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QUANTUM ROOTS FOR KAC–MOODY ROOT SYSTEMS AND FINITENESS PROPERTIES OF THE KAC–MOODY AFFINIZED BRUHAT ORDER

BY AUGUSTE HÉBERT & PAUL PHILIPPE

ABSTRACT. — The notion of quantum root comes up in the study of the quantum cohomology of generalized flag varieties, quantum Bruhat graphs or in the study of the affinized Bruhat order on an affinized Weyl semi-group. In this paper, we provide a complete classification of quantum roots associated to general Kac–Moody root systems. In particular we prove that the set of quantum roots of a given root system is always finite. We then deduce several finiteness properties of the affinized Bruhat order, generalizing results of A. Welch. In particular, we prove that its intervals are finite. This affinized Bruhat order was introduced by A. Braverman, D. Kazhdan and M. Patnaik in their study of Kac–Moody groups over discretely valued fields, and of the associated Iwahori–Hecke algebras.

RÉSUMÉ (Racines quantiques pour les systèmes de racines de Kac–Moody et propriétés de finitude de l'ordre de Bruhat affine de Kac–Moody)

Au sein d'un système de racines, les racines quantiques jouent un rôle particulier. Dans le cadre réductif elles apparaissent notamment dans l'étude de la cohomologie quantique des variétés de drapeaux, et la structure combinatoire des groupes de Weyl affines. Nous classifions les racines quantiques des systèmes de racines de Kac–Moody de type général, nous montrons en particulier que chaque système de racines n'en contient qu'un nombre fini. Nous en déduisons plusieurs propriétés de finitude pour l'ordre de Bruhat affinisé sur le semi-groupe de Weyl affinisé, généralisant des résultats de A. Welch. Nous montrons notamment que les intervalles sont finis. Cet ordre de Bruhat affinisé avait été introduit par A. Braverman, D. Kazhdan et M. Patnaik pour étudier les groupes de Kac–Moody sur les corps discrètement valués, et les algèbres de Iwahori–Hecke associées.

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1. INTRODUCTION

Kac–Moody root systems are particular arrangements of vectors in a free $\mathbb{Z}$ -lattice, whose structure is encoded in a generalized Cartan matrix; they appear in various contexts in combinatorics and algebra. For instance, finite Kac–Moody root systems arise from the theory of reductive Lie algebras and reductive groups (such as $\mathfrak{sl}_n$ and SL_n),

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and the infinite Kac–Moody root systems allow one to define their infinite-dimensional counterparts, namely Kac–Moody Lie algebras and Kac–Moody groups. Certain roots within a given root system, called quantum roots, are of particular interest. Without being named as such, quantum roots of reductive root systems first came up in the computation of the quantum cohomology of flag manifolds and related varieties ([BFP99, BMO11]). They are intrinsically connected to the combinatorics of affine Weyl groups (this is already apparent in Peterson’s lectures on quantum cohomology [Pet97, Lect. 12]) and they are essential to the definition of quantum Bruhat graphs (introduced by Postnikov [Pos04]). The term “quantum root” seems to first appear in the work of Lenart et al. [LNS⁺15], in which they apply the quantum Bruhat graph to the analysis of certain Kirillov–Reshetikhin crystals. More recently, in his doctoral thesis, Schremmer [Sch22] established a wide range of further applications of quantum Bruhat graphs associated to reductive root systems.

The notion of quantum root is also related to the notion of minuscule element defined by D. Peterson ([Ste01]); see Proposition 2.6. The latter concept has applications in various areas, for example in the theory of Kac–Moody homogeneous spaces (see [CP12]) or in the study of Schubert varieties (see [BL00, Ch. 9]), to cite but a few.

In this paper, we study quantum roots associated with (infinite) Kac–Moody (infinite) root systems. We provide a complete classification of these roots, and, in particular, we prove that the set of quantum roots is always finite. As observed by the second author in a previous work [Phi24, Prop. 3.19], describing these roots is important for understanding the affinized Bruhat order on the Kac–Moody affine Weyl semi-group. This is a combinatorial object which naturally replaces the extended affine Weyl group when looking at non-reductive Kac–Moody groups over discretely valued fields. In the second part of this paper, we deduce several finiteness properties of the affinized Weyl semi-group, which we use in a follow-up work to define and compute affine Kazhdan–Lusztig R -polynomials for Kac–Moody groups [HP26]. Independently, Dean has considered quantum Bruhat graphs associated to Kac–Moody root systems [Dea25], mainly motivated by the definition of a Demazure product on affinized Weyl semi-groups. This relies on properties of the quantum Bruhat graph which remain conjectural in the non-reductive setting, but have been proven by Dean in affine type A_1 affine. This type can be worked out very explicitly, especially because only the two simple roots are quantum (this is a particular case of Lemma 2.42). We expect a good understanding of quantum roots to be necessary to extend these results to the general Kac–Moody setting.

Let us now introduce more precisely the main objects of study and the results presented in this paper.

Quantum roots. — Let $\Phi, \Phi^\vee$ be a (free and cofree) Kac–Moody real root system and its dual. Let $(\alpha_i)_{i \in I}$ and $(\alpha_i^\vee)_{i \in I}$ be its simple roots and coroots, where I is a finite index set. There is a natural bijection $\Phi \rightarrow \Phi^\vee : \alpha \mapsto \alpha^\vee$ which sends α_i to $\alpha_i^\vee$. Each coroot $\alpha^\vee$ decomposes uniquely as $\alpha^\vee = \varepsilon \sum_{i \in I} n_i \alpha_i^\vee$ with $\varepsilon \in \{1, -1\}$ and $(n_i)_{i \in I} \in \mathbb{N}^I$. The height of $\alpha^\vee$ is then defined as $\text{ht}(\alpha^\vee) := \varepsilon \sum_{i \in I} n_i$. Intuitively,

the height measures the size of a given (co)root; for instance, a root is simple if and only if its coroot has height 1. Another quantity associated with such a system can be defined via its vectorial Weyl group W^v . This object is a Coxeter group whose set of simple reflections is in bijection with $(\alpha_i)_{i \in I}$. It acts on Φ (and dually on $\Phi^\vee$) and we have $\Phi = W^v \cdot \{\alpha_i\}_{i \in I}$. To any root α is associated a unique reflection s_α , whose length $\ell(s_\alpha)$ also measures the size of α .

In general, we have that

$$(1.1) \quad 2\text{ht}(\alpha^\vee) \geq \ell(s_\alpha) + 1$$

by Lemma 2.8. We say that a root is quantum when equality holds.

While there is a concise characterization of quantum roots in reductive root systems (see [LNS⁺15, Lem. 4.2]), the situation seems very different in the non-reductive case. For example, every positive root is quantum in a reductive simply laced root system, but this property fails even for the simplest infinite root systems. Nonetheless, we provide a complete classification of quantum roots using sequences of Dynkin diagrams.

As a corollary, we obtain:

THEOREM 1.1 (see Theorem 2.30). — *Any Kac–Moody root system admits finitely many quantum roots.*

From Theorem 1.1 we deduce finiteness properties for the affinized Bruhat order. In the following paragraphs we introduce this affinized Bruhat order and explain its connection with quantum roots.

Affine Weyl semi-group W^+ . — In their study of Kac–Moody groups over non-Archimedean local fields, A. Braverman, D. Kazhdan and M. Patnaik consider a semi-group W^+ , called the affinized Weyl semi-group. It is defined as follows. When Φ is the root system of a Kac–Moody group $\mathbb{G}$ (or of a Kac–Moody Lie algebra $\mathfrak{g}$), $\Phi^\vee$ lies in the cocharacter lattice Y and we have $\Phi \subset \text{Hom}_{\mathbb{Z}}(Y, \mathbb{Z})$. The action of W^v extends to Y . The cocharacter lattice admits a W^v -stable subcone Y^+ , called its integral Tits cone, which plays a role in the representation theory of the associated Kac–Moody Lie algebra. The affinized Weyl semi-group is then defined as $W^+ := Y^+ \rtimes W^v$.

When Φ is associated to a reductive group (and hence is finite), then $Y^+ = Y$ and W^+ is the extended affine Weyl group associated to $\mathbb{G}$. It is a finite extension of a Coxeter group, which is very useful for studying $\mathbb{G}$ over non-Archimedean local fields. In the general Kac–Moody case, we expect W^+ to play an analogous role but its structure is less well understood. It is not a group, so it does not admit any Coxeter structure, but a few key notions of Coxeter theory have been extended to W^+ . One of these notions is the Bruhat-like partial pre-order $<$ defined by Braverman, Kazhdan and Patnaik in [BKP16, App. B2]. In [Mut18], D. Muthiah proves that this relation is a partial order, and in [MO19] Muthiah and D. Orr define a strictly compatible $\mathbb{Q}$ -valued length function $\ell^a : W^+ \rightarrow \mathbb{Q}$. We call this partial order the affinized Bruhat order.

Covers for the affinized Bruhat order and quantum roots. — The affinized Bruhat length gives a characterization of covers: Muthiah and Orr (in the affine ADE case) and the second-named author (in the general case) have proved that $\mathbf{y}$ covers $\mathbf{x}$ (see Definition 3.1) if and only if $\mathbf{y} > \mathbf{x}$ and $\ell^a(\mathbf{y}) = \ell^a(\mathbf{x}) + 1$ ([MO19], [Phi24]). The notion of quantum roots naturally comes up in a more explicit description of covers for the affinized Bruhat order. This is sketched in Lemma 3.2 (see [Phi24, Prop. 3.20] for a complete statement). Therefore our classification of quantum roots also provides insight into the covers for the affinized Bruhat order. In particular, using Theorem 1.1, we prove:

THEOREM 1.2 (see Theorem 3.3 and Theorem 3.7). — *Let $\mathbf{x} \in W^+$. Then $\mathbf{x}$ admits finitely many covers. If furthermore $\mathbf{x}$ is spherical, i.e., the fixator in W^v of the associated cocharacter is finite, then $\mathbf{x}$ also admits finitely many co-covers.*

COROLLARY 1.3. — *Let $\mathbf{x}, \mathbf{y} \in W^+$. Then the interval $[\mathbf{x}, \mathbf{y}]$ is finite.*

In the reductive case, these theorems are easily obtained by considering reduced decompositions. In the Kac–Moody framework, these theorems were first proved by A. Welch in the untwisted affine ADE case, using different methods in [Wel22].

When W^v admits an infinite proper parabolic subgroup (this implies that $\mathbb{G}$ is not affine), we also prove the existence of elements of W^+ admitting infinitely many co-covers, see Lemma 3.9.

Our main motivation for the study of the combinatorics of W^+ is the construction of an affine Kazhdan–Lusztig theory for Kac–Moody groups. This theory was initiated by Muthiah in [Mut19], where he uses Corollary 1.3 to give a conjectural definition of spherical R -polynomials in the affine ADE case. We use our results to define an Iwahori version of Muthiah’s polynomials in the general Kac–Moody framework (see [HP26]).

Organization of the paper. — In Section 2, we define the quantum roots and we prove that the set of quantum roots of a Kac–Moody datum is finite (see Theorem 2.30). We also obtain a general classification of quantum roots (see Theorem 2.39). We then describe the set of quantum roots in particular cases.

In Section 3, we explain the link between covers for the affinized Bruhat order and quantum roots. We then prove Theorem 1.2. We also show that the spherical condition in Theorem 1.2 is necessary by constructing non-spherical elements with infinitely many co-covers (see Proposition 3.12). We end the section with an explicit construction of covers associated to any given quantum root, thus showing that the notion of quantum root is well-suited. Note that this section uses the finiteness of the set of quantum roots (Theorem 1.1) but is largely independent of Section 2: for the reader willing to admit this theorem, the only prerequisites for this section are contained in Section 2.1.

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2. QUANTUM ROOTS

2.1. GENERAL DEFINITIONS AND NOTATIONS. — We let $\mathbb{N} \subset \mathbb{Z}$ denote the set of non-negative integers. For $a, b \in \mathbb{Z}$, we denote by $\llbracket a, b \rrbracket$ the integer interval $[a, b] \cap \mathbb{Z}$.

Generalized Cartan matrices and Dynkin diagrams

DEFINITION 2.1. — Let I be a finite set. A generalized Cartan matrix over I is a matrix $A = (a_{i,j})_{(i,j) \in I \times I} \in M_I(\mathbb{Z})$ with integral coefficients satisfying the following properties:

- (1) $\forall i \in I \ a_{i,i} = 2$
- (2) $\forall (i,j) \in I^2, i \neq j \implies a_{i,j} \leq 0$
- (3) $\forall (i,j) \in I^2, a_{i,j} = 0 \implies a_{j,i} = 0$.

DEFINITION 2.2. — A Dynkin diagram is a tuple (I, E, w) where I is a finite set (its vertices), E is a symmetric subset of $I \times I \setminus \{(i,i) \mid i \in I\}$ (its edges) and w is a map $E \rightarrow \mathbb{Z}_{>0}$ (its weight function). In other words, it is a positively weighted directed graph such that $(i,j) \in E \iff (j,i) \in E$.

Let $\Gamma = (I, E, w)$ be a Dynkin diagram. Then any subset $J \subset I$ admits an induced Dynkin diagram structure: $(J, E \cap (J \times J), w|_J)$, where $w|_J$ is the restriction of w to $E \cap (J \times J)$. By a subdiagram (or subgraph) of Γ , we mean any Dynkin diagram of this form.

A leaf in a Dynkin diagram is any vertex which lies in at most one unoriented edge. That is to say, it is any $i \in I$ for which there is at most one $j \in I$ such that $(i,j) \in E$.

Dynkin diagrams are graphical representations of generalized Cartan matrices: suppose that $A = (a_{i,j})_{(i,j) \in I \times I}$ is a generalized Cartan matrix indexed by I . Then, setting $E = \{(i,j) \in I^2 \mid a_{i,j} < 0\}$ and $w : (i,j) \mapsto -a_{i,j}$, we obtain a Dynkin diagram (I, E, w) . Conversely, to any Dynkin diagram (I, E, w) we associate a generalized Cartan matrix $(a_{i,j})_{(i,j) \in I \times I}$ indexed by I , setting

$$a_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ -w(i,j) & \text{if } (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

DEFINITION 2.3 (Connectivity and 1-star-convexity). — A Dynkin diagram (I, E, w) is said to be connected if it is connected as a graph. That is to say, if, for any $i, j \in I$, there is a sequence $i = i_0, i_1, \dots, i_n = j$ such that $(i_k, i_{k+1}) \in E$ for all $k \in \llbracket 0, n-1 \rrbracket$.

For a general Dynkin diagram Γ , a connected component of Γ is any connected sub-diagram which is maximal for the inclusion. Therefore a Dynkin diagram is connected if and only if it has exactly one connected component.

A connected Dynkin diagram is a Dynkin tree if it does not contain any closed circuit: for any sequence $i_0, \dots, i_n$ such that $(i_k, i_{k+1}) \in E$ for all $k \in \llbracket 0, n-1 \rrbracket$ and $i_{k-1} \neq i_{k+1}$ for all $k \in \llbracket 1, n-1 \rrbracket$, we have $i_0 \neq i_n$. A Dynkin tree is a Dynkin segment if there are at most two edges going out of any given vertex.

We introduce a refinement of the notion of connectivity: we say that a Dynkin diagram $\Gamma = (I, E, w)$ is 1-star-convex if there exists a vertex $i_0 \in I$ such that, for any $j \in I$, there is a sequence $(i_1, \dots, i_n)$ with $i_n = j$ such that, for all $k \in \llbracket 0, n-1 \rrbracket$,

$$(2.1) \quad (i_k, i_{k+1}) \in E \quad \text{and} \quad w(i_k, i_{k+1}) = 1.$$

In this case, we say that Γ is 1-star-convex at i_0 .

By symmetry of E , a 1-star-convex Dynkin diagram is always connected, but the converse is not true.

Kac–Moody root systems. — Let $\mathcal{D} = (A, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$ be a Kac–Moody root datum in the sense of [Rém02, §8]. It is a quintuple such that:

- I is a finite indexing set and $A = (a_{ij})_{(i,j) \in I \times I}$ is a generalized Cartan matrix.
- X and Y are two dual free $\mathbb{Z}$ -modules of finite rank, we write $\langle, \rangle$ for the duality bracket.
- $(\alpha_i)_{i \in I}$ (resp. $(\alpha_i^\vee)_{i \in I}$) is a family of linearly independent elements of X (resp. Y), the simple roots (resp. simple coroots).
- For all $(i, j) \in I^2$ we have $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$.

Let (I, E, w) be the Dynkin diagram corresponding to A , with vertices I , which is fixed once and for all. In this article, every Dynkin diagram we will consider will be a subdiagram of (I, E, w) . Since they are identified with subsets of I , any $J \subseteq I$ now denotes either the subset J of I , or the corresponding subdiagram of (I, E, w) .

Height function. — We define a height function on Y as follows: let $\rho \in X \otimes \mathbb{Q}$ be any element such that $\langle \alpha_i^\vee, \rho \rangle = 1$ for all $i \in I$, then, for any $\lambda \in Y$, the *height* of λ is: $\text{ht}(\lambda) = \langle \lambda, \rho \rangle$. This definition depends on the choice of ρ , but its restriction to $Q^\vee = \bigoplus_{i \in I} \mathbb{Z} \alpha_i^\vee$ does not: $\text{ht}(\sum_{i \in I} n_i \alpha_i^\vee) = \sum_{i \in I} n_i$.

Vectorial Weyl group. — For every $i \in I$, set $r_i \in \text{Aut}_{\mathbb{Z}}(X) : x \mapsto x - \langle \alpha_i^\vee, x \rangle \alpha_i$. The generated group $W^v = \langle r_i \mid i \in I \rangle$ is the *vectorial Weyl group* of the Kac–Moody root datum. The duality bracket $\langle Y, X \rangle$ induces a contragredient action of W^v on Y , explicitly $r_i(y) = y - \langle y, \alpha_i \rangle \alpha_i^\vee$. By construction the duality bracket is then W^v -invariant.

The group W^v is a Coxeter group; in particular, it has a Bruhat order $<$ and a length function ℓ compatible with the Bruhat order.

Real roots. — Let $\Phi = W^v \cdot \{\alpha_i \mid i \in I\}$ be the set of real roots of $\mathcal{D}$; it is a, possibly infinite, root system (see [Kum02, 1.2.2 Def.]). In particular, let $\Phi_+ = \Phi \cap \bigoplus_{i \in I} \mathbb{N}\alpha_i$ be the set of positive real roots, then $\Phi = \Phi_+ \sqcup -\Phi_+$. We write $\Phi_- = -\Phi_+$ for the set of negative roots.

The set $\Phi^\vee = W^v \cdot \{\alpha_i^\vee \mid i \in I\}$ is the set of *coroots*, and its subset $\Phi_+^\vee = \Phi^\vee \cap \bigoplus_{i \in I} \mathbb{N}\alpha_i^\vee$ is the set of *positive coroots*.

To each root β corresponds a unique coroot $\beta^\vee$: if $\beta = w(\alpha_i)$ then $\beta^\vee = w(\alpha_i^\vee)$. This map $\beta \mapsto \beta^\vee$ is well defined, is a bijection between Φ and $\Phi^\vee$, and sends positive roots to positive coroots. Note that $\langle \beta^\vee, \beta \rangle = 2$ for all $\beta \in \Phi$.

Moreover, to each root β one associates a reflection $s_\beta \in W^v$: if $\beta = w(\pm\alpha_i)$ then $s_\beta := wr_iw^{-1}$. It is well-defined, independently of the choices of w and i . Explicitly, it is the map $x \mapsto x - \langle \beta^\vee, x \rangle \beta$. We have $s_\beta = s_{-\beta}$ and the map $\beta \mapsto s_\beta$ is a bijection between the set of positive roots and the set $\{wr_iw^{-1} \mid (w, i) \in W^v \times I\}$ of reflections of W^v .

Dynkin sequence associated to a positive root. — Let $\beta \in \Phi_+$. Since $(\alpha_i^\vee)_{i \in I}$ are linearly independent, there is a unique decomposition $\beta^\vee = \sum_{i \in I} N_i(\beta)\alpha_i^\vee$, and for all $i \in I$, $N_i(\beta) \in \mathbb{Z}_{\geq 0}$. For any $n \in \mathbb{Z}_{\geq 0}$, let us define

$$(2.2) \quad I_n(\beta) = \{i \in I \mid N_i(\beta) \geq n\}.$$

We consider $I_n(\beta)$ as a subdiagram of Γ . We call $(I_n(\beta))_{n \geq 1}$ the *Dynkin sequence* of β . The Dynkin sequence associated to a root is, by definition, non-increasing (for the inclusion) and finitely supported (in the sense that $I_n = \emptyset$ for n large enough). Note that the Dynkin sequence of a simple root α_i is given by $I_1(\alpha_i) = \{i\}$ and $I_n(\alpha_i) = \emptyset$ for $n \geq 2$. We will provide a classification of quantum roots using Dynkin sequences.

Note that the Dynkin sequence $(I_n(\beta))_{n \geq 1}$ fully determines β . Indeed, we have $\beta^\vee = \sum_{n=1}^{+\infty} \sum_{i \in I_n(\beta)} \alpha_i^\vee$, which proves that $\beta^\vee$, and thus β , is determined by its Dynkin sequence.

Inversion sets. — For any $w \in W^v$, let

$$\text{Inv}(w) = \Phi_+ \cap w^{-1} \cdot \Phi_- = \{\alpha \in \Phi_+ \mid w(\alpha) \in \Phi_-\}.$$

These sets are closely related to the Bruhat order since, by [Kum02, 1.3.13],

$$\forall \alpha \in \Phi_+, \alpha \in \text{Inv}(w) \iff ws_\alpha < w \iff s_\alpha w^{-1} < w^{-1}.$$

Moreover, they are related to the length ℓ on W : $\ell(w) = |\text{Inv}(w)|$ ([Kum02, 1.3.14]).

2.2. QUANTUM ROOTS: DEFINITION, MOTIVATION AND NOTATIONS

DEFINITION 2.4. — Let $\beta \in \Phi_+$. We say that β is *quantum* if for every $\gamma \in \text{Inv}(s_\beta) \setminus \{\beta\}$, we have $\langle \beta^\vee, \gamma \rangle = 1$. We denote by $\mathcal{Q}(\Phi_+)$ the set of quantum roots of Φ_+ .

Note that $\text{Inv}(r_i) = \{\alpha_i\}$ for every $i \in I$, and thus every simple root is quantum.

REMARK 2.5. — This definition is not the one we presented in the introduction. By Lemma 2.9 below, the two definitions are equivalent.

Note that the notion of quantum root is related to the notion of minuscule element attributed to Peterson (see [Ste01, 2]). J. Stembridge assumes the generalized Cartan matrix to be symmetrizable, but this assumption is not needed for this definition to make sense, so we drop it here.

PROPOSITION 2.6

- (1) Let $\beta \in \Phi_+$. Write $\beta = w(\alpha_i)$, with $i \in I$ and $w \in W^\vee$ of minimal possible length. Then $\{\langle \beta^\vee, \gamma \rangle \mid \gamma \in \text{Inv}(s_\beta) \setminus \{\beta\}\} = \{-\langle \alpha_i^\vee, \gamma \rangle \mid \gamma \in \text{Inv}(w)\}$.
- (2) Let $\beta \in \Phi_+$. Write $\beta = w(\alpha_i)$, with $i \in I$ and $w \in W^\vee$ of minimal possible length. Then $\beta \in \mathcal{Q}(\Phi_+)$ if and only if w is $\alpha_i^\vee$ -minuscule in the sense of [Ste01, (2.1)].

Proof

- (1) Write $w = r_{i_n} \dots r_{i_1}$, with $n = \ell(w)$ and $i_1, \dots, i_n \in I$. Then by [Kum02, 1.3.14 Lem.], we have:

$$\text{Inv}(s_\beta) = \{\alpha_{i_n}, r_{i_n}(\alpha_{i_{n-1}}), \dots, r_{i_n} \dots r_{i_2}(\alpha_{i_1}), w(\alpha_i), wr_i(\alpha_{i_1}), \dots, wr_i r_{i_1} \dots r_{i_{n-1}}(\alpha_i)\}.$$

Set

$$E_1 = \{r_{i_n} \dots r_k(\alpha_{i_{k-1}}) \mid k \in \llbracket 2, n+1 \rrbracket\}$$

and

$$E_2 = \{wr_i r_{i_1} \dots r_{i_{k-1}}(\alpha_{i_{k-1}}) \mid k \in \llbracket 0, n-1 \rrbracket\}.$$

Then $\text{Inv}(s_\beta) = E_1 \sqcup \{\beta\} \sqcup E_2$.

Let $\gamma \in E_1$. Write $\gamma = r_{i_n} \dots r_{i_k}(\alpha_{i_{k-1}})$, where $k \in \llbracket 2, n+1 \rrbracket$. Then

$$\begin{aligned} \langle \beta^\vee, \gamma \rangle &= \langle w(\alpha_i^\vee), \gamma \rangle \\ &= \langle \alpha_i^\vee, w^{-1} r_{i_n} \dots r_{i_k}(\alpha_{i_{k-1}}) \rangle \\ &= \langle \alpha_i^\vee, r_{i_1} \dots r_{i_{k-1}}(\alpha_{i_{k-1}}) \rangle \\ &= -\langle \alpha_i^\vee, r_{i_1} \dots r_{i_{k-2}}(\alpha_{i_{k-1}}) \rangle. \end{aligned}$$

Therefore $\{\langle \beta^\vee, \gamma \rangle \mid \gamma \in E_1\} = \{-\langle \alpha_i^\vee, \gamma \rangle \mid \gamma \in \text{Inv}(w)\}$.

