Stein manifold $C_{i,j}$ the complex $(\Gamma(C_{i,j},\Omega^{\bullet}_{C_{i,j}})\otimes\mathbb{C}^r,d_{\mathscr{L}})$ calculates $H^{\bullet}(C_{i,j};\mathscr{L})$. This fact and Theorem 2.1 imply Theorem 2.2. □

3. Logarithmic residue complex of Orlik-Solomon forms

3.1. Affine arrangements. — Let $\mathscr{A}=\{H_i\}_{i\in I}$ be an affine arrangement of hyperplanes, i.e., $\{H_i\}_{i\in I}$ is a finite nonempty collection of distinct hyperplanes in the affine complex space $\mathbb{C}^k$. Denote $U=\mathbb{C}^k\smallsetminus\bigcup_{i\in I}H_i$. We denote by Ω^{ℓ}_U the sheaf of holomorphic ℓ-forms on U.

For any $i\in I$, choose a degree one polynomial function f_i on $\mathbb{C}^k$ whose zero locus equals H_i. Define $\omega_i=d\log f_i=df_i/f_i\in\Gamma(U,\Omega^1_U)$. Given a natural number r, we choose matrices $P_i\in\operatorname{End}(\mathbb{C}^r)$, $i\in I$. Denote

$$\omega=\sum_{i\in I}\omega_i\otimes P_i\ \in\ \Gamma(U,\Omega^1_U)\otimes\operatorname{End}(\mathbb{C}^r).$$

The form ω defines the connection $d+\omega$ on the trivial bundle $U\times\mathbb{C}^r\to U$. We suppose that $d+\omega$ is flat. Let $\mathscr{L}$ be the sheaf on U of flat sections. Then $(\Omega^{\bullet}_U\otimes\mathbb{C}^r,d_{\mathscr{L}})$ is the complex of sheaves of $\mathbb{C}^r$-valued holomorphic differential forms on U with differential $d_{\mathscr{L}}=d+\omega$.

Define finite dimensional *Orlik-Solomon subspaces* $A^p(\mathscr{A})\subset\Gamma(U,\Omega^p_U)$ as the $\mathbb{C}$-linear subspaces generated by all forms $\omega_{i_1}\wedge\cdots\wedge\omega_{i_p}$. Then the exterior multiplication by ω defines the complex

$$0\longrightarrow A^0\otimes\mathbb{C}^r\xrightarrow{\ \omega\ }A^1\otimes\mathbb{C}^r\xrightarrow{\ \omega\ }A^2\otimes\mathbb{C}^r\xrightarrow{\ \omega\ }\cdots$$

as a subcomplex of $(\Gamma(U,\Omega^{\bullet}_U\otimes\mathbb{C}^r),d_{\mathscr{L}})$. We call $(A^{\bullet}\otimes\mathbb{C}^r,\omega)$ the *Aomoto complex* of $(U,d+\omega)$.

Let Y be any smooth compactification of $\mathbb{C}^k$ such that H_∞ is a divisor. Write $H=H_\infty\cup(\bigcup_{i\in I}H_i)$. Then $U=Y\smallsetminus H$. (Typical examples for Y include the complex projective space $\mathbb{P}^k$, $(\mathbb{P}^1)^k$ and any toric compactification of $\mathbb{C}^k$.) Note that ω can be uniquely extended to be an $\operatorname{End}(\mathbb{C}^r)$-valued rational 1-form ω on Y.

Theorem 3.1 ([ESV92, STV95]). — *Suppose $\pi:X\to Y$ is a blow-up of Y with centers in H such that 1) X is nonsingular, 2) $\pi^{-1}H$ is a normal crossing divisor,* *3) none of the eigenvalues of the residue of $\pi^*\omega$ along any component of $\pi^{-1}H$ is a positive integer. Then the inclusion $(A^\bullet \otimes \mathbb{C}^r, \omega) \hookrightarrow (\Gamma(U, \Omega^\bullet_U) \otimes \mathbb{C}^r, d_{\mathscr{L}})$ is a quasi-isomorphism.*

Remark. — Assume that the pair (X, ω) satisfies conditions 1) and 2) of Theorem 3.1 but not condition 3). Then for almost all $\kappa \in \mathbb{C}^\times$, the pair $(X, \omega/\kappa)$ satisfies all of the conditions 1)-3) of Theorem 3.1.

3.2. Orlik-Solomon manifolds. — Let X be a smooth connected complex analytic manifold, $\dim X = k$. Let $D \subset X$ be a divisor with normal crossings and $\mathscr{Z} = \{X = Z_0 \supset D = Z_1 \supset Z_2 \supset \cdots\}$ the associated filtration of X by closed subsets. We denote by $C_{i,j}$, $j = 1, 2, \ldots$, the connected components of $Z_i \smallsetminus Z_{i+1}$ and set $C_{0,1} = X \smallsetminus D$.

Assume that for any $C_{i,j}$ we have:

(i) An affine arrangement $\mathscr{A}_{i,j} = \{H_m\}_{m \in I_{i,j}}$ in $\mathbb{C}^{k-i}$ with complement $U_{i,j} = \mathbb{C}^{k-i} \smallsetminus \bigcup_{m \in I_{i,j}} H_m$ and an analytic isomorphism $\varphi_{i,j} : U_{i,j} \to C_{i,j}$.

Assume that these objects have the following property.

(ii) For any i, j, denote by $A^\bullet(U_{i,j})$ the Orlik-Solomon spaces of $U_{i,j}$. Let $C_{i+1,j'}$ lie in the closure $\overline{C_{i,j}}$ and

$$\mathrm{res} : \Gamma(C_{i,j}, \Omega^\ell_{C_{i,j}}) \longrightarrow \Gamma(C_{i+1,j'}, \Omega^{\ell-1}_{C_{i+1,j'}})$$

the residue map. Then the image of $A^\bullet(U_{i,j})$ under the composition $(\varphi_{i+1,j'})^* \circ \mathrm{res} \circ ((\varphi_{i,j})^{-1})^*$ lies in $A^\bullet(U_{i+1,j'})$

We say that (X, D) is an *Orlik-Solomon manifold* if it has charts (i) with property (ii).

The images of Orlik-Solomon spaces $A^\bullet(U_{i,j})$ under the isomorphism $\varphi_{i,j}$ give finite-dimensional subspaces of $\Gamma(C_{i,j}, \Omega^\bullet_{C_{i,j}})$. We call these subspaces the *Orlik-Solomon spaces of* $C_{i,j}$ and denote by $A^\bullet(C_{i,j})$.

Remark. — Denote by $K = \{(0,1), \ldots\}$ the set of all pairs (i, j) appearing as indices of components $C_{i,j}$ in the decomposition of the pair (X, D). Let $K_0 \subset K$ be any subset which does not contain $(0,1)$. Denote $C_{K_0} \subset X$ the closure of $\bigcup_{(i,j) \in K_0} C_{i,j}$. Denote $X_{K_0} = X \smallsetminus C_{K_0}$, $D_{K_0} = D \smallsetminus C_{K_0}$. Then X_{K_0} is a smooth connected complex analytic manifold and $D_{K_0} \subset X_{K_0}$ is a divisor with normal crossings. If (X, D) is an Orlik-Solomon manifold, then (X_{K_0}, D_{K_0}) has the induced structure of an Orlik-Solomon manifold.

We describe examples of Orlik-Solomon manifolds in Section 4.2.

3.3. Aomoto complexes. — Assume that (X, D) is an Orlik-Solomon manifold and $\nabla = d_{\mathscr{L}} = d + \omega$ is a holomorphic flat connection on $X \times \mathbb{C}^r \to X$. We say that ∇ is a *flat connection with logarithmic singularities on the Orlik-Solomon manifold* if the following property holds.

(iii) For any i, j, the induced flat connection $\nabla_{i,j} := (\varphi_{i,j})^* \nabla$ on $U_{i,j}$ has the form described in Section 3.1. Namely, $\nabla_{i,j} = d + \omega_{i,j}$, where

$$\omega_{i,j} = \sum_{m \in I_{i,j}} \omega_m \otimes P_m$$

for suitable matrices $P_m \in \mathrm{End}(\mathbb{C}^r)$.

If ∇ is a flat connection with logarithmic singularities on the Orlik-Solomon manifold (X, D), then the exterior multiplication by ω defines a finite-dimensional complex $(A^\bullet(C_{i,j}) \otimes \mathbb{C}^r, \omega)$ as a subcomplex of $(\Gamma(C_{i,j}, \Omega^\bullet_{C_{i,j}}) \otimes \mathbb{C}^r, d_{\mathscr{L}} = d + \omega)$.

We denote

$$A^\ell(X, \mathscr{Z}; \mathbb{C}^r) = \bigoplus_{i,j} A^{\ell - 2i}(C_{i,j}) \otimes \mathbb{C}^r.$$

The map $\omega + \mathrm{res}$ realizes the complex

$$0 \longrightarrow A^0(X, \mathscr{Z}; \mathbb{C}^r) \xrightarrow{\omega + \mathrm{res}} A^1(X, \mathscr{Z}; \mathbb{C}^r) \xrightarrow{\omega + \mathrm{res}} A^2(X, \mathscr{Z}; \mathbb{C}^r) \xrightarrow{\omega + \mathrm{res}} \cdots$$

as a subcomplex of $(\Gamma^\bullet(X, \mathscr{Z}; \mathbb{C}^r), \widetilde{d})$.

