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# TOWARD THE STRUCTURE OF FIBERED FUNDAMENTAL GROUPS OF PROJECTIVE VARIETIES

by Donu Arapura

Abstract. — The fundamental group of a smooth projective variety is *fibered* if it maps onto the fundamental group of a smooth curve of genus 2 or more. The goal of this paper is to establish some strong restrictions on these groups, and in particular on the fundamental groups of Kodaira surfaces. In the specific case of a Kodaira surface, these results are in the form of restrictions on the monodromy representation into the mapping class group. When the monodromy is composed with certain standard representations, the images are Zariski dense in a semisimple group of Hermitian type.

Résumé (Vers la structure des groupes fondamentaux fibrés des variétés projectives)

Le groupe fondamental d'une variété projective lisse est dit *fibré* s'il s'envoie surjectivement sur celui d'une courbe de genre 2 ou plus. Le but de cet article est d'établir des restrictions fortes sur ces groupes, et en particulier sur ceux des surfaces de Kodaira. Dans le cas spécifique d'une surface de Kodaira, ces résultats se présentent sous la forme de restrictions sur la représentation de monodromie dans le 'mapping class group'. Lorsque la représentation de monodromie se compose de certaines représentations standard, les images sont Zariski denses dans un groupe semi-simple de type hermitien.

## Contents



A useful dichotomy for groups is to subdivide them into large groups and small, where a group is large for our purposes if it surjects onto a nonabelian free group. We want to study the large groups in the class $\mathscr{P}$ of fundamental groups of complex smooth projective varieties. Standard tricks of the trade, going back to Beauville,

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Catanese and Siu [ABC$^+$96, Chap. 2] and [Cat08, §5.1], show that such a group is *fibered* in the sense that it is given as an extension of an orbifold group

$$\Gamma_{g;\vec{m}} = \big\langle \alpha_1, \ldots, \alpha_{2g}, \beta_1, \ldots, \beta_n \mid \\ [\alpha_1, \alpha_2] \cdots [\alpha_{2g-1}, \alpha_{2g}]\beta_1 \cdots \beta_n = \beta_1^{m_1} = \cdots = \beta_n^{m_n} = 1 \big\rangle$$

by a finitely generated group $K$. So we now come to the main question that motivated this paper: given an action $\Gamma_{g;\vec{m}} \to \mathrm{Aut}(K)$, or perhaps only an outer action, with $K$ finitely generated, when can we expect the semidirect product or some other associated extension to lie in $\mathscr{P}$? The group $\Gamma_{g;\vec{m}}$ will act on the finite dimensional vector space $V = K/[K,K] \otimes \mathbb{Q}$. Let $G$ be the identity component of the Zariski closure of the image of $\Gamma_{g;\vec{m}}$ in $\mathrm{GL}(V)$. We establish the following necessary conditions for an extension of $\Gamma_{g;\vec{m}}$ by $K$ to lie in $\mathscr{P}$:

– The dimension of the space of invariants $V^{\Gamma_{g;\vec{m}}}$ must be even.

– The $\mathbb{Q}$-algebraic group $G$ is semisimple, and the associated real group lies in the small list of the groups arising from Hermitian symmetric domains of classical type [Hel78, p. 518].

– The last result applies more generally to $V = H/[H,H] \otimes \mathbb{Q}$ for any finite index subgroup $H \subset K$. (A finite index subgroup $\Gamma' \subset \Gamma_{g,\vec{m}}$ will act on $V$, and $G$ can be defined as above using the $\Gamma'$-action.)

In the positive direction, we show that many semisimple groups of classical Hermitian type $G$ actually arise in this way from groups in $\mathscr{P}$.

Here is a more detailed summary of the contents of the paper. In the first section, we construct a homomorphism $\rho : \pi_1(Y, y) \to O^+(X_y)$, that we call nonabelian monodromy, where $f : X \to Y$ is an oriented $C^\infty$ fibre bundle and $O^+(X_y) = \mathrm{Out}^+(\pi_1(X_y))$ is the group of orientation preserving outer automorphisms of the fundamental group of a fibre. When $Z = X_y$ is a curve, $O^+(X_y)$ is just the mapping class group. This has a well known representation given by its action on the first homology of $Z$. More generally, a number of authors have studied the action of subgroups of $O^+(Z)$ on the first homology of finite (unramified) coverings of $Z$ [GLLM15, Kob12, Loo97]. All of these extend to the more general situation, and we refer to these as generalized Prym representations. In the second section, we study the nonabelian monodromy of a family of smooth projective varieties. Our main result here is that the Zariski closure of the image of the composite of a generalized Prym with monodromy is semisimple of classical Hermitian type. In a nutshell, this is deduced from the fact that this is the monodromy representation of a polarized variation of Hodge structure of a specific type. (And this is the main reason we work with projective manifolds rather than compact Kähler manifolds.) In the third section, we deduce the results stated in the first paragraph by extending the monodromy theorem to families with singular fibres. In the penultimate section, we go in a different direction. By combing the above techniques with some work of Grunewald, Larsen, Lubotzky, and Malestein [GLLM15], we compute Mumford-Tate groups of some unramified covers of generic curves. The conclusion is that the Hodge structure of an unramified cover looks very different from the Hodge structure of the underlying curve. The final section contains examples, involving pencils of abelian varieties and Kodaira surfaces, with interesting monodromy groups.

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## 1. Nonabelian monodromy

Suppose that $F$ is a connected manifold. Let $\delta$ be a path connecting $x_0 \in F$ to $x_1 \in F$. A self diffeomorphism $\phi : F \to F$, with $\phi(x_0) = x_1$, induces an automorphism $\pi_1(F, x_0) \to \pi_1(F, x_0)$ defined by $g \mapsto \delta^{-1}\phi_*(g)\delta$, where multiplication is taken in the fundamental groupoid. The corresponding outer automorphism is independent of $\delta$. Now suppose that $f : X \to Y$ is a locally trivial $C^\infty$ fibre bundle with fibre $F$ and connected base $Y$. Then, after choosing a Riemannian metric on $X$, we have a holonomy representation of the fundamental group

$$\widetilde{\rho} : \pi_1(Y, y) \longrightarrow \mathrm{Isom}(F) \subset \mathrm{Diffeo}(F)$$

to the group of isometries and therefore diffeomorphisms of $F$. Thus $\rho$ induces a homomorphism

$$\rho : \pi_1(Y, y) \longrightarrow \mathrm{Out}(\pi_1(F, x_0)) = \mathrm{Aut}(\pi_1(F, x_0))/\operatorname{InnerAut}(\pi_1(F, x_0)).$$

We will refer to this as *nonabelian monodromy*.

This can be described more topologically. Given $\gamma \in \pi_1(Y)$, represent it by a $C^\infty$ map $S^1 \to Y$. Then we have an exact sequence

$$1 \longrightarrow \pi_1(F, x_0) \longrightarrow \pi_1(X \times_Y S^1, x_0) \longrightarrow \pi_1(S^1, 0) = \mathbb{Z} \longrightarrow 1$$

which necessarily splits (noncanonically). Let $\widetilde{\gamma} \in \pi_1(X \times_Y S^1)$ denote a lift of $1 \in \mathbb{Z}$.