Let γ be in E_2 . Write $\gamma = wr_i r_{i_1} \dots r_{i_{k-1}}(\alpha_{i_{k-1}})$, with $k \in \llbracket 0, n-1 \rrbracket$. Then

$$\langle \beta^\vee, \gamma \rangle = \langle \alpha_i^\vee, r_i r_{i_1} \dots r_{i_{k-1}}(\alpha_{i_{k-1}}) \rangle = -\langle \alpha_i^\vee, r_{i_1} \dots r_{i_{k-1}}(\alpha_{i_{k-1}}) \rangle.$$

Therefore

$$\{\langle \beta^\vee, \gamma \rangle \mid \gamma \in E_2\} = \{-\langle \alpha_i^\vee, \gamma \rangle \mid \gamma \in \text{Inv}(w)\}.$$

This proves (1).

- (2) If $k \in \llbracket 0, n-1 \rrbracket$, then $\langle \alpha_i^\vee, r_{i_1} \dots r_{i_{k-1}}(\alpha_{i_{k-1}}) \rangle = \langle r_{i_k} \dots r_{i_1}(\alpha_i^\vee), \alpha_{i_{k+1}} \rangle$ and thus Definition 2.4 for $w(\alpha_i)$ is equivalent to [Ste01, (2.1)] for w and $\alpha_i^\vee$. $\square$

REMARK 2.7. — In [Pro99b] and [Pro99a], R. Proctor classifies λ -minuscule elements, for $\lambda \in Y$ dominant (i.e., $\langle \lambda, \alpha_i \rangle \geq 0$, for all $i \in I$), in the simply laced case. The case we study here is somewhat different: in general simple coroots are not dominant, and furthermore, when the generalized Cartan matrix is indecomposable and not of finite type, they do not lie in the Tits cone. It would be interesting to have a theory unifying these two situations.

The aim of this section is to study quantum roots and in particular to prove that for every Kac–Moody datum, the number of quantum roots is finite (see Theorem 2.30). This result is a key step in proving our finiteness results on covers and co-covers (see Theorems 3.3 and 3.7). We then classify the quantum roots. We end up this section with examples: we describe the set of quantum roots for particular choices of generalized Cartan matrices.

Let us describe the main steps of the proofs of Theorem 2.30 and Theorem 2.39. Let $\beta \in \mathcal{Q}(\Phi_+)$. We write

$$(2.3) \quad \beta = r_{i_L} \dots r_{i_2}(\alpha_{i_1}),$$

with $i_1, \dots, i_L \in I$ and L as small as possible.

(1) We begin by proving that, up to renumbering, we can assume that, if $L_1 = |I_1(\beta)|$, then $\{i_1, \dots, i_{L_1}\} = I_1(\beta)$ (see Lemma 2.12).

(2) We prove that $I_1(\beta)$ is a 1-star-convex tree (see Proposition 2.14).

(3) We prove that the “orders of appearance” in (2.3) are highly constrained. If $i, j \in I_2(\beta)$ belong to the same connected component and if the second appearance of i (when we read the right-hand side of (2.3) from right to left) precedes the second appearance of j , then for $n \in \mathbb{Z}_{\geq 3}$,

$$j \in I_n(\beta) \implies i \in I_n(\beta),$$

and the n -th appearance of i precedes the n -th appearance of j (see Lemma 2.18).

(4) We prove that the connected components of $I_2(\beta)$ are as follows. Let $\mathcal{C}$ be a connected component of $I_2(\beta)$ and j be the element of $\mathcal{C}$ whose second appearance in (2.3) is the earliest. Then $\mathcal{C}$ is the union of at most three segments based at j (see Lemma 2.15 and Proposition 2.16).

(5) Using the results described above, we obtain restrictions on the form of the Dynkin sequence $(I_n(\beta))$. We deduce that if $k = \max\{n \in \mathbb{Z} \mid I_n(\beta) \neq \emptyset\}$, then $k \leq \max(6, k' + 1)$, where k' is the maximal length of a segment appearing in point (4) (see Proposition 2.29). As k' is bounded by the size of I , we deduce an upper bound on the height of $\beta^\vee$ and hence that $\mathcal{Q}(\Phi_+)$ is finite.

(6) Refining the results of (5), we describe all the possible forms of the Dynkin sequences. We then construct, for each of these Dynkin sequences, a quantum root to which it is associated (see Theorem 2.39).

Let $\beta \in \Phi_+$. We write $\beta = w(\alpha_{i_1})$, with $i_1 \in I$ and $w \in W^v$ and we assume that $L := \ell(w) + 1$ is as small as possible. We fix a reduced decomposition $\underline{w} = (r_{i_L}, \dots, r_{i_2})$ of w , with $i_L, \dots, i_2 \in I$.

By [Kum02, 1.3.14 Lem.] we have

$$(2.4) \quad \text{Inv}(s_\beta) = \left\{ \alpha_{i_L}, r_{i_L}(\alpha_{i_{L-1}}), \dots, r_{i_L} \dots r_{i_2}(\alpha_{i_1}), r_{i_L} \dots r_{i_1}(\alpha_{i_2}), \dots, r_{i_L} \dots r_{i_1} \dots r_{i_{L-1}}(\alpha_{i_L}) \right\}.$$

Set

$$\gamma_1^\vee = \alpha_{i_1}^\vee, \gamma_2^\vee = r_{i_2}(\alpha_{i_1}^\vee), \dots, \gamma_L^\vee = r_{i_L} \dots r_{i_2}(\alpha_{i_1}^\vee) = \beta^\vee.$$

LEMMA 2.8. — *Let $\beta \in \Phi_+$. Then $\ell(s_\beta) \leq 2\text{ht}(\beta^\vee) - 1$.*

Proof. — By [Phi24, Cor. 1.10], we have

$$2\text{ht}(\beta^\vee) = 2 + \sum_{\gamma \in \text{Inv}(s_\beta) \setminus \{\beta\}} \langle \beta^\vee, \gamma \rangle$$

and all the terms on the right-hand side are positive integers. Therefore $2\text{ht}(\beta^\vee) - 2 \geq |\text{Inv}(s_\beta)| - 1 = \ell(s_\beta) - 1$. $\square$

LEMMA 2.9. — Let $\beta \in \Phi_+$. Using the same notation as above, set $\beta_1 = r_{i_L} \dots r_{i_3}(\alpha_{i_2})$, $\beta_2 = r_{i_L} \dots r_{i_4}(\alpha_{i_3}), \dots, \beta_{L-1} = \alpha_{i_L}$. Then the following conditions are equivalent:

- (1) β is quantum,
- (2) for every $t \in \llbracket 1, L-1 \rrbracket$, $\langle \beta^\vee, \beta_t \rangle = 1$,
- (3) $\forall t \in \llbracket 1, L \rrbracket$, $\gamma_t^\vee = \sum_{k=1}^t \alpha_{i_k}^\vee$ and $\langle \gamma_{t-1}^\vee, \alpha_{i_t} \rangle = -1$, if $t \neq 1$,
- (4) $\ell(s_\beta) = 2\text{ht}(\beta^\vee) - 1$.

Proof. — Let $t \in \llbracket 2, L \rrbracket$. We have:

$$\begin{aligned} \langle \beta^\vee, \beta_{t-1} \rangle &= \langle r_{i_L} \dots r_{i_2}(\alpha_{i_1}^\vee), r_{i_L} \dots r_{i_{t+1}}(\alpha_{i_t}) \rangle \\ &= \langle r_{i_t} \dots r_{i_2}(\alpha_{i_1}^\vee), \alpha_{i_t} \rangle = -\langle r_{i_{t-1}} \dots r_{i_2}(\alpha_{i_1}^\vee), \alpha_{i_t} \rangle. \end{aligned}$$

We thus have:

$$\begin{aligned} \langle r_{i_L} \dots r_{i_2}(\alpha_{i_1}^\vee), r_{i_L} \dots r_{i_1} \dots r_{i_{t-1}}(\alpha_{i_t}) \rangle &= \langle \alpha_{i_1}^\vee, r_{i_1} \dots r_{i_{t-1}}(\alpha_{i_t}) \rangle \\ &= -\langle r_{i_{t-1}} \dots r_{i_2}(\alpha_{i_1}^\vee), \alpha_{i_t} \rangle = \langle \beta^\vee, \beta_{t-1} \rangle. \end{aligned}$$

Therefore,

$$\{\langle \beta^\vee, \gamma \rangle \mid \gamma \in \text{Inv}(s_\beta) \setminus \{\beta\}\} = \{\langle \beta^\vee, \beta_t \rangle \mid t \in \llbracket 2, L \rrbracket\},$$

which proves the equivalence between (1) and (2).

Let $t \in \llbracket 1, L-1 \rrbracket$. We have:

$$\begin{aligned} \langle \beta^\vee, \beta_t \rangle &= \langle r_{i_L} \dots r_{i_2}(\alpha_{i_1}^\vee), r_{i_L} \dots r_{i_{t+2}}(\alpha_{i_{t+1}}) \rangle \\ &= \langle r_{i_{t+1}} \dots r_{i_2}(\alpha_{i_1}^\vee), \alpha_{i_{t+1}} \rangle = -\langle \gamma_t^\vee, \alpha_{i_{t+1}} \rangle. \end{aligned}$$

Therefore,

$$\gamma_{t+1}^\vee = r_{i_{t+1}}(\gamma_t^\vee) = \gamma_t^\vee - \langle \gamma_t^\vee, \alpha_{i_{t+1}} \rangle \alpha_{i_{t+1}}^\vee = \gamma_t^\vee + \langle \beta^\vee, \beta_t \rangle \alpha_{i_{t+1}}^\vee.$$

The equivalence between (1) and (3) follows.

Let $\beta \in \Phi_+$ satisfying (3). We have $\gamma_L^\vee = \beta^\vee$ and $\text{ht}(\gamma_L^\vee) = L$. As $\ell(s_\beta) = 2L - 1$, we deduce that (3) implies (4). Let $\beta^\vee \in \Phi_+^\vee$. Assume that $\beta^\vee$ satisfies (4). Then by [Phi24, Cor. 1.10], we have $2\text{ht}(\beta^\vee) = \sum_{\gamma \in \text{Inv}(s_\beta)} \langle \beta^\vee, \gamma \rangle = 2 + \sum_{\gamma \in \text{Inv}(s_\beta) \setminus \{\beta^\vee\}} \langle \beta^\vee, \gamma \rangle = \ell(s_\beta) + 1$ and all the terms of this sum are positive. Therefore $\langle \beta^\vee, \gamma \rangle = 1$ for all $\gamma \in \text{Inv}(s_\beta) \setminus \{\beta^\vee\}$, which proves that β is quantum. $\square$

REMARK 2.10. — By Lemma 2.9, for all $t \in \llbracket 1, L \rrbracket$, $\gamma_t = r_{i_t} \dots r_{i_2}(\alpha_{i_1})$ is quantum and $\text{ht}(\gamma_t^\vee) = t$.

The following proposition shows that the set $\mathcal{Q}(\Phi_+)$ can be constructed recursively. This will be used extensively in Section 2.6 in order to prove Theorem 2.39.

PROPOSITION 2.11

- (1) If $\beta \in \mathcal{Q}(\Phi_+)$ is a quantum root which is not simple, then there exists $i \in I$ such that $r_i(\beta) \in \mathcal{Q}(\Phi_+)$ and $\text{ht}(r_i(\beta^\vee)) = \text{ht}(\beta^\vee) - 1$.
- (2) Let $\beta \in \mathcal{Q}(\Phi_+)$ and $i \in I$. Then $r_i(\beta) \in \mathcal{Q}(\Phi_+)$ if and only if $|\langle \beta^\vee, \alpha_i \rangle| \leq 1$.

Proof

(1) We simply take $i = i_L$ with the notation of Formula (2.4); then, by Remark 2.10, $r_i(\beta) = \gamma_{L-1}$ is a quantum root.

(2) Let $\beta \in \mathcal{Q}(\Phi_+)$ and $i \in I$. Let $\tilde{\beta} = r_i(\beta)$. We have $s_{\tilde{\beta}} = r_i s_\beta r_i$ and thus:

$$(2.5) \quad \ell(s_{\tilde{\beta}}) \in \{\ell(s_\beta) - 2, \ell(s_\beta), \ell(s_\beta) + 2\}.$$

We have $\text{ht}(\tilde{\beta}^\vee) = \text{ht}(\beta^\vee) - \langle \beta^\vee, \alpha_i \rangle$. If $r_i(\beta) \in \mathcal{Q}(\Phi_+)$, then by Lemma 2.9, we have $\ell(s_{\tilde{\beta}}) = 2\text{ht}(\tilde{\beta}^\vee) - 1 = 2\text{ht}(\beta^\vee) - 1 - 2\langle \beta^\vee, \alpha_i \rangle = \ell(s_\beta) - 2\langle \beta^\vee, \alpha_i \rangle$. Using (2.5), we deduce that if $r_i(\beta) \in \mathcal{Q}(\Phi_+)$, then $|\langle \beta^\vee, \alpha_i \rangle| \leq 1$.

Now let $\beta \in \mathcal{Q}(\Phi_+)$ and $i \in I$. If $\langle \beta^\vee, \alpha_i \rangle = 0$, then $r_i(\beta) = \beta \in \mathcal{Q}(\Phi_+)$. Assume now $\langle \beta^\vee, \alpha_i \rangle = 1$. Set $\tilde{\beta} = r_i(\beta)$. We have $\text{ht}(\tilde{\beta}^\vee) = \text{ht}(\beta^\vee) - 1$ and by Lemma 2.8,

$$\ell(s_{\tilde{\beta}}) \leq 2\text{ht}(\tilde{\beta}) - 1 = 2\text{ht}(\beta^\vee) - 3 = \ell(s_\beta) - 2.$$

By (2.5) we deduce $\ell(s_{\tilde{\beta}}) = \ell(s_\beta) - 2 = 2\text{ht}(\tilde{\beta}) - 1$, which proves that $\tilde{\beta} = r_i(\beta)$ is quantum by Lemma 2.9.

Finally, assume that $\langle \beta^\vee, \alpha_i \rangle = -1$. Let $\tilde{\beta} := r_i(\beta)$. Note that the associated coroot is $\tilde{\beta}^\vee = r_i(\beta^\vee) = \beta^\vee + \alpha_i^\vee$. Therefore $\text{ht}(\tilde{\beta}^\vee) = \text{ht}(\beta^\vee) + 1$ and, by Lemma 2.9, $\tilde{\beta}$ is quantum if and only if

$$\ell(s_{\tilde{\beta}}) = \ell(r_i s_\beta r_i) = \ell(s_\beta) + 2.$$

Since r_i is a simple reflection, $\{\ell(r_i s_\beta), \ell(s_\beta r_i)\} \subset \{\ell(s_\beta) + 1, \ell(s_\beta) - 1\}$. Moreover since s_β is a reflection, $\ell(r_i s_\beta) = \ell((r_i s_\beta)^{-1}) = \ell(s_\beta r_i)$.

We have $\langle \beta^\vee, \alpha_i \rangle < 0$, so $s_\beta(\alpha_i) \in \Phi_+$. Hence $\alpha_i \notin \text{Inv}(s_\beta)$, $s_\beta r_i > s_\beta$ and thus $\ell(s_\beta r_i) = \ell(r_i s_\beta) = \ell(s_\beta) + 1$.

Note that $s_\beta(\alpha_i) = \alpha_i - \langle \beta^\vee, \alpha_i \rangle \beta = \beta + \alpha_i$. Moreover since $\langle \beta^\vee, \alpha_i \rangle < 0$, by [Bar96, Lem. 1.1.10] we also have $\langle \alpha_i^\vee, \beta \rangle < 0$, that is to say $\langle \alpha_i^\vee, \beta \rangle + 1 \leq 0$ (because it is an integer). Therefore,

$$r_i s_\beta(\alpha_i) = r_i(\beta + \alpha_i) = r_i(\beta) - \alpha_i = \beta - (\langle \alpha_i^\vee, \beta \rangle + 1)\alpha_i \in \Phi_+.$$

So $\alpha_i \notin \text{Inv}(r_i s_\beta)$ and $\ell(r_i s_\beta r_i) = \ell(r_i s_\beta) + 1 = \ell(s_\beta) + 2$, which proves the proposition. $\square$

From now on we assume that β is quantum. For $i \in I$ and $t \in \llbracket 1, L \rrbracket$, set $N_i(t) = |\{t' \in \llbracket 1, t \rrbracket \mid i_{t'} = i\}|$. Then, by Lemma 2.9(3),

$$(2.6) \quad \gamma_t^\vee = \sum_{i \in I_1} N_i(t) \alpha_i^\vee.$$

Note that, by Lemma 2.9(3), N_i is non-decreasing for every $i \in I_1(\beta)$, and for every $t \in \llbracket 1, L-1 \rrbracket$, $N_i(t) = N_i(t+1)$ for all but one element k of $I_1(\beta)$, and we have $N_k(t+1) = N_k(t) + 1$.

Note that the Dynkin sequence of β can be defined by

$$I_n(\beta) := \{i \in I \mid \max N_i \geq n\},$$

with the induced diagram structure from I . The set $I_n(\beta)$ is also the set of elements which appear at least n times in $(\underline{w}, r_{i_1})$.

In this section, we have fixed a quantum root β ; therefore, to simplify notation, we set

$$(2.7) \quad I_n := I_n(\beta).$$

For $j \in I$ and $n \in \mathbb{Z}_{\geq 0}$, we set

$$(2.8) \quad \mathbf{t}(j, n) = \min\{t \in \llbracket 1, L \rrbracket \mid N_j(t) = n\} - 1 \in \llbracket 0, L - 1 \rrbracket \cup \{\infty\}.$$

We then have $N_j(\mathbf{t}(j, n)) = n - 1$ and $N_j(\mathbf{t}(j, n) + 1) = n$, if $\mathbf{t}(j, n) \neq \infty$.

LEMMA 2.12. — *Let $L_1 = |I_1|$. Then up to changing the reduced decomposition of w , we may assume $I_1 = \{i_1, \dots, i_{L_1}\}$.*

Proof. — Assume $\{i_1, \dots, i_{L_1}\} \neq I_1$. Then $(i_j)_{j \in \llbracket 1, L_1 \rrbracket}$ is not injective. Let $k \in \llbracket 1, L_1 \rrbracket$ be minimal such that $i_k \in \{i_1, \dots, i_{k-1}\}$. Let $k' \in \llbracket k + 1, L \rrbracket$ be minimal such that $i_{k'} \notin \{i_1, \dots, i_{k'-1}\}$. Then for every $k'' \in \llbracket k, k' - 1 \rrbracket$, $N_{i_{k''}}(k') = N_{i_{k''}}(k' - 1) \geq 2$ and $N_{i_{k'}}(k' - 1) = 0 = N_{i_{k'}}(k') - 1$. By assumption on β , we have $\langle r_{i_{k'-1}} \dots r_{i_2}(\alpha_{i_1}^\vee), \alpha_{i_{k'}} \rangle = -1$. Using (2.6), we deduce $\sum_{i \in I_1} N_i(k' - 1) a_{i, i_{k'}} = -1$. As $N_{i_{k'}}(k' - 1) = 0$, all the terms appearing on the left-hand side are non-positive integers. Therefore exactly one is nonzero, and it is equal to -1 . Therefore $a_{i_{k''}, i_{k'}} = 0$ for every $k'' \in \llbracket k, k' - 1 \rrbracket$. In other words, $r_{i_{k''}}$ commutes with $r_{i_{k'}}$ for every $k'' \in \llbracket k, k' - 1 \rrbracket$. Therefore we can replace $r_{i_{k'}} r_{i_{k'-1}} \dots r_{i_k} r_{i_{k-1}} \dots r_{i_2}$ by $r_{i_{k'-1}} \dots r_{i_k} r_{i_{k'}} r_{i_{k-1}} \dots r_{i_2}$ in the reduced decomposition of w . Repeating this process as many times as necessary, we end up with a reduced decomposition $\underline{w}$ such that $(i_j)_{j \in \llbracket 1, L_1 \rrbracket}$ is injective and then $\{i_1, \dots, i_{L_1}\} = I_1$. $\square$

From now on, we assume that the reduced decomposition $\underline{w}$ is chosen so that $\{i_1, \dots, i_{L_1}\} = I_1$, which is possible by Lemma 2.12.

2.3. GENERAL STRUCTURE OF $I_1(\beta)$. — In this section, we study the Dynkin diagram structure of $I_1(\beta)$ when β is a quantum root. We have fixed a quantum root β and we have set $I_1 = I_1(\beta)$.

For $i \in I$, set

$$(2.9) \quad \deg(i) = - \sum_{j \in I_1 \setminus \{i\}} a_{j, i} \in \mathbb{Z}_{\geq 0}$$

and

$$\text{Supp}(i) = \{j \in I_1 \mid a_{j, i} \neq 0\}.$$

LEMMA 2.13. — *The graph I_1 is a tree: it is connected and has no circuit.*

Proof. — By [Kac90, Lem. 1.6], I_1 is connected (I_1 is the support of β for the terminology of [Kac90]). Assume, for a contradiction, that I_1 contains a circuit. Let $n \in \mathbb{Z}_{>0}$ and $\tau : \llbracket 0, n \rrbracket \rightarrow I_1$ be such that $\tau(0) = \tau(n)$ and $\{\tau(i), \tau(i+1)\}$ is an edge of I_1 for all $i \in \llbracket 0, n-1 \rrbracket$. We assume moreover that $\tau(i) \neq \tau(0)$ for $i \in \llbracket 1, n-1 \rrbracket$. Then every element of $\tau(\llbracket 0, n \rrbracket)$ has at least two neighbors in I_1 . Let $i \in \llbracket 0, n-1 \rrbracket$ be such that $\mathbf{t}(\tau(i), 1) = t := \max\{\mathbf{t}(\tau(j), 1) \mid j \in \llbracket 0, n-1 \rrbracket\}$. Then there exist $i_1, i_2 \in I_1 \setminus \{\tau(i)\}$ such that $a_{i_1, \tau(i)}, a_{i_2, \tau(i)} \neq 0$ and $\gamma_t^\vee = \alpha_{i_1}^\vee + \alpha_{i_2}^\vee + x$, for some $x \in \bigoplus_{j \in I_1 \setminus \{\tau(i)\}} \mathbb{N}\alpha_j^\vee$. Then $\alpha_{\tau(i)}(\gamma_t^\vee) \leq \alpha_{\tau(i)}(\alpha_{i_1}^\vee + \alpha_{i_2}^\vee) \leq -2 < -1$: a contradiction. The lemma follows. $\square$

A first step towards determining $\mathcal{Q}(\Phi_+)$ is to understand what the support of quantum roots can be. That is to say, to determine $\{I_1(\beta) \mid \beta \in \mathcal{Q}(\Phi_+)\}$. Proposition 2.14 below characterizes this set.

PROPOSITION 2.14. — *Let T be a subset of I . Then the following properties are equivalent:*

- (1) *As a subdiagram of I , T is a 1-star-convex Dynkin tree.*
- (2) *There exists $\beta \in \mathcal{Q}(\Phi_+)$ such that $T = I_1(\beta)$.*
- (3) *There exists $\beta \in \mathcal{Q}(\Phi_+)$ such that $T = I_1(\beta)$ and $I_2(\beta) = \emptyset$. More precisely, β is the unique element of Φ such that $\beta^\vee = \sum_{i \in T} \alpha_i^\vee$.*

Proof

(2) $\Rightarrow$ (1) Let $\beta \in \mathcal{Q}(\Phi_+)$ and $T = I_1(\beta)$. By Lemma 2.12 and Remark 2.10, $\beta'^\vee := \sum_{i \in T} \alpha_i^\vee$ is a quantum root. Therefore, up to replacing β by β' , we can assume $\beta^\vee = \sum_{i \in T} \alpha_i^\vee$. Write $\beta = r_{i_L} \dots r_{i_2}(\alpha_{i_1})$, where $L = |T|$ and $\{i_L, \dots, i_1\} = T$. Let us prove that T is 1-star-convex at i_1 . Let $t \in \llbracket 1, L-1 \rrbracket$ and assume that, as a subdiagram of T , $\{i_1, \dots, i_t\}$ is 1-star-convex at i_1 . Then we have $r_{i_t} \dots r_{i_2}(\alpha_{i_1}^\vee) = \sum_{s=1}^t \alpha_{i_s}^\vee$ and $r_{i_{t+1}} \cdot (r_{i_t} \dots r_{i_2}(\alpha_{i_1}^\vee)) = (r_{i_t} \dots r_{i_2}(\alpha_{i_1}^\vee)) + \alpha_{i_{t+1}}^\vee$, by Lemma 2.9. Therefore $\langle \sum_{s=1}^t \alpha_{i_s}^\vee, \alpha_{i_{t+1}}^\vee \rangle = \sum_{s=1}^t \langle \alpha_{i_s}^\vee, \alpha_{i_{t+1}}^\vee \rangle = -1$. As this is a sum of non-positive integers, exactly one term is -1 and the others are 0. Let $t_1 \in \llbracket 1, t \rrbracket$ be such that $a_{i_{t_1}, i_{t+1}} = -1$. By assumption, there exist $t' \in \llbracket 1, t \rrbracket$ and a sequence $(u_i)_{i \in \llbracket 1, t' \rrbracket} \in \{i_1, \dots, i_t\}^{\llbracket 1, t' \rrbracket}$ such that $a_{u_i, u_{i+1}} = -1$ for all $i \in \llbracket 1, t'-1 \rrbracket$, $u_1 = i_1$ and $u_{t'} = i_{t_1}$. Then $((u_i)_{i \in \llbracket 1, t' \rrbracket}, i_{t+1})$ is a path in I_1 relating i_1 and i_{t+1} , which proves the 1-star-convexity of $\{i_1, \dots, i_{t+1}\}$ at i_1 . By induction, we deduce that T is 1-star-convex. Using Lemma 2.13, we deduce that I_1 is a 1-star-convex tree.