Theorem 3.2. — *Assume that $\nabla = d + \omega$ is a flat connection with logarithmic singularities on the Orlik-Solomon manifold (X, D). Assume that for any i, j, the form $\omega_{i,j}$ on $U_{i,j}$ satisfies the conditions of Theorem* 3.1 *for a suitable resolution of singularities mentioned in Theorem* 3.1*. Then the embedding $(A^\bullet(X, \mathscr{Z}; \mathbb{C}^r), \omega + \mathrm{res}) \hookrightarrow (\Gamma^\bullet(X, \mathscr{Z}; \mathbb{C}^r), \widetilde{d})$ is a quasi-isomorphism.*

Proof. — By Theorem 3.1, the embedding

$$(A^\bullet(C_{i,j}) \otimes \mathbb{C}^r, \omega) \longrightarrow (\Gamma(C_{i,j}, \Omega^\bullet_{C_{i,j}}) \otimes \mathbb{C}^r, d_{\mathscr{L}})$$

is a quasi-isomorphism. This implies Theorem 3.2. □

Corollary 3.3. — *Assume that $\nabla = d + \omega$ is a flat connection with logarithmic singularities on the Orlik-Solomon manifold (X, D). For $\kappa \in \mathbb{C}^\times$, consider the flat connection $\nabla_\kappa = d + \omega/\kappa$ and the associated embedding $(A^\bullet(X, \mathscr{Z}; \mathbb{C}^r), \omega/\kappa + \mathrm{res}) \hookrightarrow (\Gamma^\bullet(X, \mathscr{Z}; \mathbb{C}^r), d + \omega/\kappa + \mathrm{res})$. Then for generic κ this embedding is a quasi-isomorphism.*

4. Resolution of singularities of arrangements

4.1. Minimal resolution of a hyperplane-like divisor. — Let Y be a smooth connected complex analytic manifold and H a divisor. The divisor H is *hyperplane-like* if Y can be covered by coordinate charts such that in each chart H is the union of hyperplanes. Such charts are called *linearizing*.

Let H be a hyperplane-like divisor, V a linearizing chart. A *local edge* of H in V is any nonempty irreducible intersection in V of hyperplanes of H in V. A local edge is *dense* if the subarrangement of all hyperplanes in V containing the edge is irreducible: the hyperplanes cannot be partitioned into nonempty sets so that, after a change of coordinates, hyperplanes in different sets are in different coordinates. In particular, each hyperplane is a dense edge. An *edge* of H is the maximal analytic continuation in Y

of a local edge. An edge is called *dense* if it is locally dense. Any edge is an immersed submanifold in Y. The irreducible components of H are considered to be dense.

Let $H \subset Y$ be a hyperplane-like divisor. Let $\pi : \widetilde{Y} \to Y$ be the minimal resolution of singularities of H in Y. The minimal resolution is constructed by first blowing-up dense vertices of H, then by blowing-up the proper preimages of dense one-dimensional edges of H and so on, see [DCP95, Var95, STV95].

We have two basic examples of pairs (Y, H).

4.1.1. *Projective arrangement*. — Let $\mathscr{A} = \{H_\ell\}_{\ell \in I}$ be a nonempty finite collection of distinct hyperplanes in the complex projective space $\mathbb{P}^k$. Denote $H = \bigcup_{\ell \in I} H_\ell$. Then $H \subset \mathbb{P}^k$ is a hyperplane-like divisor. Denote $U = \mathbb{P}^k \smallsetminus H$.

For any $\ell, m \in I$, we have $H_\ell \smallsetminus H_m = \operatorname{div}(f_{\ell,m})$ for some rational function $f_{\ell,m}$ on $\mathbb{P}^k$. Define $\omega_{\ell,m} = d \log f_{\ell,m}$. For $1 \leqslant p \leqslant k$, we define the Orlik-Solomon space $A^p(U)$ as the $\mathbb{C}$-linear span of $\omega_{\ell_1,m_1} \wedge \cdots \wedge \omega_{\ell_p,m_p}$.

Given a natural number r, we choose matrices $P_\ell \in \operatorname{End}(\mathbb{C}^r)$, $\ell \in I$, such that $\sum_\ell P_\ell = 0$. Fix $m \in I$ and define

$$\omega = \sum_{\ell \in I} \omega_{\ell,m} \otimes P_\ell.$$

The form ω defines the connection $d + \omega$ on $U \times \mathbb{C}^r \to U$. We call $d + \omega$ a *connection with logarithmic singularities* on the complement of the projective arrangement.

4.1.2. *Discriminantal arrangement*. — Let $Y = (\mathbb{P}^1)^k$. For $\ell = 1, \ldots, k$, we fix an affine coordinate t_ℓ on the ℓ-th factor of Y. For $1 \leqslant \ell < m \leqslant k$, the subset $H_{\ell,m} \subset Y$ defined by the equation $t_\ell - t_m = 0$ is called a *diagonal* hyperplane. For ℓ, $1 \leqslant \ell \leqslant k$ and $z \in \mathbb{C} \cup \{\infty\}$, the subset $H_\ell(z) \subset Y$ defined by the equation $t_\ell - z = 0$ is called a *coordinate* hyperplane. If $z \in \mathbb{C}$ (resp. $z = \infty$), we call the coordinate hyperplane *finite* (resp. *infinite*).

A *discriminantal* arrangement in Y is a finite collection of diagonal and coordinate hyperplanes, which includes all infinite coordinate hyperplanes $H_\ell(\infty)$, $\ell = 1, \ldots, k$, see [SV91]. Define by H the union of all of the hyperplanes of the arrangement. Then $H \subset Y$ is a hyperplane-like divisor. Denote $U = Y \smallsetminus H$.

To every diagonal hyperplane $H_{\ell,m}$ we assign the 1-form $\omega_{H_{\ell,m}} = d \log(t_\ell - t_m)$. To every finite coordinate hyperplane $H_\ell(z)$ we assign the 1-form $\omega_{H_\ell(z)} = d \log(t_\ell - z)$. These are holomorphic forms on U. We define the Orlik-Solomon spaces $A^\bullet(U)$ as the graded components of the exterior $\mathbb{C}$-algebra generated by the 1-forms associated with the diagonal and finite coordinate hyperplanes.

Fix a natural number r. For any diagonal or finite coordinate hyperplane H of the arrangement we choose a matrix $P_H \in \operatorname{End}(\mathbb{C}^r)$. Define

$$\omega = \sum \omega_H \otimes P_H,$$

where the sum is over all diagonal and finite coordinate hyperplanes of the discriminantal arrangement. This form ω defines the connection $d + \omega$ on the trivial bundle $U \times \mathbb{C}^r \to U$. We call $d + \omega$ a *connection with logarithmic singularities* on the complement of the discriminantal arrangement.

4.2. Examples of Orlik-Solomon manifolds

4.2.1. *Minimal resolution of singularities of a projective arrangement*

Let $\mathscr{A} = \{H_\ell\}_{\ell\in I}$ be a projective arrangement of hyperplanes in $\mathbb{P}^k$. Denote $Y = \mathbb{P}^k$ and $H = \bigcup_{\ell\in I} H_\ell$. Let $\pi : \widetilde{Y} \to Y$ be the minimal resolution of singularities of H in Y and $\widetilde{H} = \pi^{-1}H$. Then $\widetilde{H} \subset \widetilde{Y}$ is a divisor with normal crossings. For the pair $(\widetilde{Y}, \widetilde{H})$, we introduce components $C_{i,j} \subset \widetilde{Y}$ as in Section 2. It is clear from the construction of the minimal resolution that each $C_{i,j}$ is naturally isomorphic to the complement of an affine arrangement and these isomorphisms have property (ii) of Section 3.2. Thus $(\widetilde{Y}, \widetilde{H})$ has the *natural structure* of an Orlik-Solomon manifold.

4.2.2. *Minimal resolution of singularities of a discriminantal arrangement*

Let $\mathscr{A} = \{H_\ell\}_{\ell\in I}$ be a discriminantal arrangement of hyperplanes in $(\mathbb{P}^1)^k$. Denote $Y = (\mathbb{P}^1)^k$ and $H = \bigcup_{\ell\in I} H_\ell$. Let $\pi : \widetilde{Y} \to Y$ be the minimal resolution of singularities of H in Y and $\widetilde{H} = \pi^{-1}H$. Then $\widetilde{H} \subset \widetilde{Y}$ is a divisor with normal crossings. For the pair $(\widetilde{Y}, \widetilde{H})$, we introduce components $C_{i,j} \subset \widetilde{Y}$ as in Section 2. It is clear from the construction of the minimal resolution that each $C_{i,j}$ is naturally isomorphic to the complement of an affine arrangement and these isomorphisms have property (ii) of Section 3.2. Thus $(\widetilde{Y}, \widetilde{H})$ has the *natural structure* of an Orlik-Solomon manifold.

5. Weighted arrangements

5.1. Weighted projective arrangement. — Let $\mathscr{A} = \{H_\ell\}_{\ell\in I}$ be a projective arrangement of hyperplanes in $Y = \mathbb{P}^k$. Denote $H = \bigcup_{\ell\in I} H_\ell$, $U = Y \smallsetminus H$.

The arrangement $\mathscr{A}$ is *weighted* if a map $a : I \to \mathbb{C}$, $\ell \mapsto a_\ell$, is given such that $\sum_{\ell\in I} a_\ell = 0$. The number a_ℓ is called the *weight* of H_ℓ. Let X_α be an edge of $\mathscr{A}$. Denote $I_\alpha = \{\ell \in I \mid H_\ell \supset X_\alpha\}$. The number $a_\alpha = \sum_{\ell\in I_\alpha} a_\ell$ is called the *weight* of X_α. The edge X_α is *resonant* if $a_\alpha = 0$.

Fix $m \in I$ and define

$$\omega_a = \sum_{\ell\in I} \omega_{\ell,m} \otimes a_\ell,$$

see Section 4.1.1. The form ω_a defines the flat connection $d + \omega_a$ on $U \times \mathbb{C} \to U$. We call $d + \omega_a$ the *connection associated with weights a*.

