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ON THE LINKS BETWEEN HOROCYCLIC AND GEODESIC ORBITS ON GEOMETRICALLY INFINITE SURFACES

by Alexandre Bellis

Abstract. — We study the intersection between an almost minimizing half-geodesic and the closure of the corresponding horocyclic orbit on a smooth geometrically infinite surface. We prove that if the half-geodesic goes through an infinite number of parts of the surface with injectivity radii bounded from above, then the intersection contains an unbounded sequence of points. We also prove that if the half-geodesic goes through arbitrarily thin parts of the surface, the intersection is the whole half-geodesic. Finally, we construct an example proving that this last condition is not necessary.

Résumé (Sur les liens entre les orbites horocycliques et géodésiques sur les surfaces géométriquement infinies)

Nous étudions l'intersection entre une demi-géodésique quasi-minimisante et l'adhérence de l'orbite horocyclique correspondante sur une surface hyperbolique lisse géométriquement infinie. Nous démontrons que si la demi-géodésique traverse un nombre infini de parties de la surface de rayons d'injectivité bornés supérieurement, alors l'intersection contient une suite non bornée d'éléments. Nous démontrons aussi que si la demi-géodésique traverse des parties arbitrairement fines de la surface, l'intersection est toute la demi-géodésique. Enfin, nous construisons un exemple montrant que cette dernière condition n'est pas nécessaire.

Contents

1. Introduction

Among the curves of constant curvature in the Poincaré half-plane $\mathbb{H}^2$ are the geodesics of curvature zero and the horocycles of curvature one. They give rise to two flows in the unitary tangent bundle $T^1\mathbb{H}^2$ which are deeply related: the geodesic

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flow and the horocycle flow respectively. Consider now a Fuchsian group Γ and the quotient surface $S := \Gamma\backslash\mathbb{H}^2$. Both flows descend to the quotient $T^1S := \Gamma\backslash(T^1\mathbb{H}^2)$. We denote them by $g_{\mathbb{R}}$ and $h_{\mathbb{R}}$ respectively.

The orbits of the geodesic flow have many different topological behaviors. By contrast, those of the horocycle flow tend to be rigid. This is illustrated by a result of G. Hedlund in [Hed36] stating that when the surface S is compact, for every u in T^1S, the orbit $h_{\mathbb{R}}u$ is dense in T^1S. This result is deduced from a fundamental link between horocyclic and geodesic orbits: if the projection on S of $g_{\mathbb{R}^+}u$, denoted by $u(\mathbb{R}^+)$, is almost minimizing (specifically: $\exists C>0, \forall t>0, d(u(0),u(t)) > t-C$), then $h_{\mathbb{R}}u$ is not dense in the non-wandering set Ω_h of the horocycle flow. In [Ebe77], P. Eberlein proves that this implication is actually an equivalence.

As a consequence of this link, one obtains that if the surface is geometrically finite (i.e., with a finitely generated fundamental group) and if u is in Ω_h, then $h_{\mathbb{R}}u$ is either dense in Ω_h or periodic.

This rigidity property was generalized by M. Ratner to Lie groups and unipotent actions in [Rat91]. However, it does not extend to geometrically infinite surfaces (i.e., not geometrically finite). Indeed, S is geometrically finite if and only if every horocyclic orbit in Ω_h is dense in Ω_h or periodic (see [Dal11]).

In this paper, we are interested in the topological dynamics of the horocycle flow on geometrically infinite surfaces, for which little is known. Untwisted hyperbolic flutes are the simplest examples of such surfaces (see [Haa96] or [CM10]). More precisely, we investigate the links between the geodesic flow and the horocycle flow. We associate to x in S the real number $\mathrm{Inj}(x)$ defined as the maximal radius of a ball centered at x without self-intersection. It is the injectivity radius of S at x.

The main result of this paper is:

Theorem 1.1. — *Let Γ be a Fuchsian group without elliptic elements and such that the quotient surface $S := \Gamma\backslash\mathbb{H}^2$ is geometrically infinite. Consider u in the non-wandering set Ω_h of $h_{\mathbb{R}}$ in T^1S. Suppose that $u(\mathbb{R}^+)$ is almost minimizing and that $h_{\mathbb{R}}u$ is not periodic and define $\underline{\mathrm{Inj}}(u(\mathbb{R}^+)) := \liminf_{t\to+\infty} \mathrm{Inj}(u(t))$.*

If $\underline{\mathrm{Inj}}(u(\mathbb{R}^+)) < +\infty$, then there exists a sequence of times t_n converging to $+\infty$ such that $g_{t_n}u \in \overline{h_{\mathbb{R}}u}$ for all n.

Moreover, if $\underline{\mathrm{Inj}}(u(\mathbb{R}^+)) = 0$, then $g_{\mathbb{R}^+}u \subset \overline{h_{\mathbb{R}}u}$.

In particular, when $\underline{\mathrm{Inj}}(u(\mathbb{R}^+)) < +\infty$ for every u in T^1S (O. Sarig calls such a surface *weakly tame*, see [Sar10]) then, if $h_{\mathbb{R}}u$ is not periodic, there always exists a positive time t such that $g_tu \in \overline{h_{\mathbb{R}}u}$. As a corollary, we get:

Corollary 1.2. — *Let Γ be a Fuchsian group with neither elliptic nor parabolic element such that the quotient surface $S := \Gamma\backslash\mathbb{H}^2$ is geometrically infinite. If S is weakly tame, then the horocycle flow does not admit any minimal set on T^1S.*

This corollary gives an easy way to construct surfaces without a minimal set for the horocycle flow. The first example of such a surface was produced by M. Kulikov in [Kul04]. Later, theorems of non-existence were obtained in [Mat16] and [GL17].

