



INSTITUT
POLYTECHNIQUE
DE PARIS

*J*ournal de l'École polytechnique *Mathématiques*

Averil AUSSÉDAT

Locality of centered tangent cones in the Wasserstein space

Tome 13 (2026), p. 1575-1606.

<https://doi.org/10.5802/jep.355>

© L'auteur, 2026.



Cet article est mis à disposition selon les termes de la licence
LICENCE INTERNATIONALE D'ATTRIBUTION CREATIVE COMMONS BY 4.0.
<https://creativecommons.org/licenses/by/4.0/>

Publié avec le soutien
du Centre National de la Recherche Scientifique



Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 2270-518X

LOCALITY OF CENTERED TANGENT CONES IN THE WASSERSTEIN SPACE

BY AVERIL AUSSÉDAT

ABSTRACT. — The geometric tangent cone to a probability measure μ is a set of measure-valued applications that are almost geodesics. This is a nonlocal condition, typically lost when conditioning the measure on a given set. We show that if one removes the barycenter of any element of the tangent cone, then the resulting set of centered measure fields is characterized by a local condition. Precisely, centered tangent fields must be concentrated on a family of vector subspaces attached to any point, and these subspaces correspond to the normal spaces to some sets of “dimension k ” on which the measure μ is concentrated.

RÉSUMÉ (Localité des cônes tangents centrés dans l’espace de Wasserstein)

Le cône tangent géométrique en une mesure μ est défini comme un ensemble d’applications à valeurs dans les mesures, chacune décrivant presque une géodésique. Cette propriété n’est pas locale et est généralement perdue quand la mesure est restreinte à un sous-ensemble. Nous montrons qu’il suffit de retirer le barycentre des éléments du cône tangent pour obtenir un ensemble de champs centrés lui-même caractérisé par une condition locale. Plus précisément, les champs tangents centrés sont concentrés sur une famille de sous-espaces vectoriels attachés en chaque point, correspondant aux espaces normaux à certains ensembles de « dimension k » sur lesquels la mesure μ est concentrée.

CONTENTS

Introduction.....	1576
1. Preliminaries.....	1578
2. Closed convex cones of centered measure fields.....	1580
3. Tangent and solenoidal measure fields.....	1586
4. Properties of the decomposition.....	1592
Appendix.....	1603
References.....	1605

MATHEMATICAL SUBJECT CLASSIFICATION (2020). — 28A15, 51FXX, 35R06.

KEYWORDS. — Wasserstein spaces, tangent cone, DC functions.

This work benefited from the support of the grant ANR-22-CE40-0010 while the author was in Rouen, and from the ERC starting grant ConFine n°101078057 in Pisa.

INTRODUCTION

In his inspired article [Lot16], Lott observed that whenever μ is the restriction of a Hausdorff measure to a k -dimensional C^2 manifold, the measures in the geometric tangent cone to μ have a particular structure. It was already known that the barycenter of such a measure must belong to the L_μ^2 closure of gradients of smooth compactly supported functions, as proved in full generality in [Gig11]. However, once the barycenter is removed, the remaining *centered part* turns out to be concentrated on the normal directions to the manifold. The purpose of this work is to investigate the general case, using a new line of reasoning that allows us to remove any assumption on the measure $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

Our main result is a combination of Theorem 3.10, Theorem 4.10 and Proposition 4.12 below, gathered here into a single statement. A set $A \subset \mathbb{R}^d$ is DC_k if, up to a permutation of variables, it is the graph of a map from $\mathbb{R}^k$ to $\mathbb{R}^{d-k}$ whose coordinates are all Differences of Convex (DC) functions. A set that can be covered by countably many DC_k sets is said to be $\sigma\text{-DC}_k$.

THEOREM. — *Let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. There exists a unique decomposition $\mu = \sum_{k=0}^d m_k \mu^k$ into mutually singular measures such that $m_k \mu^k$ gives 0 mass to DC_{k-1} sets and is concentrated on a $\sigma\text{-DC}_k$ set A_k . Moreover, the centered tangent cone Tan_μ^0 splits into $\sum_{k=0}^d m_k \text{Tan}_{\mu^k}^0$, and a centered measure ξ^k belongs to $\text{Tan}_{\mu^k}^0$ if and only if it is concentrated on the normal spaces to A_k , which exist μ^k -almost everywhere.*

For instance, in dimension $d = 2$, let $\mu = \frac{1}{3}\mu^0 + \frac{1}{3}\mu^1 + \frac{1}{3}\mu^2$, where μ^0 is atomic, μ^1 nonatomic and supported on $[0, 1] \times \{0\}$, and μ^2 is absolutely continuous. Let ξ be a probability measure on $\text{T}\mathbb{R}^d \simeq \mathbb{R}^d \times \mathbb{R}^d$ with finite second moment, first marginal μ and barycenter 0 in each fiber. Since the μ^k are mutually singular, there is a unique way to write $\xi = \frac{1}{3} \sum_{k=0}^2 \xi^k$ with $\pi_{x\#} \xi^k = \mu^k$ for each k . The theorem then states that ξ is tangent to μ if and only if ξ^1 is concentrated on pairs (x, v) with v orthogonal to $[0, 1] \times \{0\}$, and ξ^2 is concentrated on (x, v) with $v \in (\mathbb{R}^2)^\perp$, hence $\xi^2 = (\text{id}, 0)_\# \mu^2$.

In dimension one, the splitting $\mu = m_0 \mu^0 + m_1 \mu^1$ reduces to the atomic/diffuse decomposition. In this case, the statement was already proved in [SW14] (see also [Aus25a]): it says that a centered field $\xi = m_0 \xi^0 + m_1 \xi^1$ is tangent if and only if $\xi^1 = (\text{id}, 0)_\# \mu^1$. This case is rather special, since $\sigma\text{-DC}_0$ sets are exactly countable sets, and the only $\sigma\text{-DC}_1$ set is $\mathbb{R}$ itself. In the general case, one has to pay attention to the regularity of the sets A_k for the intermediate dimensions $0 < k < d$.

The relation between DC_{d-1} sets and optimal transport plans for the 2-Wasserstein distance appeared in the work of [Gig11], which characterizes the measures μ satisfying the conclusion of the Brenier–McCann theorem [Bre91]. Precisely, a measure μ that gives 0 mass to any DC_{d-1} set (and consequently, to any DC_k set for $k \leq d-1$) does not admit optimal plans that *split mass*, in the sense that for any other measure ν and optimal transport plan α between μ and ν , there exists a map $f \in L_\mu^2(\mathbb{R}^d; \mathbb{R}^d)$ such that $\alpha = (\text{id}, f)_\# \mu$. The argument runs as follows: optimal plans are known to be concentrated on c -cyclically monotone sets, which, in the case of $c(x, y) = |x - y|^2$,

coincide with subdifferentials of convex functions. If an optimal plan splits mass at some point x , then the convex functions in question must admit several elements in their subdifferential at x , and hence fail to be differentiable there. A remarkable theorem of Zajíček [Zaj79] shows that the non-differentiability set of a convex function can be covered by countably many DC_{d-1} sets, and that any DC_{d-1} set is contained in the non-differentiability set of some convex function. Hence μ allows mass splitting if and only if it charges a DC_{d-1} set.

This work generalizes Gigli's theorem by identifying which part of μ allows optimal plans to split mass in exactly $d-k$ directions, again using Zajíček's theorem. Moreover, and this is the main point, each part is concentrated on a set A_k which admits normal spaces of dimension $d-k$ at μ -almost every point, and the centered tangent measure fields are exactly the measure fields concentrated on these normal spaces. The intuition behind this decomposition is that tangent measures try to get away from μ as fast as possible, and therefore choose normal directions to the concentration sets.

It may seem surprising that no regularity assumption on A_k is needed: this is still a consequence of Zajíček's theorem, since, quite miraculously, singularities of convex functions are described using (differences of) convex functions. Indeed, the regularity of A_k is precisely the one that allows a convex function to fail to be differentiable in $d-k$ independent directions. By Zajíček's theorem, the singular subset of A_k (at which the normal directions are not defined) has the "same size" as non-differentiability sets of convex functions *in at least one more direction*. In the decomposition, these sets are seen by the lower-order measures μ^j for $j < k$, so that each A_k has a tangent plane μ^k -almost everywhere.

Since many properties of Tan_μ^0 depend only on the fact that this set is convex in a certain sense, we first focus on a generic horizontally convex closed cone of centered fields. However, the tangent cone is constructed in a canonical way for any μ , and can be compared for different measures. As a by-product, we show that the metric orthogonal Sol_μ of Tan_μ is closed with respect to the Wasserstein distance over $\mathbb{T}\mathbb{R}^d$.

We point out that the statement only concerns centered fields. This is a natural assumption, since the opposite case of barycentric fields (induced by a map) cannot be described by a local condition. However, the following can be deduced for any measure field $\xi \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$. First, one can always write $\xi = (\pi_x, \pi_v + b(\pi_x))_\# \xi^0$, where $b \in L_\mu^2$ is the barycenter map, and ξ^0 is centered. It is known that $\xi \in \text{Tan}_\mu$ if and only if both $(\text{id}, b)_\# \mu$ and ξ^0 are tangent measure fields: hence the theorem applies to ξ^0 . Moreover, anticipating the notation of Section 3, we can further decompose $b = b_D + b_{D^\perp}$, where $b_D(x)$ is the projection of $b(x)$ onto the vector space $D(x)$, and $b_{D^\perp} = b - b_D$. Corollary 3.3 implies that the measure field $\frac{1}{2}[(\text{id}, -b_{D^\perp})_\# \mu + (\text{id}, b_{D^\perp})_\# \mu]$ is tangent, regardless of whether b induces a tangent field or not. From this, one recovers that $(\text{id}, b_{D^\perp})_\# \mu$ is itself tangent, once again regardless of b (see [Aus25b, Prop. 5.4.4]). Hence, the fact that $(\text{id}, b)_\# \mu$ is tangent restricts only its component b_D . This extends the observation of [Lot16] that the barycenter of a tangent map is only constrained through its tangential component.

This article is organized as follows: Section 2 is devoted to closed convex cones of centered measure fields. Section 3 applies the previous results to the centered geometric tangent cone Tan_μ^0 and its metric orthogonal Sol_μ^0 , providing the candidate decomposition $\mu = \sum m_k \mu^k$. The fact that each μ^k is concentrated on a $\sigma\text{-DC}_k$ set, and gives 0 mass to DC_{k-1} sets, is proved in Section 4.2. Section 4.3 shows that the normal directions are well-defined μ^k -almost everywhere, and characterize centered tangent measures. In addition, it is shown that the tangent measures in the sense of Preiss are supported on planes associated to Sol_μ^0 at μ -almost every point. The appendix collects some lengthy proofs.

Acknowledgements. — The author would like to thank Adolfo Arroyo-Rabasa for pointing out relevant literature, as well as Guy Bouchitté for comments on a preliminary version of this work.

1. PRELIMINARIES

In the sequel, $\Omega = \mathbb{R}^d$ for $d \geq 1$. We keep the notation Ω to distinguish points $x \in \Omega$ from vectors $v \in \mathbb{R}^d$. Open balls with center x and radius $r \geq 0$ are denoted $\mathcal{B}(x, r)$. The closure of a set $A \subset \Omega$ is denoted $\overline{A}$, and its complement A^c .

Wasserstein spaces. — The set of Borel probability measures on a Polish space X is denoted $\mathcal{P}(X)$. If $X = X_1 \times X_2$ is a product space and $\mu \in \mathcal{P}(X)$, we write $\mu = \mu(dx_1, dx_2)$ to name the variables of X , to be used in the canonical projections $\pi_{x_i} : (x_1, x_2) \rightarrow x_i$ for $i \in \{1, 2\}$. The set $\mathcal{P}_2(X) \subset \mathcal{P}(X)$ collects the measures with finite second moment, i.e., with $\int_{x \in X} d^2(x, o) d\mu < \infty$ for some $o \in X$.

A measurable function $f : X \rightarrow Y$ between two Polish spaces X, Y induces a function $f_\# : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ by $(f_\#\mu)(A) = \mu(f^{-1}(A))$ for any measurable $A \subset Y$. We refer to $f_\#\mu$ as the *pushforward* of μ by f .

Given $(\mu, \nu) \in \mathcal{P}_2(X) \times \mathcal{P}_2(Y)$, let the set of *transport plans* between μ and ν be

$$\Gamma(\mu, \nu) := \{\alpha = \alpha(dx, dy) \in \mathcal{P}(X \times Y) \mid \pi_{x\#}\alpha = \mu \text{ and } \pi_{y\#}\alpha = \nu\}.$$

The 2-Wasserstein distance $W(\mu, \nu)$ between μ and ν is given by

$$W^2(\mu, \nu) := \inf_{\alpha \in \Gamma(\mu, \nu)} \int_{(x, y) \in X \times Y} d^2(x, y) d\alpha(x, y).$$

We refer the reader to [San15] for an introduction to this distance, and only mention that the infimum is attained on a set of *optimal transport plans* denoted $\Gamma_o(\mu, \nu)$.

Measure fields. — Let $\text{T}\Omega := \{(x, v) \mid x \in \Omega, v \in \text{T}_x\Omega\}$ be the tangent bundle of Ω , isometric to $\Omega \times \mathbb{R}^d$. When useful, we also denote

$$\text{T}^n\Omega := \{(x, v_1, v_2, \dots, v_n) \mid x \in \Omega, v_i \in \text{T}_x\Omega \text{ for } i \in \llbracket 1, n \rrbracket\}.$$

For any measure $\mu \in \mathcal{P}_2(\Omega)$, denote by $\mathcal{P}_2(\text{T}\Omega)_\mu$ the set of Borel probability measures $\xi = \xi(dx, dv)$ on $\text{T}\Omega$, with finite second moment, and satisfying the marginal condition $\pi_{x\#}\xi = \mu$. These elements can be seen as measure-valued maps in L^2_μ , generalizing vector fields; for this reason, we refer to them as *measure fields*. The

measure fields of the form $\xi = (\pi_x, \pi_y - \pi_x)_{\#} \eta$ for $\eta \in \Gamma_o(\mu, \nu)$ and $\nu \in \mathcal{P}_2(\Omega)$ are called *velocities of geodesics*. Following [Gig08], we introduce a metric structure on $\mathcal{P}_2(T\Omega)_\mu$ that takes into account the common marginal μ . Namely, given $\xi, \zeta \in \mathcal{P}_2(T\Omega)_\mu$, define

$$\Gamma_\mu(\xi, \zeta) := \{ \alpha = \alpha(dx, dv, dw) \in \mathcal{P}_2(T^2\Omega) \mid (\pi_x, \pi_v)_{\#} \alpha = \xi \text{ and } (\pi_x, \pi_w)_{\#} \alpha = \zeta \}.$$

These plans only move mass between pairs (x, v) and (y, w) such that $x = y$. They define a distance W_μ by

$$W_\mu^2(\xi, \zeta) := \inf_{\alpha \in \Gamma_\mu(\xi, \zeta)} \int_{(x, v, w) \in T^2\Omega} |v - w|^2 d\alpha(x, v, w).$$

The distance to the zero measure field is abbreviated as $\|\xi\|_\mu = \left(\int_{(x, v)} |v|^2 d\xi \right)^{1/2}$. The distance W_μ induces a *metric scalar product* $\langle \cdot, \cdot \rangle_\mu : (\mathcal{P}_2(T\Omega)_\mu)^2 \rightarrow \mathbb{R}$ by

$$\langle \xi, \zeta \rangle_\mu := \frac{1}{2} [\|\xi\|_\mu^2 + \|\zeta\|_\mu^2 - W_\mu^2(\xi, \zeta)] = \sup_{\alpha \in \Gamma_\mu(\xi, \zeta)} \int_{(x, v, w) \in T^2\Omega} \langle v, w \rangle d\alpha(x, v, w).$$

Subsets of measure fields. — The *barycenter* of a measure field ξ is the unique element $b \in L_\mu^2(\Omega; \mathbb{R}^d)$ satisfying $\int \varphi(x, v) d\xi = \int \varphi(x, b(x)) d\mu$ for any quadratically growing $\varphi \in \mathcal{C}(T\Omega; \mathbb{R})$ that is linear in v . The measure fields with barycenter 0 are called *centered*, and the set of centered measure fields is denoted $\mathcal{P}_2(T\Omega)_\mu^0$.

For $\lambda \in \mathbb{R}$ and $\xi \in \mathcal{P}_2(T\Omega)_\mu$, define $\lambda \cdot \xi := (\pi_x, \lambda\pi_v)_{\#} \xi$. We say that a subset $\text{Set}_\mu \subset \mathcal{P}_2(T\Omega)_\mu$ is

- a (positive) cone if $\lambda \cdot \xi \in \text{Set}_\mu$ whenever $\xi \in \text{Set}_\mu$ and $\lambda \geq 0$;
- (horizontally) convex if for any $\lambda \in [0, 1]$, $\xi, \zeta \in \text{Set}_\mu$ and $\alpha \in \Gamma_\mu(\xi, \zeta)$, the measure field given by $(\pi_x, (1 - \lambda)\pi_v + \lambda\pi_w)_{\#} \alpha$ also belongs to Set_μ .

We often omit the adjective “positive” in the sequel. Horizontal convexity is stronger than geodesic convexity, since any transport plan is allowed to produce interpolating curves. It should also be distinguished from convexity in the Banach space of measures, which involves curves of the form $(1 - \lambda)\xi + \lambda\zeta$.

Grassmannian sections. — For brevity, let us agree that a *Grassmannian section* $D : \Omega \rightrightarrows \mathbb{R}^d$ is a measurable multivalued map such that $D(x)$ is a vector subspace, possibly reduced to $\{0\}$, for all $x \in \Omega$. Here measurability is understood in the classical sense, i.e., if for any open set $\mathcal{O} \subset \mathbb{R}^d$, the set $\{x \in \Omega \mid D(x) \cap \mathcal{O} \neq \emptyset\}$ is measurable. The *graph* of D is the set $\text{Graph } D := \{(x, v) \mid v \in D(x)\}$, which is a measurable subset of $T\Omega$ [Roc69, Cor. 2.2]. The following lemma will be used.