Lemma 1.1. — *The outer automorphism of $\pi_1(F)$ determined by $g \mapsto \widetilde{\gamma} g \widetilde{\gamma}^{-1}$ coincides with $\rho(\gamma)$.*

*Proof*. — We may replace $Y$ by $S^1$ and $X$ by $X \times_Y S^1$. Let $\gamma \in \pi_1(Y)$ denote a generator. Let us say that a $C^\infty$ path in $X$ is horizontal if its tangent vectors lie in $\ker df_x^\perp$. Through any $x \in F$, there is a unique horizontal lift $\varepsilon_x$ of $\gamma$ with initial point $\varepsilon_x(0) = x$. This is generally not closed. The holonomy $\phi : F \to F$ sends $x$ to the end point $\varepsilon_x(1)$. Let $\delta$ be a path in $F$ connecting $x_0$ to $x_1 = \phi(x_0)$. The element $\varepsilon_{x_0}^{-1}\delta \in \pi_1(X, x_0)$ maps to $\gamma^{-1}$. So it must be conjugate to $\widetilde{\gamma}^{-1}$. One easily checks that

$$\rho(g) = \delta^{-1}\phi_*(g)\delta = (\varepsilon_{x_0}^{-1}\delta)^{-1} g (\varepsilon_{x_0}^{-1}\delta). \qquad \square$$

Corollary 1.2. — *The outer action of* $\pi_1(Y)$ *on* $\operatorname{im}\pi_1(F)$*, by conjugation via a set-theoretic section of* $f_*$ *in the sequence below*

$$1 \longrightarrow \operatorname{im}\pi_1(F) \longrightarrow \pi_1(X) \xrightarrow{f_*} \pi_1(Y) \longrightarrow 1,$$

*is compatible with* $\rho$*.*

We set $O(F) = \operatorname{Out}(\pi_1(F))$. If $F$ is oriented, then there is an induced orientation on $H^1(F,\mathbb{R}) = \operatorname{Hom}(\pi_1(F),\mathbb{R})$. Let $O^+(F) \subset O(F)$ denote the subgroup preserving the orientation on $\operatorname{Hom}(\pi_1(F),\mathbb{R})$. If $f$ is a fibre bundle of oriented manifolds, then the image of holonomy lies in the group of orientation preserving diffeomorphisms $\operatorname{Diffeo}^+(F)$. It follows that $\rho(\pi_1(Y)) \subseteq O^+(F)$. Let $O^+(F,\omega) = O(F,\omega) \cap O^+(F)$. When $F$ is a compact oriented 2-manifold, then $O^+(F)$ is just the mapping class group [FM12]. Note that $O^+(F) = O(F,\omega)$, where $\omega \in H^2(F)$ is the fundamental class.

Many of the familiar representations of the mapping class group generalize to $O(F)$. The group $\operatorname{Aut}(\pi_1(F))$ has an obvious representation $\tau_{\mathbb{Z}}$ on

$$H_1(F,\mathbb{Z}) = \pi_1(F)^{\mathrm{ab}} := \pi_1(F)/[\pi_1(F),\pi_1(F)].$$

Since inner automorphisms act trivially on $\pi_1(F)^{\mathrm{ab}}$, $\tau_{\mathbb{Z}}$ factors through $O(F)$. Let $\tau$ denote the corresponding rational representation $\tau_{\mathbb{Z}} \otimes \mathbb{Q}$. In the case of the mapping class group, the kernel of $\tau$, called the Torelli group, is rather large and somewhat mysterious. Thus we want to consider some additional representations in order to detect elements of the Torelli group. It is convenient to adopt the viewpoint of [GLLM15] that, given a group $\Gamma$, a representation $\sigma : \Gamma_1 \to \mathrm{GL}(V)$ of a finite index subgroup should be treated on the same footing as a representation of $\Gamma$. We will refer to $\sigma$ as a *partial representation* of $\Gamma$, and call $\Gamma_1$ the domain and denote it by $\operatorname{Dom}(\sigma)$. Let us say that two partial representations are *commensurable* if they agree after restriction to a finite index subgroup of the intersection of their domains. We can always induce a partial representation to an honest representation, but it is better for our purposes not to do so. We will mainly be concerned with properties of partial representations which depend only on the commensurability class, so we will occasionally shrink the domains when it is convenient. Given $H \subset \pi_1(F)$ a subgroup of finite index, the stabilizer $\operatorname{Stab}(H) = \{\sigma \in \operatorname{Aut}(\pi_1(F)) \mid \sigma(H) = H\}$, which has finite index in $\operatorname{Aut}(\pi_1(F))$, acts on $H^{\mathrm{ab}} \otimes \mathbb{Q} = H^{\mathrm{ab}}_{\mathbb{Q}}$. Thus this is a partial representation of the automorphism group which we denote by $\tau^H$. If $H$ is characteristic, then $\operatorname{Stab}(H) = \operatorname{Aut}(\pi_1(F))$, so $\tau^H$ is an honest representation. When $H$ is normal, then $G = \pi_1(X)/H$ acts on $H^{\mathrm{ab}}$. We can break the vector space $H^{\mathrm{ab}}_{\mathbb{Q}} = H^{\mathrm{ab}} \otimes \mathbb{Q}$ up into a sum $\bigoplus_\chi (H^{\mathrm{ab}}_{\mathbb{Q}})^\chi$ of isotypic components parameterized by the irreducible $\mathbb{Q}[G]$-modules $\chi$. In more explicit terms, $(H^{\mathrm{ab}}_{\mathbb{Q}})^\chi$ is the sum of all $\mathbb{Q}[G]$-submodules isomorphic to $\chi$. This is a representation of the subgroup $\operatorname{Stab}(r) = \{\alpha \in \operatorname{Aut}(\pi_1(F)) \mid \alpha \circ r = r\} \subseteq \operatorname{Stab}(H)$ where $r : \pi_1(X) \to G$ denotes the projection. The family of partial representations obtained this way will be referred to as generalized Prym representations, and denoted by $\tau^{H,\chi}$. In the case of the mapping class group, the study of these representations (for nontrivial $H$) seems to have been initiated by Looijenga [Loo97], and continued by Koberda [Kob12], and Grunewald, Larsen, Lubotzky, and Malestein [GLLM15]. Koberbda [Kob12] has shown that $\bigoplus \tau^H$, as $H$ runs over characteristic subgroups, is a faithful representation of the based mapping class group. So in particular, the nontriviality of elements of the based Torelli group can be detected using these representations.

