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From Morse functions to Lefschetz fibrations on cotangent bundles

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FROM MORSE FUNCTIONS TO LEFSCHETZ FIBRATIONS ON COTANGENT BUNDLES

BY EMMANUEL GIROUX

ABSTRACT. — We prove that, for any Morse function on a compact manifold and any adapted gradient satisfying the Morse–Smale condition, there is a homotopically unique complex-valued symplectic Lefschetz fibration on the cotangent bundle whose restriction to the zero-section is the given function, whose imaginary part is the evaluation of covectors on the gradient, and which is equivariant under the actions of the fiberwise antipodal involution and the complex conjugation. Then we study the topology and symplectic geometry of the regular fibers of this fibration, which are well-defined Weinstein manifolds.

RÉSUMÉ (Des fonctions de Morse aux fibrations de Lefschetz sur les fibrés cotangents)

On démontre que, pour toute fonction de Morse sur une variété compacte et tout gradient adapté, vérifiant la condition de Morse–Smale, il existe sur le fibré cotangent une fibration de Lefschetz symplectique à valeurs complexes, unique à homotopie près, dont la restriction à la section nulle est la fonction donnée, dont la partie imaginaire est l'évaluation des covecteurs sur le gradient et qui est équivariante sous les actions de l'involution antipodale dans les fibres et de la conjugaison complexe. On étudie ensuite la topologie et la géométrie symplectique des fibres régulières de cette fibration, qui sont des variétés de Weinstein bien définies.

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Although there exist many (closed integral) symplectic manifolds beyond complex projective manifolds, it was demonstrated by S. Donaldson that all of them admit Lefschetz-type pencils with symplectic fibers [Don99], and that these can be useful for investigating their geometry. These ideas were further developed by his school,

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especially by P. Seidel who built on them to study Fukaya categories [Sei08], with a slight change of framework: in place of closed symplectic manifolds, he had to consider Liouville/Weinstein domains, and he consequently replaced Lefschetz pencils by Lefschetz fibrations. The existence of symplectic Lefschetz fibrations on Weinstein domains was then established in [GP17] by adapting Donaldson’s asymptotic methods. These fibrations are easy to define but their geometry looks quite subtle. Actually, except maybe in dimension 4, the abundant literature on Lefschetz fibrations describes rather few significant concrete examples, and the proof of the general existence result mentioned above is not very instructive. The goal of this elementary paper is to produce and analyze explicit Lefschetz fibrations on cotangent bundles:

EXTENSION THEOREM 1 (for closed manifolds). — *Let M be a closed manifold, $\varphi: M \rightarrow \mathbb{R}$ a Morse function, and ν an adapted gradient of φ which satisfies the Morse–Smale condition. Then φ extends to a (homotopically unique) symplectic Lefschetz fibration $h = f + ig: T^*M \rightarrow \mathbb{C}$ whose imaginary part is the function*

$$g: T^*M \longrightarrow \mathbb{R}, \quad (p, q) \longmapsto g(p, q) = \langle p, \nu(q) \rangle$$

and whose real part f is 1-homogeneous near infinity. In addition, h can be chosen equivariant under the actions of the fiberwise antipodal involution and the complex conjugation.

This statement requires some clarifications:

- throughout the paper, M is identified with the zero-section of T^*M , and h extends φ in the sense that $h|_M = \varphi$;
- a function $f: T^*M \rightarrow \mathbb{R}$ is d -homogeneous near infinity if there is a compact set $W_0 \subset T^*M$ such that $f(tp, q) = t^d f(p, q)$ for all $(p, q) \in T^*M \setminus W_0$ and all $t \geq 1$;
- a symplectic Lefschetz fibration is a map satisfying the axioms of Definition 3;
- “homotopically unique” should be understood in the very strong sense that the symplectic Lefschetz fibrations $h: T^*M \rightarrow \mathbb{C}$ satisfying the properties stated in Theorem 1 form a contractible space (see Remark 9);
- a vector field ν is an adapted gradient of φ if $\nu \cdot \varphi > 0$ away from the critical points, and near each critical point a , there are local coordinates $(q_1, \dots, q_n)$ centered at a (called Morse coordinates) in which

$$\varphi(q) = \varphi(a) + \sum_1^n \varepsilon_j q_j^2, \quad \varepsilon_j \in \{-1, 1\},$$

and

$$\nu(q) = 2 \sum_1^n \varepsilon_j q_j \partial_{q_j}.$$

Theorem 1 most likely still holds for an arbitrary gradient ν satisfying the Morse–Smale assumption; however, choosing an adapted gradient leads to simpler calculations and a nicer overall picture. In contrast, Theorem 1 fails if the (adapted) gradient ν violates the Morse–Smale condition (see Remark 10).

A well-known instance of a Morse function which extends to a simple Lefschetz fibration is the following:

EXAMPLE 1 (the sphere case). — Let $M = \mathbb{S}^n$ denote the unit sphere

$$M = \{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_1^{n+1} x_j^2 = 1\}$$

and consider the Morse function $\varphi: M \rightarrow \mathbb{R}$ given by the coordinate x_{n+1} restricted to M . Then (as explained in Section A) the cotangent bundle T^*M is symplectomorphic to the complex affine quadric

$$W = \{z = x + iy = (z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : \sum_1^{n+1} z_j^2 = 1\}$$

(with the symplectic form induced by the standard Kähler form of $\mathbb{C}^{n+1}$), and the restriction of the coordinate z_{n+1} to W is a holomorphic (hence symplectic) Lefschetz fibration h extending φ . Its regular fiber is the cotangent bundle of $\mathbb{S}^{n-1}$. This fibration does not quite satisfy the homogeneity condition of Theorem 1 but its real and imaginary parts have the same growth on the cotangent fibers, which is the key requirement for ensuring the correct behavior near infinity.

Theorem 1 can be easily generalized to exhausting Morse functions on non-compact manifolds; the construction is exactly the same but the Weinstein manifolds we obtain as regular fibers of the Lefschetz extension are no longer of finite type. There is also a version for Morse functions on cobordisms which we state below. By convention, a Morse function $\varphi: M \rightarrow \mathbb{R}$, where M is a compact manifold with boundary, is a function whose critical points are non-degenerate and which is locally constant and regular along ∂M . We then denote by $\partial^- M$ (resp. $\partial^+ M$) the union of the boundary components where the gradient $\nu = \nabla \varphi$ points inward (resp. outward), and we regard M as a cobordism from $\partial^- M$ to $\partial^+ M$.

EXTENSION THEOREM 2 (for cobordisms). — *Let M be a compact manifold with boundary, $\varphi: M \rightarrow \mathbb{R}$ a Morse function, and ν an adapted gradient of φ satisfying the Morse–Smale condition. Then φ extends to a (homotopically unique) map $h = f + ig: T^*M \rightarrow \mathbb{C}$ with the following properties, for $B := \varphi(M) \times i\mathbb{R} \subset \mathbb{C}$:*

- $g(p, q) = \langle p, \nu(q) \rangle$ for all $(p, q) \in T^*M$, and f is 1-homogeneous near infinity;
- $h|_{h^{-1}(B)}: h^{-1}(B) \rightarrow B$ is a symplectic Lefschetz fibration;
- T^*M retracts onto $h^{-1}(B)$ along the orbits of the Hamiltonian field of g .

To be precise, when M has a nonempty boundary, the set $h^{-1}(B)$ in the above statement is not a genuine Liouville manifold (at least as meant in this work: see the next section), but $h|_{h^{-1}(B)}: h^{-1}(B) \rightarrow \mathbb{C}$ is a symplectic Lefschetz fibration in the sense that it satisfies the three axioms of Definition 3.

Here are a few geometric properties of h which are direct consequences of the statements of Theorems 1 and 2:

- The critical points of h lie in $M \subset T^*M$ and coincide with those of φ .

- For every critical point a , the Lefschetz thimble of h over the real path reaching $\varphi(a)$ from below (resp. from above) is the conormal bundle of the stable (resp. unstable) disk of a for ν (cf. Lemma 6).

- For any regular fiber F of h , the composite map $F \hookrightarrow T^*M \rightarrow M$ is $(n-1)$ -connected, where $n := \dim M$ (the reason is that $T^*M \sim M$ is homotopically obtained from $F \times \mathbb{D}^2$ by attaching n -handles).

- For every regular value u of φ , the antipodal involution of T^*M preserves the Lefschetz fiber $F_u := h^{-1}(u)$ and reverses its symplectic form. Thus, it defines a real structure on F_u whose real locus is the regular level set $Q_u := \{\varphi = u\} \subset M$. Hence, this real structure changes drastically when u crosses a critical value of φ .

Additional properties of h will be established in Section F, namely:

- if u is again a real regular value, then F_u is a Weinstein manifold for the 1-form induced by the canonical Liouville form of T^*M , and the homotopy class of this Weinstein structure does not depend on the real regular value u (cf. Proposition 15);

- for $|u|$ sufficiently large, the projection $T^*M \setminus M \rightarrow ST^*M$ along the Liouville flow of T^*M identifies F_u with one half of the sphere conormal bundle $SN^*(\nu)$ of ν (see Section B for the definition and significance of this set);

- the whole cotangent bundle T^*M can be obtained from a product neighborhood of a regular fiber of h by attaching Weinstein handles which are exactly the Lefschetz thimbles of h over suitably chosen real arcs.

As a consequence of Theorem 1 and Proposition 15, to every “upgraded Morse function” (φ, ν) on a closed manifold M^n , one can canonically associate a Weinstein manifold F^{2n-2} of finite type, its “Lefschetz fiber”, which is the regular fiber of the (homotopically unique) symplectic Lefschetz fibration h extending φ . The geometry of F is quite interesting; it contains the vanishing cycles of the critical points along with the regular level sets of φ as exact Lagrangian submanifolds, and it can be described pretty explicitly from those objects and their incidence relations (see Section F). In short, Morse theory says that, when the real parameter u passes through a critical value, the associated regular level set $Q_u := \{\varphi = u\}$ undergoes a surgery of some index k , which means that a copy of $\mathbb{S}^{k-1} \times \mathbb{D}^{n-k}$ is removed and replaced with a copy of $\mathbb{D}^k \times \mathbb{S}^{n-k-1}$. As we will show in Lemma 5, these two copies actually live together in the Lefschetz fiber $F_u := h^{-1}(u)$ where they form an embedded Lagrangian sphere $\mathbb{S}^{n-1} = (\mathbb{S}^{k-1} \times \mathbb{D}^{n-k}) \cup (\mathbb{D}^k \times \mathbb{S}^{n-k-1})$ which is the vanishing cycle of the involved critical point of h . Furthermore, each vanishing cycle comes tagged with a Morse index. Thus, the symplectic invariants of F can be viewed as invariants of the pair (φ, ν) . This correspondence of invariants has been studied by J. Johns and M. Srecković [Joh10, Sre24] following suggestions of P. Seidel; in particular, Srecković showed that, for a closed surface M , the directed Donaldson–Fukaya category of F is equivalent to the homological flow category of (φ, ν) .