In the setting of Theorem 1.1, we can ask:

Question. — *Is it possible that* $g_{\mathbb{R}^+}u \subset \overline{h_{\mathbb{R}}u}$ *if* $0 < \underline{\mathrm{Inj}}(u(\mathbb{R}^+)) < +\infty$*?*

Clearly, if $h_{\mathbb{R}}u$ is not recurrent (i.e., it does not accumulate on itself), then $g_{\mathbb{R}^+}u \not\subset \overline{h_{\mathbb{R}}u}$. This implies in particular that $h_{\mathbb{R}}u$ is locally closed (there exists a neighbourhood V of u such that $V \cap (\overline{h_{\mathbb{R}}u} - h_{\mathbb{R}}u) = \varnothing$) even though it is not closed. In [Sta95], A.N. Starkov gives an example of a surface S satisfying the hypotheses of Theorem 1.1 such that $0 < \underline{\mathrm{Inj}}(u(\mathbb{R}^+)) < +\infty$ and $h_{\mathbb{R}}u$ is not recurrent. F. Ledrappier claimed that this example could be generalized to manifolds with *bounded geometry* (see [Led97, Prop. 3]): *let M be a manifold with bounded geometry and u in T^1M such that $u(\mathbb{R}^+)$ is asymptotically almost minimizing. Then, for every $t \in \mathbb{R}$, the strong stable leaf*

$$W^{ss}(g_t u) := \{v \in T^1M \mid d(g_{t+s}u, g_s v) \underset{s\to+\infty}{\longrightarrow} 0\}$$

is locally closed.

When S is a hyperbolic surface, this assertion is equivalent to saying that if the injectivity radius of S is uniformly bounded from below by a positive constant, then $h_{\mathbb{R}}u$ is locally closed provided that $u \in \Omega_h$ and that $u(\mathbb{R}^+)$ is almost minimizing.

Actually, this proposition is false and I construct in Section 4 the following counterexample which also answers the previous question:

Theorem 1.3. — *There exists a geometrically infinite surface S with an injectivity radius everywhere bigger than some positive constant C (i.e., S has bounded geometry), with u in Ω_h satisfying:*

(i) $u(\mathbb{R}^+)$ *is almost minimizing.*
(ii) $g_{\mathbb{R}^+}u \subset \overline{h_{\mathbb{R}}u}$.
(iii) $\underline{\mathrm{Inj}}(u(\mathbb{R}^+)) < +\infty$.

In particular, $h_{\mathbb{R}}u$ is not locally closed.

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2. Notation and tools

For two points z and z' in $\mathbb{H}^2$ and two points ξ and η in $\partial\mathbb{H}^2 := \mathbb{R} \cup \{\infty\}$, we denote by $[z, z']$ the hyperbolic segment between z and z', by $[z, \xi)$ the half-geodesic joining z to ξ and by (η, ξ) the geodesic between η and ξ. We denote by $g_{\mathbb{R}}$ and $h_{\mathbb{R}}$ the geodesic, respectively horocycle, flow in the unitary tangent bundle $T^1\mathbb{H}^2$. For any u in $T^1\mathbb{H}^2$, the function symbol $u(t)$ refers to the projection on $\mathbb{H}^2$ of $g_t u$ and u^+ refers to the endpoint in $\partial\mathbb{H}^2$ of the half-geodesic $u(\mathbb{R}^+)$.

Consider now two elements u and v in $T^1\mathbb{H}^2$. Denote by z and z' the base points of u and v respectively and suppose that $u^+ = v^+ = \xi$. Then there exists t in $\mathbb{R}$ such that $g_t h_{\mathbb{R}} u = h_{\mathbb{R}} v$. The Busemann cocycle $B_\xi(z, z')$ centered at ξ between z and z' is by definition the number $B_\xi(z, z') = t$. Thus, the set $\{z' \mid B_\xi(z, z') = 0\}$ is the horocycle centered at ξ passing through z.

Level sets of isometries. — The group $\mathrm{PSL}_2(\mathbb{R})$ acts by orientation preserving isometries on $(\mathbb{H}^2, d)$, where the distance d is defined by the measure $dz^2 := (dx^2 + dy^2)/y^2$. Let $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ be a hyperbolic isometry. We denote by γ^- and γ^+ respectively the repelling and attractive fixed points of γ in $\partial\mathbb{H}^2$. Observe that a point $z \in \mathbb{H}^2$ is moved by γ along the hypercycle (which is either a piece of an Euclidean circle, or a straight half-line if $\infty = \gamma^-$ or $\infty = \gamma^+$) passing through γ^-, z and γ^+. We denote that hypercycle by $C_\gamma(z)$. For a positive integer k, the point $\gamma^k z$ belongs to the portion of $C_\gamma(z)$ between z and γ^+.

Let $\ell_\gamma := \inf_{z\in\mathbb{H}^2} d(z, \gamma z)$ be the translation length of γ, realized on its axis (γ^-, γ^+). We have:

Proposition 2.1 ([PP15, §5]). — *For any hyperbolic isometry γ and any z in $\mathbb{H}^2$, we have:*

$$\sinh \frac{d(z, \gamma z)}{2} = \cosh s \, \sinh \frac{\ell_\gamma}{2},$$

where $s = d(z, (\gamma^-, \gamma^+))$.

When γ is parabolic, we denote by $C_\gamma(z)$ the horocycle centered at the unique fixed point $\gamma^+ = \gamma^-$ of γ and passing through z. We have:

Proposition 2.2 ([PP15, §6]). — *Consider a parabolic isometry γ and pick any z_0 in $\mathbb{H}^2$. Denote by $\ell_\gamma(z_0)$ the distance $\ell_\gamma(z_0) = d(z_0, \gamma z_0)$. For any z in $\mathbb{H}^2$, we have:*

$$\sinh \frac{d(z, \gamma z)}{2} = e^s \sinh \frac{\ell_\gamma(z_0)}{2},$$

where $s = B_{\gamma^+}(z, C_\gamma(z_0))$.

