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A NON-RESIDUALLY FINITE GROUP ACTING UNIFORMLY PROPERLY ON A HYPERBOLIC SPACE

by Rémi Coulon & Denis Osin

Abstract. — In this article we produce an example of a non-residually finite group which admits a uniformly proper action on a Gromov hyperbolic space.

Résumé (Un exemple de groupe non résiduellement fini muni d'une action uniformément propre sur un espace hyperbolique)

Dans cet article nous construisons un exemple de groupe qui n'est pas résiduellement fini et qui est muni d'une action uniformément propre sur un espace hyperbolique au sens de Gromov.

Contents

1. Introduction

By default, all actions of groups on metric spaces considered in this paper are by isometries. Recall that a group is *hyperbolic* if and only if it acts properly and cocompactly on a hyperbolic metric space. It is natural to ask what kind of groups we get if we remove the requirement of cocompactness from this definition. However, it turns out that every countable group admits a proper action on a hyperbolic space,

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namely the parabolic action on a combinatorial horoball [13]. Thus to obtain an interesting class of groups we have to strengthen our properness assumptions.

In this paper we propose to study the class of groups that admit a uniformly proper action on a hyperbolic length space. We denote this class of groups by $\mathscr{P}$. Recall that an action of a group G on a metric space X is *uniformly proper* if for every $r \geqslant 0$, there exists $N \in \mathbb{N}$, such that for all $x \in X$,

$$|\{g \in G \mid d_X(x, gx) \leqslant r\}| \leqslant N.$$

Having a uniformly proper action on a hyperbolic space is a rather restrictive condition. For instance, [16, Th. 1.2] implies that every group $G \in \mathscr{P}$ (as well as each of its subgroups) is either virtually cyclic or acylindrically hyperbolic, which imposes strong restrictions on the algebraic structure of G. In this article we actually focus on a smaller class $\mathscr{P}_0 \subset \mathscr{P}$ which is easier to manipulate. It consists of all groups G having an action on a hyperbolic graph with bounded valence, whose restriction to the vertex set is free.

Hyperbolic groups and their subgroups obviously belong to $\mathscr{P}_0$. Indeed if H is a subgroup of a hyperbolic group G, then the action of H on a Cayley graph of G satisfies the above properties. In general, groups from the class $\mathscr{P}_0$ have many properties similar to those of hyperbolic groups. In fact, we do not know the answer to the following question: *Does $\mathscr{P}_0$ coincide with the class of all subgroups of hyperbolic groups?* Although the affirmative answer seems unlikely, we are not aware of any counterexamples.

This paper is inspired by the well-known open problem of whether every hyperbolic group is residually finite. Our main result shows that the answer to this question is negative if one replaces the class of hyperbolic groups with the class $\mathscr{P}_0$.

Theorem 1.1. — *There exists a finitely generated non-trivial group G with an action on a hyperbolic graph of bounded valence whose restriction to the vertex set is free such that every amenable quotient of G is trivial. In particular, $G \in \mathscr{P}_0$ and G is not residually finite.*

In the process of constructing such a group G, we show that $\mathscr{P}_0$ is closed under taking certain small cancellation quotients (see Section 4). This result seems to be of independent interest and can potentially be used to construct other interesting examples of groups from the class $\mathscr{P}_0$.

The proof of the second claim of Theorem 1.1 can be illustrated as follows. We first use a variant of the Rips construction given in [1] to construct a subgroup N of a torsion-free hyperbolic group H and two elements $a, b \in N$ which are "sufficiently independent" in N (more precisely, non-commensurable – see Section 2 for the definition) but are conjugate in every finite quotient of N. The fact that these elements are "sufficiently independent" together with the result about small cancellation quotients mentioned above imply that the quotient group $G = N/\langle\langle a^p, b^q \rangle\rangle$ belongs to $\mathscr{P}_0$ for some (in fact, all sufficiently large) primes p and q. If $p \neq q$, the images of a and b are clearly trivial in every finite quotient of G. In particular, G is not residually finite.

A slightly more elaborated version of this idea involving Kazhdan's property (T) leads to the proof of the first claim of the theorem.

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2. A short review of hyperbolic geometry

In this section we recall a few notations and definitions regarding hyperbolic spaces in the sense of Gromov. For more details, refer the reader to Gromov's original article [11] or [4, 10].

