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ON THE PERSISTENCE OF HÖLDER REGULAR PATCHES OF DENSITY FOR THE INHOMOGENEOUS NAVIER-STOKES EQUATIONS

by Raphaël Danchin & Xin Zhang

Abstract. — In our recent work dedicated to the Boussinesq equations [15], we established the persistence of solutions with piecewise constant temperature along interfaces with Hölder regularity. We here address the same question for the inhomogeneous Navier-Stokes equations satisfied by a viscous incompressible and inhomogeneous fluid. We prove that, indeed, in the slightly inhomogeneous case, patches of densities with $\mathcal{C}^{1,\varepsilon}$ regularity propagate for all time. Our result follows from the conservation of Hölder regularity along vector fields moving with the flow. The proof of that latter result is based on commutator estimates involving para-vector fields, and multiplier spaces. The overall analysis is more complicated than in [15], since the coupling between the mass and velocity equations in the inhomogeneous Navier-Stokes equations is *quasilinear* while it is linear for the Boussinesq equations.

Résumé (Persistance de la régularité höldérienne des poches de densité pour les équations de Navier-Stokes inhomogène)

Dans notre travail récent consacré aux équations de Boussinesq [15], on a établi la persistance de solutions avec température constante par morceaux le long d'interfaces à régularité höldérienne. On aborde ici la même question pour les équations de Navier-Stokes inhomogène satisfaites par un liquide visqueux incompressible à densité variable. On démontre que, dans le cas légèrement non homogène, les poches de densité avec régularité $\mathcal{C}^{1,\varepsilon}$ se propagent pour tout temps. Notre résultat est conséquence de la conservation de la régularité höldérienne le long des champs de vecteurs transportés par le flot de la solution. La preuve de ce dernier résultat repose sur des estimations de commutateur mettant en jeu des para-champs et des espaces de multiplicateurs. L'analyse est plus compliquée que dans [15], dans la mesure où le couplage entre les équations de la masse et de la vitesse dans les équations de Navier-Stokes inhomogène est quasilinéaire alors qu'il est linéaire pour les équations de Boussinesq.

Contents

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Introduction

We are concerned with the following *inhomogeneous incompressible Navier-Stokes equations* in the whole space $\mathbb{R}^N$ with $N \geqslant 2$:

$$\text{(INS)} \qquad \begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho(\partial_t u + u \cdot \nabla u) - \mu \Delta u + \nabla P = 0, \\ \operatorname{div} u = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases}$$

Above, the unknowns $(\rho, u, P) \in \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}$ stand for the density, velocity vector field and pressure, respectively, and the so-called viscosity coefficient μ is a positive constant.

A number of recent works have been dedicated to the mathematical analysis of the system (INS). In particular, it is well-known that if ρ_0 is positive and bounded, and $\sqrt{\rho_0}\, u_0$ is in $L^2(\mathbb{R}^N)$, then the system (INS) admits a global weak solution with finite energy (see [3] and the references therein). That result has been extended by J. Simon in [25] to the case $\rho_0 \geqslant 0$, and by P.-L. Lions in [23] to viscosity coefficients depending on the density.

In [23], P.-L. Lions raised the so-called *density patch problem*. It may be stated as follows: assume that $\rho_0 = \mathbb{1}_{\mathcal{D}_0}$ for some domain $\mathcal{D}_0$. Can we find conditions on u_0 that ensure that

$$\rho(t) = \mathbb{1}_{\mathcal{D}_t} \quad \text{for all} \quad t \geqslant 0 \tag{0.1}$$

for some domain $\mathcal{D}_t$ *with the same regularity as the initial one* ?

Whenever $\sqrt{\rho_0}\, u_0$ is in $L^2(\mathbb{R}^N)$, the renormalized solutions theory in [16] by R. DiPerna and P.-L. Lions for transport equations ensures that the global weak solutions mentioned above have a volume preserving generalized flow ψ and that we do have (0.1) with $\mathcal{D}_t$ being the image of $\mathcal{D}_0$ by $\psi(t, \cdot)$. However, without assuming more on u_0, it is very unlikely that one can get information on the persistence of regularity of $\mathcal{D}_t$ for positive times.

The present paper aims at making one more step toward solving Lions' question, by considering the case where

$$\rho_0 = \eta_1 \mathbb{1}_{\mathcal{D}_0} + \eta_2 \mathbb{1}_{\mathcal{D}_0^c}, \tag{0.2}$$

for some simply connected bounded domain $\mathcal{D}_0$ of class $\mathcal{C}^{1,\varepsilon}$.

Our goal is to find as general as possible conditions on u_0, that guarantee that for all time $t \geqslant 0$, the domain $\mathcal{D}_t := \psi(t, \mathcal{D}_0)$ remains $\mathcal{C}^{1,\varepsilon}$, and the density reads

$$\rho(t, \cdot) = \eta_1 \mathbb{1}_{\mathcal{D}_t} + \eta_2 \mathbb{1}_{\mathcal{D}_t^c}. \tag{0.3}$$

Several recent works give a partial answer to that issue if $|\eta_1 - \eta_2|$ is small enough. Indeed, the paper by the first author with P. B. Mucha [11] ensures that if ρ_0 is given by (0.2) and u_0 belongs to the critical Besov space $\dot{B}^{(N/p)-1}_{p,1}(\mathbb{R}^N)$ (see the definition below in (1.2)), then (0.3) is fulfilled for small time (and for all time if u_0 is small) and the $\mathcal{C}^1$ regularity is preserved. Likewise, according to the work [18] by J. Huang, M. Paicu and P. Zhang (see also [14]), one may solve (INS) if the initial density is close (for the L^∞ norm) to some positive constant and u_0 belongs to

$$\dot{B}^{(N/p)-1}_{p,r}(\mathbb{R}^N) \cap \dot{B}^{(N/p)+\delta-1}_{p,r}(\mathbb{R}^N)$$

for some $1 < r < \infty$. As the flow of the corresponding solution is $\mathcal{C}^{1,\delta'}$ for all $\delta' < \delta$, one can deduce that if ρ_0 is given by (0.2) then the $\mathcal{C}^{1,\varepsilon}$ regularity of the boundary is preserved *provided that* $\varepsilon < \delta$ (as the flow need not be in $\mathcal{C}^{1,\delta}$).

Finally, as noticed in [12] then improved by M. Paicu, P. Zhang and Z. Zhang [24], in the 2D case, if working *within the energy framework*, then one may avoid the smallness condition on the density and solve (INS) globally if ρ_0 and u_0 just satisfy

$$0 < \eta_1 \leqslant \rho_0 \leqslant \eta_2, \quad u_0 \in H^s(\mathbb{R}^2) \quad \text{for some } s > 0. \tag{0.4}$$

As the constructed velocity field therein admits a $\mathcal{C}^1$ flow, one can readily deduce that, if ρ_0 is given by (0.2) with $\mathcal{D}_0 \subset \mathbb{R}^2$ then the $\mathcal{C}^1$ regularity of the boundary is preserved.

The common point between the above works is that the hypotheses on u_0 do not take into account the non-isotropic structure of ρ_0. Consequently, the maximal regularity that can be propagated for the patch is limited by the overall regularity of the initial velocity. In two recent papers [22, 21] devoted to the 2D case (see also [20] for the 3-D case), X. Liao and P. Zhang pointed out that *only tangential regularity along the boundary of* $\mathcal{D}_0$ was needed to propagate high Sobolev regularity of the patch. They followed J.-Y. Chemin's approach in his work [7] dedicated to the vortex patches problem for the 2-D incompressible Euler equations, and characterized the regularity of the boundary of the domain by means of one (or several) tangent vector fields that evolve according to the flow of the velocity field.

More precisely, assume with no loss of generality that $\partial\mathcal{D}_0$ coincides with the level set $f_0^{-1}(\{0\})$ of some (at least $\mathcal{C}^1$) function $f_0 : \mathbb{R}^N \to \mathbb{R}$ that does not degenerate in a neighborhood of $\partial\mathcal{D}_0$, namely there exists some open neighborhood V_0 of $\mathcal{D}_0$ such that

$$\mathcal{D}_0 = f_0^{-1}(\{0\}) \cap V_0 \quad \text{and} \quad \nabla f_0 \text{ does not vanish on } V_0. \tag{0.5}$$

Then $\mathcal{D}_t$ coincides with $f_t^{-1}(\{0\})$, where $f_t \equiv f(t,\cdot) := f_0 \circ \psi_t^{-1}$ with $\psi_t := \psi(t,\cdot)$ and ψ being the solution of the (integrated) ordinary differential equation:

$$\psi(t,x) = x + \int_0^t u\big(\tau, \psi(\tau,x)\big)\, d\tau. \tag{0.6}$$

Now, the tangent vector field $X_t := \nabla^\perp f_t$ coincides with the evolution of $X_0 := \nabla^\perp f_0$ along the flow of u, namely:(1)

$$X(t,\cdot) := (\partial_{X_0}\psi_t) \circ \psi_t^{-1}, \tag{0.7}$$

(1)For any vector field $Y = Y^k(x)\partial_k$ and function f in $\mathcal{C}^1(\mathbb{R}^N;\mathbb{R})$, we denote by $\partial_Y f$ the *directional derivative* of f along Y, that is, with the Einstein summation convention, $\partial_Y f := Y^k \partial_k f = Y \cdot \nabla f$.

and thus satisfies the transport equation

$$
\begin{cases}\partial_t X + u\cdot\nabla X = \partial_X u,\\ X|_{t=0} = X_0.\end{cases} \tag{0.8}
$$

Consequently, the problem of persistence of regularity for the patch reduces to that of the vector field X solution to (0.8).

In their first paper [22], X. Liao and P. Zhang justified that heuristics in the case where the jump $|\eta_1-\eta_2|$ is small enough, and $u_0\in\big(W^{1,p}(\mathbb{R}^2)\big)^2$ for $2<p<4$. Their proof was essentially based on weighted L_p-L_q estimates for the velocity and allowed to propagate Sobolev regularity $W^{k,p}$ of the boundary, with k large enough (in particular the boundary is at least $\mathcal{C}^{2,\varepsilon}$ for some $\varepsilon>0$). In a second paper [21], after revisiting the approach of [24] (that is Sobolev spaces H^s with $s>0$ and thus finite energy framework), X. Liao and P. Zhang succeeded in proving a similar result for *general* positive η_1 and η_2 in (0.2). The corresponding level set function f_0 has to be in $W^{2+k,p}(\mathbb{R}^2)$ for some integer number $k\geqslant 1$ and $p\in\,]2,4[$, hence $\mathcal{D}_0$ is still at least $\mathcal{C}^{2,\varepsilon}$. As regards the initial velocity field u_0, it has to satisfy the following *striated regularity* property along the vector field $X_0:=\nabla^\perp f_0$:

$$
\big(\partial^\ell_{X_0}u_0\in B^{s+\varepsilon(k-\ell)/k}_{2,1}(\mathbb{R}^2)\big)^2 \ \text{for all}\ \ \ell\in\{0,\dots,k\}
$$

with $0<s<1-\varepsilon$ and (s,p) in $]0,1[\,\times\,]2,\min\{4,2/(1-s)\}[$.

In the present paper, we propose a simpler approach that allows to propagate just $\mathcal{C}^{1,\varepsilon}$ Hölder regularity (for all $\varepsilon\in\,]0,1[$), within a *critical* regularity framework. By critical, we mean that the solution space that we shall consider has the same scaling invariance by time and space dilations as (INS) itself, namely:

$$
(\rho,u,P)(t,x)\longrightarrow(\rho,\lambda u,\lambda^2P)(\lambda^2t,\lambda x)\quad\text{and}\quad(\rho_0,u_0)(x)\longrightarrow(\rho_0,\lambda u_0)(\lambda x). \tag{0.9}
$$

That framework is by now classical for the homogeneous Navier-Stokes equations (that is ρ is a positive constant in (INS)) in the whole space $\mathbb{R}^N$ (see e.g. [4, 19] and the references therein). As observed by the first author in [10] (see also H. Abidi in [1] and H. Abidi and M. Paicu in [2]), working in a suitable critical functional framework is still relevant in the inhomogeneous situation.

Acknowledgements. — We are grateful to the referees for pointing out the work [17] by F. Gancedo and E. García-Juárez that has been posted on arXiv a few weeks after we submitted our paper. There, in the 2D case, the $\mathcal{C}^{1,\varepsilon}$ regularity of the interface is propagated whenever the density is given by (0.2) with $\eta_1>0$ and $\eta_2>0$, and u_0 is in $H^{\varepsilon'}$ for some $\varepsilon'>\varepsilon$. That result relies on the nice observation that in that particular case, the flow of the solution constructed in [24] has $\mathcal{C}^{1,\varepsilon}$ regularity *in all directions*. In the present paper, we are able to consider densities which are not piecewise constant (like in (0.2) with a smooth variable η_1 for example), and velocity fields that have just critical regularity but we need the density to be close to some positive constant.

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1. Results

Before stating our main result, we need to introduce a few notations. First, we recall the definition of Besov spaces: following [4, §2.2], we consider two smooth radial functions χ and φ supported in $\{\xi\in\mathbb{R}^N : |\xi|\leqslant 4/3\}$ and $\{\xi\in\mathbb{R}^N : 3/4\leqslant|\xi|\leqslant 8/3\}$, respectively, and satisfying

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1,\quad \forall\,\xi\in\mathbb{R}^N\smallsetminus\{0\},\quad \chi(\xi)+\sum_{j\geqslant 0}\varphi(2^{-j}\xi)=1,\quad \forall\,\xi\in\mathbb{R}^N. \tag{1.1}$$

Then we define Fourier truncation operators as follows:

$$\dot\Delta_j:=\varphi(2^{-j}D),\ \dot S_j:=\chi(2^{-j}D),\ \forall j\in\mathbb{Z};\quad \Delta_j:=\varphi(2^{-j}D),\ \forall j\geqslant 0,\quad \Delta_{-1}:=\chi(D).$$

For all triplet $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$, the homogeneous Besov space $\dot B^s_{p,r}(\mathbb{R}^N)$ (just denoted by $\dot B^s_{p,r}$ if the value of the dimension is clear from the context) is defined by

$$\dot B^s_{p,r}(\mathbb{R}^N):=\big\{u\in\mathcal{S}'_h(\mathbb{R}^N):\|u\|_{\dot B^s_{p,r}}:=\big\|2^{js}\|\dot\Delta_j u\|_{L^p}\big\|_{\ell^r(\mathbb{Z})}<\infty\big\}, \tag{1.2}$$

where $\mathcal{S}'_h(\mathbb{R}^N)$ is the subspace of tempered distributions $\mathcal{S}'(\mathbb{R}^N)$ defined by

$$\mathcal{S}'_h(\mathbb{R}^N):=\big\{u\in\mathcal{S}'(\mathbb{R}^N):\lim_{j\to-\infty}\dot S_j u=0\,\big\}\cdot$$

We shall also use sometimes the following inhomogeneous Besov spaces:

$$B^s_{p,r}(\mathbb{R}^N):=\big\{u\in\mathcal{S}'(\mathbb{R}^N):\|u\|_{B^s_{p,r}}:=\big\|2^{js}\|\Delta_j u\|_{L^p}\big\|_{\ell^r(\mathbb{N}\cup\{-1\})}<\infty\big\}. \tag{1.3}$$

Throughout, we adopt the common notation $b^s_{p,r}(\mathbb{R}^N)$ to denote $B^s_{p,r}(\mathbb{R}^N)$ or $\dot B^s_{p,r}(\mathbb{R}^N)$.

It is well-known that Sobolev or Hölder spaces belong to the Besov spaces hierarchy. For instance $\dot B^s_{2,2}(\mathbb{R}^N)$ coincides with the homogeneous Sobolev space $\dot H^s(\mathbb{R}^N)$, and we have

$$B^s_{\infty,\infty}(\mathbb{R}^N)=\mathcal{C}^{0,s}(\mathbb{R}^N)=L^\infty(\mathbb{R}^N)\cap\dot B^s_{\infty,\infty}(\mathbb{R}^N)\quad\text{if}\quad s\in\,]0,1[. \tag{1.4}$$

To emphasize that connection, we shall often use the notation $\dot{\mathscr{C}}^s:=\dot B^s_{\infty,\infty}$ (or $\mathscr{C}^s:=B^s_{\infty,\infty}$) for *any* $s\in\mathbb{R}$.