To prove our theorems, we will translate the dynamics of the horocycle flow on $\Gamma\backslash(T^1\mathbb{H}^2)$ in terms of the action of Γ on $\mathbb{H}^2$ and $\partial\mathbb{H}^2$ using the following proposition.

Proposition 2.3 ([Dal11, Chap. 5, Prop. 2.1]). — *Take a vector u in T^1S and a positive real number r. Denote by $\widetilde{u}$ a lift of u in $T^1\mathbb{H}^2$ and suppose that $\widetilde{u}^+$ is not fixed by any element of Γ except the identity. Then*

$$\big(g_r u \in \overline{h_{\mathbb{R}} u} - h_{\mathbb{R}} u\big)$$
$$\Updownarrow$$
$$\Big(\exists\,(\alpha_n)_{\mathbb{N}} \in \Gamma^{\mathbb{N}} \text{ s.t. } \alpha_n \widetilde{u}^+ \underset{n\to+\infty}{\longrightarrow} \widetilde{u}^+ \text{ and } B_{\widetilde{u}^+}(\alpha_n^{-1} i, \widetilde{u}(0)) \underset{n\to+\infty}{\longrightarrow} B_{\widetilde{u}^+}(i, \widetilde{u}(r))\Big).$$

3. Proof of Theorem 1.1

Definition 3.1. — Let $S := \Gamma\backslash\mathbb{H}^2$ be a hyperbolic surface. The *injectivity radius* of S at p is:

$$\mathrm{Inj}(p) := \min_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}} d(\widetilde{p}, \gamma\widetilde{p}),$$

where $\widetilde{p}$ is any lift of p to $\mathbb{H}^2$.

We shall now prove Theorem 1.1 in several steps. Consider a lift $\widetilde{u}$ of u in $T^1\mathbb{H}^2$. Up to conjugacy, we can suppose $\widetilde{u}^+ = \infty$ and $\widetilde{u}(0) = i$. Note that with our choice of lift $\widetilde{u}$ of u, the equivalence of Proposition 2.3 becomes:

$$(g_r u \in \overline{h_{\mathbb{R}}u} - h_{\mathbb{R}}u) \Longleftrightarrow \big(\exists\, (\alpha_n)_{\mathbb{N}} \in \Gamma^{\mathbb{N}},\ \alpha_n \to \infty \text{ and } B_\infty(\alpha_n^{-1}i, i) \to r\big)\,. \tag{1}$$

The key is to find elements α_n in Γ on which to apply this equivalence.

Lemma 1. — *There is a sequence of points $(q_n)_n$ going to ∞ on the half-geodesic $[i,\infty) = [\widetilde{u}(0), \widetilde{u}^+)$ and a sequence of elements γ_n in Γ that are all distinct such that*

(i) $d(q_n, \gamma_n q_n) \underset{n\to+\infty}{\longrightarrow} \underline{\mathrm{Inj}}(u(\mathbb{R}^+))$.

(ii) *For every sequence of positive integers $(k_n)_{\mathbb{N}}$ we have $\gamma_n^{k_n}\infty \underset{n\to+\infty}{\longrightarrow} \infty$.*

Proof. — The hypothesis $\underline{\mathrm{Inj}}(u(\mathbb{R}^+)) < +\infty$ of Theorem 1.1 gives us a sequence of points q_n going to ∞ in $\partial\mathbb{H}^2$ along the half-geodesic $[i,\infty)$ and a sequence of elements $(\gamma_n)_{n\geqslant 0}$ in $\Gamma - \{\mathrm{Id}\}$ satisfying Property (i) of Lemma 1. Since neither $g_{\mathbb{R}}u$ nor $h_{\mathbb{R}}u$ is periodic, these elements γ_n are all distinct.

Let us consider a subsequence of $(\gamma_n)_{\mathbb{N}}$, that we still denote by $(\gamma_n)_{\mathbb{N}}$, such that $\lim_{n\to+\infty}\gamma_n^- = \eta$ and $\lim_{n\to+\infty}\gamma_n^+ = \xi$.

Suppose first that $\eta \neq \xi$. In this case, for any z in $\mathbb{H}^2$, the distance $s_n = d(z, (\gamma_n^-, \gamma_n^+))$ is bounded from above. Thus, since $\ell_{\gamma_n} \leqslant d(q_n, \gamma_n q_n)$ for every n, Proposition 2.1 implies that for any z in $\mathbb{H}^2$, the elements $\gamma_n z$ stay in a compact. This contradicts the discreteness of Γ.

Suppose now that $\eta = \xi \neq \infty$. If the elements γ_n are hyperbolic, consider the half-geodesic $[p,\xi)$ starting from a point p on $(0,\infty)$ and orthogonal to $(0,\infty)$ (if $\eta = \xi = 0$, consider any point p on $(0,\infty)$). For n big enough, we have

$$d(p, (\gamma_n^-, \gamma_n^+)) < d(q_n, (\gamma_n^-, \gamma_n^+)).$$

Thus, Proposition 2.1 implies that $d(p, \gamma_n p) < d(q_n, \gamma_n q_n)$. Since the latter is bounded from above, we get again a contradiction with the discreteness of Γ. Finally, if the elements γ_n are parabolic, for any z and any z_0 in $\mathbb{H}^2$, we eventually have $B_{\gamma_n^+}(z, C_{\gamma_n}(z_0)) < B_{\gamma_n^+}(q_n, C_{\gamma_n}(z_0))$. Thus, Proposition 2.2 implies that $d(z, \gamma_n z) < d(q_n, \gamma_n q_n)$, which again gives a contradiction with the discreteness of Γ.

In conclusion,

$$\eta = \xi = \infty. \tag{2}$$

Choose now the following orientation for the elements γ_n.