The four point inequality. — Let (X, d) be a length space. Recall that the *Gromov product* of three points $x, y, z \in X$ is defined by

$$\langle x, y \rangle_z = \frac{1}{2}\left\{ d(x,z) + d(y,z) - d(x,y) \right\}.$$

In the remainder of this section, we assume that X is *δ-hyperbolic*, i.e., for every $x, y, z, t \in X$,

$$\langle x, z \rangle_t \geqslant \min\left\{ \langle x, y \rangle_t, \langle y, z \rangle_t \right\} - \delta. \tag{1}$$

We denote by ∂X the boundary at infinity of X, see [4, Chap. 2].

Quasi-convex subsets. — Let Y be a subset of X. Recall that Y is *α-quasi-convex* if for every $x \in X$, for every $y, y' \in Y$, we have $d(x, Y) \leqslant \langle y, y' \rangle_x + \alpha$. If Y is path-connected, we denote by d_Y the length pseudo-metric on Y induced by the restriction of d_X on Y. The set Y is *strongly quasi-convex* if Y is 2δ-quasi-convex and for every $y, y' \in Y$ we have

$$d_X(y, y') \leqslant d_Y(y, y') \leqslant d_X(y, y') + 8\delta.$$

We denote by $Y^{+\alpha}$, the *α-neighborhood* of Y, i.e., the set of points $x \in X$ such that $d(x, Y) \leqslant \alpha$.

Isometries of a hyperbolic space. — Let G be a group acting uniformly properly on X. An element $g \in G$ is either *elliptic* (it has bounded orbits, hence finite order) or *loxodromic* (it has exactly two accumulation points in ∂X) [2, Lem. 2.2]. A subgroup of G is either *elementary* (it is virtually cyclic) or contains a copy of the free group $\mathbb{F}_2$ [11, §8.2]. In order to measure the action of g on X, we use the translation length defined as follows

$$\|g\|_X = \inf_{x \in X} d(gx, x).$$

If there is no ambiguity, we omit the space X in the notation. A loxodromic element $g \in G$ fixes exactly two points g^- and g^+ in ∂X. We denote by $E(g)$ the stabilizer of $\{g^-, g^+\}$. It is the maximal elementary subgroup containing g. Moreover $\langle g \rangle$ has finite index in $E(g)$ [8, Lem. 6.5].

Given a loxodromic element $g \in G$, there exists a g-invariant strongly quasi-convex subset Y_g of X which is quasi-isometric to a line; its stabilizer is $E(g)$ and the quotient $Y_g/E(g)$ is bounded [7, Def. 3.12 and Lemma 3.13]. We call this set Y_g the *cylinder* of g.

We say that two elements $g, h \in G$ are *commensurable*, if there exist $n, m \in \mathbb{Z} \setminus \{0\}$ and $u \in G$ such that $g^n = uh^m u^{-1}$. Every loxodromic element is contained in a unique maximal elementary subgroup [7, Lem. 3.28]. Hence two loxodromic elements g and h are commensurable if and only if there exists $u \in G$ such that g and uhu^{-1} generate an elementary subgroup.

Lemma 2.1. — *Let S be a finite collection of pairwise non commensurable loxodromic elements of G. There exists $\Delta \geqslant 0$ with the following property. For every $g, g' \in S$, for every $u \in G$, if*

$$\operatorname{diam}\bigl(Y_g^{+5\delta} \cap uY_{g'}^{+5\delta}\bigr) > \Delta,$$

then $g = g'$ and $u \in E(g)$.

Proof. — Since the action of G on X is uniformly proper, it is also acylindrical – see for instance [2, p. 284] for a definition. According to [7, Prop. 3.44 and Lem. 6.14] there exist constants $A, B > 0$ with the following property: if $h, h' \in G$ are two loxodromic elements generating a non-elementary subgroup, then

$$\operatorname{diam}\left(Y_h^{+5\delta} \cap Y_{h'}^{+5\delta}\right) \leqslant A \max\{\|h\|, \|h'\|\} + B.$$

We now let

$$\Delta = A \max_{g \in S} \|g\| + B.$$

Let $g, g' \in S$, and $u \in G$ such that

$$\operatorname{diam}\left(Y_g^{+5\delta} \cap uY_{g'}^{+5\delta}\right) > \Delta.$$