The construction of the Lefschetz fibration h extending φ roughly goes as follows: by a coarse complexification process, we first extend φ to an approximately holomorphic map $h^0: W_\delta \rightarrow \mathbb{C}$ on the small closed δ -tube W_δ about the zero-section in T^*M .

The critical points of h^0 have the required shape and its fibers are symplectic submanifolds away from critical points. This map, however, is not a fibration at all (most fibers over real values are cotangent tubes about the corresponding level sets of φ), and we need to extend it over larger tubes in T^*M in order to complete the fibers until they all have the same topology. It turns out that this can be achieved by a very simple trick: a suitable reordering of the Morse–Bott function $(\operatorname{re} h^0)|_{\partial W_\delta}$.

Credits and methods. — This work is tightly related to the work of J. Johns [Joh09], and so a few comments are in order. In his PhD thesis, Johns obtained a weak version of the extension result stated above (see [Joh09, Th. A]): for any self-indexing Morse function $\varphi: M \rightarrow \mathbb{R}$ whose critical indices, besides 0 and $n := \dim M$, lie in the interval $[(n-1)/2, (n+1)/2]$, he built a Weinstein manifold W , a symplectic Lefschetz fibration $h: W \rightarrow \mathbb{C}$, and an exact Lagrangian embedding $\iota: M \rightarrow W$ whose image contains all the critical points of h and roughly satisfies $h \circ \iota = \varphi$. He also sketched a proof that ι should be a homotopy equivalence, but he was not able to identify W with T^*M . Still, Theorem A of [Joh09] yields a Lefschetz fibration which is definitely similar to ours, and in [Joh10] Johns used it to compare the flow category of φ with the directed Donaldson–Fukaya category of $h: W \rightarrow \mathbb{C}$.

More recently, S. Lee [Lee25] (in its arXiv version v2) also proposed an algorithm producing a Lefschetz fibration on the disk cotangent bundle of a closed manifold M out of a handle decomposition of M . Presumably his construction is almost equivalent to ours, although his solution to the main problem encountered by Johns remains unclear (cf. [Lee25, §6.3 of the arXiv version v2]).

Our approach in this paper is much more direct than those of Johns and Lee, and it originated in the study of contact convexity. Example I.4.8 in [Gir91] shows that the canonical contact structure on the sphere cotangent bundle of any closed manifold is “convex” in the sense of Eliashberg–Gromov [EG91], which means that it is invariant under some gradient flow. Years later, I realized that this condition is equivalent to the existence of a “supporting open book”, and the Lefschetz fibration we construct here is a natural filling of this open book.

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A. LEFSCHETZ FIBRATIONS AND THEIR LOCAL BEHAVIOR

Let W be a manifold equipped with a Liouville form λ , that is, a 1-form whose differential $\omega := d\lambda$ is symplectic; the associated Liouville vector field $\vec{\lambda}$ is defined by $\vec{\lambda} \lrcorner \omega = \lambda$. The pair (W, λ) is called:

- a Liouville domain if W is compact and λ induces a positive contact form on ∂W oriented as the boundary of (W, ω) (the latter condition is equivalent to $\vec{\lambda}$ pointing transversely outward along ∂W);

- a Liouville manifold if W is exhausted by Liouville domains $(W_k, \lambda|_{W_k})$, $k \in \mathbb{N}$, and if the Liouville field $\vec{\lambda}$ is complete;
- a Weinstein manifold if it is a Liouville manifold and if the Liouville field $\vec{\lambda}$ is gradientlike for some (unspecified but unique up to homotopy) exhausting function; in this case, the form λ is called a Weinstein structure.

As explained in [Cie24], the term gradientlike here has the following meaning: $\vec{\lambda}$ is gradientlike (for a function $\rho: W \rightarrow \mathbb{R}$) if there is an isomorphism $L: TW \rightarrow T^*W$ which is positive ($\langle L\eta, \eta \rangle > 0$ for every nonzero tangent vector η) and which sends $\vec{\lambda}$ to the differential of some function (namely, $d\rho$); alternatively, ρ is called a Lyapunov function for $\vec{\lambda}$.

The cotangent bundle T^*M of the manifold M , endowed with its canonical Liouville form λ and symplectic structure $\omega := d\lambda$, is a Weinstein manifold: the (fiberwise radial) Liouville field $\vec{\lambda}$ is gradientlike for the kinetic energy ρ of any Riemannian metric on M . Our convention is that, if $(q_1, \dots, q_n)$ are local coordinates on M and $(p_1, \dots, p_n)$ denote the associated cotangent coordinates, then

$$\lambda = - \sum_1^n p_j dq_j, \quad \text{and} \quad \omega = \sum_1^n dq_j \wedge dp_j.$$

Symplectic Lefschetz fibrations were first introduced by S. Donaldson [Don99] together with symplectic Lefschetz pencils on closed manifolds. Nowadays, there is a specific notion of symplectic Lefschetz fibration attached to each class of symplectic manifolds (such as closed symplectic manifolds, Liouville manifolds, ...), the basic extra requirement being that the regular fiber lies in the same class as the total space. Here is the notion we use in this paper:

DEFINITION 3 (Lefschetz fibrations on Liouville manifolds). — Let (W, λ) be a Liouville manifold of dimension $2n$. A map $h: W \rightarrow \mathbb{C}$ is a *symplectic Lefschetz fibration* if the following properties hold:

- (1) each critical point of h is of complex Morse-type: it is the origin of a local complex coordinate system $(z_1, \dots, z_n)$ in which $d\lambda = \sum_1^n dx_j \wedge dy_j$, $z_j = x_j + iy_j$, and $h(z) = h(0) + \sum_1^n z_j^2 + o(|z|^2)$;
- (2) the distribution $\text{Ker } dh \subset TW$ consists of symplectic subspaces (of corank 2 except at critical points), and the singular connection formed by its symplectic orthogonal complement is complete: parallel transport does not escape to infinity in finite time, but it does collapse some points onto the critical points of h (in a way prescribed by the quadratic local model);
- (3) the manifold W is exhausted by Liouville domains $(W_k, \lambda|_{W_k})$ such that, for every $w \in \mathbb{C}$ and for all sufficiently large $k \geq k_w$, the fiber $F_w := h^{-1}(w)$ intersects ∂W_k transversely along a positive contact submanifold of ∂W_k , and in addition the Liouville field on F_w dual to $\lambda|_{F_w}$ is complete.

The latter axiom above ensures that every regular fiber of h is a Liouville manifold. Moreover, because the Liouville forms of the fibers are induced by the global

1-form λ of W , the holonomy maps given by parallel transport along the (complete) connection are exact symplectomorphisms; as a result, there is a consistent notion of exact Lagrangian submanifolds in the fibers F_w .

REMARK 4 (Lefschetz fibrations and Lyapunov functions). — Let W be a Liouville manifold, $g: W \rightarrow \mathbb{R}$ a Morse function, and η the Hamiltonian vector field of g . Then finding a map $h: W \rightarrow \mathbb{C}$ whose imaginary part equals g , whose critical points are of complex Morse-type and whose differential has a symplectic kernel is equivalent to finding a Lyapunov function f for $-\eta$. This Lyapunov viewpoint on Lefschetz fibrations will be very useful in this work.

To see that the two problems are equivalent, we first observe that, if $h: W \rightarrow \mathbb{C}$ has the desired properties, then $f = \operatorname{re} h$ is a Lyapunov function for $-\eta$. Indeed, $\operatorname{Ker} dh_a$ is a symplectic subspace of $T_a W$ at a regular point a if and only if df_a does not vanish on $\eta(a)$, and the specific shape of h near critical points imposes the sign $\eta \cdot f \leq 0$ (if g has no critical points, we may have to replace it by $-g$).

For the converse, we assume that, near each critical point a of g , there exist Darboux coordinates $(x_1, y_1, \dots, x_n, y_n)$ centered at a in which the Hessian of g at a is $2 \sum_1^n x_j y_j$, namely the imaginary part of $\sum_1^n z_j^2$, $z_j = x_j + iy_j$. (This hypothesis is not necessary but it will be satisfied in our case.) If f is a Lyapunov function for $-\eta$ and if we set $h = f + ig$, then f and g have the same critical points, and $\operatorname{Ker} dh$ is a symplectic distribution away from these critical points. Moreover, near each critical point a , the special shape of g and a simple cut-off argument (see for example the proof of [GP17, Lem. 6.6]) allow us to deform f locally, among Lyapunov functions for $-\eta$, to make its Hessian at a equal to the quadratic form $\sum_1^n x_j^2 - y_j^2 = \operatorname{re} (\sum_1^n z_j^2)$ in the Darboux coordinates adapted to g . After this modification, h has critical points of complex Morse-type.

The rest of this section is devoted to the study, under various angles, of the standard local model of symplectic Lefschetz fibrations.

Consider $\mathbb{C}^n$ with its standard symplectic form $\omega := \sum_1^n dx_j \wedge dy_j$, where the complex coordinates are $z_j = x_j + iy_j$, $1 \leq j \leq n$. This is a Weinstein manifold (actually, a Stein manifold); indeed, $\omega = d\lambda$ where $\lambda := \frac{1}{2} \sum_1^n (x_j dy_j - y_j dx_j)$ and $\lambda = d^c(|z|^2/4)$, so the Liouville field $\vec{\lambda}$ is the gradient of the function $z \mapsto |z|^2/4$. (Our convention in this paper is that $d^c \rho(_) = -d\rho(i_)$ for any function $\rho: \mathbb{C}^n \rightarrow \mathbb{R}$.)

With the above definition, the quadratic function

$$h: \mathbb{C}^n \longrightarrow \mathbb{C}, \quad z = (z_1, \dots, z_n) \longmapsto h(z) := \sum_1^n z_j^2,$$

is a symplectic Lefschetz fibration. For every $w \in \mathbb{C}$, the fiber $F_w := h^{-1}(w)$ is the complex affine quadric

$$F_w = \{z \in \mathbb{C}^n : \sum_1^n z_j^2 = w\}.$$

It has a nodal singularity at the origin if $w = 0$. Otherwise, F_w is smooth and symplectomorphic to the cotangent bundle of the sphere $\mathbb{S}^{n-1}$. Indeed, each rotation $z \mapsto e^{i\alpha} z$

preserves ω and takes F_w to $F_{e^{2i\alpha}w}$, and if $u := e^{2i\alpha}w$ is a positive real number, the map

$$F_u \longrightarrow T^*\mathbb{S}^{n-1}, \quad z = x + iy \longmapsto (p, q) = (|x|y, x/|x|),$$

is a symplectomorphism which pulls back the canonical Liouville form of $T^*\mathbb{S}^{n-1}$ to the 1-form induced by the (rotationally invariant) Liouville form λ . The $(n-1)$ -sphere

$$Z_u := F_u \cap \mathbb{R}^n = \{x \in \mathbb{R}^n : \sum_1^n x_j^2 = u\},$$

collapses to 0 as $u \rightarrow 0$ and is called the vanishing cycle of F_u . Its inverse image under the rotation $z \mapsto e^{i\alpha}z$ is the vanishing cycle Z_w in F_w ; it can be characterized as the minimum locus of the function $|z|^2$ on F_w .