– If γ_n is hyperbolic, choose $|\gamma_n^-| \leqslant |\gamma_n^+|$.

– If γ_n is parabolic, choose it such that $|\gamma_n\infty| > |\gamma_n^+|$.

This choice of orientation combined with (2) enables us to conclude the proof. □

Lemma 2. — *For every positive integer n big enough and every nonpositive integer a, there exists a point $p_{n,a}$ in $\mathbb{H}^2$ satisfying the two following conditions.*

(i) $d(p_{n,a},\gamma_n p_{n,a}) = d(q_n,\gamma_n q_n)$.

(ii) $B_\infty(p_{n,a},\gamma_n^a i) = -d(\gamma_n^a i, p_{n,a})$.

Proof. — Take an isometry γ_n as in Lemma 1. Observe that a point in $C_{\gamma_n}(q_n)\cap[\gamma_n^a i,\infty)$ would satisfy Property (i) and also Property (ii) according to Proposition 2.1 and Proposition 2.2. We prove that this intersection is not empty.

Suppose that γ_n is hyperbolic. There are two cases. The first case is when the signs of γ_n^- and γ_n^+ are opposed. Since $\lim_{n\to+\infty}\gamma_n^- = \lim_{n\to+\infty}\gamma_n^+ = \infty$, the graph of $C_{\gamma_n}(i)$ lies below that of $C_{\gamma_n}(q_n)$. So every geodesic starting from a point z on $C_{\gamma_n}(i)$ and ending at ∞ has an intersection with $C_{\gamma_n}(q_n)$. This is true in particular when $z=\gamma_n^a i$.

The second case is when γ_n^- and γ_n^+ have the same sign. Observe then that we eventually have $d(i,(\gamma_n^-,\gamma_n^+)) > d(q_n,(\gamma_n^-,\gamma_n^+))$, because the converse would contradict the discreteness of Γ, as seen by using Proposition 2.1. Thus, if n is big enough, the graph of $C_{\gamma_n}(i)$ is not contained in the same component of $\mathbb{H}^2 - C_{\gamma_n}(q_n)$ as (γ_n^-,γ_n^+). Observe now that as a is a nonpositive integer, the point $\gamma_n^a i$ belongs to the portion of $C_{\gamma_n}(i)$ between i and γ_n^-, and that for any point z in this portion of $C_{\gamma_n}(i)$, the intersection $C_{\gamma_n}(q_n)\cap[z,\infty)$ is not empty.

If γ_n is parabolic, the proof is similar to the second case, when replacing $d(i,(\gamma_n^-,\gamma_n^+))$ and $d(q_n,(\gamma_n^-,\gamma_n^+))$ by $B_{\gamma_n^+}(i,C_{\gamma_n}(z_0))$ and $B_{\gamma_n^+}(q_n,C_{\gamma_n}(z_0))$ respectively, and using Proposition 2.2 instead of Proposition 2.1. □

Lemma 3. — *Fix $\varepsilon>0$ and an interval I of $\mathbb{R}^+$ of length $\underline{\mathrm{Inj}}(u(\mathbb{R}^+))+\varepsilon$. If n is big enough, there exists an integer k_n such that $B_\infty(\gamma_n^{-k_n}i,i)$ belongs to I.*

Proof. — For every positive integers n and k we put:

$$r_{n,k} := B_\infty(\gamma_n^{-k}i,i) = \sum_{\ell=0}^{k-1} B_\infty(\gamma_n^{-k+\ell}i,\gamma_n^{-k+\ell+1}i).$$

Let us prove that for n big enough, each step $B_\infty(\gamma_n^{-k+\ell}i,\gamma_n^{-k+\ell+1}i)$ is smaller than $\underline{\mathrm{Inj}}(u(\mathbb{R}^+))+\varepsilon$.

For convenience, let us set $s_{n,a} := d(\gamma_n^a i,p_{n,a})$ and $A_{k,\ell} := B_\infty(\gamma_n^{-k+\ell}i,\gamma_n^{-k+\ell+1}i)$. Using Lemma 2, we compute:

$$\begin{aligned}A_{k,\ell} &= B_\infty(\gamma_n^{-k+\ell}i,\gamma_n^{-1}p_{n,-k+\ell+1}) + B_\infty(\gamma_n^{-1}p_{n,-k+\ell+1},p_{n,-k+\ell+1})\\ &\qquad\qquad + B_\infty(p_{n,-k+\ell+1},\gamma_n^{-k+\ell+1}i)\\ &\leqslant d(\gamma_n^{-k+\ell}i,\gamma_n^{-1}p_{n,-k+\ell+1}) + d(\gamma_n^{-1}p_{n,-k+\ell+1},p_{n,-k+\ell+1}) - s_{n,-k+\ell+1}\\ &= s_{n,-k+\ell+1} + d(q_n,\gamma_n q_n) - s_{n,-k+\ell+1}\\ &= d(q_n,\gamma_n q_n).\end{aligned}$$

As $d(q_n,\gamma_n q_n)$ is eventually smaller than $\underline{\mathrm{Inj}}(u(\mathbb{R}^+))+\varepsilon$, we obtain that for n big enough:

$$A_{k,\ell}=B_\infty(\gamma_n^{-k+\ell}i,\gamma_n^{-k+\ell+1}i)\leqslant \underline{\mathrm{Inj}}(u(\mathbb{R}^+))+\varepsilon. \tag{3}$$

Thus, $r_{n,k}$ is a sum of k terms $A_{k,\ell}$, for $\ell=0,\dots,k-1$, all smaller than $\underline{\mathrm{Inj}}(u(\mathbb{R}^+))+\varepsilon$ which is the length of the interval I of $\mathbb{R}^+$. We now use the fact that since $\lim_{k\to+\infty}\gamma_n^{-k}i=\gamma_n^-\neq\infty$ for every n, we have $\lim_{k\to+\infty}r_{n,k}=+\infty$ for every n. Thus, there exists an integer k_n such that r_{n,k_n} belongs to I. $\square$