Let $\pi : \widetilde{Y} \to Y$ be the minimal resolution of singularities of H. Denote $\widetilde{H} = \pi^{-1}H$. The irreducible components of $\widetilde{H}$ are labeled by dense edges X_α of H. Such a component will be denoted by $\widetilde{H}_\alpha$. Consider $(\widetilde{Y}, \widetilde{H})$ with its natural structure of an Orlik-Solomon manifold, see Section 4.2.1.

Denote $\widetilde{\omega}_a = \pi^*\omega_a$. The form $\widetilde{\omega}_a$ is regular on an irreducible component of $\widetilde{H}$ if and only if the corresponding dense edge of H is resonant.

Let J be the set of all nonresonant dense edges of H and $\widetilde{J}$ any set of dense edges such that $J \subseteq \widetilde{J}$. Denote $\widetilde{H}_{\widetilde{J}} = \bigcup_{X_\alpha\in\widetilde{J}} \widetilde{H}_\alpha$, $X = \widetilde{Y} \smallsetminus \widetilde{H}_{\widetilde{J}}$, $D = \widetilde{H} \smallsetminus \widetilde{H}_{\widetilde{J}}$. Then (X, D) is the Orlik-Solomon manifold with respect to the structure induced from $(\widetilde{Y}, \widetilde{H})$, see Section 3.2. The form $\widetilde{\omega}_a$ is regular on X and $d+\widetilde{\omega}_a$ is a flat connection with logarithmic singularities on the Orlik-Solomon manifold (X, D). Thus we may construct the associated complex $(A^\bullet(X,\mathscr{X}),\widetilde{\omega}_a+\mathrm{res})$ and apply Theorem 3.2 and Corollary 3.3 to the triple $(X,D,d+\widetilde{\omega}_a)$. The complex $(A^\bullet(X,\mathscr{X}),\widetilde{\omega}_a+\mathrm{res})$ will be called the *Aomoto complex of the weighted Orlik-Solomon manifold* (X,D).

5.2. Weighted discriminantal arrangement. — Let $\mathscr{A}=\{H_\ell\}_{\ell\in I}$ be a discriminantal arrangement of hyperplanes in $Y=(\mathbb{P}^1)^k$. Denote $H=\bigcup_{\ell\in I}H_\ell$, $U=Y\smallsetminus H$.

According to the definition in Section 4.1.2, the discriminantal arrangement contains the infinite coordinate hyperplanes $H_p(\infty)$, $p=1,\dots,k$. Let $I_{\mathrm{fin}}\subset I$ be the set of indices of the remaining hyperplanes of $\mathscr{A}$.

The discriminantal arrangement $\mathscr{A}$ is *weighted* if a map $a:I_{\mathrm{fin}}\to\mathbb{C}$, $\ell\mapsto a_\ell$, is given. The number a_ℓ is the *weight* of $H_\ell,\ell\in I_{\mathrm{fin}}$. We also write $a(H_\ell):=a_\ell$.

We extend this map to the map $a:I\to\mathbb{C}$ as follows. We set the weight of an infinite coordinate hyperplane $H_p(\infty)$ to be the number $-\sum a_q$ where the sum is over all $q\in I_{\mathrm{fin}}$ such that H_q is of the form $t_p-t_i=0$ for some i or of the form $t_p-z=0$ for some $z\in\mathbb{C}$.

Let X_α be an edge of $\mathscr{A}$. Denote $I_\alpha=\{\ell\in I\mid H_\ell\supset X_\alpha\}$. The number $a_\alpha=\sum_{\ell\in I_\alpha}a_\ell$ is the *weight* of X_α. The edge X_α is *resonant* if $a(X_\alpha)=0$.

We define
$$\omega_a=\sum_{\ell\in I_{\mathrm{fin}}}\omega_{H_\ell}\otimes a_\ell,$$
see Section 4.1.2. The form ω_a defines the flat connection $d+\omega_a$ on $U\times\mathbb{C}\to U$. We call $d+\omega_a$ the *connection associated with weights* a.

Let $\pi:\widetilde{Y}\to Y$ be the minimal resolution of singularities of H. Denote $\widetilde{H}=\pi^{-1}H$. The irreducible components of $\widetilde{H}$ are labeled by dense edges X_α of H. Such a component component will be denoted by $\widetilde{H}_\alpha$. Consider $(\widetilde{Y},\widetilde{H})$ as the Orlik-Solomon manifold, see Section 4.2.2.

Denote $\widetilde{\omega}_a=\pi^*\omega_a$. The form $\widetilde{\omega}_a$ is regular on an irreducible component of $\widetilde{H}$ if and only if the corresponding dense edge of H is resonant.

Let J be the set of all nonresonant dense edges of H and $\widetilde{J}$ any subset of dense edges such that $J\subseteq\widetilde{J}$. Denote $\widetilde{H}_{\widetilde{J}}=\bigcup_{X_\alpha\in\widetilde{J}}\widetilde{H}_\alpha$, $X=\widetilde{Y}\smallsetminus\widetilde{H}_{\widetilde{J}}$, $D=\widetilde{H}\smallsetminus\widetilde{H}_{\widetilde{J}}$. Then (X,D) is the Orlik-Solomon manifold with respect to the structure induced from $(\widetilde{Y},\widetilde{H})$, see Section 3.2. The form $\widetilde{\omega}_a$ is regular on X and $d+\widetilde{\omega}_a$ is a flat connection with logarithmic singularities on the Orlik-Solomon manifold (X,D). Thus we may construct the associated complex $A^\bullet(X,\mathscr{X},\widetilde{\omega}_a+\mathrm{res})$ and apply Theorem 3.2 and Corollary 3.3 to the triple $(X,D,d+\widetilde{\omega}_a)$. The complex $A^\bullet(X,\mathscr{X},\widetilde{\omega}_a+\mathrm{res})$ will be called the *Aomoto complex of the weighted Orlik-Solomon manifold* (X,D).

6. Highest weight representations of $\mathfrak{sl}_2$

6.1. Modules. — Consider the complex Lie algebra $\mathfrak{sl}_2$ with standard basis e,f,h such that $[e,f]=h$, $[h,e]=2e$, $[h,f]=-2f$. We have $\mathfrak{sl}_2=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$, where $\mathfrak{n}_-=\mathbb{C}f,\mathfrak{h}=\mathbb{C}h$, $\mathfrak{n}_+=\mathbb{C}e$.

Let V be an $\mathfrak{sl}_2$-module. For $\lambda\in\mathbb{C}$, let $V[\lambda]=\{v\in V\mid hv=\lambda v\}$ be the subspace of weight λ. Assume that V has weight decomposition $V=\bigoplus_\lambda V[\lambda]$ with finite-dimensional spaces $V[\lambda]$. Then the *restricted dual* of V is $V^* := \bigoplus_\lambda V[\lambda]^*$. The restricted dual has the *contragredient* $\mathfrak{sl}_2$-module structure: for $\varphi \in V^*$, we have $\langle e\varphi, v\rangle = \langle \varphi, fv\rangle$, $\langle f\varphi, v\rangle = \langle \varphi, ev\rangle$, $\langle h\varphi, v\rangle = \langle \varphi, hv\rangle$. We have $V[\lambda]^* = V^*[\lambda]$ for any λ.

For the Lie algebra $\mathfrak{n}_-$ and a module V we denote $C_\bullet(\mathfrak{n}_-, V)$ the standard complex of $\mathfrak{n}_-$ with coefficients in V,

$$0 \longrightarrow C_0(\mathfrak{n}_-, V) \longrightarrow C_1(\mathfrak{n}_-, V) \longrightarrow 0,$$

where $C_0(\mathfrak{n}_-, V) = \mathfrak{n}_- \otimes V$, $C_1(\mathfrak{n}_-, V) = V$, and the map is $f \otimes v \mapsto fv$. We have the weight decomposition

$$C_\bullet(\mathfrak{n}_-, V) = \bigoplus_\lambda C_\bullet(\mathfrak{n}_-, V)[\lambda],$$

where $C_\bullet(\mathfrak{n}_-, V)[\lambda]$ is

$$0 \longrightarrow \mathfrak{n}_- \otimes V[\lambda+2] \longrightarrow V[\lambda] \longrightarrow 0. \tag{6.1}$$

6.2. Verma modules. — For $m \in \mathbb{C}$, the Verma module M_m is the infinite dimensional $\mathfrak{sl}_2$-module generated by a single vector v_m such that $hv_m = mv_m$ and $ev_m = 0$. The vectors $f^j v_m$, $j = 0, 1, \dots$, form a basis of M_m. The action is given by the formulas

$$f \cdot f^j v_m = f^{j+1} v_m, \quad h \cdot f^j v_m = (m-2j) f^j v_m, \quad e \cdot f^j v_m = j(m-j+1) f^{j-1} v_m.$$

Consider the contragredient module M_m^* with the basis φ_m^j, $j \in \mathbb{Z}_{\geqslant 0}$, dual to the basis $f^j v_m$ of M_m. We have

$$f \cdot \varphi_m^j = (j+1)(m-j)\varphi_m^{j+1}, \quad h \cdot \varphi_m^j = (m-2j)\varphi_m^j, \quad e \cdot \varphi_m^j = \varphi_m^{j-1}.$$

The Shapovalov symmetric bilinear form on M_m is defined by the conditions

$$S(v_m, v_m) = 1, \qquad S(fx, y) = S(x, ey),$$

for all $x, y \in M_m$. The Shapovalov form defines the morphism of modules

$$S : M_m \longrightarrow M_m^*, \qquad x \longmapsto S(x, \cdot).$$

The image $L_m := \operatorname{Im}(S) \hookrightarrow M_m^*$ is irreducible.