(1) $\Rightarrow$ (3) Let T be a subdiagram of I which is 1-star-convex and which is a tree. We assume that $|T| \geq 2$ and that for all $T' \subset I$ such that $|T'| < |T|$ and for which T' is 1-star-convex, there exists $\beta' \in \mathcal{Q}(\Phi_+)$ such that $T' = I_1(\beta')$.

Let j be a leaf of T . Since a finite tree with at least two vertices has at least two leaves, after possibly changing j , we can assume that T is 1-star-convex at a node j_0 distinct from j . Let $\{j, k\}$ be the only edge containing j , then any path from j_0 to j contains $\{k, j\}$ and so $a_{k, j} = -1$ by 1-star-convexity of T at j_0 . Moreover, $T \setminus \{j\}$ is 1-star-convex (at j_0). Let $\beta_1 \in \mathcal{Q}(\Phi_+)$ be such that $(\beta_1)^\vee = \sum_{i \in T \setminus \{j\}} \alpha_i^\vee$.

We have $\langle (\beta_1)^\vee, \alpha_j \rangle = -1$, thus $\beta^\vee = r_j(\beta_1^\vee)$ and thus by Proposition 2.11, $\beta = r_j(\beta_1)$ is quantum. Therefore (1) implies (3). As (3) obviously implies (2), we deduce the proposition. $\square$

2.4. ORDER OF APPEARANCE AND GENERAL STRUCTURE OF $I_2(\beta)$. — Let $n \in \mathbb{N}$. For $i, j \in I_n$, we write $i \triangleleft_n j$ if (i, j) is an edge of I_n , and if $\mathbf{t}(i, n) < \mathbf{t}(j, n)$. The first condition means that $a_{i,j} \neq 0$ and the second condition simply means that, if we read the decomposition of w from right to left, we meet i n times before we meet j for the n -th time. We denote by $\leq_n$ the preorder generated by $\triangleleft_n$. Note that for $i, j \in I_n$, if $i \leq_n j$ and $j \leq_n i$, then i and j appear at the same time for the n -th time when we read $(\underline{w}, i_1)$ from right to left, and thus $i = j$. Therefore $\leq_n$ is an order on I_n . Note that, although this is not obvious from the definition, this order is independent of the chosen decomposition of w , as we will see in Remark 2.19. As I_n is finite, every chain is contained in a maximal one.

In this section, we use $\leq_2$ to characterize the diagram I_2 . In Lemma 2.18 we then describe chains for $\leq_n$. This will allow us to characterize the diagrams I_n in Section 2.6.2.

LEMMA 2.15. — Assume $I_2 \neq \emptyset$. Let $j_1, \dots, j_k \in I_2$ be a maximal chain for $\leq_2$, i.e.:

- (1) j_1 is minimal in I_2 for $\leq_2$,
- (2) j_k is maximal in I_2 for $\leq_2$,
- (3) $j_1 \triangleleft_2 j_2 \triangleleft_2 \dots \triangleleft_2 j_k$.

Then:

- (1) $\deg(j_1) = 3$, $|\text{Supp}(j_1)| \leq 4$.
- (2) for every $t \in \llbracket 2, k-1 \rrbracket$, $\text{Supp}(j_t) = \{j_{t-1}, j_t, j_{t+1}\}$, $\deg(j_t) = 2$ and $a_{j_{t-1}, j_t} = a_{j_{t+1}, j_t} = -1$.
- (3) We have $\deg(j_k) \leq 2$, $|\text{Supp}(j_k)| \leq 3$ and $\text{Supp}(j_k) \supset \{j_{k-1}, j_k\}$.
 - (a) If $|\text{Supp}(j_k)| = 3$, then $\text{Supp}(j_k) = \{j_{k-1}, j_k, j_{k+1}\}$ for some $j_{k+1} \in I_1 \setminus I_2$, and we have $a_{j_{k-1}, j_k} = a_{j_{k+1}, j_k} = -1$.
 - (b) If $|\text{Supp}(j_k)| = 2$, then $\text{Supp}(j_k) = \{j_{k-1}, j_k\}$, then

$$(2.10) \quad a_{j_{k-1}, j_k} = -1 \quad \text{and} \quad \mathbf{t}(j_{k-1}, 3) < \mathbf{t}(j_k, 2) < \mathbf{t}(j_{k-1}, 4).$$

In particular, if $j \in I_2$ is such that $|\text{Supp}(j)| = 2$, then j is either minimal or maximal for $\leq_2$.

Proof. — Let $t_1 = \mathbf{t}(j_1, 2)$. By (2.6), we have: $\gamma_{t_1}^\vee = \sum_{i \in I_1} N_i(t_1) \alpha_i^\vee$ and $N_i(t_1) \geq 1$ for every $i \in I_1$ by choice of $\underline{w}$. Then $\alpha_{i_{t_1}}(\gamma_{t_1}^\vee) = -1$. Let $i \in I_1$ and assume that $N_i(t_1) \geq 2$. Then as $i \not\triangleleft_2 j_1$, we have $a_{i, j_1} = 0$. Therefore

$$\begin{aligned} \alpha_{j_1}(\gamma_{t_1}^\vee) &= -1 = \alpha_{j_1} \left(\sum_{i \in \text{Supp}(j_1)} N_i(t_1) \alpha_i^\vee \right) = \alpha_{j_1} \left(\sum_{i \in \text{Supp}(j_1)} \alpha_i^\vee \right) \\ &= 2 + \sum_{i \in \text{Supp}(j_1) \setminus \{j_1\}} a_{i, j_1} = 2 - \deg(j_1), \end{aligned}$$

and thus $\deg(j_1) = 3$.

Now let $x \in \llbracket 2, k \rrbracket$. Let $t_x = \mathbf{t}(j_x, 2)$. We have:

$$\alpha_{j_x}(\gamma_{t_x}^\vee) = -1 = \alpha_{j_x} \left(\sum_{i \in \text{Supp}(j_x)} N_i(t_x) \alpha_i^\vee \right) = 2 + \sum_{i \in \text{Supp}(j_x) \setminus \{j_x\}} N_i(t_x) a_{i, j_x}.$$

Therefore $\sum_{i \in \text{Supp}(j_x) \setminus \{j_x\}} N_i(t_x) a_{i, j_x} = -3$, and this is a sum of non-positive integers. Moreover, as $N_{j_{x-1}}$ is non-decreasing and $t_x > t_{x-1}$, we have $N_{j_{x-1}}(t_x) \geq 2$. Consequently, $|\text{Supp}(j_x)| \leq 3$. Assume $x \leq k-1$. As $j_{x-1} \triangleleft_2 j_x \triangleleft_2 j_{x+1}$, we have $\text{Supp}(j_x) = \{j_{x-1}, j_x, j_{x+1}\}$ and $a_{j_{x+1}, j_x} = a_{j_{x-1}, j_x} = -1$. In particular, $\deg(j_x) = 2$.

Assume now $x = k$. Then if $|\text{Supp}(j_k)| = 3$, we have $a_{j_{k+1}, j_k} = -1$, for some $j_{k+1} \in I_1$. If $j_{k+1} \in I_2$, then we would have $j_k \triangleleft_2 j_{k+1}$, which would contradict our maximality assumption. Therefore, $j_{k+1} \in I_1 \setminus I_2$. Assume $|\text{Supp}(j_k)| = 2$. Then $\text{Supp}(j_k) = \{j_{k-1}, j_k\}$. Let $t = \mathbf{t}(j_k, 2)$. Then

$$\alpha_{j_k}(\gamma_t^\vee) = -1 = \alpha_{j_k} (N_{j_{k-1}}(t) \alpha_{j_{k-1}}^\vee + \alpha_{j_k}^\vee) = 2 - 3,$$

which proves (2.10). $\square$

In terms of Dynkin diagrams, Lemma 2.15 implies that the form of the connected components of I_2 have a very constrained form, see Proposition 2.16 and Figure 2.1.

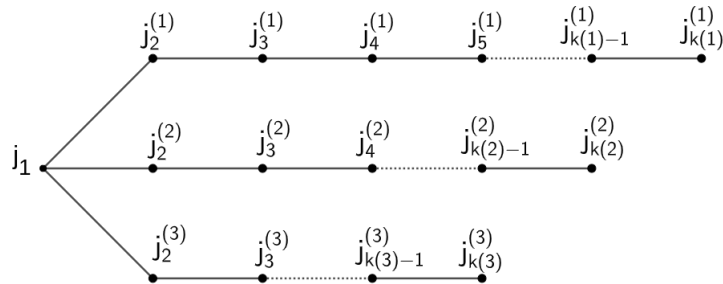


FIGURE 2.1. Generic form of a connected component of I_2 , when $m = 3$. Actually, as we shall see in Lemma 2.25, if $j_1 \in I_3$, then we have $\min\{k(1), k(2), k(3)\} \leq 2$.

PROPOSITION 2.16. — *Let $\mathcal{C}$ be a connected component of I_2 . Then:*

- (1) $\mathcal{C}$ admits a minimum j_1 and $m := |\text{Supp}(j_1)| - 1 \leq 3$.
- (2) Write $\text{Supp}(j_1) \setminus \{j_1\} = \{j_2^{(i)} \mid i \in \llbracket 1, m \rrbracket\}$. For $i \in \llbracket 1, m \rrbracket$, $\{x \in I_2 \mid x \geq_2 j_2^{(i)}\}$ is a segment: we can write it as $\{j_2^{(i)}, j_3^{(i)}, \dots, j_{k(i)}^{(i)}\}$, for some $j_2^{(i)}, \dots, j_{k(i)}^{(i)} \in I_2$ and the edges are the $(j_t^{(i)}, j_{t+1}^{(i)})$ for $t \in \llbracket 1, k(i) - 1 \rrbracket$.
- (3) We have $\mathcal{C} = \{j_1\} \sqcup \bigsqcup_{i=1}^m \{j_2^{(i)}, j_3^{(i)}, \dots, j_{k(i)}^{(i)}\}$.
- (4) If $j \in \mathcal{C} \setminus \{j_1\}$ then $|\text{Supp}(j)| \leq 3$ and $a_{j', j} = -1$ for all $j' \in \text{Supp}(j)$.

Proof. — Let $j_1 \in \mathcal{C}$ be the minimum for $\leq_2$. By Lemma 2.15, $m := |\text{Supp}(j_1)| - 1 \leq 3$. Write $\text{Supp}(j_1) = \{j_1\} \cup \{j_2^{(i)} \mid i \in \llbracket 1, m \rrbracket\}$. Let $i \in \llbracket 1, m \rrbracket$. By Lemma 2.15,

the set E_i of the elements of I_2 which are greater than $j_2^{(i)}$ for $\leq_2$ is of the form $\{j_2^{(i)}, j_3^{(i)}, \dots, j_{k(i)}^{(i)}\}$ for some $k(i) \in \mathbb{Z}_{\geq 2}$, and as a subgraph of I_2 , E_i is the line segment $[j_2^{(i)}, j_{k(i)}^{(i)}]$. In particular, if $\mathcal{C}' = \{j_1\} \cup \bigcup_{i=1}^m E_i$, then for each $x \in \mathcal{C}'$, we have $\text{Supp}(x) \cap I_2 \subset \mathcal{C}'$. Therefore $\mathcal{C}'$ is connected and $\mathcal{C} = \mathcal{C}'$. Moreover $\mathcal{C}$ is a connected subgraph of the tree I_1 (by Proposition 2.14) and thus it is a tree. Therefore the sets E_i are disjoint. Finally, by Lemma 2.15 (2) & (3), for all $j \in \mathcal{C}'$ we have $|\text{Supp}(j)| \leq 3$ and, for all $j' \in \text{Supp}(j)$, $a_{j',j} = -1$. The lemma follows. $\square$

REMARK 2.17. — The order $\leq_2$ does not depend on the choice of the reduced decomposition made at the beginning of Section 2.2, provided that it satisfies Lemma 2.12. Indeed, if two elements of I_2 are comparable for $\leq_2$, they belong to the same connected component of I_2 . Thus it suffices to prove that for each connected component $\mathcal{C}$ of I_2 the restriction of $\leq_2$ to $\mathcal{C}$ is independent of the choice of the reduced decomposition. By Lemma 2.15 and Proposition 2.16, the minimum j_1 for $\leq_2$ is the unique element of $\mathcal{C}$ whose degree is 3. Then the elements $j_2^{(1)}, \dots, j_2^{(m)}$ (where $m = |\text{Supp}(j_1) - 1|$) that cover j_1 are the elements of $\text{Supp}(j_1) \setminus \{j_1\}$. If $i \in \llbracket 1, m \rrbracket$, the only element (if it exists) covering $j_2^{(i)}$ is the element of $\text{Supp}(j_2^{(i)}) \setminus \{j_1, j_2^{(i)}\}$ etc. All these elements are defined uniquely from the graph I_2 , which is independent of the choice of the reduced decomposition.

With the above notation, we set:

$$(2.11) \quad k' = \begin{cases} k, & \text{if } |\text{Supp}(j_k)| = 2, \\ k+1 & \text{if } \text{Supp}(j_k) = \{j_{k-1}, j_k, j_{k+1}\}. \end{cases}$$

The following lemma implies that

$$(2.12) \quad N_{j_1}(t) \geq N_{j_2}(t) \geq \dots \geq N_{j_{k'}}(t), \quad \forall t \in \llbracket L_1 + 1, L \rrbracket.$$

LEMMA 2.18. — Let $j_1, \dots, j_k \in I_2$ be a maximal chain for $\leq_2$. Let $n \in \mathbb{Z}_{\geq 3}$. Then:

- (1) there exists $k_n \in \llbracket 0, k \rrbracket$ such that $\{i \in \llbracket 1, k \rrbracket \mid j_i \in I_n\} = \llbracket 1, k_n \rrbracket$,
- (2) we have $j_1 \triangleleft_n j_2 \triangleleft_n \dots \triangleleft_n j_{k_n}$,
- (3) the sequence (k_n) is non-increasing.

Suppose moreover that for all $n \geq 2$, $\mathbf{t}(j_1, n) \geq \mathbf{t}(j_2, n-1)$. Then:

- (4) we have $\text{Supp}(j_k) = \{j_{k-1}, j_k, j_{k+1}\}$,
- (5) for all $i \in \llbracket 1, k \rrbracket$ and $n \geq 2$

$$(2.13) \quad \mathbf{t}(j_i, n) \geq \mathbf{t}(j_{i+1}, n-1),$$

- (6) the sequence (k_n) is decreasing until it reaches 0. In particular $\max N_{j_1} \leq k+1$.

Proof. — We assume that $E := \{x \in \llbracket 1, k \rrbracket \mid j_x \in I_n\}$ is non-empty. Let $y = |E|$. For $x \in \llbracket 1, y \rrbracket$, let $\tilde{t}_x \in \llbracket 2, L \rrbracket$ be such that $|\{x' \in \llbracket 1, k \rrbracket \mid N_{j_{x'}}(\tilde{t}_x) \geq n\}| \geq x$ and such that $\tilde{t}_x$ is minimal for this property. By definition, $\tilde{t}_1 < \tilde{t}_2 < \dots < \tilde{t}_y$.

Let $x \in \llbracket 1, k \rrbracket$ be such that $N_{j_x}(\tilde{t}_1) = n$. We then have $N_{j_x}(\tilde{t}_1 - 1) = n - 1$ and $\alpha_{j_x}(\gamma_{\tilde{t}_1-1}^\vee) = -1$. Assume $x \in \llbracket 2, k' - 1 \rrbracket$. Then by Lemma 2.15 we have

$$\text{Supp}(j_x) = \{j_{x-1}, j_x, j_{x+1}\} \quad \text{and} \quad a_{j_{x-1}, j_x} = -1 = a_{j_{x+1}, j_x}.$$

We have $N_{j_{x-1}}(\tilde{t}_1 - 1), N_{j_{x+1}}(\tilde{t}_1 - 1) \leq n - 1$. Then:

$$\begin{aligned} \alpha_{j_x}(\gamma_{\tilde{t}_1-1}^\vee) &= \alpha_{j_x} \left(N_{j_{x-1}}(\tilde{t}_1 - 1) \alpha_{j_{x-1}}^\vee + N_{j_x}(\tilde{t}_1 - 1) \alpha_{j_x}^\vee + N_{j_{x+1}}(\tilde{t}_1 - 1) \alpha_{j_{x+1}}^\vee \right) \\ &\geq 2(n - 1) - 2(n - 1) = 0 : \end{aligned}$$

a contradiction. Therefore $x \notin \llbracket 2, k' - 1 \rrbracket$. Assume $k = k'$ and $x = k$. Then $N_{j_{x-1}}(\tilde{t}_1 - 1) \leq n - 1$. By (2.10), $a_{j_{k-1}, j_k} = -1$ and we reach a contradiction again. Therefore, $x = 1$.

Let $x \in \llbracket 2, y \rrbracket$ and $i \in \llbracket 1, k \rrbracket$ be such that $N_{j_i}(\tilde{t}_x - 1) = n - 1 = N_{j_i}(\tilde{t}_x) - 1$. We assume that for all $x' \in \llbracket 1, x - 1 \rrbracket$,

$$\{z \in \llbracket 1, k \rrbracket \mid N_{j_z}(\tilde{t}_{x'}) \geq n\} = \llbracket 1, x' \rrbracket \quad \text{and} \quad \{z \in \llbracket 1, k \rrbracket \mid N_{j_z}(\tilde{t}_{x'}) \leq n - 1\} = \llbracket x' + 1, k \rrbracket.$$

Then $i \geq x$. Assume $i \in \llbracket x + 1, y \rrbracket$ and $i < k'$. Then:

$$\begin{aligned} \alpha_{j_i}(\gamma_{\tilde{t}_x-1}^\vee) &= -1 \\ (2.14) \quad &= \alpha_{j_i} \left(N_{j_{i-1}}(\tilde{t}_x - 1) \alpha_{j_{i-1}}^\vee + N_{j_i}(\tilde{t}_x - 1) \alpha_{j_i}^\vee + N_{j_{i+1}}(\tilde{t}_x - 1) \alpha_{j_{i+1}}^\vee \right). \end{aligned}$$

By choice of i , $N_{j_i}(\tilde{t}_x - 1) = n - 1$ and by our assumption $N_{j_{i-1}}(\tilde{t}_x - 1), N_{j_{i+1}}(\tilde{t}_x - 1) \leq n - 1$. Therefore $\alpha_{j_i}(\gamma_{\tilde{t}_x-1}^\vee) \geq 2(n - 1) - 2(n - 1) = 0$: a contradiction. Therefore $i = x$ (the case $i = k' = k$ is similar). We have thus proved that, for every $x \in \llbracket 1, y \rrbracket$, we have:

$$\{z \in \llbracket 1, k \rrbracket \mid N_{j_z}(\tilde{t}_x) \geq n\} = \llbracket 1, x \rrbracket \quad \text{and} \quad \{z \in \llbracket 1, k \rrbracket \mid N_{j_z}(\tilde{t}_x) \leq n - 1\} = \llbracket x + 1, k \rrbracket.$$

In other words, $\mathbf{t}(j_1, n) < \mathbf{t}(j_2, n) < \dots < \mathbf{t}(j_y, n)$. As $a_{j_z, j_{z+1}} \neq 0$ for every $z \in \llbracket 1, k - 1 \rrbracket$, we have $j_1 \triangleleft_n j_2 \triangleleft_n \dots \triangleleft_n j_y$. We thus set $k_n = y = |\{i \in \llbracket 1, k \rrbracket \mid j_i \in I_n\}|$ and we have (1) and (2). Then (3) is clear since $I_n \subset I_{n-1}$.

We prove (2.13) by induction on i , as the assumption corresponds to the case $i = 1$. Let $i \in \llbracket 1, k - 1 \rrbracket$ and suppose that (2.13) is satisfied for $(i - 1, n + 1)$, that is to say $\mathbf{t}(j_{i-1}, n + 1) \geq \mathbf{t}(j_i, n)$. Let $t = \mathbf{t}(j_i, n)$. If it is infinite, then (2.13) is clear, so we suppose it is finite. Then $n \leq N_{j_{i-1}}(t)$, since $j_{i-1} \triangleleft_n j_i$ by (2). We also have $N_{j_{i-1}}(t) < n + 1$ by assumption, so $N_{j_{i-1}}(t) = n$. Therefore,

$$-1 = \alpha_{j_i}(\gamma_t^\vee) = 2N_{j_i}(t) - N_{j_{i-1}}(t) - N_{j_{i+1}}(t) = 2(n - 1) - n - N_{j_{i+1}}(t),$$

so $N_{j_{i+1}}(t) = n - 1$. Thus $t \geq \mathbf{t}(j_{i+1}, n - 1)$, which completes the induction. For $(i, n) = (k, 2)$, since $\mathbf{t}(j_k, 2)$ is finite (because $j_k \in I_2$) the same computation proves that $N_{j_{k+1}}(\mathbf{t}(j_k, 2)) = 1$, hence $|\text{Supp}(j_k)| = 3$. Thus we have proved (4) and (5). In particular, if $\mathbf{t}(j_{i+1}, n - 1) = \infty$ then $\mathbf{t}(j_i, n) = \infty$. If $k_n \neq 0$, applying this to $(i, n) = (k_n, n + 1)$ we deduce that $j_{k_n} \notin I_{n+1}$, so $k_{n+1} < k_n$. Therefore the sequence (k_n) is decreasing until it reaches zero, $k_{k+2} = 0$ and $j_1 \notin I_{k+2}$, which proves (6). $\square$

REMARK 2.19. — Let $\mathcal{C}$ be a connected component of I_2 . Write

$$\mathcal{C} = \{j_1\} \sqcup \bigsqcup_{i=1}^m \{j_2^{(i)}, \dots, j_{k(i)}^{(i)}\},$$

with the same notation as in Proposition 2.16. Then Lemma 2.18 implies that if $I_n \cap \mathcal{C} \neq \emptyset$, then there exists $(k_m(i))_{i \in \llbracket 1, m \rrbracket} \in \prod_{i=1}^m \llbracket 1, k(i) \rrbracket$ such that $\mathcal{C} \cap I_n = \{j_1\} \sqcup \bigsqcup_{i=1}^m \{j_2^{(i)}, \dots, j_{k_m(i)}^{(i)}\}$ (where $\{j_2^{(i)}, \dots, j_{k_m(i)}^{(i)}\} = \emptyset$, if $k_m(i) = 1$). Moreover, the order $\leq_n$ is simply the restriction of $\leq_2$ to I_n , and it is independent of the choice of a reduced decomposition for β , by Remark 2.17.

2.5. OPTIMAL BOUND ON $\max N_{j_1}$. — In this section, we fix $j_1 \in I_2$ minimal for $\leq_2$, and we fix a maximal chain $j_1 \triangleleft_2 j_2 \triangleleft_2 \dots \triangleleft_2 j_k$ as in Lemma 2.15. In particular, k always denotes the length of a maximal $\leq_2$ -chain starting at j_1 .

By Lemma 2.15(1), we have $\deg(j_1) = 3$, therefore $|\text{Supp}(j_1)| \in \{2, 3, 4\}$. We treat each case separately, to obtain a bound on $\max N_{j_1}$ which depends on k . Along the way, we obtain more precise information on the $\leq_n$ -chains starting at j_1 . First, let us state a technical lemma on the difference $N_{j_k} - N_{j_{k-1}}$.

LEMMA 2.20. — *Let $t \in \llbracket L_1 + 1, L \rrbracket$ be such that $N_{j_k}(t) \geq 2$. Then:*

- (1) *If $\text{Supp}(j_k) = \{j_{k-1}, j_k, j_{k+1}\}$ (i.e., if $k' = k+1$), then $2N_{j_k}(t) \leq N_{j_{k-1}}(t) + 2$. If moreover $N_{j_k}(t) < \max N_{j_k}$, then $N_{j_{k-1}}(t) \leq 2N_{j_k}(t)$.*
- (2) *If $\text{Supp}(j_k) = \{j_{k-1}, j_k\}$ (i.e., if $k' = k$), then $2N_{j_k}(t) \leq N_{j_{k-1}}(t) + 1$. If moreover $N_{j_k}(t) < \max N_{j_k}$, then $2N_{j_k}(t) \geq N_{j_{k-1}}(t) - 1$. In particular, if $N_{j_{k-1}}(t)$ is even, $N_{j_k}(t) = \frac{1}{2}N_{j_{k-1}}(t)$ and if $N_{j_{k-1}}(t)$ is odd, $N_{j_k}(t) \in \left\{ \frac{N_{j_{k-1}}(t)-1}{2}, \frac{N_{j_{k-1}}(t)+1}{2} \right\}$.*

Proof. — Let $t' = \mathbf{t}(j_k, N_{j_k}(t))$ and let $t'' = \mathbf{t}(j_k, N_{j_k}(t) + 1)$, if $N_{j_k}(t) < \max N_{j_k}$. We have $t' < t \leq t''$.

(1) If $\text{Supp}(j_k) = \{j_{k-1}, j_k, j_{k+1}\}$, by Lemma 2.15 we have $j_{k+1} \in I_1 \setminus I_2$ and thus $N_{j_{k+1}}(t') = 1$. We also have

$$2N_{j_k}(t') = N_{j_{k-1}}(t') + N_{j_{k+1}}(t') - 1 = N_{j_{k-1}}(t')$$

and thus $2N_{j_k}(t) - 2 = N_{j_{k-1}}(t') \leq N_{j_{k-1}}(t)$,

since $N_{j_{k-1}}$ is non-decreasing. We also have

$$2N_{j_k}(t'') = 2N_{j_k}(t) = N_{j_{k-1}}(t'') \geq N_{j_{k-1}}(t),$$

which proves (1).