Proof of Theorem 1.1. — Take the elements γ_n given by Lemma 1. Fix an $\varepsilon>0$, chosen arbitrarily small, and an interval I of $\mathbb{R}^+$ of length $\underline{\mathrm{Inj}}(u(\mathbb{R}^+))+\varepsilon$. Consider the sequence of positive integers $(k_n)_\mathbb{N}$ given by Lemma 3. Since the numbers $B_\infty(\gamma_n^{-k_n}i,i)$ eventually all belong to I, the sequence of numbers $(B_\infty(\gamma_n^{-k_n}i,i))_\mathbb{N}$ admits an accumulation point r in I. Thus, setting $\alpha_n:=\gamma_n^{k_n}$ and applying the equivalence (1), we get that $g_r u\in\overline{h_\mathbb{R}u}-h_\mathbb{R}u$.

Now, applying the same argument to a partition $(I_\ell)_\mathbb{N}$ of $\mathbb{R}^+$ in intervals of lengths $\underline{\mathrm{Inj}}(u(\mathbb{R}^+))+\varepsilon$, we get a sequence of times $(t_\ell)_\mathbb{N}$, where each t_ℓ is in I_ℓ, and such that $g_{t_\ell}u$ belongs to $\overline{h_\mathbb{R}u}$ for every ℓ. Moreover, if $\underline{\mathrm{Inj}}(u(\mathbb{R}^+))=0$, as the intervals I_ℓ will be of length $0+\varepsilon$ for an $\varepsilon>0$ arbitrarily small, we get $g_{\mathbb{R}^+}u\subset\overline{h_\mathbb{R}u}$. $\square$

4. An example to prove Theorem 1.3

The Dirichlet domain centered at i of a Fuchsian group Γ with no elliptic element fixing i is defined by:

$$D_i(\Gamma):=\bigcap_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}}\mathbb{H}_i(\gamma),$$

where $\mathbb{H}_i(\gamma):=\{z\in\mathbb{H}^2\mid d(z,i)\leqslant d(z,\gamma(i))\}$. The following classical result (see [Dal11, Chap. I, Prop. 4.9]) shows a link between the ideal boundary of the Dirichlet domain and the almost minimizing character of geodesics.

Proposition 4.1. — *If $u\in\Gamma\backslash(T^1\mathbb{H}^2)$ and if for some lift $\widetilde{u}$ in $T^1\mathbb{H}^2$ the point $\widetilde{u}^+$ belongs to $\overline{D_i(\Gamma)}\cap\partial\mathbb{H}^2$, then $u(\mathbb{R}^+)$ is almost minimizing.*

Let us now construct our example. Consider, for any rational number $q\in[4,+\infty)$ and any n in $\mathbb{N}$, the hyperbolic isometry:

$$g_{q,n}:=\begin{pmatrix}\sqrt{q} & (1-q)r_n\\ -1/r_n & \sqrt{q}\end{pmatrix},$$

where $(r_n)_\mathbb{N}$ is the sequence of real numbers defined by:

$$\begin{cases} r_1=2\\ r_n=3r_{n-1},\ \forall\, n\geqslant 2.\end{cases}$$

Let $F_1:=\{g_{q,n}\mid q\in\mathbb{Q}\cap[4,+\infty),\ n\in\mathbb{N}\}$. We now conjugate the isometries $g_{4,n}$ by the isometries $T_{q,n}:=\left(\begin{smallmatrix}1 & t_{q,n}\\ 0 & 1\end{smallmatrix}\right)$, for any rational number $q\in(1,4)$ and any n in $\mathbb{N}$, with $t_{q,n} := -r_n(\sqrt{q}-2)$. We have:

$$h_{q,n} := T_{q,n}^{-1}\, g_{4,n}\, T_{q,n} = \begin{pmatrix} 4-\sqrt{q} & r_n[(\sqrt{q}-2)^2-3] \\ -1/r_n & \sqrt{q} \end{pmatrix}.$$

Set $F_2 := \{h_{q,n} \mid q \in \mathbb{Q}\cap(1,4),\, n\in\mathbb{N}\}$. For every $q\in\mathbb{Q}\cap(1,+\infty)$, we define the hyperbolic isometry $f_{q,n}$ by:

$$f_{q,n} := \begin{cases} h_{q,n} & \text{if } q\in\mathbb{Q}\cap(1,4), \\ g_{q,n} & \text{if } q\in\mathbb{Q}\cap[4,+\infty), \end{cases}$$

and set $F := F_1\cup F_2 = \{f_{q,n}, q\in\mathbb{Q}\cap(1,+\infty)\}$.

For any non-elliptic isometry γ in $\mathrm{PSL}_2(\mathbb{R})$, define $\partial\mathbb{H}_i(\gamma)$ to be the perpendicular bisector of the segment $[i,\gamma i]$. Also denote by $c(\gamma)$ the centre of the Euclidean half-circle $\partial\mathbb{H}_i(\gamma)$ and by $e_\ell(\gamma)$, with $\ell=1,2$, the endpoints in $\partial\mathbb{H}^2$ of $\partial\mathbb{H}_i(\gamma)$. Finally, denote by $C_{(\eta,\xi)}(z)$ the hypercycle with endpoints η and ξ in $\partial\mathbb{H}^2$ and passing through z in $\mathbb{H}^2$. The following key proposition gives us all the necessary information about the perpendicular bisectors $\partial\mathbb{H}_i(\gamma)$ (see Section 5 for the proof).