LEMMA 1.1. — *Let $\mu \in \mathcal{P}_2(\Omega)$ and $D : \Omega \rightrightarrows \mathbb{R}^d$ be a Grassmannian section. The set of $\xi \in \mathcal{P}_2(T\Omega)_\mu$ such that $\xi(\text{Graph } D) = 1$ is closed with respect to the Wasserstein distance $W_{T\Omega}$ on the tangent bundle.*

Proof. — Let $(\xi_n)_n \subset \mathcal{P}_2(T\Omega)_\mu$ be a sequence such that $\xi_n(\text{Graph } D) = 1$ which converges towards $\xi \in \mathcal{P}_2(T\Omega)_\mu$ with respect to $W_{T\Omega}(\cdot, \cdot)$. Using a Castaing representation of D [Roc69, Th. 3], one can find d measurable functions $u_1, \dots, u_d : \Omega \rightarrow \mathbb{R}^d$

such that $D(x) = \text{span}\{u_1(x), \dots, u_d(x)\}$ for any $x \in \Omega$. Let $\varepsilon > 0$. By Lusin's theorem [Fed96, 2.3.5], there exists a measurable set $B_\varepsilon \subset \Omega$ such that $\mu(B_\varepsilon) \geq 1 - \varepsilon$ and each u_j coincides with a continuous function on B_ε . Since $\mu(B_\varepsilon) = \sup_{\overline{C_\varepsilon} \subset B_\varepsilon} \mu(\overline{C_\varepsilon})$, there is a closed set $C_\varepsilon \subset B_\varepsilon$ closed such that $\mu(C_\varepsilon) \geq 1 - 2\varepsilon$. The set

$$G_\varepsilon := \bigcup_{x \in C_\varepsilon} \{x\} \times D(x) = \text{Graph } D \cap C_\varepsilon \times \mathbb{R}^d = \bigcup_{x \in C_\varepsilon} \{x\} \times \text{span}\{u_1(x), \dots, u_d(x)\}$$

is measurable and closed by the first and second equalities, respectively. Moreover, $\xi_n(G_\varepsilon) = \mu(C_\varepsilon) \geq 1 - 2\varepsilon$ for all $n \in \mathbb{N}$. Since $\zeta \mapsto \int \mathbb{1}_{G_\varepsilon} d\zeta$ is upper semicontinuous with respect to $W_{T\Omega}$ [AGS05, Lem. 5.1.7],

$$\xi(\text{Graph } D) \geq \xi(G_\varepsilon) \geq \limsup_{n \rightarrow \infty} \xi_n(G_\varepsilon) \geq 1 - 2\varepsilon.$$

Letting $\varepsilon \searrow 0$, we conclude that $\xi(\text{Graph } D) = 1$. □

2. CLOSED CONVEX CONES OF CENTERED MEASURE FIELDS

Throughout this section, we consider a set $\text{Set}_\mu^0 \subset \mathcal{P}_2(T\Omega)_\mu^0$ with the following properties.

ASSUMPTION 2.1. — The set Set_μ^0 is a W_μ -closed, horizontally convex cone of centered measure fields.

Here and below, “cone” means “positive cone”. The main motivation is the study of Tan_μ^0 and Sol_μ^0 carried out in Section 3, which explains our notation.

PROPOSITION 2.2 (Local characterization). — *A set Set_μ^0 satisfies Assumption 2.1 if and only if there exists a Grassmannian section $D : \Omega \rightrightarrows \mathbb{R}^d$ such that*

$$\xi \in \text{Set}_\mu^0 \iff [\xi \in \mathcal{P}_2(T\Omega)_\mu^0 \text{ and } \xi(\text{Graph } D) = 1].$$

As an example, in dimension $d = 2$, the set of centered ξ satisfying $\langle v, (1, 0) \rangle = 0$ almost everywhere is W_μ -closed, horizontally convex, stable under multiplication by any scalar, and characterized by the constant map $D \equiv \text{vect}\{(0, 1)\}$.

REMARK 2.3 (Negative cone). — A direct consequence of Proposition 2.2 is that Set_μ^0 is stable under multiplication by any scalar, including negative ones. Such stability was already noticed in the case of the tangent cone in [Gig08], using ad hoc arguments; in this case, the property even holds for map-induced fields. Here it is essential to restrict to centered fields; indeed, already in dimension $d = 1$, the set of measure fields concentrated on (x, v) with $v \geq 0$ provides a closed and horizontally convex cone that cannot be characterized by concentration on the graph of a Grassmannian section.

REMARK 2.4 (Vertical convexity). — A somewhat surprising implication of Proposition 2.2 is that Set_μ^0 is convex as a subset of the Banach space of measures, in the sense that $(1 - \lambda)\xi_0 + \lambda\xi_1 \in \text{Set}_\mu^0$ whenever $\xi_0, \xi_1 \in \text{Set}_\mu^0$ and $\lambda \in [0, 1]$. Indeed, the superposition $(1 - \lambda)\xi_0 + \lambda\xi_1$ remains centered, and concentrated on the graph of D .

The aim of this section is to prove Proposition 2.2. Our strategy is to represent Set_μ^0 as the set of centered measure fields that are orthogonal to a family of “simple” measure fields, defined in (3) below. These fields will be used to construct D . The key observation, which justifies our interest in centered fields, is the following.

LEMMA 2.5 (Centered is local). — *Let ξ, ζ be measure fields, with ξ centered. Then $\langle \xi, \zeta \rangle_\mu \geq 0$, and equality holds if and only if any disintegrations $(\xi_x)_x, (\zeta_x)_x$ satisfy $\langle \xi_x, \zeta_x \rangle_{\delta_x} = 0$ for μ -almost every x .*

Proof. — Owing to [Gig08, Prop. 4.2], the metric scalar product can be written as

$$(1) \quad \langle \xi, \zeta \rangle_\mu = \int_{x \in \Omega} \langle \xi_x, \zeta_x \rangle_{\delta_x} d\mu(x).$$

Fix $x \in \Omega$. Identifying ξ_x, ζ_x with measures on $\mathcal{P}_2(T_x \Omega)_{\delta_x} \simeq \mathcal{P}_2(\mathbb{R}^d)$, we may consider the product plan $\alpha_x := \xi_x \otimes \zeta_x$ in the definition of the metric scalar product to get

$$\langle \xi_x, \zeta_x \rangle_{\delta_x} \geq \int_{v, w \in \mathbb{R}^d} \langle v, w \rangle d\alpha_x = \int_{v \in \mathbb{R}^d} v d\xi_x(v) \int_{w \in \mathbb{R}^d} w d\zeta_x(w).$$

As the first term of the right-hand side vanishes μ -almost everywhere, the integrand in (1) is nonnegative. Hence $\langle \xi, \zeta \rangle_\mu \geq 0$, and $\langle \xi, \zeta \rangle_\mu = 0$ implies $\langle \xi_x, \zeta_x \rangle_{\delta_x} = 0$ for μ -a.e. x . The converse follows directly from (1). $\square$

This observation is used as follows. In the sequel, one often has to prove that $\langle \xi, \zeta \rangle_\mu = 0$ for some ξ, ζ lying in cones of interest. By specific arguments, one first obtains the inequality $\langle \xi, \zeta \rangle_\mu \leq 0$. In the Hilbertian case, when working with two-sided cones, one would typically take $-1 \cdot \zeta$ in place of ζ , and conclude with the same inequality that $\langle \xi, \zeta \rangle_\mu = 0$. However, we do not a priori assume that Set_μ^0 is stable under multiplication by a negative scalar, and the missing inequality is provided by Lemma 2.5.

2.1. THE METRIC ORTHOGONAL COMPLEMENT. — Our aim here is to write Set_μ^0 as the orthogonal complement of its orthogonal complement. For convenience, we restrict to centered measure fields, since any measure field induced by a map is orthogonal to any centered measure field with respect to $\langle \cdot, \cdot \rangle_\mu$.

LEMMA 2.6. — *Let Set_μ^0 satisfy Assumption 2.1. Then so does*

$$(\text{Set}_\mu^0)^{\perp 0} := \{ \zeta \in \mathcal{P}_2(T\Omega)_\mu^0 \mid \langle \xi, \zeta \rangle_\mu = 0 \text{ for all } \xi \in \text{Set}_\mu^0 \},$$

i.e., $(\text{Set}_\mu^0)^{\perp 0}$ is a W_μ -closed horizontally convex cone of centered measure fields.

Proof. — By construction, the scalar product is continuous with respect to W_μ , so that $(\text{Set}_\mu^0)^{\perp 0}$ is W_μ -closed. Let $\zeta_0, \zeta_1 \in (\text{Set}_\mu^0)^{\perp 0}$, $\alpha \in \Gamma_\mu(\zeta_0, \zeta_1)$ and $\lambda \in [0, 1]$. The measure field $\zeta_\lambda := (\pi_x, (1 - \lambda)\pi_v + \lambda\pi_w)_\# \alpha$ is centered. To show that $\zeta_\lambda \in (\text{Set}_\mu^0)^{\perp 0}$, let $\xi \in \text{Set}_\mu^0$, and $\beta \in \Gamma_\mu(\zeta_\lambda, \xi)$. We construct a plan $\omega = \omega(dx, dv_0, dv_1, dw) \in \Gamma_\mu(\zeta_0, \zeta_1, \xi)$ such that $(\pi_x, \pi_{v_0}, \pi_{v_1})_\# \omega = \alpha$ and $(\pi_x, (1 - \lambda)\pi_{v_0} + \lambda\pi_{v_1}, \pi_w)_\# \omega = \beta$ as follows: first change variables to consider $\tilde{\alpha} := (\pi_x, \pi_v, (1 - \lambda)\pi_v + \lambda\pi_w)_\# \alpha \in$

$\Gamma_\mu(\zeta_0, \zeta_\lambda)$, then glue $\tilde{\alpha}$ and β along their common marginal ζ_λ by [AGS05, Lem. 5.3.2] to produce $\tilde{\omega} = \tilde{\omega}(dx, dv, dv_\lambda, dw)$, and return to the original variables with $\omega := (\pi_x, \pi_v, \lambda^{-1}(\pi_{v_\lambda} - (1 - \lambda)\pi_v), \pi_w)_\# \tilde{\omega}$. Then

$$\int \langle v, w \rangle d\beta = \int \langle (1 - \lambda)v_0 + \lambda v_1, w \rangle d\omega \leq (1 - \lambda)\langle \zeta_0, \xi \rangle_\mu + \lambda\langle \zeta_1, \xi \rangle_\mu = 0.$$

Passing to the supremum over β , we get that $\langle \zeta_\lambda, \xi \rangle_\mu \leq 0$. By Lemma 2.5, the inequality $\langle \zeta_\lambda, \xi \rangle_\mu \geq 0$ holds since both fields are centered. Hence $\zeta_\lambda \in (\text{Set}_\mu^0)^{\perp 0}$. Lastly, if $\zeta \in (\text{Set}_\mu^0)^{\perp 0}$ and $\lambda \geq 0$, we have $\langle \xi, \lambda \cdot \zeta \rangle_\mu = \langle \lambda \cdot \xi, \zeta \rangle_\mu = 0$ for all $\xi \in \text{Set}_\mu^0$, so that $\lambda \cdot \xi \in (\text{Set}_\mu^0)^{\perp 0}$. $\square$

The next result uses elementary tools from the pseudo-Hilbertian structure of $\mathcal{P}_2(\text{T}\Omega)_\mu$, and is very close to similar statements in [Gig08, Aus25a]. For this reason, we defer the proof to the appendix.

PROPOSITION 2.7 (Projection). — *Let C satisfy Assumption 2.1. Then any $\xi \in \mathcal{P}_2(\text{T}\Omega)_\mu^0$ admits a unique metric projection $\pi_C^\mu \xi \in C$, realizing $\inf_{\zeta \in C} W_\mu(\xi, \zeta)$. Moreover, for any $\alpha \in \Gamma_{\mu, o}(\xi, \pi_C^\mu \xi)$, the measure field $(\pi_x, \pi_v - \pi_w)_\# \alpha$ is the metric projection of ξ on $C^\perp := \{\zeta \in \mathcal{P}_2(\text{T}\Omega)_\mu \mid \langle \zeta, \gamma \rangle_\mu = 0 \ \forall \gamma \in C\}$, and we have*

$$(2) \quad \langle \xi, \zeta \rangle_\mu \leq \langle \pi_C^\mu \xi, \zeta \rangle_\mu \quad \forall \zeta \in C,$$

with equality if $\zeta = -1 \cdot \zeta = (\pi_x, -\pi_v)_\# \zeta$.

REMARK 2.8. — The conclusions of Proposition 2.7 are not sharp; as a consequence of Proposition 2.2, equality will hold in (2) for any $\zeta \in C$. One could additionally prove that $\Gamma_{\mu, o}(\xi, \pi_C^\mu \xi)$ reduces to a singleton that is induced by a map in a certain sense, following the reasoning of [Gig08, Th. 4.33].

LEMMA 2.9. — *We have $\text{Set}_\mu^0 = ((\text{Set}_\mu^0)^{\perp 0})^{\perp 0}$.*

Proof. — The inclusion $\text{Set}_\mu^0 \subset ((\text{Set}_\mu^0)^{\perp 0})^{\perp 0}$ holds by definition. Conversely, let $\xi \in ((\text{Set}_\mu^0)^{\perp 0})^{\perp 0}$, and denote by $\pi^\mu \xi$ its metric projection on Set_μ^0 . By Proposition 2.7, for any $\alpha = \alpha(dx, dv, dw) \in \Gamma_{\mu, o}(\xi, \pi^\mu \xi)$, the measure field $(\pi_x, \pi_v - \pi_w)_\# \alpha$ is the projection of ξ on $(\text{Set}_\mu^0)^{\perp 0}$. However, this projection is 0_μ , since $W_\mu^2(\xi, \zeta) = \|\xi\|_\mu^2 + \|\zeta\|_\mu^2$ for any $\zeta \in (\text{Set}_\mu^0)^{\perp 0}$. Hence $v = w$ for α -a.e. (x, v, w) , and $\xi = \pi^\mu \xi \in \text{Set}_\mu^0$. $\square$

2.2. REDUCTION TO SYMMETRIC MEASURE FIELDS. — It will be useful to introduce the following notation: with any $f \in L_\mu^2(\Omega; \mathbb{R}^d)$, we associate

$$(3) \quad \gamma_f := \frac{1}{2}[(\text{id}, -f)_\# \mu + (\text{id}, f)_\# \mu] \in \mathcal{P}_2(\text{T}\Omega)_\mu^0.$$

Note that $(\pi_x, -\pi_v)_\# \gamma_f = \gamma_f$. For any $f, g \in L_\mu^2(\Omega; \mathbb{R}^d)$, one can use the transport plan $\frac{1}{2}[(\text{id}, f, g)_\# \mu + (\text{id}, -f, -g)_\# \mu]$ to get the estimate $W_\mu(\gamma_f, \gamma_g) \leq \|f - g\|_{L_\mu^2}$. Our interest in the measure fields of this form stems from the following lemma.

LEMMA 2.10. — *Let $f \in L_\mu^2(\Omega; \mathbb{R}^d)$. A centered measure field ξ is orthogonal to γ_f if and only if for some (thus any) measurable $f^* : \Omega \rightarrow \mathbb{R}^d$ in the equivalence class f , we have $\langle v, f^*(x) \rangle = 0$ for ξ -almost every (x, v) .*

Proof. — If $\langle v, f^\cdot(x) \rangle = 0$ for ξ -almost all (x, v) , then any transport plan between ξ and γ is concentrated on (x, v, w) with $w = \pm f^\cdot(x)$, and necessarily $\langle v, w \rangle = 0$ almost everywhere. This shows that $\langle \xi, \gamma \rangle_\mu = 0$. Conversely, assume that ξ is centered and $\langle \xi, \gamma \rangle_\mu = 0$. Let $(\xi_x)_{x \in \Omega}$ be a disintegration of ξ , which can be chosen so that $\int_v v d\xi_x = 0$ for all $x \in \Omega$. By Lemma 2.5, we have, for μ -almost every x ,

$$(4) \quad \int_{(v,w) \in (\mathbb{R}^d)^2} \langle v, w \rangle d\alpha_x = 0 \quad \text{for all } \alpha_x \in \Gamma\left(\xi_x, \frac{1}{2} [\delta_{f^\cdot(x)} + \delta_{(-f^\cdot(x))}]\right).$$

If we show that (4) implies $\langle v, f^\cdot(x) \rangle = 0$ for ξ_x -almost every v , then the equality $\int |\langle v, f^\cdot(x) \rangle| d\xi = \int_x \int_v |\langle v, f^\cdot(x) \rangle| d\xi_x(v) d\mu(x)$ will ensure that $\langle v, f^\cdot(x) \rangle = 0$ for ξ -almost every (x, v) . In the rest of the proof, we simplify the notation by letting $\zeta := \xi_x \in \mathcal{P}_2(\mathbb{R}^d)^0$ and $\bar{w} := f^\cdot(x) \in \mathbb{R}^d$.

Assume by contradiction that there is $\varepsilon_+ > 0$ such that $m_+ := \zeta \{ \langle v, \bar{w} \rangle > \varepsilon_+ \} > 0$. Then, as $\int_v \langle v, \bar{w} \rangle d\zeta = 0$, there exists $\varepsilon_- > 0$ such that $m_- := \zeta \{ \langle v, \bar{w} \rangle < -\varepsilon_- \} > 0$. Let $m := \min(m_-, m_+) > 0$. Construct a transport plan $\alpha \in \Gamma(\zeta, \frac{1}{2} [\delta_{-\bar{w}} + \delta_{\bar{w}}])$ by sending a mass m/m_+ from $\zeta_+ := \zeta \llcorner \{ \langle v, \bar{w} \rangle > \varepsilon_+ \}$ to $\bar{w}$, a mass m/m_- from $\zeta_- := \zeta \llcorner \{ \langle v, \bar{w} \rangle < -\varepsilon_- \}$ to $-\bar{w}$, and splitting the rest evenly; explicitly,

$$\alpha := m \left[\left(\frac{1}{m_+} \zeta_+ \right) \otimes \delta_{\bar{w}} + \left(\frac{1}{m_-} \zeta_- \right) \otimes \delta_{-\bar{w}} \right] + \left(\zeta - \frac{m}{m_+} \zeta_+ - \frac{m}{m_-} \zeta_- \right) \otimes \left[\left(\frac{1}{2} - m \right) (\delta_{\bar{w}} + \delta_{-\bar{w}}) \right].$$

Then $\pi_{v\#} \alpha = \zeta$, and $\pi_{w\#} \alpha = \frac{1}{2} [\delta_{\bar{w}} + \delta_{-\bar{w}}]$. Moreover,

$$\int_{(v,w)} \langle v, w \rangle d\alpha = \frac{m}{m_+} \int_v \langle v, \bar{w} \rangle d\zeta_+ + \frac{m}{m_-} \int_v \langle v, -\bar{w} \rangle d\zeta_- + 0 \geq m\varepsilon_+ + m\varepsilon_- > 0,$$

contradicting (4). Hence $\zeta \{ \langle v, \bar{w} \rangle > \varepsilon_+ \} = 0$ for all $\varepsilon_+ > 0$. Using that ζ is centered, we conclude that $\zeta = \xi_x$ is concentrated on the orthogonal complement of $\bar{w} = f^\cdot(x)$. $\square$

Lemma 2.10 will allow us to reduce a “global” orthogonality to a “local” one. Consider again a W_μ -closed, horizontally convex cone $\text{Set}_\mu^0 \subset \mathcal{P}_2(T\Omega)_\mu^0$, and introduce

$$\mathcal{F} := \{ f \in L_\mu^2(\Omega; \mathbb{R}^d) \mid \gamma_f \in \text{Set}_\mu^0 \}.$$

The following result shows that the set $\mathcal{F}$ suffices to characterize $(\text{Set}_\mu^0)^{\perp 0}$.