(2) If $\text{Supp}(j_k) = \{j_{k-1}, j_k\}$, we have $2N_{j_k}(t') = N_{j_{k-1}}(t') - 1$ and thus

$$2(N_{j_k}(t) - 1) = 2N_{j_k}(t') = N_{j_{k-1}}(t') - 1 \leq N_{j_{k-1}}(t) - 1.$$

Therefore $2N_{j_k}(t) \leq N_{j_{k-1}}(t) + 1$.

We also have $2N_{j_k}(t'') = N_{j_{k-1}}(t'') - 1$ and thus $2N_{j_k}(t) \geq N_{j_{k-1}}(t) - 1$. Consequently we have

$$N_{j_{k-1}}(t) - 1 \leq 2N_{j_k}(t) \leq N_{j_{k-1}}(t) + 1,$$

which proves (2). □

We can now obtain a bound on N_{j_1} .

2.5.1. Case where $|\text{Supp}(j_1)| \leq 3$

LEMMA 2.21. — Let $j_1 \triangleleft_2 j_2 \triangleleft_2 \cdots \triangleleft_2 j_k \in I_2$ be as in Lemma 2.15. We assume that $\text{Supp}(j_1) = \{j_1, j_2\}$. Then $\max N_{j_1} \leq 2$.

Proof. — By Lemma 2.15, $\deg(j_1) = -a_{j_2, j_1} = 3$. Assume $j_1 \in I_3$ and set $t = \mathbf{t}(j_1, 3)$. Then

$$\alpha_{j_1}(\gamma_t^\vee) = \alpha_{j_1}(2\alpha_{j_1}^\vee + N_{j_2}(t)\alpha_{j_2}^\vee) = 4 - 3N_{j_2}(t) = -1 :$$

this is impossible. The lemma follows. $\square$

LEMMA 2.22. — Let $j_1 \in I_2$ be minimal for $\leq_2$ such that $|\text{Supp}(j_1)| = 3$. Then $\max N_{j_1} \leq k + 1$.

More precisely, write $\text{Supp}(j_1) = \{j_1, j_2, j'_2\}$ with $a_{j'_2, j_1} = -1$ and $a_{j_2, j_1} = -2$. Then:

- (1) if $\mathbf{t}(j'_2, 2) \leq \mathbf{t}(j_1, 3)$ then $\mathbf{t}(j_1, 3) = \infty$,
- (2) if $\mathbf{t}(j_1, 3) < \mathbf{t}(j'_2, 2) \neq \infty$ then $\text{Supp}(j'_2) = \{j_1, j'_2\}$ and $\mathbf{t}(j_1, 4) = \mathbf{t}(j'_2, 3) = \infty$,
- (3) otherwise $j'_2 \notin I_2$ and, for all $n \geq 2$, $\mathbf{t}(j_2, n-1) \leq \mathbf{t}(j_1, n) \leq \mathbf{t}(j_2, n)$. The chain $j_1 \triangleleft_2 j_2 \triangleleft_2 \cdots \triangleleft_2 j_k$ then satisfies properties 4-6 of Lemma 2.18 and in particular $\max N_{j_1} \leq k + 1$.

Proof. — Note that, by minimality of j_1 for $\leq_2$, $\mathbf{t}(j_1, 2) \leq \mathbf{t}(j'_2, 2)$ and $\mathbf{t}(j_1, 2) \leq \mathbf{t}(j_2, 2)$. Moreover by Lemma 2.18, the same holds with 2 replaced by $n \geq 2$.

Suppose first that $\mathbf{t}(j'_2, 2) \leq \mathbf{t}(j_1, 3)$. If $\mathbf{t}(j'_2, 2) = \infty$ then clearly $\mathbf{t}(j_1, 3) = \infty$ is clear. Otherwise, for $t \geq \mathbf{t}(j'_2, 2) + 1$ such that $N_{j_1}(t) = 2$, $\alpha_{j_1}(\gamma_t^\vee)$ is even and therefore $\mathbf{t}(j_1, 3) = \infty$.

Suppose now that $\mathbf{t}(j_1, 3) < \mathbf{t}(j'_2, 2) \neq \infty$. Then at time $t = \mathbf{t}(j'_2, 2)$, we have $-1 = \alpha_{j'_2}(\gamma_t^\vee) \geq 2 - N_{j_1}(t) \geq 2 - 3$. Therefore $N_{j_1}(t) = 3$ and there is no other element in $\text{Supp}(j'_2)$. In particular $\mathbf{t}(j'_2, 2) < \mathbf{t}(j_1, 4) \leq \mathbf{t}(j'_2, 3)$. Let $t \in [\mathbf{t}(j'_2, 2) + 1, \mathbf{t}(j_1, 4)]$. Then $N_{j'_2}(t) = 2$ and $\alpha_{j_1}(\gamma_t^\vee) = 2N_{j_1}(t) - 2N_{j_2}(t) - 2$ is even, so $\mathbf{t}(j_1, 4) = \infty$. Therefore $\max N_{j_1} \leq 3$. Moreover $k \geq 2$, since $|\text{Supp}(j_1)| > 1$, which proves that $\max N_{j_1} \leq k + 1$.

Finally, suppose that $\mathbf{t}(j'_2, 2) = \infty$, that is to say $j'_2 \notin I_2$. Let $n \geq 2$ and suppose that $t := \mathbf{t}(j_1, n)$ is finite. Then $-1 = \alpha_{j_1}(\gamma_t^\vee) = 2(n-1) - 2N_{j_2}(t) - 1$, so $N_{j_2}(t) = n-1$. Hence $\mathbf{t}(j_2, n-1) \leq \mathbf{t}(j_1, n)$. The second inequality is given by Lemma 2.18(2). By Lemma 2.18(6), we deduce $\max N_{j_1} \leq k + 1$. $\square$

2.5.2. Case where $|\text{Supp}(j_1)| = 4$

PROPOSITION 2.23. — Let $j_1 \in I_2$ be minimal for $\leq_2$ such that $|\text{Supp}(j_1)| = 4$. Let k denote the maximal length of a $\leq_2$ -chain starting at j_1 . Then $\max N_{j_1} \leq \max(6, k+1)$.

Proof. — If $j_1 \notin I_3$ then $\max N_{j_1} \leq 2$ and the result is clear, so we can assume $j_1 \in I_3$. If $|\text{Supp}(j_1)| = 4$ and $\text{Supp}(j_1) \not\subset I_2$, then by Lemma 2.24 $\max N_{j_1} \leq k + 1$. Otherwise, since we assume $j_1 \in I_3$, either there is $j \in \text{Supp}(j_1)$ such that $\text{Supp}(j) \not\subset I_2$ and, by Lemma 2.26, $\max N_{j_1} \leq k + 1$; or $\text{Supp}(j) \subset I_2$ for all $j \in \text{Supp}(j_1)$ and, by Lemma 2.27, we have $\max N_{j_1} \leq 6$. $\square$

LEMMA 2.24. — *Let $j_1 \in I_2$ be minimal for $\leq_2$ such that $|\text{Supp}(j_1)| = 4$ and suppose $\text{Supp}(j_1) \not\subset I_2$. Then, any maximal chain $j_1 \triangleleft_2 j_2 \cdots \triangleleft_2 j_k$ with $k \geq 2$ satisfies properties of Lemma 2.18 (4), (5) & (6) and in particular $\max N_{j_1} \leq k + 1$.*

Proof. — Let us write $\text{Supp}(j_1) = \{j_1, j_2, j'_2, j''_2\}$ with $j''_2 \notin I_2$. By Lemma 2.18, to prove the result it suffices to prove that $\mathbf{t}(j_1, n) \geq \mathbf{t}(j_2, n - 1)$ for all $n \geq 2$.

If $\mathbf{t}(j_1, n) = \infty$ then the inequality is clear, so suppose $t = \mathbf{t}(j_1, n) \neq \infty$. Since $N_{j''_2}(t) = 1$ we have

$-1 = \alpha_{j_1}(\gamma_t^\vee) = 2N_{j_1}(t) - N_{j_2}(t) - N_{j'_2}(t) - N_{j''_2}(t) = 2(n - 1) - 1 - N_{j_2}(t) - N_{j'_2}(t)$,
so $N_{j_2}(t) + N_{j'_2}(t) = 2(n - 1)$. Moreover, $\max(N_{j_2}(t), N_{j'_2}(t)) \leq n - 1$, since, by Lemma 2.18(2), j_1 is also minimal for $\leq_n$. Therefore $N_{j_2}(t) + N_{j'_2}(t) = 2(n - 1)$ implies $N_{j_2}(t) = N_{j'_2}(t) = n - 1$, which proves $\mathbf{t}(j_1, n) = t > \mathbf{t}(j_2, n - 1)$. $\square$

LEMMA 2.25. — *Let $j_1 \in I_2$ be minimal for $\leq_2$ such that $|\text{Supp}(j_1)| = 4$, $\text{Supp}(j_1) \subset I_2$ and $j_1 \in I_3$. Then there is a unique $j \in \text{Supp}(j_1)$ such that $|\text{Supp}(j)| = 2$, and for any $j_2 \in \text{Supp}(j_1) \setminus \{j_1, j\}$, we have $|\text{Supp}(j_2)| = 3$ and*

$$(2.15) \quad \mathbf{t}(j_2, 2) < \mathbf{t}(j_1, 3) < \mathbf{t}(j, 2) \leq \mathbf{t}(j_1, 4).$$

Proof. — Write $\text{Supp}(j_1) \setminus \{j_1\} = \{j_2^{(1)}, j_2^{(2)}, j_2^{(3)}\}$. Fix $i \in \llbracket 1, 3 \rrbracket$. Let $t = \mathbf{t}(j_1, 3) \neq \infty$. Since j_1 is minimal for $\leq_2$, it is also minimal for $\leq_3$. Then

$$N_{j_2^{(1)}}(t) + N_{j_2^{(2)}}(t) + N_{j_2^{(3)}}(t) = 5 \quad \text{and} \quad N_{j_2^{(i)}}(t) \leq 2 \quad \text{for all } i,$$

so, upon relabeling, we can suppose $(N_{j_2^{(1)}}(t), N_{j_2^{(2)}}(t), N_{j_2^{(3)}}(t)) = (2, 2, 1)$.

For $i \in \{1, 2\}$, $t_i := \mathbf{t}(j_2^{(i)}, 2)$ satisfies

$$t_i < t \quad \text{and} \quad -1 = \alpha_{j_2^{(i)}}(\gamma_{t_i}^\vee) < 0 = 2N_{j_2^{(i)}}(t_i) - N_{j_1}(t_i).$$

Therefore $\text{Supp}(j_2^{(i)})$ contains exactly one other element and, hence has cardinality 3. Now let $t = \mathbf{t}(j_2^{(3)}, 2) < \mathbf{t}(j_1, 3)$. Then

$$\alpha_{j_2^{(3)}}(\gamma_t^\vee) = -1 \quad \text{and} \quad 2N_{j_2^{(3)}}(t) - N_{j_1}(t) \leq 2 - 3 = -1,$$

so $N_{j_1}(t) = 3$ and $\text{Supp}(j_2^{(3)}) = \{j_1, j_2^{(3)}\}$, which concludes the proof. $\square$

LEMMA 2.26. — *Let $j_1 \in I_2$ be minimal for $\leq_2$ such that $|\text{Supp}(j_1)| = 4$, $\text{Supp}(j_1) \subset I_2$ and $j_1 \in I_3$. Let j'_2 be the unique element of $\text{Supp}(j_1)$ with $|\text{Supp}(j'_2)| = 2$, given by Lemma 2.25.*

Suppose that there exists $j'_2 \in \text{Supp}(j_1)$ with $\text{Supp}(j'_2) \not\subset I_2$. Then the following properties are satisfied:

(1) *For any $n \geq 1$,*

$$(2.16) \quad \mathbf{t}(j_1, 2n) \leq \mathbf{t}(j'_2, n + 1) \leq \mathbf{t}(j_1, 2n + 1) \leq \mathbf{t}(j'_2, n + 1) \leq \mathbf{t}(j_1, 2n + 2).$$

(2) *The unique maximal chain $j_1 \triangleleft_2 j_2 \cdots \triangleleft_2 j_k$ such that $j_2 \notin \{j'_2, j''_2\}$ satisfies properties (4), (5), (6) of Lemma 2.18, and in particular $\max N_{j_1} \leq k + 1$.*

Proof. — By assumption, $\text{Supp}(j_2'') = \{j_1, j_2''\}$ and $\text{Supp}(j_2') \not\subset I_2$, hence both j_2', j_2'' are maximal for $\leq_2$. Let j_2 denote the remaining element of $\text{Supp}(j_1)$.

Fix $n \geq 2$. Since j_2' is maximal for $\leq_2$ and has support of cardinality 3, Lemma 2.20 applied to $t := \mathbf{t}(j_2', n+1) + 1$ (if it is finite, otherwise it is clear) gives $\mathbf{t}(j_1, 2n) \leq \mathbf{t}(j_2', n+1)$. Similarly, Lemma 2.20(2) applied to $t := \mathbf{t}(j_2'', n+1) + 1$ gives $\mathbf{t}(j_1, 2n+1) \leq \mathbf{t}(j_2'', n+1)$.

Let $t = \mathbf{t}(j_1, 2n+1)$, which we suppose finite. Then $N_{j_1}(t) = 2n$, $N_{j_2''}(t) \leq n$ and, since j_1 is minimal for $\triangleleft_{2n+1}$, $N_{j_2}(t) \leq 2n$. Therefore

$$-1 = \alpha_{j_1}(\gamma_t^\vee) \geq 4n - n - 2n - N_{j_2'}(t) = n - N_{j_2'}(t),$$

so $N_{j_2'}(t) \geq n+1$ and $t > \mathbf{t}(j_2', n+1)$.

Let $t = \mathbf{t}(j_1, 2n+2)$ which we suppose finite, then $N_{j_1}(t) = 2n+1$, $N_{j_2'}(t) \leq n+1$ (by Lemma 2.20) and $N_{j_2}(t) \leq 2n+1$. Therefore

$$-1 = \alpha_{j_1}(\gamma_t^\vee) \geq 4n+2 - (2n+1) - (n+1) - N_{j_2''}(t) = n - N_{j_2''}(t),$$

so $N_{j_2''}(t) \geq n+1$ and $\mathbf{t}(j_2'', n+1) \leq t$. This concludes the proof of Formula (2.16).

Let us check that $\mathbf{t}(j_1, n) \geq \mathbf{t}(j_2, n-1) \geq \mathbf{t}(j_1, n-1)$ for any $n \geq 3$. The inequality $\mathbf{t}(j_2, n-1) \geq \mathbf{t}(j_1, n-1)$ holds because $j_1 \triangleleft_2 j_2 \Rightarrow j_1 \triangleleft_{n-1} j_2$ by Lemma 2.18(2) (if they belong to I_{n-1}). For the remaining inequality, suppose $t := \mathbf{t}(j_1, n) \neq \infty$:

– If n is even, write $n = 2p$. Then by Formula (2.16), $\max(N_{j_2'}(t), N_{j_2''}(t)) \leq p$, so $-1 = \alpha_{j_1}(\gamma_t^\vee) \geq 2(2p-1) - 2p - N_{j_2}(t)$. We deduce $N_{j_2}(t) \geq n-1$.

– Otherwise, write $n = 2p+1$. Then similarly $N_{j_2''}(t) \leq p$ and $N_{j_2'}(t) \leq p+1$, so $-1 = \alpha_{j_1}(\gamma_t^\vee) \geq 2(2p) - 2p - 1 - N_{j_2}(t)$, so $N_{j_2}(t) \geq 2p = n-1$.

Either way, we have proved $\mathbf{t}(j_1, n) \geq \mathbf{t}(j_2, n-1)$. Then (2) follows from Lemma 2.18. $\square$

LEMMA 2.27. — Let $j_1 \in I_2$ be minimal for $\leq_2$ with $|\text{Supp}(j_1)| = 4$, $j_1 \in I_3$ and, for all $j \in \text{Supp}(j_1)$, $\text{Supp}(j) \subset I_2$. Then $\max N_{j_1} \leq 6$.

More precisely, using Lemma 2.25, we write $\text{Supp}(j_1) = \{j_1, j_2, j_2', j_2''\}$ such that

$$|\text{Supp}(j_2'')| = 2 \quad \text{and} \quad \mathbf{t}(j_2, 3) \leq \mathbf{t}(j_2', 3),$$

and write $\text{Supp}(j_2') = \{j_1, j_2', j_3'\}$. Then:

- (1) If $\mathbf{t}(j_2', 3) \leq \mathbf{t}(j_1, 4)$, then $j_1 \notin I_4$ and $j_2'' \notin I_3$
- (2) If $\mathbf{t}(j_1, 4) \neq \infty$ and $\mathbf{t}(j_2', 3) = \infty$, then $j_1 \notin I_5$ and $j_2'' \notin I_3$
- (3) Otherwise, if $\mathbf{t}(j_1, 4) < \mathbf{t}(j_2', 3) < \infty$, then $\mathbf{t}(j_2, n-1) \leq \mathbf{t}(j_1, n)$ for all $n \in \llbracket 2, 5 \rrbracket$, $\text{Supp}(j_3') = \{j_2', j_3'\}$ and $j_1 \notin I_7$, $j_2' \notin I_5$, $j_3' \notin I_3$.

Proof. — Suppose first that $\mathbf{t}(j_2', 3) \leq \mathbf{t}(j_1, 4)$ and assume, for a contradiction, that $t := \mathbf{t}(j_1, 4)$ is finite. By Lemma 2.25, we have $N_{j_2''}(t) \geq 2$, and since $\mathbf{t}(j_2, 3) \leq \mathbf{t}(j_2', 3) \leq \mathbf{t}(j_1, 4)$, we have $N_{j_2}(t) \geq 3$, $N_{j_2'}(t) \geq 3$. Therefore $N_{j_2}(t) + N_{j_2'}(t) + N_{j_2''}(t) \geq 8$, so $-1 = \alpha_{j_1}(\gamma_t^\vee) \leq -2$, a contradiction. Hence in this case, $\mathbf{t}(j_1, 4) = \infty$ and $j_1 \notin I_4$.

Suppose now that $\mathbf{t}(j_1, 4) \neq \infty$ and $\mathbf{t}(j_2', 3) = \infty$ and let $t \geq \mathbf{t}(j_1, 4) + 1$. Then $N_{j_1}(t) = 4$ so $N_{j_2}(t) \leq 4$ and $N_{j_2''}(t) \leq 2$ by Lemma 2.20. Moreover, by assumption $N_{j_2'}(t) \leq 2$ and thus $\alpha_{j_1}(\gamma_t^\vee) \geq 8 - 4 - 2 - 2 = 0$ cannot be equal to -1 , so N_{j_1} is

stuck at 4: $j_1 \notin I_5$. Moreover, in these two cases, $N_{j_1} \leq 4$ and thus by Lemma 2.20, $N_{j_2''} \leq 2$.

Finally, assume that $\mathbf{t}(j_1, 4) < \mathbf{t}(j_2', 3) \neq \infty$. Note that $\mathbf{t}(j_2, 3) < \mathbf{t}(j_1, 4)$ is necessary in order to have $\alpha_{j_1}(\gamma_{\mathbf{t}(j_1, 4)}^\vee) = -1$. Let $t = \mathbf{t}(j_2', 3)$, by assumption $N_{j_1}(t) \geq 4$ and $N_{j_3'}(t) \geq 1$. So $\alpha_{j_2'}(\gamma_t^\vee) = -1$ forces $N_{j_1}(t) = 4$ and $N_{j_3'}(t) = 1$. The first consequence is that $\mathbf{t}(j_2', 3) < \mathbf{t}(j_3', 2)$. Since we suppose $\text{Supp}(j_2') \subset I_2$, a computation at time $\mathbf{t}(j_3', 2)$ then proves that

$$\text{Supp}(j_3') = \{j_2', j_3'\}.$$

We then have

$$j_3' \in I_3 \implies j_2' \in I_5 \implies j_1 \in I_7.$$

Indeed, the first implication is a direct consequence of Lemma 2.20(2). Now assume that $t := \mathbf{t}(j_2', 5) < \infty$. Then by Lemma 2.20, we have $N_{j_3'}(t) \leq 2$ and thus $\alpha_{j_2'}(\gamma_t^\vee) = -1 = 8 - N_{j_1}(t) - N_{j_3'}(t)$ and so $N_{j_1}(t) \geq 7$ and

$$(2.17) \quad \mathbf{t}(j_1, 7) \leq \mathbf{t}(j_2', 5).$$

We now show that, under the assumptions of the statement, $j_1 \notin I_7$, which is enough to conclude.

We computed $N_{j_1}(\mathbf{t}(j_2', 3)) = 4$ and $N_{j_3'}(\mathbf{t}(j_2', 3)) = 1$, so $\mathbf{t}(j_2', 3) < \mathbf{t}(j_1, 5)$. Let $t = \mathbf{t}(j_1, 5)$ and suppose it is finite. For $\mathbf{t}(j_2', 3) < t' \leq t$ we have $\alpha_{j_2'}(\gamma_{t'}^\vee) \geq 6 - 4 - 2 \geq 0$, so $N_{j_2'}(t) = 3$. Moreover since $N_{j_1}(t) = 4$, we have $N_{j_2''}(t) = 2$ and $N_{j_2}(t) \leq 4$. Therefore, $\alpha_{j_1}(\gamma_t^\vee) = -1$ forces $N_{j_2}(t) = 4$. This proves $\mathbf{t}(j_2, n - 1) \leq \mathbf{t}(j_1, n)$ for $n = 5$. For $n = 2$ it is clear, for $n = 3$ it is given by Lemma 2.25 and we have already checked it for $n = 4$, hence the first statement of (3) is verified. Moreover, at time $t' = t + 1$ we have

$$(2.18) \quad (N_{j_1}(t'), N_{j_2}(t'), N_{j_2'}(t'), N_{j_2''}(t')) = (5, 4, 3, 2).$$

Now, set $t = \mathbf{t}(j_1, 6)$. If it is finite, then at this time exactly two out of the three values $N_{j_2}, N_{j_2'}, N_{j_2''}$ have increased by one. Indeed $N_{j_2} + N_{j_2'} + N_{j_2''}$ needs to increase by two. By Lemma 2.20, $N_{j_2} \leq N_{j_1}$ and $2N_{j_2''} \leq N_{j_1} + 1$ so these two terms cannot increase by two. Moreover, by the same lemma, as long as $N_{j_2'} \leq 4$, $N_{j_3'} \leq 2$. Thus if $t' = \mathbf{t}(j_2', 5)$, we have

$$\alpha_{j_2'}(\gamma_{t'}^\vee) = -1 = 8 - N_{j_1}(t') - N_{j_3'}(t') \geq 6 - N_{j_1}(t'),$$

so $t' \geq \mathbf{t}(j_1, 7) \geq t$ and $N_{j_2'}(t) \leq 4$: $N_{j_2'}$ cannot increase by two either. Thus two out of the three values $N_{j_2}, N_{j_2'}, N_{j_2''}$ have increased by one, we can actually check that all the three cases are possible. We deal with each case separately.

(1) Suppose that

$$(N_{j_2}(t), N_{j_2'}(t), N_{j_2''}(t)) = (5, 3, 3).$$

Then at any time t' such that $\mathbf{t}(j_2', 4) \geq t' > t$ we have $\alpha_{j_2'}(\gamma_{t'}^\vee) \leq 6 - 6 - 2 < -1$ by Lemma 2.20), so $\mathbf{t}(j_2', 4) = \infty$. Therefore if $t' = \mathbf{t}(j_1, 7)$ is finite, necessarily $(N_{j_1}(t'), N_{j_2''}(t')) = (6, 4)$, which contradicts Lemma 2.20(2). Hence $\mathbf{t}(j_1, 7)$ is infinite and $j_1 \notin I_7$, which implies $j_2' \notin I_5$, $j_3' \notin I_3$.

(2) Suppose that $(N_{j_2}(t), N_{j'_2}(t), N_{j''_2}(t)) = (5, 4, 2)$. The situation is similar: in this case $N_{j'_2}$ is stuck at 2, so $\mathbf{t}(j'_2, 5) \leq \mathbf{t}(j_1, 7)$. However by Lemma 2.20(2), $\mathbf{t}(j'_2, 5) \leq \mathbf{t}(j'_3, 3)$. We deduce that, for any $\mathbf{t}(j_1, 6) < t' \leq \mathbf{t}(j'_2, 5)$, $\alpha_{j'_2}(\gamma_{t'}^\vee) = 0$, so $\mathbf{t}(j'_2, 5) = \mathbf{t}(j_1, 7) = \mathbf{t}(j'_3, 3) = \infty$.