The map h will play a crucial role in our construction since it provides the desired extension for any Morse function near a critical point. In the rest of this section, we review some important geometric properties of h ; though these considerations do not formally enter in the proofs of Theorems 1 and 2, they are helpful for understanding the geometry of the Lefschetz fiber we will obtain.

A-1. RECOLLECTIONS ON PARALLEL TRANSPORT. — We briefly explain here how to determine the parallel transport between fibers of h . The connection being the symplectic orthogonal complement of $\text{Ker } dh$, it is the complex line field $z \mapsto \mathbb{C}\bar{z}$ on $\mathbb{C}^n \setminus \{0\}$. Therefore, given any real vector subspace $P \subset \mathbb{R}^n$, its complex span $P_{\mathbb{C}}$ is preserved by parallel transport since it is invariant under complex conjugation. Moreover, the parallel transport in $P_{\mathbb{C}}$ has the same behavior for all P of any fixed dimension because the map h and the symplectic form ω (hence the connection) are invariant under the action of the orthogonal group $O_n \subset U_n$, which also acts transitively on the Grassmannian. Finally, the subspaces $P_{\mathbb{C}}$ with $\dim P = 2$ cover $\mathbb{C}^n$ (every point $z \in \mathbb{C}^n$ is in the complex span of $z + \bar{z}$ and $i(z - \bar{z})$), so it suffices to study parallel transport in a complex plane $P_{\mathbb{C}}$. Actually, since every fiber F_w of h meets such a $P_{\mathbb{C}}$ along a copy of $T^*\mathbb{S}^1$, the parallel transport between any two fibers can be described as a family (parameterized by P) of annulus diffeomorphisms. (This reflects the $(T^*\mathbb{S}^1 \setminus \mathbb{S}^1)$ -bundle structure of $T^*\mathbb{S}^{n-1} \setminus \mathbb{S}^{n-1}$ over the Grassmannian of planes in $\mathbb{R}^n$.)

We now focus on parallel transport over (arcs of) circles about the origin in $\mathbb{C}$, which is generated by the Hamiltonian field of the function $|h|^2$. Given a plane $P \subset \mathbb{R}^n$, we can find (appropriate and temporary) coordinates (z_1, z_2) on $P_{\mathbb{C}}$ (which are actually unitary up to a factor $\sqrt{2}$) in which $h = h|_{P_{\mathbb{C}}}$ takes the form

$$h(z_1, z_2) = z_1 z_2.$$

Then, setting $z_j = x_j + iy_j$, $j \in \{1, 2\}$, the Hamiltonian field of $|h|^2 = |z_1|^2 |z_2|^2$ reads

$$2(x_2^2 + y_2^2)(x_1 \partial_{y_1} - y_1 \partial_{x_1}) + 2(x_1^2 + y_1^2)(x_2 \partial_{y_2} - y_2 \partial_{x_2}).$$

Observing that the functions $|z_1|^2$ and $|z_2|^2$ are first integrals of this vector field, we see that its flow is given by

$$(z_1, z_2, t) \in P_{\mathbb{C}} \times \mathbb{R} \longmapsto (e^{2i|z_2|^2 t} z_1, e^{2i|z_1|^2 t} z_2) \in P_{\mathbb{C}}.$$

Thus, every solution $t \in \mathbb{R} \mapsto (z_1(t), z_2(t))$ with initial condition (z_1, z_2) at $t = 0$ satisfies

$$h(z_1(t), z_2(t)) = z_1(t)z_2(t) = e^{2i(|z_1|^2 + |z_2|^2)t} z_1 z_2 \quad \text{for all } t \in \mathbb{R}.$$

Hence, the time necessary for $h(z_1, z_2)$ to rotate by an angle α is $t = \alpha/2(|z_1|^2 + |z_2|^2)$. Combining this with the expression of the flow, we get an explicit formula for the parallel transport τ_α from a fiber $F_w \cap P_{\mathbb{C}}$ to the fiber $F_{e^{i\alpha}w} \cap P_{\mathbb{C}}$:

$$\tau_\alpha(z_1, z_2) = (e^{is\alpha} z_1, e^{i(1-s)\alpha} z_2), \quad \text{where } s := |z_2|^2 / (|z_1|^2 + |z_2|^2).$$

We can also parameterize $F_w \cap P_{\mathbb{C}}$ by the map

$$\psi_w: \mathbb{C}^* \longrightarrow P_{\mathbb{C}}, \quad z \longmapsto (z_1, z_2) = \left(|w|^{1/2} z, \frac{w}{|w|^{1/2} z} \right),$$

(normalized so as to take the unit circle to $Z_w \cap P_{\mathbb{C}}$). Then, writing $z = re^{i\theta}$, we obtain

$$\psi_{e^{i\alpha}w}^{-1} \circ \tau_\alpha \circ \psi_w(r, \theta) = \left(r, \theta + \frac{\alpha}{1+r^4} \right).$$

For $\alpha = 2\pi$, we recover the fact that the monodromy is a right-handed Dehn twist.

We henceforth reset our coordinates $z_j = x_j + iy_j$.

A-2. THE REAL FORMS OF THE MODEL. — For $0 \leq k \leq n$, let $M_k \subset \mathbb{C}^n$ denote the Lagrangian plane spanned by the coordinates $(y_1, \dots, y_k, x_{k+1}, \dots, x_n)$, namely

$$M_k := \{z = x + iy \in \mathbb{C}^n : x_1 = \dots = x_k = y_{k+1} = \dots = y_n = 0\}.$$

On M_k , the map h is real-valued and $\varphi_k := h|_{M_k}: M_k \rightarrow \mathbb{R}$ is the function

$$\varphi_k(y_1, \dots, y_k, x_{k+1}, \dots, x_n) = -y_1^2 - \dots - y_k^2 + x_{k+1}^2 + \dots + x_n^2,$$

which is the standard model for a Morse function near a critical point of index k . Accordingly, the level sets of φ_k ,

$$Q_{k,u} := F_u \cap M_k = \{\varphi_k = u\} \subset M_k, \quad u \in \mathbb{R}, \quad 0 \leq k \leq n,$$

represent the various real forms of the complex quadric F_u .

We (symplectically) identify $\mathbb{C}^n$ with T^*M_k using the map

$$z = (x' + iy', x'' + iy'') \in \mathbb{C}^k \times \mathbb{C}^{n-k} \longmapsto (p, q) = ((-x', y''), (y', x'')) \in T^*M_k.$$

In the coordinates (p, q) , the function $g(z) := \text{im } h(z) = 2 \sum_1^n x_j y_j$ takes the form $g(p, q) = \langle p, \nabla \varphi_k(q) \rangle$, as required in Theorems 1 and 2. On the other hand, in the coordinates $z_j = x_j + iy_j$, the canonical 1-form λ_k of T^*M_k reads

$$\lambda_k := \sum_1^k x_j dy_j - \sum_{k+1}^n y_j dx_j = d^c \rho_k,$$

where $\rho_k(z) := \frac{1}{2}(\sum_1^k x_j^2 + \sum_{k+1}^n y_j^2)$.

It follows that the form $\lambda_{k,w} := \lambda_k|_{F_w}$ has a dual field $\vec{\lambda}_{k,w}$ which is the gradient of $\rho_{k,w} := \rho_k|_{F_w}$. Moreover, for any $w \in \mathbb{C}^*$, a calculation shows that $\rho_{k,w}$ is a Morse–Bott function whose critical locus is the union of the following sets:

- the $(k-1)$ -sphere $Z_w \cap (\mathbb{C}^k \times \{0\})$;

- the $(n - k - 1)$ -sphere $Z_w \cap (\{0\} \times \mathbb{C}^{n-k})$;
- the intersection $F_w \cap M_k$, which is empty as soon as $w \notin \mathbb{R}$.

It is worth noting that the function $\rho_{k,w}$ is not proper if $1 \leq k \leq n - 1$, so $\lambda_{k,w}$ is not a Weinstein structure. Still, the behavior of $\rho_{k,w}$ near the vanishing cycle Z_w , especially when $w = u \in \mathbb{R}$, is helpful for analyzing the bifurcation of the Weinstein structure which occurs at a critical point (see Lemma 17).

A-3. THE VANISHING CYCLE AND THE LEVEL SETS. — We now assume that $w = u \in \mathbb{R}$. Then the fiber F_u contains two obvious exact Lagrangian submanifolds:

- the level set $Q_{k,u} = \{\varphi_k = u\} = F_u \cap M_k$, which is diffeomorphic to $\mathbb{S}^{k-1} \times \mathbb{R}^{n-k}$ if $u < 0$ and to $\mathbb{R}^k \times \mathbb{S}^{n-k-1}$ if $u > 0$;
- the vanishing cycle Z_u , which is invariant under the flow of the gradient $\vec{\lambda}_{k,u}$ of the Morse–Bott function $\rho_{k,u}$ (indeed, $\lambda_k|_{Z_u} = 0$).

These two submanifolds intersect cleanly, along $Z_u \cap (\mathbb{C}^k \times \{0\}) \simeq \mathbb{S}^{k-1}$ if $u < 0$ and along $Z_u \cap (\{0\} \times \mathbb{C}^{n-k}) \simeq \mathbb{S}^{n-k-1}$ if $u > 0$, the intersection being the attaching sphere in $Q_{k,u}$. To complete this picture, we have the following result (see also [Sre24, Th. 1.8.4]):

LEMMA 5 (parallel transport and Lagrangian surgery). — *For $u > 0$, the parallel transport $\tau_\pi: F_{-u} \rightarrow F_u$ maps the level set $Q_{k,-u}$ to an exact Lagrangian submanifold of F_u which is symplectically isotopic to the one obtained from $Q_{k,u}$ and Z_u by (the Morse–Bott version of the right-handed) Lagrangian surgery.*

Actually, the parameters involved in the Lagrangian surgery can be chosen so as to produce a Lagrangian submanifold which is Hamiltonian isotopic to $\tau_\pi(Q_{k,-u})$.

Proof. — We return to the method and the notation used in Section A-1 to study parallel transport.