When investigating evolutionary equations in critical Besov spaces, it is wise to use the following *tilde homogeneous Besov spaces* first introduced by J.-Y. Chemin in [8]: for any $T\in\,]0,+\infty]$ and $(s,p,r,\gamma)\in\mathbb{R}\times[1,+\infty]^3$, we set(2)

$$\widetilde L^\gamma_T(\dot B^s_{p,r}):=\Big\{u\in\mathcal{S}'(]0,T[\times\mathbb{R}^N):\lim_{j\to-\infty}\dot S_j u=0\text{ in }L^\gamma_T(L^\infty)\text{ and }\|u\|_{\widetilde L^\gamma_T(\dot B^s_{p,r})}<\infty\Big\},$$

(2)For $T\in\,]0,+\infty[$, $p\in[1,+\infty]$ and E a Banach space, the notation $L^p_T(E)$ denotes the space of L^p functions on $]0,T[$ with values in E, and $L^p(\mathbb{R}_+;E)$ corresponds to the case $T=+\infty$. We keep the same notation for vector or matrix-valued functions.

where

$$\|u\|_{\widetilde{L}^\gamma_T(\dot{B}^s_{p,r})} := \big\|2^{js}\|\dot{\Delta}_j u\|_{L^\gamma_T(L^p)}\big\|_{\ell^r(\mathbb{Z})} < \infty.$$

The index T will be omitted if equal to $+\infty$, and we shall denote

$$\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot{B}^s_{p,r}) := \widetilde{L}^\infty(\mathbb{R}_+;\dot{B}^s_{p,r}) \cap \mathcal{C}(\mathbb{R}_+;\dot{B}^s_{p,r}).$$

We also need to introduce the following spaces for $(\sigma, p, T) \in \mathbb{R} \times [1,\infty] \times]0,\infty]$:

$$\dot{E}^\sigma_p(T) := \big\{(v,\nabla Q) : v \in \widetilde{\mathcal{C}}_b\big([0,T[;\dot{B}^{(N/p)-1+\sigma}_{p,1}\big),\ (\partial_t v, \nabla^2 v, \nabla Q) \in L^1_T\big(\dot{B}^{(N/p)-1+\sigma}_{p,1}\big)\big\},$$

endowed with the norm

$$\|(v,\nabla Q)\|_{\dot{E}^\sigma_p(T)} := \|v\|_{\widetilde{L}^\infty_T(\dot{B}^{(N/p)+\sigma-1}_{p,1})} + \|(\partial_t v, \nabla^2 v, \nabla Q)\|_{L^1_T(\dot{B}^{(N/p)+\sigma-1}_{p,1})}.$$

For notational simplicity, we shall omit σ or T in the notation $\dot{E}^\sigma_p(T)$ whenever σ is zero or $T=\infty$. For instance, $\dot{E}_p := \dot{E}^0_p(\infty)$.

Finally, we shall make use of *multiplier spaces* associated to pairs (E,F) of Banach spaces included in the set of tempered distributions. The definition goes as follows:

Definition. — Let E and F be two Banach spaces embedded in $\mathcal{S}'(\mathbb{R}^N)$. The *multiplier space* $\mathcal{M}(E\to F)$ (simply denoted by $\mathcal{M}(E)$ if $E=F$) is the set of those functions φ satisfying $\varphi u \in F$ for all u in E and, additionally,

$$\|\varphi\|_{\mathcal{M}(E\to F)} := \sup_{\substack{u\in E\\ \|u\|_E \leqslant 1}} \|\varphi u\|_F < \infty. \tag{1.5}$$

It goes without saying that $\|\cdot\|_{\mathcal{M}(E\to F)}$ is a norm on $\mathcal{M}(E\to F)$ and that one may restrict the supremum in (1.5) to any *dense* subset of E.

The following result that has been proved in [11] is the starting point of our analysis:[(3)]

Theorem 1.1. — *Let $p\in[1,2N[$ and u_0 be a divergence-free vector field with coefficients in $\dot{B}^{(N/p)-1}_{p,1}$. Assume that ρ_0 belongs to the multiplier space $\mathcal{M}\big(\dot{B}^{(N/p)-1}_{p,1}\big)$. There exist two constants c and C depending only on p and on N such that if*

$$\|\rho_0 - 1\|_{\mathcal{M}(\dot{B}^{(N/p)-1}_{p,1})} + \|u_0\|_{\dot{B}^{(N/p)-1}_{p,1}} \leqslant c$$

then the system (INS) *in $\mathbb{R}^N$ with $N\geqslant 2$ has a unique solution $(\rho,u,\nabla P)$ satisfying*

$$\rho \in L^\infty\Big(\mathbb{R}_+;\mathcal{M}\big(\dot{B}^{(N/p)-1}_{p,1}\big)\Big) \quad \textit{and} \quad (u,\nabla P)\in \dot{E}_p.$$

Furthermore, the following inequality is fulfilled:

$$\|u\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{(N/p)-1}_{p,1})} + \|\partial_t u, \nabla^2 u, \nabla P\|_{L^1(\mathbb{R}^+;\dot{B}^{(N/p)-1}_{p,1})} \leqslant C\|u_0\|_{\dot{B}^{(N/p)-1}_{p,1}}. \tag{1.6}$$

(3) As the viscosity coefficient μ will be fixed once and for all, we shall set it to 1 for notational simplicity. Likewise, we shall assume the reference density at infinity to be 1.

By classical embedding, having $\nabla^2 u$ in $L^1(\mathbb{R}_+;\dot B^{(N/p)-1}_{p,1})$ implies that ∇u is in $L^1(\mathbb{R}_+;\mathcal{C}_b)$. Therefore the flow ψ of u defined by (0.6) is in $\mathcal{C}^1$. Now, it has been observed in [11] that for any uniformly $\mathcal{C}^1$ bounded domain $\mathcal{D}_0$, the function $\mathbb{1}_{\mathcal{D}_0}$ belongs to $\mathcal{M}\big(\dot B^s_{p,1}\big)$ whenever $-1+1/p<s<1/p$. Therefore, one may deduce from Theorem 1.1 that if ρ_0 is given by (0.2), if u_0 is in $\dot B^{(N/p)-1}_{p,1}$ for some $N-1<p<2N$ and if

$$\|u_0\|_{\dot B^{(N/p)-1}_{p,1}}+|\eta_2-\eta_1|\quad\text{is small enough}$$

then the system (INS) admits a unique global solution $(\rho,u,\nabla P)$ with $(u,\nabla P)$ in $\dot E_p$ and ρ given by (0.3) with $\mathcal{D}_t$ in $\mathcal{C}^1$ for all time $t\geqslant 0$.

The (parabolic type) gain of regularity for u pointed out in Theorem 1.1 is optimal, as well as the embedding of $\dot B^{N/p}_{p,1}$ in the set of continuous bounded functions. Therefore, one cannot expect the flow of u given by Theorem 1.1 to be in any Hölder space $\mathcal{C}^{1,\alpha}$ for some $\alpha>0$, which prevents our propagating more than $\mathcal{C}^1$ regularity. Following [22, 21], it is natural to make an additional tangential regularity assumption on u_0. This motivates the following general result of persistence of geometric structures for (INS).

Theorem 1.2. — *Let ε be in $]0,1[$ and p satisfy*

$$N/2<p<\min\big\{N/(1-\varepsilon),2N\big\}\cdot \tag{1.7}$$

Let u_0 be a divergence-free vector field with coefficients in $\dot B^{(N/p)-1}_{p,1}$. Assume that the initial density ρ_0 is bounded and belongs to the multiplier space

$$\mathcal{M}\big(\dot B^{(N/p)-1}_{p,1}\big)\cap\mathcal{M}(\dot B^{(N/p)+\varepsilon-2}_{p,1}).$$

There exists a constant c depending only on p and N such that if

$$\|\rho_0-1\|_{\mathcal{M}(\dot B^{(N/p)-1}_{p,1})\cap\mathcal{M}(\dot B^{(N/p)+\varepsilon-2}_{p,1})\cap L^\infty}+\|u_0\|_{\dot B^{(N/p)-1}_{p,1}}\leqslant c, \tag{1.8}$$

then the system (INS) *in $\mathbb{R}^N$ has a unique global solution $(\rho,u,\nabla P)$ with*

$$\rho\in L^\infty\Big(\mathbb{R}_+;L^\infty\cap\mathcal{M}\big(\dot B^{(N/p)-1}_{p,1}\big)\cap\mathcal{M}(\dot B^{(N/p)+\varepsilon-2}_{p,1})\Big)\quad\textit{and}\quad(u,\nabla P)\in\dot E_p.$$

Moreover, for any vector field X_0 with $\mathcal{C}^{0,\varepsilon}$ regularity (assuming in addition that $\varepsilon>2-N/p$ if $\operatorname{div}X_0\not\equiv 0$), if the following conditions are fulfilled

$$\partial_{X_0}\rho_0\in\mathcal{M}\big(\dot B^{(N/p)-1}_{p,1}\longrightarrow\dot B^{(N/p)+\varepsilon-2}_{p,1}\big)\quad\textit{and}\quad\partial_{X_0}u_0\in\dot B^{(N/p)+\varepsilon-2}_{p,1},$$

then the system (0.8) *in $\mathbb{R}^N$ has a unique global solution $X\in C_w(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon})$, and we have*

$$\partial_X\rho\in L^\infty\Big(\mathbb{R}_+;\mathcal{M}\big(\dot B^{(N/p)-1}_{p,1}\longrightarrow\dot B^{(N/p)+\varepsilon-2}_{p,1}\big)\Big)\quad\textit{and}\quad(\partial_X u,\partial_X\nabla P)\in\dot E^{\varepsilon-1}_p.$$

Some comments are in order:

– The divergence-free property on X_0 is conserved during the evolution because if one takes the divergence of (0.8), then we get, remembering that $\operatorname{div} u = 0$,

$$\begin{cases} \partial_t \operatorname{div} X + u \cdot \nabla \operatorname{div} X = 0, \\ \operatorname{div} X|_{t=0} = \operatorname{div} X_0. \end{cases} \tag{1.9}$$

– In the case $\operatorname{div} X_0 \neq 0$, the additional constraint on (ε, p) is due to the fact that the product of a general $\mathcal{C}^{0,\varepsilon}$ function with a $\dot{B}^{(N/p)-2}_{p,1}$ distribution need not be defined if the sum of regularity coefficients, namely $\varepsilon + (N/p) - 2$, is negative.

– The vector field X given by (0.8) has the Finite Propagation Speed Property. Indeed, from the definitions of the flow and of the space $\dot{E}_p$, and from the embedding of $\dot{B}^{N/p}_{p,1}(\mathbb{R}^N)$ in $\mathcal{C}_b(\mathbb{R}^N)$, we readily get[(4)] for all $t \geqslant 0$ and $x \in \mathbb{R}^N$,

$$\big|\psi(t,x) - x\big| \lesssim \sqrt{t}\, \|u\|_{L^2_t(\dot{B}^{N/p}_{p,1})} \leqslant C\sqrt{t}\, \|u_0\|_{\dot{B}^{(N/p)-1}_{p,1}}.$$

Therefore, if the initial vector field X_0 is supported in the set K_0 then $X(t)$ is supported in some set K_t such that

$$\operatorname{diam}(K_t) \leqslant \operatorname{diam}(K_0) + C\sqrt{t}\, \|u_0\|_{\dot{B}^{(N/p)-1}_{p,1}}.$$

– One can prove a similar result (only local in time) for *large* u_0 in $\dot{B}^{(N/p)-1}_{p,1}$. Moreover, we expect our method to be appropriate for handling Hölder regularity $\mathcal{C}^{k,\varepsilon}$ if making suitable assumptions on $\partial^j_{X_0}\rho_0$ and $\partial^j_{X_0} u_0$ for $j = 0, \ldots, k$. We refrained from writing out here this generalization to keep the presentation as short as possible.

In the density patch situation (that is if ρ_0 is given by (0.2)) the condition on $\partial_{X_0}\rho_0$ is trivially satisfied as the derivative of the density along any continuous vector field that is tangent to $\partial\mathcal{D}_0$, vanishes. This implies the following statement of propagation of Hölder regularity of density patches for (INS) in the plane:

Theorem 1.3. — *Let $\mathcal{D}_0$ be a simply connected bounded domain of $\mathbb{R}^2$ satisfying* (0.5) *for some function $f_0 \in \mathcal{C}^{1,\varepsilon}(\mathbb{R}^2;\mathbb{R})$ with ε in $]0,1[$. There exists a constant η_0 depending only on $\mathcal{D}_0$ and such that if*

$$\rho_0 := (1+\eta)\mathbb{1}_{\mathcal{D}_0} + \mathbb{1}_{\mathcal{D}_0^c} \quad \textit{with} \quad \eta \in\,]-\eta_0, \eta_0[\tag{1.10}$$

and if the divergence free vector-field $u_0 \in \mathcal{S}'_h(\mathbb{R}^2)$ has vorticity $\omega_0 := \partial_1 u_0^2 - \partial_2 u_0^1$ given by

$$\omega_0 = \overline{\omega}_0 + \widetilde{\omega}_0\, \mathbb{1}_{\mathcal{D}_0} \quad \textit{with} \quad \operatorname{div}(\overline{\omega}_0 \nabla^\perp f_0) = 0 \quad \textit{and} \quad \int_{\mathbb{R}^2} \omega_0\, dx = 0 \tag{1.11}$$

for some small enough compactly supported functions $(\overline{\omega}_0, \widetilde{\omega}_0)$ in $L^p(\mathbb{R}^2) \times \mathcal{C}^{\varepsilon'}(\mathbb{R}^2)$ with $0 < \varepsilon' < \varepsilon$ and $1 < p < 2/(2-\varepsilon)$, then the system (INS) *has a unique solution*

(4) All over the paper, we agree that $A \lesssim B$ means $A \leqslant CB$ for some harmless "constant" C, the meaning of which may be guessed from the context.

$(\rho, u, \nabla P)$ with the properties listed in Theorem 1.1*. Moreover, if we denote by $\psi(t,\cdot)$ the flow of u then for all $t \geqslant 0$, we have*

$$\rho(t,\cdot) := (1+\eta)\mathbb{1}_{\mathcal{D}_t} + \mathbb{1}_{\mathcal{D}_t^c} \quad \textit{with} \quad \mathcal{D}_t := \psi(t,\mathcal{D}_0), \tag{1.12}$$

and $\mathcal{D}_t$ remains a simply connected bounded domain of class $\mathcal{C}^{1,\varepsilon}$.

Remark 1.4. — Of course, one can take $\overline{\omega}_0 \equiv 0$ or $\widetilde{\omega}_0 \equiv 0$ in the above statement.

The zero average condition guarantees that u_0 belongs to some homogeneous Besov space $\dot{B}^{(2/p)-1}_{p,1}$ (so as to apply Theorem 1.2). It is not needed if we have constant vortex pattern in $\mathbb{R}^2$ (see Theorem 2.3) or if the dimension $N=3$ (see Theorem 2.4).

Remark 1.5. — We imposed the particular structure of the vorticity in Theorem 1.3 just to give an explicit example for which (0.3) with regularity $\mathcal{C}^{1,\varepsilon}$ holds true. It goes without saying that one can consider a much more general class of initial velocities: according to Theorem 1.2, it suffices that u_0 satisfies the smallness condition of Theorem 1.1 and that $\operatorname{div}(\nabla^\perp f_0 \otimes u_0)$ is in $\dot{B}^{(2/p)+\varepsilon-2}_{p,1}$ for some $1 < p < \min\{4, 2/(1-\varepsilon)\}$. In other words, we just need "u_0 to have ε more regularity in the direction that is tangential to the patch of density." This is of course satisfied if u_0 vanishes in a neighborhood of $\mathcal{D}_0$. However, one may consider much more singular examples like the case where u_0 is compactly supported and behaves locally near some $x_0 \notin \partial\mathcal{D}_0$, like the function $|x-x_0|^{-1}(-\log|x-x_0|)^{-(1+\delta)}$ with $\delta > 0$.

Remark 1.6. — Similar results, only local in time, hold true for large u_0 with critical regularity.

We end this section with a short presentation of the main ideas of the proof of Theorem 1.2. From Theorem 1.1, we have a global solution $(\rho, u, \nabla P)$ such that $\rho \in L^\infty\big(\mathbb{R}_+; \mathcal{M}\big(\dot{B}^{(N/p)-1}_{p,1}\big)\big)$ and $(u, \nabla P) \in \dot{E}_p$. As already explained, our main task is to prove that $X(t,\cdot)$ remains in $\mathcal{C}^{0,\varepsilon}$ for all time. Now, in light of (0.8), we have

$$X(t,x) = X_0\big(\psi_t^{-1}(x)\big) + \int_0^t \partial_X u\big(t', \psi_{t'}\big(\psi_t^{-1}(x)\big)\big)\,dt'.$$

Therefore, because ψ_t is a $\mathcal{C}^1$ diffeomorphism of $\mathbb{R}^N$, it suffices to show that $\partial_X u$ is in $L^1_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{C}^{0,\varepsilon})$. To this end, it is natural to look for a suitable evolution equation for $\partial_X u$. Since (0.8) means that $[D_t, \partial_X] = 0$, where $D_t := \partial_t + u\cdot\nabla$ stands for the material derivative associated to u, differentiating the momentum equation of (INS) along X yields

$$\rho D_t \partial_X u + \partial_X \rho D_t u - \partial_X \Delta u + \partial_X \nabla P = 0. \tag{1.13}$$

Even though (1.13) has some similarities with the Stokes system, it is not clear that it does have the same smoothing properties, as its coefficients have very low regularity. One of the difficulties lies in the product of the discontinuous function ρ with $D_t\partial_X u$, as having only $\partial_X u$ in $\mathcal{C}^{0,\varepsilon}$ suggests that $D_t \partial_X u$ has *negative* regularity. At the same time, the term with $\partial_X\rho$ is harmless as, owing to $[D_t,\partial_X]=0$ and to the mass equation, we have

$$D_t\partial_X\rho=0. \tag{1.14}$$

Hence any (reasonable) regularity assumption for $\partial_{X_0}\rho_0$ persists through the evolution.