(3) Suppose that $(N_{j_2}(t), N_{j'_2}(t), N_{j''_2}(t)) = (4, 4, 3)$. Then by (2.17) and Lemma 2.20, we have

$$\mathbf{t}(j_2, 6) \leq \mathbf{t}(j_1, 7) \leq \min(\mathbf{t}(j'_2, 5), \mathbf{t}(j''_2, 4)).$$

Hence if $t := \mathbf{t}(j_1, 7) < \infty$, we have $N_{j_2}(t) + N_{j'_2}(t) + N_{j''_2}(t) = 13$, with $N_{j'_2}(t) + N_{j''_2}(t) \leq 7$. Consequently $\mathbf{t}(j_2, 6) < \mathbf{t}(j_1, 7) < \infty$. But this contradicts Lemma 2.28 below. Therefore $j_1 \notin I_7$, $j'_2 \notin I_5$ and $j'_3 \notin I_3$. $\square$

LEMMA 2.28. — *Let j_1, j_2 be the first terms of a maximal chain for $\leq_2$. Then there exists no pair $(x, y) \in \llbracket 1, L \rrbracket^2$ such that $(N_{j_1}(x), N_{j_2}(x)) = (6, 4)$ and $(N_{j_1}(y), N_{j_2}(y)) = (6, 6)$.*

Proof. — We assume that there exists $(x, y) \in \llbracket 1, L \rrbracket^2$ such that $(N_{j_1}(x), N_{j_2}(x)) = (6, 4)$ and $(N_{j_1}(y), N_{j_2}(y)) = (6, 6)$. Let $z \in \llbracket 1, L \rrbracket$. Assume $(N_{j_1}(z), N_{j_2}(z)) = (6, 4)$. If we had $\text{Supp}(j_2) = \{j_1, j_2\}$, then j_2 would be maximal for $\leq_2$ and Lemma 2.20(2) would lead to a contradiction. Therefore we have $\text{Supp}(j_2) = \{j_1, j_2, j_3\}$, for some $j_3 \in I_1$, and by Lemma 2.15, we have $a_{j_3, j_2} = a_{j_3, j_1} = -1$.

Let $t = \mathbf{t}(j_2, 6)$. We have $N_{j_1}(t) = 6$ by assumption. We have $10 - 6 - N_{j_3}(t) = -1$ and thus

$$(2.19) \quad \mathbf{t}(j_3, 4) < \mathbf{t}(j_3, 5) < \mathbf{t}(j_2, 6).$$

Let $t = \mathbf{t}(j_2, 5)$. We have $t < \mathbf{t}(j_1, 7)$ and thus we have $N_{j_3}(t) = 3$ (in order to have $8 - 6 - 3 = -1$). Therefore $(N_{j_1}(t+1), N_{j_2}(t+1), N_{j_3}(t+1)) = (6, 5, 3)$.

Now let $t \in \llbracket 1, L \rrbracket$ be such that $(N_{j_1}(t), N_{j_2}(t), N_{j_3}(t)) = (6, 5, 3)$. As $\mathbf{t}(j_3, 4) < \infty$ and $6 - 5 \neq -1$, we have $\text{Supp}(j_3) \neq \{j_2, j_3\}$. So by Lemma 2.15, we can write $\text{Supp}(j_3) = \{j_2, j_3, j_4\}$, for some $j_4 \in I_1$, and we have $a_{j_2, j_3} = a_{j_4, j_2} = -1$. If $t = \mathbf{t}(j_3, 4)$, we have $N_{j_4}(t) = 2$ (since $6 - 5 - 2 = -1$). So we have

$$(2.20) \quad (N_{j_1}(t), N_{j_2}(t), N_{j_3}(t), N_{j_4}(t)) = (6, 5, 3, 2)$$

and

$$(2.21) \quad (N_{j_1}(t+1), N_{j_2}(t+1), N_{j_3}(t+1), N_{j_4}(t+1)) = (6, 5, 4, 2).$$

Let $t = \mathbf{t}(j_3, 5)$. By (2.19), we have $\gamma_t^\vee = 6\alpha_{j_1}^\vee + 5\alpha_{j_2}^\vee + 4\alpha_{j_3}^\vee + a\alpha_{j_4}^\vee + \delta^\vee$, for some $a \in \llbracket 1, 4 \rrbracket$ and $\delta^\vee \in \bigoplus_{i \in I \setminus \{j_1, j_2, j_3, j_4\}} \mathbb{N}\alpha_i^\vee$. We have $\alpha_{j_3}(\gamma_t^\vee) = 8 - 5 + a\alpha_{j_3}(\alpha_{j_4}^\vee) = -1$ and thus $a\alpha_{j_3}(\alpha_{j_4}^\vee) = aa_{j_4, j_3} = -4$ and thus $a = 4$. Consequently

$$\mathbf{t}(j_4, 4) < \mathbf{t}(j_3, 5) < \infty.$$

Let $\tilde{t} = \mathbf{t}(j_4, 4)$. We have $2N_{j_4}(\tilde{t}) = 6 > N_{j_3}(\tilde{t}) + 1$ since $N_{j_3}(\tilde{t}) \leq 4$. By Lemma 2.20, this implies $\text{Supp}(j_4) \neq \{j_3, j_4\}$. Let j_5 be such that $\text{Supp}(j_4) = \{j_3, j_4, j_5\}$. Then $6 - N_{j_3}(\tilde{t}) - N_{j_5}(\tilde{t}) = -1 = 6 - 4 - N_{j_5}(\tilde{t})$ and thus

$$(2.22) \quad \mathbf{t}(j_5, 3) < \mathbf{t}(j_4, 4) < \infty.$$

Let $t = \mathbf{t}(j_4, 3)$. By (2.21), we have $4 - 4 - N_{j_5}(t) = -1$ and thus $N_{j_5}(t) = 1$. We thus have

$$(N_{j_1}(t), N_{j_2}(t), N_{j_3}(t), N_{j_4}(t), N_{j_5}(t)) = (6, 5, 4, 2, 1)$$

and then

$$(N_{j_1}(t+1), N_{j_2}(t+1), N_{j_3}(t+1), N_{j_4}(t+1), N_{j_5}(t+1)) = (6, 5, 4, 3, 1).$$

By (2.22), $t := \mathbf{t}(j_5, 2) < \infty$. Then $\gamma_t^\vee = z\alpha_{j_4}^\vee + \alpha_{j_5}^\vee + \delta^\vee$, for some $z \in \mathbb{Z}_{\geq 3}$ and $\delta^\vee \in Q_N^\vee = \bigoplus_{i \in I \setminus \{j_4, j_5\}} \mathbb{N}\alpha_i^\vee$. By Lemma 2.15 $a_{j_4, j_5} = -1$, thus $2 - z - \alpha_{j_5}(\delta^\vee) = -1$ and hence $z = 3$ and $\alpha_{j_5}(\delta^\vee) = 0$. This proves that $\text{Supp}(j_5) = \{j_4, j_5\}$. Then if $t = \mathbf{t}(j_5, 2)$, we have

$$(N_{j_1}(t), N_{j_2}(t), N_{j_3}(t), N_{j_4}(t), N_{j_5}(t)) = (6, 5, 4, 3, 1)$$

and

$$(N_{j_1}(t+1), N_{j_2}(t+1), N_{j_3}(t+1), N_{j_4}(t+1), N_{j_5}(t+1)) = (6, 5, 4, 3, 2).$$

But then $2N_{j_i}(t+1) - N_{j_{i+1}}(t+1) - N_{j_{i-1}}(t+1) = 0$ for every $i \in \llbracket 2, 4 \rrbracket$, and $2N_{j_5}(t+1) - N_{j_4}(t+1) = 1$. Therefore the situation can no longer evolve, which proves the lemma. $\square$

2.5.3. Conclusion: Finiteness of the set of quantum roots. — With the preceding results, we are already able to give an upper bound on the number of quantum roots in a given Kac–Moody root system. In particular, we prove that the set $\mathcal{Q}(\Phi_+)$ is finite for any Kac–Moody root system.

PROPOSITION 2.29. — *Let $k_{\max}$ be the length of a maximal chain for $\leq_2$ and $n = |I|$ (we have $k_{\max} \leq n$). Let β be a quantum root. Write $\beta^\vee = \sum_{i \in I} x_i \alpha_i^\vee$. Then we have $x_i \leq \max(k_{\max} + 1, 6)$ for every $i \in I$. In particular, $\text{ht}(\beta^\vee) \leq n(\max(6, k_{\max} + 1))$.*

Proof. — Let $i \in I$. If $i \notin I_2$, we have $x_i \leq 1$. Assume $i \in I_2$. Let $j \in I_2$ be such that $j \leq_2 i$ and such that j is minimal for $\leq_2$. Then by (2.12), we have $x_i \leq x_j$. Moreover, by Lemma 2.22, Lemma 2.21 and Proposition 2.23, $x_j \leq \max(6, k_{\max} + 1)$. Therefore $\text{ht}(\beta^\vee) = \sum_{i' \in I_1} x_{i'} \leq |I_1| \max(k_{\max} + 1, 6) \leq n \max(k_{\max} + 1, 6)$. $\square$

THEOREM 2.30. — *Let $n \in \mathbb{Z}_{\geq 2}$. Then for every generalized Cartan matrix A of size n , if Φ is the real root system associated to A , then the set of quantum roots is finite, with cardinality at most $(n + 6)^n$.*

Proof. — Let $\mathcal{Q}(\Phi_+)$ be the set of quantum roots. By Proposition 2.29, for every $\beta \in \mathcal{Q}(\Phi_+)$, we can write $\beta^\vee = \sum_{i \in I} x_i \alpha_i^\vee$, with $x_i \in \llbracket 0, n + 5 \rrbracket$ for every $i \in I$. Therefore $|\mathcal{Q}(\Phi_+)| = |\{\beta^\vee \mid \beta \in \mathcal{Q}(\Phi_+)\}| \leq (n + 6)^n$. $\square$

REMARK 2.31

(1) Note that the bound given in Theorem 2.30 is very crude and is never sharp. A way to improve it would be to first count the number of possible 1-star-convex subtrees of I (whose cardinality is at most $2^{|I|}$). Then for a given such subtree I_1 of I , the constraints of Lemma 2.15, Lemma 2.18 and Proposition 2.29 should yield a much better upper bound on the number of quantum β satisfying $I_1(\beta) = I_1$. Moreover if

a generalized Cartan matrix has few -1 coefficients, then the associated root system will have few quantum roots, see Lemma 2.42 for example.

(2) Note that in the reductive case, in ADE type, the set of quantum roots is the entire set Φ_+ (see Proposition 2.40). By contrast, when Φ_+ is infinite, “almost all” the roots are not quantum, by the theorem above.

2.6. CLASSIFICATION OF QUANTUM ROOTS THROUGH DYNKIN SEQUENCES

In Sections 2.5.1 and 2.5.2 we have obtained finer structure results on quantum roots than those we used to prove the finiteness of $\mathcal{Q}(\Phi_+)$. In this section, we reformulate these results in terms of Dynkin sequences. We then obtain a complete classification of quantum roots through Dynkin sequences (see Theorem 2.39). This section is independent of Section 3.

2.6.1. Standard Dynkin diagrams. — Certain classes of Dynkin diagrams have standard names, which we now recall. Most of our notation follows the standard classification of Dynkin diagrams (see e.g., [Bou81]). In this section, Γ is a Dynkin diagram with set of edges E and weight function w . Recall that, if $A = (a_{i,j})_{i,j \in I}$ is a generalized Cartan matrix, then the associated Dynkin diagram is $\Gamma = (I, E, w)$ with

$$E = \{(i, j) \in I^2 \mid a_{i,j} < 0\} \quad \text{and} \quad w(i, j) = -a_{i,j} \text{ for all } (i, j) \in E.$$

We identify a Dynkin diagram Γ with its set of vertices.

Among Dynkin segments, we have the following classes:

(A) For $n \geq 1$, we say that Γ is a Dynkin diagram of type A_n if it is a segment with n vertices such that $w(i, j) = 1$ for all $(i, j) \in E$.

(C) For $n \geq 2$, we say that Γ is a Dynkin diagram of type C_n if it is a segment with a leaf j_0 such that $\Gamma \setminus \{j_0\}$ is a diagram of type A_{n-1} and, for j_1 the unique vertex adjacent to j_0 , $(w(j_0, j_1), w(j_1, j_0)) = (1, 2)$. We call j_0 the *special vertex* of Γ .

(F) For $n \geq 4$, we say that Γ is a Dynkin diagram of type F_n if it is a Dynkin segment with a leaf j_{-1} such that $\Gamma \setminus \{j_{-1}\}$ is a Dynkin diagram of type C_{n-1} and its special vertex j_0 is the unique neighbor of j_{-1} .

(G) For $n \geq 2$, we say that Γ is a Dynkin diagram of type G_n if it is a segment with a leaf j_0 such that $\Gamma \setminus \{j_0\}$ is a diagram of type A_{n-1} and, for j_1 the unique vertex adjacent to j_0 , $(w(j_0, j_1), w(j_1, j_0)) = (1, 3)$.

We also have classes of Dynkin trees which are not Dynkin segments, but Dynkin trees with a unique branching point:

(D) For $n \geq 4$, we say that Γ is a Dynkin diagram of type D_n if it has two distinct leaves j'_2, j''_2 sharing their unique neighbor j_1 , such that $\Gamma \setminus \{j'_2, j''_2\}$ is a Dynkin diagram of type A_{n-2} with leaf j_1 .

(E) For $n \geq 6$, we say that Γ is a Dynkin diagram of type E_n if it has a unique branching point j_1 with three neighbors, it is not of type D_n but there exists $j'_3 \in \Gamma$ such that $\Gamma \setminus \{j'_3\}$ is a Dynkin diagram of type D_{n-1} .

Note that the Dynkin diagrams associated to finite root systems are exactly the Dynkin diagrams of type $(A_n)_{n \geq 1}$, $(B_n)_{n \geq 2}$ (which is dual to type C), $(C_n)_{n \geq 2}$, $(D_n)_{n \geq 4}$ and the exceptional types E_6 , E_7 , E_8 , F_4 and G_2 .

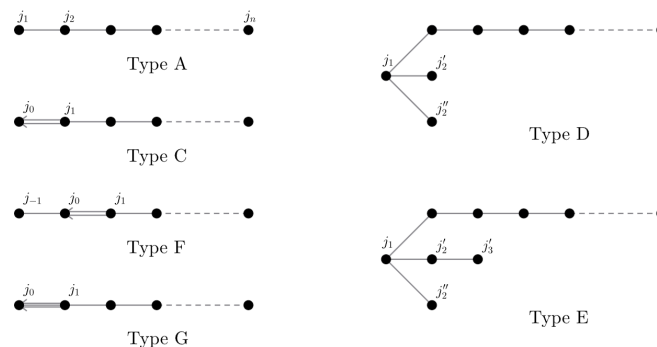


FIGURE 2.2. The different types of Dynkin diagrams

2.6.2. Dynkin sequences associated to quantum roots. — In this section, we fix a quantum root β . For $n \geq 2$ and $j \in I_2(\beta)$ such that $\deg(j) = 3$, let $I_n(\beta, j)$ denote the connected component of $I_n(\beta)$ containing j (if $j \notin I_n(\beta)$ then we set $I_n(\beta, j) = \emptyset$). We have the following classification for $(I_n(\beta, j))_{n \geq 2}$.

PROPOSITION 2.32. — *Let $j \in I_2(\beta)$ be such that $\deg(j) = 3$ and $|\text{Supp}(j)| = 2$. Then*

(2G) $I_2(\beta, j)$ is a Dynkin diagram of type G , j is the only leaf of $I_2(\beta, j)$ which is a leaf in $I_1(\beta)$. Moreover $I_3(\beta, j)$ is empty.

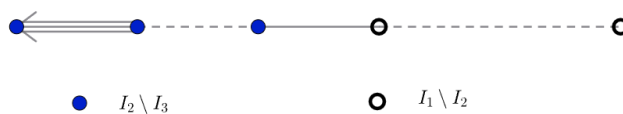


FIGURE 2.3. Dynkin diagram of type (2G)

Proof. — By Lemma 2.21, $j \notin I_3$, so $I_3(\beta, j)$ is empty. Moreover, since $|\text{Supp}(j)| = 2$, by Lemma 2.15 $I_2(\beta, j)$ is a Dynkin segment of type G , with j as special vertex. If $I_2(\beta, j) \neq \{j\}$, by Lemma 2.18(4), the other leaf of $I_2(\beta, j)$ has support of size three, hence is not a leaf in $I_1(\beta)$. $\square$

PROPOSITION 2.33. — *Let $j \in I_2(\beta)$ be such that $\deg(j) = 3$ and $|\text{Supp}(j)| = 3$. Then, one of the following is satisfied:*

(3S) $I_3(\beta, j) = \emptyset$ and $I_2(\beta, j) \setminus \{j\}$ is the disjoint union of two non-empty Dynkin diagrams of type A. Moreover the leaves of $I_2(\beta, j)$ are not leaves in $I_1(\beta)$.

(3C) $(I_n(\beta, j))_{n \geq 2}$ is a decreasing sequence of Dynkin diagrams of type C with leaf j . The leaves of $I_2(\beta, j)$ are not leaves of $I_1(\beta)$.

(3F) Let $j'_2 \in \text{Supp}(j)$ be such that $a_{j'_2, j} = -1$. Then j'_2 is a leaf of both $I_1(\beta)$ and $I_2(\beta, j)$, $I_2(\beta, j) \setminus \{j'_2\}$ is of type C, $I_3(\beta, j)$ is of type C and $I_4(\beta, j) = \emptyset$. The leaves of $I_3(\beta, j)$ (resp. $I_2(\beta, j) \setminus \{j'_2\}$) are not leaves in $I_2(\beta, j)$ (resp. $I_1(\beta)$).

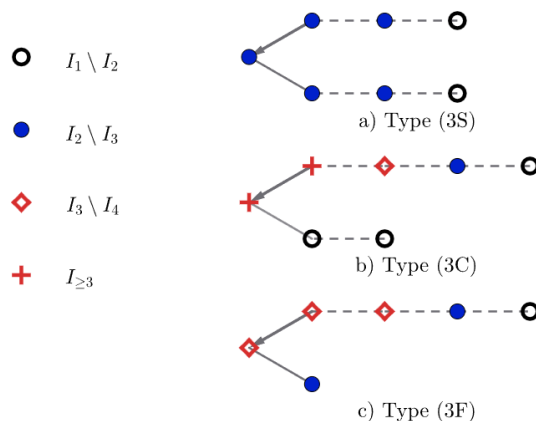


FIGURE 2.4. Dynkin sequences of type (3)

Proof. — This is a reformulation of Lemma 2.22. More precisely, write $\text{Supp}(j) = \{j, j_2, j'_2\}$ with $a_{j_2, j_1} = -2$ and $a_{j'_2, j_1} = -1$. By Proposition 2.16, $I_2(\beta, j) \setminus \{j\}$ is the disjoint union of two diagrams of type A. The case (1) of Lemma 2.22 therefore corresponds to type (3S), the case (2) corresponds to the type (3F), and the case (3) corresponds to type (3C). Moreover if $j_2 \in I_2(\beta, j)$, in every case we can apply Lemma 2.18 (4) & (5) to the chain starting with $j_1 \triangleleft_2 j_2$ to deduce that, for every $n \geq 2$, the leaves of $I_n(\beta, j)$ are not leaves of $I_{n-1}(\beta, j)$ (except j'_2 for $n = 2$ in type (3F), and j for $n \geq 3$ in type (3C)). $\square$

PROPOSITION 2.34. — *Let $j_1 \in I_2(\beta)$ be such that $\deg(j_1) = 3$ and $|\text{Supp}(j_1)| = 4$. Then the Dynkin sequence $(I_n(\beta, j_1))_{n \geq 2}$ is of one of the following forms.*

(4S) The diagram $I_2(\beta, j_1) \setminus \{j_1\}$ is the disjoint union of three non-empty diagrams of type A. Moreover the leaves of $I_2(\beta, j_1)$ are not leaves in $I_1(\beta)$, and $I_3(\beta, j_1) = \emptyset$.

(4A) The Dynkin sequence $(I_n(\beta, j_1))_{n \geq 2}$ is a decreasing sequence of diagrams of type A such that, for all $n \geq 2$, the leaves of $I_n(\beta, j_1)$ are not leaves of $I_{n-1}(\beta, j_1)$.

(4D) Let $K = N_{j_1}(\beta) = \max\{n \in \mathbb{Z}_{\geq 0} \mid j_1 \in I_K(\beta)\}$. Then $K \geq 5$ and there exist $j'_2, j''_2 \in \text{Supp}(j_1)$ such that:

- j''_2 is a leaf in $I_1(\beta)$ and j'_2 is a leaf of $I_2(\beta, j_1)$ but not of $I_1(\beta)$
- $(I_n(\beta, j_1))_{n \in \llbracket 1, K \rrbracket}$ is a decreasing sequence of diagrams. If K is even, then $I_n(\beta, j_1)$ is of type D for $n \in \llbracket 1, K/2 \rrbracket$ and $I_n(\beta, j_1)$ is of type A for $n \in \llbracket K/2 + 1, K \rrbracket$. If K is odd, there exists $\eta \in \{0, 1\}$ such that $I_n(\beta, j_1)$ is of type D for $n \in \llbracket 1, (K-1)/2 + \eta \rrbracket$ and $I_n(\beta, j_1)$ is of type A for $n \in \llbracket (K-1)/2 + \eta + 1, K \rrbracket$.
- j'_2 and j''_2 are leaves of the diagrams containing them. For $n \in \llbracket 1, K-1 \rrbracket$, if $\tilde{j}$ is a leaf of $I_n(\beta, j_1)$ and $\tilde{j} \notin \{j_1, j'_2, j''_2\}$, then $\tilde{j} \notin I_{n+1}(\beta, j_1)$.
- $I_{K-1}(\beta, j_1) \neq \{j_1\}$.

(4EA) $I_2(\beta, j_1)$ is a Dynkin diagram of type E, and there exist $j'_2, j''_2 \in \text{Supp}(j_1)$ such that:

- j''_2 is a leaf in $I_1(\beta)$ and does not lie in $I_3(\beta, j_1)$
- $\text{Supp}(j'_2) = \{j_1, j'_2, j'_3\}$ for some leaf j'_3 of both $I_1(\beta)$, $I_2(\beta)$ and $j'_3 \notin I_3(\beta, j_1)$
- $(I_n(\beta, j_1))_{n \in \{3, 4, 5\}}$ is a decreasing sequence of diagrams of type A, j'_2 is a leaf of $I_3(\beta, j_1)$ and $j'_2 \notin I_5(\beta, j_1)$
- $I_7(\beta, j_1) = \emptyset$ and, if $I_5(\beta, j_1) = \emptyset$ then $j'_2 \notin I_4(\beta, j_1)$
- For $n \geq 1$, if j is a leaf of $I_n(\beta, j_1)$ and $j \notin \{j_1, j'_2, j'_3, j''_2\}$ then $j \notin I_{n+1}(\beta)$.

(4ED) $I_2(\beta, j_1)$ is a Dynkin diagram of type E, and there exist $j'_2, j''_2 \in \text{Supp}(j_1)$ such that:

- $\text{Supp}(j'_2) = \{j_1, j'_2, j'_3\}$ for some leaf j'_3 of $I_1(\beta)$ and $j'_3 \notin I_3(\beta, j_1)$
- $I_3(\beta, j_1)$ is a diagram of type D, two of its leaves are j'_2, j''_2 and $j''_2 \notin I_4(\beta, j_1)$
- $I_4(\beta, j_1), I_5(\beta, j_1)$ are non-empty diagrams of type A, j_1 is a leaf of $I_5(\beta, j_1)$
- $I_7(\beta, j_1) = \emptyset$
- Except in the case (4ED2) below, for $n \geq 1$ if j is a leaf of $I_n(\beta, j_1)$ and $j \notin \{j_1, j'_2, j'_3, j''_2\}$ then $j \notin I_{n+1}(\beta, j_1)$.

Moreover, exactly one of the following holds:

(4ED1) $j'_2 \in I_4(\beta, j_1)$, $I_6(\beta, j_1) = \emptyset$.

(4ED2) $j'_2 \in I_4(\beta, j_1)$, $I_6(\beta, j_1) = \{j_1\}$ and there exists $n_0 \in \llbracket 2, 6 \rrbracket$ such that $|I_{n-1}(\beta, j_1) \setminus (I_n(\beta, j_1) \cup \{j'_2, j''_2, j'_3\})| = 1$ for $n \in \llbracket n_0 + 1, 5 \rrbracket$, and

$$I_{n_0-1}(\beta, j_1) \setminus \{j'_2, j''_2, j'_3\} = I_{n_0}(\beta, j_1) \setminus \{j'_2, j''_2, j'_3\}.$$

Moreover for $n < n_0$, the leaf of $I_n(\beta, j_1)$ not in $\{j_1, j'_2, j''_2, j'_3\}$ is not a leaf of $I_{n-1}(\beta, j_1)$. In particular if $n_0 = 2$ then β is the highest root of an E_8 -subsystem of Φ .

(4ED3) $j'_2 \notin I_4(\beta, j_1)$ and $I_5(\beta, j_1) \neq I_6(\beta, j_1)$.

(4SA) There is a unique $j_2'' \in \text{Supp}(j_1)$ which is a leaf of $I_2(\beta, j_1)$ and it is the unique leaf of $I_2(\beta, j_1)$ which is also a leaf of $I_1(\beta, j_1)$; $I_3(\beta, j_1)$ is a non-empty diagram of type A. Moreover exactly one of the following holds:

(4SA1) The vertex j_1 is not a leaf of $I_3(\beta, j_1)$. Then $I_4(\beta, j_1) = \emptyset$, and leaves of $I_3(\beta, j_1)$ are not leaves of $I_2(\beta, j_1)$.

(4SA2) The vertex j_1 is a leaf of $I_3(\beta, j_1)$. Then $I_4(\beta, j_1)$ is a diagram of type A, either empty or having j_1 as a leaf, and $I_5(\beta, j_1) = \emptyset$. Moreover j_1 is the unique leaf of $I_4(\beta, j_1)$ which is also a leaf of $I_3(\beta, j_1)$, and leaves of $I_3(\beta, j_1)$ are not leaves of $I_2(\beta, j_1)$.