Lemma 1.3. — *The representations $\tau^H$ and $\tau^{H,\chi}$ descend to partial representations of $\mathrm{Out}(F)$ after possibly shrinking their domains.*

*Proof*. — We focus on $\tau^H$, the argument for the second case is the same. Let $V = H_{\mathbb{Q}}^{\mathrm{ab}}$ and let $S = \mathrm{Dom}(\tau^H)$. We have an exact sequence

$$\pi_1(F) \longrightarrow \mathrm{Aut}(\pi_1(F)) \longrightarrow O(F) \longrightarrow 1.$$

Since inner automorphism by elements of $H$ act trivially on $V$, the image $\tau^H(\pi_1(F) \cap S)$ is finite. Since the image $\tau^H(S)$ is finitely generated linear, and therefore residually finite, we can choose a finite index subgroup of $\Gamma_1 \subset S$ such that $\tau^H(\Gamma_1 \cap \pi_1(F)) = \{1\}$. $\square$

## 2. Smooth projective families

Now suppose that $f : X \to Y$ is a smooth projective morphism of smooth varieties. We assume furthermore that the fibres of $f$ are connected. By assumption, we have a relatively ample line bundle $\mathscr{L}$ on $X$ with first Chern class $\omega$. By Ehresmann's theorem, $f$ is a $C^\infty$ fibre bundle. Thus we get a homomorphism $\rho : \pi_1(Y, y) \to O(X_y, \omega)$, where $X_y$ is the fibre over $y$. It is worth observing that $\operatorname{im} \rho \subseteq O^+(X_y, \omega)$. *From now on, the representation $\tau^H$ will denote the restriction of the previous $\tau^H$ to* $O^+(X_y, \omega)$. Let $n$ denote the dimension of $X_y$. If $\pi : \widetilde{X}_y \to X_y$ is the finite unramified covering corresponding to a finite index subgroup $H \subset \pi_1(X_y)$, then $X_y$ is projective with an ample class $\widetilde{\omega} = \pi^*\omega$. By the hard Lefschetz theorem, we have a symplectic form

$$(\ ,\ ) : H^1(\widetilde{X}_y, \mathbb{Q}) \times H^1(\widetilde{X}_y, \mathbb{Q}) \longrightarrow H^2(\widetilde{X}_y, \mathbb{Q}) \xrightarrow{\ \cup\, \widetilde{\omega}^{n-2}\ } H^{2n}(\widetilde{X}_y, \mathbb{Q}) \cong \mathbb{Q}.$$

This induces a dual pairing denoted by the same symbol on $H_1(\widetilde{X}_y, \mathbb{Q}) \cong H_{\mathbb{Q}}^{\mathrm{ab}}$. The action of $\mathrm{Stab}(H)$ preserves this, so $\tau^H$ is a representation into the corresponding symplectic group.

In order to analyze the Zariski closures of these representations, we need to recall some basic facts about Mumford-Tate groups; we refer to [Mil94, §1] or [Moo] for a more detailed treatment. Recall that a rational Hodge structure $H$ consists of a finite dimensional $\mathbb{Q}$-vector space and a decomposition $H_{\mathbb{C}} = H \otimes \mathbb{C} = \oplus H^{pq}$ with $H^{qp} = \overline{H}^{pq}$. The bigrading determines and is determined by the homomorphism of $h : \mathbb{C}^* \to \mathrm{GL}(H_{\mathbb{R}})$, given by $h(\lambda)v = \sum \lambda^q \overline{\lambda}^p v^{pq}$. The Mumford-Tate group $\mathrm{MT}(H) \subseteq \mathrm{GL}(H)$ is the smallest $\mathbb{Q}$-algebraic subgroup whose real points contain the image of $\mathbb{C}^*$. For our purposes, it is more convenient to work with a slightly smaller group called the Hodge group or the special Mumford-Tate group $\mathrm{SMT}(H)$ given as the identity component of $\mathrm{MT}(H)\cap\mathrm{SL}(H)$. This is the smallest $\mathbb{Q}$-algebraic subgroup $\mathrm{SMT}(H)\subset\mathrm{GL}(H)$, whose real points contain the image of the unit circle $h(\mathrm{U}(1))$.

Let us say that a real algebraic group $G$ is of Hermitian type if it is connected, reductive and the quotient of it by a maximal compact subgroup $K$ is a Hermitian symmetric space. Say that $G$ is of symplectic Hermitian type if in addition $G/K$ has a totally geodesic holomorphic embedding into a Siegel upper half-plane. By Cartan's classification, a noncompact simple group of Hermitian type is isogenous to $\mathrm{SU}(p,q),\mathrm{SO}(2,p)^o,\mathrm{SO}^*(2p),\mathrm{Sp}(2g,\mathbb{R})$ or certain real forms of $E_6$ or $E_7$ [Hel78, p. 518]. By Satake [Sat65], only the first four are symplectic. A $\mathbb{Q}$-algebraic group $G$ will be called (symplectic) Hermitian if $G(\mathbb{R})$ has these properties.

Theorem 2.1 (Mumford). — *Suppose that $H$ is polarizable of type $\{(-1,0),(0,-1)\}$; in other words, suppose that $H$ is the first homology of an abelian variety. Then $M=\mathrm{SMT}(H)$ is of symplectic Hermitian type.*

*Proof*. — This is stated in Mumford [Mum66, pp. 348-350] without proof, so we give a brief explanation here. The connectedness of $M$ is clear from the definition. The polarizability of $H$ shows that $M$ leaves a positive definite form invariant, and this implies reductivity. The Hermitianness of $M$ can be deduced from [Mil05, Th. 1.21] (the homomorphism $h$ satisfies conditions (a), (b), (c) of that theorem). Furthermore, since $H$ has a polarization $\psi$, we have a homomorphism $\mathrm{U}(1)\to\mathrm{Sp}(H,\psi)$, whence an inclusion $M\subset\mathrm{Sp}(H,\psi)$ satisfying the $(H_2)$ condition of [Sat65]. Therefore, we have an embedding of the symmetric space associated to $M$ into the symmetric space associated to $\mathrm{Sp}(H,\psi)$. □

As noted above, the result puts very strong restrictions on the possible values for $M$. Note that $\mathrm{SMT}(H)=\mathrm{SMT}(H^*)$. So we switch to the dual when it is convenient.

Here is the first main result.

Theorem 2.2. — *Suppose that $X\to Y$ is a smooth projective family with ample class $\omega$ over a smooth quasiprojective base. Let $\rho:\pi_1(Y,y)\to O(X_y,\omega)$ denote the nonabelian monodromy. Then for any finite index normal subgroup $H\subset\pi_1(X_y)$ and character $\chi$ of the quotient, the identity component of the Zariski closure of the image of $\tau^{H,\chi}\circ\rho$ is semisimple of symplectic Hermitian type.*

The proof will rely on the following lemmas.

Lemma 2.3. — *Suppose that we are given a homomorphism of groups $r:\Gamma\to G$ and an action of another group $\Pi$ on $\Gamma$ preserving $r$, i.e., $\mathrm{Stab}(r)=\Pi$. Then $r$ extends to a homomorphism $\widetilde{r}:\Gamma\rtimes\Pi\to G$.*

The proof is immediate from the standard formulas for the semidirect product.

Lemma 2.4. — *With the same notation as in Theorem 2.2, there exist a surjective generically finite morphism of smooth varieties $p:\widetilde{Y}\to Y$ such that $p_*(\pi_1(\widetilde{Y}))\subset\pi_1(Y)$*

*has a finite index and is contained in the domain of* $\tau^{H,\chi}$. *Furthermore* $\tau^{H,\chi}\circ\rho\circ p_*$ *is the monodromy representation of a polarizable variation of Hodge structures on* $\widetilde{Y}$ *of type* $\{(-1,0),(0,-1)\}$.