We fix a plane $P \subset \mathbb{R}^n$ and use ψ_{-u} to identify $F_{-u} \cap P_{\mathbb{C}}$ with $\mathbb{C}^*$. The complexified plane $P_{\mathbb{C}}$ intersects $Z_{\pm u}$ in the circle $\psi_{\pm u}(\mathbb{S}^1)$, while $P_{\mathbb{C}}$ meets $Q_{k,\pm u} \setminus Z_{\pm u}$ if and only if $P = P' \oplus P''$, where P' and P'' are lines in $\mathbb{R}^k \times \{0\}$ and $\{0\} \times \mathbb{R}^{n-k}$, respectively, and in this case, $Q_{k,\pm u} \cap P_{\mathbb{C}}$ consists of the two rays $\psi_{\pm u}(\{\theta = 0\} \cup \{\theta = \pi\})$. The formula

$$\psi_u^{-1} \circ \tau_\pi \circ \psi_{-u}(r, \theta) = \left(r, \theta + \frac{\pi}{1 + r^4}\right),$$

and a little drawing then show that τ_π sends $Q_{k,-u} \cap P_{\mathbb{C}}$ to a curve in $F_u \cap P_{\mathbb{C}}$ which, up to isotopy, is obtained by the right-handed surgery of $Q_{k,u} \cap P_{\mathbb{C}}$ and $Z_u \cap P_{\mathbb{C}}$.

These observations, applied to all planes P , prove the lemma. $\square$

B. THE TWO LIFTS OF A VECTOR FIELD

An important basic fact in our construction is that every vector field ν on M has two natural lifts:

- a vector field $\tilde{\nu}$ on T^*M which preserves the canonical Liouville form λ ; this property (namely, the Lie derivative $\tilde{\nu} \cdot \lambda$ being zero) and the Cartan formula then imply that $\tilde{\nu}$ is Hamiltonian, with Hamiltonian function $(\tilde{\nu} \lrcorner \lambda)(p, q) = -\langle p, \nu(q) \rangle$;
- a vector field $\bar{\nu}$ on ST^*M which preserves the contact structure defined by λ ; actually, $\bar{\nu}$ is just the image of $\tilde{\nu}$ under the projection $T^*M \setminus M \rightarrow ST^*M$, where $T^*M \setminus M$ can be viewed as the symplectization of ST^*M .

The next two lemmas describe elementary properties of these two lifts.

LEMMA 6 (properties of $\tilde{\nu}$). — *Let ν be a vector field on M with non-degenerate singularities, and let $\tilde{\nu}$ denote its Hamiltonian lift on T^*M .*

(a) *The singularities of $\tilde{\nu}$ lie on the zero-section M , coincide with those of ν and are non-degenerate. Actually, $\tilde{\nu}$ is tangent to each fiber T_a^*M over a zero a of ν , and on T_a^*M , it agrees with the negative transpose of the linearization $L_a\nu$ of ν at a .*

(b) *A singularity a is hyperbolic for ν if and only if it is for $\tilde{\nu}$, and its stable (resp. unstable) manifold for $\tilde{\nu}$ is the conormal bundle of its stable (resp. unstable) manifold for ν .*

Proof. — Everything is obvious except maybe the assertion about the stable and unstable manifolds of a for $\tilde{\nu}$ when a is a hyperbolic singularity of ν . The conormal bundle of an invariant manifold for ν is an invariant manifold for $\tilde{\nu}$ but we have to show that the conormal bundle $N^*E^-(a)$ of the stable manifold $E^-(a)$ of a for ν is the stable manifold of a for $\tilde{\nu}$.

If a is a hyperbolic singularity of ν then T_a^*M splits as the direct sum of a stable subspace P_a^- and an unstable subspace P_a^+ for $\tilde{\nu}|_{T_a^*M} = -L_a\nu^\top$. Thus a is a hyperbolic zero of $\tilde{\nu}$. Next, the tangent space of $N^*E^-(a)$ at a is the direct sum of $T_aE^-(a) \subset T_aM$ and the conormal of this subspace in T_a^*M , which is P_a^- . This shows that $\tilde{\nu}$ is contracting on $T_a(N^*E^-(a))$ and implies that $N^*E^-(a)$ is contained in the stable manifold of a for $\tilde{\nu}$. Now, the stable manifold of a for $\tilde{\nu}$ is a Lagrangian submanifold, so it equals $N^*E^-(a)$. $\square$

Before describing the dynamics of $\bar{\nu}$, we recall that a contact vector field η on a contact manifold (V, ξ) has a special invariant manifold, called its “dividing hypersurface”, which is the set of points $a \in V$ where $\eta(a) \in \xi_a$. This dividing hypersurface is empty if and only if η is a Reeb vector field, it contains all the singularities of η , and it is smooth when these singularities are non-degenerate (see [Gir91]). For $\bar{\nu}$, the dividing hypersurface is the sphere conormal bundle $SN^*(\nu)$ of ν , which, by definition, is the union of the sphere conormal bundles of its orbits. It has a natural projection $\pi: SN^*(\nu) \rightarrow M$ and can be viewed as a singular sphere bundle over M : it is a smooth $\mathbb{S}^{n-2}$ -bundle over $M \setminus \{\nu = 0\}$ compactified by the fibers $ST_a^*M \simeq \mathbb{S}^{n-1}$, $a \in \{\nu = 0\}$. It can alternatively be described as the projection to ST^*M of the zero set of the Hamiltonian function g of $-\tilde{\nu}$ restricted to $T^*M \setminus M$.

LEMMA 7 (properties of $\bar{\nu}$). — *Let ν be a vector field on M and $\bar{\nu}$ its contact lift on ST^*M . We assume that each singularity a of ν is nondegenerate and that the linearization $L_a\nu$ is diagonalizable over $\mathbb{C}$.*

(a) *The singularities of $\bar{\nu}$ in ST^*M lie in the fibers over the singularities of ν . For each zero a of ν , they form spheres in ST_a^*M along which $\bar{\nu}$ is transversely non-degenerate. These are the spheres of the eigenspaces of $-L_a\nu^\top$ associated with the real eigenvalues. The non-real eigenvalues give rise to invariant spheres filled up with periodic orbits along which $\bar{\nu}$ is transversely non-degenerate provided the eigenvalue is not pure imaginary.*

(b) *Let a be a hyperbolic singularity of ν with k attracting directions, and denote by $C_a^-, C_a^+ \subset \text{ST}_a^*M$ the respective projections of the stable and unstable subspaces of $-L_a\nu^\top = \tilde{\nu}|_{T_a^*M}$. Then C_a^- is an invariant $(n - k - 1)$ -sphere which, inside the hypersurface $\text{SN}^*(\nu)$, is transversely hyperbolic with stable manifold $\text{SN}^*E^-(a)$ and unstable manifold $\pi^{-1}(E^+(a)) \setminus \text{SN}^*E^+(a)$; moreover, $\bar{\nu}$ is expanding along C_a^- in the direction normal to $\text{SN}^*(\nu)$. Similarly, C_a^+ is an invariant $(k - 1)$ -sphere which, inside $\text{SN}^*(\nu)$, is transversely hyperbolic with unstable manifold $\text{SN}^*E^+(a)$ and stable manifold $\pi^{-1}(E^-(a)) \setminus \text{SN}^*E^-(a)$, and $\bar{\nu}$ is contracting along C_a^+ in the direction normal to $\text{SN}^*(\nu)$.*

Proof. — Here again, everything is obvious except maybe the properties of the spheres C_a^- and C_a^+ over a hyperbolic singularity a of ν . First of all, we note that the sphere conormal bundle of any invariant manifold for ν lies in $\text{SN}^*(\nu)$. Hence, $\text{SN}^*(\nu)$ contains $\text{SN}^*E^-(a)$ and $\text{SN}^*E^+(a)$. Next, $\bar{\nu}$ being the projection of $\tilde{\nu}$ implies that $\text{SN}^*E^-(a)$ and $\pi^{-1}(E^+(a)) \setminus \text{SN}^*E^+(a)$ are respectively included in the stable and unstable manifolds of C_a^- , and for dimensional reasons, they equal these manifolds inside the invariant manifold $\text{SN}^*(\nu)$. Finally the behavior of $\bar{\nu}$ in the direction normal to $\text{SN}^*(\nu)$ along C_a^- follows also from the behavior of $\tilde{\nu}$ on T_a^*M along P_a^- by projection to ST_a^*M . $\square$

In the next sections, we shall apply the above considerations to an adapted gradient ν of a given Morse function $\varphi: M \rightarrow \mathbb{R}$. In this case, it is useful to note that, for any regular value u of φ , the inverse image $\pi^{-1}(Q_u) \subset \text{SN}^*(\nu)$ of the level set $Q_u := \{\varphi = u\}$ can be canonically identified with ST^*Q_u : at a point of Q_u , hyperplanes of TQ_u correspond one-to-one to the hyperplanes of TM which contain ν . The dividing hypersurface $\text{SN}^*(\nu)$ is actually a major character in our story since, as we will see at the end of the paper, it is the double of the Lefschetz fiber F of (φ, ν) . Another important remark is that, as a consequence of the Morse–Smale property of ν , the sphere conormal bundles of the stable and unstable manifolds of ν are disjoint in ST^*M (see the proof of Proposition 13).

C. FIBRATIONS WITH PRESCRIBED IMAGINARY PART

The next three sections are devoted to the proof of Theorems 1 and 2. As in the statements of those results, $\varphi: M \rightarrow \mathbb{R}$ is a Morse function and ν an adapted gradient.

We choose a Riemannian metric on M for which $\nu = \nabla\varphi$ and which is the Euclidean metric in some Morse coordinates near each critical point (such a metric is easily constructed with a partition of unity). We denote by $\rho: T^*M \rightarrow \mathbb{R}$ the associated kinetic energy:

$$\rho(p, q) := \frac{1}{2}|p|^2 \quad \text{for all } q \in M, p \in T_q^*M.$$

For every $r > 0$, we also set:

$$\begin{aligned} W_r &:= \{(p, q) \in T^*M : |p| \leq r\} \subset T^*M, \\ V_r &:= \{(p, q) \in T^*M : |p| = r\} \simeq ST^*M. \end{aligned}$$

Finally, we write $\Gamma_\varphi \subset M$ for the critical locus of φ and $\Delta_\varphi := \varphi(\Gamma_\varphi) \subset \mathbb{R}$ for its discriminant locus, and, to avoid inessential complications, we systematically assume that no two critical points have equal values: φ induces a bijection $\Gamma_\varphi \rightarrow \Delta_\varphi$.