Recall that $uY_{g'}$ is the cylinder of $ug'u^{-1}$. It follows from our choice of Δ, that g and $ug'u^{-1}$ generate an elementary subgroup. Since the elements of S are pairwise non-commensurable it forces $g = g'$ and $u \in E(g)$. ☐

3. The class $\mathscr{P}_0$

Definition 3.1. — A subset S of a metric space X is *r-separated* if for every distinct points $s, s' \in S$, $d(s, s') \geqslant r$. Given a subset Y of X and $r > 0$, we define the *r-capacity* of Y, denoted by $C_r(Y)$, as the maximal number of points in an r-separated subset of Y. We say that X has *r-bounded geometry* if for every $R > 0$, there is an integer N bounding from above the r-capacity of every ball of radius R. If there exists $r > 0$ such that X has r-bounded geometry we simply say that X has *bounded geometry.*

The class $\mathscr{P}_0$ we are interested in consists of all groups admitting a uniformly proper action on a hyperbolic length space with bounded geometry. It is clear that $\mathscr{P}_0 \subseteq \mathscr{P}$. We will show that the class $\mathscr{P}_0$ is closed under certain small cancellation quotients. Before we discuss the precise statements and proofs, a few remarks are in order. First, we do not know whether $\mathscr{P}_0$ is indeed a proper subclass of $\mathscr{P}$. Second, it is possible to prove the results of the next section for the whole class $\mathscr{P}$. Nevertheless, the proofs become much easier for $\mathscr{P}_0$. Therefore we restrict our attention to this subclass.

We start with a few equivalent characterizations of the class $\mathscr{P}_0$. In this article all the graphs are undirected, we refer to [17] for a precise definition. Observe that a graph $\Gamma = (V, E)$ has bounded geometry whenever it has *uniformly bounded valence* i.e., there exists $d \in \mathbb{N}$, such that the valence of any vertex $v \in V$ is at most d.

Remark 3.2. — The converse statement is false. Indeed, consider the real line, which we think of as a graph with the vertex set $\mathbb{Z}$ and the obvious edges; to each vertex, attach infinitely many edges of length 1. The resulting graph has 3-bounded geometry while some vertices have infinite valence.

If Γ is a graph with uniformly bounded valence, the action of a group G on Γ is uniformly proper if and only if there exists $N \in \mathbb{N}$ such that the stabilizer of any vertex contains at most N elements. Recall that a group G acts on a graph Γ *without inversion* if there is no element $g \in G$ sending an edge $e \in E$ to $\bar{e}$ (where $\bar{e}$ is the same edge with the reverse orientation).

Proposition 3.3. — *Let G be a group. The following assertions are equivalent.*

(1) *G belongs to $\mathscr{P}_0$.*

(2) *G acts uniformly properly without inversion on a hyperbolic graph Γ with uniformly bounded valence.*

(3) *G acts on a hyperbolic graph Γ with uniformly bounded valence such that the action of G is free when restricted to the vertex set of Γ.*

Proof. — To show that (3) $\Rightarrow$ (2) one simply takes the barycentric subdivision of the graph. The implication (2) $\Rightarrow$ (1) directly follows from the definition. We now focus on (1) $\Rightarrow$ (3). By definition there exists $r > 0$ such that G acts uniformly properly on a hyperbolic length space X with r-bounded geometry. Using Zorn's Lemma we choose an r-separated subset $\overline{S}$ of $\overline{X} = X/G$ which is maximal for this property. We denote by S the pre-image of $\overline{S}$ in X. We fix $S_0 \subset S$ to be a set of representatives for the action of G on S. Let $R = 2r + 1$. We now define a graph $\Gamma = (V, E)$ as follows. Its vertex set is $V = G \times S_0$. The edge set E is the set of pairs $((u, s), (u', s')) \in V \times V$ such that $d_X(us, u's') \leqslant R$. The initial and terminal vertices of such an edge are (u, s) and (u', s') respectively. The group G acts freely on V as follows: for every $g \in G$, for every $(u, s) \in V$, we have $g \cdot (u, s) = (gu, s)$. This action induces an action by isometries of G on Γ. Recall that $R > 2r$. A variation on the Milnor-Svarč Lemma implies that the map $V \to X$ sending (u, s) to us induces a (G-equivariant) quasi-isometry from Γ to X. In particular Γ is hyperbolic. We are left to prove that Γ has uniformly bounded valence.