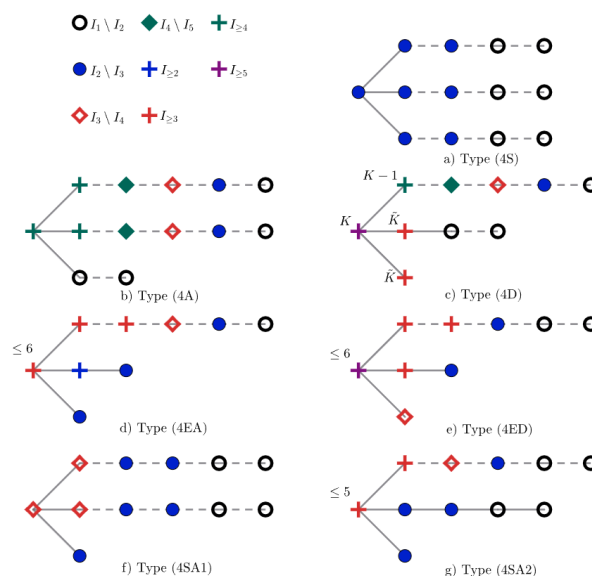


FIGURE 2.5. Dynkin sequences of type (4)

Proof. — Fix $j_1 \in I_2(\beta)$ such that $|\text{Supp}(j_1)| = 4$ and $\deg(j_1) = 3$. Then the connected components $(I_n(\beta, j_1))_{n \geq 2}$ of j_1 in $I_n(\beta)$ form a sequence of Dynkin diagrams ordered by inclusion. By Proposition 2.16, $I_2(\beta, j_1) \setminus \{j_1\}$ (and therefore by Lemma 2.18, $I_n(\beta, j_1) \setminus \{j_1\}$ for all $n \geq 2$) is a disjoint union of 3 diagrams of type A (possibly empty), each one starting at a distinct element of $\text{Supp}(j_1) \setminus \{j_1\}$. Moreover, Lemmas 2.24 to 2.28 give more information on their form:

(1) If $\text{Supp}(j_1) \not\subset I_2$, then by Lemma 2.24, the sequence of Dynkin diagrams $(I_n(\beta, j_1))_{n \geq 2}$ is of the form (4A).

(2) Otherwise, if $|\text{Supp}(j)| = 3$ for all $j \in \text{Supp}(j_1) \subset I_2$, by Lemma 2.25 it is of the form (4S). Moreover, by Lemma 2.18 4-6), the leaves of $I_2(\beta, j_1)$ are not leaves of $I_1(\beta)$.

(3) Otherwise, if there is $j \in \text{Supp}(j_1)$ such that $\text{Supp}(j) \not\subset I_2$, by Lemma 2.26, it is of the form (4D). Indeed, let $K = N_{j_1}(\beta)$. First assume that K is even. Let $\tilde{K} = K/2$. Then by (2.16), we have

$$t(j'_2, \tilde{K}) \leq t(j''_2, \tilde{K}) \leq t(j_1, 2\tilde{K}) \leq t(j'_2, \tilde{K} + 1) \leq t(j_1, 2\tilde{K} + 1) = \infty = t(j''_2, \tilde{K} + 1),$$

with $t(j_1, 2\tilde{K}) < \infty$.

We have $I_{\tilde{K}}(\beta, j_1) \supset \{j'_2, j''_2\}$ and $j'_2 \notin I_{\tilde{K}+1}(\beta, j_1)$. Therefore $I_{\tilde{K}}(\beta, j_1)$ is of type D and $I_{\tilde{K}+1}(\beta, j_1)$ is of type A . By Lemma 2.26, $j_2 \in I_{2\tilde{K}-1}(\beta, j_1) \neq \{j_1\}$.

Assume now that K is odd. Set $\tilde{K} = (K-1)/2$. Then by (2.16), we have

$$t(j'_2, \tilde{K}) \leq t(j'_2, \tilde{K} + 1) \leq t(j_1, 2\tilde{K} + 1) \leq t(j'_2, \tilde{K} + 1) \leq t(j_1, 2\tilde{K} + 2) = \infty,$$

$$t(j_1, 2\tilde{K} + 1) < \infty \quad \text{and} \quad \infty = t(j_1, 2\tilde{K} + 2) = t(j'_2, \tilde{K} + 2) = t(j''_2, \tilde{K} + 2).$$

Therefore $j'_2 \in I_{\tilde{K}+1}(\beta, j_1)$, $j'_2, j''_2 \notin I_{\tilde{K}+2}(\beta, j_1)$ and thus $I_{\tilde{K}+1}(\beta, j_1)$ is of type D or A and $I_{\tilde{K}+2}(\beta, j_1)$ is of type A . By Lemma 2.24, $j_2 \in I_{2\tilde{K}}(\beta, j_1) \neq \{j_1\}$.

(4) Otherwise, we can apply Lemma 2.27. If $j_1 \notin I_4(\beta, j_1)$ we are in the case (1) of Lemma 2.27 and we obtain the form (4SA1); otherwise, if $j'_2 \notin I_3(\beta, j_1)$ we are in the cases (1) or (2) of Lemma 2.27 and we obtain the form (4SA2).

Finally case (3) of Lemma 2.27 corresponds to the forms (4EA) or (4ED), according to whether j'_2 lies in I_3 . We obtain the more specific forms (4ED1) to (4ED3) through a more careful study in these cases, as it is done in the proofs of Lemma 2.27 and Lemma 2.28; we now develop these cases.

Recall the notation of Lemma 2.27 for j'_3 . Assume, for a contradiction, that $j_1 \notin I_5(\beta, j_1)$ and $j'_2 \in I_4(\beta, j_1)$. Then as we are in the case of Lemma 2.27(3), $j'_3 \notin I_3(\beta, j_1)$. But then if $t = t(j'_2, 4)$, we have $\alpha_{j'_2}(\gamma_t^\vee) = 6 - N_{j_1}(\gamma_t^\vee) - N_{j'_3}(\gamma_t^\vee) = -1$, with $N_{j'_3}(\gamma_t^\vee) \leq 2$ and thus $N_{j_1}(\gamma_t^\vee) \geq 5$: $j_1 \in I_5$, a contradiction. Therefore if $j_1 \notin I_5$, we have $j_2 \notin I_4(\beta, j_1)$. Moreover by Lemma 2.20, we have $j'_2 \notin I_3(\beta, j_1)$. Thus if $j_1 \notin I_5$, we have $j'_2 \notin I_4(\beta, j_1)$, $j'_3 \notin I_3(\beta, j_1)$, $j'_2 \notin I_3(\beta, j_1)$ so the sequence is of the form (4EA). Otherwise, $j_1 \in I_5(\beta, j_1)$ and by Formula (2.18), there is a time t such that

$$(N_{j_1}(t), N_{j_2}(t), N_{j'_2}(t), N_{j'_3}(t)) = (5, 4, 3, 2).$$

We can also suppose that $N_{j'_3}(t) = 2$. Indeed, $j'_3 \in I_2$ so $t(j'_3, 2)$ is finite. Since $\text{Supp}(j'_3) = \{j'_2, j'_3\}$, by Lemma 2.20 $t(j'_2, 3) < t(j'_3, 2) < t(j'_2, 4)$, $r_{j'_3}$ commutes with every other reflection applied between $t(j'_2, 3)$ and $t(j'_2, 4)$, so we can suppose that $t(j'_3, 2) = t(j'_2, 3) + 1$.

– If $N_{j'_2}$ no longer increases, the sequence is of the form (4EA).

– Otherwise, by Lemma 2.20, $N_{j'_2}$ increases before N_{j_1} , and if $N_{j'_2}$ does not increase the sequence is of the form (4ED3). Indeed, by Lemma 2.27(3), $t(j_2, n-1) \leq t(j_1, n)$ for $n \in \llbracket 2, 5 \rrbracket$ and it is also satisfied for $n \geq 7$, since $t(j_1, 7) = \infty$. If $t := t(j_1, 6)$ is finite, since $N_{j'_2}(t) + N_{j'_3}(t) \leq 6$ necessarily $N_{j_2}(t) \geq 5$ hence $t(j_2, 5) \leq t(j_1, 6)$. Therefore by Lemma 2.18(6), the corresponding sequence $(k_n)_{n \geq 2}$ is decreasing, in particular $I_5(\beta, j_1) \neq I_6(\beta, j_1)$.

– Otherwise, if $N_{j'_2}$ increases and $j_1 \notin I_6$, we obtain the form (4ED1).

– The final case is when $N_{j'_2}$ increases and $j_1 \in I_6(\beta, j_1)$, then there exists a time t such that $(N_{j_1}(t), N_{j_2}(t), N_{j'_2}(t), N_{j''_2}(t)) = (6, 4, 4, 3)$. Following the analysis done in Lemma 2.28, we obtain the form (4ED2): if N_{j_2} stays at 4, it is of the form (4ED2) with $n_0 = 6$ (since $j'_3 \notin I_3$ and $\max N_{j_1} = 6$, we have $j'_2 \notin I_5$).

– Otherwise (using the same reasoning as in the proof of Lemma 2.28 to obtain Formula (2.20)) there is a time t such that

$$(N_{j_1}(t), N_{j_2}(t), N_{j_3}(t), N_{j_4}(t)) \in \{(6, 5, 3, 2), (6, 5, 3, 3)\}.$$

If $j_3 \notin I_4(\beta, j_1)$ we obtain the form (4ED2) with $n_0 = 5$; otherwise there is a time for which Equality (2.21) holds, and the argument can be repeated. This process necessarily ends with $n_0 = 2$, in which case $j_5 \in I_2(\beta, j_1) \setminus I_3(\beta, j_1)$ has support of size 2, and the root is the highest root of E_8 . $\square$

2.6.3. General classification of quantum roots

DEFINITION 2.35. — Let I be a Dynkin diagram, and let $(I_n)_{n \in \mathbb{Z}_{\geq 1}}$ be a sequence of Dynkin subdiagrams. We say that $(I_n)_{n \in \mathbb{Z}_{\geq 1}}$ is a quantum Dynkin sequence if it is the Dynkin sequence of a quantum root. That is to say, $(I_n)_{n \in \mathbb{Z}_{\geq 1}}$ is a quantum Dynkin sequence if and only if there exists $\beta \in \mathcal{Q}(\Phi_+)$ such that $\beta^\vee = \sum_{n \geq 1} \sum_{i \in I_n} \alpha_i^\vee$.

In this section, we obtain a classification of quantum Dynkin sequences, and therefore a classification of quantum roots.

DEFINITION 2.36. — Let I be a Dynkin diagram and let $(I_n)_{n \in \mathbb{Z}_{\geq 1}}$ and $(I'_n)_{n \in \mathbb{Z}_{\geq 1}}$ be two Dynkin sequences of I . We say that (I_n) and (I'_n) are mergeable if $I_1 = I'_1$ and if I_2, I'_2 are disconnected. That is to say, if $I_2 \cap I'_2 = \emptyset$ and if there is no edge (i, i') such that $i \in I_2, i' \in I'_2$.

LEMMA 2.37. — Suppose that $(I_n)_{n \in \mathbb{Z}_{\geq 1}}$ and $(I'_n)_{n \in \mathbb{Z}_{\geq 1}}$ are two quantum Dynkin sequences. Then, if $(I_n)_{n \in \mathbb{Z}_{\geq 1}}$ and $(I'_n)_{n \in \mathbb{Z}_{\geq 1}}$ are mergeable, $(I_n \cup I'_n)_{n \in \mathbb{Z}_{\geq 1}}$ is also quantum.

Proof. — Let $\beta \in \mathcal{Q}(\Phi_+)$ (resp. $\beta' \in \mathcal{Q}(\Phi_+)$) be the quantum root such that $(I_n)_{n \in \mathbb{Z}_{\geq 1}} = (I_n(\beta))_{n \in \mathbb{Z}_{\geq 1}}$ (resp. $(I'_n)_{n \in \mathbb{Z}_{\geq 1}} = (I_n(\beta'))_{n \in \mathbb{Z}_{\geq 1}}$), which exists since, by assumption, the two sequences are quantum. Since the two sequences are also mergeable, we have $I_1 = I'_1$ and this is a 1-star-convex tree by Proposition 2.14.

Let $\beta_0 \in \mathcal{Q}(\Phi_+)$ be the quantum root such that $I_1(\beta_0) = I_1$ and $I_2(\beta_0) = \emptyset$, which is given by Proposition 2.14. Note that, since $I_1 = I'_1$ and I_n, I'_n are disjoint for $n \geq 2$, we have $\sum_{n \geq 1} \sum_{i \in I_n \cup I'_n} \alpha_i^\vee = \beta^\vee + (\beta')^\vee - \beta_0^\vee$. It therefore suffices to prove that this is the coroot associated to a quantum root.

By Lemma 2.12, we can write $\beta = w(\beta_0)$ (resp. $\beta' = w'(\beta_0)$), where

$$w \in \langle r_i \mid i \in I_2 \rangle \subset W^v \quad (\text{resp. } w' \in \langle r_i \mid i \in I'_2 \rangle \subset W^v) \quad \text{is minimal.}$$

Fix a reduced decomposition $r_{i_1} \dots r_{i_n}$ of w' . Note that for all $k \in \llbracket 1, n \rrbracket$, $r_{i_k} \dots r_{i_n}(\beta_0)$ is quantum with coroot $\beta_0^\vee + \sum_{j=k}^n \alpha_{i_j}^\vee$ (in particular $(\beta')^\vee = \beta_0^\vee + \sum_{j=1}^n \alpha_{i_j}^\vee$). Since I_2

and I'_2 are disconnected, we have $w^{-1}(\alpha_i) = \alpha_i$ for all $i \in \{i_1, \dots, i_n\} \subset I'_2$. Therefore, by applying Lemma 2.38 to the triple (β_0, i_n, w) , $r_{i_n}w(\beta_0)$ is quantum with coroot $\beta^\vee + \alpha_{i_n}^\vee$. Since $r_{i_n}w(\beta_0) = wr_{i_n}(\beta_0)$ we can iterate the application of Lemma 2.38 to the triple $(r_{i_n}(\beta_0), i_{n-1}, w)$ and we obtain that $r_{i_{n-1}}r_{i_n}w(\beta_0)$ is quantum with coroot $\beta^\vee + \alpha_{i_{n-1}}^\vee + \alpha_{i_n}^\vee$. After n successive applications of Lemma 2.38, we obtain that $w'w(\beta_0) = r_{i_1} \dots r_{i_n}w(\beta_0)$ is quantum with coroot $\beta^\vee + \sum_{j=1}^n \alpha_{i_j}^\vee = \beta^\vee + (\beta')^\vee - \beta_0^\vee$. Therefore $(I_n \cup I'_n)_{n \in \mathbb{Z}_{\geq 1}} = (I_n(w'w(\beta_0)))_{n \in \mathbb{Z}_{\geq 1}}$ is a quantum Dynkin sequence. $\square$

LEMMA 2.38. — *Suppose that $\beta \in \Phi_+$, $i \in I$ and $w \in W^v$ are such that $w^{-1}(\alpha_i) = \alpha_i$. Then if β , $r_i(\beta)$, $w(\beta)$ are quantum, $r_iw(\beta)$ is also quantum. Moreover $r_iw(\beta^\vee) = w(\beta^\vee) + r_i(\beta^\vee) - \beta^\vee$.*

Proof. — Suppose that β and $w(\beta)$ are quantum. By Proposition 2.11, $r_iw(\beta)$ is quantum if and only if $|\langle w(\beta^\vee), \alpha_i \rangle| = |\langle \beta^\vee, w^{-1}(\alpha_i) \rangle| \leq 1$. However, $w^{-1}(\alpha_i) = \alpha_i$. Therefore this is satisfied if and only if $|\langle \beta^\vee, \alpha_i \rangle| \leq 1$ that is to say, by Proposition 2.11, if and only if $r_i(\beta)$ is quantum.

Moreover since $r_i(\beta^\vee) - \beta^\vee = -\langle \beta^\vee, \alpha_i \rangle \alpha_i^\vee$, we check that

$$r_iw(\beta^\vee) = w(\beta^\vee) - \langle w(\beta^\vee), \alpha_i \rangle \alpha_i^\vee = w(\beta^\vee) + r_i(\beta^\vee) - \beta^\vee. \quad \square$$

THEOREM 2.39. — *Let I be a Dynkin diagram with associated root system Φ , and let $(I_n)_{n \in \mathbb{Z}_{\geq 1}}$ be a sequence of Dynkin subdiagrams of I . Then there exists a quantum root $\beta \in \mathcal{Q}(\Phi_+)$ such that $(I_n)_{n \geq 1} = (I_n(\beta))_{n \geq 1}$ (see Formula (2.2)) if and only if the sequence $(I_n)_{n \in \mathbb{Z}_{\geq 1}}$ satisfies the following properties:*

- (1) *For any $n \in \mathbb{Z}_{\geq 1}$, I_{n+1} is a subdiagram of I_n .*
- (2) *The diagram I_1 is a 1-star-convex Dynkin tree.*
- (3) *For any $n \geq 2$, for any vertex $j \in I_n$ we have $\deg(j) \leq 3$ (see Formula (2.9)).*

Moreover each connected component of I_n admits a unique vertex of degree 3.

- (4) *For any vertex $j \in I_2$ such that $\deg(j) = 3$, let $I_n(j)$ denote the corresponding connected component (with $I_n(j) = \emptyset$ if $j \notin I_n$), given by point (3). Then:*

- (a) *If $|\text{Supp}(j)| = 2$, then $(I_n(j))_{n \geq 2}$ is a sequence of type (2G) (see Proposition 2.32).*
- (b) *If $|\text{Supp}(j)| = 3$, then $(I_n(j))_{n \geq 2}$ is a sequence of type (3S), (3C) or (3F) (see Proposition 2.33).*
- (c) *If $|\text{Supp}(j)| = 4$, then $(I_n(j))_{n \geq 2}$ is a sequence of type (4S), (4A), (4D), (4EA), (4ED) or (4SA) (see Proposition 2.34).*

Proof. — Let $\beta \in \mathcal{Q}(\Phi_+)$. Then by the analysis done in the preceding sections, $(I_n(\beta))_{n \geq 1}$ satisfies properties (1) to (4): this is Proposition 2.14 for point (2), Proposition 2.16 for point (3) and Propositions 2.32, 2.33 and 2.34 for point (4). Therefore it remains to associate a quantum root β to each Dynkin sequence sequence satisfying the properties (4).

Let $(I_n)_{n \geq 1}$ be a quantum Dynkin sequence. Let $J_3 := \{j \in I_2 \mid \deg(j) = 3\}$. For any $j \in J_3$ and $n \geq 1$, as in Definition 2.35 let $I_n(j)$ denote the connected component of j in I_n , which is of one of the types given by Definition 2.35(4), and either $I_n(j) = \emptyset$,

or $I_n(j) \cap J_3 = \{j\}$. We have $I_1(j) = I_1$ for every $j \in J_3$. The graphs $I_2(j)$ and $I_2(j')$ are disconnected for $j, j' \in J_3$ such that $j \neq j'$, and thus $(\bigcup_{j \in E_1} I_n(j))_{n \in \mathbb{Z}_{\geq 1}}$ and $(\bigcup_{j \in E_2} I_n(j))_{n \in \mathbb{Z}_{\geq 1}}$ are mergeable for all subsets E_1, E_2 of J_3 such that $E_1 \cap E_2 = \emptyset$. Therefore using Lemma 2.37, we can assume $|J_3| \leq 1$. We have already proved the case where J_3 is empty in Proposition 2.14.

We now assume that J_3 is a singleton $\{j\}$. We have $I_n(j) = I_n$ for all $n \in \mathbb{Z}_{\geq 1}$.

By Proposition 2.14 there exists $\beta_\emptyset \in \mathcal{Q}(\Phi_+)$ such that

$$I_1(\beta_\emptyset) = I_1 \quad I_n(\beta_\emptyset) = \emptyset \text{ for } n \geq 2.$$

Note that, since $(I_n)_{n \geq 2}$ is among the types given in Propositions 2.32, 2.33 and 2.34, it has the same properties as the connected component $\mathcal{C}$ in Proposition 2.16 and Remark 2.19. That is to say, with $m = |I_2 \cap \text{Supp}(j)| - 1$, $I_2 \setminus \{j\}$ is the disjoint union of m non-empty Dynkin segments of type A , one for each neighbor of j in I_1 . As in Proposition 2.16, let us write $(I_2 \cap \text{Supp}(j)) \setminus \{j\} = \{j_2^{(i)} \mid i \in \llbracket 1, m \rrbracket\}$ and, for all $i \in \llbracket 1, m \rrbracket$ write $j_2^{(i)}, j_3^{(i)}, \dots, j_{k(2,i)}^{(i)}$ the ordered elements of the segment starting at $j_2^{(i)}$. Therefore, $k(2, i) - 1$ is the length of the segment of type A starting at $j_2^{(i)}$ in $I_2 \setminus \{j\}$. Upon relabeling, we may suppose $k(2, i) \geq k(2, i+1)$ for all $i \in \llbracket 1, m-1 \rrbracket$. Moreover by Remark 2.19, $[j_2^{(i)}, j_{k(2,i)}^{(i)}] \cap I_n = [j_2^{(i)}, j_{k(n,i)}^{(i)}]$ for some non-increasing sequence $(k(n, i))_{n \geq 2}$ (with the convention that $k(n, i) = 1$ if $j_2^{(i)} \notin I_n$, and $[j_2^{(i)}, j_1^{(i)}] = \emptyset$).

Finally, for each $i \in \llbracket 1, m \rrbracket$, $n \geq 2$, let us define the following element of W^v :

$$(2.23) \quad w_n^{(i)} = \begin{cases} r_{j_{k(n,i)}^{(i)}} \dots r_{j_2^{(i)}} & \text{if } k(n, i) \geq 2, \\ 1_{W^v} & \text{otherwise,} \end{cases} \quad \text{and} \quad w_n = \left(\prod_{i=1}^m w_n^{(i)} \right) r_j$$

With this notation, we can now construct β from $\beta_\emptyset$.

Suppose for now that $(I_n)_{n \geq 2}$ is of the form (2G), (3S), (3C), (4S) or (4A). In all these cases, $j_{k(n,i)}^{(i)}$ is not a leaf of I_{n-1} , so the sequences $(k(n, i))_{n \geq 2}$ are decreasing until they reach 1, and $|\text{Supp}(j_{k(2,i)}^{(i)})| = 3$.

By Proposition 2.11, as $\deg(j) = 3$, $r_j(\beta_\emptyset)$ is quantum, and $I_2(r_j(\beta_\emptyset)) = \{j\}$. If $m = 0$, $\text{Supp}(j) \cap I_2 = \{j\}$, so we are done. Otherwise, recursively applying Proposition 2.11 $\ell(w_2)$ times, we similarly check that $w_2(\beta_\emptyset)$ is quantum (we crucially use that $|\text{Supp}(j_{k(2,i)}^{(i)})| = 3$ when we apply $r_{j_{k(2,i)}^{(i)}}$). Indeed, for any $i \in \llbracket 1, m \rrbracket$ and $p \in \llbracket 2, k(2, i) \rrbracket$, at time $t := t(j_p^{(i)}, 2)$, we have $(N_{j_{p-1}^{(i)}}(t), N_{j_p^{(i)}}(t), N_{j_{p+1}^{(i)}}(t)) = (2, 1, 1)$ therefore we can apply Proposition 2.11. Moreover

$$I_2(w_2(\beta_\emptyset)) = I_2 \quad \text{and} \quad I_3(w_2(\beta_\emptyset)) = \emptyset.$$

We iterate the process: if $j \in I_n$ we apply w_n to $w_{n-1} \dots w_2(\beta_\emptyset)$, which is quantum, and we obtain a quantum root (we need to use that $k(n, i) < k(n-1, i)$ if $w_n^{(i)} \neq 1_{W^v}$). When we reach n_0 such that $I_{n_0+1} = \emptyset$ and $I_{n_0} \neq \emptyset$, we stop the process, and $\beta := w_{n_0} \dots w_2(\beta_\emptyset)$ is a quantum root such that $I_n(\beta) = I_n$ for all $n \geq 2$.

It remains to deal with the forms (3F), (4D), (4EA), (4ED), (4SA). Let us start with the form (4D); it is the only one among these for which I_n may be non-empty

for arbitrarily large n . Let $K = \max\{n \in \mathbb{Z} \mid I_n \neq \emptyset\}$. Let $\tilde{K} = \lfloor (K-1)/2 \rfloor$. In this case $m = 3$ but $k(2, 2) = k(2, 3) = 2$, and upon relabeling we can suppose that $j_2^{(3)}$ is a leaf of I_1 , but $j_2^{(2)}$ is not.

By construction, the root $\beta^0 := \beta_\emptyset$ is quantum and has the property (H_0) :

$$(H_0) \quad I_1(\beta^0) = I_1, \quad I_2(\beta^0) = \emptyset.$$

For $p \in \llbracket 1, \tilde{K} - 1 \rrbracket$, we say that a root β^p has the property (H_p) if:

$$(H_p) \quad N_{j_k^{(1)}}(\beta^p) = N_j(\beta^p) = 2p + 1 \text{ for } k \in \llbracket 1, k(2p+1, 1) \rrbracket, N_{j_{k(2p+1, 1)+1}^{(1)}}(\beta^p) = 2p \\ \text{and } N_{j_2^{(i)}}(\beta^p) = p + 1 \text{ for } i \in \{2, 3\}.$$

Suppose that $p \in \llbracket 0, \tilde{K} - 1 \rrbracket$ is such that β^p is constructed, quantum and satisfies (H_p) .