PROPOSITION 8 (fibration criterion). — *Let M be a connected manifold, $\varphi: M \rightarrow \mathbb{R}$ a Morse function, and ν an adapted gradient of φ . Assume that the Hamiltonian lift $\tilde{\nu}$ of ν admits a Lyapunov function $f: T^*M \rightarrow \mathbb{R}$ which extends φ and is 1-homogeneous near infinity. As usual, g is the function $g(p, q) := \langle p, \nu(q) \rangle$.*

(a) *If M is closed then, after a local deformation near the critical points, the map $h := f + ig: T^*M \rightarrow \mathbb{C}$ is a symplectic Lefschetz fibration.*

(b) *If M is compact with nonempty boundary, suppose in addition that $\pm \tilde{\lambda} \cdot f \geq 0$ on $\partial^\pm T^*M := T^*M|_{\partial^\pm M}$. Then, again after a local deformation near the critical points, the map $h := f + ig: T^*M \rightarrow \mathbb{C}$ has the following properties, where $B := \varphi(M) \times i\mathbb{R} \subset \mathbb{C}$:*

- $h|_{h^{-1}(B)}: h^{-1}(B) \rightarrow B$ is a symplectic Lefschetz fibration;
- T^*M retracts onto $h^{-1}(B)$ along the orbits of $\tilde{\nu}$.

This statement does not explicitly require ν to satisfy the Morse–Smale condition because, as explained in Remark 10 below, this property is a consequence of $\tilde{\nu}$ admitting a Lyapunov function which is homogeneous near infinity.

REMARK 9 (homotopical uniqueness of the Lefschetz extension). — Since the space of Lyapunov functions for $\tilde{\nu}$ which extend φ and are 1-homogeneous near infinity is a convex set, it is contractible provided it is nonempty. The homotopical uniqueness of the Lefschetz fibration extending φ will therefore readily follow from its mere existence.

Proof. — We first observe that the critical points of $g: T^*M \rightarrow \mathbb{R}$ are the zeros of $\tilde{\nu}$, and hence coincide with the critical points of φ (see Lemma 6). Moreover, since ν is an adapted gradient, near each critical point a of φ , one can find coordinates $(q_1, \dots, q_n)$

centered at a in M in which

$$\varphi(q) = \varphi(a) + \sum_1^n \varepsilon_j q_j^2, \quad \varepsilon_j \in \{-1, 1\},$$

and

$$\nu(q) = 2 \sum_1^n \varepsilon_j q_j \partial_{q_j}.$$

If $(p_1, \dots, p_n)$ denote the associated cotangent coordinates, then

$$g(p, q) = \langle p, \nu(q) \rangle = 2 \sum_1^n \varepsilon_j p_j q_j.$$

Thus, the Morse function g satisfies the property used in Remark 4 to deform the Lyapunov function f locally near $\Gamma_\varphi \subset M \subset T^*M$ so that every critical point of $h = f + ig$ is of complex Morse-type and every fiber $F_w = h^{-1}(w)$ is a symplectic submanifold of (T^*M, ω) away from Γ_φ . Moreover, since ω is exact, the induced symplectic form on F_w is exact too. We set $\lambda_w := \lambda|_{F_w \setminus \Gamma_\varphi}$ for every $w \in \mathbb{C}$, and we denote by η the vector field on $T^*M \setminus \Gamma_\varphi$ equal to the Liouville field $\vec{\lambda}_w$ on each fiber F_w .

Now we have two completeness issues to address: the completeness of the vector field η and the completeness of the (singular) connection ζ which is the symplectic orthogonal complement of the distribution $\text{Ker } dh$. For every $t \in \mathbb{R}_{>0}$, we write $\sigma_t: T^*M \rightarrow T^*M$ the fiberwise dilation by t . Then the relations $\sigma_t^* \lambda = t\lambda$ and $\sigma_t^* h = th$ near infinity imply that η is invariant by every σ_t , $t > 0$, and hence is complete. On the other hand, $\sigma_t^*(d\rho/\rho) = d\rho/\rho$ on $T^*M \setminus M$, so there exists a constant $C > 0$ such that

$$d\rho_{(p,q)}(v) \leq C\rho(p, q) |dh_{(p,q)}(v)| \quad \text{for all } (p, q) \in T^*M \text{ and } v \in \zeta_{(p,q)}.$$

Then the completeness of ζ follows from the divergence of the integral $\int_1^\infty dx/x$; indeed, this divergence shows that any horizontal curve in T^*M along which ρ goes to infinity is mapped by h to a path of infinite length in $\mathbb{C}$.

We will prove that, for all sufficiently large $r \geq r_w$, the intersection $F_w \cap V_r$ is a positive contact submanifold of V_r . If M is closed, this implies that F_w is a Liouville manifold for every $w \in \mathbb{C} \setminus \Delta_\varphi$. If M has boundary, this conclusion remains valid for $w \in B = (\varphi(M) \oplus i\mathbb{R}) \setminus \Delta_\varphi$ due to the behavior of φ on ∂M and to the assumption that $\pm(\vec{\lambda} \cdot f) \geq 0$ on $\partial^\pm T^*M$: this shows that $F_w \cap \partial T^*M$, even though it may be nonempty, consists of points where F_w remains a smooth submanifold of T^*M and has no boundary there.

CLAIM. — *The zero set of f meets the complement of every compact subset of T^*M .*

Proof of the claim. — Choose $r > 0$ large enough that f is 1-homogeneous outside W_r , meaning that $\vec{\lambda} \cdot f = f$.

If M is closed, suppose (arguing by contradiction) that f is positive on $T^*M \setminus W_r$. The homogeneity condition $\vec{\lambda} \cdot f = f$ then implies that the level sets of f are transverse to $\vec{\lambda}$ in $T^*M \setminus W_r$. Now choose a level set $X_s := \{f = s\}$ with $s > 0$ so large that X_s is contained in $T^*M \setminus W_r$ (and is therefore diffeomorphic to ST^*M). The

condition $\tilde{\nu} \cdot f > 0$ then says that $\tilde{\nu}$ is pointing transversely upward along X_s , which is obviously impossible for a Hamiltonian vector field.

If M has boundary and ∂^-M, ∂^+M are both nonempty, the homogeneity condition $\vec{\lambda} \cdot f = f$ in $T^*M \setminus W_r$ and the sign condition $\pm(\vec{\lambda} \cdot f)|_{\partial^\pm T^*M} \geq 0$ imply that $\pm f|_{\partial^\pm T^*M \setminus W_r} \geq 0$. Hence, f vanishes in $T^*M \setminus W_r$ since M is connected.

If M has boundary but either ∂^-M or ∂^+M is empty then a mix of the two previous arguments yields the result. Assume for instance that ∂^+M is not empty but ∂^-M is. Then $f \geq 0$ on $\partial^+T^*M \setminus W_r$. If f does not vanish, we can construct a level set $X_s = \{f = s\}$ (diffeomorphic to ST^*M) which encloses a compact region of T^*M . On the boundary of this region (made up of X_s and a piece of ∂^+T^*M), the vector field $\tilde{\nu}$ points transversely outward, which is again impossible. $\square$

CLAIM. — *If $r > 0$ is so large that $\vec{\lambda} \cdot f = f$ near V_r then $F_0 \cap V_r$ is a nonempty positive contact submanifold of V_r .*

Proof of the claim. — Since f and g are homogeneous, the Liouville field $\vec{\lambda}$ is tangent to F_0 near V_r . On the other hand, $\vec{\lambda}$ points transversely outward along V_r , so it also points transversely out of $F_0 \cap W_r$ along $F_0 \cap V_r$. This means exactly that $F_0 \cap V_r$ is a (nonempty) positive contact submanifold of V_r . $\square$

To complete the proof of Proposition 8, we fix an $r_0 > 0$ such that f is homogeneous in $T^*M \setminus W_{r_0}$. Since $F_0 \cap V_{r_0}$ is a contact submanifold of V_{r_0} and this property is “open”, there exists an $\varepsilon > 0$ such that, for all $|u|, |v| \leq \varepsilon$, the intersection $F_w \cap V_{r_0}$, $w = u + iv$, is a positive contact submanifold of V_{r_0} .

Now pick any number $s > 0$ and set $r(s) = r_0 s / \varepsilon$. We claim that, for all $|u|, |v| \leq s$ and $r \geq r(s)$, the intersection $F_w \cap V_r$ is a positive contact submanifold of V_r . Indeed, the radial projection $V_r \rightarrow V_{r_0}$ takes $F_w \cap V_r$ to $F_{r_0 w/r} \cap V_{r_0}$ (due to the homogeneity of f and g), and the hypotheses $r \geq r(s) = r_0 s / \varepsilon$ and $|u|, |v| \leq s$ imply that $|r_0 u/r|, |r_0 v/r| \leq \varepsilon$, and so $F_{r_0 w/r} \cap V_{r_0}$ is a positive contact submanifold of V_{r_0} . $\square$

REMARK 10 (on the Morse–Smale condition). — We briefly explain here why the Morse–Smale condition is necessary in Theorems 1 and 2.

Assume that ν violates the Morse–Smale condition. This means that ν has an orbit γ , running from a critical point a to a critical point b , along which the unstable manifold $E^+(a)$ and the stable manifold $E^-(b)$ of ν are not transverse. Therefore, given a point $c \in \gamma$, the subspace $T_c E^+(a) + T_c E^-(b)$ lies in some hyperplane $\tau_c \subset T_c M$. Spreading τ_c by the flow of ν , we get a hyperplane field τ along γ which contains $TE^+(a) + TE^-(b)$ at every point of γ and which extends up to the end-points a and b of γ (this can be seen from the shape of ν in Morse coordinates near a and b). We then consider, over the segment $C := \gamma \cup \{a, b\}$, the real line bundle

$$R := \bigcup_{q \in C} \tau_q^\perp \subset T^*M|_C.$$

We denote by $\partial_a R$ and $\partial_b R$ the components of ∂R containing a and b , respectively. By construction (see also Lemma 6), $\tilde{\nu}$ is tangent to R as well as to ∂R , and the dynamics of $\tilde{\nu}|_R$ is very simple:

- $\tilde{\nu}|_{\partial_a R}$ (resp. $\tilde{\nu}|_{\partial_b R}$) is a dilating (resp. contracting) linear vector field on a real line;
- all orbits of $\tilde{\nu}$ in $R \setminus \partial R$ go from a to b .

Viewing C as the zero-section of R , we choose one of the two components of $R \setminus C$ and denote its closure by $R^+ \subset R$. We also define $\partial_q^+ R = \partial_q R \cap R^+$ for $q \in \{a, b\}$.