*Proof*. — Let $\Gamma=\pi_1(X_y)$ and let $r:\Gamma\to G=\Gamma/H$ denote projection. Let $e\in\mathbb{Q}[G]$ be the central idempotent whose image is $A_\chi$. After passing to an étale cover $p:Y_1\to Y$, we can assume $\rho(p_*\pi_1(Y_1))\subseteq \mathrm{Dom}(\tau^{H,\chi})$. The generic fibre of $X\times_Y Y_1\to Y_1$ admits a rational point defined over some finite extension $K$ of the function field $\mathbb{C}(Y_1)$. Let $\widetilde{Y}$ be a desingularization of the normalization of $Y_1$ in $K$. Let $\widetilde{X}=X\times_Y\widetilde{Y}$. Then the map $\widetilde{X}\to\widetilde{Y}$ possesses a section. Therefore $\pi_1(\widetilde{X})$ is the semidirect product $\Gamma\rtimes\pi_1(\widetilde{Y})$. Let $C\subset Y$ be a curve given as a complete intersection of ample divisors in general position. Let $\widetilde{C}\subset\widetilde{Y}$ denote an irreducible component of the preimage of $C$. Let $U$ be the complement of the set of branch points of $\widetilde{C}\to C$, and let $\widetilde{U}\subset\widetilde{C}$ denote the preimage. Consider the diagram

$$\begin{tikzcd} \pi_1(\widetilde{U}) \arrow[r] \arrow[d, "\alpha"'] & \pi_1(\widetilde{Y}) \arrow[d, "p_*"] \\ \pi_1(U) \arrow[r, "\beta"] & \pi_1(Y) \end{tikzcd}$$

By a suitable Lefschetz hyperplane theorem [GM88, p. 153], we obtain a surjection $\pi_1(C)\to\pi_1(Y)$. We have a surjection $\pi_1(U)\to\pi_1(C)$. Combining these two assertions shows that $\beta$ is surjective. Covering space theory shows the image of $\alpha$ has finite index in $\pi_1(U)$. After chasing the diagram the other way, we can conclude $p_*\pi_1(\widetilde{Y})$ has finite index in $\pi_1(Y)$.

Applying Lemma 2.3 yields normal subgroup $T\subset\pi_1(\widetilde{X})$ with $\pi_1(\widetilde{X})/T=G$. Let $p:Z\to\widetilde{X}$ be the corresponding Galois étale cover. Then $\tau^H\circ\rho$ is the monodromy representation of the local system $\bigcup_y H_1(Z_y,\mathbb{Q})$ which can be identified with $R^1(f\circ p)_*\mathbb{Q}^\vee$. The latter clearly underlies a polarized variation of Hodge structure of type $\{(-1,0),(0,-1)\}$. Note that $G$ acts on this by automorphisms. The representation $\tau^{H,\chi}\circ\eta$ corresponds to the sub variation of Hodge structure $e(R^1(f\circ p)_*\mathbb{Q}^\vee)$. $\square$

*Proof of Theorem* 2.2. — Let $Z$ be the identity component of the Zariski closure of the image of $\tau^{H,\chi}\circ\rho$, and let $\mathscr{H}$ be the corresponding variation of Hodge structure. By the theorem of André [And92, §5,Th. 1], $Z$ is a normal subgroup of the derived group $\mathrm{DMT}(\mathscr{H}_x)$ of the Mumford-Tate group of a very general fibre $\mathscr{H}_x$. Observe that $\mathrm{DMT}(\mathscr{H}_x)=\mathrm{DSMT}(\mathscr{H}_x)$ and this is isogenous to $\mathrm{SMT}(\mathscr{H}_x)$ because the last group is semisimple. Hence $\mathrm{DSMT}(\mathscr{H}_X)$ is semisimple of symplectic Hermitian type by Theorem 2.1. Therefore $Z$ also has the same property. $\square$

In the previous set up, given $\gamma\in\pi_1(Y)$ some positive power $\gamma^n$ will lie $\mathrm{Dom}(\tau^H)$. By the same technique, we get further constraints.

Proposition 2.5. — *With the same notation as in Theorem 2.2, suppose that $\gamma\in\pi_1(Y)$ is a loop around a smooth boundary divisor of some smooth compactification. Then $\tau^{H,\chi}\circ\rho(\gamma^n)$ is quasi-unipotent for all $n$ as above.*

*Proof.* — After replacing $n$ by a multiple, we can assume that $\gamma^n\in p_*\pi_1(\widetilde{Y})$, where $p:\widetilde{Y}\to Y$ is as in Lemma 2.4. The result now follows from [Sch73, Lem. 4.5]. $\square$

## 3. Fibered fundamental groups

Let $f:X\to Y$ be a projective map of smooth quasiprojective varieties such that $f$ has connected fibres. Then we have a surjection $\pi_1(f):\pi_1(X)\to\pi_1(Y)$. The kernel of this map may be quite large, however. We first want to factor this through a map with better properties. Given a divisor $D\subset Y$ with simple normal crossings, the restriction $\pi_1(Y-D)\to\pi_1(Y)$ is surjective. The kernel is the normal subgroup generated by loops $\gamma_i$ about the components $D_i$. If $m_i>1$ are integers, define the orbifold fundamental group $\pi_1^{\mathrm{orb}}(Y,\sum m_iD_i)$ as the quotient of $\pi_1(Y-D)$ be the normal subgroup generated by $\gamma_i^{m_i}$. This can be interpreted as the fundamental group of $Y$ with a suitable orbifold structure, but we won't need this. After removing a closed subset $Z\subset Y$ of codimension at least 2, we can suppose that the discriminant of $f$ is a smooth divisor $D=\sum D_i$, and that $f^{-1}D$ is a divisor with normal crossings such that the restriction of $f$ to the intersections of components are submersions over $D$. Let $m_i$ denote the greatest common divisor of the multiplicities of the components of $f^{-1}D_i$. The following is proved in [Ara11, Lem. 3.5].

Proposition 3.1. — *Let $y_0\in Y-D-Z$. Then $\pi_1(f)$ factors through a surjection $\phi:\pi_1(X)\to\pi_1^{\mathrm{orb}}(Y,\sum m_iD_i)$ such that*

$$\begin{tikzcd}
\pi_1(f^{-1}(y_0)) \arrow[r] \arrow[d, "r_1"] & \pi_1(X-f^{-1}(D\cup Z)) \arrow[r, "\psi"] \arrow[d, "r_2"] & \pi_1(Y-D-Z) \arrow[r] \arrow[d, "r_3"] & 1 \\
\ker(\phi) \arrow[r] & \pi_1(X-f^{-1}Z) \arrow[r, "\phi"] & \pi_1^{\mathrm{orb}}(Y,\sum m_iD_i) \arrow[r] & 1
\end{tikzcd}$$

*commutes and has exact rows. The map $r_1:\pi_1(f^{-1}(y_0))\to\ker\phi$ is surjective. In particular, $\ker\phi$ is finitely generated.*

If $f$ is flat then $f^{-1}Z$ has codimension $\geqslant 2$. Consequently $\pi_1(X-f^{-1}Z)\cong\pi_1(X)$ et cetera.

Corollary 3.2. — *Assuming flatness of $f$, $Z$ can be omitted in the statement of the proposition.*

Therefore if $f$ is flat, we have an exact sequence

$$1\longrightarrow\ker(\phi)\longrightarrow\pi_1(X)\longrightarrow\pi_1^{\mathrm{orb}}(Y)\longrightarrow 1$$

with finitely generated kernel, where we write $\pi_1^{\mathrm{orb}}(Y)=\pi_1^{\mathrm{orb}}(Y,\sum m_iD_i)$ for simplicity. This gives an outer action of $\rho:\pi_1^{\mathrm{orb}}(Y)\to\mathrm{Out}(\ker\phi)$. Given a finite index subgroup $H \subset \ker(\phi)$. Let $\sigma^H$ denote the partial representation of $\mathrm{Out}(\ker\phi)$ on $H^{\mathrm{ab}}_{\mathbb{Q}}$ with domain $\mathrm{Stab}(H)$. We write $\sigma = \sigma^H$ when $H = \ker(\phi)$ is the full group.