Since X has r-bounded geometry, there exists $N_1 \in \mathbb{N}$ such that the r-capacity of any ball of radius R in X is at most N_1. The group G acting uniformly properly on X, there exists $N_2 \in \mathbb{N}$ such that for every $x \in X$, the cardinality of the set

$$U(x) = \{g \in G \mid d_X(x, gx) \leqslant 2R\}$$

is bounded above by N_2. We now fix a vertex $v_0 = (u_0, s_0)$ of Γ. We fix a subset S_1 of $B(u_0 s_0, R)$ such that any G-orbit of S intersecting $B(u_0 s_0, R)$ contains exactly one point in S_1. It follows from our choice of S that if $s, s' \in S$ belong to distinct G-orbits, then $d_X(s, s') \geqslant r$. Consequently the cardinality of S_1 is bounded above by the r-capacity of this ball, i.e., N_1. By construction for every $s \in S_1$, there exists $u_s \in G$ such that $u_s s$ belongs to S_0. It follows from the definition of Γ combined with the triangle inequality that any neighbor of v_0 belongs to the set

$$\left\{(uu_s^{-1}, u_s s) \mid s \in S_1, u \in U(s)\right\}.$$

The cardinality of this set is bounded above by $d = N_1 N_2$, which does not depend on v_0, hence Γ has uniformly bounded valence. $\square$

4. Stability of the class $\mathscr{P}_0$

We now explain how $\mathscr{P}_0$ behaves under small cancellation. To that end we first review the geometric theory of small cancellation as it has been introduced by M. Gromov [12] and further developed in [9, 5, 8]. For a detailed exposition we refer the reader to [6, §§4-6].

The setting. — Let X be a δ-hyperbolic length space and G a group acting on X. Let $\mathscr{Q}$ be a family of pairs (H, Y) such that Y is a strongly quasi-convex subset of X and H a subgroup of $\mathrm{Stab}(Y)$. We assume that $\mathscr{Q}$ is closed under the following action of G: for every $(H, Y) \in \mathscr{Q}$, for every $g \in G$, $g(H, Y) = (gHg^{-1}, gY)$. In addition we require that $\mathscr{Q}/G$ is finite. We denote by K the (normal) subgroup generated by the union of the subgroups H such that $(H, Y) \in \mathscr{Q}$. The goal is to study the quotient $\overline{G} = G/K$ and the corresponding quotient map $\pi\colon G \to \overline{G}$. To that end we define the following two small cancellation parameters

$$\begin{aligned}
\Delta(\mathscr{Q}, X) &= \sup\left\{\operatorname{diam}\left(Y_1^{+5\delta} \cap Y_2^{+5\delta}\right) \mid (H_1, Y_1) \neq (H_2, Y_2) \in \mathscr{Q}\right\}, \\
\operatorname{inj}(\mathscr{Q}, X) &= \inf\left\{\|h\| \mid h \in H \setminus \{1\},\ (H, Y) \in \mathscr{Q}\right\}.
\end{aligned}$$

They play the role of the lengths of the longest piece and the shortest relation respectively. We now fix a number $\rho > 0$ whose value will be specified later. It should be thought of as a very large parameter.

Hyperbolic cones. — Let $(H, Y) \in \mathscr{Q}$. The *cone of radius ρ over Y*, denoted by $Z(Y)$, is the quotient of $Y \times [0, \rho]$ by the equivalence relation that identifies all the points of the form $(y, 0)$. The equivalence class of $(y, 0)$, denoted by a, is called the *apex* of the cone. By abuse of notation, we still write (y, r) for the equivalence class of (y, r). The map $\iota\colon Y \to Z(Y)$ that sends y to (y, ρ) provides a natural embedding form Y to $Z(Y)$. This space can be endowed with a metric as described below. For the geometric interpretation of the distance see [6, §4.1].