If there exists a Lefschetz fibration h of the form $h = f + ig$, then $\tilde{\nu}$ is a pseudo-gradient of f , and it follows from the dynamical behavior of $\tilde{\nu}$ on R that:

- $f(R)$ is contained in $I := [\varphi(a), \varphi(b)]$ (observe that this bound already prevents f from being homogeneous of degree 1 or more);
- $f(\partial_a^+ R)$ (resp. $f(\partial_b^+ R)$) is an open interval of I containing $\varphi(a)$ (resp. $\varphi(b)$), and these two open intervals are disjoint. For the latter claim, we argue as follows: if $f(p_0, a) = f(p_1, b) = u$, then near (p_0, a) (resp. (p_1, b)) the set $\{f = u\} \cap R$ is an arc transverse to $\partial_a^+ R$ (resp. $\partial_b^+ R$); hence, there are plenty of interior orbits which intersect both arcs, so f takes the same value twice on every such orbit, contradicting that $\tilde{\nu}$ is a pseudogradient of f .

Now take $u := f(p, a)$ for some $(p, a) \in \partial_a^+ R$ with $p \neq 0$. The above observations show that, for the symplectic connection defined by h (which is spanned by $\tilde{\nu}$ over $\mathbb{R}$), the arc $[u, \varphi(b)]$ has no horizontal lift stemming from $(p, a) \in \partial_a^+ R$. Hence, the connection is not complete, so h is not a fibration.

To summarize this discussion, if ν violates the Morse–Smale condition, then its Hamiltonian lift $\tilde{\nu}$ provides a necessarily incomplete connection between the level sets of any Lyapunov function it admits.

D. COARSE COMPLEXIFICATION OF A MORSE FUNCTION

We recall that our goal is to extend the Morse function $\varphi: M \rightarrow \mathbb{R}$ to a symplectic Lefschetz fibration $h: T^*M \rightarrow \mathbb{C}$, and in this section we define h in a neighborhood of the zero-section $M \subset T^*M$. As before, M is endowed with a Riemannian metric for which $\nu = \nabla \varphi$, and we denote by $\tilde{\nu}$ and $\bar{\nu}$ the Hamiltonian and contact lifts of ν . We set

$$\begin{aligned} W_r &:= \{(p, q) \in T^*M : |p| \leq r\} \subset T^*M, \\ V_r &:= \{(p, q) \in T^*M : |p| = r\} \simeq ST^*M, \end{aligned}$$

and we use the associated splitting

$$T^*M \setminus M = ST^*M \times \mathbb{R}_{>0}, \quad \text{with} \quad ST^*M \times \{r\} = V_r \quad \text{for all } r > 0,$$

to view $\bar{\nu}$ as a vector field on $T^*M \setminus M$ tangent to each hypersurface V_r .

Very explicitly, we consider the map $h^0: T^*M \rightarrow \mathbb{C}$ defined by

$$h^0(p, q) := \varphi(q) - \frac{1}{2}\chi(q)\nabla^2\varphi_q(p, p) + i\langle p, \nu(q) \rangle, \quad \text{for all } (p, q) \in T^*M.$$

Here $\nabla^2\varphi_q$ is the covariant second derivative of φ at q , regarded as a symmetric pairing on $T_q^*M \cong T_qM$, and $\chi: M \rightarrow [0, 1]$ is a cut-off function equal to 1 at least in a neighborhood of Γ_φ ; if M is closed, $\chi \equiv 1$ is perfectly fine, but if M has boundary it is technically convenient to take $\chi \equiv 0$ near ∂M . As a matter of fact, h^0 is a “first-order complexification” of φ . The imaginary part $g := \operatorname{im} h^0$ is the function whose Hamiltonian field is $-\tilde{\nu}$ (cf. Section B) and it will remain globally the imaginary part of our final Lefschetz fibration h . As for the real part $f^0 := \operatorname{re} h^0$, we will have to modify it far away from the zero-section (we will mostly rearrange its critical values) but it has nice basic properties near the zero-section:

LEMMA 11 (properties of h^0). — *Let M be a compact manifold, $\varphi: M \rightarrow \mathbb{R}$ a Morse function, and ν an adapted gradient of φ . Then there exists a radius $\delta > 0$ such that:*

- (1) *in W_δ , the critical points of h^0 coincide with those of φ and are of complex Morse-type;*
- (2) *in W_δ , the real part $f^0 = \operatorname{re} h^0$ is a Lyapunov function for the Hamiltonian field $\tilde{\nu}$;*
- (3) *on V_r with $0 < r \leq \delta$, the restriction $f_r^0 := f^0|_{V_r}$ is a Morse–Bott Lyapunov function for the contact field $\bar{\nu}$.*

As already explained, property (2) says that, in W_δ , the fibers of h^0 are symplectic submanifolds away from Γ_φ . On the other hand, property (3) implies that the critical submanifolds of each function f_r^0 , $0 < r \leq \delta$, are the components of the zero set of $\bar{\nu}$. Near each critical point a of φ , since the chosen metric is the standard Euclidean metric in Morse coordinates $(q_1, \dots, q_n)$ centered at a , we have

$$\nu(q) = \nabla\varphi(q) = -2q_1\partial_{q_1} - \dots - 2q_k\partial_{q_k} + 2q_{k+1}\partial_{q_{k+1}} + \dots + 2q_n\partial_{q_n}.$$

Thus, all positive (resp. negative) eigenvalues of $L_a\nu$ are equal, and the components of the zero set of $\bar{\nu}$ in ST_a^*M are the two spheres $C_a^\pm$ defined in Lemma 7(b). The corresponding critical values of f_r^0 are $f_r^0(C_a^\pm) = \varphi(a) \pm r^2$, $0 < r \leq \delta$.

Proof of the lemma. — The reason for adding $-\frac{1}{2}\nabla^2\varphi$ to φ in the real part f^0 of h^0 is that, at every critical point a of φ , the gradient of the quadratic form $-\frac{1}{2}\nabla^2\varphi_a$ on T_a^*M for the Euclidean metric is the linear vector field $-L_a\nu^\top = \tilde{\nu}|_{T_a^*M}$. Then, by orthogonal projection to any sphere $S_{a,r} := T_a^*M \cap V_r$, the gradient of the Morse–Bott function $f^0|_{S_{a,r}}$ for the round metric is $\bar{\nu}|_{S_{a,r}}$.

Furthermore, the function $(p, q) \mapsto \nabla^2\varphi_q(p, p)$ vanishes identically together with its 1-jet along the zero-section $M \subset T^*M$, so f^0 and (the pullback of) φ are arbitrarily $\mathcal{C}^1$ -close near M . As a result, for any open subset U of M containing Γ_φ , the positivity of $\nu \cdot \varphi$ on the compact set $M \setminus U$ implies that $\tilde{\nu} \cdot f^0$ and $\bar{\nu} \cdot f^0$ (where $\bar{\nu}$ is viewed as a vector field on $T^*M \setminus M$ tangent to the hypersurfaces V_r) are both positive on $W_{\delta_0} \cap T^*(M \setminus U)$ for some $\delta_0 > 0$.

To complete the proof, we study h^0 more carefully near the critical points of φ . For $a \in \Gamma_\varphi$, let $(q_1, \dots, q_n)$ be coordinates centered at a , on a neighborhood U_a , in which

the metric is Euclidean and

$$\varphi(q) = \varphi(0) + \sum_1^n \varepsilon_j q_j^2, \quad \varepsilon_j \in \{-1, +1\}.$$

In the associated cotangent coordinates, we have:

$$\begin{aligned} \nu(q) &= 2 \sum_1^n \varepsilon_j q_j \partial_{q_j}, & \nabla^2 \varphi_q(p, p) &= 2 \sum_1^n \varepsilon_j p_j^2, \\ \tilde{\nu}(p, q) &= 2 \sum_1^n \varepsilon_j (q_j \partial_{q_j} - p_j \partial_{p_j}), & g(p, q) &= 2 \sum_1^n \varepsilon_j p_j q_j. \end{aligned}$$

Thus, in the complex coordinates $z_j = q_j + ip_j$ on T^*U_a , $1 \leq j \leq n$, the map h^0 is given by

$$h^0(z) = \varphi(0) + \sum_1^n \varepsilon_j z_j^2.$$

Hence, a is the only critical point of h^0 in T^*U_a and it is of complex Morse-type, which proves (1). Furthermore,

$$(\tilde{\nu} \cdot f^0)(z) = 4 \sum_1^n |z_j|^2,$$

which proves (2).

Finally, the contact lift $\bar{\nu}$, viewed again as a vector field on $T^*M \setminus M$, has the form $\bar{\nu} = \tilde{\nu} - v\vec{\lambda}$, where the function v can be computed by evaluating the differential of the kinetic energy ρ : since $\bar{\nu} \cdot \rho = 0$ and $\vec{\lambda} \cdot \rho = 2\rho$,

$$v = \frac{\tilde{\nu} \cdot \rho}{2\rho} = \tilde{\nu} \cdot \log \sqrt{\rho}.$$

In $T^*U_a \setminus U_a$, we have $\rho = \frac{1}{2} \sum_1^n p_j^2$, and so

$$v(p, q) = -2 \frac{\sum_1^n \varepsilon_j p_j^2}{\sum_1^n p_j^2}.$$

As a result, again in $T^*U_a \setminus U_a$,

$$\begin{aligned} (\bar{\nu} \cdot f^0)(p, q) &= 4 \sum_1^n q_j^2 + 4 \frac{(\sum_1^n p_j^2)^2 - (\sum_1^n \varepsilon_j p_j^2)^2}{\sum_1^n p_j^2} \\ &= 4 \sum_1^n q_j^2 + 4 \frac{(\sum_1^n (1 + \varepsilon_j) p_j^2) \cdot (\sum_1^n (1 - \varepsilon_j) p_j^2)}{\sum_1^n p_j^2}. \end{aligned}$$

This function is non-negative and vanishes exactly (and identically) along the spheres $C_a^\pm$. The transverse non-degeneracy of these critical submanifolds is clear in the q -directions and, in the p -directions (lying in the fiber T_a^*M), it follows from the observation that every non-degenerate quadratic form on a Euclidean space restricts to a Morse–Bott function on any sphere. This concludes the proof of (3). $\square$

REMARK 12 (properties of v). — We quickly work out here a few properties of the function v which we will need to prove Proposition 13. For any point $a \in \Gamma_\varphi$, in the same complex coordinates $z_j = q_j + ip_j$ as before, $1 \leq j \leq n$, the spheres $C_a^\pm \subset \text{ST}_a^*M$ are given by

$$C_a^\pm = \{p \in \mathbb{S}^{n-1} \simeq \text{ST}_a^*M : p_j = 0 \text{ if } \varepsilon_j = \pm 1\} \subset \text{ST}_a^*M = \{q = 0\}.$$

Then the explicit expression of v yields $v|_{C_a^\pm} = \pm 2$. Furthermore, a little calculation (similar to the previous ones) shows that, on ST_a^*M ,

$$(\bar{\nu} \cdot v)(z) = 8 \frac{(\sum_1^n (1 + \varepsilon_j) p_j^2) \cdot (\sum_1^n (1 - \varepsilon_j) p_j^2)}{(\sum_1^n p_j^2)^2}.$$

This function is clearly non-negative and vanishes exactly on the zero set of $\bar{\nu}|_{\text{ST}_a^*M}$, namely $C_a^- \cup C_a^+$.