Proposition 4.2. — *For every $q\in(1,+\infty)$ we have the following:*

(i) $\lim_{n\to+\infty} c(f_{q,n}) = -\infty$ *and* $\lim_{n\to+\infty} c(f_{q,n}^{-1}) = +\infty$.

(ii) $\lim_{n\to+\infty} e_\ell(f_{q,n}) = -\infty$ *and* $\lim_{n\to+\infty} e_\ell(f_{q,n}^{-1}) = +\infty$ *for* $\ell=1,2$.

(iii) *If n is big enough, the perpendicular bisector $\partial\mathbb{H}_i(f_{q,n}^{-1})$ do not meet the hypercycle $C_{(0,\infty)}(1+2i/\sqrt{5})$.*

By means of Proposition 4.2, we shall choose a sequence of elements γ_m of F such that all the perpendicular bisectors $\partial\mathbb{H}_i(\gamma_m)$ are disjoint and which contains an infinite number of elements $f_{q,n}$ for every rational number $q\in(1,+\infty)$.

Consider any bijection $\psi:\mathbb{N}\mapsto\mathbb{Q}\cap(1,+\infty)\times\mathbb{N}$ and set $\psi(m)=(q_m,\psi_2(m))$. Observe that for every q in $(1,+\infty)$, there is an infinite number of elements q_m such that $q_m=q$. We can now write $F=\{f_{q_m,\psi_2(m)} \mid m\in\mathbb{N}\}$.

Choice of the elements γ_m. — Put $\gamma_0 := f_{q_0,0}$ and set $2C$ to be the distance between the geodesic $(0,\infty)$ and the hypercycle $C_{(0,\infty)}(1+2i/\sqrt{5})$. Then, for every $m\geqslant 1$, we define γ_m by induction. We ask that $\gamma_m = f_{q_m,n_m}$ where n_m is the smallest integer among the integers p satisfying the following properties:

(i) $|e_1(f_{q_m,p})| > |e_2(\gamma_{m-1})|$ and $|e_1(f_{q_m,p}^{-1})| > |e_2(\gamma_{m-1}^{-1})|$.

(ii) $d(\partial\mathbb{H}_i(f_{q_m,p}), \partial\mathbb{H}_i(f_{q_m,p}^{-1})) \geqslant 2C$.

(iii) $d(\partial\mathbb{H}_i(f_{q_m,p}), \partial\mathbb{H}_i(\gamma_s^{\pm1})) \geqslant C$ and $d(\partial\mathbb{H}_i(f_{q_m,p}^{-1}), \partial\mathbb{H}_i(\gamma_s^{-1})) \geqslant C$ for all integers $s<m$.

Such a sequence $(\gamma_m)_{\mathbb{N}}$ exists according to Proposition 4.2. We set $\Gamma := \langle\gamma_m \mid m\in\mathbb{N}\rangle$ and prove that $S=\Gamma\backslash\mathbb{H}^2$ fulfills the properties of Theorem 1.3.

By a classic ping-pong argument (see [Dal11]), the group Γ is discrete and free. Moreover, its Dirichlet domain centered at i is:

$$D_i(\Gamma) = \bigcap_{m\in\mathbb{N}} \mathbb{H}_i(\gamma_m) \cap \bigcap_{m\in\mathbb{N}} \mathbb{H}_i(\gamma_m^{-1}).$$

Fix $\widetilde{u}$ in $T^1\mathbb{H}^2$ such that $\widetilde{u}(0) = i$ and $\widetilde{u}^+ = \infty$ and consider its projection u to $T^1S := \Gamma\backslash(T^1\mathbb{H}^2)$. Since ∞ is in $\overline{D_i(\Gamma)} \cap \partial\mathbb{H}^2$, according to Proposition 4.1, the half-geodesic $u(\mathbb{R}^+)$ is almost minimizing. Hence (i) of Theorem 1.3.

To prove condition (ii), we use Proposition 2.3. Observe that it follows directly from the definition of $f_{q,n}$ that for every rational number q in $(1,+\infty)$, we have

$$f_{q,n}^{-1}\infty \underset{n\to+\infty}{\longrightarrow} \infty. \tag{4}$$

Moreover, since $\lim_{n\to+\infty} \operatorname{im}(f_{q,n}i) = 1/q$, we have:

$$B_\infty(f_{q,n}i, i) \underset{n\to+\infty}{\longrightarrow} B_\infty(i/q, i) = |\ln(q)|. \tag{5}$$

Now, for every $q \in \mathbb{Q}\cap(1,+\infty)$ there is an infinite number of elements $f_{q,n}$ in $(\gamma_m)_{\mathbb{N}}$, thus in Γ. So, according to (4), (5) and Proposition 2.3, it follows that all the elements $g_{|\ln(q)|}u$ belong to $\overline{h_{\mathbb{R}}u}$. Hence, $g_{\mathbb{R}^+}u$ is included in $\overline{h_{\mathbb{R}}u}$ and we get (ii) of Theorem 1.3.

Finally, fix z in the interior of $D_i(\Gamma)$ and any $\gamma = \gamma_{m_1}^{i_1}\cdots\gamma_{m_k}^{i_k}$ in Γ different from the identity, written as a reduced word in the letters γ_m. If $k=1$, then $\gamma = \gamma_{m_1}^{i_1}$ and

$$d(z,\gamma z) = d(z,\gamma^{-1}z) \geqslant \max(d(z,\partial\mathbb{H}_i(\gamma_{m_1})), d(z,\partial\mathbb{H}_i(\gamma_{m_1}^{-1})))$$