LEMMA 2.11. — *If $\xi \in \mathcal{P}_2(T\Omega)_\mu^0$, then ξ belongs to $(\text{Set}_\mu^0)^{\perp 0}$ if and only if it is orthogonal to all γ_f for $f \in \mathcal{F}$.*

The proof uses a construction that makes it somewhat lengthy, but not complicated.

Proof. — One implication being immediate, we show that if $\langle \xi, \gamma_f \rangle_\mu = 0$ for all $f \in \mathcal{F}$, then $\xi \in (\text{Set}_\mu^0)^{\perp 0}$. This is equivalent to $\pi^\mu \xi = 0_\mu$, where $\pi^\mu \xi$ is the metric projection of ξ on Set_μ^0 , given by Proposition 2.7. In particular, as $\gamma_f = (\pi_x, -\pi_v)_\# \gamma_f$ for any $f \in \mathcal{F}$, we have $\langle \xi, \gamma_f \rangle_\mu = \langle \pi^\mu \xi, \gamma_f \rangle_\mu$. Assume by contradiction that $\pi^\mu \xi \neq 0_\mu$: we construct $f \in \mathcal{F}$ such that $\langle \pi^\mu \xi, \gamma_f \rangle_\mu > 0$, contradicting the assumption on ξ .

First consider the constant vector fields $g_i(x) \equiv e_i$, where e_i is the i^{th} element of the canonical basis of $\mathbb{R}^d$. By Lemma 2.10, if $\langle \pi^\mu \xi, \gamma_{g_i} \rangle_\mu = 0$ for all i , then $\langle v, e_i \rangle = 0$ for $\pi^\mu \xi$ -almost every (x, v) and each $i \in \llbracket 1, d \rrbracket$, so that $\pi^\mu \xi = 0_\mu$. Consequently, there must be $i \in \llbracket 1, d \rrbracket$ such that $\langle \pi^\mu \xi, \gamma_{g_i} \rangle_\mu > 0$.

Pick $\alpha \in \Gamma_{\mu, o}(\pi^\mu \xi, \gamma_{g_i})$. Then $\alpha = \frac{1}{2}(\pi_x, \pi_v, -e_i)_\# \zeta_-^0 + \frac{1}{2}(\pi_x, \pi_v, e_i)_\# \zeta_+^0$ for some $\zeta_\pm^0 \in \mathcal{P}_2(\text{T}\Omega)_\mu$ such that $\pi^\mu \xi = \frac{1}{2}\zeta_+^0 + \frac{1}{2}\zeta_-^0 \in \text{Set}_\mu^0$. Construct inductively a sequence $(\zeta_+^k, \zeta_-^k)_{k \in \mathbb{N}}$ as follows: assuming that $\frac{1}{2}\zeta_+^k + \frac{1}{2}\zeta_-^k$ belongs to Set_μ^0 , consider the transport plan $\beta := \frac{1}{2}\zeta_+^k \otimes_\mu \zeta_+^k + \frac{1}{2}\zeta_-^k \otimes_\mu \zeta_-^k$, where for any $\zeta \in \mathcal{P}_2(\text{T}\Omega)_\mu$, the “pointwise product plan” is defined as

$$\zeta \otimes_\mu \zeta := \int_{x \in \Omega} [\delta_x \otimes \zeta_x \otimes \zeta_x] d\mu(x) \in \mathcal{P}_2(\text{T}^2\Omega)_\mu.$$

Let $\zeta_\pm^{k+1} := (\pi_x, \frac{1}{2}[\pi_v + \pi_w])_\# \zeta_\pm^k \otimes_\mu \zeta_\pm^k$. Then

$$\frac{1}{2}\zeta_+^{k+1} + \frac{1}{2}\zeta_-^{k+1} = \left(\pi_x, \frac{\pi_v + \pi_w}{2}\right)_\# \beta \in \text{Set}_\mu^0$$

by horizontal convexity. The barycenter of each $\zeta_\pm^k$ is preserved along the sequence: indeed, for any $\varphi \in \mathcal{C}(\text{T}\Omega; \mathbb{R})$ linear in its second argument and with quadratic growth,

$$\begin{aligned} \int_{(x,v)} \varphi(x, v) d\zeta_+^{k+1} &= \int_{(x,v,w)} \varphi\left(x, \frac{v+w}{2}\right) d[\zeta_+^k \otimes_\mu \zeta_+^k] \\ &= \frac{1}{2} \int_{(x,v)} \varphi(x, v) d\zeta_+^k + \frac{1}{2} \int_{(x,w)} \varphi(x, w) d\zeta_+^k, \end{aligned}$$

hence $\text{Bary}(\zeta_\pm^{k+1}) = \text{Bary}(\zeta_\pm^k) =: f_\pm$. In fact

$$\frac{1}{2}f_+ + \frac{1}{2}f_- = \text{Bary}\left(\frac{1}{2}\zeta_+^0 + \frac{1}{2}\zeta_-^0\right) = \text{Bary}(\pi^\mu \xi) = 0,$$

so that $f_- = -f_+$. Now, the sequence $(\zeta_+^k)_{k \in \mathbb{N}}$ converges to $(\text{id}, f_+)_{\#} \mu$: indeed,

$$\begin{aligned} W_\mu^2(\zeta_+^{k+1}, (\text{id}, f_+)_{\#} \mu) &= \int_{(x,v,w)} \left| \frac{v+w}{2} - f_+(x) \right|^2 d[\zeta_+^k \otimes_\mu \zeta_+^k] \\ &= 2 \times \frac{1}{4} \int_{(x,v)} |v - f_+(x)|^2 d\zeta_+^k + 2 \int_{(x,v,w)} \left\langle \frac{v - f_+(x)}{2}, \frac{w - f_+(x)}{2} \right\rangle d[\zeta_+^k \otimes_\mu \zeta_+^k] \\ &= \frac{1}{2} W_\mu^2(\zeta_+^k, (\text{id}, f_+)_{\#} \mu) + 0, \end{aligned}$$

where we used that $\Gamma_\mu(\xi, (\text{id}, g)_{\#} \mu)$ always reduces to a singleton whenever $g \in L_\mu^2$, and the definition of the pointwise product measure. The same argument implies that $\zeta_-^k \rightarrow_k (\text{id}, f_-)_{\#} \mu = (\text{id}, -f_+)_{\#} \mu$ with respect to W_μ . Consequently, the sequence $(\frac{1}{2}\zeta_+^k + \frac{1}{2}\zeta_-^k)_{k \in \mathbb{N}} \subset \text{Set}_\mu^0$ converges with respect to W_μ towards the centered field $\frac{1}{2}(\text{id}, f_+)_{\#} \mu + \frac{1}{2}(\text{id}, f_-)_{\#} \mu = \gamma_{f_+}$, which must belong to Set_μ^0 . Now, recalling the definition of $\zeta_\pm^0$, we have

$$0 < \langle \pi^\mu \xi, \gamma_{g_i} \rangle_\mu = \frac{1}{2} \int_{(x,v)} \langle v, -e_i \rangle d\zeta_-^0 + \frac{1}{2} \int_{(x,v)} \langle v, e_i \rangle d\zeta_+^0 = \int \langle f_+(x), e_i \rangle d\mu,$$

so that $\|f_+\|_\mu > 0$; on the other hand, using that $\langle \zeta, (\text{id}, g)_\# \rangle_\mu = \langle \text{Bary}(\zeta), g \rangle_{L_\mu^2}$ for any measure field $\zeta \in \mathcal{P}_2(T\Omega)_\mu$ and vector field $g \in L_\mu^2(\Omega; \mathbb{R}^d)$,

$$\begin{aligned} \langle \pi^\mu \xi, \gamma_{f_+} \rangle_\mu &\geq \frac{1}{2} \langle \zeta_+^0, (\text{id}, f_+)_\# \mu \rangle_\mu + \frac{1}{2} \langle \zeta_-^0, (\text{id}, f_-)_\# \mu \rangle_\mu \\ &= \frac{1}{2} \|f_+\|_{L_\mu^2}^2 + \frac{1}{2} \|f_-\|_{L_\mu^2}^2 = \|f_+\|_{L_\mu^2}^2 > 0. \end{aligned}$$

In conclusion, if $\pi^\mu \xi \neq 0_\mu$, we have constructed $f_+ \in \mathcal{F}$ such that $0 < \langle \pi^\mu \xi, \gamma_{f_+} \rangle_\mu = \langle \xi, \gamma_f \rangle$, contradicting the assumption. Hence $\pi^\mu \xi = 0$, and $\xi \in (\text{Sol}_\mu^0)^{\perp 0}$. $\square$

2.3. CHARACTERIZATION BY A GRASSMANNIAN SECTION. — We can now turn to the proof of Proposition 2.2. Our aim is to construct a Grassmannian section $D : \Omega \rightrightarrows \mathbb{R}^d$ such that $\xi \in \text{Set}_\mu^0$ if and only if ξ is centered and concentrated on Graph D .

Proof of Proposition 2.2. — Assume first that a Grassmannian section $D : \Omega \rightrightarrows \mathbb{R}^d$ is given, and let Set_μ^0 be the set of centered measure fields concentrated on Graph D . Clearly, Set_μ^0 is a convex cone of centered measure fields. Moreover, for each $x \in \Omega$, the set of measures in $\mathcal{P}_2(T_x \Omega) \simeq \mathcal{P}_2(\mathbb{R}^d)$ which are concentrated on the closed set $D(x)$ is closed with respect to the Wasserstein distance. Hence Set_μ^0 is closed with respect to $W_\mu \simeq L_\mu^2(\Omega; (\mathcal{P}_2(\mathbb{R}^d); W))$.

Now let Set_μ^0 be any W_μ -closed convex cone of centered measure fields. By Lemma 2.9, $\text{Set}_\mu^0 = ((\text{Set}_\mu^0)^{\perp 0})^{\perp 0}$. Let $\mathcal{F}_\perp := \{f \in L_\mu^2(\Omega; \mathbb{R}^d) \mid \gamma_f \in (\text{Set}_\mu^0)^{\perp 0}\}$. Using that $W_\mu(\gamma_f, \gamma_g) \leq \|f - g\|_{L_\mu^2}$ and that $(\text{Set}_\mu^0)^{\perp 0}$ is W_μ -closed, the set $\mathcal{F}_\perp$ is closed in L_μ^2 , hence separable. Consider a countable dense set $(f_n)_{n \in \mathbb{N}} \subset \mathcal{F}_\perp$. From Lemma 2.11 and the continuity of the scalar product, $\xi \in \text{Set}_\mu^0 = ((\text{Set}_\mu^0)^{\perp 0})^{\perp 0}$ if and only if $\langle \xi, \gamma_{f_n} \rangle_\mu = 0$ for all $n \in \mathbb{N}$. By Lemma 2.10, the latter condition is equivalent to $\langle v, f_n^*(x) \rangle = 0$ for ξ -almost all (x, v) , where $f_n^* : \Omega \rightarrow \mathbb{R}^d$ is a measurable map in the L_μ^2 -equivalence class f_n . For every $x \in \Omega$, define

$$D(x) := \{v \in \mathbb{R}^d \mid \langle v, f_n^*(x) \rangle = 0 \text{ for all } n \in \mathbb{N}\}.$$

The map D depends on the precise choice of $(f_n)_n$, but only up to a μ -negligible subset. Each $D(x)$ is a vector space, and by [Roc69, Th. 3.(e)], D is measurable as a multivalued map. If ξ is concentrated on the graph of D , then $\langle v, f_n^*(x) \rangle = 0$ for all n ξ -almost everywhere. Conversely, if for any n , there exists B^n such that $\xi(B^n) = 0$ and $\langle v, f_n^*(x) \rangle = 0$ for any $(x, v) \notin B^n$, then $B := \bigcup_n B^n$ is still ξ -negligible, and $v \in D(x)$ for any $x \notin B$. Hence $\xi \in \text{Set}_\mu^0$ if and only if $\xi(\text{Graph } D) = 1$. $\square$

An interesting corollary of Proposition 2.2 is that Set_μ^0 is closed with respect to the (weaker) topology of the Wasserstein distance over the tangent bundle $W_{T\Omega}$, i.e., with cost $c((x, v), (y, w)) := \sqrt{|x - y|^2 + |v - w|^2}$. Here, it is necessary to restrict to centered measure fields: the geometric tangent cone Tan_μ is a W_μ -closed, horizontally convex and (two-sided) cone, but is not closed with respect to $W_{T\Omega}$ in general (an example can be found below Proposition 2.10 in [Aus25a]).

COROLLARY 2.12 ($W_{T\Omega}$ -closedness of Set_μ^0). — Assume that Set_μ^0 satisfies Assumption 2.1. Let $(\xi_n)_{n \in \mathbb{N}} \subset \text{Set}_\mu^0$ be a sequence converging to $\xi \in \mathcal{P}_2(T\Omega)_\mu$ with respect to $W_{T\Omega}$. Then $\xi \in \text{Set}_\mu^0$.

Proof. — By [Vil09, Th. 6.9], convergence with respect to $W_{T\Omega}$ is equivalent to the convergence of $\int_{(x,v)} \varphi(x,v) d\xi_n \rightarrow \int_{(x,v)} \varphi(x,v) d\xi$ for any continuous and quadratically growing $\varphi : T\Omega \rightarrow \mathbb{R}$. Consequently, $\int \varphi(x, \text{Bary}(\xi)(x)) d\mu = 0$ for any φ linear with respect to its second argument, and $\text{Bary}(\xi) = 0$. By Lemma 1.1, the set of ξ such that $\xi(\text{Graph } D) = 1$ is $W_{T\Omega}$ -closed, hence $\xi \in \text{Set}_\mu^0$. $\square$

The characterization of Set_μ^0 provides many examples of closed convex cones of centered fields; one just has to choose the map D . However, the following section focuses on two particular subsets of measure fields which are not a priori constructed from such maps, but can be proved to be closed and horizontally convex, yielding an additional structure.

3. TANGENT AND SOLENOIDAL MEASURE FIELDS

We introduce the geometric tangent cone Tan_μ in its classical definition, as well as its metric orthogonal, and immediately restrict our attention to their centered subsets.

DEFINITION 3.1 (Tan_μ^0 and Sol_μ^0). — The geometric tangent cone is defined as

$$\text{Tan}_\mu := \overline{\{\lambda \cdot \xi \mid \lambda \geq 0, (\pi_x, \pi_x + \pi_v)_\# \xi \in \Gamma_o(\mu, \nu), \nu \in \mathcal{P}_2(\Omega)\}}^{W_\mu} \subset \mathcal{P}_2(T\Omega)_\mu.$$

The solenoidal cone Sol_μ is defined as $(\text{Tan}_\mu)^\perp$, i.e.,

$$\text{Sol}_\mu := \{\zeta \in \mathcal{P}_2(T\Omega)_\mu \mid \langle \xi, \zeta \rangle_\mu = 0 \text{ for all } \xi \in \text{Tan}_\mu\}.$$

The sets Tan_μ^0 and Sol_μ^0 are defined as the intersections of $\text{Tan}_\mu, \text{Sol}_\mu$ with $\mathcal{P}_2(T\Omega)_\mu^0$.

REMARK 3.2 (Alternative definition). — One can show that for any $\xi, \zeta \in \mathcal{P}_2(T\Omega)_\mu$,

$$(5) \quad \langle \xi, \zeta \rangle_\mu = \langle \xi^0, \zeta^0 \rangle_\mu + \langle \text{Bary}(\xi), \text{Bary}(\zeta) \rangle_{L_\mu^2},$$

where ξ^0, ζ^0 are the respective centered components of ξ, ζ , i.e.,

$$\xi^0 = (\pi_x, \pi_v - \text{Bary}(\xi))_\# \xi.$$

Therefore the sets Tan_μ^0 and Sol_μ^0 coincide with the sets of metric projections of $\text{Tan}_\mu, \text{Sol}_\mu$ onto the closed, convex cone $\mathcal{P}_2(T\Omega)_\mu^0$. The orthogonal decomposition (5) also implies that ξ is tangent (resp. ζ solenoidal) if and only if ξ^0 and $(\text{id}, \text{Bary}(\xi))_\# \mu$ are tangent (resp. ζ^0 and $(\text{id}, \text{Bary}(\zeta))_\# \mu$ solenoidal).

Our first result on Tan_μ^0 and Sol_μ^0 is an application of Proposition 2.2. If $D : \Omega \rightrightarrows \mathbb{R}^d$ is a Grassmannian section, denote by $D^\perp$ the map such that $D^\perp(x)$ is the orthogonal complement of $D(x)$ in $\mathbb{R}^d$.

COROLLARY 3.3. — *Let $\mu \in \mathcal{P}_2(\Omega)$. There exists a Grassmannian section $D : \Omega \rightrightarrows \mathbb{R}^d$ such that*

$$(6) \quad \begin{aligned} \zeta \in \text{Sol}_\mu^0 &\iff [\zeta \in \mathcal{P}_2(\text{T}\Omega)_\mu^0 \text{ and } \zeta(\text{Graph } D) = 1], \\ \xi \in \text{Tan}_\mu^0 &\iff [\xi \in \mathcal{P}_2(\text{T}\Omega)_\mu^0 \text{ and } \xi(\text{Graph } D^\perp) = 1]. \end{aligned}$$

In the sequel, D will often be denoted D^{Sol} , and $D^\perp$ will be denoted D^{Tan} .