Our strategy is to assume that ρ belongs to the multiplier space corresponding to the space to which $D_t\partial_X u$ is expected to belong. As the flow is $\mathcal{C}^1$, propagating multiplier information is straightforward (see Lemma A.3). This new viewpoint spares us the tricky energy estimates and iterated differentiation along vector fields (requiring higher regularity of the patch) that were the cornerstone of the work by X. Liao and P. Zhang. In fact, *under the smallness assumption* (1.8) which, unfortunately, forces the fluid to have small density variations, we succeed in closing the estimates using *only one* differentiation along X. This makes the proof rather elementary and allows us to propagate low Hölder regularity.

Whether one can differentiate terms like Δu or ∇P along X within our critical regularity framework is not totally clear, though. Therefore, as in our recent work [15] dedicated to the incompressible Boussinesq system, we shall replace differentiation along vector-fields by *para-differentiation*.

Let us briefly recall how it works. Fix some suitably large integer N_0 and introduce the following *paraproduct* and *remainder* operators (after J.-M. Bony in [5]):

$$\dot T_u v:=\sum_{j\in\mathbb{Z}}\dot S_{j-N_0}u\dot\Delta_j v\quad\text{and}\quad \dot R(u,v)\equiv\sum_{j\in\mathbb{Z}}\dot\Delta_j u\widetilde{\dot\Delta}_j v:=\sum_{\substack{j\in\mathbb{Z}\\ |j-k|\leqslant N_0}}\dot\Delta_j u\dot\Delta_k v.$$

Then any product may be formally decomposed as follows:

$$uv=\dot T_u v+\dot T_v u+\dot R(u,v). \tag{1.15}$$

To overcome the problem with the definition of $\partial_X\Delta u$ and $\partial_X\nabla P$, we shall change the vector field X to the para-vector field operator $\dot{\mathcal{T}}_X\cdot:=\dot T_{X^k}\partial_k\cdot$ which, in our regularity framework, will turn out to be *the principal part* of operator ∂_X. Indeed, for any pair (X,f), the decomposition (1.15) ensures that

$$(\dot{\mathcal{T}}_X-\partial_X)f=\dot T_{\partial_k f}X^k+\partial_k\dot R(f,X^k)-\dot R(f,\operatorname{div}X).$$

Therefore, taking advantage of classical continuity results for operators $\dot T$ and $\dot R$ (see [4]), we discover that

$$\|(\dot{\mathcal{T}}_X-\partial_X)f\|_{\dot B^{(N/p)+\varepsilon-2}_{p,1}}\lesssim\|f\|_{\dot B^{(N/p)-1}_{p,1}}\|X\|_{\dot{\mathscr{C}}^\varepsilon} \tag{1.16}$$

whenever $(\varepsilon,p)\in\,]0,1[\,\times[1,+\infty]$ fulfills:

$$N/p\in\,]\,1-\varepsilon,\,2\,[\quad\text{if}\quad\operatorname{div}X=0,\qquad\text{and}\quad N/p\in\,]\,2-\varepsilon,\,2\,[\quad\text{otherwise.} \tag{1.17}$$

In our situation, we will apply (1.16) with $f=\nabla P$ or Δu, which are in $L^1\big(\mathbb{R}_+;\dot B^{(N/p)-1}_{p,1}(\mathbb{R}^N)\big)$.

Now, incising the term $\partial_X u$ by the scalpel $\dot{\mathcal{T}}_X$ in (1.13) and applying $\dot{\mathcal{T}}_X$ to the third equation of (INS) yield

$$\begin{cases}\rho D_t \dot{\mathcal{T}}_X u - \Delta \dot{\mathcal{T}}_X u + \nabla \dot{\mathcal{T}}_X P = g,\\ \operatorname{div} \dot{\mathcal{T}}_X u = \operatorname{div}(\dot{T}_{\partial_k X} u^k - \dot{T}_{\operatorname{div} X} u),\\ \dot{\mathcal{T}}_X u|_{t=0} = \dot{\mathcal{T}}_{X_0} u_0\end{cases} \tag{1.18}$$

with

$$\begin{aligned} g := -\rho[\dot{\mathcal{T}}_X, D_t]u + [\dot{\mathcal{T}}_X, \Delta]u - [\dot{\mathcal{T}}_X, \nabla]P + (\partial_X - \dot{\mathcal{T}}_X)(\Delta u - \nabla P)\\ - \partial_X \rho D_t u + \rho(\dot{\mathcal{T}}_X - \partial_X) D_t u.\end{aligned} \tag{1.19}$$

This surgery leading to (1.18) is effective for three reasons. First, all the commutator terms in (1.19) are under control (see the Appendix). Second, $D_t \partial_X u$ and $D_t \dot{\mathcal{T}}_X u$ are in the same Besov space, and the multiplier type regularity on the density that was pointed out before is thus appropriate. Last, the condition (1.8) ensures that $(\dot{\mathcal{T}}_X - \partial_X)u$ is a remainder term. Of course, the divergence free condition need not be satisfied by $\dot{\mathcal{T}}_X u$, but one can further modify (1.18) so as to enter in the standard maximal regularity theory. Then, under the smallness condition (1.8), one can close the estimates involving $\partial_X u$ or $\partial_X \rho$, globally in time.

The rest of the paper unfolds as follows. In the next section, we show that Theorem 1.2 entails a general (but not so explicit) result of persistence of Hölder regularity for patches of density in any dimension. We shall then obtain Theorem 1.3, and an analogous result in dimension $N = 3$. Section 3 is devoted to the proof of our general result of all-time persistence of striated regularity (Theorem 1.2). Some technical results pertaining to commutators and multiplier spaces are postponed in appendix.

2. The density patch problem

This section is devoted to the proof of results of persistence of regularity for patches of constant densities, taking Theorem 1.2 for granted. Throughout this section we shall use repeatedly the fact (proved in e.g. [11, Lem. A.7]) that for any (not necessarily bounded) domain $\mathcal{D}$ of $\mathbb{R}^N$ with uniform $\mathcal{C}^1$ boundary, we have

$$\mathbb{1}_{\mathcal{D}} \in \mathcal{M}\big(\dot{B}^s_{p,r}(\mathbb{R}^N)\big) \quad \text{whenever} \quad (s,p,r) \in](1/p)-1, 1/p[\times]1,\infty[\times [1,\infty].$$

From that property, we deduce that if $(\varepsilon, p) \in]0,1[\times]N-1, \frac{N-1}{1-\varepsilon}[$, then the density ρ_0 given by (1.10) belongs to $\mathcal{M}\big(\dot{B}^{(N/p)-1}_{p,r}(\mathbb{R}^N)\big) \cap \mathcal{M}\big(\dot{B}^{(N/p)+\varepsilon-2}_{p,r}(\mathbb{R}^N)\big)$.

As a start, let us give a result of persistence of regularity, under rather general hypotheses.

Proposition 2.1. — *Assume that ρ_0 is given by* (1.10) *with small enough η and some $\mathcal{C}^{1,\varepsilon}$ domain $\mathcal{D}_0$ of $\mathbb{R}^N$ satisfying* (0.5). *Let u_0 be a small enough divergence free vector field with coefficients in $\dot{B}^{(N/p)-1}_{p,1}$ for some*

$$N-1 < p < \min\big\{(N-1)/(1-\varepsilon), 2N\big\}\cdot \tag{2.1}$$

Consider a family $(X_{\lambda,0})_{\lambda\in\Lambda}$ *of* $\mathcal{C}^{0,\varepsilon}$ *divergence free vector fields tangent to* $\mathcal{D}_0$ *and such that* $\partial_{X_{\lambda,0}}u_0\in\dot B^{(N/p)+\varepsilon-2}_{p,1}$ *for all* $\lambda\in\Lambda$.

Then the unique solution $(\rho,u,\nabla P)$ *of* (INS) *given by Theorem* 1.1 *satisfies the following additional properties:*

– $\rho(t,\cdot)$ *is given by* (1.12),

– *all the time-dependent vector fields* X_λ *solutions to* (0.8) *with initial data* $X_{\lambda,0}$ *are in* $L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon})$ *and remain tangent to the patch for all time.*

Proof. — Assumptions (1.10) and (2.1) guarantee that ρ_0 is in

$$\mathcal{M}(\dot B^{(N/p)-1}_{p,1})\cap\mathcal{M}(\dot B^{(N/p)-2+\varepsilon}_{p,1}),$$

and that (1.8) is fulfilled if η and u_0 are small enough. Of course, $\partial_{X_{\lambda,0}}\rho_0\equiv 0$ for all $\lambda\in\Lambda$ because the vector fields $X_{\lambda,0}$ are tangent to the boundary. Therefore, applying Theorem 1.2 ensures that all the vector fields X_λ are in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon})$. Now, if we consider a level set function f_0 in $\mathcal{C}^{1,\varepsilon}$ associated to $\mathcal{D}_0$ as in (0.5), then $f_t:=f_0\circ\psi_t$ is associated to the transported domain $\mathcal{D}_t=\psi_t(\mathcal{D}_0)$, and we have

$$D_t\nabla f=-\nabla u\cdot\nabla f\quad\text{with}\quad(\nabla u)_{ij}=\partial_i u^j.\tag{2.2}$$

Therefore, as X_λ satisfies (0.8), we have

$$D_t(X_\lambda\cdot\nabla f)=(D_tX_\lambda)\cdot\nabla f+X_\lambda\cdot(D_t\nabla f)=0,$$

which ensures that X_λ remains tangent to the patch for all time. □

Example. As a consequence of Bony decomposition and of $\operatorname{div}X_{\lambda,0}\equiv 0$, we have

$$\partial_{X_{\lambda,0}}u_0=\dot{\mathcal{T}}_{X_{\lambda,0}}u_0+\dot T_{\partial_k u_0}X^k_{\lambda,0}+\operatorname{div}\dot R(u_0,X_{\lambda,0}).$$

Hence, if $u_0\in\dot B^{(N/p)-1}_{p,1}\cap\dot B^{(N/p)+\varepsilon-1}_{p,1}$ with p satisfying (2.1), then the conditions of Proposition 2.1 are fulfilled. In fact, the additional regularity $\dot B^{(N/p)+\varepsilon-1}_{p,1}$ of u_0 implies that the flow ψ_t is in $\mathcal{C}^{1,\varepsilon}$, because the solution $(u,\nabla P)$ lies in $\dot E^\varepsilon_q$ for some $q>(N-1)/(1-\varepsilon)$. This is a consequence of the following result that may be obtained along the lines of the proof of Theorem 1.2.

Proposition 2.2. — *If the initial data* (ρ_0,u_0) *are as in Theorem* 1.1 *and if in addition* u_0 *is in* $\dot B^{(N/q)+\varepsilon-1}_{q,1}$ *and*

$$\|\rho_0-1\|_{\mathcal{M}\left(\dot B^{(N/q)+\varepsilon-1}_{q,1}\right)}\leqslant c\quad\textit{for small constant } c,\ \ 0<\varepsilon<1\ \ \textit{and}\ \ \frac{N-1}{1-\varepsilon}<q\leqslant\infty,$$

then, beside the properties listed in Theorem 1.1, *the unique global solution* $(\rho,u,\nabla P)$ *of the system* (INS) *satisfies*

$$\rho\in L^\infty\big(\mathbb{R}_+;\mathcal{M}\big(\dot B^{(N/q)+\varepsilon-1}_{q,1}\big)\big)\quad\textit{and}\quad(u,\nabla P)\in\dot E^\varepsilon_q.$$

2.1. The two-dimensional case. — Here we prove Theorem 1.3. As a start, we have to show that if the vorticity ω_0 is given by (1.11) then u_0 is in $\dot{B}^{(2/p)-1}_{p,1}(\mathbb{R}^2)$. This will be achieved by using the fact that u_0 can be computed from ω_0 by means of the following *Biot-Savart law*:

$$u_0 = K_2 * \omega_0, \quad \text{with} \quad K_2(x) := \frac{1}{2\pi|x|^2}\begin{pmatrix}-x_2\\ x_1\end{pmatrix}\cdot \tag{2.3}$$

Recall that ω_0 is in $L^p(\mathbb{R}^2)$ and is supported in some ball $B(0,R)$. Now, on the one hand, one may write for all x in $B(0,2R)$,

$$|u_0(x)| \leqslant \frac{1}{2\pi}\int_{B(0,3R)} |\omega_0(x-y)|\,\frac{dy}{|y|},$$

which, by convolution inequalities and our choice of p, implies that $|u_0|\mathbb{1}_{B(0,2R)}$ is in $L^r(\mathbb{R}^2)$ for any r satisfying

$$r \in \big[1, 2p/(2-p)\big[\subset \big[1, 2/(1-\varepsilon)\big[. \tag{2.4}$$

On the other hand, for $|x| \geqslant 2R$, owing to the zero average condition for ω_0, we have

$$u_0(x) = \frac{1}{2\pi}\int_{|y|\leqslant R} \big(K_2(x-y) - K_2(x)\big)\omega_0(y)\,dy.$$

Therefore, by computing $K_2(x-y) - K_2(x)$, it is not difficult to see that we have for some constant C_R depending only on R,

$$|u_0(x)| \leqslant \frac{C_R}{|x|^2}\,\|\omega_0\|_{L^1} \quad \text{for all} \quad x \quad \text{such that} \quad |x| \geqslant 2R.$$

Then putting the two information together, we get u_0 in L^r for all r given by (2.4).

Next, let us write that

$$u_0 = \dot{S}_0 u_0 + (\mathrm{Id} - \dot{S}_0)u_0.$$

To handle the first term, we infer from the embedding of L^r in $\dot{B}^{2/p-2/r}_{p,\infty}$ for all $1 < r < p < 2$,

$$\|\dot{S}_0 u_0\|_{\dot{B}^{(2/p)-1}_{p,1}} \lesssim \|\dot{S}_0 u_0\|_{\dot{B}^{2/p-2/r}_{p,\infty}} \lesssim \|\dot{S}_0 u_0\|_{L^r} \lesssim \|u_0\|_{L^r} \lesssim \|\omega_0\|_{L^p}.$$

As regards the high frequency part of u_0, the Fourier multiplier $(\mathrm{Id} - \dot{S}_0)\nabla^\perp(-\Delta)^{-1}$ is homogeneous of degree -1 away from a neighborhood of 0, which yields

$$\begin{aligned}\|(\mathrm{Id} - \dot{S}_0)u_0\|_{\dot{B}^{(2/p)-1}_{p,1}} &= \|(\mathrm{Id} - \dot{S}_0)\nabla^\perp(-\Delta)^{-1}\omega_0\|_{\dot{B}^{(2/p)-1}_{p,1}}\\ &\lesssim \|(\mathrm{Id} - \dot{S}_0)\omega_0\|_{\dot{B}^{(2/p)-2}_{p,1}} \lesssim \|\omega_0\|_{L^p}.\end{aligned} \tag{2.5}$$

Next, consider the divergence free vector field $X_0 = \nabla^\perp f_0$, where f_0 is given by (0.5) and is (with no loss of generality) compactly supported. If it is true that

$$\partial_{X_0} u_0 \in \dot{B}^{(2/p)-2+\varepsilon}_{p,1}, \tag{2.6}$$

then one can apply Proposition 2.1 which ensures that the transported vector field X_t remains in $\mathcal{C}^{0,\varepsilon}$ for all $t \geqslant 0$. Now, it is classical that we have $X_t = (\nabla f_t)^\perp$ with $f_t = f_0 \circ \psi_t^{-1}$. Hence $\mathcal{D}_t$ has a $\mathcal{C}^{1,\varepsilon}$ boundary.