Proposition 3.3. — *With the assumptions and notation of the previous paragraph, the identity component of the Zariski closure of the image of $\sigma^H \circ \rho$ is semisimple of symplectic Hermitian type.*

*Proof*. — From Proposition 3.1, we deduce a diagram

$$\begin{tikzcd}
1 \arrow[r] & \operatorname{im}\pi_1(X_{y_0}) \arrow[r] \arrow[d, "\overline{r}_1"] & \pi_1(X - f^{-1}D) \arrow[r, "\psi"] \arrow[d, "r_2"] & \pi_1(Y - D) \arrow[r] \arrow[d, "r_3"] & 1 \\
1 \arrow[r] & \ker(\phi) \arrow[r] & \pi_1(X) \arrow[r, "\phi"] & \pi_1^{\mathrm{orb}}(Y) \arrow[r] & 1
\end{tikzcd}$$

where $\overline{r}_1$ is surjective. Corollary 1.2 shows that nonabelian monodromy on $\operatorname{im}\pi_1(X_{y_0})$ coincides with the conjugation action of $\pi_1(Y - D)$ coming from the first row above. Let $K \subset \operatorname{im}\pi_1(X_{y_0})$ be the preimage of $H$ under the map $\overline{r}_1$. Then $K^{\mathrm{ab}}_{\mathbb{Q}}$ surjects onto $H^{\mathrm{ab}}_{\mathbb{Q}}$, and this is compatible with the partial actions of $\pi_1(Y - D)$ and $\pi_1^{\mathrm{orb}}(Y)$ given by conjugation.

Therefore the Zariski closure of the image of $\sigma^H \circ \rho$ is a quotient of the Zariski closure of the image of $\mathrm{Dom}(\tau^K) \subseteq \pi_1(Y - D)$ in $\mathrm{GL}(K^{\mathrm{ab}}_{\mathbb{Q}})$. A quotient of a semisimple group of symplectic Hermitian type is again semisimple of symplectic Hermitian type. As a consequence, the proposition follows from Theorem 2.2. □

For the remainder of this section, we focus on the case where $Y$ is a smooth projective curve of genus $g$. Its fundamental group is given by

$$\Gamma_g = \big\langle \alpha_1, \dots, \alpha_{2g} \mid [\alpha_1, \alpha_2] \cdots [\alpha_{2g-1}, \alpha_{2g}] = 1 \big\rangle.$$

Given integers $m_1, \dots, m_n > 1$, let

$$\begin{aligned}\Gamma_{g;m_1,\dots,m_n} = \Gamma_{g;\vec{m}} = \big\langle \alpha_1, \dots, \alpha_{2g}, \beta_1, \dots, \beta_n \mid \\ [\alpha_1, \alpha_2] \cdots [\alpha_{2g-1}, \alpha_{2g}]\beta_1 \cdots \beta_n = \beta_1^{m_1} = \cdots = \beta_n^{m_n} = 1 \big\rangle.\end{aligned}$$

This is $\pi_1^{\mathrm{orb}}(Y, \sum m_i p_i)$ for some $p_i \in Y$. By [Fox52], there exists a torsion free normal subgroup $\Gamma' \subset \Gamma_{g;\vec{m}}$ of finite index. In more geometric terms, $\Gamma'$ is the ordinary fundamental group of a curve $Y'$, where $r : Y' \to Y$ is a Galois cover with ramification divisor $r^*(\sum m_i p_i)$. Consequently $\Gamma' = \Gamma_h$ for some $h$, and $2h - 2$ is a positive integer multiple of

$$2g - 2 + \sum \frac{m_i - 1}{m_i}.$$

Let us say that $\Gamma_{g;\vec{m}}$ is hyperbolic if the above expression is greater than zero.

Lemma 3.4. — *The following statements hold.*

(a) $\dim H^1(\Gamma_{g;\vec{m}}, \mathbb{Q}) = 2g$.
(b) $H^2(\Gamma_{g;\vec{m}}, \mathbb{Q}) \cong H^2(\Gamma', \mathbb{Q})$ *is one dimensional.*

*Proof*. — The first statement follows immediately from the presentation. Let $G = \Gamma/\Gamma'$. The Hochschild-Serre spectral sequence gives an isomorphism

$$H^2(\Gamma_{g;\vec{m}}, \mathbb{Q}) \cong H^2(\Gamma', \mathbb{Q})^G.$$

The generator of $H^2(\Gamma', \mathbb{Q}) = H^2(Y', \mathbb{Q})$ is invariant under $G$. This proves (b). □

Suppose that $f : X \to Y$ is surjective holomorphic map from a smooth projective variety. Recall that a fibre is a multiple fibre if the greatest common divisor of the multiplicities of the components is greater than 1. Suppose $f$ has $n$ multiple fibres with multiplicity $m_i$. Then from the previous discussion, we obtain a surjective homomorphism $\phi : \pi_1(X) \to \Gamma_{g;\vec{m}}$ with finitely generated kernel.

Theorem 3.5 (Catanese [Cat08, Th. 5.14]). — *Conversely, any surjective homomorphism $\phi : \pi_1(X) \to \Gamma_{g;m_1,\ldots,m_n}$ with finitely generated kernel must arise in the above manner from a holomorphic map $X \to Y$ to a genus $g$ curve with exactly $n$ multiple fibres of multiplicity $m_1, \ldots, m_n$.*

We can now prove the main results announced in the introduction. If $X$ is a smooth projective variety such that $\pi_1(X)$ surjects onto a nonabelian free group, then by [Cat08, Cor. 5.4, prop 5.13] it surjects onto a hyperbolic $\Gamma_{g;\vec{m}}$ with finitely generated kernel. In fact, the hyperbolicity condition is not needed for the results below.

Theorem 3.6. — *Let $X$ be a smooth complex projective variety. Suppose that $\phi : \pi_1(X) \to \Gamma_{g;\vec{m}}$ is a surjective homomorphism such that $\ker\phi$ is finitely generated. For any finite index subgroup $H \subset \ker(\phi)$, the identity component of the Zariski closure of the image of $\sigma^H \circ \rho$ is semisimple symplectic Hermitian, where $\rho : \Gamma_{g;\vec{m}} \to \mathrm{Out}(\ker\phi)$ is the representation associated to the extension.*

*Proof*. — This follows from Proposition 3.3, and Theorem 3.5. □

Corollary 3.7. — *The partial representation $\sigma^H \circ \rho$ is semisimple.*

*Proof*. — The theorem implies that the Zariski closure has a compact real form. So the corollary follows from Weyl's unitary trick. □

Remark 3.8. — The groups $\Gamma_{g,\vec{m}}$ which are not hyperbolic are either finite or abelian. The theorem is vacuous in the finite case. In the abelian case, the theorem implies that it acts through a finite quotient.