Proof. — The fact that Tan_μ^0 is a W_μ -closed positive cone of centered fields follows directly from the definition. The fact that it is horizontally convex is proved in [Gig08, Prop. 4.25], first by considering optimal plans and using cyclical monotonicity, then by approximation. The decomposition of the metric scalar product in (5) yields that $\text{Sol}_\mu^0 = (\text{Tan}_\mu^0)^{\perp 0}$, so that by Lemma 2.6, Sol_μ^0 is also a closed convex cone of centered measure fields. Hence the existence of D characterizing Sol_μ^0 as in (6) follows from Proposition 2.2. The same result yields a Grassmannian section $\tilde{D} : \Omega \rightrightarrows \mathbb{R}^d$ characterizing Tan_μ^0 ; to conclude, it only remains to show that $\tilde{D} = D^\perp$ μ -almost everywhere. But this is implied by the following equivalences for ξ centered:

$$\begin{aligned} \xi \text{ concentrated on } (x, v) \text{ such that } v \in \tilde{D}(x) &\iff \xi \in \text{Tan}_\mu^0 \quad (\text{Prop. 2.2}) \\ &\iff \langle \xi, \gamma_f \rangle_\mu = 0 \text{ for any } f \in L_\mu^2 \text{ such that } \gamma_f \in \text{Sol}_\mu^0 \quad (\text{Lem. 2.9 and 2.11}) \\ &\iff \xi \text{ concentrated on } (x, v) \text{ such that } \langle v, f^*(x) \rangle = 0 \quad (\text{Prop. 2.2}), \end{aligned}$$

where in the last line, f^* ranges over all measurable functions such that $f^*(x) \in D(x)$ for all $x \in \Omega$, and $\int |f^*|^2 d\mu < \infty$. The fact that sufficiently many of such f^* can be found follows from the Castaing representation of the measurable map D , for instance in [Roc69, Th. 3.(d)]. $\square$

By Corollary 2.12, both Tan_μ^0 and Sol_μ^0 are closed with respect to the Wasserstein distance $W_{\text{T}\Omega}$ on the tangent bundle. In addition, the set of solenoidal measure fields (possibly with nonzero barycenter) is closed in the same topology.

COROLLARY 3.4 (Sol_μ is $W_{\text{T}\Omega}$ -closed). — *Let $\mu \in \mathcal{P}_2(\Omega)$. The set Sol_μ is $W_{\text{T}\Omega}$ -closed.*

Proof. — We first show that any $\zeta \in \text{Sol}_\mu$ is concentrated on $\text{Graph } D$. By Remark 3.2, if $\zeta \in \text{Sol}_\mu$, then $g := \text{Bary}(\zeta) \in L_\mu^2(\Omega; \mathbb{R}^d)$ is such that $(\text{id}, g)_\# \mu \in \text{Sol}_\mu$. By [Aus25a, Lem. 2.4 & 2.7], both measure fields $(\text{id}, -g)_\# \mu$ and $\frac{1}{2}[(\text{id}, g)_\# \mu + (\text{id}, -g)_\# \mu]$ are solenoidal. Applying Corollary 3.3 to the latter, we get that $g(x) \in D^{\text{Sol}}(x)$ for μ -almost every x . As the centered field $\zeta^0 := (\pi_x, \pi_v - g(\pi_x))_\# \zeta$ is solenoidal, it is also concentrated on $\text{Graph } D$, thus so is $\zeta = (\pi_x, \pi_v + g(\pi_x))_\# \zeta^0$.

Now let $(\zeta_n)_{n \in \mathbb{N}} \subset \text{Sol}_\mu$ be a Cauchy sequence with respect to $W_{\text{T}\Omega}$, and let $\zeta \in \mathcal{P}_2(\text{T}\Omega)_\mu$ be its limit. By Lemma 1.1, $\zeta(\text{Graph } D) = 1$. Consequently, the centered component $\zeta^0 := (\pi_x, \pi_v - \text{Bary}(\zeta)(\pi_x))_\# \zeta$ is concentrated on $\text{Graph } D$, hence solenoidal. To show that $\text{Bary}(\zeta)$ also induces a solenoidal field, it is enough to prove that $\langle f, \text{Bary}(\zeta) \rangle_{L_\mu^2} = 0$ for any $f \in L_\mu^2(\Omega; \mathbb{R}^d)$ such that $(\text{id}, f)_\# \mu$ is tangent (by (5)).

In addition, we may take $f = \nabla\varphi$ for $\varphi \in \mathcal{C}_c^\infty(\Omega; \mathbb{R})$ [Gig08, Th. 4.14]. In particular, $(x, v) \mapsto \langle f(x), v \rangle$ is continuous and has quadratic growth, so that

$$\langle f, \text{Bary}(\zeta) \rangle_{L_\mu^2} = \int \langle f(x), v \rangle d\zeta = \lim_{n \rightarrow \infty} \int \langle f(x), v \rangle d\zeta_n = \lim_{n \rightarrow \infty} \langle f, \text{Bary}(\zeta_n) \rangle_{L_\mu^2} = 0.$$

In conclusion, both ζ^0 and $(\text{id}, \text{Bary}(\zeta))_{\#}\mu$ belong to Sol_μ , so that ζ is solenoidal. $\square$

One might be tempted to think that Tan_μ is $W_{T\Omega}$ -closed by the same argument. This is not correct: it does not hold that $\xi(\text{Graph } D^{\text{Tan}}) = 1$ for every tangent ξ . It *does* hold that

- Tan_μ^0 is $W_{T\Omega}$ -closed (by Corollary 2.12),
- the set of $f \in L_\mu^2(\Omega; \mathbb{R}^d)$ such that $(\text{id}, f)_{\#}\mu \in \text{Tan}_\mu$ is weakly closed in L_μ^2 (by the previous proof),

but the latter set of vector fields is not strongly closed in L_μ^2 , and the oscillations captured by the $W_{T\Omega}$ -limit may leave D^{Tan} . On the other hand, any map inducing a solenoidal field is already valued in D^{Sol} , so that oscillations can only produce measure fields that are again concentrated on D^{Sol} .

3.1. STABILITY WITH RESPECT TO RESTRICTION. — The aim of this section is to investigate how Tan_μ^0 and Sol_μ^0 depend on the local properties of the underlying measure μ . We start by showing that the centered solenoidal spaces are stable under restriction of measures. This is valid only for centered measure fields; Remark 3.9 below provides a counterexample in the case of map-induced fields.

Given a measurable set $A \subset \Omega$, denote by $\mu \llcorner A$ the measure given by $(\mu \llcorner A)(B) := \mu(A \cap B)$ for any measurable B , and by $T A \subset T\Omega$ the set $\{(x, v) \mid x \in A, v \in T_x \Omega\}$.

PROPOSITION 3.5 (Restriction of centered solenoidal measure fields). — *Let $\mu \in \mathcal{P}_2(\Omega)$ and $A \subset \Omega$ be measurable with $\mu(A) \in (0, 1)$. Denote $\mu_A := (\mu \llcorner A)/\mu(A)$. Then*

$$\text{Sol}_{\mu_A}^0 = \{(\zeta \llcorner T A)/\mu(A) \mid \zeta \in \text{Sol}_\mu^0\}.$$

Proposition 3.5 relies on the following intermediate results, whose proofs are deferred to the appendix. The statements are formulated for Sol_μ instead of Sol_μ^0 , since centeredness plays no role there.

LEMMA 3.6. — *Let $\mu = (1 - \lambda)\mu_1 + \lambda\mu_2 \in \mathcal{P}_2(\Omega)$ for $\mu_i \in \mathcal{P}_2(\Omega)$ and $\lambda \in [0, 1]$. Let $\eta \in \mathcal{P}_2(T\Omega)_{\mu_1}$ be the velocity of a geodesic, with $\nu := (\pi_x + \pi_v)_{\#}\eta$ compactly supported. There exists $\gamma \in \mathcal{P}_2(T\Omega)_{\mu_2}$ such that*

$$(7) \quad \xi = (1 - \lambda)\eta + \lambda\gamma$$

is the velocity of a geodesic starting from μ .

Lemma 3.6 does not stand without the assumption that ν has compact support. Indeed, consider for instance $\mu_1 = \delta_0$, $\mu_2 = \delta_1$ and ν a Gaussian measure in dimension one. There is only one η such that $(\pi_x, \pi_x + \pi_v)_{\#}\eta$ is optimal between μ_1 and ν , and its support is not bounded in the velocity variable. To construct a γ satisfying (7), one should be able to ensure that for any $(x, v) \in \text{supp } \eta$ and $(y, w) \in \text{supp } \gamma$, the

monotonicity condition $|v|^2 + |w|^2 \leq |x + v - y|^2 + |y + w - x|^2$ holds. Equivalently, using that $x = 0$ and $y = 1$, one should have $2v - 1 \leq 1 + 2w$; since v can be arbitrarily large, no finite w satisfies this. Consequently, in order to use Lemma 3.6, we characterize Sol_μ by orthogonality with respect to velocities of geodesics ending at compactly supported measures.

LEMMA 3.7. — *Let $\zeta \in \mathcal{P}_2(\text{T}\Omega)_\mu$ satisfy $\langle \zeta, (\pi_x, \pi_y - \pi_x)_\# \gamma \rangle_\mu = 0$ for any $\gamma \in \Gamma_o(\mu, \nu)$ with $\nu \in \mathcal{P}_2(\Omega)$ compactly supported. Then $\zeta \in \text{Sol}_\mu$.*

We now turn to our original claim. In the proof, we use the formula given by [Gig08, Prop. 4.2] to deduce a Chasles relation for the metric scalar product, stating that for $\mu \in \mathcal{P}_2(\Omega)$ and $A \subset \Omega$ measurable,

$$\langle \xi, \zeta \rangle_\mu = \int_{x \in \Omega} \langle \xi_x, \zeta_x \rangle_{\delta_x} d\mu(x) = \int_{x \in A} \langle \xi_x, \zeta_x \rangle_{\delta_x} d\mu(x) + \int_{x \in A^c} \langle \xi_x, \zeta_x \rangle_{\delta_x} d\mu(x).$$

Proof of Proposition 3.5. — We proceed by double inclusion. Consider $\zeta \in \text{Sol}_\mu^0$, and let $\zeta_A := (\zeta \llcorner A)/\mu(A)$ and $\zeta_{A^c} := (\zeta \llcorner A^c)/\mu(A^c)$. By Lemma 3.7, it suffices to show that $\langle \zeta_A, \xi^A \rangle_{\mu_A} = 0$ for any $\xi^A \in \mathcal{P}_2(\text{T}\Omega)_{\mu_A}$ which induces a geodesic between μ_A and a compactly supported measure. Consider such a ξ^A . Applying Lemma 3.6 with $\mu_1 = \mu_A$, $\mu_2 = \mu_{A^c}$ and $1 - \lambda = \mu(A)$, we obtain a measure field $\xi^{A^c} \in \mathcal{P}_2(\text{T}\Omega)_{\mu_{A^c}}$ such that $\xi = \mu(A)\xi^A + \mu(A^c)\xi^{A^c}$ belongs to Tan_μ . As μ_A and μ_{A^c} are mutually singular, we have $\xi^A = (\xi \llcorner A)/\mu(A)$. Hence

$$(8) \quad 0 = \langle \zeta, \xi \rangle_\mu = \mu(A)\langle \zeta_A, \xi^A \rangle_{\mu_A} + \mu(A^c)\langle \zeta_{A^c}, \xi^{A^c} \rangle_{\mu_{A^c}} \geq \mu(A)\langle \zeta_A, \xi^A \rangle_{\mu_A} \geq 0,$$

where we used that ζ_A and ζ_{A^c} are centered (see Lemma 2.5). Hence $\langle \zeta_A, \xi^A \rangle_{\mu_A} = 0$, and $\zeta_A \in \text{Sol}_{\mu_A}^0$.

Conversely, let $\zeta^A \in \text{Sol}_{\mu_A}^0$, and consider $\zeta := \mu(A)\zeta^A + \mu(A^c)0_{\mu_{A^c}}$. In particular, $\zeta^A = (\zeta \llcorner A)/\mu(A)$. Let $\xi \in \mathcal{P}_2(\text{T}\Omega)_\mu$ be the velocity of a geodesic. By restriction of optimality [Vil09, Th. 4.6], $\xi_A := (\xi \llcorner A)/\mu(A)$ is also the velocity of a geodesic; moreover, by Chasles,

$$\langle \zeta, \xi \rangle_\mu = \mu(A)\langle \zeta^A, \xi_A \rangle_{\mu_A} + \mu(A^c)\langle 0_{\mu_{A^c}}, \xi_{A^c} \rangle_{\mu_{A^c}} = \mu(A)\langle \zeta^A, \xi_A \rangle_{\mu_A} = 0.$$

Hence $\zeta \in \text{Sol}_\mu^0$, and any solenoidal measure field in $\text{Sol}_{\mu_A}^0$ can be written as the restriction of an element of Sol_μ^0 . $\square$

As a corollary, we deduce the corresponding statement for Tan_μ^0 .

COROLLARY 3.8 (Restriction of centered tangent measure fields). — *With the same notation as in Proposition 3.5,*

$$\text{Tan}_{\mu_A}^0 = \{(\xi \llcorner A)/\mu(A) \mid \xi \in \text{Tan}_\mu^0\}.$$

Proof. — First let $\xi \in \text{Tan}_\mu^0$, and denote $\xi_A := (\xi \llcorner A)/\mu(A)$. By Proposition 3.5, any $\zeta^A \in \text{Sol}_{\mu_A}^0$ can be written as $(\zeta \llcorner A)/\mu(A)$ for some $\zeta \in \text{Sol}_\mu^0$. Hence, using Chasles as in (8), $0 = \langle \xi, \zeta \rangle_\mu \geq \mu(A)\langle \xi_A, \zeta^A \rangle_{\mu_A} \geq 0$, so that $\xi_A \in \text{Tan}_{\mu_A}^0$. Conversely, if $\xi^A \in \text{Tan}_{\mu_A}^0$, define $\xi := \mu(A)\xi^A + \mu(A^c)0_{\mu_{A^c}} \in \mathcal{P}_2(\text{T}\Omega)_\mu^0$. For any $\zeta \in \text{Sol}_\mu^0$,

one has $\langle \xi, \zeta \rangle_\mu = \mu(A) \langle \xi^A, \zeta_A \rangle_{\mu_A}$, where $\zeta_A := (\zeta \llcorner T A) / \mu(A)$ belongs to $\text{Sol}_{\mu_A}^0$ by Proposition 3.5. Hence $\langle \xi, \zeta \rangle_\mu = 0$, and $\xi \in \text{Tan}_\mu^0$, completing the proof. $\square$

We emphasize that the argument is not exactly symmetric between Tan_μ^0 and Sol_μ^0 ; the difficulty lies in Lemma 3.6, where an *optimal* plan attached to a measure is “extended” to an optimal plan attached to another measure. Despite many attempts, the author could not find a direct proof of an extension result for *tangent* measure fields: when letting the optimal time decrease to 0, there is no guarantee that the narrow/Wasserstein limit remains tangent. However, optimal plans are sufficient to characterize solenoidal measure fields, so we can first prove the restriction property for Sol_μ^0 , and then transfer it to Tan_μ^0 .

REMARK 3.9 (Necessity of the centered assumption). — Let $\mu \in \mathcal{P}_2(\Omega)$ be the normalized 1-Hausdorff measure restricted to the boundary of the unit square $S := \partial[0, 1]^2$. Parametrize S by a constant-speed closed curve $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ rotating clockwise, and let $\zeta := (\gamma, \dot{\gamma})_\# \mathcal{L}_{[0, 1]}$. We claim that ζ is solenoidal; since it is induced by a map, this is equivalent to $\langle \xi, \zeta \rangle_\mu = 0$ for any map-induced tangent ξ . Any such ξ can be approximated arbitrarily well with respect to W_μ by $(\text{id}, \nabla \varphi)_\# \mu$ for some $\varphi \in \mathcal{C}_c^\infty(\Omega; \mathbb{R})$ [Gig11]. As $\langle (\text{id}, \nabla \varphi)_\# \mu, \zeta \rangle_\mu = \int_{[0, 1]} \frac{d}{dt} \varphi \circ \gamma dt = 0$, the measure field ζ is solenoidal.

However, if $\nu := 4\mu \llcorner [0, 1] \times \{1\}$ is the (normalized) restriction of μ to the top side of the square, then the corresponding restriction $\varsigma := (\text{id}, (1, 0))_\# \nu$ belongs to Tan_ν , since it induces a geodesic. The reader may check that the centered fields $\frac{1}{2} [\zeta + (\pi_x, -\pi_v)_\# \zeta]$ and $\frac{1}{2} [\varsigma + (\pi_x, -\pi_v)_\# \varsigma]$ are both solenoidal.

3.2. DECOMPOSITION ACCORDING TO THE DIMENSION OF SPLITTING. — With the above material, we can now state our first main result.

THEOREM 3.10. — Let $\mu \in \mathcal{P}_2(\Omega)$. There exists a decomposition $\mu = \sum_{k=0}^d m_k \mu^k$, where $m_k \in [0, 1]$ sum to one and $\mu^k \in \mathcal{P}_2(\Omega)$ are mutually singular measures, with the following properties. For each $k \in \llbracket 0, d \rrbracket$, there exists a Grassmannian section $D_k : \Omega \rightrightarrows \mathbb{R}^d$ such that $\dim D_k \equiv k$, and

- (i) $\zeta \in \text{Sol}_\mu^0$ if and only if $\zeta = \sum_{k=0}^d m_k \zeta^k$ with $\zeta^k \in \text{Sol}_{\mu^k}^0$ for $k \in \llbracket 0, d \rrbracket$.
- (ii) $\xi \in \text{Tan}_\mu^0$ if and only if $\xi = \sum_{k=0}^d m_k \xi^k$ with $\xi^k \in \text{Tan}_{\mu^k}^0$ for $k \in \llbracket 0, d \rrbracket$.
- (iii) If $m_k > 0$, $\zeta^k \in \text{Sol}_{\mu^k}^0$ if and only if ζ^k is centered and concentrated on $\text{Graph } D_k$.
- (iv) If $m_k > 0$, $\xi^k \in \text{Tan}_{\mu^k}^0$ if and only if ξ^k is centered and concentrated on $\text{Graph } D_k^\perp$.

In addition, the measures $m_k \mu^k$ in the decomposition are unique. This follows directly from the explicit formula given in Theorem 4.10. Figure 1 illustrates Theorem 3.10.

Proof of Theorem 3.10. — By Corollary 3.3, there exists a Grassmannian section $D : \Omega \rightrightarrows \mathbb{R}^d$ such that $\zeta \in \text{Sol}_\mu^0$ if and only if ζ is centered and concentrated on $\text{Graph } D$,



FIGURE 1. A measure $\mu \in \mathcal{P}_2(\mathbb{R}^2)$ and its section $D^{\text{Tan}} = D^\perp$ characterizing Tan_μ^0 .