Let us establish (2.6). First note that

$$X_0 \in \mathscr{C}_c^\varepsilon \hookrightarrow b^\varepsilon_{p,\infty} \cap b^\alpha_{p,r} \quad \text{provided} \quad 1 \leqslant r \leqslant \infty \text{ and } -2/p' < \alpha < \varepsilon \tag{2.7}$$

due to Proposition A.2 and Proposition A.1. Now, (1.16) ensures that

$$\|\dot{\mathcal{T}}_{X_0} u_0 - \partial_{X_0} u_0\|_{\dot B^{(2/p)+\varepsilon-2}_{p,1}} \lesssim \|u_0\|_{\dot B^{(2/p)-1}_{p,1}} \|X_0\|_{\mathscr{C}^\varepsilon} \quad \text{for any } 1 < p < 2/(1-\varepsilon)\cdot \tag{2.8}$$

Then thanks to (2.3), we obtain

$$\dot{\mathcal{T}}_{X_0} u_0 = \dot{\mathcal{T}}_{X_0}(-\Delta)^{-1}\nabla^\perp\omega_0 = (-\Delta)^{-1}\nabla^\perp \dot{\mathcal{T}}_{X_0}\omega_0 + [\dot{\mathcal{T}}_{X_0}, (-\Delta)^{-1}\nabla^\perp]\omega_0,$$

whence using Lemma B.1 and (2.5),

$$\|\dot{\mathcal{T}}_{X_0} u_0 - (-\Delta)^{-1}\nabla^\perp\dot{\mathcal{T}}_{X_0}\omega_0\|_{\dot B^{(2/p)+\varepsilon-2}_{p,1}} \lesssim \|X_0\|_{\mathscr{C}^\varepsilon}\|\omega_0\|_{L^p}. \tag{2.9}$$

Next, we notice that

$$\dot{\mathcal{T}}_{X_0}\omega_0 - \operatorname{div}(X_0\omega_0) = -\operatorname{div}\big(\dot T_{\omega_0} X_0 + \dot R(\omega_0, X_0)\big).$$

Therefore, taking advantage of standard continuity results for $\dot T$ and $\dot R$, we have

$$\|\dot{\mathcal{T}}_{X_0}\omega_0 - \operatorname{div}(X_0\omega_0)\|_{\dot B^{(2/p)+\varepsilon-3}_{p,1}} \lesssim \|\omega_0\|_{L^p}\|X_0\|_{\mathscr{C}^\varepsilon}, \quad \text{as } 1 < p < 2/(2-\varepsilon)\cdot \tag{2.10}$$

Since $\operatorname{div}(X_0\,\overline{\omega}_0) \equiv 0$ by assumption, it is sufficient to study $\operatorname{div}(X_0\,\widetilde{\omega}_0\,\mathbb{1}_{\mathcal{D}_0})$. Recall that $\widetilde{\omega}_0 \in \mathscr{C}_c^{\varepsilon'}$ for some $0 < \varepsilon' < \varepsilon$, and that $\operatorname{div} X_0 = 0$. As we have $\partial_{X_0}\mathbb{1}_{\mathcal{D}_0} = 0$, Corollary B.5 implies that

$$\operatorname{div}(X_0\,\widetilde{\omega}_0\,\mathbb{1}_{\mathcal{D}_0}) \in \dot B^{(2/p)+\varepsilon-3}_{p,1}.$$

Putting (2.8), (2.9) and (2.10) together, we conclude that (2.6) is fulfilled, which completes the proof of Theorem 1.3. □

If dropping off the zero average condition for the function ω_0 in Theorem 1.3, then the corresponding initial velocity field u_0 cannot be in $L^r(\mathbb{R}^2)$ for any $r \in \,]1,2]$. Still, one can get a similar statement in the particular case where $(\overline{\omega}_0, \widetilde{\omega}_0) \equiv (0,\eta')$ for some small enough η'. Indeed, from (2.3) and Hardy-Littlewood-Sobolev inequality, we deduce that u_0 belongs to all spaces $L^r(\mathbb{R}^2)$ with $r \in \,]2,\infty[$. Repeating the first part of the proof of Theorem 1.3 thus yields $u_0 \in \dot B^{(2/p)-1}_{p,1}(\mathbb{R}^2)$ for any $2 < p < \infty$. Now, as ω_0 is bounded and compactly supported, it is in $\dot B^{(2/q)+\varepsilon-2}_{q,1}(\mathbb{R}^2)$ for any $0 < \varepsilon < 1$ and $1 < q < \infty$, which implies that $u_0 \in \dot B^{(2/q)+\varepsilon-1}_{q,1}$. Hence, applying Proposition 2.2, and using the fact that the flow of the solution constructed therein is in $\mathcal{C}^{1,\varepsilon}$, we conclude to the following generalization of of [22, Rem. 1.1].

Theorem 2.3. — *Let $\mathcal{D}_0$ satisfy (0.5) for some ε in $]0,1[$. There exists a constant η_0 depending only on $\mathcal{D}_0$ so that for all $\eta, \eta' \in \,]-\eta_0, \eta_0[$ if*

$$\rho_0 := (1+\eta)\mathbb{1}_{\mathcal{D}_0} + \mathbb{1}_{\mathcal{D}_0^c},$$

and if the divergence free vector-field u_0 in $W^{1,p}(\mathbb{R}^2)$ for some $p > 2$ is given by

$$u_0 := (-\Delta)^{-1}\nabla^\perp(\eta'\mathbb{1}_{\mathcal{D}_0}),$$

then the system (INS) *has a unique solution* $(\rho, u, \nabla P)$ *with the properties listed in Proposition* 2.2 *for some suitable In addition,* (0.3) *is fulfilled for all* $t \geqslant 0$*, and* $\mathcal{D}_t$ *remains a simply connected bounded domain of class* $\mathcal{C}^{1,\varepsilon}$*.*

2.2. The three-dimensional case. — As another application of Proposition 2.1, one can generalize Theorem 1.3 to the three-dimensional case. Our result reads as follows.

Theorem 2.4. — *Let* $\mathcal{D}_0$ *be a* $\mathcal{C}^{1,\varepsilon}$ *simply connected bounded domain of* $\mathbb{R}^3$ *with* $\varepsilon \in]0,1[$*. Let* ρ_0 *be given by* (1.10) *with small enough* η*. Assume that the initial velocity* u_0 *has coefficients in* $\mathcal{S}'_h(\mathbb{R}^3)$ *and vorticity*[5]

$$\Omega_0 := \nabla \wedge u_0 = \widetilde{\Omega}_0 \mathbb{1}_{\mathcal{D}_0},$$

for some small enough $\widetilde{\Omega}_0$ *in* $\mathcal{C}^{0,\delta}(\mathbb{R}^3;\mathbb{R}^3)$ $(\delta \in]0,\varepsilon[)$ *with* $\operatorname{div} \widetilde{\Omega}_0 = 0$ *and* $\widetilde{\Omega}_0 \cdot \vec{n}_{\mathcal{D}_0}|_{\partial \mathcal{D}_0} \equiv 0$ *(here* $\vec{n}_{\mathcal{D}_0}$ *denotes the outwards unit normal of the domain* $\mathcal{D}_0$*).*

There exists a unique solution $(\rho, u, \nabla P)$ *to the system* (INS) *with the properties listed in Theorem* 1.1 *for some suitable* p *satisfying*

$$2 < p < \min\Big\{2/(1-\varepsilon), 6\Big\}\cdot \tag{2.11}$$

Furthermore, for all $t \geqslant 0$*, we have* (1.12) *and* $\mathcal{D}_t$ *remains a simply connected bounded domain of class* $\mathcal{C}^{1,\varepsilon}$*.*

Proof. — With no loss of generality, one may assume that $\widetilde{\Omega}_0$ is compactly supported. Like in the 2D case, we first have to check that u_0 fulfills the assumptions of Proposition 2.1. As it is divergence free and decays at infinity (recall that $u_0 \in \mathcal{S}'_h$), it is given by the Biot-Savart law:

$$u_0 = (-\Delta)^{-1} \nabla \wedge \Omega_0, \quad \text{with } \Omega_0 = \widetilde{\Omega}_0 \, \mathbb{1}_{\mathcal{D}_0}. \tag{2.12}$$

We claim that u_0 belongs to $\dot{B}^{(3/p)-1}_{p,1}$ for some p satisfying (2.11). Indeed, the characteristic function of any bounded domain with $\mathcal{C}^1$ regularity belongs to all Besov spaces $B^{1/q}_{q,\infty}$ with $1 \leqslant q \leqslant \infty$ (see e.g. [26]). Hence combining Proposition A.1 and the embedding (A.1) gives

$$\mathbb{1}_{\mathcal{D}_0} \in \mathcal{E}' \cap B^{1/q}_{q,\infty} \hookrightarrow \dot{B}^{(3/q)-2}_{q,1} \quad \text{for any } q \in]1,\infty[. \tag{2.13}$$

Now, using Bony's decomposition and standard continuity results for operators $\dot{R}$ and $\dot{T}$, we discover that

$$\widetilde{\Omega}_0 \in \mathscr{C}^\delta_c \hookrightarrow \mathcal{M}\big(\dot{B}^{(3/q)-2}_{q,1}\big) \quad \text{for any } q \in \big]3/2,\, 3/(2-\delta)\big[.$$

Hence the definition of Multiplier space and (2.13) yield

$$\Omega_0 = \widetilde{\Omega}_0 \, \mathbb{1}_{\mathcal{D}_0} \in \dot{B}^{(3/q)-2}_{q,1} \quad \text{for any } q \in \big]3/2,\, 3/(2-\delta)\big[. \tag{2.14}$$

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[5] For any point $Y \in \mathbb{R}^3$, we set $X \wedge Y := (X^2Y^3 - X^3Y^2,\, X^3Y^1 - X^1Y^3,\, X^1Y^2 - X^2Y^1)$, where X stands for an element of $\mathbb{R}^3$ or for the ∇ operator.

As u_0 is in $\mathcal{S}'_h$ and $(-\Delta^{-1})^{-1}\nabla\wedge$ in (2.12) is a homogeneous multiplier of degree -1, one can conclude that

$$u_0 \in \dot{B}^{(3/q)-1}_{q,1} \hookrightarrow \dot{B}^{(3/p)-1}_{p,1}, \quad \text{for any } p \geqslant q.$$

Note that for any δ in $]0,1[$, one can find some p satisfying the above conditions and (2.11) altogether.

Next, consider some (compactly supported) level set function f_0 associated to $\partial\mathcal{D}_0$, and the three $\mathcal{C}^{0,\varepsilon}$ vector-fields $X_{k,0} := e_k \wedge \nabla f_0$ with (e_1, e_2, e_3) being the canonical basis of $\mathbb{R}^3$. It is clear that those vector-fields are divergence free and tangent to $\partial\mathcal{D}_0$. Let us check that we have $\partial_{X_{k,0}} u_0 \in \dot{B}^{(3/p)-2+\varepsilon}_{p,1}$ for some p satisfying (2.11). As in the two-dimensional case, this will follow from Biot-Savart law and the special structure of Ω_0. Indeed, from (1.16) and $\operatorname{div} X_{k,0} = 0$, we have

$$\|\dot{\mathcal{T}}_{X_{k,0}} u_0 - \partial_{X_{k,0}} u_0\|_{\dot{B}^{(3/p)+\varepsilon-2}_{p,1}} \lesssim \|u_0\|_{\dot{B}^{(3/p)-1}_{p,1}} \|X_0\|_{\mathscr{C}^\varepsilon}, \ \ \forall\, p \in \big]3/2, 3/(1-\varepsilon)\big[.$$

Then (2.12) yields

$$\dot{\mathcal{T}}_{X_{k,0}} u_0 = \dot{\mathcal{T}}_{X_{k,0}} (-\Delta)^{-1}\nabla\wedge\Omega_0 = (-\Delta)^{-1}\nabla\wedge\dot{\mathcal{T}}_{X_{k,0}}\Omega_0 + [\dot{\mathcal{T}}_{X_{k,0}}, (-\Delta)^{-1}\nabla\wedge]\,\Omega_0.$$

Thanks to Lemma B.1 and homogeneity of $(-\Delta^{-1})^{-1}\nabla\wedge$, it is thus sufficient to verify that $\dot{\mathcal{T}}_{X_{k,0}}\Omega_0$ belongs to $\dot{B}^{(3/p)+\varepsilon-3}_{p,1}$ for some p satisfying (2.11). In fact, from the decomposition

$$\dot{\mathcal{T}}_{X_{k,0}}\Omega_0 - \operatorname{div}(X_{k,0}\Omega_0) = -\operatorname{div}\big(\dot{T}_{\Omega_0} X_{k,0} + \dot{R}(\Omega_0, X_{k,0})\big),$$

and continuity results for $\dot{R}$ and $\dot{T}$, we get

$$\|\dot{\mathcal{T}}_{X_{k,0}}\Omega_0 - \operatorname{div}(X_{k,0}\Omega_0)\|_{\dot{B}^{(3/q)+\varepsilon-3}_{q,1}} \lesssim \|\Omega_0\|_{\dot{B}^{(3/q)-2}_{q,1}} \|X_{k,0}\|_{\mathscr{C}^\varepsilon}, \ \ \forall\, q \in \big]3/2, 3/(2-\varepsilon)\big[.$$

Thus, remembering (2.14) and $0 < \delta < \varepsilon$, we have to choose some p satisfying (2.11), such that the following standard embedding holds

$$\dot{B}^{(3/q)+\varepsilon-3}_{q,1} \hookrightarrow \dot{B}^{(3/p)+\varepsilon-3}_{p,1} \text{ for some } q \in \big]3/2, 3/(2-\delta)\big[\text{ with } q \leqslant p. \tag{2.15}$$

Now, because $\partial_{X_{k,0}} \mathbb{1}_{\mathcal{D}_0} \equiv 0$ and $\widetilde{\Omega}_0$ is in $\mathcal{C}^\delta$, Corollary B.5 yields for all $0 < \delta_\star < \delta$,

$$\partial_{X_{k,0}}\Omega_0 = \operatorname{div}(X_{k,0}\otimes\Omega_0) = \operatorname{div}(X_{k,0}\otimes\widetilde{\Omega}_0\,\mathbb{1}_{\mathcal{D}_0}) \in \dot{B}^{\delta_\star-1}_{q,1} \ \text{ for all } \ q \geqslant 1.$$

One can thus conclude that $\partial_{X_{k,0}} u_0 \in \dot{B}^{(3/p)-2+\varepsilon}_{p,1}$ for any index p satisfying $p \geqslant q$ with q satisfying the condition (2.15) and $(3/q) + \varepsilon - 2 = \delta^* \in]0, \delta[$.

As one can require in addition p to fulfill (2.11), Proposition 2.1 applies with the family $(X_{k,0})_{1\leqslant k\leqslant 3}$. Denoting by $(X_k)_{1\leqslant k\leqslant 3}$ the corresponding family of divergence free vector fields in $\mathcal{C}^{0,\varepsilon}$ given by (0.8) with initial data $X_{0,k}$, and introducing $Y_1 := X_3 \wedge X_1$, $Y_2 := X_3 \wedge X_1$ and $Y_3 = X_1 \wedge X_2$, we discover that for $\alpha = 1, 2, 3$,

$$\begin{cases} \partial_t Y_\alpha + u\cdot\nabla Y_\alpha = -\nabla u \cdot Y_\alpha, \\ (Y_\alpha)|_{t=0} = \partial_\alpha f_0\, \nabla f_0. \end{cases} \tag{2.16}$$

From (2.2), it is clear that the time-dependent vector field $\big(\partial_\alpha f_0(\psi_t^{-1})\big)\nabla f_t$ also satisfies (2.16), hence we have, by uniqueness, $Y_\alpha(t,\cdot)=\big((\partial_\alpha f_0)(\psi_t^{-1})\big)\nabla f_t$. So finally,

$$\big|\nabla f_0\circ\psi_t^{-1}\big|^2\,\nabla f_t=\sum_{\alpha=1}^{3}Y_\alpha(t,\cdot)\,\partial_\alpha f_0\circ\psi_t^{-1}.$$

As ψ_t^{-1} is $\mathcal{C}^1$ and as both Y_α and ∇f_0 are in $\mathcal{C}^{0,\varepsilon}$, one can conclude that ∇f_t is $\mathcal{C}^{0,\varepsilon}$ in some neighborhood of $\partial\mathcal{D}_0$. Therefore $\mathcal{D}_t$ remains of class $\mathcal{C}^{1,\varepsilon}$ for all time. □

Remark 2.5. — In contrast with the 2D case, one cannot consider constant vortex patterns for the condition $\widetilde{\Omega}_0\cdot\vec{n}_{\mathcal{D}_0}|_{\partial\mathcal{D}_0}\equiv 0$ is not fulfilled. One can define directly u_0 through $u_0=(-\Delta)^{-1}\nabla\wedge e$, where e is a constant vector of $\mathbb{R}^3$ (as we did for the Boussinesq system in [15]), but then, $\nabla\wedge u_0$ does not coincides with e.

3. The proof of persistence of striated regularity

That section is devoted to the proof of Theorem 1.2. The first step is to apply Theorem 1.1. From it, we get a unique global solution $(\rho,u,\nabla P)$ with $\rho\in\mathcal{C}_b\big(\mathbb{R}_+;\mathcal{M}(\dot B^{(N/p)-1}_{p,1})\big)$ and $(u,\nabla P)\in\dot E_p$, satisfying (1.6). Because the product of functions maps $\dot B^{(N/p)-1}_{p,1}\times\dot B^{N/p}_{p,1}$ to $\dot B^{(N/p)-1}_{p,1}$, we deduce that $D_tu=\partial_tu+u\cdot\nabla u$ is also bounded by the right-hand side of (1.6). So finally,

$$\|(u,\nabla P)\|_{\dot E_p}+\|D_tu\|_{L^1_t(\dot B^{(N/p)-1}_{p,1})}\leqslant C\|u_0\|_{\dot B^{(N/p)-1}_{p,1}}.\tag{3.1}$$

In order to complete the proof of the theorem, it is only a matter of showing that the additional multiplier and striated regularity properties are conserved for all positive times. We shall mainly concentrate on the proof of a priori estimates for the corresponding norms, just explaining at the end how a suitable regularization process allows to make it rigorous.

3.1. Bounds involving multiplier norms.

— As already pointed out in the introduction, because ∇u is in $L^1(\mathbb{R}_+;\dot B^{N/p}_{p,1})$ and $\dot B^{N/p}_{p,1}$ is embedded in $\mathcal{C}_b$, the flow ψ of u is $\mathcal{C}^1$ and we have for all $t\geqslant 0$, owing to (1.6),

$$\|\nabla\psi_t^{\pm1}\|_{L^\infty}\leqslant\exp\bigg(\int_0^t\|\nabla u\|_{L^\infty}\,d\tau\bigg)\leqslant C\tag{3.2}$$

for a suitably large universal constant C.