Then, using the fact that $(k(n, 1))$ is decreasing and applying Proposition 2.11 we check that

$$\beta^{p+1} := r_{j_2^{(3)}} w_{2p+3}^{(1)} r_j r_{j_2^{(2)}} w_{2p+2}^{(1)} r_j(\beta^p)$$

is a quantum root which satisfies (H_{p+1}) . We obtain $\beta^{\tilde{K}}$ satisfying $(H_{\tilde{K}})$. Then if K is odd we are done. Otherwise, we set $\beta = r_j(\beta^{\tilde{K}})$, which concludes the case (4D).

In the remaining cases, $I_n(j) = \emptyset$ for all $n \geq 7$, so they can be checked by hand in a similar fashion, following the arguments in the proofs of Lemma 2.22 (for the case (3F)) or Lemma 2.27 (for the cases (4EA), (4ED), (4SA)). For completeness, we now carry out these verifications explicitly.

In the case (3F), $m = 2$ and upon relabeling, suppose that

$$(a_{j_2^{(1)}, j}, a_{j_2^{(2)}, j}) = (-2, -1).$$

Then we check that $\beta := r_{j_2^{(2)}} w_3^{(1)} r_j w_2^{(1)} r_j(\beta_\emptyset)$ is a quantum root such that $I_n(\beta) = I_n$ for all $n \geq 2$.

For the case (4SA), $m = 3$ and denote by $j_2^{(3)}$ the unique leaf of I_2 in $\text{Supp}(j)$; by definition, it is also a leaf of I_1 . In the case (4SA2), since j is a leaf of I_3 , there is another $j_2^{(2)} \in \text{Supp}(j)$ which does not belong to I_3 . We set $w = r_{j_2^{(3)}} w_3^{(2)} w_3^{(1)} r_j w_2^{(2)} w_2^{(1)} r_j$. Then, in the case (4SA1), $\beta := w(\beta_\emptyset)$ is the required quantum root. In the case (4SA2), we instead set $\beta := w_4^{(1)} r_j w(\beta_\emptyset)$ (note that $w_3^{(2)} = 1_{W^\vee}$ hence $\langle w(\beta_\emptyset^\vee), \alpha_j \rangle = -1$ as required to apply Proposition 2.11).

It remains to deal with the cases (4EA) and (4ED). Again $m = 3$ and by definition we can suppose that $j_2^{(3)}, j_3^{(2)}$ are leaves of I_1 (they correspond to j_2'' and j_3' respectively, with the notation of Proposition 2.34). Suppose that $(I_n)_{n \geq 2}$ is not of the form (4ED2).

Let

$$\beta^1 := r_{j_3^{(2)}} r_{j_2^{(2)}} w_4^{(1)} r_j r_{j_2^{(3)}} w_3^{(1)} r_j r_{j_2^{(2)}} w_2^{(1)} r_j(\beta_\emptyset),$$

then by Proposition 2.11, β^1 is a quantum root and satisfies

$$(N_j, N_{j_2^{(1)}}, N_{j_2^{(2)}}, N_{j_2^{(3)}}, N_{j_3^{(2)}}) = (4, 4, 3, 2, 2).$$

If $(I_n)_{n \geq 2}$ is of the form (4EA) and $j \notin I_5$, then we set $\beta := \beta^1$. If $(I_n)_{n \geq 2}$ is of the form (4EA) but $j \in I_5$, we instead set $\beta := w_5^{(1)} r_j(\beta^1)$. It remains to treat the cases (4ED).

Let $\beta^2 := w_4^{(2)} r_{j_2^{(3)}} w_5^{(1)} r_j(\beta^1)$. If $j \notin I_5$ (hence in the case (4ED1) or a particular case of (4ED3)), we set $\beta := \beta^2$. Otherwise, $j \in I_5$ and we set $\beta := r_j(\beta^2)$ (note that this is indeed a quantum root by Proposition 2.11 because either $w_4^{(2)} = 1_{W^v}$ either or $w_5^{(1)} = 1_{W^v}$).

The only remaining case is the form (4ED2). It corresponds to the analysis done in the proof of Lemma 2.28, and we construct β following this proof: Recall the notation for $n_0 \in \llbracket 2, 6 \rrbracket$, such that the leaf $j_{k(n_0,1)}^{(1)}$ of I_{n_0} is also a leaf of I_{n_0-1} , note that $k(n_0, 1) = 7 - n_0$. Let us start with β_0 such that $(I_n(\beta_0))_{n \geq 0}$ is of the form (4ED1), with $I_k(\beta_0) = I_k$ for $k \leq n_0 - 1$ and $I_k(\beta_0) \setminus \{j_2^{(2)}, j_3^{(2)}, j_2^{(3)}\} = [j, j_{6-k}^{(1)}]$ for $k \in \llbracket n_0, 6 \rrbracket$ (in particular $I_6(\beta_0) = \emptyset$ and $I_5(\beta_0) = \{j\}$). Then $\alpha_j(\beta_0^\vee) = -1$, so $r_j(\beta_0)$ is quantum with Dynkin sequence of the form (4ED2). If $n_0 = 6$, we can set $\beta := r_j(\beta_0)$. Otherwise, we check that $\beta := r_{j_{7-n_0}^{(1)}} \dots r_{j_2^{(1)}} r_j(\beta_0)$ is a quantum root with Dynkin sequence $(I_n)_{n \geq 1}$.

This concludes the proof of Theorem 2.39. $\square$

2.7. EXAMPLES OF SETS OF QUANTUM ROOTS. — As shown by Lemma 3.2 and Proposition 3.13, quantum roots naturally appear in the study of covers for the affinized Bruhat order. In this subsection, we determine the sets of quantum roots in some particular cases (type A_n , $A_n^{(1)}$, $D_n^{(1)}$, etc.). Besides giving examples, we want to illustrate that the methods developed in this section to prove Theorem 2.30 are constructive and make it possible to determine the set of quantum roots in particular examples.

Let $\beta \in \Phi_+$. Write $\beta = r_{i_L} \dots r_{i_2}(\alpha_{i_1})$, with L minimal, $i_1, \dots, i_L \in I$. For $\ell \in \mathbb{Z}_{\geq 0}$, set

$$I_\ell(\beta) = \{j \in I \mid |\{t \in \llbracket 1, L \rrbracket \mid i_t = j\}| \geq \ell\}.$$

2.7.1. Finite type. — The following result was explained to us by Dinakar Muthiah.

PROPOSITION 2.40. — *Assume that the root system Φ is of type A, D or E. Then the set of quantum roots of Φ is Φ_+ .*

Proof. — By [Bou81, Pl. I to X], we can embed Φ in $\mathbb{R}^n$, for some $n \in \mathbb{N}$, equipped with the canonical scalar product $(\cdot | \cdot)$ and Φ is a subset of elements of norm $\sqrt{2}$. Let $\alpha \in \Phi$. We have $\alpha^\vee = 2\alpha/(\alpha|\alpha) = \alpha$ for every $\alpha \in \Phi$. Thus if $\alpha, \beta \in \Phi$, we have $\langle \alpha^\vee, \beta \rangle = (\alpha|\beta) \in \mathbb{Z}$ and $|(\alpha|\beta)| \leq |\alpha||\beta| \leq 2$, so $\langle \alpha^\vee, \beta \rangle \in \llbracket -2, 2 \rrbracket$. Now if α and β are not proportional, we have $|(\alpha|\beta)| < |\alpha||\beta|$ and thus $\langle \alpha^\vee, \beta \rangle \in \llbracket -1, 1 \rrbracket$. For every $\alpha \in \text{Inv}(s_\beta)$, we have $\langle \alpha^\vee, \beta \rangle > 0$ (see [Phi24, Cor. 1.10] for example) and thus $\langle \alpha^\vee, \beta \rangle = 1$ for all $\alpha \in \text{Inv}(s_\beta)$, which proves that β is quantum. $\square$

REMARK 2.41. — Proposition 2.40 can also be proved using Theorem 2.39: Dynkin diagrams of type ADE are trees which are 1-star-convex at any given vertex. Fix a root system Φ of type ADE, and $\beta \in \Phi_+$. Then the Dynkin diagram $I_1(\beta)$ is of type

ADE as well. If it is of type A, it has no vertex of degree 3 or larger, so $I_2(\beta) = \emptyset$. Hence $(I_n(\beta))_{n \geq 1}$ is quantum and $\beta \in \mathcal{Q}(\Phi_+)$. If $I_1(\beta)$ is of type D or E, it admits a unique vertex j of degree 3 and no vertex of higher degree, so $I_2(\beta)$ is non-empty if and only if it contains j . We check that if this is the case, $(I_n(\beta))_{n \geq 1}$ must be of the form (4A), with $I_3(\beta) = \emptyset$, when $I_1(\beta)$ is of type D; or of the forms (4A), (4D), (4EA), (4ED) when $I_1(\beta)$ is of type E.

However, for every other indecomposable Cartan matrix (of finite type), there exists a positive root which is not quantum. Indeed, there exist $i, j \in I$ such that $\langle \alpha_i^\vee, \alpha_j \rangle \leq -2$. Set $\beta = r_j(\alpha_i)$. Then $\alpha_i^\vee \in \text{Inv}(s_\beta)$ and $\langle \beta^\vee, \alpha_j \rangle = -\langle \alpha_i^\vee, \alpha_j \rangle \geq 2$ and thus $r_j(\alpha_i)$ is not quantum. The set of quantum roots can in fact be explicitly described using the notions of long and short roots [LNS⁺15, Lem. 4.2].

Note that when Φ is infinite, then Φ_+ strictly contains the set of quantum roots, which is finite.

2.7.2. Case where the generalized Cartan matrix has no coefficient -1

LEMMA 2.42. — Let $\beta \in \Phi_+$ be a quantum root. Write $\beta = r_{i_L} \dots r_{i_2}(\alpha_{i_1})$, with $i_1, \dots, i_L \in I$ and $L \in \mathbb{N}$ minimal. Then we have $a_{i_1, i_2} = -1$ (where $A = (a_{i,j})_{i,j \in I}$ is the generalized Cartan matrix of the Kac–Moody datum). In particular, if $a_{i,j} \neq -1$ for every $i, j \in I$, the set of quantum roots is exactly the set of simple roots.

Proof. — We have $s_\beta = r_{i_L} \dots r_{i_2} r_{i_1} r_{i_2} \dots r_{i_L}$. We have $r_{i_L} \dots r_{i_3} \cdot \alpha_{i_2} \in \text{Inv}(s_\beta) \setminus \{\beta\}$ and

$$\langle r_{i_L} \dots r_{i_2}(\alpha_{i_1}^\vee), r_{i_L} \dots r_{i_3}(\alpha_{i_2}) \rangle = -\langle \alpha_{i_1}^\vee, \alpha_{i_2} \rangle = -a_{i_1, i_2} = 1. \quad \square$$

2.7.3. Type $A_n^{(1)}$, $n \geq 2$

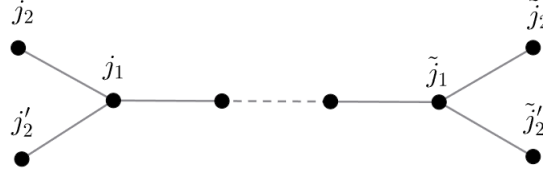
PROPOSITION 2.43. — We assume that the generalized Cartan matrix A is of type $A_n^{(1)}$, with $n \in \mathbb{Z}_{\geq 2}$ (see [Kac90, Ch. 4] for the classification). We regard I as a diagram as in Section 2.1 and we denote by $\text{Seg}(I)$ the set of non-empty subsets of I which are a segment when regarded as a subdiagram of I . Then $\mathcal{Q}(\Phi_+) = \{\sum_{i \in J} \alpha_i \mid J \in \text{Seg}(I)\}$.

Proof. — Let $\beta \in \mathcal{Q}(\Phi_+)$. We have $\deg(j) = 2$ for every $j \in I$ (we defined the degree in (2.9)). By Lemma 2.15, we deduce that $I_2(\beta) = \emptyset$. Therefore by Proposition 2.14, $\{\beta^\vee \mid \beta \in \mathcal{Q}(\Phi_+)\} = \{\sum_{j \in J} \alpha_j^\vee \mid J \in \text{Seg}(I)\}$. As the generalized Cartan matrix only contains 0 or -1 coefficients off the diagonal, we have $\beta^\vee = \sum_{i \in I} n_i \alpha_i^\vee$, for all $\beta = \sum_{i \in I} n_i \alpha_i \in \Phi$, and the proposition follows. $\square$

Note that the case $A_1^{(1)}$ is a particular case of Section 2.7.2.

2.7.4. Type $D_n^{(1)}$, $n \geq 4$. — Let $n \in \mathbb{Z}_{\geq 5}$. Let $I = I_{D_n^{(1)}}$ be the Dynkin diagram of type $D_n^{(1)}$ (see Figure 2.6).

It admits two vertices of valency 3, which we denote $j_1, \tilde{j}_1$. Let j_2, j'_2 (resp. $\tilde{j}_2, \tilde{j}'_2$) denote the two leaves of I in $\text{Supp}(j_1)$ (resp. $\text{Supp}(\tilde{j}_1)$). The Dynkin diagram I contains four distinct Dynkin diagrams of type D_n , obtained by removing one of the vertices among $j_2, j'_2, \tilde{j}_2, \tilde{j}'_2$. Let $\Phi \subset \oplus_{i \in I} \mathbb{Z} \alpha_i$ be the root system associated to I .


 FIGURE 2.6. Dynkin diagram of type $D_n^{(1)}$

From Theorem 2.39 we deduce that $\beta \in \oplus_{i \in I} \mathbb{N}\alpha_i$ is a quantum root if and only if one of the following conditions is satisfied:

- (1) The element β is a root in one of the four standard root subsystems of type D_n (in particular $I_1(\beta) \neq I$ and $I_3(\beta) = \emptyset$).
- (2) We have that $I_1(\beta) = I$ and $I_2(\beta) \subset I \setminus \{j_2, j'_2, \tilde{j}_2, \tilde{j}'_2\}$ admits at most two connected components, each of them containing exactly one element among $\{j_1, \tilde{j}_1\}$. Moreover $I_3(\beta) = \emptyset$.

Note that both of these cases correspond to Dynkin sequences of type (4A).

If $n = 4$, then $I_{D_4^{(1)}}$ consists of a unique vertex of valency 4 and four leaves. Similarly a root β is quantum if and only if either it is a root supported on a subdiagram of type D_4 , or if $I_1(\beta) = I_{D_4^{(1)}}$ and $I_2(\beta) = \emptyset$.

3. FROM QUANTUM ROOTS TO COVERS FOR THE AFFINIZED BRUHAT ORDER

In this section, we use the finiteness of the set of quantum roots to obtain finiteness results on the affinized Bruhat order on the Weyl semi-group associated to a Kac–Moody datum. We start this section by introducing this object and of the affinized Bruhat order.

3.1. DEFINITION AND NOTATIONS: AFFINE WEYL SEMI-GROUP

Let $\mathcal{D} = (A, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$ be a Kac–Moody root datum. Let W^v be the vectorial Weyl group and Φ the root system attached to it.

Dominant coweights and Tits cone. — Elements of Y are called coweights; a coweight λ is *regular* if $\langle \lambda, \beta \rangle \neq 0$ for all $\beta \in \Phi$, or equivalently if its fixator in W^v is trivial. We say that a coweight $\lambda \in Y$ is *dominant* if $\langle \lambda, \alpha_i \rangle \geq 0$ for all $i \in I$, or equivalently if $\langle \lambda, \beta \rangle \geq 0$ for all $\beta \in \Phi_+$. The set of dominant coweights is denoted by $Y^{++} = \{\lambda \in Y \mid \langle \lambda, \alpha_i \rangle \geq 0 \ \forall i \in I\}$ and the set of dominant regular coweights is denoted $Y_{\text{reg}}^{++} = \{\lambda \in Y^{++} \mid \langle \lambda, \alpha_i \rangle > 0 \ \forall i \in I\}$.

The orbit of Y^{++} by W^v is the *integral Tits cone*, $Y^+ = \bigcup_{w \in W^v} w.Y^{++}$; it is a sub-cone of Y and it is a proper sub-cone if and only if W^v is infinite, if and only if A is not of finite type (see [Kum02, 1.4.2]).

The set Y^{++} is a fundamental domain for the action of W^v on Y^+ , and for any coweight $\lambda \in Y^+$, we denote by λ^{++} to be the unique element of Y^{++} in its W^v -orbit.

For $\lambda, \mu \in Y$, we set

$$[\lambda, \mu]_Y = \{\nu \in Y \mid \exists t \in \mathbb{Q}, \nu = (1-t)\lambda + t\mu\} \quad \text{and} \quad]\lambda, \mu[_Y = [\lambda, \mu]_Y \setminus \{\lambda, \mu\}.$$

Parabolic subgroups and minimal coset representatives. — Let $\lambda \in Y^{++}$ be a dominant coweight. Then the fixator of λ in W^v is a standard parabolic subgroup, which we denote by W_λ . Explicitly, if $J = \{i \in I \mid \langle \lambda, \alpha_i \rangle = 0\}$, then $W_\lambda = W_J := \langle r_j \mid j \in J \rangle$.

Any class $[w] \in W^v/W_\lambda$ admits a unique element of minimal length, and we denote by W^λ the set of coset representatives of minimal length. Therefore

$$w \in W^\lambda \iff \ell(w) \leq \ell(\tilde{w}) \quad \forall \tilde{w} \in wW_\lambda.$$

Suppose now that $\lambda \in Y^+$ (not necessarily dominant), we denote by v^λ the unique element of minimal length such that $\lambda = v^\lambda \lambda^{++}$; it is an element of $W^{\lambda^{++}}$. The coweight λ also admits a parabolic fixator in W^v , which is $W_\lambda = v^\lambda W_{\lambda^{++}} (v^\lambda)^{-1}$.

A coweight λ is *spherical* if its fixator W_λ is finite, and it is *regular* if W_λ is trivial. These notions only depend on the dominant part λ^{++} of λ . We denote by Y_{sph}^+ the set of spherical coweights.

Affine Weyl semi-group. — Through the left action $W^v \curvearrowright Y$ we can form the semidirect product $Y \rtimes W^v$; for $(\lambda, w) \in Y \rtimes W^v$ we denote by $\varpi^\lambda w$ the corresponding element of $Y \rtimes W^v$. If W^v is finite, this semi-direct product is (a finite extension of) a Coxeter group, hence admits a Bruhat order, a Bruhat length and a set of simple reflections. However, when W^v is infinite, it does not have a Coxeter structure, and there is no known analog of the Bruhat order on it.

By definition, $Y^+ \subset Y$ is stable under the action of W^v , therefore we can form $W^+ = Y^+ \rtimes W^v$ which is a sub semi-group of $Y \rtimes W^v$. This semi-group is called *the affine Weyl semi-group*; it is sometimes referred to as “the affine Weyl group” in the literature, even though it is not a group.

We denote by proj^{Y^+} the projection function $W^+ \rightarrow Y^+$ and $\text{proj}^{Y^{++}}$ the projection on the dominant coweight, so if $\mathbf{x} = \varpi^\lambda w \in W^+$, then $\text{proj}^{Y^+}(\mathbf{x}) = \lambda$ and $\text{proj}^{Y^{++}}(\mathbf{x}) = \lambda^{++}$. We say that an element $\mathbf{x} \in W^+$ is *spherical* if its coweight $\text{proj}^{Y^+}(\mathbf{x})$ is spherical.

Affine roots. — Let $\Phi^a = \Phi \times \mathbb{Z}$; it is *the set of affine roots* of $\mathcal{D}$. Let $(\beta, n) \in \Phi^a$, we say that (β, n) is positive if $n > 0$ or $(n = 0 \text{ and } \beta \in \Phi_+)$; we write Φ_+^a for the set of positive affine roots. We also have $\Phi^a = \Phi_+^a \sqcup -\Phi_+^a$. We refer to [Phi24, 1.2] for a geometric interpretation of affine roots and of the affinized Bruhat order.

The semidirect product $Y \rtimes W^v$ acts on Φ^a by:

$$(3.1) \quad \varpi^\lambda w \cdot (\beta, n) = (w\beta, n + \langle \lambda, w\beta \rangle).$$

For any $n \in \mathbb{Z}$, its sign is denoted $\text{sgn}(n) \in \{-1, +1\}$, with the convention that $\text{sgn}(0) = +1$.

For $n \in \mathbb{Z}$ and $\beta \in \Phi_+$, we set:

$$(3.2) \quad \beta[n] = (\operatorname{sgn}(n)\beta, |n|\varpi) \in \Phi_+^a$$

$$(3.3) \quad s_{\beta[n]} = \varpi^{n\beta^\vee} s_\beta \in Y \rtimes W^v.$$

Note that the element $s_{\beta[n]}$ does not belong to W^+ in general, but it is an element of order 2 in $Y \rtimes W^v$, the *affine reflection associated to $\beta[n]$* .

Affinized Bruhat order. — Even though $Y \rtimes W^v$ is a priori not a Coxeter group when W^v is infinite, Braverman, Kazhdan and Patnaik have defined a preorder on W^+ in [BKP16, B.2] using the action of $Y \rtimes W^v$ on Φ^a . It is called the *affinized Bruhat order on W^+* . It is defined as the transitive closure of the relation $<$ defined by, for $\mathbf{x} \in W^+$ and $\beta[n] \in \Phi_+^a$: $s_{\beta[n]}\mathbf{x} < \mathbf{x} \iff \mathbf{x}^{-1}(\beta[n]) \notin \Phi_+^a$. In some sense this means that $\beta[n]$ belongs to the inversion set of $\mathbf{x}^{-1}$. Note in particular that if $\mathbf{x}^{-1}(\beta[n]) \notin \Phi_+^a$ and $\mathbf{x}$ lies in W^+ , then so does $s_{\beta[n]}\mathbf{x}$.

Explicitly, if $\mathbf{x} = \varpi^\lambda w \in W^+$, then $\mathbf{x}^{-1}(\beta[n]) \notin \Phi_+^a$ if and only if $|n| < \operatorname{sgn}(n)\langle \lambda, \beta \rangle$, or $|n| = \operatorname{sgn}(n)\langle \lambda, \beta \rangle$ and $\operatorname{sgn}(n)w^{-1}\beta \in \Phi_-$.

D. Muthiah (see [Mut18]) has proved that the transitive closure of $<$ is anti-symmetric, hence it is an order. Moreover, together with D. Orr, he defines in [MO19] a $\mathbb{R}$ -valued length function on W^+ strictly compatible with the affinized Bruhat order which we now introduce.

Affinized Bruhat length. — If $\varpi^\lambda w \in W^+$, the *affinized Bruhat length* is defined by:

$$(3.4) \quad \ell^a(\varpi^\lambda w) = 2\operatorname{ht}(\lambda^{++}) + \ell_{v^\lambda}(w).$$

The function ℓ_{v^λ} is the relative Bruhat length on W^v : $\ell_v(w) = \ell(v^{-1}w) - \ell(v)$ for all $v, w \in W^v$. The equivalence between this definition of the affinized Bruhat length and its definition in [MO19] is given by [Phi24, Prop. 2.10]. This length is strictly compatible with the affinized Bruhat order by [MO19, Th. 3.3]. Note that it depends on the choice of height function, and since ht may not take values in $\mathbb{Z}$, it is not integral in general. However, since two comparable elements $\mathbf{x}, \mathbf{y}$ satisfy

$$\operatorname{proj}^{Y^{++}}(\mathbf{x}) - \operatorname{proj}^{Y^{++}}(\mathbf{y}) \in Q^\vee,$$

the length difference $\ell^a(\mathbf{x}) - \ell^a(\mathbf{y})$ is an integer independent of this choice of height function.

Covers and gradation of partially ordered sets

DEFINITION 3.1. — Let $(X, <)$ be a partially ordered set.

- For $x, y \in X$, we say that y covers x if: $x < y$ and $\{z \in X \mid x < z < y\} = \emptyset$. We write $x \triangleleft y$ if y covers x . We also say that x cocovers y .
- A length function $\ell : X \rightarrow \mathbb{R}$ is said to be strictly compatible with $<$ if, for every comparable pair $(x, y) \in X^2$, we have $x < y \iff \ell(x) < \ell(y)$. Moreover it is a $\mathbb{Z}$ -grading of $(X, <)$ if: y covers x if and only if $y > x$ and $\ell(y) = \ell(x) + 1$.

The Bruhat length on W^v is an example of a grading (actually $\mathbb{N}$ -valued) of $(W^v, <)$, where $<$ is the Bruhat order; we refer to [BB05, Ch. 2] for general properties of the Bruhat order.

The affinized Bruhat length is a $\mathbb{Z}$ -grading of $(W^+, <)$ (by [MO19, Th. 6.2] in particular cases and [Phi24] in the general case): for all $\mathbf{x}, \mathbf{y} \in W^+$ such that $\mathbf{x} < \mathbf{y}$,

$$(3.5) \quad \mathbf{x} \triangleleft \mathbf{y} \iff \ell^a(\mathbf{y}) = \ell^a(\mathbf{x}) + 1.$$

The next lemma is a weaker reformulation of [Phi24, Prop. 3.20], which is a more explicit classification of covers in W^+ . In the reductive case, this is a reformulation of the classification of covers given by Schremmer in terms of quantum Bruhat graph (see [Sch24, Prop. 4.5]).

LEMMA 3.2. — *Let $\mathbf{x}, \mathbf{y} \in W^+$ and suppose that $\mathbf{y}$ covers $\mathbf{x}$. Then there are unique elements $\lambda \in Y^{++}$, $v, w \in W^v$, $\beta \in \Phi_+$ and $n \in \mathbb{Z}$ such that:*

- v is a minimal coset representative: $v \in W^\lambda$
- $\mathbf{x} = \varpi^{v\lambda} w$
- $\mathbf{y} = s_{v(\beta)[n]} \mathbf{x} = \varpi^{v(s_\beta \lambda + n\beta^\vee)} s_{v(\beta)} w$.