The measure μ^0 is the sum of the three atoms, and D_0^{Tan} is 2-dimensional. The measure μ^1 is supported on a countable union of DC_1 sets, which, in dimension 2, are graphs of DC functions up to permuting the axes. The direction of the one-dimensional Grassmannian section D_1^{Tan} is represented by the arrows, whose length is irrelevant. The fact that D_1^{Tan} is orthogonal to the set on which μ^1 is concentrated will be proved in Proposition 4.12. The remaining measure μ^2 is transport-regular in the sense that it gives 0 mass to any DC_1 set, and satisfies the conclusion of the Brenier–McCann theorem. Consequently, no optimal plan splits mass, and $D_2^{\text{Tan}} \equiv \{0\}$.

and $\xi \in \text{Tan}_\mu^0$ if and only if $\xi \in \mathcal{P}_2(\text{T}\Omega)_\mu^0$ and ξ is concentrated on $\text{Graph } D^\perp$. For each $k \in \llbracket 0, d \rrbracket$, define

$$A_k := \{x \in \Omega \mid \dim D(x) = k\}.$$

Each set A_k is measurable; indeed, by [Roc69, Th. 3.(d)], we have

$$D(x) = \overline{\text{conv} \{g_n(x) \mid n \in \mathbb{N}\}}$$

for some countable family $(g_n)_{n \in \mathbb{N}}$ of measurable maps. Then A_k can be written as the set of x such that any choice of $k+1$ vectors $g_n(x)$ is linearly dependent, and there exist k linearly independent vectors $(g_{n_j}(x))_{j \in \llbracket 1, k \rrbracket}$, i.e.,

$$\bigcap_{\sigma: \llbracket 1, k+1 \rrbracket \rightarrow \mathbb{N}} \{\det_{k+1}(g_{\sigma_1}, \dots, g_{\sigma_{k+1}}) = 0\} \cap \bigcup_{\theta: \llbracket 1, k \rrbracket \rightarrow \mathbb{N}} \{\det_k(g_{\theta(1)}, \dots, g_{\theta(k)}) \neq 0\}.$$

Here $\det_j : (\mathbb{R}^d)^j \rightarrow \mathbb{R}$ is the j -determinant, which is continuous. As each g_n is measurable, the set A_k is itself measurable.

If $m_k := \mu(A_k) > 0$, define $\mu^k := (\mu \llcorner A^k) / \mu(A_k)$. Since $(A_k)_{k=0}^d$ is a partition of Ω , the measures μ^k are mutually singular, and $\mu = \sum_{k=0}^d m_k \mu^k$. Let

$$D_k(x) := \begin{cases} D(x) & x \in A_k, \\ \text{vect}\{e_1, \dots, e_k\} & \text{otherwise.} \end{cases}$$

Clearly, D_k is measurable. We now show that μ^k and D_k satisfy the claims.

Let $\zeta \in \text{Sol}_\mu^0$, and write it as $\sum_{k=0}^d m_k \zeta^k$, where $m_k \zeta^k := \zeta \llcorner A_k$. In particular, ζ^k is centered. For each k such that $m_k > 0$, Proposition 3.5 yields that $\zeta^k \in \text{Sol}_{\mu^k}^0$. This proves the first implication of point (i). Conversely, let $(\zeta^k)_{k \in \llbracket 0, d \rrbracket}$ be a family of centered measure fields such that $\zeta^k \in \text{Sol}_{\mu^k}^0$, and define $\zeta := \sum_{k=0}^d m_k \zeta^k$. To show

that $\zeta \in \text{Sol}_\mu^0$, let $\xi \in \text{Tan}_\mu^0$, decomposed as $\sum_{k=0}^d m_k \xi^k$. By Chasles,

$$\langle \zeta, \xi \rangle_\mu = \sum_{k=0}^d m_k \langle \zeta^k, \xi^k \rangle_{\mu^k}.$$

By Corollary 3.8, each ξ^k belongs to $\text{Tan}_{\mu^k}^0$, so that every term of the sum vanishes. Hence $\zeta \in (\text{Tan}_\mu^0)^{\perp 0} = \text{Sol}_\mu^0$. This proves (i); the argument is completely symmetric for (ii), with Proposition 3.5 in place of Corollary 3.8.

We turn to point (iii). If $\zeta^k \in \text{Sol}_{\mu^k}^0$, then, by (i), the measure field $\zeta := m_k \zeta^k + \sum_{j=0, j \neq k}^d m_j 0_{\mu^j}$ belongs to Sol_μ^0 . As such, it is concentrated on $\text{Graph } D$. Therefore so is ζ^k , and since $D = D_k$ for μ^k -almost every x , we get that $\zeta^k(\text{Graph } D_k) = 1$. On the other hand, let $\zeta^k \in \mathcal{P}_2(\text{T}\Omega)_{\mu^k}^0$ be concentrated on $\text{Graph } D_k$. As $D_k = D$ μ^k -almost everywhere, ζ^k is concentrated on $\text{Graph } D$. Therefore $\zeta := m_k \zeta^k + \sum_{j=0, j \neq k}^d m_j 0_{\mu^j}$ is also concentrated on $\text{Graph } D$, and must belong to Sol_μ^0 . By restriction, $\zeta^k = (\zeta \llcorner \text{T} A_k)/m_k$ belongs to $\text{Sol}_{\mu^k}^0$. The point (iv) is proved by repeating the argument with $D^\perp$ in place of D . $\square$

4. PROPERTIES OF THE DECOMPOSITION

Theorem 3.10 relies only on the algebraic properties of tangent and solenoidal cones. We now give a refined description of the measures μ^k in terms of concentration on particular subsets. The argument is based on Zajíček's theorem, recalled below, which characterizes non-differentiability points of convex functions using DC_k sets. We start with some lemmas specific to DC_k and $\sigma\text{-DC}_k$ sets, and then return to the measures $(\mu^k)_k$.

4.1. PRELIMINARIES ON DC_k SETS

DEFINITION 4.1 (DC_k and $\sigma\text{-DC}_k$ sets). — Let $k \in \llbracket 0, d \rrbracket$. A set $A \subset \mathbb{R}^d$ is a *difference of convex functions of dimension k* , denoted DC_k , if $A = \Phi(\mathbb{R}^k)$ for some function $\Phi : \mathbb{R}^k \rightarrow \mathbb{R}^d$ which, up to a permutation of coordinates, can be written as

$$(9) \quad \Phi(X_k) = (X_k, (\varphi_{k+1} - \psi_{k+1})(X_k), \dots, (\varphi_d - \psi_d)(X_k))$$

where $\varphi_j, \psi_j : \mathbb{R}^k \rightarrow \mathbb{R}$ are convex functions for $j \in \llbracket k+1, d \rrbracket$. A set that can be covered by countably many DC_k sets will be called $\sigma\text{-DC}_k$.

DC_k sets are called “c-c hypersurfaces of dimension k ” in the original work of Zajíček [Zaj79], and “ δ -convex surfaces of dimension k ” by [Pav04]. By convention, DC_0 sets are points, and the only DC_d set is $\mathbb{R}^d$. The parametrization yields that DC_k sets are closed; with the notation of (9), a Cauchy sequence in $A = \Phi(\mathbb{R}^k)$ necessarily has its first k components converging in $\mathbb{R}^k$, hence converges to an element of A by continuity of the functions φ_j, ψ_j .

REMARK 4.2 (Composition). — Let $0 \leq j \leq k \leq d$. If $A \subset \mathbb{R}^k$ is $\sigma\text{-DC}_j$, and $\Phi : \mathbb{R}^k \rightarrow \mathbb{R}^d$ can be written as a permutation of $(\text{id}_{\mathbb{R}^k}, \varphi_{k+1} - \psi_{k+1}, \dots, \varphi_d - \psi_d)$ for convex functions $\varphi_j, \psi_j : \mathbb{R}^k \rightarrow \mathbb{R}$, then $\Phi(A)$ is $\sigma\text{-DC}_j$ as well. Indeed, this is

trivial if $j = 0$ (since A is countable) or $j = k$ (since $A = \mathbb{R}^k$). Otherwise, let $(A_n)_n$ be a countable family of DC_j sets covering A , each written as $\Phi_n(\mathbb{R}^j)$ for some Φ_n of the form (9). The composition $\Phi \circ \Phi_n$ coincides with the identity on j of its coordinates. Since compositions of DC functions are still DC by [Har59, (I) & (II)], the remaining coordinates of $\Phi \circ \Phi_n$ are DC functions from $\mathbb{R}^j$ to $\mathbb{R}$. Therefore the sets $(\Phi \circ \Phi_n(A_n))_{n \in \mathbb{N}}$ are a countable family of DC_j sets covering A , i.e., A is σ - DC_j .

THEOREM (Zajíček's theorem). — *Let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a convex function. Then each set*

$$(10) \quad J_k(\varphi) := \{x \in \mathbb{R}^d \mid \dim \partial_x \varphi \geq d - k\}$$

is σ - DC_k , i.e., can be covered by countably many DC_k sets. Conversely, if $A \subset \mathbb{R}^d$ is σ - DC_k , there exists $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ convex such that $A \subset J_k(\varphi)$.

If one is given a single DC_k set A , then the convex function can be chosen “uniformly non-differentiable” over A . This is merely an observation on the construction of [Zaj79], but it will be used in the sequel.

LEMMA 4.3. — *Let $A = \Phi(\mathbb{R}^k) \subset \mathbb{R}^d$ be a DC_k set, with Φ as in Definition 4.1. There exist a convex function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ and $d - k + 1$ measurable vector fields $f_k, \dots, f_d$ such that $f_j(x) \in \partial_x \phi$ for each x , and whenever $x \in A$, the vectors $f_k(x), \dots, f_d(x)$ are at distance at least one from each other.*

Proof. — We may permute the coordinates of the space so as to write

$$\Phi = (\text{id}, \varphi_{k+1} - \psi_{k+1}, \dots, \varphi_d - \psi_d)$$

for some convex functions $\varphi_j, \psi_j : \mathbb{R}^k \rightarrow \mathbb{R}$. Let $\phi := \sum_{j=k+1}^d \phi_j$, where each ϕ_j is given by

$$\phi_j(x_1, \dots, x_d) := \max(x_j + \psi_j(x_1, \dots, x_k), \varphi_j(x_1, \dots, x_k)).$$

Then ϕ_j and ϕ are convex. For convenience, write $x = (X^k, x_{k+1}, \dots, x_d)$ whenever $x \in \mathbb{R}^d$. For $j \in \llbracket k+1, d \rrbracket$, let $v_j, w_j : \mathbb{R}^k \rightarrow \mathbb{R}^k$ be measurable selections of $\partial \varphi_j$ and $\partial \psi_j$ respectively. Then $\partial_x \phi_j$ contains the vectors $(v_j(X^k), 0) \in \mathbb{R}^d$ and $(w_j(X^k), 0) + e_j$, with $e_j \in \mathbb{R}^d$ the j^{th} vector of the canonical basis. Indeed, since $x_j = \varphi_j(X^k) - \psi_j(X^k)$, we have, for any $y = (Y^k, y_{k+1}, \dots, y_d)$,

$$\begin{aligned} \phi_j(y) &= \max(y_j + \psi_j(Y^k), \varphi_j(Y^k)) \\ &\geq \max(x_j + \psi_j(X^k) + (y_j - x_j) + \langle w_j(X^k), Y^k - X^k \rangle, \\ &\quad \varphi_j(X^k) + \langle v_j(X^k), Y^k - X^k \rangle) \\ &= \phi_j(x) + \max_{z \in \{(w_j(X^k), 0) + e_j, (v_j(X^k), 0)\}} \langle z, y - x \rangle. \end{aligned}$$

Recall that $\partial_x \phi = \bigoplus_{j=k+1}^d \partial_x \phi_j$ [Roc70, 10.9 & 7.27]. Hence $\partial_x \phi$ contains the vector $\sum_{j=k+1}^d (v_j(X^k), 0)$ and each $\sum_{j=k+1, j \neq \ell}^d (v_j(X^k), 0) + (w_\ell(X^k), 0) + e_\ell$ for $\ell \in \llbracket k+1, d \rrbracket$, which are at pairwise distance at least one. $\square$

The next lemma allows us to pass from the direction of the non-differentiability sets of a convex function to that of the subdifferential. A function φ is semiconvex if $\varphi + \lambda|\cdot|^2/2$ is convex for some $\lambda \in \mathbb{R}$.

LEMMA 4.4. — *Let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ be semiconvex, $A \subset J_k(\varphi)$ be DC_k , and $x \in A$ such that*

- *the dimension of $\partial_x \varphi$ is exactly $d - k$,*
- *the function Φ parametrizing A as in (9) is differentiable at x ,*
- *there exist $\varepsilon > 0$ and $r > 0$ such that for any $y \in A \cap \overline{\mathcal{B}}(x, r)$, the set $\partial_y \varphi$ contains $d - k + 1$ vectors at pairwise distance at least ε .*

Then $\text{Im } \nabla \Phi(x)$ is orthogonal to $\text{span}\{\partial_x \varphi - p\}$ for any $p \in \partial_x \varphi$.

Note that the vector space $\text{span}\{\partial_x \varphi - p\}$ is independent of the point $p \in \partial_x \varphi$. Intuitively, Lemma 4.4 generalizes the observation that $\partial_x \varphi$ is orthogonal to any differentiable surface contained in its set of minimum points. The assumption that $\partial_y \varphi$ is sufficiently large for y close to x could be replaced, for instance, by requiring that all $\partial_y \varphi$ near x contain a given (relative) interior point of $\partial_x \varphi$.

Proof. — The function $\Phi : \mathbb{R}^k \rightarrow \mathbb{R}^d$ is injective, since it coincides with the identity on k of its coordinates. Let $X := \Phi^{-1}(x) \in \mathbb{R}^k$ and $(h_n)_n \subset (0, 1)$ be a sequence converging to 0. Fix $e \in \mathbb{R}^k$, and denote $x_n := \Phi(X + h_n e) \in A$. By assumption, for sufficiently large n , there exist vectors $(w_n^\ell)_{\ell \in \llbracket k, d \rrbracket} \subset \partial_{x_n} \varphi$ at distance at least ε from each other. Since φ is locally Lipschitz, the sequences $(w_n^\ell)_n$ are relatively compact. Extracting subsequences successively, we may assume that $w_n^\ell \rightarrow w^\ell$. By upper semicontinuity, each w^ℓ belongs to $\partial_x \varphi$, and is still at distance at least ε from $w^{\ell'}$ for $\ell' \neq \ell$. Hence the vectors $(w^\ell - w^k)_{\ell \in \llbracket k+1, d \rrbracket}$ span a space of dimension $d - k$, which must coincide with $\text{span}\{\partial_x \varphi - w^k\}$ since the latter has dimension $d - k$.

Denoting by λ the semiconvexity constant of φ , we have, for any $\ell, \ell' \in \llbracket k, d \rrbracket$,

$$\begin{aligned} \varphi(x_n) &\geq \varphi(x) + \langle w^\ell, x_n - x \rangle - \frac{\lambda}{2} |x_n - x|^2 \\ &\geq \varphi(x_n) + \langle w_n^{\ell'}, x - x_n \rangle + \langle w^\ell, x_n - x \rangle - \lambda |x_n - x|^2. \end{aligned}$$

Dividing by $h_n > 0$ and letting $n \rightarrow \infty$, we get

$$\begin{aligned} 0 &\geq \lim_{n \rightarrow \infty} \left\langle w^\ell - w_n^{\ell'}, \frac{\Phi(X + h_n e) - \Phi(X)}{h_n} \right\rangle - \frac{\lambda}{h_n} |\Phi(X + h_n e) - \Phi(X)|^2 \\ &= \langle w^\ell - w^{\ell'}, \nabla \Phi(X)(e) \rangle. \end{aligned}$$

Interchanging ℓ and ℓ' , we obtain that $\text{Im } \nabla \Phi(X)$ is orthogonal to $\text{span}\{\partial_x \varphi - w^k\}$. $\square$

We consider the following definition of differentiability, tailored to σ - DC_k sets.

DEFINITION 4.5 (Tangent plane to a σ - DC_k set). — Let A be a σ - DC_k set contained in $\bigcup_n A_n$ for DC_k sets A_n . Let $\mathbb{N}_x \subset \mathbb{N}$ be the subset of $n \in \mathbb{N}$ such that $x \in A_n$, and for $n \in \mathbb{N}_x$, denote by Φ_n a permutation of $(\text{id}_{\mathbb{R}^k}, \varphi_{k+1}^n - \psi_{k+1}^n, \dots, \varphi_d^n - \psi_d^n)$ such

that $A_n = \Phi_n(\mathbb{R}^k)$. Then A admits P as a tangent plane at x if for each $n \in \mathbb{N}_x$ and $j \in \llbracket k+1, d \rrbracket$, each φ_j^n, ψ_j^n is differentiable at $\Phi_n^{-1}(x)$, and $\text{Im } \nabla \Phi_n(x) = P$.

LEMMA 4.6. — *Let $(A_n)_{n \in \mathbb{N}}$ be a family of DC_k sets, and denote $A := \bigcup_{n \in \mathbb{N}} A_n$. Then there exists a $\sigma\text{-}DC_{k-1}$ set $B \subset A$ such that A admits a tangent plane P at any point $x \in A \setminus B$ in the sense of Definition 4.5.*

Proof. — For each n , denote by $\Phi_n : \mathbb{R}^k \rightarrow \mathbb{R}^d$ a function as in (9) such that $A_n = \Phi_n(\mathbb{R}^k)$. Let $B^{(n)} \subset \mathbb{R}^k$ be the union of the non-differentiability sets of the functions φ_j, ψ_j for $j \in \llbracket k+1, d \rrbracket$. Each set $B^{(n)}$ is $\sigma\text{-}DC_{k-1}$, and by Remark 4.2, the image $\Phi_n(B^{(n)}) \subset \mathbb{R}^d$ is still a $\sigma\text{-}DC_{k-1}$ subset of $\mathbb{R}^d$. Denote by $P_n(x)$ the image of $\nabla \Phi_n(x)$ at any point $x \in A_n \setminus B^{(n)}$.