Now, from the mass conservation equation and (1.14), we gather that

$$\rho(t,\cdot)=\rho_0\circ\psi_t^{-1}\quad\text{and}\quad(\partial_X\rho)(t,\cdot)=(\partial_{X_0}\rho_0)\circ\psi_t^{-1}.$$

Hence $\|\rho(t,\cdot)\|_{L^\infty}$ is time independent, and Lemma A.3 (keeping in mind the condition (1.7)) guarantees that for all $t\in\mathbb{R}_+$,

$$\|\rho(t)-1\|_{\mathcal{M}(\dot B^{(N/p)-1}_{p,1})}\leqslant C\|\rho_0-1\|_{\mathcal{M}(\dot B^{(N/p)-1}_{p,1})},\tag{3.3}$$

$$\|\rho(t)-1\|_{\mathcal{M}(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\leqslant C\|\rho_0-1\|_{\mathcal{M}(\dot B^{(N/p)+\varepsilon-2}_{p,1})},\tag{3.4}$$

$$\|(\partial_X\rho)(t)\|_{\mathcal{M}(\dot B^{(N/p)-1}_{p,1}\to\dot B^{(N/p)+\varepsilon-2}_{p,1})}\leqslant C\|\partial_{X_0}\rho_0\|_{\mathcal{M}(\dot B^{(N/p)-1}_{p,1}\to\dot B^{(N/p)+\varepsilon-2}_{p,1})}.\tag{3.5}$$

3.2. Estimates for the striated regularity. — Recall that $\dot{\mathcal{T}}_X u$ satisfies the Stokes-like system (1.18). As $\dot{\mathcal{T}}_X u$ need not be divergence free, to enter into the standard theory, we set

$$v := \dot{\mathcal{T}}_X u - w \quad \text{with} \quad w := \dot{T}_{\partial_k X} u^k - \dot{T}_{\operatorname{div} X} u.$$

Denoting $\widetilde{g} := g - \rho u \cdot \nabla \dot{\mathcal{T}}_X u - (\rho \partial_t w - \Delta w)$ with g defined in (1.19), we see that v satisfies:

$$\text{(S)} \qquad \begin{cases} \rho \partial_t v - \Delta v + \nabla \dot{\mathcal{T}}_X P = \widetilde{g}, \\ \operatorname{div} v = 0, \\ v|_{t=0} = v_0. \end{cases}$$

We shall decompose the proof of a priori estimates for striated regularity into three steps. The first one is dedicated to bounding $\widetilde{g}$ (which mainly requires the commutator estimates of the appendix). In the second step, we take advantage of the smoothing effect of the heat flow so as to estimate v. In the third step, we revert to $\dot{\mathcal{T}}_X u$ and eventually bound X.

First step: bounds of $\widetilde{g}$. — Recall that $\widetilde{g} := g - \rho u \cdot \nabla \dot{\mathcal{T}}_X u - (\rho \partial_t w - \Delta w)$ with

$$g = -\rho[\dot{\mathcal{T}}_X, D_t]u + [\dot{\mathcal{T}}_X, \Delta]u - [\dot{\mathcal{T}}_X, \nabla]P + (\partial_X - \dot{\mathcal{T}}_X)(\Delta u - \nabla P) \\ - \partial_X \rho D_t u + \rho(\dot{\mathcal{T}}_X - \partial_X) D_t u.$$

The first term of g may be bounded according to Proposition B.3 and to the definition of multiplier spaces. We get, under assumption (1.17),

$$\text{(3.6)} \quad \|\rho[\dot{\mathcal{T}}_X, D_t]u\|_{\dot{B}^{(N/p)+\varepsilon-2}_{p,1}} \lesssim \|\rho\|_{\mathcal{M}(\dot{B}^{(N/p)+\varepsilon-2}_{p,1})} \Big(\|u\|_{\dot{\mathscr{C}}^{-1}} \|\dot{\mathcal{T}}_X u\|_{\dot{B}^{(N/p)+\varepsilon}_{p,1}} \\ + \|u\|_{\dot{B}^{(N/p)+1}_{p,1}} \|\dot{\mathcal{T}}_X u\|_{\dot{\mathscr{C}}^{\varepsilon-2}} + \|u\|_{\dot{B}^{(N/p)+1}_{p,1}} \|u\|_{\dot{B}^{(N/p)-1}_{p,1}} \|X\|_{\dot{\mathscr{C}}^{\varepsilon}} \Big).$$

Next, thanks to the commutator estimates in Lemma B.1, we have

$$\text{(3.7)} \qquad \|[\dot{\mathcal{T}}_X, \Delta]u\|_{\dot{B}^{(N/p)+\varepsilon-2}_{p,1}} \lesssim \|\nabla X\|_{\dot{\mathscr{C}}^{\varepsilon-1}} \|\nabla u\|_{\dot{B}^{N/p}_{p,1}},$$

$$\text{(3.8)} \qquad \|[\dot{\mathcal{T}}_X, \nabla]P\|_{\dot{B}^{(N/p)+\varepsilon-2}_{p,1}} \lesssim \|\nabla X\|_{\dot{\mathscr{C}}^{\varepsilon-1}} \|\nabla P\|_{\dot{B}^{(N/p)-1}_{p,1}}.$$

Bounding the fourth term of g stems from (1.16): we have

$$\text{(3.9)} \qquad \|(\dot{\mathcal{T}}_X - \partial_X)(\Delta u - \nabla P)\|_{\dot{B}^{(N/p)+\varepsilon-2}_{p,1}} \lesssim \|(\Delta u, \nabla P)\|_{\dot{B}^{(N/p)-1}_{p,1}} \|X\|_{\dot{\mathscr{C}}^{\varepsilon}}.$$

Then the definition of multiplier spaces yields

$$\text{(3.10)} \qquad \|\partial_X \rho D_t u\|_{\dot{B}^{(N/p)+\varepsilon-2}_{p,1}} \lesssim \|\partial_X \rho\|_{\mathcal{M}(\dot{B}^{(N/p)-1}_{p,1} \to \dot{B}^{(N/p)+\varepsilon-2}_{p,1})} \|D_t u\|_{\dot{B}^{(N/p)-1}_{p,1}}.$$

Finally, using again (1.16) and the definition of multiplier spaces, we may write

$$\text{(3.11)} \qquad \|\rho(\dot{\mathcal{T}}_X - \partial_X) D_t u\|_{\dot{B}^{(N/p)+\varepsilon-2}_{p,1}} \lesssim \|\rho\|_{\mathcal{M}(\dot{B}^{(N/p)+\varepsilon-2}_{p,1})} \|X\|_{\dot{\mathscr{C}}^{\varepsilon}} \|D_t u\|_{\dot{B}^{(N/p)-1}_{p,1}}.$$

Putting together (3.6) – (3.11) and integrating with respect to time, we end up with

$$\begin{aligned}(3.12)\quad &\|g\|_{L^1_t(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\\ &\lesssim\int_0^t \|\rho\|_{\mathcal{M}(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\big(\|u\|_{\dot{\mathscr{C}}^{-1}}\|\dot{\mathcal{T}}_X u\|_{\dot B^{(N/p)+\varepsilon}_{p,1}}+\|\nabla u\|_{\dot B^{N/p}_{p,1}}\|\dot{\mathcal{T}}_X u\|_{\dot{\mathscr{C}}^{\varepsilon-2}}\big)\,dt'\\ &\quad+\int_0^t \|X\|_{\dot{\mathscr{C}}^{\varepsilon}}\Big(\big(\|\nabla u\|_{\dot B^{N/p}_{p,1}}\|u\|_{\dot B^{(N/p)-1}_{p,1}}+\|D_t u\|_{\dot B^{(N/p)-1}_{p,1}}\big)\|\rho\|_{\mathcal{M}(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\\ &\qquad\qquad+\|(\nabla^2 u,\nabla P)\|_{\dot B^{(N/p)-1}_{p,1}}\Big)dt'\\ &\quad+\int_0^t \|\partial_X\rho\|_{\mathcal{M}(\dot B^{(N/p)-1}_{p,1}\to\dot B^{(N/p)+\varepsilon-2}_{p,1})}\|D_t u\|_{\dot B^{(N/p)-1}_{p,1}}\,dt'.\end{aligned}$$

Bounding the second term of $\widetilde{g}$ is obvious: taking advantage of Bony's decomposition (1.15) and remembering that $(N/p)+\varepsilon>1$ and that $\operatorname{div} u=0$, we get

$$\begin{aligned}(3.13)\quad &\|\rho u\cdot\nabla\dot{\mathcal{T}}_X u\|_{L^1_t(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\\ &\lesssim\int_0^t \|\rho\|_{\mathcal{M}(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\big(\|u\|_{\dot{\mathscr{C}}^{-1}}\|\dot{\mathcal{T}}_X u\|_{\dot B^{(N/p)+\varepsilon}_{p,1}}+\|u\|_{\dot B^{(N/p)+1}_{p,1}}\|\dot{\mathcal{T}}_X u\|_{\dot{\mathscr{C}}^{\varepsilon-2}}\big)\,dt'.\end{aligned}$$

To bound the last term of $\widetilde{g}$, we use the decomposition

$$\rho\partial_t w-\Delta w=\rho(W_1+W_2)+W_3,\quad\text{with}\quad\begin{cases}W_1:=\dot T_{\partial_k X}\partial_t u^k-\dot T_{\operatorname{div}X}\partial_t u,\\ W_2:=\dot T_{\partial_k\partial_t X}u^k-\dot T_{\operatorname{div}\partial_t X}u,\\ W_3:=\Delta\big(\dot T_{\operatorname{div}X}u-\dot T_{\partial_k X}u^k\big).\end{cases}$$

Continuity results for the paraproduct and the definition of $\mathcal{M}(\dot B^{(N/p)+\varepsilon-2}_{p,1})$ ensure that

$$(3.14)\qquad \|\rho W_1\|_{L^1_t(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\lesssim\int_0^t\|\rho\|_{\mathcal{M}(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\|\nabla X\|_{\dot{\mathscr{C}}^{\varepsilon-1}}\|\partial_t u\|_{\dot B^{(N/p)-1}_{p,1}}\,dt',$$

$$(3.15)\qquad \|\rho W_2\|_{L^1_t(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\lesssim\int_0^t\|\rho\|_{\mathcal{M}(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\|\partial_t X\|_{\dot{\mathscr{C}}^{\varepsilon-2}}\|u\|_{\dot B^{(N/p)+1}_{p,1}}\,dt',$$

$$(3.16)\qquad \|W_3\|_{L^1_t(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\lesssim\int_0^t\|\nabla X\|_{\dot{\mathscr{C}}^{\varepsilon-1}}\|u\|_{\dot B^{(N/p)+1}_{p,1}}\,dt'.$$

To estimate $\partial_t X$ in (3.15), we use the fact that

$$\partial_t X=-u\cdot\nabla X+\partial_X u=-\operatorname{div}(u\otimes X)+\partial_X u.$$

Hence using (1.15), and continuity results for the remainder and paraproduct operators, we get under the condition (1.17),

$$\|\partial_t X\|_{\dot{\mathscr{C}}^{\varepsilon-2}}\lesssim\|u\|_{\dot B^{(N/p)-1}_{p,1}}\|X\|_{\dot{\mathscr{C}}^{\varepsilon}}+\|\partial_X u\|_{\dot{\mathscr{C}}^{\varepsilon-2}}.$$

Therefore, taking advantage of (1.16) yields

$$\begin{aligned}(3.17)\quad &\|\rho W_2\|_{L^1_t(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\\ &\lesssim\int_0^t\|\rho\|_{\mathcal{M}(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\big(\|X\|_{\dot{\mathscr{C}}^{\varepsilon}}\|u\|_{\dot B^{(N/p)-1}_{p,1}}+\|\dot{\mathcal{T}}_X u\|_{\dot{\mathscr{C}}^{\varepsilon-2}}\big)\|\nabla u\|_{\dot B^{N/p}_{p,1}}\,dt'.\end{aligned}$$

Combining (3.14), (3.15) and (3.17), we eventually obtain

$$
\begin{aligned}
(3.18)\quad &\|\rho\partial_t w-\Delta w\|_{L^1_t(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\\
&\lesssim\int_0^t\|\dot{\mathcal T}_X u\|_{\dot{\mathscr C}^{\varepsilon-2}}\|\nabla u\|_{\dot B^{N/p}_{p,1}}\|\rho\|_{\mathcal M(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\,dt'\\
&\quad+\int_0^t\|X\|_{\dot{\mathscr C}^{\varepsilon}}\Big(\big(\|\rho\|_{\mathcal M(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\|u\|_{\dot B^{(N/p)-1}_{p,1}}+1\big)\|\nabla u\|_{\dot B^{N/p}_{p,1}}\\
&\qquad+\|\rho\|_{\mathcal M(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\|\partial_t u\|_{\dot B^{(N/p)-1}_{p,1}}\Big)\,dt',
\end{aligned}
$$

whence, putting together estimate (3.12), (3.13) and (3.18),

$$
\begin{aligned}
(3.19)\quad &\|\widetilde g\|_{L^1_t(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\\
&\lesssim\int_0^t\|\rho\|_{\mathcal M(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\big(\|u\|_{\dot{\mathscr C}^{-1}}\|\dot{\mathcal T}_X u\|_{\dot B^{(N/p)+\varepsilon}_{p,1}}+\|\nabla u\|_{\dot B^{N/p}_{p,1}}\|\dot{\mathcal T}_X u\|_{\dot{\mathscr C}^{\varepsilon-2}}\big)\,dt'\\
&\quad+\int_0^t\|X\|_{\dot{\mathscr C}^{\varepsilon}}\big(\|\nabla u\|_{\dot B^{N/p}_{p,1}}\|u\|_{\dot B^{(N/p)-1}_{p,1}}+\|(\partial_t u,D_t u)\|_{\dot B^{(N/p)-1}_{p,1}}\big)\|\rho\|_{\mathcal M(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\,dt'\\
&\quad+\int_0^t\|X\|_{\dot{\mathscr C}^{\varepsilon}}\|(\nabla^2 u,\nabla P)\|_{\dot B^{(N/p)-1}_{p,1}}\,dt'\\
&\quad+\int_0^t\|\partial_X\rho\|_{\mathcal M(\dot B^{(N/p)-1}_{p,1}\to\dot B^{(N/p)+\varepsilon-2}_{p,1})}\|D_t u\|_{\dot B^{(N/p)-1}_{p,1}}\,dt'.
\end{aligned}
$$

Second step: bounds of v. — We now want to bound v in

$$\widetilde L^\infty_t(\dot B^{(N/p)+\varepsilon-2}_{p,1})\cap L^1_t(\dot B^{(N/p)+\varepsilon}_{p,1}),$$

knowing (3.19). This will follow from the smoothing properties of the heat flow. More precisely, introduce the projector $\mathbb P$ over divergence-free vector fields, and apply $\mathbb P\dot\Delta_j$ (with $j\in\mathbb Z$) to the equation (S). We get

$$\begin{cases}\partial_t\dot\Delta_j v-\Delta\dot\Delta_j v=\mathbb P\dot\Delta_j(\widetilde g+(1-\rho)\partial_t v)\\ \dot\Delta_j v|_{t=0}=\dot\Delta_j v_0.\end{cases}$$

Lemma 2.1 in [8] implies that if $p\in[1,\infty]$,

$$\|\dot\Delta_j v(t)\|_{L^p}\leqslant e^{-ct2^{2j}}\|\dot\Delta_j v_0\|_{L^p}+C\int_0^t e^{-c(t-t')2^{2j}}\|\dot\Delta_j(\widetilde g+(1-\rho)\partial_t v)(t')\|_{L^p}\,dt'.$$

Therefore, taking the supremum over $j\in\mathbb Z$, using the fact that

$$\partial_t v=\Delta v+\mathbb P\big(\widetilde g+(1-\rho)\partial_t v\big)$$

and that $\mathbb P:\dot B^{(N/p)+\varepsilon-2}_{p,1}\to\dot B^{(N/p)+\varepsilon-2}_{p,1}$, we find that

$$
\begin{aligned}
(3.20)\quad &\|v\|_{\widetilde L^\infty_t(\dot B^{(N/p)+\varepsilon-2}_{p,1})}+\|v\|_{L^1_t(\dot B^{(N/p)+\varepsilon}_{p,1})}+\|\partial_t v\|_{L^1_t(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\\
&\lesssim\|v_0\|_{\dot B^{(N/p)+\varepsilon-2}_{p,1}}+\|\widetilde g\|_{L^1_t(\dot B^{(N/p)+\varepsilon-2}_{p,1})}+\|(1-\rho)\partial_t v\|_{L^1_t(\dot B^{(N/p)+\varepsilon-2}_{p,1})}.
\end{aligned}
$$