If $m \notin \mathbb{Z}_{\geqslant 0}$, then M_m is irreducible, otherwise the subspace with basis $f^j v_m$, $j \geqslant m+1$, is a submodule which is identified with the Verma module M_{-m-2} under the map $M_{-m-2} \hookrightarrow M_m$, $f^j v_{-m-2} \mapsto f^{j+m+1} v_m$. The quotient M_m/M_{-m-2} is an irreducible module with basis induced by $v_m, fv_m, \dots, f^m v_m$. The submodule $M_{-m-2} \hookrightarrow M_m$ is the kernel of the Shapovalov form. The induced Shapovalov form on M_m/M_{-m-2} identifies M_m/M_{-m-2} and $L_m \hookrightarrow M_m^*$.

We have the exact sequence of $\mathfrak{sl}_2$-modules

$$0 \longrightarrow L_m \longrightarrow M_m^* \longrightarrow M_{-m-2}^* \longrightarrow 0,$$

which is called the *BGG resolution* of the irreducible $\mathfrak{sl}_2$-module L_m, see [BGG75]. We will keep two terms of this sequence

$$M_m^* \xrightarrow{\iota} M_{-m-2}^* \tag{6.2}$$

in which the epimorphism is denoted by ι. We consider this map as a complex with terms in degree 0 and 1.

6.3. Tensor product of Verma modules. — For a vector $\boldsymbol{m} = (m_1, \dots, m_n) \in \mathbb{C}^n$, denote $|\boldsymbol{m}| = m_1 + \dots + m_n$. Consider the tensor product $\bigotimes_{a=1}^n M_{m_a}$ of Verma modules. For $J = (j_1, \dots, j_n) \in \mathbb{Z}_{\geqslant 0}^n$, let

$$f^J v_{\boldsymbol{m}} := f^{j_1} v_{m_1} \otimes \dots \otimes f^{j_n} v_{m_n}.$$

The vectors $f^J v_{\boldsymbol{m}}$ form a basis of $\bigotimes_{a=1}^n M_{m_a}$. We have

$$f \cdot f^J v_{\boldsymbol{m}} = \sum_{a=1}^n f^{J+1_a} v_{\boldsymbol{m}}, \quad h \cdot f^J v_{\boldsymbol{m}} = (|\boldsymbol{m}| - 2|J|) f^J v_{\boldsymbol{m}},$$
$$e \cdot f^J v_{\boldsymbol{m}} = \sum_{a=1}^n j_a(m_a - j_a + 1) f^{J-1_a} v_{\boldsymbol{m}},$$

where $J \pm 1_a = (j_1, \dots, j_a \pm 1, \dots, j_n)$.

We have the weight decomposition

$$\bigotimes_{a=1}^n M_{m_a} = \bigoplus_{k=0}^\infty \Big(\bigotimes_{a=1}^n M_{m_a}\Big)[|\boldsymbol{m}| - 2k].$$

The basis in $\big(\bigotimes_{a=1}^n M_{m_a}\big)[|\boldsymbol{m}| - 2k]$ is formed by the monomials $f^J v_{\boldsymbol{m}}$ with $|J| = k$.

Consider the restricted dual space $\big(\bigotimes_{a=1}^n M_{m_a}\big)^*$ with the weight decomposition

$$\Big(\bigotimes_{a=1}^n M_{m_a}\Big)^* = \bigoplus_{k=0}^\infty \Big(\bigotimes_{a=1}^n M_{m_a}\Big)^*[|\boldsymbol{m}| - 2k].$$

The basis of $\big(\bigotimes_{a=1}^n M_{m_a}\big)^*[|\boldsymbol{m}| - 2k]$ is formed by vectors

$$\varphi_{\boldsymbol{m}}^J := \varphi_{m_1}^{j_1} \otimes \dots \otimes \varphi_{m_n}^{j_n}$$

with $|J| = k$.

The $\mathfrak{sl}_2$-action is given by the formulas

$$f\cdot\varphi_{\boldsymbol{m}}^J = \sum_{a=1}^n (j_1+1)(m-j_a)\varphi_{\boldsymbol{m}}^{J+1_a}, \quad h\cdot\varphi_{\boldsymbol{m}}^J = (|\boldsymbol{m}|-2|J|)\varphi_{\boldsymbol{m}}^{J+1_a}, \quad e\cdot\varphi_{\boldsymbol{m}}^J = \sum_{a=1}^n \varphi_{\boldsymbol{m}}^{J-1_a}.$$

6.4. Tensor product of complexes. — Let coordinates of $\boldsymbol{m} = (m_1, \dots, m_n)$ be positive integers. For $a = 1, \dots, n$, denote by $A_{m_a}^0 \xrightarrow{\iota_a} A_{m_a}^1$ the complex $M_{m_a}^* \xrightarrow{\iota_a} M_{-m_a-2}^*$. Consider the tensor product $(A_{\boldsymbol{m}}^\bullet, \iota)$ of these complexes, where

$$A_{\boldsymbol{m}}^i = \bigoplus_{i_1+\dots+i_n=i} A_{m_1}^{i_1} \otimes \dots \otimes A_{m_n}^{i_n}, \qquad i = 0, \dots, n,$$

with differential

$$\iota : x_1 \otimes \dots \otimes x_n \longmapsto \sum_{a=1}^n (-1)^{\deg x_1 + \dots + \deg x_{a-1}} x_1 \otimes \dots \otimes \iota_a x_a \otimes \dots \otimes x_n.$$

The differential is a morphism of $\mathfrak{sl}_2$-modules. We have

$$\bigotimes_{a=1}^n L_a = \ker(\iota : A_{\boldsymbol{m}}^0 \longrightarrow A_{\boldsymbol{m}}^1).$$

At all other degrees the complex $(A^\bullet_{\boldsymbol{m}}, \iota)$ is acyclic. Thus $(A^\bullet_{\boldsymbol{m}}, \iota)$ gives the resolution of $\bigotimes_{a=1}^n L_a$ which we will call the *BGG resolution* of $\bigotimes_{a=1}^n L_a$.

Consider the complex $C_\bullet(\mathfrak{n}_-, A^\bullet_{\boldsymbol{m}})$,

$$\mathfrak{n}_- \otimes A^\bullet_{\boldsymbol{m}} \xrightarrow{\ f\ } A^\bullet_{\boldsymbol{m}}. \tag{6.3}$$

The differential f of this complex commutes with the differential ι acting on $A^\bullet_{\boldsymbol{m}}$. Consider the complex $(B^\bullet_{\boldsymbol{m}}, \widetilde{d})$, where

$$B^i_{\boldsymbol{m}} = (\mathfrak{n}_- \otimes A^i_{\boldsymbol{m}}) \oplus A^{i-1}_{\boldsymbol{m}}, \qquad i = 0, \ldots, n+1,$$

and

$$\widetilde{d}\colon\ f \otimes x + y \ \longmapsto\ fx - f \otimes \iota x + \iota y.$$

The embeddings

$$\begin{aligned} \mathfrak{n}_- \otimes \bigotimes_{a=1}^n L_{m_a} &\hookrightarrow \mathfrak{n}_- \otimes \Big(\bigotimes_{a=1}^n M_{m_a}\Big)^* = B^0_{\boldsymbol{m}}, \\ \bigotimes_{a=1}^n L_{m_a} &\hookrightarrow \Big(\bigotimes_{a=1}^n M_{m_a}\Big)^* \hookrightarrow B^1_{\boldsymbol{m}} \end{aligned} \tag{6.4}$$

define the morphism of complexes

$$C_\bullet(\mathfrak{n}_-, \bigotimes_{a=1}^n L_{m_a}) \longrightarrow (B^\bullet_{\boldsymbol{m}}, \widetilde{d}). \tag{6.5}$$

Lemma 6.1. — *This morphism is a quasi-isomorphism.*

This quasi-isomorphism will be called the *BGG resolution of* $C_\bullet(\mathfrak{n}_-, \bigotimes_{a=1}^n L_{m_a})$. The complex $(B^\bullet_{\boldsymbol{m}}, \widetilde{d})$ has weight decomposition. For any $\lambda \in \mathbb{C}$ we have

$$B^i_{\boldsymbol{m}}[\lambda] = (\mathfrak{n}_- \otimes A^i_{\boldsymbol{m}}[\lambda+2]) \oplus A^{i-1}_{\boldsymbol{m}}[\lambda].$$

In the next section we identify the complex $(B^\bullet_{\boldsymbol{m}}[|\boldsymbol{m}| - 2k], \widetilde{d})$ with the skew-symmetric part of the Aomoto complex of a suitable weighted Orlik-Solomon manifold.

7. Discriminantal arrangements with $\mathfrak{sl}_2$ weights

7.1. Weighted discriminantal arrangement in $\mathbb{C}^k$. — Fix $\boldsymbol{m} = (m_1, \ldots, m_n)$ with positive integer coordinates and a positive integer k. We assume that $m_j \leqslant k-1$ for $j = 1, \ldots, n_0$ and $m_j > k-1$ for $j = n_0+1, \ldots, n$. Fix $(z_1, \ldots, z_n) \in \mathbb{C}^n$ with distinct coordinates. Fix a generic nonzero number κ.

Consider $\mathbb{C}^k$ with coordinates $t_1, \ldots, t_k$ and the weighted discriminantal arrangement $\mathscr{A}$ consisting of the following hyperplanes: $H_{i,j}$ defined by the equation $t_i - t_j = 0$ for $1 \leqslant i < j \leqslant k$, H^j_i defined by the equation $t_i - z_j = 0$ for $i = 1, \ldots, k$, $j = 1, \ldots, n$. The weights are $a_{i,j} = 2/\kappa$, $a^j_i = -m_j/\kappa$. We denote by $H \subset \mathbb{C}^k$ the union of all hyperplanes of $\mathscr{A}$. Set $U = \mathbb{C}^k \smallsetminus H$.