Proposition 3.9. — *Let $X$ be a smooth projective variety. Given an exact sequence*

$$1 \longrightarrow K \longrightarrow \pi_1(X) \longrightarrow \Gamma_{g,\vec{m}} \longrightarrow 1$$

*with $K$ finitely generated, $\dim V^{\Gamma_{g;\vec{m}}}$ is even, where $V = K/[K,K] \otimes \mathbb{Q}$.*

*Proof*. — Let $\Gamma = \Gamma_{g,\vec{m}}$. From the Hochschild-Serre spectral sequence, we deduce an exact sequence

$$\begin{aligned}0 \longrightarrow H^1(\Gamma,\mathbb{Q}) \longrightarrow H^1(\pi_1(X),\mathbb{Q}) \longrightarrow H^0(\Gamma, H^1(K,\mathbb{Q}))\\ \longrightarrow H^2(\Gamma,\mathbb{Q}) \longrightarrow H^2(\pi_1(X),\mathbb{Q}).\end{aligned}$$

By Theorem 3.5, the map $\pi_1(X) \to \Gamma$ is realized by a surjective holomorphic map $f : X \to Y$ to a curve. We claim that $H^2(\Gamma,\mathbb{Q}) \to H^2(\pi_1(X),\mathbb{Q})$ is injective. This together with Lemma 3.4, will imply that the Betti number $b_1(X) = 2g + \dim H^0(\Gamma, H^1(K,\mathbb{Q}))$. It would follow that the dimension of coinvariants $\dim V_\Gamma = \dim H^0(\Gamma, H^1(K,\mathbb{Q}))$ is even. Since $\Gamma$ acts semisimply, this is also the dimension of the space of invariants.

To prove the claim, choose $\Gamma' \subset \Gamma$ as above. Then $\Gamma'$ is the (usual) fundamental group of some finite Galois cover $Y' \to Y$. Let $X'$ be a desingularization of $X \times_Y Y'$. Since by Lemma 3.4,

$$H^2(\Gamma,\mathbb{Q}) \cong H^2(\Gamma',\mathbb{Q}) \cong H^2(Y',\mathbb{Q}),$$

it suffices to prove to prove that the composite of

$$H^2(\Gamma',\mathbb{Q}) \longrightarrow H^2(\pi_1(X'),\mathbb{Q}) \longrightarrow H^2(X',\mathbb{Q})$$

is injective. But this is clear because the image contains the fundamental class of the fibre of $X' \to Y'$. □

Remark 3.10. — The same proof shows that $\dim V_{\Gamma_{g;\vec{m}}}$ is even, when $X$ is compact Kähler.

## 4. Étale covers of general curves

Let $\Gamma_g$ be the fundamental group of a genus $g$ curve. Fix a finite index normal subgroup $H \subset \Gamma_g$ with quotient $G$. This determines a Galois étale cover $\widetilde{C} \to C$ of any curve of genus $g$. Our goal is to compute the special Mumford-Tate group of $H^1(\widetilde{C})$ when $C$ is a *very general curve*. This means that $C$ occurs in the complement of a countable union of proper subvarieties of the moduli space of curves $M_g(\mathbb{C})$. Since $G$ acts on $H^1(\widetilde{C})$, we can decompose it as sum of isotypic components $\bigoplus_\chi H^1(\widetilde{C})^\chi$, as $\chi$ runs over (isomorphism classes of) irreducible $\mathbb{Q}[G]$-modules. It suffices to compute $\mathrm{SMT}(H^1(\widetilde{C})^\chi)$.

In order to state the main result, we need to recall some terminology from [GLLM15]. The space $V = H^1(\widetilde{C},\mathbb{Q})$ carries a $\mathbb{Q}[G]$-valued pairing given by

$$\langle u, v\rangle = \sum_{g\in G} (u, gv)g,$$

where $(\ ,\ )$ is the usual intersection pairing on $V$. This is sesquilinear and skew-Hermitian with respect to the involution $g^* = g^{-1}$, i.e., $\langle gu, hv\rangle = g\langle u,v\rangle h^*$ and $\langle u,v\rangle = -\langle v,u\rangle^*$ [GLLM15, Lem. 3.1]. We have

$$\operatorname{im}\tau^H \subseteq \operatorname{Aut}(V, \langle\ ,\ \rangle).$$

We can break up the group on the right into simpler pieces. Let $\chi$ be the class of an irreducible $\mathbb{Q}[G]$-module in the Grothendieck group. The $\chi$-isotypic submodule of $\mathbb{Q}[G]$ is a subalgebra $A_\chi$ which is a matrix algebra over a division ring $D_\chi$. The algebra $A_\chi$ is stable under $*$ [GLLM15, Lem. 3.2]. Let $L_\chi$ denote the center of $A_\chi$, and let $K_\chi$ the fixed field for the involution. The isotypic component $V^\chi = H_{\mathbb{Q}}^{\mathrm{ab},\chi}$ becomes a $A_\chi$-submodule which is also stable under the action of $\mathrm{Stab}(r)$. The restriction of the pairing $\langle\,,\rangle$ to $V_\chi$ is $A_\chi$-valued [GLLM15, Lem. 3.3]. The group $\mathrm{Aut}(V_\chi, \langle\,,\rangle)$ is naturally an algebraic group over $K_\chi$, but we wish to regard it as an algebraic groups over $\mathbb{Q}$. More formally, we apply Weil restriction $\mathbb{G}_{H,\chi} = \mathrm{Res}_{K_\chi/\mathbb{Q}} \mathrm{Aut}(V_\chi, \langle\,,\rangle)$. We will need to consider the subgroup $\mathbb{G}^1_{H,\chi} \subset \mathbb{G}_{H,\chi}$ of elements with reduced norm equal to 1. The associated complex group $\mathbb{G}_{H,\chi}(\mathbb{C})$ is symplectic, orthogonal or general linear according to whether $A_\chi \otimes_{K_\chi} \mathbb{R}$ becomes a matrix algebra over $\mathbb{R}, \mathbb{C}$ or the quaternions. Proofs of this and more can be found in [GLLM15]. Let $\tau^{H,\chi}$ denote the representation corresponding to $V^\chi$. The image of this is in $\mathbb{G}_{H,\chi}(\mathbb{Q})$. We can decompose $\tau^H = \sum_\chi \tau^{H,\chi}$. We say that the quotient map $r : \Gamma_g \to G$ is redundant if it factors as

$$\Gamma_g \xrightarrow{r'} F_g \xrightarrow{r''} G,$$

where $F_g$ is a free group on $g$ generators, $r'$ is a surjection and, $r''$ contains a free generator in its kernel. Clearly, there is a redundant homomorphism onto $G$ if and only if it is generated by fewer than $g$ elements.

Theorem 4.1. — *Let $g \geqslant 3$. Suppose that $r : \Gamma_g \to G$ is a redundant surjective homomorphism. Let $C$ be a very general curve of genus $g$ and let $\widetilde{C}$ be the corresponding étale $G$-cover. For each irreducible $\mathbb{Q}[G]$-module $\chi$, the special Mumford-Tate group of the isotoypic component* $\mathrm{SMT}(H^1(\widetilde{C})^\chi) = \mathbb{G}^1_{H,\chi}$.

The main ingredient is the following theorem of Grunewald, Larsen, Lubotzky, and Malestein [GLLM15, Th. 1.6].

Theorem 4.2. — *Suppose that $g \geqslant 3$ and $r$ is redundant. Then for any irreducible $\mathbb{Q}[G]$-module $\chi$,* $\operatorname{im} \tau^{H,\chi}$ *is an arithmetic subgroup of* $\mathbb{G}^1_{H,\chi}$.