The smallness condition (1.8) combined with Inequality (3.4) ensure that the last term of (3.20) may be absorbed by the left-hand side, and we thus end up with

$$\|v\|_{\widetilde{L}^\infty_t(\dot{B}^{(N/p)+\varepsilon-2}_{p,1})\cap L^1_t(\dot{B}^{(N/p)+\varepsilon}_{p,1})} + \|\partial_t v\|_{L^1_t(\dot{B}^{(N/p)+\varepsilon-2}_{p,1})} \lesssim \|v_0\|_{\dot{B}^{(N/p)+\varepsilon-2}_{p,1}} + \|\widetilde{g}\|_{L^1_t(\dot{B}^{(N/p)+\varepsilon-2}_{p,1})}.$$

Next, we use the fact that by definition of v_0,

$$\begin{aligned} v_0 &= \dot{\mathcal{T}}_{X_0}u_0 - \dot{T}_{\partial_k X_0}u_0^k + \dot{T}_{\operatorname{div} X_0}u_0 \\ &= \partial_{X_0}u_0 - \dot{T}_{\partial_k u_0}X_0^k - \partial_k\dot{R}(X_0^k, u_0) + \dot{R}(\operatorname{div} X_0, u_0) - \dot{T}_{\partial_k X_0}u_0^k + \dot{T}_{\operatorname{div} X_0}u_0. \end{aligned}$$

Hence continuity results for the paraproduct yield, under the condition (1.17),

$$\|v_0\|_{\dot{B}^{(N/p)+\varepsilon-2}_{p,1}} \lesssim \|\partial_{X_0}u_0\|_{\dot{B}^{(N/p)+\varepsilon-2}_{p,1}} + \|X_0\|_{\dot{\mathscr{C}}^\varepsilon}\|u_0\|_{\dot{B}^{(N/p)-1}_{p,1}}.$$

Thus

$$\begin{aligned} \|v\|_{\widetilde{L}^\infty_t(\dot{B}^{(N/p)+\varepsilon-2}_{p,1})\cap L^1_t(\dot{B}^{(N/p)+\varepsilon}_{p,1})} &+ \|\partial_t v\|_{L^1_t(\dot{B}^{(N/p)+\varepsilon-2}_{p,1})} \\ &\lesssim \|\partial_{X_0}u_0\|_{\dot{B}^{(N/p)+\varepsilon-2}_{p,1}} + \|X_0\|_{\dot{\mathscr{C}}^\varepsilon}\|u_0\|_{\dot{B}^{(N/p)-1}_{p,1}} + \|\widetilde{g}\|_{L^1_t(\dot{B}^{(N/p)+\varepsilon-2}_{p,1})}. \end{aligned} \tag{3.21}$$

Third step: bounds for striated regularity. — Remembering that

$$\dot{\mathcal{T}}_X u = v + w \quad \text{with} \quad w = \dot{T}_{\partial_k X}u^k - \dot{T}_{\operatorname{div} X}u,$$

it is now easy to bound the following quantity:

$$\mathscr{H}(t) := \|\dot{\mathcal{T}}_X u\|_{\widetilde{L}^\infty_t(\dot{B}^{(N/p)+\varepsilon-2}_{p,1})} + \|\dot{\mathcal{T}}_X u\|_{L^1_t(\dot{B}^{(N/p)+\varepsilon}_{p,1})} + \|\nabla\dot{\mathcal{T}}_X P\|_{L^1_t(\dot{B}^{(N/p)+\varepsilon-2}_{p,1})}.$$

Indeed, we have

$$\nabla\dot{\mathcal{T}}_X P = (\mathrm{Id} - \mathbb{P})(\widetilde{g} - \rho\partial_t v), \tag{3.22}$$

and thus $\|\nabla\dot{\mathcal{T}}_X P\|_{L^1_t(\dot{B}^{(N/p)+\varepsilon-2}_{p,1})}$ may be bounded by the right-hand side of (3.21). Note also that continuity results for paraproduct operators guarantee that

$$\begin{aligned} \|w\|_{\widetilde{L}^\infty_t(\dot{B}^{(N/p)+\varepsilon-2}_{p,1})} &\lesssim \|u\|_{\widetilde{L}^\infty_t(\dot{B}^{(N/p)-1}_{p,1})}\|X\|_{L^\infty_t(\dot{\mathscr{C}}^\varepsilon)}, \\ \|w\|_{L^1_t(\dot{B}^{(N/p)+\varepsilon}_{p,1})} &\lesssim \int_0^t \|u\|_{\dot{B}^{(N/p)+1}_{p,1}}\|\nabla X\|_{\dot{\mathscr{C}}^{\varepsilon-1}}\,dt'. \end{aligned}$$

Hence we have

$$\begin{aligned} \mathscr{H}(t) \lesssim \|\partial_{X_0}u_0\|_{\dot{B}^{(N/p)+\varepsilon-2}_{p,1}} &+ \|X_0\|_{\dot{\mathscr{C}}^\varepsilon}\|u_0\|_{\dot{B}^{(N/p)-1}_{p,1}} + \|\widetilde{g}\|_{L^1_t(\dot{B}^{(N/p)+\varepsilon-2}_{p,1})} \\ &+ \|u\|_{\widetilde{L}^\infty_t(\dot{B}^{(N/p)-1}_{p,1})\cap L^1_t(\dot{B}^{(N/p)+1}_{p,1})}\|X\|_{L^\infty_t(\dot{\mathscr{C}}^\varepsilon)}. \end{aligned} \tag{3.23}$$

Because X satisfies (0.8), standard Hölder estimates for transport equations imply that

$$\|X\|_{L^\infty_t(\dot{\mathscr{C}}^\varepsilon)} \leqslant \|X_0\|_{\dot{\mathscr{C}}^\varepsilon} + \int_0^t \|\nabla u\|_{L^\infty}\|X\|_{\dot{\mathscr{C}}^\varepsilon}\,dt' + \int_0^t \|\partial_X u\|_{\dot{\mathscr{C}}^\varepsilon}\,dt'.$$

Now, recall that

$$\partial_X u - \dot{\mathcal{T}}_X u = \dot{T}_{\partial_k u}X^k + \dot{R}(\partial_k u, X^k).$$

Hence, using standard continuity results for operators $\dot T$ and $\dot R$, and embedding,

$$\|\dot{\mathcal{T}}_X u-\partial_X u\|_{\dot{\mathscr{C}}^\varepsilon}\lesssim\|\dot{\mathcal{T}}_X u-\partial_X u\|_{\dot B^{(N/p)+\varepsilon}_{p,1}}\lesssim\|\nabla u\|_{\dot B^{N/p}_{p,1}}\|X\|_{\dot{\mathscr{C}}^\varepsilon}.\tag{3.24}$$

Therefore we have

$$\|X\|_{L^\infty_t(\dot{\mathscr{C}}^\varepsilon)}\leqslant\|X_0\|_{\dot{\mathscr{C}}^\varepsilon}+\int_0^t\|\nabla u\|_{\dot B^{N/p}_{p,1}}\|X\|_{\dot{\mathscr{C}}^\varepsilon}\,dt'+\|\dot{\mathcal{T}}_X u\|_{L^1_t(\dot B^{(N/p)+\varepsilon}_{p,1})}.\tag{3.25}$$

Then, using (3.1) and plugging the above inequality in (3.23), we get

$$\begin{aligned}\mathscr{H}(t)\lesssim\|\partial_{X_0}u_0\|_{\dot B^{(N/p)+\varepsilon-2}_{p,1}}+\|X_0\|_{\dot{\mathscr{C}}^\varepsilon}\|u_0\|_{\dot B^{(N/p)-1}_{p,1}}+\|\widetilde g\|_{L^1_t(\dot B^{(N/p)+\varepsilon-2}_{p,1})}&\\ +\|u_0\|_{\dot B^{(N/p)-1}_{p,1}}\bigg(\|\dot{\mathcal{T}}_X u\|_{L^1_t(\dot B^{(N/p)+\varepsilon}_{p,1})}+\int_0^t\|\nabla u\|_{\dot B^{N/p}_{p,1}}\|X\|_{\dot{\mathscr{C}}^\varepsilon}\,dt'\bigg)&\cdot\end{aligned}$$

Choosing c small enough in (1.8), we see that the first term of the second line may be absorbed by the left-hand side. Therefore, setting

$$\mathscr{K}(t):=\mathscr{H}(t)+\|X\|_{L^\infty_t(\dot{\mathscr{C}}^\varepsilon)}$$

and using again (3.25) and the smallness of u_0,

$$\mathscr{K}(t)\lesssim\|\partial_{X_0}u_0\|_{\dot B^{(N/p)+\varepsilon-2}_{p,1}}+\|X_0\|_{\dot{\mathscr{C}}^\varepsilon}+\|\widetilde g\|_{L^1_t(\dot B^{(N/p)+\varepsilon-2}_{p,1})}+\int_0^t\|\nabla u\|_{\dot B^{N/p}_{p,1}}\|X\|_{\dot{\mathscr{C}}^\varepsilon}\,dt'.$$

In order to close the estimates, it suffices to bound $\widetilde g$ by means of (3.19). Then the above inequality becomes, after using (3.4) and (3.5) (and the fact that $\|\rho_0-1\|_{\mathcal{M}(\dot B^{(N/p)+\varepsilon-2}_{p,1})}$ is small implies that $\|\rho_0\|_{\mathcal{M}(\dot B^{(N/p)+\varepsilon-2}_{p,1})}$ is of order one),

$$\begin{aligned}\mathscr{K}(t)\lesssim\|\partial_{X_0}u_0\|_{\dot B^{(N/p)+\varepsilon-2}_{p,1}}+\|X_0\|_{\dot{\mathscr{C}}^\varepsilon}&\\ +\int_0^t\big(\|u\|_{\dot{\mathscr{C}}^{-1}}\|\dot{\mathcal{T}}_X u\|_{\dot B^{(N/p)+\varepsilon}_{p,1}}+\|\nabla u\|_{\dot B^{N/p}_{p,1}}\|\dot{\mathcal{T}}_X u\|_{\dot{\mathscr{C}}^{\varepsilon-2}}\big)\,dt'&\\ +\int_0^t\|X\|_{\dot{\mathscr{C}}^\varepsilon}\big(\|\nabla u\|_{\dot B^{N/p}_{p,1}}\|u\|_{\dot B^{(N/p)-1}_{p,1}}+\|(\partial_t u,D_t u,\nabla^2u,\nabla P)\|_{\dot B^{(N/p)-1}_{p,1}}\big)\,dt'&\\ +\|\partial_{X_0}\rho_0\|_{\mathcal{M}(\dot B^{(N/p)-1}_{p,1}\to\dot B^{(N/p)+\varepsilon-2}_{p,1})}\int_0^t\|D_t u\|_{\dot B^{(N/p)-1}_{p,1}}\,dt'.&\end{aligned}$$

The smallness of u_0 and (1.6) imply that all the terms of the right-hand side (except for the ones pertaining to the data), may be absorbed by the left-hand side. Therefore using the bounds for $D_t u$ in (3.1), we eventually get

$$\begin{aligned}\mathscr{K}(t)\lesssim\|\partial_{X_0}u_0\|_{\dot B^{(N/p)+\varepsilon-2}_{p,1}}+\|X_0\|_{\dot{\mathscr{C}}^\varepsilon}&\\ +\|\partial_{X_0}\rho_0\|_{\mathcal{M}(\dot B^{(N/p)-1}_{p,1}\to\dot B^{(N/p)+\varepsilon-2}_{p,1})}\|u_0\|_{\dot B^{(N/p)-1}_{p,1}}&.\end{aligned}\tag{3.26}$$

From (3.24), we gather that $\partial_X u$ is bounded by the right-hand side of (3.26). Next, in order to control the whole nonhomogeneous Hölder norm of X, it suffices to remember that

$$\|X\|_{\mathcal{C}^{0,\varepsilon}}=\|X\|_{L^\infty}+\|X\|_{\dot{\mathscr{C}}^\varepsilon}$$

and that Relation (0.7) together with (3.2) directly yield

$$\|X_t\|_{L^\infty}\leqslant\|\partial_{X_0}\psi_t\|_{L^\infty}\leqslant C\|X_0\|_{L^\infty}.$$

Finally, to estimate $\partial_X\nabla P$, we use Inequality (1.16) and get

$$\|\partial_X\nabla P-\nabla\dot{\mathcal{T}}_XP\|_{L^1_t(\dot B^{(N/p)+\varepsilon-2}_{p,1})}\lesssim\|X\|_{L^\infty_t(\dot{\mathscr{C}}^\varepsilon)}\|\nabla P\|_{L^1_t(\dot B^{(N/p)-1}_{p,1})}.$$

Therefore $\|\partial_X\nabla P\|_{L^1_t(\dot B^{(N/p)+\varepsilon-2}_{p,1})}$ may be bounded like $\mathscr{K}(t)$.

3.3. The regularization process. — In all the above computations, we implicitly assumed that X and ∂_Xu were in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon})$ and $L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon})$, respectively. However, Theorem 1.1 just ensures continuity of those vector-fields, not Hölder regularity.

To overcome that difficulty, one may smooth out the initial velocity (not the density, not to destroy the multiplier hypotheses) by setting for example $u_0^n:=\dot S_nu_0$. Then the condition (1.8) is satisfied by (ρ_0,u_0^n) and, as in addition u_0^n belongs to all Besov spaces $\dot B^{(N/p)-1}_{\widetilde p,r}$ with $\widetilde p\geqslant p$ and $r\geqslant1$, one can apply[(6)] [14, Th. 1.1] for solving (INS) with initial data (ρ_0,u_0^n). This provides us with a unique global solution $(\rho^n,u^n,\nabla P^n)$ which, among others, satisfies

$$\nabla u^n\in L^r(\mathbb{R}_+;\dot B^{N/p}_{\widetilde p,r})\quad\text{for all}\quad r\in\,]1,\infty[\quad\text{and}\quad\max\Big(p,\frac{Nr}{3r-2}\Big)\leqslant\widetilde p\leqslant\frac{Nr}{r-1}\cdot$$

By taking r sufficiently close to 1 and using embedding, we see that this implies that ∇u^n is in $L^1_{\mathrm{loc}}(\mathbb{R}_+;\dot{\mathscr{C}}^{0,\delta})$ for all $0<\delta<1$ and thus the corresponding flow ψ^n is (in particular) in $\mathcal{C}^{1,\varepsilon}$. This ensures, thanks to (0.7), that X^n is in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon})$ and thus that $\partial_{X^n}u^n$ is in $L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon})$.

From the previous steps and the fact that the data (ρ_0,u_0^n) satisfy (1.8) uniformly, we get uniform bounds for ρ^n, u^n, ∇P^n and X^n, and standard arguments thus allow to show that u^n tends to u in $L^1_{\mathrm{loc}}(\mathbb{R}_+;L^\infty)$ and thus $(\psi^n-\psi)\to0$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^\infty)$. Interpolating with the uniform bounds and using standard functional analysis arguments, one can conclude that $X^n\to X$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon'})$ for all $\varepsilon'<\varepsilon$ (and similar results for $(u^n)_{n\in\mathbb{N}}$) and that all the estimates of the previous steps are satisfied. The details are left to the reader. □

Appendix A. Multiplier spaces

The following relationship between the nonhomogeneous Besov spaces $B^s_{p,r}(\mathbb{R}^N)$ and the homogeneous Besov spaces $\dot B^s_{p,r}(\mathbb{R}^N)$ for *compactly supported* functions or distributions has been established in [13, §2.1].

Proposition A.1. — *Let $(p,r)\in[1,\infty]^2$ and $s>-N/p':=-N(1-1/p)$ (or just $s\geqslant-N/p'$ if $r=\infty$). For any u in the set $\mathcal{E}'(\mathbb{R}^N)$ of compactly supported distributions on $\mathbb{R}^N$, we have*

$$u\in B^s_{p,r}(\mathbb{R}^N)\Longleftrightarrow u\in\dot B^s_{p,r}(\mathbb{R}^N).$$

Moreover, there exists a constant $C=C(s,p,r,N,\operatorname{Supp}u)$ such that

$$C^{-1}\|u\|_{\dot B^s_{p,r}}\leqslant\|u\|_{B^s_{p,r}}\leqslant C\|u\|_{\dot B^s_{p,r}}.$$

(6) That paper concerns the half-space; having the same result in the whole space setting is much easier.