The symmetric group S_k acts on $\mathbb{C}^k$ by permutation of coordinates. The action preserves the weighted arrangement $\mathscr{A}$.

For $j = 1, \ldots, n_0$, let $I \subset \{1, \ldots, n\}$ be a subset with m_j+1 elements. The edge X^j_I of $\mathscr{A}$ defined by equations $t_i = z_j$ for $i \in I$, is resonant.

Lemma 7.1. — *The edges X_I^j, $j=1,\dots,n_0$, $|I|=m_j+1$, are the only resonant dense edges of $\mathscr{A}$.*

Proof. — The dense edges of $\mathscr{A}$ have the form $t_{i_1}=\cdots=t_{i_\ell}$ or $t_{i_1}=\cdots=t_{i_\ell}=z_j$ for $2\leqslant\ell\leqslant k$. One checks that the edges of the former type are not resonant, and edges of the latter type are resonant if and only if $\ell=m_j+1\leqslant k$. □

7.2. Skew-symmetric part of Aomoto complex of U. — The symmetric group S_k naturally acts on the Orlik-Solomon spaces $A^\bullet(U)$. The *skew-symmetrization* of a form $\eta\in A^\bullet(U)$ is the form $\operatorname{Skew}\eta:=\sum_{\sigma\in S_k}(-1)^{|\sigma|}\sigma\eta$. The form $\operatorname{Skew}\eta$ is skew-symmetric. More generally, if $G\subset S_k$ is a subgroup, then the *G-skew-symmetrization* of a form $\eta\in A^\bullet(U)$ is the form $\operatorname{Skew}_G\eta:=\sum_{\sigma\in G}(-1)^{|\sigma|}\sigma\eta$.

The skew-symmetric part $A_-^\bullet(U)$ of the Orlik-Solomon spaces $A^\bullet(U)$ is described in [SV91]. We have $A_-^p(U)\neq 0$ only if $p=k-1,k$. Let $J=(j_1,\dots,j_n)$ be a vector with nonnegative integer coordinates and $|J|=k$. Define $\ell_0(J)=0$ and $\ell_i(J)=j_1+\cdots+j_i$ for $i=1,\dots,n$, and

$$\eta_{J,i}=d\log(t_{\ell_{i-1}(J)+1}-z_i)\wedge\cdots\wedge d\log(t_{\ell_i(J)}-z_i)$$

for $i=1,\dots,n$. Let ω_J be the skew-symmetrization of the k-form $\alpha_J\,\eta_{J,1}\wedge\cdots\wedge\eta_{J,n}$, where $\alpha_J=(\kappa^{|J|}j_1!\dots j_n!)^{-1}$.

Let $J=(j_1,\dots,j_n)$ be a vector with nonnegative integer coordinates and $|J|=k-1$. Define the $(k-1)$-form $\eta_J=\alpha_J\,\eta_{J,1}\wedge\cdots\wedge\eta_{J,n}$ as above, and then ω_J as the skew-symmetrization of $(-1)^k\eta_J$.

Lemma 7.2 ([SV91]). — *The forms $\{\omega_J\}_{|J|=k}$ form a basis of $A_-^k(U)$. The forms $\{\omega_J\}_{|J|=k-1}$ form a basis of $A_-^{k-1}(U)$.*

Define the form

$$\omega_a=\sum_{H\in\mathscr{A}}a_H\,d\log f_H\ \in\ A^1(U). \tag{7.1}$$

The form ω_a is symmetric with respect to the S_k-action.

Lemma 7.3 ([SV91]). — *For any $\boldsymbol{m}\in\mathbb{C}^n$, the complex $\wedge\omega_a:A_-^{k-1}(U)\to A_-^k(U)$ is isomorphic to the weight component of weight $|\boldsymbol{m}|-2k$ of the complex*

$$\mathfrak{n}_-\otimes\Big(\bigotimes_{a=1}^n M_{m_a}\Big)^*\longrightarrow\Big(\bigotimes_{a=1}^n M_{m_a}\Big)^*.$$

The isomorphism sends ω_J to $f\otimes\varphi_{\boldsymbol{m}}^J$ if $|J|=k-1$ and to $\varphi_{\boldsymbol{m}}^J$ if $|J|=k$.

7.3. Skew-symmetric forms on $\mathbb{P}^m$. — For a positive integer m, consider a subset $I=\{1\leqslant i_0<\cdots<i_m\leqslant k\}$ and the space $\mathbb{C}^{m+1}$ with coordinates $t_i,i\in I$. Consider the central arrangement in $\mathbb{C}^{m+1}$ consisting of coordinate hyperplanes and all diagonal hyperplanes. This arrangement is preserved by the action of the symmetric group S_{m+1} which permutes the coordinates.

Consider the projectivization in $\mathbb{P}^m$ of the initial arrangement. The functions $u_{i_\ell} = t_{i_\ell}/t_{i_0}$, $\ell = 1, \dots, m$ are coordinates on an affine chart on $\mathbb{P}^m$. In these coordinates the projectivization of the initial arrangement consists of hyperplanes $u_{i_\ell} = 0$, $u_{i_\ell} - 1 = 0$, $u_{i_\ell} - u_{i_q} = 0$ and the hyperplane at infinity. Denote $U \subset \mathbb{P}^m$ the complement to the arrangement. Let $A^\bullet_-(U)$ denote the skew-symmetric part of the Orlik-Solomon space $A^\bullet(U)$ with respect to the S_{m+1}-action.

Lemma 7.4. — *$A^p_-(U) = 0$ if $p \neq m$, and $\dim A^m_-(U) = 1$. The form*

$$\mu_I = d\log u_{i_1} \wedge \cdots \wedge d\log u_{i_m}$$

generates $A^m_-(U)$.

Proof. — Let $\widetilde{U}$ denote the complement of the original central arrangement in $\mathbb{C}^{m+1}$. The skew-symmetric part of $A^\bullet(\widetilde{U})$ is two-dimensional, $\dim A^p_-(\widetilde{U}) = 1$ for $p = k, k+1$. The skew-symmetrizations of

$$\eta_{I,m} = d\log t_{i_1} \wedge \cdots \wedge d\log t_{i_m} \quad \text{and} \quad \eta_I = d\log t_{i_0} \wedge \cdots \wedge d\log t_{i_m}$$

form a basis in $A^\bullet_-(\widetilde{U})$.

Using the identity $d\log u_{i_\ell} = d\log t_{i_\ell} - d\log t_{i_0}$, one identifies $A^\bullet(U)$ with a subspace of the Orlik-Solomon space $A^\bullet(\widetilde{U})$ of the initial central arrangement in $\mathbb{C}^{m+1}$. By [Dim92, §6.1], the contraction along the Euler vector field $\varepsilon = \sum_{\ell=0}^m t_{i_\ell}\partial/\partial t_{i_\ell}$ defines an epimorphism $\partial\colon A^\bullet(\widetilde{U}) \to A^\bullet(U)$, which restricts to an epimorphism $A^\bullet_-(\widetilde{U}) \to A^\bullet_-(U)$ of skew-symmetric forms. The map ∂ is the boundary map in the acyclic complex studied in [OT92, §3.1], and also coincides with the residue map along the exceptional divisor in the blow-up of $\mathbb{C}^{m+1}$ at the origin.

Under this identification, the skew-symmetrization of the form $\eta_{I,m}$ equals a nonzero multiple of the form μ_I considered as an element of $A^\bullet(\widetilde{U})$. The form η_I is skew-symmetric and its contraction along ε equals μ_I. The contraction of μ_I along ε is trivial since $\partial^2 = 0$. Then $A^p_-(U) = 0$ for $p \neq m$ and $A^m_-(U)$ is spanned by μ_I. □

7.4. Weighted Orlik-Solomon manifold. — Consider the minimal resolution $\pi : \widetilde{Y} \to \mathbb{C}^k$ of singularities of H, see Section 7.1. Denote $\widetilde{H} = \pi^{-1}H$. The irreducible components of $\widetilde{H}$ are labeled by dense edges of H. We denote by X the manifold obtained from $\widetilde{Y}$ by deleting the union of all irreducible components of H corresponding to nonresonant dense edges. We set $D = \widetilde{H} \cap X$. Then $D \subset X$ is a divisor with normal crossings and (X, D) is a weighted Orlik-Solomon manifold, see Sections 5.1 and 5.2. The symmetric group S_k acts on the Orlik-Solomon manifold (X, D). The action preserves the weights.

Let $\mathscr{Z} = \{X = Z_0 \supset D = Z_1 \supset Z_2 \supset \cdots\}$ be the associated filtration by closed subsets, and $U = Z_0 \smallsetminus Z_1 = X \smallsetminus D$.