We now construct a $\sigma\text{-}DC_{k-1}$ subset of A outside of which the surfaces A_n cannot intersect transversely. Precisely, for each $n \neq m$, let $B^{(n,m)}$ be the set of $x \in A_n \cap A_m \setminus (B^{(n)} \cup B^{(m)})$ such that $P_n(x) \neq P_m(x)$. We construct a convex function containing $B^{(n,m)}$ in its non-differentiability set J_{k-1} . By Lemma 4.3, there exist convex functions $\phi^{(n)}, \phi^{(m)} : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $A_n \subset J_k(\phi^{(n)})$, $A_m \subset J_k(\phi^{(m)})$, and the subdifferentials of $\phi^{(n)}, \phi^{(m)}$ on A_n, A_m contain uniformly separated points. Denote $B_{n,m} := J_{k-1}(\phi^{(n)}) \cup J_{k-1}(\phi^{(m)})$, which is $\sigma\text{-}DC_{k-1}$. By Lemma 4.4, for any $x \in B^{(n,m)} \setminus B_{n,m}$, the affine space containing $\partial_x \phi^{(n)}$ is orthogonal to $P_n(x)$, and similarly for $\partial_x \phi^{(m)}$ and $P_m(x)$. Since $P_n(x) \neq P_m(x)$, we deduce that the affine hulls of $\partial_x \phi^{(n)}$ and $\partial_x \phi^{(m)}$ are not parallel. As a consequence, the convex set

$$\partial_x(\phi^{(n)} + \phi^{(m)}) = \{v + w \mid v \in \partial_x \phi^{(n)}, w \in \partial_x \phi^{(m)}\}$$

has dimension at least $d-k+1$. Hence $B^{(n,m)} \setminus B_{n,m}$ is contained in $J_{k-1}(\phi^{(n)} + \phi^{(m)})$, which is $\sigma\text{-}DC_{k-1}$ since $\phi^{(n)} + \phi^{(m)}$ is convex. Adding back $B_{n,m}$, we obtain that $B^{(n,m)}$ is $\sigma\text{-}DC_{k-1}$.

Denote $B := \bigcup_{n \in \mathbb{N}} B^{(n)} \cup \bigcup_{n,m \in \mathbb{N}^2} B^{(n,m)}$. Then B is $\sigma\text{-}DC_{k-1}$, and if $x \in A \setminus B$, all sets A_n containing x admit a plane $P_n(x)$ as before. Additionally, $P_n(x) = P_m(x) =: P(x)$ for any $n \neq m$ such that $x \in A_n \cap A_m$, since otherwise x would be in $B^{(n,m)} \subset B$. $\square$

4.2. CHARACTERIZATION OF μ^k BY CONCENTRATION ON DC_k SETS. — We return to measures on $\Omega = \mathbb{R}^d$, with two intermediate results relating the dimension of the map $D^{\text{Tan}} = D^\perp$ appearing in Theorem 3.10 to the size of the sets on which μ can be concentrated.

PROPOSITION 4.7 (Large D^{Tan} implies thin concentrations). — *Let $0 \leq k \leq d-1$, and $\mu \in \mathcal{P}_2(\Omega)$ be a measure such that the Grassmannian section $D^{\text{Tan}} : \Omega \rightrightarrows \mathbb{R}^d$ characterizing Tan_μ^0 has dimension at least $d-k$ at μ -almost every point. Then μ is concentrated on a $\sigma\text{-}DC_k$ set.*

For $\varphi : \Omega \rightarrow \mathbb{R}$ semiconvex, denote again $J_k(\varphi) := \{x \in \Omega \mid \dim \partial_x \varphi \geq d-k\}$.

Proof. — We first consider the case of a compactly supported measure, then proceed by exhaustion. Precisely, in the first two steps, we show that whenever μ is compactly

supported and such that $D^{\text{Tan}} = D_\mu^{\text{Tan}}$ has dimension $d - k$, then there exists a measurable $\sigma\text{-DC}_k$ set B such that $\mu(B) \geq 1/2$.

Compact case: construction of B . — Let $(f_j)_{j \in \llbracket k+1, d \rrbracket}$ be measurable selections of D^{Tan} such that $(f_{k+1}(x), \dots, f_d(x))$ is an orthonormal basis of $D^{\text{Tan}}(x)$ for any $x \in \Omega$. Such maps can be constructed by the Gram–Schmidt orthogonalization algorithm applied to a Castaing representation of D^{Tan} , provided by [Roc69, Th. 3.(d)]. By Proposition 2.2, the measure field

$$\xi := \frac{1}{d-k} \sum_{j=k+1}^d \frac{1}{2} [(\text{id}, -f_j)_\# \mu + (\text{id}, f_j)_\# \mu]$$

belongs to Tan_μ^0 . We show using a Chebyshev-type inequality that if η is close to ξ with respect to W_μ , then it must put mass on balls around each $f_j(x)$ for any x belonging to a set of large μ -measure.

Let $j \in \llbracket k+1, d \rrbracket$ and $s \in \{\pm 1\}$. Since $(x, v) \mapsto |v - sf_j(x)|$ is measurable from $\text{T}\Omega$ to $\mathbb{R}$, the sets

$$A_{j,s} := \{(x, v) \in \text{T}\Omega \mid |v - sf_j(x)| \leq 1/2\} \subset \text{T}\Omega$$

are measurable. Given a set $A \subset \text{T}\Omega$, denote $A^x := \{v \mid (x, v) \in A\}$. By the disintegration theorem of [Bog07, 10.4.15], for any $\eta \in \mathcal{P}_2(\text{T}\Omega)_\mu$, there exist measures $(\eta_x)_{x \in \Omega}$ with $\eta_x \in \mathcal{P}(\mathbb{R}^d)$ such that for any $A \subset \text{T}\Omega$ measurable, $x \mapsto \eta_x(A^x)$ is measurable, and $\eta(A) = \int_{x \in \Omega} \eta_x(A^x) d\mu(x)$. Therefore, the set

$$C := \left\{ x \in \Omega \mid \begin{array}{l} \exists (j, s) \in \llbracket k+1, d \rrbracket \times \{\pm 1\} \text{ such that} \\ \eta_x(A_{j,s}^x) = \eta_x(\overline{\mathcal{B}}(sf_j(x), 1/2)) \leq \frac{1/2}{2(d-k)} \end{array} \right\}$$

is a measurable subset of Ω . Let $\xi_x = \frac{1}{d-k} \sum_{j=k+1}^d \frac{1}{2} [\delta_{(x, -f_j(x))} + \delta_{(x, f_j(x))}]$ be a particular disintegration of ξ . Whenever $x \in C$, the measure ξ_x places a mass $1/2(d-k)$ on each $\pm f_j(x)$, whereas η_x places less than $(1/2)/2(d-k)$ on at least one ball of radius $1/2$ centered at one of these points. Consequently, any transport plan α_x between η_x and ξ_x will force a mass of at least $(1/2)/2(d-k)$ to travel a distance greater than $1/2$. In integral form, $\int_{v \in \mathbb{R}^d} |v|^2 d\alpha_x \geq (1/2)/2(d-k)(1/2)^2$ for any $x \in C$. Taking the infimum over α_x , this yields $W^2(\eta_x, \xi_x) \geq 1/16(d-k)$, and integrating over x , we get that

$$(11) \quad \mu(C) = \int_{x \in \Omega} \mathbb{1}_C(x) d\mu(x) \leq \int_{x \in \Omega} \frac{W^2(\eta_x, \xi_x)}{1/(16(d-k))} d\mu(x) \leq 16(d-k)W^2(\eta, \xi).$$

Here the last inequality holds since $W^2(\xi, \eta) = \int_{x \in \Omega} W^2(\xi_x, \eta_x) d\mu$ [Gig08, Prop. 4.2]. Since ξ is tangent, there exists $\eta \in \mathcal{P}_2(\text{T}\Omega)_\mu$ inducing a geodesic on $[0, \tau]$ for some $\tau > 0$, and such that $16(d-k)W^2(\eta, \xi) \leq \varepsilon$. Let $B := C^c$. By (11), $\mu(B) \geq 1 - \varepsilon$. By definition of C , η_x puts a mass of at least $(1/2)/2(d-k)$ on each $\overline{\mathcal{B}}(\pm f_j(x), 1/2)$ for $x \in B$, so that the restricted measure

$$\gamma := \eta \llcorner \bigcup_{j \in \llbracket k+1, d \rrbracket}^{s \in \{\pm 1\}} A_{j,s}$$

still puts mass on $T B$, i.e., $B \subset \text{supp } \pi_{x\#} \gamma$. Moreover, by restriction of optimality [Vil09, Th. 4.6], γ induces a geodesic on $[0, \tau]$.

Compact case: B is σ -DC_k. — Define φ as the classical explicit Kantorovich potential [Vil09, (5.13)], i.e.,

$$\varphi(x) := \frac{1}{2\tau} \sup \left\{ |\tau v_N|^2 - |x_N + \tau v_N - x|^2 + \sum_{i=0}^{N-1} (|\tau v_i|^2 - |x_i + \tau v_i - x_{i+1}|^2) \right\},$$

where the supremum runs over $N \in \mathbb{N}$ and sequences $(x_i, v_i)_{i=1}^N \subset \text{supp } \gamma$. Then $\varphi(x) \geq (|\tau v_0|^2 - |x_0 + \tau v_0 - x|^2)/(2\tau)$ by taking $N = 0$. In particular, $\varphi(x_0) \geq 0$, and $\varphi(x_0) \leq 0$ by cyclical monotonicity. Moreover, $\varphi(x) + |x|^2/(2\tau)$ is a supremum of convex functions, hence φ is semiconvex. Using that $\sup_{a \in A} f(a) - \sup_{a \in A} g(a) \leq \sup_{a \in A} f(a) - g(a)$ whenever $\sup_A g < \infty$, we have

$$\begin{aligned} \varphi(x) - \varphi(x_0) &\leq \frac{1}{2\tau} \sup_{(x_N, v_N) \in \text{supp } \gamma} |x_N + \tau v_N - x_0|^2 - |x_N + \tau v_N - x|^2 \\ &= \frac{|x_0|^2 - |x|^2}{2\tau} + \frac{|x - x_0|}{\tau} \sup_{(x_N, v_N) \in \text{supp } \gamma} |x_N + \tau v_N|, \end{aligned}$$

with the last supremum bounded by $\text{diam supp } \mu + \tau(1 + 1/2) < \infty$. Therefore, φ is real-valued. We now show that if $(x, v) \in \text{supp } \gamma$, then $v \in \partial_x \varphi$. For any $\iota > 0$, let $(x_i, v_i)_{i=1}^N \subset \text{supp } \gamma$ be ι -optimal for the definition of $\varphi(x)$. Appending (x, v) to the sequence $(x_i, v_i)_i$, we get that for any $y \in \Omega$,

$$\begin{aligned} \varphi(y) &\geq \frac{1}{2\tau} \left[|\tau v|^2 - |x + \tau v - y|^2 + |\tau v_N|^2 - |x_N + \tau v_N - x|^2 \right. \\ &\quad \left. + \sum_{i=0}^{N-1} (|\tau v_i|^2 - |x_i + \tau v_i - x_{i+1}|^2) \right] \\ &\geq \langle v, y - x \rangle - \frac{|x - y|^2}{2\tau} + \varphi(x) - \iota. \end{aligned}$$

Letting $\iota \searrow 0$, we obtain that $v \in \partial_x \varphi$. As γ_x puts mass on balls of radius $1/2$ around each $\pm f_j(x)$ for any $x \in B \subset \text{supp } \mu$, with $|\pm f_j(x)| = 1$, the convex set $\partial_x \varphi$ has dimension at least $d - k$, so that $B \subset J_k(\varphi)$. By Zajíček's theorem, B is σ -DC_k.

General case: exhaustion. — Consider now $\mu \in \mathcal{P}_2(\Omega)$ arbitrary. Let $R_0 > 0$ be such that $\mu(\overline{\mathcal{B}}(0, R_0)) \geq 1/2$. By the restriction formula of Corollary 3.8, the centered tangent cone to $\mu_0 := \mu \llcorner \overline{\mathcal{B}}(0, R_0)/\mu(\overline{\mathcal{B}}(0, R_0))$ shares the map D^{Tan} with μ . By the previous steps, there exists a σ -DC_k set $B_0 \subset \overline{\mathcal{B}}(0, R_0)$ such that $\mu_0(B_0) \geq 1/2$, which implies $\mu(B_0) \geq 1/4$.

Assume that an increasing sequence of radii R_j and measurable σ -DC_k sets B_j have been constructed such that $\mu(B_j) \geq \mu(B_{j-1}) + (1 - \mu(B_{j-1}))/4$ for $j \in \llbracket 1, n \rrbracket$. Let R_{n+1} be such that $\mu(B_n^c \cap \overline{\mathcal{B}}(0, R_{n+1})) \geq \mu(B_n^c)/2$, and apply the previous steps to the measure $\mu_{n+1} := \mu \llcorner (B_n^c \cap \overline{\mathcal{B}}(0, R_{n+1}))/\mu(B_n^c \cap \overline{\mathcal{B}}(0, R_{n+1}))$. This yields a measurable σ -DC_k set B such that $\mu_{n+1}(B) \geq 1/2$. Up to restriction, we may assume

that $B \subset \overline{\mathcal{B}}(0, R_{n+1})$ and is disjoint from B_n . Define $B_{n+1} := B_n \cup B$. Then

$$\begin{aligned} \mu(B_{n+1}) &= \mu(B_n) + \mu(B_n^c \cap \mathcal{B}(0, R_{n+1})) \times \mu_{n+1}(B) \geq \mu(B_n) + \frac{\mu(B_n^c)}{4} \\ &= \mu(B_n) + \frac{1 - \mu(B_n)}{4}. \end{aligned}$$

The sequence $p_n := \mu(B_n)$ satisfies $p_0 \geq \frac{1}{4}$ and $p_{n+1} \geq \frac{1+3p_n}{4}$, so $\lim_{n \rightarrow \infty} p_n = 1$. Therefore, the limit B_∞ of the increasing sequence $(B_n)_n$ satisfies $\mu(B_\infty) = 1$. As a countable union of σ -DC_k sets, B_∞ is σ -DC_k. $\square$

REMARK 4.8 (Support). — The conclusion of Proposition 4.7 cannot be improved to covering the support of μ by DC_k sets. Indeed, consider $\mu = \sum_{n \in \mathbb{N}} \alpha_n \delta_{x_n}$ for a sequence $(x_n)_{n \in \mathbb{N}}$ dense in $[0, 1]$. By [Aus25a, Prop. 2.9], Tan_μ^0 is equal to $\mathcal{P}_2(\Gamma \Omega)_\mu^0$, so that D^{Tan} can be taken identically equal to $\mathbb{R}$. The measure μ is concentrated on the σ -DC₀ (countable) set $\bigcup_{n \in \mathbb{N}} \{x_n\}$, but its support is $[0, 1]$.

The following result is a partial converse to Proposition 4.7.

LEMMA 4.9 (Thin concentration implies large D^{Tan}). — Let $0 \leq k \leq d - 1$, and let $\mu \in \mathcal{P}_2(\Omega)$ give positive mass to a measurable DC_k set $A \subset \Omega$. Then the map D^{Tan} characterizing Tan_μ^0 in Proposition 2.2 satisfies $\dim D^{\text{Tan}}(x) \geq d - k$ for μ -almost every $x \in A$.

Proof. — By Lemma 4.3, there exist a convex function $\varphi : \Omega \rightarrow \mathbb{R}$ and $d - k + 1$ measurable vector fields $f_k, \dots, f_d$ such that $f_j(x) \in \partial_x \varphi$ for all x , and for any $x \in A$, $|f_j(x) - f_\ell(x)| \geq 1$ for $j \neq \ell \in \llbracket k, d \rrbracket$. Let

$$\xi := \frac{1}{d - k + 1} \sum_{j=k+1}^d \frac{(\text{id}, f_j - \text{id})_\# \mu + (\text{id}, f_k - \text{id})_\# \mu}{2}.$$

Then $(\pi_x, \pi_x + \pi_y)_\# \xi$ is concentrated on the graph of the subdifferential of the convex function φ , and ξ is the velocity of a geodesic. By Remark 3.2, the centered measure field $\xi^0 := (\pi_x, \pi_v - \text{Bary}(\xi)(\pi_x))_\# \xi$ is also tangent, and concentrated on $(x, \pm \frac{f_j(x) - f_k(x)}{2})$ for $j \in \llbracket k + 1, d \rrbracket$. Since the latter vectors are independent, ξ^0 splits mass in at least $d - k$ directions, so that $\dim D^{\text{Tan}}(x) \geq d - k$ for μ -almost every $x \in A$. $\square$

Combining Proposition 4.7 and Lemma 4.9, we arrive at the following statement.

THEOREM 4.10. — Let $\mu \in \mathcal{P}_2(\Omega)$ be decomposed as $\mu = \sum_{k=0}^d m_k \mu^k$ according to Theorem 3.10. Then, for each $k \in \llbracket 0, d \rrbracket$ such that $m_k > 0$, the measure μ^k is concentrated on a σ -DC_k set and gives 0 mass to σ -DC_{k-1} sets. Moreover, for any $k \in \llbracket 0, d \rrbracket$ and measurable set $A \subset \Omega$,

$$(12) \quad m_k \mu^k(A) = \max_{\substack{B \text{ } \sigma\text{-DC}_k \\ \text{measurable}}} \min_{\substack{C \text{ } \sigma\text{-DC}_{k-1} \\ \text{measurable}}} \mu(A \cap B \setminus C),$$

and the order of min and max can be reversed.

Proof. — For each k such that $m_k > 0$, the Grassmannian section D_k characterizing $\text{Tan}_{\mu^k}^0$ has images of dimension $d - k$. Hence, whenever $k < d$, Proposition 4.7 implies that μ^k is concentrated on a $\sigma\text{-DC}_k$ set. If $k = d$, the only $\sigma\text{-DC}_d$ set is Ω , and the statement is vacuous. For $k = 0$, we get that μ^0 is purely atomic. Moreover, if $k > 0$, then μ^k must give 0 mass to any DC_{k-1} set, since otherwise Lemma 4.9 would imply that $\dim D_k \geq d - (k - 1) = d - k + 1$ on a set of positive μ^k -measure. For each k such that $m_k > 0$, let A_k be a measurable $\sigma\text{-DC}_k$ set on which μ^k is concentrated, and define $A_k = \emptyset$ if $m_k = 0$.