A simple consequence of Proposition A.1 and of standard embeddings for nonhomogeneous Besov spaces is that for any (s,p,r) as above, we have

$$\mathcal{E}'(\mathbb{R}^N)\cap \dot B^{s+\delta}_{p,r}(\mathbb{R}^N)\hookrightarrow \mathcal{E}'(\mathbb{R}^N)\cap \dot B^{s}_{p,r}(\mathbb{R}^N)\quad\text{for any}\quad \delta>0. \tag{A.1}$$

We also used the following statement:

Proposition A.2. — *Let (s,p,r) be arbitrary in $\mathbb{R}\times[1,\infty]^2$. Then for all $u\in B^s_{\infty,r}(\mathbb{R}^N)\cap\mathcal{E}'(\mathbb{R}^N)$, we have $u\in B^s_{p,r}(\mathbb{R}^N)$ and there exists $C=C(s,p,\operatorname{Supp}u)$ such that*

$$\|u\|_{B^s_{p,r}}\leqslant C\|u\|_{B^s_{\infty,r}}.$$

Proof. — Let u be in $B^s_{\infty,r}(\mathbb{R}^N)$ with compact support. Fix some smooth cut-off function ϕ so that $\phi\equiv 1$ on $\operatorname{Supp}u$. Being compactly supported and smooth, ϕ belongs to any nonhomogeneous Besov space. Then, using (the nonhomogeneous version of) the decomposition (1.15) and that $u=\phi u$, we get

$$u=T_\phi u+T_u\phi+R(u,\phi).$$

Because ϕ is in L^p and u in $B^s_{\infty,r}$, standard continuity results for the paraproduct ensure that $T_\phi u$ is in $B^s_{p,r}$. For the second term, we just use that u is in $\mathscr{C}^{-|s|-1}$ and ϕ in $B^{|s|+1+s}_{p,r}$ hence $T_u\phi$ is in $B^s_{p,r}$. For the remainder term, we use for instance the fact that ϕ is in $\mathscr{C}^{|s|+1}$. Putting all those information together completes the proof. □

The following result was the key to bounding the density terms in our study of (INS).

Lemma A.3. — *Let $(s,s_k,p,p_k,r,r_k)\in\,]-1,1[^2\times[1,\infty]^4$ with $k=1,2$, and $Z:\mathbb{R}^N\to\mathbb{R}^N$ be a $\mathcal{C}^1$ measure preserving diffeomorphism such that DZ and DZ^{-1} are bounded. When we consider the homogeneous Besov space $\dot B^s_{p,r}(\mathbb{R}^N)$ or $\dot B^{s_k}_{p_k,r_k}(\mathbb{R}^N)$, we assume in addition that $s\in\,]-N/p',N/p[$ and $s_k\in\,]-N/p_k',N/p_k[$ for $k=1,2$. Then we have:*

(i) *If $b^s_{p,r}(\mathbb{R}^N)$ stands for $B^s_{p,r}(\mathbb{R}^N)$ or $\dot B^s_{p,r}(\mathbb{R}^N)$, then the mapping $u\mapsto u\circ Z$ is continuous on $b^s_{p,r}(\mathbb{R}^N)$: there is a positive constant $C_{Z,s,p,r}$ such that*

$$\|u\circ Z\|_{b^s_{p,r}}\leqslant C_{Z,s,p,r}\|u\|_{b^s_{p,r}}. \tag{A.2}$$

(ii) *If $b^{s_k}_{p_k,r_k}$ with $k=1,2$, denote the same type of Besov spaces, then the mapping $\varphi\mapsto\varphi\circ Z$ is continuous on $\mathcal{M}\big(b^{s_1}_{p_1,r_1}(\mathbb{R}^N)\to b^{s_2}_{p_2,r_2}(\mathbb{R}^N)\big)$, that is*

$$\|\varphi\circ Z\|_{\mathcal{M}(b^{s_1}_{p_1,r_1}\to b^{s_2}_{p_2,r_2})}\leqslant C_{Z^{-1},s_1,p_1,r_1}C_{Z,s_2,p_2,r_2}\|\varphi\|_{\mathcal{M}(b^{s_1}_{p_1,r_1}\to b^{s_2}_{p_2,r_2})}.$$

(iii) *We have the following equivalence for any $\varphi\in\mathcal{E}'(\mathbb{R}^N)$,*

$$\varphi\in\mathcal{M}\big(B^{s_1}_{p_1,r_1}(\mathbb{R}^N)\longrightarrow B^{s_2}_{p_2,r_2}(\mathbb{R}^N)\big)\iff\varphi\in\mathcal{M}\big(b^{s_1}_{p_1,r_1}(\mathbb{R}^N)\longrightarrow b^{s_2}_{p_2,r_2}(\mathbb{R}^N)\big).$$

Here $b^{s_1}_{p_1,r_1}$ and $b^{s_2}_{p_2,r_2}$ can be different type of Besov spaces but obey our convention on the index s_k for homogeneous Besov space.

Proof. — Item (i) in the case $b=\dot B$ has been proved in [13, Lem. 2.1.1]. One may easily modify the proof to handle nonhomogeneous Besov spaces: use the finite difference characterization of [26, p. 98] if $s>0$, argue by duality if $s<0$ and interpolate for the case $s=0$. We get $C_{Z,s,p,r}\approx 1+\|DZ\|_{L^\infty}^{s+N/r}$ if $s>0$, and $C_{Z,s,p,r}\approx 1+\|DZ^{-1}\|_{L^\infty}^{-s+N/r'}$ if $s<0$.

Part (ii) is immediate according to (1.5) and (A.2). Indeed we may write:

$$\begin{aligned}\|\varphi\circ Z\|_{\mathcal{M}\left(b^{s_1}_{p_1,r_1}\to b^{s_2}_{p_2,r_2}\right)}&=\sup_{\|u\|_{b^{s_1}_{p_1,r_1}}\leqslant 1}\|(\varphi\circ Z)\,u\|_{b^{s_2}_{p_2,r_2}}\\&=\sup_{\|u\|_{b^{s_1}_{p_1,r_1}}\leqslant 1}\|(\varphi\,(u\circ Z^{-1}))\circ Z\|_{b^{s_2}_{p_2,r_2}}\\&\leqslant C_{Z,s_2,p_2,r_2}\sup_{\|u\|_{b^{s_1}_{p_1,r_1}}\leqslant 1}\|\varphi\,(u\circ Z^{-1})\|_{b^{s_2}_{p_2,r_2}}\\&\leqslant C_{Z,s_2,p_2,r_2}\|\varphi\|_{\mathcal{M}(b^{s_1}_{p_1,r_1}\to b^{s_2}_{p_2,r_2})}\sup_{\|u\|_{b^{s_1}_{p_1,r_1}}\leqslant 1}\|u\circ Z^{-1}\|_{b^{s_1}_{p_1,r_2}}\\&\leqslant C_{Z^{-1},s_1,p_1,r_1}C_{Z,s_2,p_2,r_2}\|\varphi\|_{\mathcal{M}(b^{s_1}_{p_1,r_1}\to b^{s_2}_{p_2,r_2})}.\end{aligned}$$

To prove the last item, it suffices to check that if φ belongs to $\mathcal{E}'\cap\mathcal{M}(B^{s_1}_{p_1,r_1}\to B^{s_2}_{p_2,r_2})$, then φ is also in the multiplier space between the general type Besov spaces. Take $u\in b^{s_1}_{p_1,r_1}$ with compact support, and some smooth and compactly supported nonnegative cut-off function ψ satisfying $\psi\equiv 1$ on $\operatorname{Supp}\varphi$. Then from Proposition A.1 and (1.5), we have

$$\begin{aligned}\|\varphi u\|_{b^{s_2}_{p_2,r_2}}=\|\varphi\psi u\|_{b^{s_2}_{p_2,r_2}}\lesssim\|\varphi\psi u\|_{B^{s_2}_{p_2,r_2}}&\lesssim\|\varphi\|_{\mathcal{M}(B^{s_1}_{p_1,r_1}\to B^{s_2}_{p_2,r_2})}\|\psi u\|_{B^{s_1}_{p_1,r_1}}\\&\lesssim\|\varphi\|_{\mathcal{M}(B^{s_1}_{p_1,r_1}\to B^{s_2}_{p_2,r_2})}\|\psi u\|_{b^{s_1}_{p_1,r_1}}\\&\lesssim\|\varphi\|_{\mathcal{M}(B^{s_1}_{p_1,r_1}\to B^{s_2}_{p_2,r_2})}\|\psi\|_{\mathcal{M}(b^{s_1}_{p_1,r_1})}\|u\|_{b^{s_1}_{p_1,r_1}}.\end{aligned}$$

For the last inequality, we used $\mathcal{C}^\infty_c\hookrightarrow\mathcal{M}(b^{s_1}_{p_1,r_1})$ (see [13, Cor. 2.1.1]). □

Appendix B. Commutator Estimates

We here recall and prove some commutator estimates that were crucial in this paper. All of them strongly rely on continuity results in Besov spaces for the paraproduct and remainder operators, and on the following classical result (see e.g. [4, §2.10]).

Lemma B.1. — *Let $A:\mathbb{R}^N\smallsetminus\{0\}\to\mathbb{R}$ be a smooth function, homogeneous of degree m. Let $(\varepsilon,s,p,r,r_1,r_2,p_1,p_2)\in\,]0,1[\,\times\mathbb{R}\times[1,\infty]^6$ with $\frac1p=\frac1{p_1}+\frac1{p_2}$, $\frac1r=\frac1{r_1}+\frac1{r_2}$ and*

$$s-m+\varepsilon<N/p\quad\text{or}\quad\{s-m+\varepsilon<N/p\text{ and }r=1\}\cdot$$

There exists a constant C depending only on s,ε,N and A such that,

$$\|[\dot T_g,A(D)]u\|_{\dot B^{s-m+\varepsilon}_{p,r}}\leqslant C\|\nabla g\|_{\dot B^{\varepsilon-1}_{p_1,r_1}}\|u\|_{\dot B^{s}_{p_2,r_2}}.$$

If the integer N_0 in the definition of Bony's paraproduct and remainder is large enough (for instance $N_0=4$ does), then the following fundamental lemma holds.

Lemma B.2 (Chemin-Leibniz Formula). — *Let* $(\varepsilon,s_k,p,p_k,p_3,r,r_k)\in]0,1[\times\mathbb{R}\times[1,\infty]^5$ *for* $k=1,2$ *satisfying*

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\quad\text{and}\quad\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}\cdot$$

(i) *If* $s_2<0$ *and* $s_1+s_2+\varepsilon-1<N/p$ *or* $\{s_1+s_2+\varepsilon-1=N/p$ *and* $r=1\}$, *then we have*

$$\|\dot{\mathcal{T}}_X\dot{T}_g f-\dot{T}_g\dot{\mathcal{T}}_X f-\dot{T}_{\dot{\mathcal{T}}_X g}f\|_{\dot{B}^{s_1+s_2+\varepsilon-1}_{p,r}}\leqslant C\|X\|_{\dot{B}^{\varepsilon}_{p_3,\infty}}\|g\|_{\dot{B}^{s_2}_{p_2,\infty}}\|f\|_{\dot{B}^{s_1}_{p_1,r}}.$$

The above inequality still holds for $s_2=0$, *if one replaces* $\|g\|_{\dot{B}^0_{p_2,\infty}}$ *by* $\|g\|_{L^{p_2}}$.

(ii) *If* $s_1+s_2+\varepsilon-1\in]0,N/p[$ *or* $\{s_1+s_2+\varepsilon-1=N/p$ *and* $r=1\}$, *then we have*

$$\|\dot{\mathcal{T}}_X\dot{R}(f,g)-\dot{R}(\dot{\mathcal{T}}_X f,g)-\dot{R}(f,\dot{\mathcal{T}}_X g)\|_{\dot{B}^{s_1+s_2+\varepsilon-1}_{p,r}}\leqslant C\|X\|_{\dot{B}^{\varepsilon}_{p_3,\infty}}\|f\|_{\dot{B}^{s_1}_{p_1,r_1}}\|g\|_{\dot{B}^{s_2}_{p_2,r_2}}.$$

The above inequality still holds for $s_1+s_2+\varepsilon-1=0$, $r=\infty$ *and* $\frac{1}{r_1}+\frac{1}{r_2}=1$.

Proof. — This is a mere adaptation of [15] to the homogeneous framework. The proof is based on a generalized Leibniz formula for para-vector field operators which was derived by J.-Y. Chemin in [6]. More precisely, define the following Fourier multipliers

$$\dot{\Delta}_{k,j}:=\varphi_k(2^{-j}D)\qquad\text{with }\ \varphi_k(\xi):=i\xi_k\varphi(\xi)\ \text{ for }\ k\in\{1,\dots,N\}\ \text{ and }\ j\in\mathbb{Z}.$$

Then we have

$$\begin{aligned}\dot{\mathcal{T}}_X\dot{T}_g f&=\sum_{j\in\mathbb{Z}}(\dot{S}_{j-N_0}g\dot{\mathcal{T}}_X\dot{\Delta}_j f+\dot{\Delta}_j f\dot{\mathcal{T}}_X\dot{S}_{j-N_0}g)+\sum_{j\in\mathbb{Z}}(\dot{T}_{1,j}+\dot{T}_{2,j})\\&=\dot{T}_g\dot{\mathcal{T}}_X f+\dot{T}_{\dot{\mathcal{T}}_X g}f+\sum_{\substack{j\in\mathbb{Z}\\ \alpha=1,\dots,4}}\dot{T}_{\alpha,j},\end{aligned}$$

where

$$\begin{aligned}\dot{T}_{1,j}&:=\sum_{\substack{j\leqslant j'\leqslant j+1\\ j-N_0-1\leqslant j''\leqslant j'-N_0-1}}2^{j'}\dot{\Delta}_{j''}X^k\big(\dot{\Delta}_{k,j'}(\dot{\Delta}_j f\dot{S}_{j-N_0}g)-\dot{\Delta}_{k,j'}\dot{\Delta}_j f\dot{S}_{j-N_0}g\big),\\ \dot{T}_{2,j}&:=\sum_{\substack{j'\leqslant j-2\\ j'-N_0\leqslant j''\leqslant j-N_0-2}}2^{j'}\dot{\Delta}_{j''}X^k(\dot{\Delta}_j f)\dot{\Delta}_{k,j'}\dot{S}_{j-N_0}g,\\ \dot{T}_{3,j}&:=\dot{S}_{j-N_0}g[\dot{T}_{X^k},\dot{\Delta}_j]\partial_k f,\\ \dot{T}_{4,j}&:=\dot{\Delta}_j f[\dot{T}_{X^k},\dot{S}_{j-N_0}]\partial_k g.\end{aligned}$$

Bounding $\dot{T}_{1,j}$ and $\dot{T}_{2,j}$ stems from the definition of Besov norms, and Lemmas 2.99, 2.100 of [4] allow to bound $\dot{T}_{3,j}$ and $\dot{T}_{4,j}$ provided $\varepsilon<1$.

In order to prove the second item, let us set

$$A_{j,j'}:=\big\{j-N_0-1,\dots,j'-N_0-1\big\}\cup\big\{j'-N_0,\dots,j-N_0-2\big\}\cdot$$

We have

$$\begin{aligned}\dot{\mathcal{T}}_X\dot{R}(f,g)&=\sum_{j\in\mathbb{Z}}(\widetilde{\dot{\Delta}}_j g\dot{\mathcal{T}}_X\dot{\Delta}_j f+\dot{\Delta}_j f\dot{\mathcal{T}}_X\widetilde{\dot{\Delta}}_j g)+\sum_{j\in\mathbb{Z}}(\dot{R}_{1,j}+\dot{R}_{2,j})\\&=\dot{R}(\dot{\mathcal{T}}_X f,g)+\dot{R}(f,\dot{\mathcal{T}}_X g)+\sum_{\substack{j\in\mathbb{Z}\\ \alpha=1,\dots,4}}\dot{R}_{\alpha,j},\end{aligned}$$

where, denoting $\widetilde{\dot{\Delta}}_j:=\dot{\Delta}_{j-N_0}+\cdots+\dot{\Delta}_{j+N_0}$,

$$\begin{aligned}\dot{R}_{1,j}&:=\sum_{\substack{|j'-j|\leqslant N_0+1\\ j''\in A_{j,j'}}}\operatorname{sgn}(j'-j+1)2^{j'}\dot{\Delta}_{j''}X^k\big(\dot{\Delta}_{k,j'}(\dot{\Delta}_j f\widetilde{\dot{\Delta}}_j g)-\dot{\Delta}_j f\dot{\Delta}_{k,j'}\widetilde{\dot{\Delta}}_j g\big)\\&\qquad+\sum_{\substack{j-1\leqslant j'\leqslant j\\ j'-N_0\leqslant j''\leqslant j-N_0}}2^{j'}\dot{\Delta}_{j''}X^k(\dot{\Delta}_{k,j'}\dot{\Delta}_j f)\widetilde{\dot{\Delta}}_j g,\\ \dot{R}_{2,j}&:=\sum_{\substack{j'\leqslant j-N_0-2\\ j'-N_0\leqslant j''\leqslant j-N_0-2}}2^{j'}\dot{\Delta}_{j''}X^k\dot{\Delta}_{k,j'}(\dot{\Delta}_j f\widetilde{\dot{\Delta}}_j g),\\ \dot{R}_{3,j}&:=\widetilde{\dot{\Delta}}_j g[\dot{T}_{X^k},\dot{\Delta}_j]\partial_k f,\\ \dot{R}_{4,j}&:=\dot{\Delta}_j f[\dot{T}_{X^k},\widetilde{\dot{\Delta}}_j]\partial_k g.\end{aligned}$$

Here again, bounding $\dot{R}_{1,j}$ and $\dot{R}_{2,j}$ follows from the definition of Besov norms, while Lemma 2.100 of [4] allows to bound $\dot{R}_{3,j}$ and $\dot{R}_{4,j}$. ☐