Proposition 4.1 ([3, Chap. I.5, Prop. 5.9]). — *The cone $Z(Y)$ is endowed with a metric characterized in the following way. Let $x = (y, r)$ and $x' = (y', r')$ be two points of $Z(Y)$ then*

$$\cosh d_{Z(Y)}(x, x') = \cosh r \cosh r' - \sinh r \sinh r' \cos \theta(y, y'),$$

where $\theta(y, y')$ is the angle at the apex *defined by $\theta(y, y') = \min\{\pi, d_Y(y, y')/\sinh \rho\}$.*

Coning-off. — The *cone-off of radius ρ over X relative to $\mathscr{Q}$* denoted by $\dot{X}_\rho(\mathscr{Q})$ (or simply $\dot{X}$) is obtained by attaching for every $(H, Y) \in \mathscr{Q}$, the cone $Z(Y)$ on X along Y according to ι. We endow $\dot{X}$ with the largest pseudo-metric $d_{\dot{X}}$ for which all the maps $X \to \dot{X}$ and $Z(Y) \to \dot{X}$ – when (H, Y) runs over $\mathscr{Q}$ – are 1-Lipschitz. It turns out that this pseudo-distance is actually a length metric on $\dot{X}$ [6, Prop. 5.10].

The action of G on X naturally extends to an action by isometries on $\dot{X}$ as follows. Let $(H, Y) \in \mathscr{Q}$. For every $x = (y, r) \in Z(Y)$, for every $g \in G$, gx is the point of $Z(gY)$ defined by $gx = (gy, r)$. The space $\overline{X}_\rho(\mathscr{Q})$ (or simply $\overline{X}$) is the quotient $\overline{X} = \dot{X}/K$. The metric on $\dot{X}$ induces a pseudo-metric on $\overline{X}$. We write $\zeta \colon \dot{X} \to \overline{X}$ for the canonical projection from $\dot{X}$ to $\overline{X}$. The quotient $\overline{G}$ naturally acts by isometries on $\overline{X}$.

Proposition 4.2. — *Assume that for every $(H, Y) \in \mathscr{Q}$, the space Y/H is bounded. Then the spaces $\overline{X}$ and X/K are quasi-isometric.*

Proof. — Recall that the embedding $X \to \dot{X}$ is 1-Lipschitz. Hence it induces a 1-Lipschitz embedding $X/K \to \overline{X}$. We claim that the map $X/K \to \overline{X}$ is actually bi-Lipschitz. For simplicity, we implicitly identify X/K with its image in $\overline{X}$. Recall that $\mathscr{Q}/G$ is finite. It follows from our assumption that there exists $D \geqslant 0$ such that for every $(H, Y) \in \mathscr{Q}$, the image of Y in X/K has diameter at most D.

Let $\overline{x}, \overline{x}' \in X/K$. Let $\eta > 0$. There exist $x, x' \in X$, respective pre-images of $\overline{x}$ and $\overline{x}'$, such that $d_{\dot{X}}(x, x') < d_{\overline{X}}(\overline{x}, \overline{x}') + \eta$. Following the construction of the metric on $\dot{X}$ – see for instance [6, §5.1] – we observe that there exists a sequence of points $(x_0, y_0, x_1, y_1, \dots, x_m, y_m)$ which approximates the distance between x and x' in the following sense:

(i) $x_0 = x$ and $y_m = x'$;
(ii) For every $i \in \{0, \dots, m-1\}$, there exists $(H_i, Y_i) \in \mathscr{Q}$ such that $y_i, x_{i+1} \in Y_i$;
(iii)

$$\sum_{i=0}^{m} d_X(x_i, y_i) + \sum_{i=0}^{m-1} d_{Z(Y_i)}(y_i, x_{i+1}) < d_{\dot{X}}(x, x') + \eta. \tag{2}$$

For every $i \in \{0, \dots, m\}$, we write $\overline{x}_i$ and $\overline{y}_i$ for the images in X/K of x_i and y_i respectively. It follows from the triangle inequality that

$$d_{X/K}(\overline{x}, \overline{x}') \leqslant \sum_{i=0}^{m} d_{X/K}(\overline{x}_i, \overline{y}_i) + \sum_{i=0}^{m-1} d_{X/K}(\overline{y}_i, \overline{x}_{i+1}). \tag{3}$$

We are going to compare the terms of the latter inequality with the ones of (2). Note first that for every $i \in \{0, \ldots, m\}$, we have

$$d_{X/K}(\overline{x}_i, \overline{y}_i) \leqslant d_X(x_i, y_i)\,. \tag{4}$$

Let $i \in \{0, \ldots, m-1\}$. In order to estimate $d_{X/K}(\overline{y}_i, \overline{x}_{i+1})$, we distinguish two cases. Assume first that $d_{Y_i}(y_i, x_{i+1}) \leqslant \pi \sinh \rho$. It follows from the definition of the metric on $Z(Y_i)$ that

$$d_{X/K}(\overline{y}_i, \overline{x}_{i+1}) \leqslant d_X(y_i, x_{i+1}) \leqslant d_{Y_i}(y_i, x_{i+1}) \leqslant \frac{\pi \sinh \rho}{2\rho}\, d_{Z(Y_i)}(y_i, x_{i+1})\,. \tag{5}$$