Let $k \in \llbracket 0, d \rrbracket$, and let $A, B, C \subset \mathbb{R}^d$ be measurable such that B is $\sigma\text{-DC}_k$ and C is $\sigma\text{-DC}_{k-1}$. Since μ^j gives 0 mass to any $\sigma\text{-DC}_{j-1}$ set, it gives 0 mass to any $\sigma\text{-DC}_\ell$ set for $\ell \leq j - 1$. Then

$$\begin{aligned} \mu(A \cap B \setminus C) &= \sum_{j=0}^d m_j \mu^j(A \cap B \setminus C) \\ (13) \qquad &= \sum_{j=0}^{k-1} m_j \mu^j(A \cap B \setminus C) + m_k \mu^k(A \cap B). \end{aligned}$$

In particular, $\mu(A \cap B \setminus C) \geq m_k \mu^k(A \cap B)$ for any such C , with equality if C is chosen as a $\sigma\text{-DC}_{k-1}$ set containing $\bigcup_{j \in \llbracket 0, k-1 \rrbracket} A_j$. We get that

$$\min_{\substack{C \text{ } \sigma\text{-DC}_{k-1} \\ \text{measurable}}} \mu(A \cap B \setminus C) = m_k \mu^k(A \cap B).$$

Since $m_k \mu^k(A \cap B) \leq m_k \mu^k(A)$, with equality if $B = A_k$, we can take the maximum over $\sigma\text{-DC}_k$ sets B to obtain (12). Using that $\mu^j(A \cap B \setminus C) \leq \mu^j(B \setminus C)$, we can also first maximize over A in (13), then minimize over C , to obtain the same result. $\square$

As a consequence of the explicit formula (12), the measures μ^k in Theorem 3.10 are uniquely defined (if $m_k > 0$). For $k = 0$, the measure μ^0 is purely atomic, and collects the atoms of μ ; for $k = d$, the measure μ^d gives 0 mass to any DC_{d-1} set, and is the transport-regular part of μ . Let us give some examples.

EXAMPLE A. — In dimension one, the decomposition reduces to $\mu = m_0 \mu^0 + m_1 \mu^1$, where μ^0 is concentrated on a $\sigma\text{-DC}_0$ set and μ^1 gives 0 mass to $\sigma\text{-DC}_0$ sets. Since $\sigma\text{-DC}_0$ sets are exactly countable sets, we get that μ^0 is the (normalized) atomic part of μ , while μ^1 is its (normalized) diffuse part. Note that the Cantor part is classified in the same component as the absolutely continuous part of μ .

EXAMPLE B. — If $\mu = \mathcal{H}^k \llcorner \mathcal{M}$ for $\mathcal{H}^k$ the Hausdorff measure of dimension k , and $\mathcal{M}$ some k -dimensional $\mathcal{C}^2$ -manifold with $\mathcal{H}^k(\mathcal{M}) = 1$, then $\mu = \mu^k$. Indeed, covering $\mathcal{M}$ by countably many $\mathcal{C}^2$ charts, and using that $\mathcal{C}^2$ maps are DC [HU85], one sees that $\mathcal{M}$ is $\sigma\text{-DC}_k$. Moreover, DC_j sets for $j < k$ are negligible for $\mathcal{H}^k$ (as non-differentiability sets of convex functions from $\mathbb{R}^k$ to $\mathbb{R}$). The fact that in this case, the centered elements of the geometric tangent cone are concentrated on the normal directions

to $\mathcal{M}$ was observed by Lott [Lot16], and will be generalized to any $\mu \in \mathcal{P}_2(\Omega)$ in the next section.

EXAMPLE C. — The regularity in the previous example cannot be weakened to $\mathcal{C}^1$. We sketch a counterexample in dimension $d = 2$, taking inspiration from [Zaj79, Jui11]. Let $f \in \mathcal{C}([0, 1]; [0, 1])$ be continuous and nowhere approximately differentiable, given for instance by [Kha06, Chap. 6]. Define $F : x \mapsto \int_{y=0}^x f(y)dy$ and $\mu := (\text{id}, F)_\# \mathcal{L}_{[0,1]}$, with $\mathcal{L}$ the Lebesgue measure. We claim that $\mu = \mu^2$, i.e., $D^{\text{Sol}} \equiv \mathbb{R}^2$.

By the above characterization, it suffices to prove that μ gives 0 mass to any DC_1 set. By [AS94], any DC_1 set can be covered by countably many $\mathcal{C}^2$ manifolds up to a set of $\mathcal{H}^1$ -measure 0. As μ is absolutely continuous with respect to $\mathcal{H}^1$, it is enough to show that $\mu(A) = 0$ for any $\mathcal{C}^2$ manifold A . Using a parametrization of A by graphs of $\mathcal{C}^2$ functions, this follows from the fact that $\mathcal{L}_{[0,1]}(\{F = \varphi\}) = 0$ for any $\varphi \in \mathcal{C}^2([0, 1]; \mathbb{R})$. Now, if $\{F = \varphi\}$ has positive Lebesgue measure, one shows that $f(x) = \varphi'(x)$ for any density point $x \in \{F = \varphi\}$. Therefore $\mathcal{L}(\{f = \varphi'\}) > 0$, and since $\varphi' \in \mathcal{C}^1$, the function f admits φ'' as an approximate differential at any density point of $\{f = \varphi'\}$, contradicting the choice of f .

REMARK 4.11 (Difference with the decomposability bundle). — In [AM16], the authors introduce the decomposability bundle of a measure μ as a Grassmannian section constructed (roughly) as follows: consider every possible representation of μ as a superposition of 1-Hausdorff measures supported on 1-rectifiable sets, and let D^{AM} be the essential union of the approximate tangent planes to each piece. If $\mu = (\text{id}, F)_\# \mathcal{L}_{[0,1]}$ for $F \in \mathcal{C}^1([0, 1]; \mathbb{R})$, then $D^{\text{AM}}(x, F(x))$ reduces to the classical tangent plane $\mathbb{R} \cdot (1, F'(x))$. This differs from D^{Sol} in example Example C. This raises the following question: can D^{AM} be linked to a Wasserstein tangent cone for another cost?

4.3. RELATIONS WITH OTHER NOTIONS OF TANGENCY. — The reader might rightfully complain that so far, no information has been provided concerning the direction of the Grassmannian section D^{Sol} in Corollary 3.3. We now remedy this in two successive steps: first, if μ is concentrated on a $\sigma\text{-DC}_k$ set A , we show that the tangent planes to A are aligned with D^{Sol} at least μ -almost everywhere. The definition of tangent plane involved is rather strong, and might be useful in applications; however, it uses information coming from a particular set on which μ is concentrated, and lacks an “intrinsic” character. Therefore, in a second step, we show that Preiss tangent measures are concentrated on D^{Sol} at μ -almost every point.

In this section, we consider only $k \in \llbracket 1, d-1 \rrbracket$, since in the extremal cases $k = 0$ and $k = d$, the map D^{Sol} is either $\{0\}$ or the whole space $\mathbb{R}^d$.

PROPOSITION 4.12 (μ^k aligns with D_k^{Sol}). — Let $k \in \llbracket 1, d-1 \rrbracket$. Assume that $\mu \in \mathcal{P}_2(\Omega)$ is concentrated on a $\sigma\text{-DC}_k$ set A and gives 0 mass to any $\sigma\text{-DC}_{k-1}$ set. Then for

μ -almost every point x , A admits a tangent plane $P(x)$ in the sense of Definition 4.5, and $P(x) = D^{\text{Sol}}(x)$.

Proof. — Let $(A_n)_n$ be a cover of A by DC_k sets. By Lemma 4.6, there exists a $\sigma\text{-DC}_{k-1}$ set $B_0 \subset A$ such that A admits a tangent plane $P = P(x)$ for any $x \in A \setminus B_0$, in the sense of Definition 4.5. Our aim is to show that P is orthogonal to $D^{\text{Tan}} \mu$ -almost everywhere.

The beginning of the argument follows that of Lemma 4.9. For each A_n , let $\varphi_n : \Omega \rightarrow \mathbb{R}$ be a convex function given by Lemma 4.3, i.e., such that $A_n \subset J_k(\varphi_n)$ and $\partial\varphi_n$ admits $d - k + 1$ measurable selections $f_k, \dots, f_d$ that are at pairwise distance at least one over A_n . Then, for each $j \in \llbracket k + 1, d \rrbracket$, the measure field $\frac{1}{2}(\text{id}, f_k - \text{id})_{\#} \mu + \frac{1}{2}(\text{id}, f_j - \text{id})_{\#} \mu$ induces a geodesic since φ is convex, so that by Remark 3.2, its projection onto centered measure fields

$$\frac{1}{2} \left(\text{id}, \frac{f_k - f_j}{2} \right)_{\#} \mu + \frac{1}{2} \left(\text{id}, \frac{f_j - f_k}{2} \right)_{\#} \mu$$

is tangent. Since centered tangent measure fields are concentrated on $\text{Graph } D^{\text{Tan}}$ by Theorem 3.10, there exists a μ -negligible set $B_0^{(n)} \subset A$ such that $f_j(x) - f_k(x) \in D^{\text{Tan}}(x)$ for any $x \in A \setminus B_0^{(n)}$. As $D^{\text{Tan}}(x)$ is a subspace of dimension $d - k$, the independent vectors $f_j(x) - f_k(x)$ span it.

We now use the particular choice of φ_n . Recall that for any $x \in A_n \setminus B_0$, the parametrization of A_n is differentiable with differential spanning P . Denote $B_1^{(n)} := J_{k-1}(\varphi_n)$ the set of x such that $\dim \partial_x \varphi_n > d - k$, which is $\sigma\text{-DC}_{k-1}$. By Lemma 4.4, for any $x \in A_n \setminus (B_0 \cup B_1^{(n)})$,

$$\text{span} \{f_j(x) - f_k(x) \mid j \in \llbracket k + 1, d \rrbracket\} = P(x)^{\perp}.$$

Hence, for any $x \in A_n \setminus (B_0 \cup B_0^{(n)} \cup B_1^{(n)})$, we have $D^{\text{Tan}}(x) = P(x)^{\perp}$. Since μ gives 0 mass to $\sigma\text{-DC}_{k-1}$ sets, both B_0 and $B_1^{(n)}$ are μ -negligible. Hence $B := \bigcup_{n \in \mathbb{N}} B_0 \cup B_0^{(n)} \cup B_1^{(n)}$ is a μ -negligible set such that $P(x) = D^{\text{Tan}}(x)^{\perp} = D^{\text{Sol}}(x)$ for any $x \in A \setminus B$, as claimed. $\square$

The following definition is taken from [Pre87], where it is shown that any measure μ admits at least one tangent measure ν at μ -almost every point.

DEFINITION 4.13 (Tangent measure in the sense of Preiss). — Let $\mu \in \mathcal{P}(\Omega)$ and $x \in \text{supp } \mu$. A locally finite measure $\nu \in \mathcal{M}_+(\Omega)$ is tangent to μ at x in the sense of Preiss if there exist a vanishing sequence $(h_n)_{n \in \mathbb{N}} \subset (0, 1]$, and a sequence $(c_n)_{n \in \mathbb{N}} \in \mathbb{R}_*^+$, such that for any $\varphi \in \mathcal{C}_c(\Omega; \mathbb{R})$,

$$\int_{x \in \Omega} \varphi(x) d\nu(x) = \lim_{n \rightarrow \infty} c_n \int_{y \in \Omega} \varphi\left(\frac{y - x}{h_n}\right) d\mu(y).$$

PROPOSITION 4.14. — Let $k \in \llbracket 1, d - 1 \rrbracket$. Assume that $\mu \in \mathcal{P}_2(\Omega)$ is concentrated on a $\sigma\text{-DC}_k$ set A and gives 0 mass to any $\sigma\text{-DC}_{k-1}$ set. Then, for μ -almost every point x , any tangent measure $\nu \in \mathcal{M}_+$ to μ at x in the sense of Definition 4.13 is supported on $D^{\text{Sol}}(x)$.

Proof. — Again, let $(A_n)_n$ be a cover of A by DC_k sets. By Proposition 4.12, there exists a μ -negligible set $B_0 \subset A$ such that for all $x \notin B_0$, A admits $D^{\text{Sol}}(x)$ as a tangent plane in the sense of Definition 4.5. For each n , let $A^{\cup n} := \bigcup_{m \leq n} A_m$. Then $A = \bigcup_{n \in \mathbb{N}} A^{\cup n}$. By the density theorem [Fed96, 2.9.13], there exists a μ -null set $B^{(n)} \subset A^{\cup n}$ such that every $x \in A^{\cup n} \setminus B^{(n)}$ is a density point of $A^{\cup n}$, i.e.,

$$(14) \quad \lim_{h \searrow 0} \frac{\mu(A^{\cup n} \cap \mathcal{B}(x, h))}{\mu(\mathcal{B}(x, h))} = 1.$$

Let $B_1 := \bigcup_{n \in \mathbb{N}} B^{(n)}$. Then B_1 is μ -negligible, and for any $x \in A \setminus B_1$, there exists $n_x \in \mathbb{N}$ such that x is a density point of $A^{\cup n_x}$.

Let $x \in A \setminus (B_0 \cup B_1)$ be a point at which the measure μ admits a Preiss tangent measure ν . Following the notations of Definition 4.13, we let $(h_k)_k$ be a vanishing sequence and $(c_k)_k \in \mathbb{R}_+^+$ be such that $\nu = \lim_{k \rightarrow \infty} c_k(\frac{\text{id} - x}{h_k})_{\#} \mu$ in C'_c . By the choice of B_0 , the set A admits a tangent plan $P := D^{\text{Sol}}(x)$ at x in the sense of Definition 4.5. To show that ν is concentrated on P , it is enough to prove that $\nu(\mathcal{B}(0, R_m) \setminus P^\varepsilon) = 0$ for a sequence $(R_m)_m$ going to $+\infty$, and any $\varepsilon > 0$. Here $P^\varepsilon := \{y \in \Omega \mid d(y, P) \leq \varepsilon\}$. Let $(R_m)_m$ be an increasing sequence going to $+\infty$ and such that $\nu(\partial \mathcal{B}(0, R_m)) = 0$ for all m , which exists by a classical disintegration argument. For the rest of the proof, $R = R_m$ and $\varepsilon > 0$ are fixed.

Let $I_x \subset \llbracket 0, n_x \rrbracket$ be the subset of indices n such that $x \in A_n$. In particular, for each $n \in I_x$, there exists $h'_n > 0$ such that $A_n \cap \mathcal{B}(x, Rs) \subset (x + P)^{\varepsilon s}$ for any $s \in (0, h'_n)$. On the other hand, for each $n \in \llbracket 0, n_x \rrbracket \setminus I_x$, the distance $\text{dist}(x, A_n) =: \bar{h}_n$ from the point x to the closed set A_n is positive. Taking the infimum over all $h'_n, \bar{h}_n$ for $n \in \llbracket 0, n_x \rrbracket$, we obtain $\bar{h}_x > 0$ such that $A^{\cup n_x} \cap \mathcal{B}(x, Rs) \subset (x + P)^{\varepsilon s}$, hence

$$(15) \quad \frac{\mu((x + P)^{\varepsilon s} \cap A^{\cup n_x} \cap \mathcal{B}(x, Rs))}{\mu(A^{\cup n_x} \cap \mathcal{B}(x, Rs))} = 1 \quad \text{for all } s \in (0, \bar{h}_x).$$

For each $\iota \in (0, 1)$, consider $\varphi \in \mathcal{C}_c^\infty(\Omega; [0, 1])$ such that $\mathbb{1}_{\mathcal{B}(0, (1-\iota)R)} \leq \varphi \leq \mathbb{1}_{\mathcal{B}(0, R)}$, and $\psi \in \mathcal{C}_c^\infty(\Omega; [0, 1])$ such that $\mathbb{1}_{P^\varepsilon \cap \mathcal{B}(0, R)} \leq \psi \leq \mathbb{1}_{P^\varepsilon \cap \mathcal{B}(0, (1+\iota)R)}$. Then

$$\begin{aligned} \frac{\nu(P^\varepsilon \cap \mathcal{B}(0, (1+\iota)R))}{\nu(\mathcal{B}(0, (1-\iota)R))} &\geq \lim_{k \rightarrow \infty} \frac{c_k \int \psi\left(\frac{y-x}{h_k}\right) d\mu}{c_k \int \varphi\left(\frac{y-x}{h_k}\right) d\mu} \geq \lim_{k \rightarrow \infty} \frac{\mu((x + P)^{\varepsilon h_k} \cap \mathcal{B}(x, Rh_k))}{\mu(\mathcal{B}(x, Rh_k))} \\ &= \lim_{k \rightarrow \infty} \frac{\mu(A^{\cup n_x} \cap \mathcal{B}(x, Rh_k))}{\mu(\mathcal{B}(x, Rh_k))} \times \frac{\mu((x + P)^{\varepsilon h_k} \cap \mathcal{B}(x, Rh_k))}{\mu(A^{\cup n_x} \cap \mathcal{B}(x, Rh_k))} = 1. \end{aligned}$$

In the last line, we used respectively (14) and (15). Sending ι to 0, and using that $\nu(\partial \mathcal{B}(0, R)) = 0$, we get

$$\nu(P^\varepsilon \cap \overline{\mathcal{B}}(0, R)) \geq \nu(\mathcal{B}(0, R)) = \nu(P^\varepsilon \cap \overline{\mathcal{B}}(0, R)) + \nu(\mathcal{B}(0, R) \setminus P^\varepsilon).$$

Thus $\nu(\mathcal{B}(0, R) \setminus P^\varepsilon) = 0$ for all $\varepsilon > 0$ and $R = R_m$, so that ν is supported on P . $\square$

It may happen that the inclusion $\text{supp } \nu \subset D^{\text{Sol}}(x)$ is strict. For instance, in example Example C at the end of Section 4.2, the measure $(\text{id}, F)_{\#} \mu$ with $F \in \mathcal{C}^1$ admits $(\text{id}, F'(x) \cdot \text{id})_{\#} \mathcal{L}_{\mathbb{R}}$ as its unique Preiss tangent measure at $(x, F(x)) \in \mathbb{R}^2$. In particular, ν is supported on a line, but D^{Sol} is equal to $\mathbb{R}^2$ at μ -almost every point.