Lemma 4.3. — *Let $H, G, r, \chi$ be as above, but with $r$ not necessarily redundant. Let $C$ be a smooth projective curve of genus $g$ and $\widetilde{C}$ be the corresponding unramified $G$-cover. The $\chi$-isotypic component of $H_1(\widetilde{C}, \mathbb{Q}) = H^1(\widetilde{C}, \mathbb{Q})^\vee$ is a sub Hodge structure with special Mumford-Tate group contained in $\mathbb{G}^1_{H,\chi}$.*

*Proof*. — Since $G$ acts holomorphically on $\widetilde{C}$, it preserves the canonically polarized Hodge structure $H^1(\widetilde{C})$. Therefore the $\chi$-isotypic component $M \subset H_1(\widetilde{C}, \mathbb{Q})$ is a polarized sub Hodge structure. Let $K = K_\chi$, $A = A_\chi$, $\mathscr{G} = \mathrm{Aut}(M, \langle\,,\rangle)$ and $\mathscr{G}^1 \subset \mathscr{G}$ subgroup of elements with reduced norm 1. Then $\mathbb{G}^1_{H,\chi} = \mathrm{Res}_{K/\mathbb{Q}} \mathscr{G}^1$. To prove the lemma, it suffices to show that the image of $h : \mathrm{U}(1) \to \mathrm{GL}(M_{\mathbb{R}})$ lies $\mathrm{Res}_{K/\mathbb{Q}} \mathscr{G}^1(\mathbb{R}) = \mathscr{G}^1(K \otimes \mathbb{R})$. Since as noted $G$ acts by automorphisms of the polarized Hodge structure, it follows that the image of $h : \mathrm{U}(1) \to \mathrm{GL}(M_{\mathbb{R}})$ lies in

$\mathrm{Aut}_G(M_{\mathbb{R}}, \langle , \rangle) = \mathrm{Aut}_A(M_{\mathbb{R}}, \langle\, , \rangle)$. This almost does it but it remains to check the reduced norm condition. After extending scalars to $\mathbb{C}$, we see that the reduced norm of $h(\lambda)$ equals

$$\prod \lambda^{(q-p)\dim M^{pq}} = 1. \qquad \square$$

*Proof of Theorem* 4.1. — Let $Y = M_g[n]$ be the moduli space of curves of genus $g$ with level $n \geqslant 3$ structure [MF82]. This is a fine moduli space, so it carries a universal curve $X \to Y$. We can identify $\pi_1(Y)$ with congruence subgroup $\ker[\mathrm{Mod}_g \to \mathrm{Sp}_{2g}(\mathbb{Z}/n\mathbb{Z})]$ of the mapping class group $\mathrm{Mod}_g$. Lemma 2.4 gives a surjective map $p : \widetilde{Y} \to Y$ such that $\pi_1(\widetilde{Y})$ has finite index in $\pi_1(Y)$ and such that $\tau^{H,\chi} \circ p_*$ comes from a polarizable variation of Hodge structure $\mathscr{H}$. Lemma 4.3 gives an inclusion $\mathrm{SMT}(\mathscr{H}_y) \subseteq \mathbb{G}^1(\mathbb{Q})$ for any $y \in \widetilde{Y}$. For very general $y \in \widetilde{Y}$, the identity component $Z$ of the Zariski closure of the monodromy group of $\mathscr{H}$ lies in $\mathrm{SMT}(\mathscr{H}_y)$ [Mil05, Th. 6.19]. By Theorem 4.2, we have $Z = \mathbb{G}^1(\mathbb{Q})$. Thus we obtain the reverse inclusion $\mathbb{G}^1(\mathbb{Q}) \subseteq \mathrm{SMT}(\mathscr{H}_y)$ for very general $y$. $\square$

When $G = \mathbb{Z}/2\mathbb{Z}$, we recover the main result of [BP02] that the special Mumford-Tate group of a very general Prym variety is the full symplectic group.

We record the following corollary of the proof for later use.

Corollary 4.4. — *Assume $g \geqslant 3$ and that $H \subset \Gamma_g$ is a finite index normal subgroup such that $\Gamma_g/H$ is generated by fewer than $g$ elements. Then the identity component of the Zariski closure of* $\mathrm{im}\,\tau^H$ *is* $\prod_\chi \mathbb{G}^1_{H,\chi}$.

## 5. Examples

5.1. Families of abelian varieties. — Theorems 2.2 and 3.6 give strong restrictions on the representations $\tau \circ \rho$ and $\sigma \circ \rho$ associated to fundamental groups of varieties. We now want to understand which representations can actually arise in this way. To simplify the statement of the proposition below, let us say that a group $\Gamma$ occurs in Theorem 2.2 (respectively Theorem 3.6) if it is isomorphic to the image of $\tau \circ \rho$ (respectively $\sigma \circ \rho$) for an example satisfying the conditions of the theorem.

Proposition 5.1. — *Let $\Gamma_1$ be an arithmetic subgroup of a special Mumford-Tate group $G$ of a polarized Hodge structure of type $\{(-1, 0), (0, -1)\}$. Then all but finitely many finite index subgroups $\Gamma \subset \Gamma_1$ occur in Theorem* 2.2. *If none of the irreducible factors of the symmetric space $G(\mathbb{R})^0/K$ are the* 1*-ball,* 2*-ball, or genus* 2 *Siegel upper half-plane, then almost all finite index subgroup $\Gamma \subset \Gamma_1$ occur in Theorem* 3.6. *Certain lattices in* $\mathrm{SL}_2(\mathbb{R})$ *occur in Theorem* 3.6.

Before we explain the examples, we briefly recall the notion of a Shimura variety of Hodge type following the original viewpoint of Mumford [Mum66, Mum69]. Let $G$ be the special Mumford-Tate group of an abelian variety, and $\Gamma \subset G$ an arithmetic subgroup. Then, roughly speaking, the Shimura variety $S$ parameterizes the set of all abelian varieties having Mumford-Tate group finer than or equal to $G$ such that $\Gamma$ fixes the lattice. Here "finer" should be understood as "is contained in". To be precise, let us fix a polarized Hodge structure $L = H_{\mathbb{Z}}$ of type $\{(-1,0),(0,-1)\}$ together with a polarization $\psi$. Let $G = \mathrm{SMT}(H)$ along with the homomorphisms $h : \mathrm{U}(1) \to G(\mathbb{R})$ and $\rho : G \to \mathrm{Sp}(H_{\mathbb{Q}}, \psi)$ that come with it. Given this, the subgroup $K = \{g \in G(\mathbb{R}) \mid \rho(g)h = h\rho(g)\}$ is a maximal compact subgroup of $G(\mathbb{R})$ such that $\widetilde{S} = G(\mathbb{R})/K$ is a Hermitian symmetric domain which embeds into $\mathbb{H}_q = \mathrm{Sp}(H_{\mathbb{R}}, \psi)/\{\text{centralizer of } h\}$. The space $\mathbb{H}_q$ can be identified with the Siegel upper half-plane of genus $q = \dim H/2$. Thus $\widetilde{S}$ carries a holomorphic family of abelian varieties $\mathscr{U} \to \widetilde{S}$ given by pulling back the universal family from $\mathbb{H}_q$. Given an arithmetic subgroup $\Gamma_1 \subset G(\mathbb{Q})$, we can choose a finite index torsion free subgroup $\Gamma \subset \Gamma_1$ stabilizing $L$. Let $S = \Gamma\backslash\widetilde{S}$. Then we can construct a family of Abelian varieties $\Gamma\backslash\mathscr{U} \to S$. We note that $S$ is quasiprojective, with a minimal projective compactification $\overline{S}$ constructed by Baily-Borel [BB66]. The following is almost immediate from the construction.