E. REARRANGEMENT OF CRITICAL VALUES

We recall that a Morse function f is said to be *ordered* if the order of critical values is consistent with that of indices: $\text{ind}(a) < \text{ind}(b)$ implies $f(a) < f(b)$ for any two critical points a, b . It is well-known that, if the gradient of f (for some metric) satisfies the Morse–Smale condition, then one can deform f through Morse functions with the same gradient (for different metrics) to an ordered Morse function [Sma61, Th. B]. This rearrangement process, applied to the Morse–Bott functions $f_r^0: V_r \simeq \text{ST}^*M \rightarrow \mathbb{R}$ of Lemma 11 (with $r \leq \delta$), is the key trick we need for constructing our symplectic Lefschetz fibration.

PROPOSITION 13 (rearrangement process). — *Let M be a compact manifold, $\varphi: M \rightarrow \mathbb{R}$ a Morse function, and ν an adapted gradient of φ which satisfies the Morse–Smale condition. Then the contact vector field $\bar{\nu}$ admits a family of Lyapunov Morse–Bott functions $f_r: \text{ST}^*M \rightarrow \mathbb{R}$ ($r > 0$) with the following properties:*

- (1) *for r sufficiently small, $f_r = f_r^0: V_r \cong \text{ST}^*M \rightarrow \mathbb{R}$;*
- (2) *for r sufficiently large, the function $f_\infty := f_r/r: \text{ST}^*M \rightarrow \mathbb{R}$ is independent of r and transverse to 0;*
- (3) *for any $r > 0$, the function $\pm \partial_r f_r$ is non-negative near $\partial^\pm \text{ST}^*M$ while it is positive on $C_a^\pm$ for every $a \in \Gamma_\varphi$;*
- (4) *the function $f: \text{T}^*M \rightarrow \mathbb{R}$ given by $f|_{V_r} := f_r$ and $f|_M := \varphi$ is a smooth Lyapunov function for the Hamiltonian field $\tilde{\nu}$;*
- (5) *the function $f: \text{T}^*M \rightarrow \mathbb{R}$ is invariant under the fiberwise antipodal involution.*

We recall here that $\partial^\pm \text{ST}^*M := \text{ST}^*M|_{\partial^\pm M}$ and $C_a^-, C_a^+ \subset \text{ST}_a^*M$ are the projections of the stable and unstable subspaces of $-\text{L}_a \nu^\top$ (the negative transpose of the linearization of ν at a).

Proof. — The key remark is that, for any $a, b \in \Gamma_\varphi$, no trajectories of $\bar{\nu}$ go from C_a^+ to C_b^- . Indeed, by Lemma 7, the stable manifold of C_b^- for $\bar{\nu}$ is the sphere conormal bundle of the stable manifold $E^-(b)$ of b for ν while the unstable manifold of C_a^+

for $\bar{\nu}$ is the sphere conormal bundle of the unstable manifold $E^+(a)$ of a for ν . But since ν satisfies the Morse–Smale condition, the submanifolds $E^+(a)$ and $E^-(b)$ are transverse to each other, and hence their sphere conormal bundles do not intersect. The next lemma is a direct consequence of this remark. We define

$$A^\pm := \partial^\pm \text{ST}^*M \cup \bigcup_{a \in \Gamma_\varphi} E^\pm(C_a^\pm),$$

where E^- and E^+ denote here the stable and unstable manifolds for $\bar{\nu}$. We also recall that the vector fields $\tilde{\nu}$ and $\bar{\nu}$ on $\text{T}^*M \setminus M$ satisfy a relation of the form

$$\bar{\nu} = \tilde{\nu} - v\vec{\lambda},$$

where $v = \tilde{\nu} \cdot \log r$ is a function independent of r (namely, the pullback of some function on ST^*M).

LEMMA 14 (slope function). — *The contact vector field $\bar{\nu}$ admits a Lyapunov function $f_\infty: \text{ST}^*M \rightarrow \mathbb{R}$ such that:*

- (i) f_∞ is negative on A^- and positive on A^+ ;
- (ii) $(\bar{\nu} \cdot f_\infty) + v f_\infty > 0$ everywhere on ST^*M ;
- (iii) $v f_\infty \geq 0$ in ST^*U for some neighborhood U of Γ_φ in M .

Proof of the lemma. — Since the sets $A^\pm$ defined above are disjoint they possess (small, and hence disjoint) neighborhoods $H^\pm$ whose boundaries are smooth hypersurfaces transverse to $\bar{\nu}$ and which retract onto $A^\pm$ along the flow lines of $\bar{\nu}$. Concretely, $H^\pm$ is obtained from a collar neighborhood of $\partial^\pm \text{ST}^*M$ by successive attachments of Morse–Bott handles, following the order of critical values for H^- and the inverse order for H^+ . Then $\partial H^\pm$ contains $\partial^\pm \text{ST}^*M$, and we set

$$K^\pm := \partial H^\pm \setminus \partial^\pm \text{ST}^*M.$$

With these notations, the closure of $\text{ST}^*M \setminus (H^- \cup H^+)$ is a cobordism in which all orbits of $\bar{\nu}$ go from K^- to K^+ , so it is a product. Let $K \subset \text{ST}^*M \setminus (H^- \cup H^+)$ be a closed hypersurface transverse to $\bar{\nu}$. For any $a \in \Gamma_\varphi$, every orbit of $\bar{\nu}$ in $\text{ST}_a^*M \setminus (C_a^- \cup C_a^+)$ meets K in one point and also contains a unique zero of the function v , with $\bar{\nu} \cdot v > 0$ at that point (see Remark 12). Hence we can slide K along the orbits of $\bar{\nu}$ to make it agree with $\{v = 0\}$ in ST^*U for some neighborhood U of Γ_φ in M . Now, using the handlebody structure of $H^\pm$, it is not hard to construct a function $f_\infty: \text{ST}^*M \rightarrow \mathbb{R}$ with the following properties:

- f_∞ is a Lyapunov function for $\bar{\nu}$ and its zero set is K , so it is negative on the component V^- of $\text{ST}^*M \setminus K$ containing H^- and positive on the other component V^+ containing H^+ ;
- f_∞ is increasing so fast along $\bar{\nu}$ that $\bar{\nu} \cdot \log |f_\infty| > -v$ in V^+ and $\bar{\nu} \cdot \log |f_\infty| < -v$ in V^- .

The construction of a function satisfying the first property is standard in Morse theory [Sma61], and its generalization to the Morse–Bott case is straightforward. The second property is a minor add-on to the first. It holds trivially on the zero set of $\bar{\nu}$

because v is negative on the spheres C_a^- and positive on the spheres C_a^+ ; it can be achieved globally because the function $\log |f_\infty|$ is not bounded below (it tends to $-\infty$ on both sides of K). In practice, the germ of f_∞ along K can be chosen a priori with $f_\infty|_K = 0$ and $(\bar{v} \cdot f_\infty)|_K > 0$, and its extensions over V^- and V^+ can then be performed independently. The properties stated in the lemma are direct consequences of the above ones. In particular, since v and f_∞ have the same sign at every point of ST^*U , their product is non-negative. $\square$

We now return to the proof of the proposition. Before defining the family f_r , we note that the function $f^1: \text{T}^*M \setminus M \rightarrow \mathbb{R}$ given by $f^1|_{V_r} = rf_\infty$ satisfies

$$\tilde{v} \cdot f^1 = r((\bar{v} \cdot f_\infty) + vf_\infty) > 0.$$

The family f_r will take the shape

$$f_r := \tau_0(r) f_r^0 + \tau_1(r) f_r^1,$$

where $f_r^0 := f^0|_{V_r}$, $f_r^1 = rf_\infty$, and $\tau_0, \tau_1: \mathbb{R}_{>0} \rightarrow \mathbb{R}_{\geq 0}$ have the following properties:

- $\tau_0(r) = 1$ for $r \leq \delta/2$ and $\tau_0(r) = 0$ for $r \geq \delta$;
- $\tau_0'(r) \leq 0$ and $|\tau_0'(r)| \leq 3/\delta$ for all $r > 0$;
- $\tau_1(r) = 0$ for $r \leq \varepsilon/2$ and $\tau_1(r) = C$ for $r \geq \varepsilon$, with $C \gg 0$ and $0 < \varepsilon \ll \delta/2$;
- $\tau_1'(r) \geq 0$ and $|\tau_1'(r)| < 3C/\varepsilon$ for all $r > 0$.

(In particular, the derivatives τ_0' and τ_1' have disjoint supports.)

The functions f_r are clearly Lyapunov functions for $\bar{v}$, and they satisfy properties (1) and (2). As for property (3), a simple calculation shows that it holds provided C (the value of $\tau_1(r)$ for $r \geq \varepsilon$) is sufficiently large. To check property (4), namely that the function $f: \text{T}^*M \rightarrow \mathbb{R}$ defined by $f|_{V_r} = f_r$ is a Lyapunov function for $\tilde{v}$, we write

$$\tilde{v} \cdot f = \tau_0(r) (\tilde{v} \cdot f^0) + \tau_1(r) (\tilde{v} \cdot f^1) + r\tau_0'(r) v f_r^0 + r^2\tau_1'(r) v f_\infty.$$

In the domain $\{r \geq \delta\}$, this quantity is positive because f equals Cf^1 . In the region $\{\delta/2 \leq r \leq \delta\}$, the function τ_1 is constant equal to C , and

$$C(\tilde{v} \cdot f^1) + r\tau_0'(r) v f_r^0 = r(C((\bar{v} \cdot f_\infty) + vf_\infty) + \tau_0'(r) v f_r^0)$$

is positive—by Lemma 14(ii)—as soon as C is sufficiently large. Finally, in the region $\{r \leq \delta/2\}$, the function τ_0 is constant equal to 1, so

$$\tilde{v} \cdot f = (\tilde{v} \cdot f^0) + \tau_1(r) (\tilde{v} \cdot f^1) + r^2\tau_1'(r) v f_\infty.$$

In the neighborhood ST^*U of the cotangent fibers over critical points, this derivative is positive because the first two terms of the right-hand side are positive while the third term is non-negative (see Lemma 14). Outside ST^*U , the first term $(\tilde{v} \cdot f^0)$ is bounded below by a constant $\kappa > 0$ and the second term $(\tau_1(r)(\tilde{v} \cdot f^1))$ is non-negative. As for the last term $(r^2\tau_1'(r)vf_\infty)$, since $r^2|\tau_1'(r)| \leq 3C\varepsilon$, it can be made smaller than κ by taking ε sufficiently small.