Proposition B.3. — *Let* (ε,p) *be in* $]0,1[\times[1,\infty]$. *Consider a pair of vector fields* (X,v) *in*

$$\big(L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\dot{\mathscr{C}}^\varepsilon)\big)^N\times\big(L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\dot{B}^{(N/p)-1}_{p,1})\cap L^1_{\mathrm{loc}}(\mathbb{R}_+;\dot{B}^{(N/p)+1}_{p,1})\big)^N,$$

satisfying $\operatorname{div}v=0$ *and the transport equation*

$$\text{(B.1)}\qquad\begin{cases}(\partial_t+v\cdot\nabla)X=\partial_X v,\\ X|_{t=0}=X_0.\end{cases}$$

If in addition

$$\text{(B.2)}\qquad N/p>2-\varepsilon,\quad\text{or}\quad N/p>1-\varepsilon\ \text{ and }\ \operatorname{div}X\equiv 0,$$

then there exists a constant C *such that:*

$$\text{(B.3)}\quad\begin{aligned}\|[\dot{\mathcal{T}}_X,\partial_t+v\cdot\nabla]v\|_{\dot{B}^{(N/p)+\varepsilon-2}_{p,1}}\leqslant C(&\|X\|_{\dot{\mathscr{C}}^\varepsilon}\|v\|_{\dot{B}^{(N/p)+1}_{p,1}}\|v\|_{\dot{B}^{(N/p)-1}_{p,1}}\\&+\|v\|_{\dot{\mathscr{C}}^{-1}}\|\dot{\mathcal{T}}_X v\|_{\dot{B}^{(N/p)+\varepsilon}_{p,1}}+\|v\|_{\dot{B}^{(N/p)+1}_{p,1}}\|\dot{\mathcal{T}}_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}}).\end{aligned}$$

Proof. — This is essentially the proof of [15, Prop. A.5]. For the reader convenience, we here give a sketch of it. Because $\operatorname{div}v=0$, we may write

$$\begin{aligned}[\dot{\mathcal{T}}_X,\partial_t+v^\ell\partial_\ell]v&=-v^\ell\partial_\ell\dot{T}_{X^k}\partial_k v-\dot{T}_{\partial_t X^k}\partial_k v+\dot{T}_{X^k}\partial_k(v^\ell\partial_\ell v)\\&=-\dot{T}_{\partial_t X^k}\partial_k v+\partial_\ell\dot{\mathcal{T}}_X(v^\ell v)-\dot{\mathcal{T}}_{\partial_\ell X}(v^\ell v)-v^\ell\partial_\ell\dot{\mathcal{T}}_X v.\end{aligned}$$

Hence, decomposing $v^\ell v$ according to Bony's decomposition, we discover that

$$[\dot{\mathcal{T}}_X, \partial_t + v^\ell \partial_\ell]v = \sum_{\alpha=1}^{\alpha=5} \dot{R}_\alpha$$

with
$$\dot{R}_1 := -\dot{T}_{\partial_t X^k}\partial_k v, \quad \dot{R}_2 := \partial_\ell(\dot{\mathcal{T}}_X \dot{T}_{v^\ell} v + \dot{\mathcal{T}}_X \dot{T}_v v^\ell), \quad \dot{R}_3 := \partial_\ell \dot{\mathcal{T}}_X \dot{R}(v^\ell, v),$$
$$\dot{R}_4 := -\dot{\mathcal{T}}_{\partial_\ell X}(v^\ell v), \quad \dot{R}_5 := -v^\ell \partial_\ell \dot{\mathcal{T}}_X v.$$

It suffices to check that all the terms $\dot{R}_\alpha$ may be bounded by the right-hand side of (B.3).

Bound of $\dot{R}_1$. — From the equation (B.1), we have

$$\dot{R}_1 = \dot{T}_{v\cdot\nabla X^k}\partial_k v - \dot{T}_{\partial_X v^k}\partial_k v.$$

Hence using standard continuity results for the paraproduct, we deduce that

$$\|\dot{R}_1\|_{\dot{B}^{(N/p)+\varepsilon-2}_{p,1}} \lesssim \|\nabla v\|_{\dot{B}^{N/p}_{p,1}}\big(\|v\cdot\nabla X\|_{\dot{\mathscr{C}}^{\varepsilon-2}} + \|\partial_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}}\big).$$

Keeping in mind (B.2), the last term may be bounded according to (1.16), after using the embedding $\dot{B}^{(N/p)+\varepsilon-2}_{p,1}(\mathbb{R}^N) \hookrightarrow \dot{\mathscr{C}}^{\varepsilon-2}(\mathbb{R}^N)$. We get

$$\|\partial_X v - \dot{\mathcal{T}}_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}} \lesssim \|\nabla v\|_{\dot{B}^{(N/p)-2}_{p,1}}\|X\|_{\dot{\mathscr{C}}^{\varepsilon}}.$$

As for the first term, we use the fact $\operatorname{div} v = 0$ and the following decomposition

$$v\cdot\nabla X = \dot{\mathcal{T}}_v X + \dot{T}_{\partial_\ell X} v^\ell + \partial_\ell \dot{R}(v^\ell, X),$$

which allow to get, as long as (B.2) holds

$$\|\dot{R}_1\|_{\dot{B}^{(N/p)+\varepsilon-2}_{p,1}} \lesssim \|\nabla v\|_{\dot{B}^{N/p}_{p,1}}\big(\|v\|_{\dot{B}^{(N/p)-1}_{p,1}}\|X\|_{\dot{\mathscr{C}}^{\varepsilon}} + \|\dot{\mathcal{T}}_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}}\big).$$

Bound of $\dot{R}_2$. — Due to Lemma B.2 (i) and continuity of paraproduct operator, we have

$$\|\dot{R}_2\|_{\dot{B}^{(N/p)+\varepsilon-2}_{p,1}} \lesssim \|X\|_{\dot{\mathscr{C}}^{\varepsilon}}\|v\|_{\dot{B}^{(N/p)+1}_{p,1}}\|v\|_{\dot{\mathscr{C}}^{-1}} + \|v\|_{\dot{\mathscr{C}}^{-1}}\|\dot{\mathcal{T}}_X v\|_{\dot{B}^{(N/p)+\varepsilon}_{p,1}} + \|v\|_{\dot{B}^{(N/p)+1}_{p,1}}\|\dot{\mathcal{T}}_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}}.$$

Bound of $\dot{R}_3$. — Applying Lemma B.2 (ii) and continuity of remainder operator under the condition $(N/p)+\varepsilon-1>0$ yields

$$\|\dot{R}_3\|_{\dot{B}^{(N/p)+\varepsilon-2}_{p,1}} \lesssim \|X\|_{\dot{\mathscr{C}}^{\varepsilon}}\|v\|_{\dot{B}^{(N/p)+1}_{p,1}}\|v\|_{\dot{\mathscr{C}}^{-1}} + \|v\|_{\dot{B}^{(N/p)+1}_{p,1}}\|\dot{\mathcal{T}}_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}}.$$

Bound of $\dot{R}_4$. — From Bony decomposition (1.15), it is easy to get

$$\|v^l v\|_{\dot{B}^{N/p}_{p,1}} \lesssim \|v\|_{\dot{\mathscr{C}}^{-1}}\|v\|_{\dot{B}^{(N/p)+1}_{p,1}}.$$

Hence

$$\|\dot{R}_4\|_{\dot{B}^{(N/p)+\varepsilon-2}_{p,1}} \lesssim \|\nabla X\|_{\dot{\mathscr{C}}^{\varepsilon-1}}\|v\|_{\dot{\mathscr{C}}^{-1}}\|v\|_{\dot{B}^{(N/p)+1}_{p,1}}.$$

Bound of $\dot{R}_5$. — Applying Bony decomposition and using that $\operatorname{div} v=0$ and $\frac{N}{p}+\varepsilon>1$ give

$$\|\dot{R}_5\|_{\dot{B}^{(N/p)+\varepsilon-2}_{p,1}} \lesssim \|v\|_{\mathscr{C}^{-1}}\|\dot{\mathcal{T}}_X v\|_{\dot{B}^{(N/p)+\varepsilon}_{p,1}} + \|v\|_{\dot{B}^{(N/p)+1}_{p,1}}\|\dot{\mathcal{T}}_X v\|_{\mathscr{C}^{\varepsilon-2}}.$$

Combining the above estimates for all $\dot{R}_\alpha$, with $\alpha=1,\dots,5$ yields (B.3). □

Another consequence of Lemma B.2 is the following estimate of $\operatorname{div}(Xfg)$:

Proposition B.4. — *Let (s,p,r) be in $]0,1[\times[1,\infty]^2$ and η in $]0,1-s[$. Consider a bounded vector field X and two bounded functions f,g satisfying*

$$X\in\big(\dot{B}^s_{p,r}(\mathbb{R}^N)\cap\mathscr{C}^{s+\eta}\big)^N,\ \ (f,g)\in\dot{B}^s_{p,r}(\mathbb{R}^N)\times\dot{B}^{-\eta}_{p,r}(\mathbb{R}^N)\ \text{ and }\ \partial_X g\in\dot{B}^{s-1}_{p,r}(\mathbb{R}^N).$$

If in addition $\operatorname{div} X$ *belongs to* $\mathcal{M}\big(\dot{B}^s_{p,r}(\mathbb{R}^N)\to\dot{B}^{s-1}_{p,r}(\mathbb{R}^N)\big)$*, and there exists some $q\in[1,p[$ such that*

$$\operatorname{div} X\in\dot{B}^{s_{p,q}}_{q,r}(\mathbb{R}^N)\quad\text{with}\quad s_{p,q}:=s-1+N(1/q-1/p)>0, \tag{B.4}$$

then we have $\operatorname{div}(Xfg)\in\dot{B}^{s-1}_{p,r}(\mathbb{R}^N)$*, and the following estimate holds true:*

$$\begin{aligned}\|\operatorname{div}(Xfg)\|_{\dot{B}^{s-1}_{p,r}} \lesssim \|X\|_{\dot{B}^s_{p,r}\cap\mathscr{C}^{s+\eta}}\|f\|_{L^\infty\cap\dot{B}^s_{p,r}}\|g\|_{L^\infty\cap\dot{B}^{-\eta}_{p,r}} + \|f\|_{L^\infty}\|\partial_X g\|_{\dot{B}^{s-1}_{p,r}}\\ + \|\operatorname{div} X\|_{\dot{B}^{s_{p,q}}_{q,r}\cap\mathcal{M}(\dot{B}^s_{p,r}\to\dot{B}^{s-1}_{p,r})}\|g\|_{L^\infty}\|f\|_{\dot{B}^s_{p,r}\cap L^\infty}.\end{aligned}$$

Proof. — In light of Bony's decomposition (1.15), and denoting $\dot{T}'_g f:=\dot{T}_g f+\dot{R}(f,g)$, we can decompose $\operatorname{div}(Xfg)$ into

$$\operatorname{div}(Xfg)=\operatorname{div}\big(\dot{T}'_{fg}X+\dot{T}_X(fg)\big)=\sum_{\alpha=1}^4\dot{F}_\alpha,\ \text{ where }\ \begin{cases}\dot{F}_1:=\operatorname{div}(\dot{T}'_{fg}X),\ \ \dot{F}_3:=\dot{\mathcal{T}}_X\dot{T}'_g f,\\ \dot{F}_2:=\dot{T}_{\operatorname{div} X}(fg),\ \ \dot{F}_4:=\dot{\mathcal{T}}_X\dot{T}_f g.\end{cases}$$

Bound of $\dot{F}_1$. — As $s>0$, standard continuity results for $\dot{T}$ and $\dot{R}$ yield

$$\|\dot{F}_1\|_{\dot{B}^{s-1}_{p,r}} \lesssim \|\dot{T}'_{fg}X\|_{\dot{B}^s_{p,r}} \lesssim \|f\|_{L^\infty}\|g\|_{L^\infty}\|X\|_{\dot{B}^s_{p,r}}.$$

Bound of $\dot{F}_2$. — Thanks to continuity results for $\dot{T}$, we have for $s<1$,

$$\|\dot{F}_2\|_{\dot{B}^{s-1}_{p,r}} \lesssim \|\operatorname{div} X\|_{\dot{B}^{s-1}_{p,r}}\|f\|_{L^\infty}\|g\|_{L^\infty}.$$

Bound of $\dot{F}_3$. — Because X and g are bounded and $s>0$, we readily have

$$\|\dot{F}_3\|_{\dot{B}^{s-1}_{p,r}} \lesssim \|X\|_{L^\infty}\|\dot{T}'_g f\|_{\dot{B}^s_{p,r}} \lesssim \|X\|_{L^\infty}\|g\|_{L^\infty}\|f\|_{\dot{B}^s_{p,r}}.$$

Bound of $\dot{F}_4$. — Because $0<s<s+\eta<1$, Lemma B.2 and continuity results for the paraproduct $\dot{T}$ imply that

$$\begin{aligned}\|\dot{\mathcal{T}}_X\dot{T}_f g\|_{\dot{B}^{s-1}_{p,r}} &\lesssim \|X\|_{\mathscr{C}^{s+\eta}}\|f\|_{L^\infty}\|g\|_{\dot{B}^{-\eta}_{p,r}} + \|\dot{T}_f\dot{\mathcal{T}}_X g\|_{\dot{B}^{s-1}_{p,r}} + \|\dot{T}_{\dot{\mathcal{T}}_X f}g\|_{\dot{B}^{s-1}_{p,r}}\\ &\lesssim \|X\|_{\mathscr{C}^{s+\eta}}\|f\|_{L^\infty}\|g\|_{\dot{B}^{-\eta}_{p,r}} + \|f\|_{L^\infty}\|\dot{\mathcal{T}}_X g\|_{\dot{B}^{s-1}_{p,r}} + \|g\|_{L^\infty}\|\dot{\mathcal{T}}_X f\|_{\dot{B}^{s-1}_{p,r}}.\end{aligned}$$

To bound the last term, one may use the decomposition

$$\dot{\mathcal{T}}_X f=\operatorname{div}(\dot{T}_X f)-f\operatorname{div} X+\dot{T}_f\ \operatorname{div} X+\dot{R}(f,\operatorname{div} X).$$

Hence using continuity results for $\dot R$ and $\dot T$ and the fact that $(s_{p,q},q)$ satisfies (B.4),

$$\|\dot{\mathcal{T}}_X f\|_{\dot B^{s-1}_{p,r}} \lesssim \|f\|_{\dot B^s_{p,r}}\big(\|X\|_{L^\infty}+\|\operatorname{div}X\|_{\mathcal{M}(\dot B^s_{p,r}\to\dot B^{s-1}_{p,r})}\big)+\|f\|_{L^\infty}\|\operatorname{div}X\|_{\dot B^{s_{p,q}}_{q,r}}.$$

Finally, to bound the term with $\dot{\mathcal{T}}_X g$, we use the fact that

$$\partial_X g-\dot{\mathcal{T}}_X g=\dot T_{\nabla g}\cdot X+\operatorname{div}\dot R(X,g)-\dot R(\operatorname{div}X,g),$$

whence

$$\|\partial_X g-\dot{\mathcal{T}}_X g\|_{\dot B^{s-1}_{p,r}}\lesssim\|g\|_{L^\infty}\big(\|X\|_{\dot B^s_{p,r}}+\|\operatorname{div}X\|_{\dot B^{s_{p,q}}_{q,r}}\big). \tag{B.5}$$

This completes the proof of the proposition. □

Proposition B.4 above reveals that the bounded function g may behave like some element in $\mathcal{M}(\dot B^{s-1}_{p,\infty})$ under a suitable additional structure assumption. If in addition g has compact support, then one can relax a bit the regularity of X and f to study $\partial_X(fg)$, and get the following generalization of [9, Lem. A.6].

Corollary B.5. — *Consider a divergence-free vector field X with coefficients in $\mathscr{C}^\varepsilon$, and some function f in $\mathscr{C}^{\varepsilon'}$ with $0<\varepsilon,\varepsilon'<1$. Let $g\in L^\infty$ be compactly supported and satisfy $\partial_X g\in\dot B^{\alpha-1}_{p,r}$ for some $(p,r)\in[1,\infty]^2$ and $\alpha\in\,]0,\min\{\varepsilon,\varepsilon'\}[$. Then $\operatorname{div}(Xfg)=\partial_X(fg)\in\dot B^{\alpha-1}_{p,r}$.*

Proof. — Let $\psi\in\mathcal{C}^\infty_c$ be a cut-off function such that $\psi\equiv1$ near $\operatorname{Supp}g$. Denote $(\widetilde X,\widetilde f):=(\psi X,\psi f)$. From Proposition A.1 and the proof of Proposition A.2, we know that

$$(\widetilde X,\widetilde f,g)\in(B^\varepsilon_{q,\infty})^N\times B^{\varepsilon'}_{q,\infty}\times B^{-\eta}_{q,r}\hookrightarrow(\dot B^\alpha_{q,1}\cap L^\infty)^{N+1}\times\dot B^{-\eta}_{q,r},$$

for any $q\in[1,\infty]$ and some $\eta\in\,]0,\min\{N/q',\varepsilon-\alpha\}[$. It is also clear that $\partial_X(fg)=\operatorname{div}(\widetilde X\widetilde f g)$ and $\partial_{\widetilde X}g=\partial_X g$. Hence applying Proposition B.4 gives the result. □

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