APPENDIX

Proof of Proposition 2.7. — The existence and uniqueness of $\pi_C^\mu \xi$ follow verbatim the steps of [Gig08, Prop. 4.30], mimicking the Hilbertian case. We show that for any $\alpha \in \Gamma_{\mu,o}(\xi, \pi_C^\mu \xi)$, the element $\gamma := (\pi_x, \pi_v - \pi_w)_\# \alpha$ belongs to $C^\perp$. Let $\zeta \in C$ be arbitrary, and $\beta = \beta(\pi_x, \pi_u, \pi_v, \pi_w) \in \Gamma_\mu(\xi, \pi_C^\mu \xi, \zeta)$ with $(\pi_x, \pi_u, \pi_v)_\# \beta = \alpha$. By horizontal convexity, $(\pi_x, (1 - \varepsilon)\pi_v + \varepsilon\pi_w)_\# \beta \in C$. Then

$$\begin{aligned} W_\mu^2(\xi, \pi_C^\mu \xi) &\leq W_\mu^2(\xi, (\pi_x, (1 - \varepsilon)\pi_v + \varepsilon\pi_w)_\# \beta) \leq \int_{(x,v,w)} |u - (1 - \varepsilon)v - \varepsilon w|^2 d\beta \\ &= W_\mu^2(\xi, \pi_C^\mu \xi) + 2\varepsilon \int \langle u - v, v - w \rangle d\beta + \varepsilon^2 \int |v - w|^2 d\beta. \end{aligned}$$

Letting ε tend to 0, we get $0 \leq \int \langle u - v, v - w \rangle d\beta$. Choosing in particular $\zeta = r \cdot \pi_C^\mu \xi \in C$ for $r > 0$, and $\beta = (\pi_x, \pi_v, \pi_w, r\pi_w)_\# \alpha$, we get that $\int \langle u - v, v \rangle d\alpha = 0$ for any $\alpha \in \Gamma_{\mu,o}(\xi, \pi_C^\mu \xi)$. Therefore, for a general $\zeta \in C$,

$$\langle \gamma, \zeta \rangle_\mu = \sup_{\substack{\beta \in \Gamma_\mu(\xi, \pi_C^\mu \xi, \zeta) \\ (\pi_x, \pi_u, \pi_v)_\# \beta = \alpha}} \int_{(x,u,v,w)} \langle u - v, w \rangle d\beta \leq \int_{(x,u,v)} \langle u - v, v \rangle d\alpha = 0.$$

As the opposite inequality holds by Lemma 2.5, we deduce that $\gamma \in C^\perp$.

Now, for any $\zeta \in C^\perp$, we introduce $\pi_C^\mu \xi$ in the definition of $W_\mu^2(\xi, \zeta)$ to get

$$\begin{aligned} W_\mu^2(\xi, \zeta) &= \inf_{\substack{\beta \in \Gamma_\mu(\xi, \pi_C^\mu \xi, \zeta) \\ (\pi_x, \pi_u, \pi_v)_\# \beta = \alpha}} \int_{(x,u,v,w)} |v + (u - v - w)|^2 d\beta \\ &= \|\pi_C^\mu \xi\|_\mu^2 + \inf_{\substack{\beta \in \Gamma_\mu(\xi, \pi_C^\mu \xi, \zeta) \\ (\pi_x, \pi_u, \pi_v)_\# \beta = \alpha}} 2 \int_{(x,u,v,w)} \langle v, u - v - w \rangle d\beta + \int_{(x,u,v,w)} |(u - v) - w|^2 d\beta \\ &\geq W_\mu^2(\xi, \gamma) + \inf_{\substack{\beta \in \Gamma_\mu(\xi, \pi_C^\mu \xi, \zeta) \\ (\pi_x, \pi_u, \pi_v)_\# \beta = \alpha}} 2 \int_{(x,u,v,w)} \langle v, -w \rangle d\beta + W_\mu^2(\gamma, \zeta). \end{aligned}$$

Here we used respectively that $(\pi_x, \pi_v, \pi_v - \pi_w)_\# \alpha$ is a transport plan between ξ and γ , that $\int \langle v, u - v \rangle d\alpha = 0$, and that $(\pi_x, \pi_u - \pi_v, \pi_w)_\# \beta$ is a transport plan between γ and ζ . The middle term is exactly $-2\langle \pi_C^\mu \xi, \zeta \rangle_\mu$, which vanishes since $\pi_C^\mu \xi \in C$ and $\zeta \in C^\perp$. Therefore, for any $\zeta \in C^\perp$, we have $W_\mu^2(\xi, \zeta) \geq W_\mu^2(\xi, \gamma) + W_\mu^2(\gamma, \zeta)$, and γ must be the metric projection of ξ onto $C^\perp$.

If now $\zeta \in C$, we have for any $\beta \in \Gamma_\mu(\xi, \pi_C^\mu \xi, \zeta)$ such that $(\pi_x, \pi_u, \pi_v)_\# \beta = \alpha$ that $\int \langle u - v, w \rangle d\beta \leq \langle \gamma, \zeta \rangle_\mu = 0$. We deduce that

$$\begin{aligned} \langle \xi, \zeta \rangle_\mu &= \sup_{\substack{\beta \in \Gamma_\mu(\xi, \pi_C^\mu \xi, \zeta) \\ (\pi_x, \pi_u, \pi_v)_\# \beta = \alpha}} \int_{(x,u,v,w)} \langle u - v + v, w \rangle d\beta \\ &\leq 0 + \sup_{\substack{\beta \in \Gamma_\mu(\xi, \pi_C^\mu \xi, \zeta) \\ (\pi_x, \pi_u, \pi_v)_\# \beta = \alpha}} \int_{(x,u,v,w)} \langle v, w \rangle d\beta = \langle \pi_C^\mu \xi, \zeta \rangle_\mu. \end{aligned}$$

If, in addition, $\zeta = -1 \cdot \zeta$, then

$$\int \langle u - v, w \rangle d\beta = - \int \langle u - v, -w \rangle d\beta \geq -\langle \gamma, -1 \cdot \zeta \rangle_\mu = -\langle \gamma, \zeta \rangle_\mu = 0,$$

and equality holds in the penultimate line, so that $\langle \xi, \zeta \rangle_\mu = \langle \pi_C^\mu \xi, \zeta \rangle_\mu$. $\square$

Proof of Lemma 3.6. — Let $M \geq 0$ be such that $\text{supp } \nu \subset \overline{\mathcal{B}}(0, M)$, and $(x_0, v_0) \in \text{supp}(\pi_x, \pi_x + \pi_v)_\# \eta$. The formula

$$\varphi(x) := \sup \left\{ \sum_{i=0}^n |x_i - y_i|^2 - \sum_{i=0}^{n-1} |x_{i+1} - y_i|^2 - |x - y_n|^2 \mid n \in \mathbb{N}, (x_i, y_i)_{i=1}^n \subset \text{supp}(\pi_x, \pi_x + \pi_v)_\# \eta \right\}$$

defines a semiconvex function from Ω to $\mathbb{R} \cup \{\infty\}$ with the property that

$$\varphi^c(y) - \varphi(x) = |x - y|^2$$

for $(\pi_x, \pi_x + \pi_v)_\# \eta$ -almost every (x, y) [Vil09, Th. 5.10]. The support of $(\pi_x, \pi_x + \pi_v)_\# \eta$ is cyclically monotone, so taking $x = x_0$, we get that $\varphi(x_0) \leq 0$. Since each y_i appearing in the supremum is contained in $\text{supp}(\pi_x + \pi_v)_\# \eta = \text{supp } \nu \subset \overline{\mathcal{B}}(0, M)$, one has

$$\begin{aligned} \varphi(x) - \varphi(x_0) &\leq \sup \left\{ -|x - y_n|^2 + |x_0 - y_n|^2 \mid \begin{array}{l} \exists x_n \in \Omega \text{ such that} \\ (x_n, y_n) \in \text{supp}(\pi_x, \pi_x + \pi_v)_\# \eta \end{array} \right\} \\ &\leq |x_0|^2 + 2M|x_0 - x| + |x|^2. \end{aligned}$$

The function φ is bounded from below by $-|x - y_0|^2 + |y_0 - x_0|^2$ by definition, hence locally bounded, and therefore locally Lipschitz since it is semiconvex. Therefore the set-valued subdifferential map $x \mapsto \partial_x (\varphi - |\cdot|^2/2)$ is compact-valued, and upper semicontinuous in the set-valued sense by [Roc70, Cor. 24.5.1]. By the selection theorem [AB06, 18.13], it admits a measurable selection $f : \Omega \rightarrow T\Omega$. Then define $\xi := (1 - \lambda)\eta + \lambda(f_\# \mu_2)$. By construction, ξ is still supported on the subdifferential of $\varphi - |\cdot|^2/2$, so $(\pi_x, \pi_x + \pi_v)_\# \xi$ is cyclically monotone, hence optimal. $\square$

Proof of Lemma 3.7

By definition, the set of measure fields of the form $\lambda \cdot (\pi_x, \pi_y - \pi_x)_\# \gamma$, where $\lambda \geq 0$ and γ is optimal, is dense in Tan_μ with respect to W_μ . The metric scalar product being continuous and positively homogeneous, one has that $\zeta \in \text{Sol}_\mu$ if and only if $\langle \zeta, (\pi_x, \pi_y - \pi_x)_\# \gamma \rangle_\mu = 0$ for any $\gamma \in \Gamma_o(\mu, \nu)$ and $\nu \in \mathcal{P}_2(\Omega)$. To prove the claim, we have to show that it suffices to consider compactly supported ν . Since we need convergence with respect to W_μ , we cannot approximate the target measure by an arbitrary compactly supported measure and use stability of optimality; hence we construct an explicit approximation. The results on c-transforms that we use may be found in [Vil09, §5], in particular Th. 5.10.

Let $\eta \in \Gamma_o(\mu, \nu)$ for some $\nu \in \mathcal{P}_2(\Omega)$. Let $\varphi : \Omega \rightarrow \mathbb{R} \cup \{\infty\}$ be a proper c-convex Kantorovich potential for the pair (μ, ν) , and $\varphi^c : y \mapsto \inf_{x \in \Omega} \varphi(x) + |x - y|^2$ its c-transform, which satisfies $\varphi(x) = \sup_{y \in \Omega} \varphi^c(y) - |x - y|^2$. Denote by $\Gamma \subset \Omega^2$ the set of (x, y) such that $\varphi^c(y) - \varphi(x) = |x - y|^2$, which contains $\text{supp } \eta$ and is cyclically monotone. Our strategy is to “truncate” Γ in the y variable, as follows.

For each $R > 0$, define

$$\varphi_R(x) := \sup_{y \in \overline{\mathcal{B}}(0, R)} \varphi^c(y) - |x - y|^2 = \sup_{y \in \Omega} \varphi^c(y) - |x - y|^2 - \chi_R(y),$$

where $\chi_R(y) = 0$ if $|y| \leq R$, and $+\infty$ otherwise. The function φ_R is c-convex by definition, bounded from above by φ , bounded from below by a quadratic polynomial, and for any $x_0 \in \text{dom } \varphi$,

$$\varphi_R(x) - \varphi_R(x_0) \leq \sup_{y \in \overline{\mathcal{B}}(0, R)} |x_0 - y|^2 - |x - y|^2 \leq |x_0|^2 + 2R|x_0 - x| - |x|^2.$$

Hence φ_R is locally bounded, and locally Lipschitz since it is semiconvex.

By a classical computation, the set $\Gamma_R \subset \Omega^2$ of pairs (x, y) such that $\varphi_R(x) = \varphi^c(y) - |x - y|^2 - \chi_R(y)$ is cyclically monotone, and contained in $\Omega \times \overline{\mathcal{B}}(0, R)$. Note that $\Gamma \cap (\Omega \times \overline{\mathcal{B}}(0, R)) \subset \Gamma_R$; indeed, whenever $(x, y) \in \Gamma$ with $|y| \leq R$, we have $\varphi(x) = \varphi^c(y) - |x - y|^2 \geq \varphi^c(z) - |x - z|^2$ for any $z \in \mathbb{R}^d$. In particular, the supremum on the ball of radius R is also attained at y , and $\varphi_R(x) = \varphi(x) = \varphi^c(y) - |x - y|^2 - \chi_R(y)$. We deduce that

$$(16) \quad \text{supp } \eta \cap (\Omega \times \overline{\mathcal{B}}(0, R)) \subset \Gamma_R.$$

For each $R > 0$, the correspondence $x \mapsto \{y \mid (x, y) \in \Gamma_R\}$ is upper semicontinuous with compact images. By [AB06, 18.13], it admits a measurable selection $f_R : \Omega \rightarrow \Omega$, which satisfies $|f_R(x)| \leq R$ for all $x \in \Omega$. Define η_R by replacing the part of η that lies outside $\Omega \times \overline{\mathcal{B}}(0, R)$ by the measurable selection f_R ; explicitly,

$$\eta_R := \eta \llcorner (\Omega \times \overline{\mathcal{B}}(0, R)) + (\pi_x, f_R(\pi_x))_{\#} \eta \llcorner (\Omega \times \overline{\mathcal{B}}(0, R)^c).$$

Recalling (16), η_R is a probability measure concentrated on the cyclically monotone set Γ_R , hence an optimal transport plan between its marginals. The measure $\pi_y \# \eta_R$ is supported on $\overline{\mathcal{B}}(0, R)$ by construction, and since $|f_R(x)| \leq R$, one has

$$\begin{aligned} W_\mu^2((\pi_x, \pi_y - \pi_x)_{\#} \eta, (\pi_x, \pi_y - \pi_x)_{\#} \eta_R) &\leq \int_{x, y \in \Omega, |y| > R} |f_R(x) - y|^2 d\eta \\ &\leq \int_{y \in \Omega, |y| > R} (R + |y|)^2 d\nu \xrightarrow{R \rightarrow \infty} 0. \end{aligned}$$

Recalling that $\langle \zeta, (\pi_x, \pi_y - \pi_x)_{\#} \eta_R \rangle_\mu = 0$ for any R by assumption, we deduce that $\langle \zeta, (\pi_x, \pi_y - \pi_x)_{\#} \eta \rangle_\mu = 0$. Since η and ν were arbitrary, ζ is solenoidal. $\square$

Conflict of interest. — For the purpose of open access, the author has applied a CC-BY public copyright license to any Author Accepted Manuscript (AAM) version arising from this submission. Data availability is not applicable to this article as no new data were created or analyzed in this study.

REFERENCES

- [AM16] G. ALBERTI & A. MARCHESI — “On the differentiability of Lipschitz functions with respect to measures in the Euclidean space”, *Geom. Funct. Anal.* **26** (2016), no. 1, p. 1–66.
- [AB06] C. ALIPRANTIS & K. BORDER — *Infinite dimensional analysis*, Springer-Verlag, Berlin, Heidelberg, 2006.

- [AGS05] L. AMBROSIO, N. GIGLI & G. SAVARÉ – *Gradient flows*, Lectures in Math. ETH Zürich, Birkhäuser-Verlag, Basel, 2005.
- [AS94] G. ANZELLOTTI & R. SERAPIONI – “Ck-rectifiable sets”, *J. reine angew. Math.* **453** (1994), p. 1–20.
- [Aus25a] A. AUSSÉDAT – “On the structure of the geometric tangent cone to the Wasserstein space”, *J. Differential Equations* **442** (2025), p. 113520.
- [Aus25b] ———, “Optimal control problems and Hamilton-Jacobi-Bellman equations in some curved metric spaces”, PhD Thesis, INSA Rouen Normandie, 2025, available at <https://theses.hal.science/tel-05133552/>.
- [Bog07] V. I. BOGACHEV – *Measure theory*, Springer, Berlin, Heidelberg, 2007.
- [Bre91] Y. BRENIER – “Polar factorization and monotone rearrangement of vector-valued functions”, *Comm. Pure Appl. Math.* **44** (1991), no. 4, p. 375–417.
- [Fed96] H. FEDERER – *Geometric measure theory*, Classics in Math., Springer, Berlin, Heidelberg, 1996.
- [Gig08] N. GIGLI – “On the geometry of the space of probability measures endowed with the quadratic optimal transport distance”, PhD Thesis, Scuola Normale Superiore di Pisa, Pisa, 2008, https://ricerca.sns.it/retrieve/e3aacdfd-ef40-4c98-e053-3705fe0acb7e/Gigli_Nicola.pdf.
- [Gig11] ———, “On the inverse implication of Brenier-McCann theorems and the structure of $(P_2(M), W_2)$ ”, *Methods Appl. Anal.* **18** (2011), no. 2, p. 127–158.
- [Har59] P. HARTMAN – “On functions representable as a difference of convex functions”, *Pacific J. Math.* **9** (1959), no. 3, p. 707–713.
- [HU85] J.-B. HIRIART-URRUTY – “Generalized differentiability/duality and optimization for problems dealing with differences of convex functions”, in *Convexity and duality in optimization* (J. Ponstein, ed.), Springer, Berlin, Heidelberg, 1985, p. 37–70.
- [Jui11] N. JUILLET – “On displacement interpolation of measures involved in Brenier’s theorem”, *Proc. Amer. Math. Soc.* **139** (2011), no. 10, p. 3623–3623.
- [Kha06] A. B. KHARAZISHVILI – *Strange functions in real analysis*, 2nd ed., Pure and Applied Math., no. 272, Chapman & Hall/CRC, Boca Raton, FL, 2006.
- [Lot16] J. LOTT – “On tangent cones in Wasserstein space”, *Proc. Amer. Math. Soc.* **145** (2016), no. 7, p. 3127–3136.
- [Pav04] D. PAVLICA – “On the points of non-differentiability of convex functions”, *Comment. Math. Univ. Carolinae* **45** (2004), no. 4, p. 727–734.
- [Pre87] D. PREISS – “Geometry of measures in $\mathbb{R}^n$: Distribution, rectifiability, and densities”, *Ann. of Math. (2)* **125** (1987), no. 3, p. 537–643.
- [Roc69] R. T. ROCKAFELLAR – “Measurable dependence of convex sets and functions on parameters”, *J. Math. Anal. Appl.* **28** (1969), no. 1, p. 4–25.
- [Roc70] ———, *Convex analysis*, Princeton University Press, Princeton, NJ, 1970.
- [San15] F. SANTAMBROGIO – *Optimal transport for applied mathematicians*, Progress in nonlinear differential equations and their appl., vol. 87, Springer International Publishing, 2015.
- [SW14] M. SEDJRO & M. WESTDICKENBERG – “Characterization of the tangent space of monotone transport plans in $\mathbb{R} \times \mathbb{R}$ with prescribed first projection”, 2014, [arXiv:1411.3836](https://arxiv.org/abs/1411.3836).
- [Vil09] C. VILLANI – *Optimal transport*, Grundlehren Math. Wissen., vol. 338, Springer Berlin Heidelberg, Berlin, Heidelberg, 2009.
- [Zaj79] L. ZAJÍČEK – “On the differentiation of convex functions in finite and infinite dimensional spaces”, *Czechoslovak Math. J.* **29** (1979), no. 3, p. 340–348.

Manuscript received 6th February 2026

accepted 2nd October 2026

AVERIL AUSSÉDAT, Dipartimento di Matematica, Università di Pisa,
 largo Pontecorvo 5, 56127 Pisa, Italy
 E-mail : averil.aussedat@dm.unipi.it
 Url : <https://averil-aussedat.github.io/>