Since $d(\partial\mathbb{H}_i(\gamma_{m_1}), \partial\mathbb{H}_i(\gamma_{m_1}^{-1})) \geqslant 2C$, we obtain that $d(z,\gamma z) \geqslant C$. If $k>1$,

$$\begin{aligned} d(z,\gamma z) = d(z,\gamma_{m_1}^{i_1}\ldots\gamma_{m_k}^{i_k}z) &= d(\gamma_{m_1}^{-i_1}z, \gamma_{m_2}^{i_2}\ldots\gamma_{m_k}^{i_k}z) \\ &\geqslant d(\partial\mathbb{H}_i(\gamma_{m_1}^{\pm1}), \partial\mathbb{H}_i(\gamma_{m_2}^{\pm1})) \geqslant C. \end{aligned}$$

It follows that the injectivity radius on S is everywhere bigger than C. Finally, since all the axes of the elements γ_m in Γ intersect the half-geodesic $[i,\infty)$, and since their translation length ℓ_{γ_n} is constant by definition of the elements $f_{q,n}$, the injectivity radius $\underline{\operatorname{Inj}}(u(\mathbb{R}^+))$ is also finite. So $C \leqslant \underline{\operatorname{Inj}}(u(\mathbb{R}^+)) < +\infty$. Hence Property (iii) of Theorem 1.3. This completes the proof. $\square$

5. Proof of Proposition 4.2

Using the classical formula

$$\forall\, a,b\in\mathbb{H}^2,\quad \sinh\frac{d(a,b)}{2} = \frac{|a-b|}{2\sqrt{\operatorname{im}(a)\operatorname{im}(b)}},$$

we get:

Proposition 5.1. — *Consider a point $P = R + iI$ in $\mathbb{H}^2$. The equation of the perpendicular bisector of the hyperbolic segment between i and P is:*

$$\Big(x + \frac{R}{I-1}\Big)^2 + y^2 = I\Big(1 + \frac{R^2}{(I-1)^2}\Big).$$

For the following calculations, we distinguish the case $q \in [4, +\infty)$ from the case $q \in (1, 4)$.

Case 1: fix $q \in [4, +\infty)$. — We have $f_{q,n}i = R_{q,n} + iI_{q,n}$ where

$$R_{q,n} := \frac{\sqrt{q}(1-q)r_n - \sqrt{q}/r_n}{q + 1/r_n^2} \quad \text{and} \quad I_{q,n} := \frac{1}{q + 1/r_n^2}.$$

Since $\lim_{n\to+\infty} r_n = +\infty$, the quantities $R_{q,n}$ and $I_{q,n}$ are equivalent to $r_n(1-q)/\sqrt{q}$ and $1/q$ respectively as n converges to $+\infty$. Applying Proposition 5.1 we get the following asymptotic equivalence:

$$c(f_{q,n}) = -\frac{R_{q,n}}{I_{q,n} - 1} \underset{n\to+\infty}{\asymp} -r_n\sqrt{q}.$$

Hence, the centres $c(f_{q,n})$ converge to $-\infty$ as n goes to $+\infty$.

Let us now study the radii of the geodesics $\partial\mathbb{H}_i(f_{q,n})$. We have

$$\sqrt{I_{q,n}}\Big(1 + \frac{R_{q,n}^2}{(I_{q,n}-1)^2}\Big)^{1/2} \underset{n\to+\infty}{\asymp} \frac{1}{\sqrt{q}}\frac{R_{q,n}}{I_{q,n}-1} = -\frac{1}{\sqrt{q}}c(f_{q,n}).$$

According to Proposition 5.1, the endpoints $e_\ell(f_{q,n})$ for $\ell = 1, 2$ of the geodesics $\partial\mathbb{H}_i(f_{q,n})$ converge to $-\infty$ as n goes to $+\infty$. Moreover, since $1/\sqrt{q} \leqslant 2/3$, all these geodesics are below the hypercycle $C_{(0,\infty)}(-1 + 2i/\sqrt{5})$.

We now study the case of $\partial\mathbb{H}_i(f_{q,n}^{-1})$. We have:

$$f_{q,n}^{-1}i = \frac{\sqrt{q}\,(q-1)r_n + \sqrt{q}/r_n}{q + 1/r_n^2} + i\frac{1}{q + 1/r_n^2}.$$

We observe that the real part of $f_{q,n}^{-1}i$ is the negative of the real part of $f_{q,n}i$. So the geodesics $\partial\mathbb{H}_i(f_{q,n}^{-1})$ and $\partial\mathbb{H}_i(f_{q,n})$ are symmetric with respect to the imaginary axis. In particular, they are below the hypercycle $C_{(0,\infty)}(1 + 2i/\sqrt{5})$.

Case 2: fix $q \in \mathbb{Q} \cap (1, 4)$. — We have $f_{q,n}i = R_{q,n} + iI_{q,n}$ where

$$R_{q,n} := \frac{r_n\sqrt{q}((\sqrt{q}-2)^2 - 3) - (4 - \sqrt{q})/r_n}{q + 1/r_n^2} \quad \text{and} \quad I_{q,n} = \frac{1}{q + 1/r_n^2}.$$

Since $q \in (1, 4)$, the number $\sqrt{q}((\sqrt{q}-2)^2 - 3)$ is different from 0. Thus, as n goes to $+\infty$, we have the equivalences:

$$R_{q,n} \underset{n\to+\infty}{\asymp} r_n\frac{(\sqrt{q}-2)^2 - 3}{\sqrt{q}} \quad \text{and} \quad I_{q,n} \underset{n\to+\infty}{\asymp} \frac{1}{q}.$$

Hence, according to Proposition 5.1,

$$c(f_{q,n}) = -\frac{R_{q,n}}{I_{q,n} - 1} \underset{n\to+\infty}{\asymp} -r_n\sqrt{q}\,\frac{(\sqrt{q}-2)^2 - 3}{1 - q},$$

where $\sqrt{q}((\sqrt{q}-2)^2 - 3)/(1-q)$ is a real number greater than 2. Thus, these centers converge to $-\infty$ as n goes to $+\infty$.