Assume now that $d_{Y_i}(y_i, x_{i+1}) > \pi \sinh \rho$. In particular $d_{Z(Y_i)}(y_i, x_{i+1}) = 2\rho$. Recall that the diameter of the image of Y_i in X/K is at most D. Hence

$$d_{X/K}(\overline{y}_i, \overline{x}_{i+1}) \leqslant \frac{D}{2\rho}\, d_{Z(Y_i)}(y_i, x_{i+1})\,. \tag{6}$$

Combining (2)–(6) we get that

$$d_{X/K}(\overline{x}, \overline{x}') \leqslant \lambda\, (d_{\dot X}(x, x') + \eta) \leqslant \lambda\, (d_{\overline{X}}(\overline{x}, \overline{x}') + 2\eta)\,,$$

where

$$\lambda = \max\left\{1, \frac{\pi \sinh \rho}{2\rho}, \frac{D}{2\rho}\right\}.$$

The previous inequality holds for every $\eta > 0$, hence $X/K \to \overline{X}$ is bi-Lipschitz, which completes the proof of our claim. Note that the diameter of the cones attached to X to form the cone-off space $\dot X$ have diameter at most 2ρ. Hence any point of $\overline{X}$ is a distance at most 2ρ from a point of X/K. Consequently the map $X/K \to \overline{X}$ is a quasi-isometry. $\square$

Small cancellation theorem. — The small cancellation theorem recalled bellow is a compilation of Proposition 6.7, Corollary 3.12, and Proposition 6.12 from [6].

Theorem 4.3. — *There exist positive constants δ_0, δ_1, Δ_0 and ρ_0 satisfying the following property. Let X be a δ-hyperbolic length space and G a group acting by isometries on X. Let $\mathcal{Q}$ be a G-invariant family of pairs (H, Y) where Y is a strongly quasi-convex subset of X and H a subgroup of G stabilizing Y. We assume that $\mathcal{Q}/G$ is finite. Let $\rho \geqslant \rho_0$. If $\delta \leqslant \delta_0$, $\Delta(\mathcal{Q}, X) \leqslant \Delta_0$ and $\operatorname{inj}(\mathcal{Q}, X) \geqslant 2\pi \sinh \rho$ then the following holds.*

(1) *The space $\overline{X} = \overline{X}_\rho(\mathcal{Q})$ is a δ_1-hyperbolic length space.*

(2) *Let $(H, Y) \in \mathcal{Q}$. Let a be the apex of $Z(Y)$ and $\overline{a}$ its image in $\overline{X}$. The quotient map $\pi\colon G \twoheadrightarrow \overline{G}$ induces an isomorphism from $\operatorname{Stab}(Y)/H$ onto $\operatorname{Stab}(\overline{a})$.*

(3) *For every $x \in X$, the quotient map $\pi\colon G \to \overline{G}$ induces a bijection from the set $\{g \in G \mid d(gx, x) \leqslant \rho/100\}$ onto its image.*

(4) *Let $\overline{F}$ be an elliptic subgroup of $\overline{G}$. Either there exists an elliptic subgroup F of G such that the quotient map $\pi\colon G \to \overline{G}$ induces an isomorphism from F onto $\overline{F}$, or there exists $(H, Y) \in \mathcal{Q}$ such that $\overline{F}$ is contained in $\operatorname{Stab}(\overline{a})$, where $\overline{a}$ stands for the image in $\overline{X}$ of the apex a of the cone $Z(Y)$.*

We are now in position to prove the following statement.