The irreducible components of D are labeled by resonant dense edges of H. For $j \in \{1, \dots, n_0\}$ and $I \subset \{1, \dots, n\}$, $|I| = m_j + 1$, we denote by $\widetilde{H}^j_I$ the component corresponding to the resonant dense edge X^j_I. We denoted by C^j_I the connected component of $Z_1 \smallsetminus Z_2$ whose closure is $\widetilde{H}^j_I$. Then C^j_I is isomorphic to the complement of the product of weighted arrangements in $\mathbb{P}^{m_j} \times \mathbb{C}^{k-m_j-1}$, with weights induced by $\mathscr{A}$. If $I = \{1 \leqslant i_0 < \cdots < i_{m_j} \leqslant k\}$, then u_{i_ℓ}, $\ell = 1, \ldots, m_j$, are coordinates on an affine chart on $\mathbb{P}^{m_j}$, see Section 7.3. The arrangement in $\mathbb{P}^{m_j}$ has hyperplanes $u_{i_\ell} = 0$, $u_{i_\ell} - 1 = 0$, $u_{i_\ell} - u_{i_q} = 0$ and the hyperplane at infinity. The weights induced by $\mathscr{A}$ are $-m_j/\kappa$ for $u_{i_\ell} = 0$ and $2/\kappa$ for $u_{i_\ell} - 1 = 0$ and $u_{i_\ell} - u_{i_q} = 0$. Coordinates on $\mathbb{C}^{k-m_j-1}$ are t_i, $i \in \{1, \ldots, n\} \smallsetminus I$. The arrangement in $\mathbb{C}^{k-m_j-1}$ is the discriminantal arrangement with hyperplanes $t_i - t_q = 0$, $i, q \in \{1, \ldots, k\} \smallsetminus I$ and $t_i - z_\ell = 0$, $i \in \{1, \ldots, k\} \smallsetminus I$, $\ell \in \{1, \ldots, n\}$. The weights of this arrangement in $\mathbb{C}^{k-m_j-1}$ induced from $\mathscr{A}$ are given by the pair $(\boldsymbol{m}^j, \kappa)$, where $\boldsymbol{m}^j = (m_1, \ldots, -m_j - 2, \ldots, m_n)$, see Section 7.1.

The set $\{C_I^j\}_{j,I}$ is the set of connected components of $Z_1 \smallsetminus Z_2$. The group S_k acts on $\{C_I^j\}_{j,I}$. For fixed j, the subset $\{C_I^j\}_I$ forms a single orbit.

For $p \geqslant 2$, the connected components $\{C_{\boldsymbol{I}}^{\boldsymbol{j}}\}_{\boldsymbol{j},\boldsymbol{I}}$ of $Z_p \smallsetminus Z_{p+1}$ are labeled by pairs $(\boldsymbol{j}, \boldsymbol{I})$, where $\boldsymbol{j}$ is a p-element subset of $\{1, \ldots, n_0\}$ and $\boldsymbol{I} = \{I_j\}_{j \in \boldsymbol{j}}$ is a set of pairwise disjoint subsets of $\{1, \ldots, k\}$ such that $|I_j| = m_j + 1$. The connected component $C_{\boldsymbol{I}}^{\boldsymbol{j}}$ is isomorphic to the complement of the product of weighted arrangements in $(\times_{j \in \boldsymbol{j}} \mathbb{P}^{m_j}) \times \mathbb{C}^{e(\boldsymbol{j})}$, where $e(\boldsymbol{j}) = k - p - \sum_{j \in \boldsymbol{j}} m_j$. For $j \in \boldsymbol{j}$, if

$$I_j = \{1 \leqslant i_0 < \cdots < i_{m_j} \leqslant k\},$$

then u_{i_ℓ}, $\ell = 1, \ldots, m_j$, are coordinates on an affine chart on $\mathbb{P}^{m_j}$, see Section 7.3. The arrangement in $\mathbb{P}^{m_j}$ has hyperplanes $u_{i_\ell} = 0$, $u_{i_\ell} - 1 = 0$, $u_{i_\ell} - u_{i_q} = 0$ and the hyperplane at infinity. The weights induced by $\mathscr{A}$ are $-m_j/\kappa$ for $u_{i_\ell} = 0$ and $2/\kappa$ for $u_{i_\ell} - 1 = 0$ and $u_{i_\ell} - u_{i_q} = 0$. The space $\mathbb{C}^{e(\boldsymbol{j})}$ has coordinates t_i, $i \in \{1, \ldots, k\} \smallsetminus \bigcup_{j \in J} I_j$. The weighted arrangement in $\mathbb{C}^{e(\boldsymbol{j})}$ is the discriminantal arrangement with weights given by the pair $(\boldsymbol{m}^{\boldsymbol{j}}, \kappa)$, where $m_i^{\boldsymbol{j}} = -m_i - 2$ if $i \in \boldsymbol{j}$ and $m_i^{\boldsymbol{j}} = m_i$ otherwise, see Section 7.1.

The group S_k acts on the set $\{C_{\boldsymbol{I}}^{\boldsymbol{j}}\}_{\boldsymbol{j},\boldsymbol{I}}$. For fixed $\boldsymbol{j}$, the subset $\{C_{\boldsymbol{I}}^{\boldsymbol{j}}\}_{\boldsymbol{I}}$ forms a single orbit.

Let $C_{\boldsymbol{I}}^{\boldsymbol{j}}$ be a connected component of $Z_p \smallsetminus Z_{p+1}$ and $C_{\widetilde{\boldsymbol{I}}}^{\widetilde{\boldsymbol{j}}}$ a connected component of $Z_{p+1} \smallsetminus Z_{p+2}$. Then $C_{\widetilde{\boldsymbol{I}}}^{\widetilde{\boldsymbol{j}}}$ lies in the closure of $C_{\boldsymbol{I}}^{\boldsymbol{j}}$ if and only if $\boldsymbol{j} \subset \widetilde{\boldsymbol{j}}$ and $I_j = \widetilde{I}_j$ for every $j \in \boldsymbol{j}$.

7.5. Skew-symmetric forms on weighted Orlik-Solomon manifold. — For $p > 0$, fix a set $\boldsymbol{j} = \{1 \leqslant j_1 < \cdots < j_p \leqslant n_0\}$. Consider the S_k-orbit $\{C_{\boldsymbol{I}}^{\boldsymbol{j}}\}_{\boldsymbol{I}}$ of connected components of $Z_p \smallsetminus Z_{p+1}$. Recall that $\boldsymbol{I} = \{I_j\}_{j \in \boldsymbol{j}}$ is a set of pairwise disjoint subsets of $\{1, \ldots, k\}$ such that $|I_j| = m_j + 1$. Each component $C_{\boldsymbol{I}}^{\boldsymbol{j}}$ is invariant with respect to the action of the subgroup $S_{\boldsymbol{I}} = S_{m_{j_1}+1} \times \cdots \times S_{m_{j_p}+1} \times S_{e(\boldsymbol{j})} \subset S_k$, where S_{m_j+1} is the group of permutations of elements of the subset I_j, $e(\boldsymbol{j}) = k - p - \sum_{\ell=1}^{p} m_{j_\ell}$ and $S_{e(\boldsymbol{j})}$ is the group of permutations of elements of the subset $\{1, \ldots, k\} \smallsetminus \bigcup_{j \in \boldsymbol{j}} I_j$.

Our goal is to describe S_k-skew-symmetric Orlik-Solomon forms on $\bigcup_{\boldsymbol{I}} C_{\boldsymbol{I}}^{\boldsymbol{j}}$. Such a form is uniquely determined by its restriction to one of the components in $\{C_{\boldsymbol{I}}^{\boldsymbol{j}}\}_{\boldsymbol{I}}$.

That restriction is $S_{\boldsymbol{I}}$-skew-symmetric. According to Sections 7.2 and 7.3, the S_k-skew-symmetric Orlik-Solomon forms on $\bigcup_{\boldsymbol{I}} C^{\boldsymbol{j}}_{\boldsymbol{I}}$ are available only in degrees $k-p$ and $k-p-1$.

Denote

$$d_{\boldsymbol{j}} = \sum_{i=1}^{p-1} i(m_{j_i}+1), \qquad s_{\boldsymbol{j}} = p + \sum_{i=1}^{p} m_{j_i}.$$

Select in $\{C^{\boldsymbol{j}}_{\boldsymbol{I}}\}_{\boldsymbol{I}}$ the component $C^{\boldsymbol{j}}_{\boldsymbol{I}^0}$, where $\boldsymbol{I}^0 = \{I^0_{j_1},\dots,I^0_{j_p}\}$ and

$$I^0_{j_i} = \Big\{1+\textstyle\sum_{\ell=1}^{i-1}(m_{j_\ell}+1),\dots, m_{j_i}+1+\sum_{\ell=1}^{i-1}(m_{j_\ell}+1)\Big\}, \qquad i=1,\dots,p.$$

Let $K=(k_1,\dots,k_n)\in\mathbb{Z}^n_{\geqslant 0}$, where $|K|$ equals $e(\boldsymbol{j})$ or $e(\boldsymbol{j})-1$. Denote $\ell_0(K)=0$ and $\ell_i(K)=k_1+\cdots+k_i$, $i=1,\dots,n$. Denote

$$\eta^{\boldsymbol{j}}_{K,i} = d\log(t_{s_{\boldsymbol{j}}+\ell_{i-1}+1}-z_i)\wedge d\log(t_{s_{\boldsymbol{j}}+\ell_{i-1}+2}-z_i)\wedge\cdots\wedge d\log(t_{s_{\boldsymbol{j}}+\ell_i}-z_i), \quad i=1,\dots,n,$$

$$\alpha^{\boldsymbol{j}}_K = (-1)^{d_{\boldsymbol{j}}}\big((m_{j_1}+1)!\cdots(m_{j_p}+1)!\,k_1!\dots k_n!\big)^{-1}.$$