Let us now study the radii of the geodesics $\partial\mathbb{H}_i(f_{q,n})$. We have

$$\sqrt{I_{n,q}}\Big(1 + \frac{R_{n,q}^2}{(I_{n,q}-1)^2}\Big)^{1/2} \underset{n\to+\infty}{\asymp} \frac{1}{\sqrt{q}}\frac{R_{n,q}}{I_{n,q}-1} = -\frac{1}{\sqrt{q}}c(f_{q,n}),$$

where $1/\sqrt{q}$ belongs to $(1/2,1)$. Thus, again, the endpoints $e_\ell(f_{q,n})$ for $\ell=1,2$ of the geodesics $\partial\mathbb{H}_i(f_{q,n})$ converge to $-\infty$ as n goes to $+\infty$.

We now study the case of $f_{q,n}^{-1}$ for $q\in(1,4)$. We have $f_{q,n}^{-1}i=R_{q,n}+iI_{q,n}$ where

$$R_{q,n}=\frac{-r_n(4-\sqrt{q})[(\sqrt{q}-2)^2-3]+\sqrt{q}/r_n}{(4-\sqrt{q})^2+1/r_n^2}\quad\text{and}\quad I_{q,n}=\frac{1}{(4-\sqrt{q})^2+1/r_n^2}.$$

Since $q\in(1,4)$, the number $(4-\sqrt{q})[(\sqrt{q}-2)^2-3]$ is different from 0. Thus, we get the equivalences

$$R_{q,n}\underset{n\to+\infty}{\asymp} r_n\frac{(\sqrt{q}-2)^2-3}{\sqrt{q}-4}\quad\text{and}\quad I_{q,n}\underset{n\to+\infty}{\asymp}\frac{1}{(4-\sqrt{q})^2},$$

from which we deduce

$$c(f_{q,n}^{-1})=-\frac{R_{q,n}}{I_{q,n}-1}\underset{n\to+\infty}{\asymp}-r_n\frac{((\sqrt{q}-2)^2-3)(\sqrt{q}-4)}{1-(4-\sqrt{q})^2}.$$

Since the number $((\sqrt{q}-2)^2-3)(\sqrt{q}-4)/(1-(4-\sqrt{q})^2)$ is negative for $q\in(1,4)$, the centers $c(f_{q,n}^{-1})$ of the geodesics $\partial\mathbb{H}_i(f_{q,n}^{-1})$ converge to $+\infty$ as n goes to $+\infty$.

We now study the radii:

$$\sqrt{I_{n,q}}\Big(1+\frac{R_{n,q}^2}{(I_{n,q}-1)^2}\Big)^{1/2}\underset{n\to+\infty}{\asymp}\frac{1}{4-\sqrt{q}}\,\frac{R_{q,n}}{I_{q,n}-1}=-\frac{1}{4-\sqrt{q}}\,c(f_{q,n}^{-1}),$$

where the number $1/(4-\sqrt{q})$ belongs to $(1/3,1/2)$. Thus, the endpoints $e_\ell(f_{q,n}^{-1})$ for $\ell=1,2$ converge to $+\infty$ as n goes to $+\infty$. Moreover, from the inequality $1/(4-\sqrt{q})<2/3$, we deduce that the geodesics $\partial\mathbb{H}_i(f_{q,n}^{-1})$ do not meet the hypercycle $C_{(0,\infty)}(1+2i/\sqrt{5})$ as claimed. □

References

[CM10] Y. Coudène & F. Maucourant – "Horocycles récurrents sur des surfaces de volume infini", *Geom. Dedicata* **149** (2010), p. 231–242.

[Dal11] F. Dal'Bo – *Geodesic and horocyclic trajectories*, Universitext, Springer-Verlag London, Ltd. & EDP Sciences, London & Les Ulis, 2011.

[Ebe77] P. Eberlein – "Horocycle flows on certain surfaces without conjugate points", *Trans. Amer. Math. Soc.* **233** (1977), p. 1–36.

[GL17] M. Gaye & C. Lo – "Sur l'inexistence d'ensembles minimaux pour le flot horocyclique", *Confluentes Math.* **9** (2017), no. 1, p. 95–104.

[Haa96] A. Haas – "Dirichlet points, Garnett points, and infinite ends of hyperbolic surfaces. I", *Ann. Acad. Sci. Fenn. Math.* **21** (1996), no. 1, p. 3–29.

[Hed36] G. A. Hedlund – "Fuchsian groups and transitive horocycles", *Duke Math. J.* **2** (1936), p. 530–542.

[Kul04] M. Kulikov – "The horocycle flow without minimal sets", *Comptes Rendus Mathématique* **338** (2004), no. 6, p. 477–480.

[Led97] F. Ledrappier – "Horospheres on abelian covers", *Bol. Soc. Brasil. Mat. (N.S.)* **28** (1997), no. 2, p. 363–375, Erratum: *Ibid.* **29** (1998), no. 1, p. 195.

[Mat16] S. Matsumoto – "Horocycle flows without minimal sets", *J. Math. Sci. Univ. Tokyo* **23** (2016), no. 3, p. 661–673.

[PP15] J. Parkkonen & F. Paulin – "On the hyperbolic orbital counting problem in conjugacy classes", *Math. Z.* **279** (2015), no. 3-4, p. 1175–1196.

[Rat91] M. Ratner – "Raghunathan's topological conjecture and distributions of unipotent flows", *Duke Math. J.* **63** (1991), no. 1, p. 235–280.

[Sar10] O. Sarig – "The horocyclic flow and the Laplacian on hyperbolic surfaces of infinite genus", *Geom. Funct. Anal.* **19** (2010), no. 6, p. 1757–1812.

[Sta95] A. N. Starkov – "Fuchsian groups from the dynamical viewpoint", *J. Dynam. Control Systems* **1** (1995), no. 3, p. 427–445.

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