Proposition 4.4. — *Let G be a group acting uniformly properly without inversion on a hyperbolic graph Γ with uniformly bounded valence. Let $\{g_1, \dots, g_m\}$ be a finite subset of G whose elements are loxodromic (with respect to the action of G on Γ) and pairwise non-commensurable. In addition, we assume that for every $i \in \{1, \dots, m\}$, the group $\langle g_i \rangle$ is normal in $E(g_i)$. Then for every finite subset $U \subseteq G$ there exists $N \in \mathbb{N}$ with the following property. Let $n_1, \dots, n_m \in \mathbb{N}$, all bounded below by N. Let K be the normal closure of $\{g_1^{n_1}, \dots, g_m^{n_m}\}$ in G. Then the quotient $\overline{G} = G/K$ belongs to $\mathscr{P}_0$. Moreover, we have the following.*

(1) *For every $i \in \{1, \dots, m\}$, the natural homomorphism $\pi\colon G \to \overline{G}$ induces an embedding of $E(g_i)/\langle g_i \rangle$ into $\overline{G}$.*

(2) *The quotient map π is injective when restricted to U.*

(3) *Let $\overline{F}$ be a finite subgroup of $\overline{G}$. Then either there exists a finite subgroup F of G such that $\pi(F) = \overline{F}$ or $\overline{F}$ is conjugate to a subgroup of $\pi(E(g_i))$ for some $i \in \{1, \dots, m\}$.*

Proof. — The constant δ_0 δ_1, Δ_0, and ρ_0 are the one given by Theorem 4.3. We choose an arbitrary $\rho \geqslant \rho_0$. We write δ for the hyperbolicity constant of Γ. According to Lemma 2.1 there exists a constant Δ such that for every $u \in G$, for every $i \neq j$ in $\{1, \dots, m\}$, if
$$\operatorname{diam}\bigl(Y_{g_i}^{+5\delta} \cap uY_{g_j}^{+5\delta}\bigr) > \Delta,$$
then $i = j$ and u belongs to $E(g_i)$. Up to replacing Γ by a rescaled version of Γ, that we denote X, we may assume that the following holds

- $\delta \leqslant \delta_0$ and $\Delta \leqslant \Delta_0$,
- there exists $x \in X$, such that for every $u \in U$ we have $d_X(ux, x) \leqslant \rho/100$.

Since the g_i's are loxodromic, there exists $N \in \mathbb{N}$ such that for every $n \geqslant N$, for every $i \in \{1, \dots, m\}$, we have $\|g_i^n\|_X \geqslant 2\pi \sinh \rho$. Let $n_1, \dots, n_m \in \mathbb{N}$, all bounded below by N. Let K be the normal closure of $\{g_1^{n_1}, \dots, g_m^{n_m}\}$ and $\overline{G}$ be the quotient $\overline{G} = G/K$.

Since G acts without inversion on Γ, the quotient of $\overline{\Gamma} = \Gamma/K$ is a graph endowed with an action without inversion of $\overline{G}$. According to our assumptions there exist $d, M \in \mathbb{N}$ such that given any vertex v of Γ, its valence is at most d and the cardinality of its stabilizer is bounded above by M. Observe that the same holds for the vertices of $\overline{\Gamma}$. By the second characterization of Proposition 3.3, to prove that $\overline{G}$ belongs to $\mathscr{P}_0$, it suffices to show that $\overline{\Gamma}$ is hyperbolic. To that end, we use small cancellation theory. Let $\mathscr{Q}$ be the following collection
$$\mathscr{Q} = \bigl\{ \bigl(\langle u g_i^{n_i} u^{-1} \rangle, uY_g\bigr) \bigm| u \in G,\ 1 \leqslant i \leqslant m \bigr\}.$$
By construction $\Delta(\mathscr{Q}, X) \leqslant \Delta_0$ and $\operatorname{inj}(\mathscr{Q}, X) \geqslant 2\pi \sinh \rho$. The cone-off space $\dot{X} = \dot{X}_\rho(\mathscr{Q})$ and the quotient $\overline{X} = \dot{X}/K$ are built as above. The parameters have been chosen in such a way so that the family $\mathscr{Q}$ satisfies the assumptions of Theorem 4.3. It follows that $\overline{X}$ is a hyperbolic length space. Note that for every $(H, Y) \in \mathscr{Q}$,

the quotient Y/H is bounded, hence $\overline{X}$ is quasi-isometric to X/K (Proposition 4.2). Nevertheless X/K is just a rescaled copy of $\overline{\Gamma}$. Thus $\overline{\Gamma}$ is quasi-isometric to $\overline{X}$, and therefore hyperbolic. Points (1)–(3) directly follow from Theorem 4.3. ☐

5. Proof of the main theorem

We begin with an auxiliary result, which is similar to [14, Prop. 4.2].