The form

$$\alpha^{\boldsymbol{j}}_K\,\mu_{I^0_{j_1}}\wedge\cdots\wedge\mu_{I^0_{j_p}}\wedge\eta^{\boldsymbol{j}}_{K,1}\wedge\cdots\wedge\eta^{\boldsymbol{j}}_{K,n}$$

is an Orlik-Solomon form on $C^{\boldsymbol{j}}_{\boldsymbol{I}^0}$. We extend it by zero to other components of $\bigcup_{\boldsymbol{I}} C^{\boldsymbol{j}}_{\boldsymbol{I}}$. If $|K|=e(\boldsymbol{j})$, we define the form $\omega^{\boldsymbol{j}}_K$ on $\bigcup_{\boldsymbol{I}} C^{\boldsymbol{j}}_{\boldsymbol{I}}$ as the S_k-skew-symmetrization of the form

$$\kappa^{-k}\alpha^{\boldsymbol{j}}_K\,\mu_{I^0_{j_1}}\wedge\cdots\wedge\mu_{I^0_{j_p}}\wedge\eta^{\boldsymbol{j}}_{K,1}\wedge\cdots\wedge\eta^{\boldsymbol{j}}_{K,n}.$$

If $|K|=e(\boldsymbol{j})-1$, we define the from $\omega^{\boldsymbol{j}}_K$ on $\bigcup_{\boldsymbol{I}} C^{\boldsymbol{j}}_{\boldsymbol{I}}$ as the S_k-skew-symmetrization of the form

$$(-1)^{k-p}\kappa^{k-1}\alpha^{\boldsymbol{j}}_K\,\mu_{I^0_{j_1}}\wedge\cdots\wedge\mu_{I^0_{j_p}}\wedge\eta^{\boldsymbol{j}}_{K,1}\wedge\cdots\wedge\eta^{\boldsymbol{j}}_{K,n}.$$

Denote by $A^\bullet_-(\bigcup_{\boldsymbol{I}} C^{\boldsymbol{j}}_{\boldsymbol{I}}) \subset \bigoplus_{\boldsymbol{I}} A^\bullet(C^{\boldsymbol{j}}_{\boldsymbol{I}})$ the skew-symmetric part of the Orlik-Solomon space $\bigoplus_{\boldsymbol{I}} A^\bullet(C^{\boldsymbol{j}}_{\boldsymbol{I}})$ of $\bigcup_{\boldsymbol{I}} C^{\boldsymbol{j}}_{\boldsymbol{I}}$. Recall the 1-form ω_a in (7.1). The form ω_a lifts to an element $\widetilde{\omega}_a=\pi^*\omega_a$ of $\bigoplus_{\boldsymbol{j},\boldsymbol{I}} A^1(C^{\boldsymbol{j}}_{\boldsymbol{I}})$ which is symmetric with respect to the S_k action. The exterior multiplication by $\widetilde{\omega}_a$ defines the complex

$$\wedge\,\widetilde{\omega}_a : A^{k-p-1}_-(\textstyle\bigcup_{\boldsymbol{I}} C^{\boldsymbol{j}}_{\boldsymbol{I}}) \longrightarrow A^{k-p}_-(\bigcup_{\boldsymbol{I}} C^{\boldsymbol{j}}_{\boldsymbol{I}}). \tag{7.2}$$

Recall the vector $\boldsymbol{m}^{\boldsymbol{j}}=(m^{\boldsymbol{j}}_1,\dots,m^{\boldsymbol{j}}_n)$ from Section 7.4.

Lemma 7.5. — *The complex in* (7.2) *is isomorphic to the weight component of weight* $|\boldsymbol{m}|-2k$ *of the complex* $\mathfrak{n}_-\otimes(\bigotimes_{i=1}^n M_{m^{\boldsymbol{j}}_i})^*\to(\bigotimes_{i=1}^n M_{m^{\boldsymbol{j}}_i})^*$, *see* (6.1). *The isomorphism sends* $\omega^{\boldsymbol{j}}_K$ *to* $(-1)^p f\otimes\varphi^K_{\boldsymbol{m}^{\boldsymbol{j}}}$ *if* $|K|=e(\boldsymbol{j})-1$ *and to* $\varphi^K_{\boldsymbol{m}^{\boldsymbol{j}}}$ *if* $|K|=e(\boldsymbol{j})$.

Lemma 7.5 is a corollary of Lemma 7.3.

7.6. Residues of skew-symmetric forms. — Consider an S_k-orbit $\{C^{\boldsymbol{j}}_{\boldsymbol{I}}\}_{\boldsymbol{I}}$ of connected components of $Z_p \smallsetminus Z_{p+1}$ and an S_k-orbit $\{C^{\widetilde{\boldsymbol{j}}}_{\widetilde{\boldsymbol{I}}}\}_{\widetilde{\boldsymbol{I}}}$ of connected components of $Z_{p+1} \smallsetminus Z_{p+2}$ such that the second orbit lies in the closure of the first orbit. This statement holds if and only if $\boldsymbol{j} \subset \widetilde{\boldsymbol{j}}$. More precisely, if $\boldsymbol{j} = \{j_1 < \cdots < j_p\}$, then $\widetilde{\boldsymbol{j}} = \{j_1 < \cdots < j_q < \widetilde{j}_{q+1} < j_{q+1} < \cdots < j_p\}$ for some $0 \leqslant q \leqslant p$.

Consider $\omega^{\boldsymbol{j}}_K \in A^\bullet_-(\bigcup_{\boldsymbol{I}} C^{\boldsymbol{j}}_{\boldsymbol{I}})$. Then the residue of $\omega^{\boldsymbol{j}}_K$ at $\bigcup_{\widetilde{\boldsymbol{I}}} C^{\widetilde{\boldsymbol{j}}}_{\widetilde{\boldsymbol{I}}}$ is an element of $A^\bullet_-(\bigcup_{\widetilde{\boldsymbol{I}}} C^{\widetilde{\boldsymbol{j}}}_{\widetilde{\boldsymbol{I}}})$. We denote this residue by $\operatorname{res}^{\widetilde{\boldsymbol{j}}}_{\boldsymbol{j}}\, \omega^{\boldsymbol{j}}_K$.

Lemma 7.6. — *Given $K = (k_1, \ldots, k_n)$, denote $\widetilde{K} = (k_1, \ldots, k_{\widetilde{j}_{q+1}} - m_{\widetilde{j}_{q+1}} - 1, \ldots, k_n)$.*

- *If $k_{\widetilde{j}_{q+1}} < m_{\widetilde{j}_{q+1}} + 1$, then*
$$\operatorname{res}^{\widetilde{\boldsymbol{j}}}_{\boldsymbol{j}}\, \omega^{\boldsymbol{j}}_K = 0.$$
- *If $k_{\widetilde{j}_{q+1}} \geqslant m_{\widetilde{j}_{q+1}} + 1$, then*
$$\operatorname{res}^{\widetilde{\boldsymbol{j}}}_{\boldsymbol{j}}\, \omega^{\boldsymbol{j}}_K = \begin{cases} (-1)^q \omega^{\boldsymbol{j}}_{\widetilde{K}} & \text{for } |K| = e(\boldsymbol{j}), \\ (-1)^{q+1} \omega^{\boldsymbol{j}}_{\widetilde{K}} & \text{for } |K| = e(\boldsymbol{j}) - 1. \end{cases}$$

Proof. — If $k_{\widetilde{j}_{q+1}} < m_{\widetilde{j}_{q+1}} + 1$, then the form $\omega^{\boldsymbol{j}}_K$ is regular on $\bigcup_{\widetilde{\boldsymbol{I}}} C^{\widetilde{\boldsymbol{j}}}_{\widetilde{\boldsymbol{I}}}$ and $\operatorname{res}^{\widetilde{\boldsymbol{j}}}_{\boldsymbol{j}}\, \omega^{\boldsymbol{j}}_K = 0$. If $k_{\widetilde{j}_{q+1}} \geqslant m_{\widetilde{j}_{q+1}} + 1$, then the statement is checked by direct calculation. □

7.7. Skew-symmetric part of Aomoto complex of weighted Orlik-Solomon manifold. — Consider the weighted Orlik-Solomon manifold (X, D) introduced in Section 7.4 and its Aomoto complex $(A^\bullet(X, \mathscr{X}), \widetilde{\omega}_a + \operatorname{res})$ introduced in Section 5.2. By Theorem 3.2, for generic nonzero κ the complex $(A^\bullet(X, \mathscr{X}), \widetilde{\omega}_a + \operatorname{res})$ calculates the cohomology $H^\bullet(X, \mathscr{L}_{\widetilde{\omega}_a})$ of X with coefficients in the rank 1 local system $\mathscr{L}_{\widetilde{\omega}_a}$ on X associated with the differential form $\widetilde{\omega}_a$, see Corollary 3.3.

The group S_k acts on the complex. Denote $(A^\bullet_-(X, \mathscr{X}), \widetilde{\omega}_a{+}\operatorname{res})$ the skew-symmetric part of the complex. For generic nonzero κ the complex $(A^\bullet_-(X, \mathscr{X}), \widetilde{\omega}_a{+}\operatorname{res})$ calculates the skew-symmetric part $H^\bullet_-(X, \mathscr{L}_{\widetilde{\omega}_a})$ of the cohomology $H^\bullet(X, \mathscr{L}_{\widetilde{\omega}_a})$.