Moreover $n \in \{0, \langle \lambda, \beta \rangle, -1, \langle \lambda, \beta \rangle + 1\}$, and $\text{proj}^{Y^{++}}(\mathbf{y}) = \text{proj}^{Y^{++}}(\mathbf{x})$ if and only if $n \in \{0, \langle \lambda, \beta \rangle\}$.

Furthermore:

- (i) If $n = \langle \lambda, \beta \rangle$ (possibly equal to zero), then $\ell(s_\beta v^{-1} w) = \ell(v^{-1} w) + 1$, so $v^{-1} w \triangleleft s_\beta v^{-1} w$
- (ii) If $n = 0 \neq \langle \lambda, \beta \rangle$, then $\ell(vs_\beta) = \ell(v) - 1$, so $vs_\beta \triangleleft v$.
- (iii) If $n \in \{-1, \langle \lambda, \beta \rangle + 1\}$ then β is a quantum root and there exists an element $u \in W^v$ such that $\text{proj}^{Y^+}(\mathbf{y}) = vu \cdot \text{proj}^{Y^{++}}(\mathbf{y})$ and $\ell(vu) = \ell(v) + \ell(u)$, in particular $v \leqslant vu$.

The first part is a direct consequence of the definition of the affinized Bruhat order and of [Phi24, Prop. 3.1], items (i) and (ii) correspond to [Phi24, Prop. 3.20(1) & (2)] whereas (iii) corresponds to [Phi24, Prop. 3.20(3)(a),(c) & (4)(a),(c)].

In this section, we use the finiteness of the set of quantum roots to show that any element $\mathbf{x} \in W^+$ has a finite number of covers, and if it is spherical, then it also has a finite number of co-covers. This extends to the general Kac–Moody case a result obtained by Welch in the affine ADE case ([Wel22, Th. 1.1]). We provide examples of non-spherical elements which admit an infinite number of co-covers, and we give a characterization of the irreducible Kac–Moody root systems for which every element admits a finite number of co-covers. At the end of this section, we construct explicit covers in W^+ associated to any given quantum root, thereby showing that the notion of quantum root is the most refined one in this context.

3.2. FINITENESS RESULTS FOR COVERS AND CO-COVERS

THEOREM 3.3. — *Suppose that $\mathbf{x}$ is an element of an affine Weyl semi-group W^+ . Then $\mathbf{x}$ has a finite number of covers for the affinized Bruhat order.*

Proof. — For any $u \in W^v$, let $n_{\text{cov}}(u)$ (resp. $n_{\text{cocov}}(u)$) denote the number of covers (resp. cocovers) of u in W^v . Since W^v is a Coxeter group of finite rank, these numbers are finite.

Let $\mathbf{x}$ be an element of W^+ . Then, by Lemma 3.2(iii), the number of covers $\mathbf{y}$ of $\mathbf{x}$ such that $\text{proj}^{Y^{++}}(\mathbf{y}) \neq \text{proj}^{Y^{++}}(\mathbf{x})$ is bounded by twice the number $N_{\text{asr}}(\Phi)$ of quantum roots of Φ , which is finite by Theorem 2.30. Moreover, write $\mathbf{x} = \varpi^{v\lambda}w$ with $\lambda = \text{proj}^{Y^{++}}(\mathbf{x})$ and $v \in W^v$ minimal, then the number of covers $\mathbf{y}$ such that $\text{proj}^{Y^{++}}(\mathbf{y}) = \text{proj}^{Y^{++}}(\mathbf{x})$ is exactly $n_{\text{cov}}(v^{-1}w) + n_{\text{cocov}}(v)$ using items (i) and (ii) of the same lemma. Hence $\mathbf{x}$ has at most $2N_{\text{asr}}(\Phi) + n_{\text{cov}}(v^{-1}w) + n_{\text{cocov}}(v)$ covers. $\square$

COROLLARY 3.4. — *Let $n \in \mathbb{Z}_{\geq 0}$ and $\mathbf{x} \in W^+$. Then*

$$\{\mathbf{y} \in W^+ \mid \exists (\mathbf{x}_1, \dots, \mathbf{x}_n) \in (W^+)^n, \mathbf{x} = \mathbf{x}_1 \triangleleft \mathbf{x}_2 \triangleleft \dots \triangleleft \mathbf{x}_{n-1} \triangleleft \mathbf{x}_n = \mathbf{y}\}$$

is finite.

COROLLARY 3.5. — *Let $\mathbf{x}, \mathbf{y} \in W^+$. Then $[\mathbf{x}, \mathbf{y}] := \{\mathbf{z} \in W^+ \mid \mathbf{x} \leq \mathbf{z} \leq \mathbf{y}\}$ is finite.*

Proof. — Assume $\mathbf{x} \leq \mathbf{y}$. As the Bruhat order is strictly compatible with ℓ^a (by [MO19, Th. 3.3]), for every $\mathbf{z} \in [\mathbf{x}, \mathbf{y}]$, there exist $k \in \llbracket 0, \ell^a(\mathbf{y}) - \ell^a(\mathbf{x}) \rrbracket$ and $\mathbf{x}_1, \dots, \mathbf{x}_k \in W^+$ such that $\mathbf{x} = \mathbf{x}_1 \triangleleft \mathbf{x}_2 \triangleleft \dots \triangleleft \mathbf{x}_k = \mathbf{z}$ and thus the result follows from Corollary 3.4. $\square$

LEMMA 3.6. — *Let $Y_{\text{in}} = \{\lambda \in Y \mid \langle \lambda, \alpha_i \rangle = 0, \forall i \in I\}$. Let $\lambda \in Y_{\text{in}}$, $\mu \in Y^+$ and $v, w \in W^v$. Then $\varpi^\mu w \leq \varpi^\lambda v$ if and only if $\mu = \lambda$ and $w \leq v$ (for the Bruhat order on W^v).*

Proof. — Let $\beta \in \Phi_+$ and $n \in \mathbb{Z}$ be such that $s_{\beta[n]}\varpi^\lambda v < \varpi^\lambda v$, note that $s_{\beta[n]}\varpi^\lambda v = \varpi^{\lambda+n\beta^\vee}s_\beta v$. Then by [Phi24, Lem. 1.7] we have $\lambda + n\beta^\vee \in [\lambda, s_\beta \cdot \lambda]_Y = \{\lambda\}$ and hence $n = 0$.

As $\lambda \in Y_{\text{in}}$, we have $W^\lambda = \{1\}$. Now $s_\beta \varpi^\lambda v = \varpi^\lambda s_\beta v < \varpi^\lambda v$ if and only if $\ell^a(\varpi^\lambda s_\beta v) < \ell^a(\varpi^\lambda v)$ if and only if $\ell(s_\beta v) < \ell(v)$ if and only if $s_\beta v < v$ for the Bruhat order on W^v . The lemma follows. $\square$

THEOREM 3.7. — *Suppose that $\mathbf{y} \in W^+$ is spherical, or that $\text{proj}^{Y^+}(\mathbf{y}) \in Y_{\text{in}}$. Then it admits a finite number of co-covers.*

Proof. — Suppose that $\mathbf{y}$ is spherical, let us write $\mathbf{y} = \varpi^\mu w$ with $\mu \in Y_{\text{sph}}^+$. Let $C(\mathbf{y})$ denote the set of co-covers of $\mathbf{y}$ in W^+ . We have $C(\mathbf{y}) = C_1(\mathbf{y}) \sqcup C_2(\mathbf{y})$, where $C_1(\mathbf{y}) = \{\mathbf{x} \in C(\mathbf{y}) \mid \text{proj}^{Y^{++}}(\mathbf{x}) = \text{proj}^{Y^{++}}(\mathbf{y}) = \mu^{++}\}$ and $C_2(\mathbf{y}) = C(\mathbf{y}) \setminus C_1(\mathbf{y})$. We show separately that both of these sets are finite. Recall that v^μ denotes the minimal element of $\{v \in W^v \mid \mu = v(\mu^{++})\}$.

— Suppose that $\mathbf{x} \in C_1(\mathbf{y})$. Then, by Lemma 3.2, we can write $\mathbf{x} = \varpi^{v(\mu^{++})}\tilde{w}$ and $\mathbf{y} = s_{v(\beta)[n]}(\mathbf{x})$ for some $\beta \in \Phi_+$, $v \in W^{\mu^{++}}$, $\tilde{w} \in W^v$ and $n \in \{0, \langle \mu^{++}, \beta \rangle\}$. The element $\tilde{w}$ is entirely determined by (v, β) since $\tilde{w} = s_{v(\beta)}w$. So we need to prove that there is a finite number of viable choices for (v, β) . We have:

– If $n = 0 \neq \langle \mu^{++}, \beta \rangle$, then $\mu = vs_\beta(\mu^{++})$, so vs_β lies in $v^\mu.W_{\mu^{++}}$, where $W_{\mu^{++}}$ is the stabilizer of μ^{++} in W^v . By assumption, μ is spherical so $W_{\mu^{++}}$ is finite, and so is $v^\mu.W_{\mu^{++}}$. Moreover, by Lemma 3.2(ii), v covers vs_β . Since W^v is a Coxeter group of finite rank, any element of $v^\mu.W_{\mu^{++}}$ admits a finite number of covers, hence the pair (v, β) can only take a finite number of values.

– If $n \neq 0$, then $\mathbf{x}$ and $\mathbf{y}$ have the same co-weight μ , so $v = v^\mu$. Moreover, $\tilde{w} = s_{v(\beta)}w = vs_\beta v^{-1}w$ and therefore by Lemma 3.2(i), $v^{-1}w = s_\beta v^{-1}\tilde{w}$ covers $v^{-1}\tilde{w} = s_\beta v^{-1}w$. So the number of viable choices of β is equal to the number of co-covers of $v^{-1}w$, which is finite.

Either way, we thus have shown that $C_1(\mathbf{y})$ is finite; let us now turn to $C_2(\mathbf{y})$.

– Suppose that $\mathbf{x} \in C_2(\mathbf{y})$. By Lemma 3.2 we can write $\mathbf{x} = \varpi^{v\lambda}\tilde{w}$ with $\lambda \in Y^{++}$, $\tilde{w} \in W^v$, $v \in W^\lambda$ and $\mathbf{y} = s_{v(\beta)[n]}\mathbf{x}$ for $\beta \in \Phi_+$ and $n \in \{-1, \langle \lambda, \beta \rangle + 1\}$. The coweight λ is either equal to $v^{-1}\mu - \beta^\vee$ (if $n \neq -1$) or to $s_\beta v^{-1}\mu - \beta^\vee$ (if $n = -1$). Moreover, $\tilde{w} = s_{v(\beta)}w$ and thus it suffices to show that the set of pairs (v, β) such that $\mathbf{y} \in \{\varpi^{v(\lambda+\beta^\vee)}w, \varpi^{vs_\beta(\lambda+\beta^\vee)}w\}$ and is a cover of $\varpi^{v\lambda}s_{v(\beta)}w$ (for one of these two choices of λ) is finite. Note that, since μ is spherical, the set

$$\{\tilde{v} \in W^v \mid \mu = \tilde{v}\mu^{++}\} = v^\mu.W_{\mu^{++}}$$

is finite, so we can define

$$L = \max\{\ell(\tilde{v}) \mid \tilde{v} \in W^v, \tilde{v}\mu^{++} = \mu\}.$$

Then by Lemma 3.2(iii), we have $\ell(v) < L$, and β is a quantum root. Therefore, since $\{v \in W^v \mid \ell(v) < L\}$ is finite and the set of quantum roots is finite as well by Theorem 2.30, we deduce that $C_2(\mathbf{y})$ is finite.

If $\mathbf{y} \in W^+$ is such that $\text{proj}^{Y^+}(\mathbf{y}) \in Y_{\text{in}} = \bigcap_{i \in I} \ker(\alpha_i)$, then by Lemma 3.6 $\text{proj}^{Y^+}(\mathbf{x}) = \text{proj}^{Y^+}(\mathbf{y})$ for all $\mathbf{x} \in W^+$ such that $\mathbf{x} < \mathbf{y}$, and there are as many co-covers of $\mathbf{y}$ as co-covers of w in W^v , hence a finite number. $\square$

REMARK 3.8. — When Y is associated with an affine generalized Cartan matrix, then every element of Y^+ is either spherical or in Y_{in} (by [Kac90, Prop. 5.8 b])). Thus the theorem above generalizes [Wel22, Cor. 24].

Recall that $A = (\alpha_j(\alpha_i^\vee))_{i,j \in I}$ is the generalized Cartan matrix of the root datum $\mathcal{D}$. Following [Kac90, 1], we say that the matrix A is *indecomposable* if for all non-empty subsets $J, K \subset I$ such that $J \sqcup K = I$, there exists $(j, k) \in J \times K$ such that $\alpha_j(\alpha_k^\vee) \neq 0$.

LEMMA 3.9. — Assume that A is indecomposable and that there exists $\mu \in Y^{++}$ such that W_μ is infinite and W^v does not fix μ . Then there exists $\mathbf{x} \in W^+$ admitting infinitely many co-covers.

Proof. — Let $J = \{i \in I \mid \alpha_i(\mu) = 0\}$. Then $J \subsetneq I$. Let $J' \subset J$ be such that $W_{J'}$ is infinite and for every $J'' \subsetneq J'$, $W_{J''}$ is finite. Let $\nu \in Y$ be such that $\alpha_j(\nu) = 3$ for every $j \in I \setminus J'$ and $\alpha_j(\nu) = 0$ for every $j \in J'$. Then $\nu \in Y^{++}$ and $W_\nu = W_{J'}$.

As A is indecomposable, there exists $(i, j) \in (I \setminus J') \times J'$ such that $\alpha_i(\alpha_j^\vee) \neq 0$. Set $\lambda = \nu - \alpha_i^\vee \in]\nu, r_i \cdot \nu[_Y$. Then

$$\alpha_j(\lambda) = \alpha_j(\nu) - \alpha_j(\alpha_i^\vee) = -\alpha_j(\alpha_i^\vee) > 0, \quad \alpha_i(\lambda) = \alpha_i(\nu) - 2 > 0$$

and for every $i' \in I \setminus \{j\}$, we have

$$\alpha_{i'}(\lambda) = \alpha_{i'}(\nu) - \alpha_{i'}(\alpha_j^\vee) \geq \alpha_{i'}(\nu) \geq 0.$$

Therefore $\lambda \in Y^{++}$ and $\{k \in I \mid \alpha_k(\lambda) = 0\} \subset J' \setminus \{j\}$. Thus $W_\lambda \subsetneq W_\nu$ and by assumption on J' , W_λ is finite.

Let $w \in W_\nu \cap W^\lambda$ (we prove below that this set is non-empty). Set $\beta = w(\alpha_i)$ and $\mathbf{x}_w = \varpi^{w \cdot \lambda} s_\beta$. We have:

$$s_{\beta[(w \cdot \lambda, \beta) + 1]} \mathbf{x}_w = \varpi^{(\langle w \cdot \lambda, \beta \rangle + 1)\beta^\vee} \varpi^{s_\beta w \cdot \lambda} = \varpi^\nu$$

and thus $\mathbf{x}_w < \varpi^\nu$ if and only if $\ell^a(\mathbf{x}_w) < \ell^a(\varpi^\nu)$. Moreover we have:

$$\begin{aligned} \ell^a(\mathbf{x}_w) &= 2\text{ht}(\lambda) + \ell_w(s_{w(\alpha_i)}) \\ &= 2(\text{ht}(\nu) - 1) + \ell(w^{-1}wr_iw^{-1}) - \ell(w) \\ &= \ell^a(\varpi^\nu) - 2 + \ell(r_iw^{-1}) - \ell(w). \end{aligned}$$

Therefore

$$(3.6) \quad \ell^a(\mathbf{x}_w) \in \{\ell^a(\varpi^\nu) - 1, \ell^a(\varpi^\nu) - 3\},$$

for every $w \in W^\lambda \cap W_\nu$.

Let $w \in W_\nu$. Then we can write uniquely $w = vu$, with $v \in W^\lambda$, $u \in W_\lambda$. But then $u \in W_\nu$ and hence $v \in W^\lambda \cap W_\nu$. Therefore $W_\nu \cdot \lambda = (W_\nu \cap W^\lambda) \cdot \lambda$. As W_λ is finite, $W_\nu \cdot \lambda$ is infinite and thus $(W^\lambda \cap W_\nu) \cdot \lambda$ and $\{\mathbf{x}_w \mid w \in W^\lambda \cap W_\nu\}$ are infinite.

For $\mathbf{y} \in W^+$, denote by $C'(\mathbf{y})$ the union of the set of co-covers of $\mathbf{y}$ with $\{\mathbf{y}\}$. By compatibility of ℓ^a with $<$ and by (3.6), for every $w \in W_\nu \cap W^\lambda$, we have $\mathbf{x}_w \in C'(C'(C'(\varpi^\nu)))$. Therefore

$$C'(C'(C'(\varpi^\nu))) \supset \{\mathbf{x}_w \mid w \in W^\lambda \cap W_\nu\},$$

and the right-hand side is infinite. Consequently there exists $\mathbf{x} \in W^+$ for which $C'(\mathbf{x})$ is infinite. $\square$

REMARK 3.10. — The situation of Lemma 3.9 can happen only when A is of indefinite type. Indeed, if A has finite type, W^v is finite and if A has affine type, then every element of Y^{++} with infinite fixator is fixed by W^v , by [Kac90, Prop. 5.8(b) & Prop. 3.12(f)].

EXAMPLE 3.11. — Assume that I has cardinality 3 and write $I = \{i, j, k\}$. Assume that $\alpha_m(\alpha_{m'}^\vee)\alpha_{m'}(\alpha_m^\vee) \geq 4$ for every $m, m' \in \{i, j, k\}$. By [Kum02, 1.3.21 Prop.], this implies that $\langle r_i, r_j \rangle$, $\langle r_j, r_k \rangle$ and $\langle r_i, r_k \rangle$ are infinite. Take $\nu \in Y$ such that $\alpha_i(\nu) \geq 3$, $\alpha_j(\nu) = \alpha_k(\nu) = 0$. Set $\lambda = \nu - \alpha_i^\vee \in Y^{++}$. Then $W_\lambda = \{1\}$ and with the same proof as in Lemma 3.9, we can prove that $\varpi^{w \cdot \lambda} wr_i w^{-1}$ is covered by ϖ^ν for every $w \in \langle r_j, r_k \rangle$.

Let $\mathbb{A} = Y \otimes \mathbb{R}$ be the *standard apartment*. The elements of Φ can naturally be regarded as elements of the dual $\mathbb{A}^*$ of $\mathbb{A}$. The Tits cone $\mathcal{T}$ is then $W^v \cdot \overline{C_f^v}$, where $\overline{C_f^v} = \{x \in \mathbb{A} \mid \langle x, \alpha_i \rangle \geq 0, \forall i \in I\}$.

PROPOSITION 3.12. — *We assume that $(\langle \alpha_j^\vee, \alpha_i \rangle)_{i,j \in I}$ is indecomposable. Let $\mathbb{A}_{\text{in}} = \bigcap_{i \in I} \ker(\alpha_i)_{\mathbb{R}}$, where $\ker(\alpha_i)_{\mathbb{R}} = \{x \in \mathbb{A} \mid \alpha_i(x) = 0\}$, for $i \in I$. The following properties are equivalent:*

- (1) *The set $\mathcal{T} \setminus \mathbb{A}_{\text{in}}$ is open.*
- (2) *Every proper parabolic subgroup of W^v is finite.*
- (3) *For every $x \in W^+$, the set of co-covers of x is finite.*

Proof

(1) $\Rightarrow$ (2). Assume (1). Let J be a proper subset of I . Let $\lambda \in Y$ be such that $\alpha_j(\lambda) = 0$ for all $j \in J$ and $\alpha_j(\lambda) > 0$ for all $j \in I \setminus J$. Then $\lambda \in \overline{C_f^v} \setminus \mathbb{A}_{\text{in}} \subset \mathcal{T} \setminus \mathbb{A}_{\text{in}} \subset \tilde{\mathcal{T}}$. Then $W_\lambda = W_J$ and by [Kac90, Prop. 3.12(f)], W_J is finite.

(2) $\Rightarrow$ (1). Assume (2). Let $x \in \mathcal{T} \setminus \mathbb{A}_{\text{in}}$. We can write $x = w.y$, for some $w \in W^v$ and $y \in \overline{C_f^v} \setminus \mathbb{A}_{\text{in}}$. Let $J = \{i \in I \mid \alpha_i(y) = 0\}$. Then $J \subsetneq I$ and thus $W_y = W_J$ is finite. By [Kac90, Prop. 3.12(f)], there exists an open neighborhood V of y contained in $\mathcal{T}$. Then the fixator of every element of V is finite and thus $V \subset \mathcal{T} \setminus \mathbb{A}_{\text{in}}$. Then $x \in w^{-1} \cdot V \subset \mathcal{T} \setminus \mathbb{A}_{\text{in}}$, which proves that $\mathcal{T} \setminus \mathbb{A}_{\text{in}}$ is open.

(3) $\Rightarrow$ (1). Assume (3). Then by Lemma 3.9, for every $\mu \in Y^+$, we have either W_μ is finite or $W_\mu = W^v$. Let $x \in \mathcal{T}$. Write $x = w.y$, with $w \in W^v$ and $y \in \overline{C_f^v}$. Let $J = \{i \in I \mid \alpha_i(y) = 0\}$. Let $\mu \in Y$ be such that for $i \in I$, $\alpha_i(\mu) = 1$ if $i \in I \setminus J$ and $\alpha_i(\mu) = 0$ if $i \in J$. Then $W_\mu = W_x$. If $W_x = W^v$, then $x \in \mathbb{A}_{\text{in}}$ and if W_x is finite, then $x \in \tilde{\mathcal{T}}$ by [Kac90, Prop. 3.12(f)], which proves that $\mathcal{T} = \mathbb{A}_{\text{in}} \sqcup \tilde{\mathcal{T}}$. Thus we have (1).

(1) $\Rightarrow$ (3) is a consequence of Theorem 3.7. □

3.3. EXPLICIT COVERS FOR A GIVEN QUANTUM ROOT. — In this section, we construct, for any quantum root, explicit covers in which it appears; hence, the restriction on $\beta \in \Phi_+$ appearing in Lemma 3.2 is optimal.

PROPOSITION 3.13. — *Let β be a quantum root, and let $\lambda \in Y_{\text{reg}}^{++}$ be such that $\lambda + \beta^\vee \in Y_{\text{reg}}^{++}$ (explicitly, $\langle \lambda, \alpha_i \rangle > -\langle \beta^\vee, \alpha_i \rangle$ for all $i \in I$). Then:*

- (1) *Let v be any element of W^v and let $w \in W^v$ be such that $\ell(s_\beta w) = \ell(s_\beta) + \ell(w)$ (in particular, this always holds for $w = 1_{W^v}$). Then $\varpi^{v(\lambda + \beta^\vee)} v w$ covers $\varpi^{v(\lambda)} v s_\beta w$.*
- (2) *Let $v \in W^v$ be such that $\ell(v s_\beta) = \ell(v) + \ell(s_\beta)$ and let w be any element of W^v . Then $\varpi^{v s_\beta(\lambda + \beta^\vee)} s_{v(\beta)} w$ covers $\varpi^{v(\lambda)} w$.*

In particular, for any quantum root β there exist covers of the form given in Lemma 3.2 with varying dominance class.

Proof. — Recall the expression of the affinized Bruhat length obtained in [Phi24, Prop. 2.10]: for any $\lambda \in Y_{\text{reg}}^{++}$, $v, w \in W^v$:

$$(3.7) \quad \ell^a(\varpi^{v(\lambda)} w) = 2\text{ht}(\lambda) + \ell(v^{-1}w) - \ell(v).$$

Thus, for any $\lambda \in Y_{\text{reg}}^{++}$ such that $\lambda + \beta^\vee \in Y_{\text{reg}}^{++}$:

$$\begin{aligned}
\ell^a(s_{v(\beta)[\langle \lambda, \beta \rangle + 1]} \varpi^{v(\lambda)} v s_\beta w) &= \ell^a(\varpi^{v(\lambda + \beta^\vee)} v w) \\
&= 2\text{ht}(\lambda + \beta^\vee) + \ell(w) - \ell(v) \\
&= \ell^a(\varpi^{v(\lambda)} v s_\beta w) + 2\text{ht}(\beta^\vee) + \ell(w) - \ell(s_\beta w). \\
\ell^a(s_{v(\beta)[-1]} \varpi^{v(\lambda)} w) &= \ell^a(\varpi^{v s_\beta(\lambda + \beta^\vee)} v s_\beta v^{-1} w) \\
&= 2\text{ht}(\lambda + \beta^\vee) + \ell(v^{-1} w) - \ell(v s_\beta) \\
&= \ell^a(\varpi^{v(\lambda)} w) + 2\text{ht}(\beta^\vee) + \ell(v) - \ell(v s_\beta).
\end{aligned}$$

Therefore, since quantum roots are characterized by the fact that $2\text{ht}(\beta^\vee) = \ell(s_\beta) + 1$ (by Lemma 2.9), we obtain that the length difference $\ell^a(s_{v(\beta)[\langle \lambda, \beta \rangle + 1]} \varpi^{v(\lambda)} v s_\beta w) - \ell^a(\varpi^{v(\lambda)} v s_\beta w)$ (resp. $\ell^a(s_{v(\beta)[-1]} \varpi^{v(\lambda)} w) - \ell^a(\varpi^{v(\lambda)} w)$) is exactly one if and only if condition (1) (resp. condition (2)) is satisfied. $\square$

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