Lemma 5.2. — *The given action of $\Gamma$ on $L$ is the monodromy of $\Gamma\backslash\mathscr{U} \to S$ on the first homology of the fibre. The fundamental group $\pi_1(\Gamma\backslash\mathscr{U}) = L \rtimes \Gamma$.*

As observed in [Tol90], if the boundary of $\overline{S}$ has codimension at least three, we can apply a suitable weak Lefschetz theorem [GM88, p. 153] to show that there is a smooth projective surface $Z \subset S$ with $\pi_1(Z) \cong \pi_1(S)$, and a smooth projective curve $C \subset S$ with $\pi_1(C) \twoheadrightarrow \pi_1(S)$. By restricting $\Gamma\backslash\mathscr{U}$ to these varieties, and using standard facts about the structure of the boundary [Tol90, Rem. 2], [WK65, Th. 4.13], we deduce:

Lemma 5.3. — *With the notation as above, and suppose that none of the irreducible factors of $\widetilde{S}$ are the 1-ball, 2-ball, or genus 2 Siegel upper half-plane, then $L \rtimes \Gamma \in \mathscr{P}$. Furthermore, there exists a fibered group $L \rtimes \Gamma_g \in \mathscr{P}$, where $\Gamma_g$ acts on $L$ via a surjective homomorphism $\Gamma_g \twoheadrightarrow \Gamma$. (For the last statement, it is only necessary to exclude the 1-ball.)*

To justify the last assertion of Proposition 5.1, we use Shimura curves associated to quaternion algebras [Shi67]. Let $D$ be an indefinite quaternion division algebra over $\mathbb{Q}$. Concretely, $D$ is a $\mathbb{Q}$ algebra with generators $i, j, k$ and relations $i^2 = a$, $j^2 = -b$ and $k = ij = -ji$ for rational numbers $a, b > 0$. Either $D$ splits which means that $D = M_2(\mathbb{Q})$ or $D$ is a division algebra. We want the latter to hold, and for this it is sufficient to assume that the projective conic $ax^2 - by^2 + z^2 = 0$ has no rational points. After extending scalars, we have an isomorphism of algebras $\psi : D \otimes \mathbb{R} \cong M_2(\mathbb{R})$. We have an involution on $D$ given by conjugation $\overline{x + yi + zj + wk} = x - yi - zj - wk$. Let $G = \{\alpha \in D \mid \alpha\overline{\alpha} = 1\}$ be viewed as an algebraic group over $\mathbb{Q}$. Under $\psi$ we have $G(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{R})$. Fix a maximal order $O \subset D$. For $\tau$ in the upper half-plane $\mathbb{H}$, $A_\tau = \mathbb{C}^2/\psi(O)\left(\begin{smallmatrix}\tau\\1\end{smallmatrix}\right)$ is an abelian variety. For general $\tau$, its special Mumford-Tate group is precisely $G$. If $\Gamma_1 = G(\mathbb{Q}) \cap O$, Shimura proves that the corresponding Shimura variety $S = \mathbb{H}_1/\Gamma_1$ is compact. (It is the moduli space of abelian surfaces $A_\tau$ having multiplication by $O$.) Therefore

Lemma 5.4. — *If $\Gamma \subset \Gamma_1$ is a torsion free subgroup of finite index, $O \rtimes \Gamma \in \mathscr{P}$.*

5.2. Kodaira surfaces. — In the previous examples, we considered only the "top" representations $\tau$ and $\sigma$ because $\tau^H, \sigma^H$ would give essentially nothing new. By contrast, let us a consider Kodaira surface. This is a smooth projective surface admitting an everywhere smooth map to a curve $f : X \to C$, such that the fibres are connected with nonconstant moduli. Let $g$ and $q$ denote the genus of $C$ and the fibres respectively. The fundamental group is a nontrivial extension

$$1 \longrightarrow \Gamma_q \longrightarrow \pi_1(X) \longrightarrow \Gamma_g \longrightarrow 1. \tag{5.1}$$

We have an associated homomorphism $\rho : \Gamma_g \to \mathrm{Mod}_q \subset O(\Gamma_q)$ into the genus $q$ mapping class group. We note that this is the sole invariant, in the sense that the extension (5.1) is uniquely determined by $\rho$. To see this, observe that by a theorem of Eilenberg and Maclane [EM47], the possible extensions with outer action $\rho$ are parameterized by $H^2(\Gamma_g, Z(\Gamma_q))$, but the centre $Z(\Gamma_q)$ is trivial [FM12, Chap. 1]. Our main theorems give strong restrictions on the possible values of $\operatorname{im} \tau^H \circ \rho$. In particular, an arbitrary $\rho$ will not come from a Kodaira surface (or any other projective manifold). Our interest now is in seeing how big these can be. By assumption the map $C \to M_q$ to the moduli space of curves is nonconstant. Therefore by Torelli's theorem, it follows that the induced map $C \to A_q$ is also nonconstant. This forces the variation of Hodge structure $R^1 f_* \mathbb{Z}$ to be nontrivial. In fact, it must have infinite monodromy. If the monodromy were finite, then we could assume, after a finite base change, that it is trivial implying that $R^1 f_* \mathbb{Z}$ is a trivial variation of Hodge structure by [Sch73, Th. 7.24]. We can apply Theorem 2.2 to strengthen the conclusion. Therefore we have proved that:

Lemma 5.5. — *For any Kodaira surface, the representation $\tau \circ \rho : \Gamma_g \to \mathrm{Sp}_{2q}(\mathbb{Q})$ has infinite image. In particular, $\operatorname{im} \rho$ is infinite. Furthermore, the identity component of the Zariski closure of $\operatorname{im} \tau \circ \rho$ is a nontrivial semisimple group of symplectic Hermitian type.*

In order to say more, we need a further assumption. Let us say that a Kodaira surface is *generic* if the image of $\rho$ has finite index in the mapping class group. Although the original examples constructed by Kodaira [Kod67] are not generic, it is easy to see that generic Kodaira surfaces exist. Let $M_q[n]$ be the fine moduli space of genus $q$ curve with level $n \geqslant 3$ structure, and let $M_q[n]^*$ denote the Satake compactification. Note that the boundary $M_q[n]^* - M_q[n]$ has codimension at least 2 in $M_q[n]^*$ provided that $q > 2$. Therefore a curve $C \subset M_q[n]^*$ given as an intersection of general ample divisors would lie entirely in $M_q[n]$. Let $X \to C$ be the pull back of the universal family. Then the map on fundamental groups $\pi_1(C) \to \pi_1(M_q[n])$ would be surjective by weak Lefschetz [GM88, p. 153]. But $\pi_1(M_q[n])$ is a finite index subgroup of the mapping class group. From Corollary 4.4, we see that:

Lemma 5.6. — *Let $X \to C$ be generic with $q \geqslant 3$ and suppose that $H \subset \Gamma_q$ is a finite index normal subgroup such that $\Gamma_q/H$ is generated by fewer than $q$ elements. Then the image $\operatorname{im}\tau^H \circ \rho$ contains a Zariski dense subgroup of $\prod_\chi \mathbb{G}^1_{H,\chi}$.*

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