Lemma 5.1. — *Let Q be a finitely presented infinite simple group and let H be a torsion-free hyperbolic group splitting as*

$$1 \longrightarrow N \longrightarrow H \longrightarrow Q \longrightarrow 1,$$

where the subgroup N is finitely generated. Let $a \in N \setminus \{1\}$. Then there exists $b \in N \setminus \{1\}$ such that a and b are not commensurable in N but are conjugate in every finite quotient of N.

Proof. — Let $C = \langle c \rangle$ be the maximal cyclic subgroup of H containing a. Note that $CN/N = C/(C \cap N)$ is a cyclic subgroup, hence either non-simple or finite. In particular, CN/N is a proper subgroup of Q, thus $CN \neq H$. Let $h \in H \setminus CN$. Let $b = h^{-1}ah$ and $a = c^n$ for some $n \in \mathbb{Z} \setminus \{0\}$.

If a and b are commensurable in N, then there exist $t \in N$ and $k, \ell \in \mathbb{Z} \setminus \{0\}$ such that $c^{kn} = t^{-1}h^{-1}c^{\ell n}ht$. Since H is torsion-free we have $k = \ell$ and by the uniqueness of roots in a torsion-free hyperbolic group – see for instance [4, Cor. 7.2] – we obtain $c = t^{-1}h^{-1}cht$. It follows that $ht \in C$ and consequently $h \in CN$, which contradicts our assumption. Thus a and b are not commensurable in N.

Assume now that there exists a finite index normal subgroup K of N such that the images of a and b are not conjugate in N/K. Since N is finitely generated, there are only finitely many subgroups of any finite index in N. Replacing K with the intersection of all subgroups of N of index $[N : K]$ if necessary, we can assume that K is normal in H. The natural action of the group H on the finite set Ω of conjugacy classes of N/K is non-trivial; indeed, the element h acts non-trivially as the images of a and b are not conjugate in N/K. Since every element of N acts on Ω trivially, the action of H on Ω gives rise to a non-trivial homomorphism $\epsilon\colon Q \to \mathrm{Sym}(\Omega)$, which contradicts the assumption that Q is infinite simple. ☐

Theorem 5.2. — *There exists a finitely generated group $G \in \mathscr{P}_0$ such that every amenable quotient of G is trivial.*

Proof. — Let H_1 be a torsion-free hyperbolic group with property (T) of Kazhdan and

$$H_2 = \left\langle x, y \mid y = x\left(y^{-1}xy\right)x^2\left(y^{-1}xy\right)\cdots x^{10}\left(y^{-1}xy\right)\right\rangle.$$

It is easy to see that H_2 satisfies the $C'(1/6)$ small cancellation condition and hence is hyperbolic. Moreover it is generated by some conjugates of x. Any two non-cyclic torsion-free hyperbolic groups have a common non-cyclic torsion-free hyperbolic quotient group [15, Th. 2]. Let H_0 denote a common non-cyclic torsion-free hyperbolic quotient of H_1 and H_2.

By [1, Cor. 1.2], there exists a short exact sequence

$$1 \longrightarrow N \longrightarrow H \longrightarrow Q \longrightarrow 1$$

such that H is torsion-free hyperbolic, N is a quotient of H_0, and Q is a finitely presented infinite simple group. Clearly N inherits property (T) from H_1. As a subgroup of a hyperbolic group, N belongs to the class $\mathscr{P}_0$. Let a denote the image of $x \in H_2$ in N. Since N is a quotient group of H_2, it is generated by conjugates of a (in N).

According to the Lemma 5.1, there exists $b \in N$ such that a and b are not commensurable in N but are conjugate in every finite quotient of N. By Proposition 4.4, there exist distinct primes p and q such that $G = N/\langle\langle a^p, b^q\rangle\rangle$ belongs to $\mathscr{P}_0$, and the images of a and b in G have orders p and q, respectively.

Let A be an amenable quotient of G. Being a quotient group of N, A has property (T) and, therefore, is finite. It follows that the images of a and b in A, denoted by $\overline{a}$ and $\overline{b}$, are conjugate. As $\overline{a}^p = \overline{b}^q = 1$ and $gcd(p, q) = 1$, we have $\overline{a} = \overline{b} = 1$. Since N is generated by conjugates of a, A is generated by conjugates of $\overline{a}$, which implies $A = \{1\}$. □

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