Recall the complex $(B^\bullet_{\boldsymbol{m}}[|\boldsymbol{m}| - 2k], \widetilde{d})$ in Section 6.4. Define the linear map

$$\gamma : A^\bullet_-(X, \mathscr{X}) \longrightarrow B^\bullet_{\boldsymbol{m}}[|\boldsymbol{m}| - 2k],$$
$$\omega^{\boldsymbol{j}}_K \longmapsto \begin{cases} f \otimes \varphi^K_{\boldsymbol{m}^{\boldsymbol{j}}} & \text{if } |K| = e(\boldsymbol{j}) - 1, \\ \varphi^K_{\boldsymbol{m}^{\boldsymbol{j}}} & \text{if } |K| = e(\boldsymbol{j}). \end{cases} \tag{7.3}$$

Theorem 7.7. — *The map γ defines an isomorphism between the complexes $(A^\bullet_-(X, \mathscr{X}), \widetilde{\omega}_a{+}\operatorname{res})$ and $(B^\bullet_{\boldsymbol{m}}[|\boldsymbol{m}| - 2k], \widetilde{d})$.*

Proof. — The theorem follows from Lemmas 7.5 and 7.6. □

The quasi-isomorphism $C_\bullet(\mathfrak{n}_-, \bigotimes_{a=1}^n L_{m_a})[|\boldsymbol{m}| - 2k] \to (B^\bullet_{\boldsymbol{m}}, \widetilde{d})[|\boldsymbol{m}| - 2k]$ in (6.5) allows us to identify the cohomology $H^\bullet_-(X, \mathscr{L}_{\widetilde{\omega}_a})$ and the cohomology of the complex

$C_\bullet(\mathfrak{n}_-, \bigotimes_{a=1}^n L_{m_a})[|\boldsymbol{m}|-2k]$. Namely, let

$$\bigotimes_{a=1}^n L_{m_a} = \bigoplus_p (L_p \otimes W_p)$$

be the decomposition of the tensor product into irreducible $\mathfrak{sl}_2$-modules, where W_p are the multiplicity spaces.

Corollary 7.8. — *If $|\boldsymbol{m}|-2k \geqslant 0$, then*

$$\dim H_-^k(X, \mathscr{L}_{\widetilde{\omega}_a}) = \dim W_{|\boldsymbol{m}|-2k} \quad \text{and} \quad H_-^q(X, \mathscr{L}_{\widetilde{\omega}_a}) = 0 \quad \text{for} \quad q \neq k.$$

If $|\boldsymbol{m}|-2k=-1$, then $H_-^\bullet(X, \mathscr{L}_{\widetilde{\omega}_a})=0$. If $|\boldsymbol{m}|-2k<-1$, then

$$\dim H_-^{k-1}(X, \mathscr{L}_{\widetilde{\omega}_a}) = \dim W_{2k-2-|\boldsymbol{m}|} \quad \text{and} \quad H_-^q(X, \mathscr{L}_{\widetilde{\omega}_a}) = 0 \quad \text{for}\, q \neq k-1.$$

7.8. BGG resolution and flag forms. — Theorem 7.7 gives a geometric interpretation of the BGG resolution given in (6.5). Namely, the embeddings

$$\mathfrak{n}_- \otimes \bigotimes_{a=1}^n L_{m_a}[|\boldsymbol{m}|-2k+2] \longhookrightarrow \mathfrak{n}_- \otimes \Big(\bigotimes_{a=1}^n M_{m_a}\Big)^*[|\boldsymbol{m}|-2k+2],$$
$$\bigotimes_{a=1}^n L_{m_a}[|\boldsymbol{m}|-2k] \longhookrightarrow \Big(\bigotimes_{a=1}^n M_{m_a}\Big)^*[|\boldsymbol{m}|-2k]$$

in (6.4) have the form: the element $f\otimes f^K v_{\boldsymbol{m}}$ is mapped to $\beta_{\boldsymbol{m}}^K f \otimes \varphi_{\boldsymbol{m}}^K$ if $|K|=k-1$ and the element $f^K v_{\boldsymbol{m}}$ is mapped to $\beta_{\boldsymbol{m}}^K \varphi_{\boldsymbol{m}}^K$ if $|K|=k$, where

$$\beta_{\boldsymbol{m}}^K = \prod_{i=1}^n k_i! \prod_{\ell=1}^{k_i}(m_i+1-\ell).$$

Under the isomorphism of Theorem 7.7, we obtain embeddings

$$\mathfrak{n}_- \otimes \bigotimes_{a=1}^n L_{m_a}[|\boldsymbol{m}|-2k+2] \longhookrightarrow A_-^{k-1}(U), \qquad f\otimes f^K v_{\boldsymbol{m}} \longmapsto \beta_{\boldsymbol{m}}^K \omega_K,$$
$$\bigotimes_{a=1}^n L_{m_a}[|\boldsymbol{m}|-2k] \longhookrightarrow A_-^k(U), \qquad f^K v_{\boldsymbol{m}} \longmapsto \beta_{\boldsymbol{m}}^K \omega_K.$$

The images

$$\mathscr{F}_-^{k-1} = \operatorname{span}\langle \beta_{\boldsymbol{m}}^K \omega_K\rangle_{|K|=k-1} \subset A_-^{k-1}(U), \qquad \mathscr{F}_-^k = \operatorname{span}\langle \beta_{\boldsymbol{m}}^K \omega_K\rangle_{|K|=k} \subset A_-^k(U)$$

of these embeddings are called the *subspaces of skew-symmetric flag forms*, see [SV91, Var95]. The exterior multiplication by ω_a gives the *complex of skew-symmetric flag forms* $\wedge\omega_a : \mathscr{F}_-^{k-1} \to \mathscr{F}_-^k$. Now the BGG resolution in (6.5) can be interpreted as the statement that the natural embedding of the complex of skew-symmetric flag forms to the complex $(A_-^\bullet(X,\mathscr{Z}), \widetilde{\omega}_a + \mathrm{res})$ is a quasi-isomorphism.

The complex of skew-symmetric flag forms can be characterized as follows.

Lemma 7.9. — *The vector space $\mathscr{F}_-^\bullet$ is the kernel of the residue map*

$$A_-^\bullet(U) \longrightarrow \bigoplus_{j=1}^{n_0} A_-^\bullet\Big(\bigcup_{|I|=m_j+1} C_I^j\Big).$$

Proof. — The lemma follows from Lemma 7.6. □

7.9. Cohomology $H^\bullet(X, \mathscr{L}_{\widetilde{\omega}_a})$ and intersection cohomology. — Let $j : U \to \mathbb{C}^k$ be the canonical embedding. Let $\mathscr{L}_{\omega_a}$ be the rank 1 local system on U associated with the form ω_a, see Section 5.2. Consider the intersection cohomology $H^\bullet(\mathbb{C}^k, j_{!*}\mathscr{L}_{\omega_a})$. By [AV12], for generic nonzero real κ, the intersection cohomology $H^\bullet(\mathbb{C}^k, j_{!*}\mathscr{L}_{\omega_a})$ is canonically isomorphic to the cohomology $H^\bullet(X, \mathscr{L}_{\widetilde{\omega}_a})$ if the following condition A from [AV12] is satisfied.

For $1 \leqslant j \leqslant n_0$, consider $\mathbb{C}^{m_j}$ with coordinates $u_1, \dots, u_{m_j}$. Consider the weighted arrangement in $\mathbb{C}^{m_j}$ consisting of the hyperplanes $u_i = 0$, $u_i - 1 = 0$, $u_i - u_p = 0$ with weights $-m_j/\kappa$ for hyperplanes $u_i = 0$ and weights $2/\kappa$ for hyperplanes $u_i - 1 = 0$ and $u_i - u_p = 0$, cf. Section 7.4. Denote by $U_j \subset \mathbb{C}^{m_j}$ the complement to the union of hyperplanes of the arrangement. Let $\mathscr{L}_j$ be the rank 1 local system on U_j associated with this weighted arrangement, see Section 5.2. The condition A is satisfied if for any $1 \leqslant j \leqslant n_0$ we have $H^\ell(U_j, \mathscr{L}_j) = 0$ for $\ell > m_j$. Clearly in this situation condition A is satisfied and $H^\bullet(\mathbb{C}^k, j_{!*}\mathscr{L}_{\omega_a})$ is canonically isomorphic to the cohomology $H^\bullet(X, \mathscr{L}_{\widetilde{\omega}_a})$ by [AV12]. In particular, this implies that for generic nonzero real κ, the skew-symmetric part $H^\bullet_-(\mathbb{C}^k, j_{!*}\mathscr{L}_{\omega_a})$ of the intersection cohomology $H^\bullet(\mathbb{C}^k, j_{!*}\mathscr{L}_{\omega_a})$ is isomorphic to the cohomology of the complex $(A^\bullet_-(X, \mathscr{Z}), \widetilde{\omega}_a + \mathrm{res})$ and, hence, to the cohomology of the complex $C_\bullet(\mathfrak{n}_-, \otimes L_{m_j})[|\boldsymbol{m}| - 2k]$, see Section 7.7, cf. [KV06, §6 of Introduction] and [KV06, Cor. 6.11].

7.10. Remark. — In the constructions of Section 7 we may assume that $\boldsymbol{m} = (m_1, \dots, m_n)$ is a vector with arbitrary complex coordinates instead of being a vector with positive integer coordinates. Then all statements of Section 7 hold. In particular, the same proofs show that in this more general situation the complex $C_\bullet(\mathfrak{n}_-, \bigotimes_{a=1}^n L_{m_a})[|\boldsymbol{m}| - 2k]$ calculates the cohomology $H^\bullet_-(X, \mathscr{L}_{\widetilde{\omega}_a})$ as well as the intersection cohomology $H^\bullet_-(\mathbb{C}^k, j_{!*}\mathscr{L}_{\omega_a})$.

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Michael Falk, Department of Mathematics and Statistics, Northern Arizona University
Flagstaff, AZ 86011, USA
E-mail : michael.falk@nau.edu
Url : http://www.cefns.nau.edu/~falk/

Vadim Schechtman, Institut de Mathématiques de Toulouse, Université Paul Sabatier
118 Route de Narbonne, 31062 Toulouse, France
E-mail : vadim.schechtman@math.univ-toulouse.fr
Url : http://www.math.univ-toulouse.fr/~schechtman

Alexander Varchenko, Department of Mathematics, University of North Carolina at Chapel Hill
Chapel Hill, NC 27599-3250, USA
E-mail : anv@email.unc.edu
Url : http://www.unc.edu/math/Faculty/av/