It remains to prove property (5), *i.e.*, that f can be chosen invariant under the action of the fiberwise antipodal involution σ of ST^*M . First we note that the vector

field $\bar{\nu}$ and the functions f_r^0 , $r \leq \delta$, are invariant. To make each function f_r invariant, we apply the same construction as above but we work on the projective cotangent bundle instead of the sphere cotangent bundle, and then we lift the resulting functions to ST^*M . Alternatively, as suggested by the referee, we replace each f_r by the function $\frac{1}{2}(f_r + \sigma^* f_r)$. $\square$

Proofs of Theorems 1 and 2. — We apply Proposition 8 with the function f provided by Proposition 13. $\square$

F. THE LEFSCHETZ FIBER

This section collects information about the topology and the symplectic geometry of the Lefschetz fiber of the upgraded function (φ, ν) , namely the regular fiber of our symplectic Lefschetz fibration extending φ .

PROPOSITION 15 (the Weinstein structures of real fibers). — *Let M be a compact manifold, $\varphi: M \rightarrow \mathbb{R}$ a Morse function, ν an adapted gradient of φ satisfying the Morse–Smale condition, and $h: \mathrm{T}^*M \rightarrow \mathbb{C}$ the symplectic Lefschetz fibration given by Propositions 8 and 13. Then every fiber $F_u = h^{-1}(u)$, $u \in \mathbb{R} \setminus \Delta_\varphi$, has a Weinstein structure induced by the canonical 1-form λ of T^*M , a Lyapunov function for the corresponding Liouville field being the Morse–Bott function $\rho_u := \rho|_{F_u}$, where ρ is the kinetic energy of the underlying metric. Moreover, these Weinstein structures belong to the same homotopy class.*

The last assertion of this proposition should be understood as follows: given any embedded arc I in $\mathbb{C} \setminus \Delta_\varphi$ joining two real regular values u_0, u_1 , the symplectic fibers F_w , $w \in I$, admit Weinstein structures which, for $w \in \partial I = \{u_0, u_1\}$, are the Weinstein structures induced by λ .

Proof. — To check that $\lambda_u := \lambda|_{F_u}$ is a Weinstein structure on F_u , we first show that ρ_u is a Morse–Bott function and then prove that each level set $\{\rho_u = r^2/2\}$ is a positive contact submanifold of V_r at any point which is non-critical for ρ_u .

Let $N^*(\nu)$ denote the hypersurface $N^*(\nu) := \{g = 0\} \subset \mathrm{T}^*M$, which is smooth away from Γ_φ (the critical points of g are the zeros of its Hamiltonian field $-\tilde{\nu}$). The regular fiber F_u and the zero-section M both sit in $N^*(\nu)$, and inside $N^*(\nu)$, they intersect transversely along the level set $Q_u = \{\varphi = u\}$. Hence $F_u \cap M$ is a transversely non-degenerate minimum submanifold of ρ_u .

Set $Y := N^*(\nu) \setminus M$. Then $f|_Y$ and $\rho|_Y$ have no critical points, the former because $df(\tilde{\nu}) > 0$ while $dg(\tilde{\nu}) = 0$, and the latter because $d\rho(\vec{\lambda}) > 0$ while $dg(\vec{\lambda}) = 0$ along Y . It then follows from the Lagrange multiplier theorem that the critical points of ρ on the fibers of h with real values (namely, the critical points of $\rho|_Y$ on the level sets of $f|_Y$) coincide with the critical points of the functions $f_r|_Y$, $r > 0$ (which are the critical points of $f|_Y$ on the level sets of $\rho|_Y$). Moreover, since f_r is a Lyapunov function for $\bar{\nu}$ and Y is an invariant submanifold containing all zeros of $\bar{\nu}$, the critical points of $f_r|_Y$

are just the critical points of f_r , and so they form transversely non-degenerate critical submanifolds. Thus, ρ_u is a Morse–Bott function.

REMARK 16 (critical submanifolds of ρ_u). — It is instructive to identify precisely the critical submanifolds of ρ_u . By Proposition 13, properties 2 and 3, for every point $a \in \Gamma_\varphi$, the critical value $f_r(C_a^+)$ (resp. $f_r(C_a^-)$) is an increasing (resp. decreasing) and unbounded function of r . Therefore, given $u \in \mathbb{R} \setminus \Delta_\varphi$ and $a \in \Gamma_\varphi$, there exists a unique r such that $f_r(C_a^+) = u$ (resp. $f_r(C_a^-) = u$) if $\varphi(a) < u$ (resp. $\varphi(a) > u$) and none otherwise. The critical submanifolds of ρ_u , besides $Q_u := F_u \cap M$, are the spheres $C_a^\pm$ sitting in the corresponding V_r . This provides many different presentations of the regular fiber as a Weinstein manifold.

Now consider a point $(p, q) \in \{\rho_u = r^2/2\} = F_u \cap V_r$. If (p, q) is non-critical for ρ_u , the above discussion ensures that $\bar{v}$ does not vanish at (p, q) , and hence $(\bar{v} \cdot f_r)(p, q) > 0$. We then observe that $\bar{v}$ spans the (contact geometric) characteristic foliation of $Y \cap V_r$ in V_r (indeed, $Y \cap V_r$ is the dividing hypersurface of the contact vector field $\bar{v}$ on V_r). Therefore, the inequality $(\bar{v} \cdot f_r)(p, q) > 0$ implies that $F_u \cap V_r$, which is contained in $Y \cap V_r$, is transverse to the characteristic foliation of $Y \cap V_r$ at (p, q) , and so is a contact submanifold of V_r at (p, q) .

It remains to show that the Weinstein structures of the real fibers lie in a common homotopy class. Let $u_0, u_1 \in \mathbb{R} \setminus \Delta_\varphi$. If u_0, u_1 are in the same component of $\mathbb{R} \setminus \Delta_\varphi$, clearly the Weinstein structures of F_{u_0} and F_{u_1} are homotopic through those of the regular fibers F_u , $u \in [u_0, u_1]$. Therefore (assuming as usual that no two critical points have the same value), it suffices to treat the case where there is exactly one point $a \in \Gamma_\varphi$ such that $u_0 < \varphi(a) < u_1$. To simplify the notations, we set $\varphi(a) = 0$. Since u_0, u_1 can be chosen arbitrarily small, the homotopy only requires a local construction near a ; we now describe the process.

The map h resulting from Propositions 8 and 13 provides local complex Darboux coordinates $(z_1, \dots, z_n) \in \mathbb{C}^n$ centered at a in which h is the model function $\sum_1^n z_j^2$ while M is represented by M_k , φ by φ_k , and ρ by ρ_k where $k := \text{ind}_\varphi(a)$ (see Section A for the notations). We use these coordinates to identify a neighborhood D of a with the ball $\{z \in \mathbb{C}^n : |z| \leq 3\delta\}$ for some $\delta > 0$. Next, we choose a function $\chi : [0, 3\delta] \rightarrow [0, 1]$ that equals 1 on $[0, \delta]$ and 0 on $[2\delta, 3\delta]$. Then, for any $w = u + iv \in \mathbb{C}$ with v sufficiently small relative to δ , we let F'_w denote the submanifold of T^*M obtained as follows:

- $F'_w \cap D$ is defined by the equation

$$\sum_1^n z_j^2 = u + iv\chi(|z|);$$

- $F'_w \setminus D := F_u \setminus D$.

Thus, $F'_u = F_u$ if u is real. For $u > 0$ sufficiently small and for $w := ue^{(1-t)i\pi}$, $t \in [0, 1]$, the manifolds F'_w connect F_{-u} to F_u , and we would expect the 1-forms induced on them by λ to provide the desired homotopy of Weinstein structures. This is roughly correct, but a slight perturbation of λ is necessary beforehand.

LEMMA 17 (perturbation of λ). — *There exists a 1-form $\lambda^\#$ on T^*M such that:*

- $\lambda^\#$ coincides with λ outside D ;
- $\lambda^\#$, inside D viewed as a ball in $\mathbb{C}^n$, equals $d^c \rho^\#$ where $\rho^\#$ is pseudoconvex and arbitrarily $\mathcal{C}^2$ close to ρ ;
- for some $\varepsilon > 0$ depending on $\rho^\#$, the 1-form induced by $\lambda^\#$ on each $F_w \cap D$, $|w| \leq \varepsilon$, is non-singular in $F_w \cap E$ where $E := \{z \in \mathbb{C}^n : \delta \leq |z| \leq 2\delta\}$.

With this lemma, we can complete the proof of the proposition as planned. First, inside D , we write $\lambda = \lambda_k = d^c \rho_k = d^c \rho$ (see Section A-2). Then we invoke the lemma, taking $\rho^\# - \rho$ so small that $\rho^\#$ is pseudoconvex in D . Thus, on each intersection $F_w \cap D$, the form $\lambda^\#$ induces a Liouville form $\lambda_w^\#$ whose Liouville field $\tilde{\lambda}_w^\#$ is gradientlike for the restriction of $\rho^\#$. Moreover, according to the third point of the lemma, $\tilde{\lambda}_w^\#$ is non-singular in the compact region $F_w \cap E$.

Take $w = u + iv \in \mathbb{C}$ with $|w| \leq \varepsilon$ and pick any point $z \in F_w' \cap E$. Then z also belongs to $F_{w'} \cap E$ where $w' = u + iv\chi(|z|)$, and the tangent spaces of $F_{w'}$ and F_w' at z are respectively defined by

$$0 = \sum_1^n z_j dz_j, \quad 0 = 2 \sum_1^n z_j dz_j - iv\chi'(|z|) d(|z|).$$

For ε sufficiently small, these spaces are so close (uniformly in $z \in E \cap \bigcup_{|w| \leq \varepsilon} F_w'$) that the 1-form λ_w' induced by $\lambda^\#$ on each F_w' with $|w| \leq \varepsilon$ is a non-singular Liouville form whose Liouville field $\tilde{\lambda}_w'$ is gradientlike for the restriction of $\rho^\#$. As a result, each (F_w', λ_w') is a Weinstein manifold. Hence, for $u \in (0, \varepsilon]$ and for $w = ue^{(1-t)i\pi}$, $t \in [0, 1]$, the Weinstein manifolds (F_w', λ_w') define a homotopy from (F_{-u}, λ'_{-u}) to (F_u, λ'_u) . Finally, we concatenate this homotopy at each end of the interval with the barycentric homotopy from $\lambda_{\pm u}$ to $\lambda'_{\pm u}$, which consists of Weinstein structures on $F_{\pm u}$, to obtain the desired homotopy from (F_{-u}, λ_{-u}) to (F_u, λ_u) . Note that the symplectic form $d\lambda_w'$ on the perturbed fiber F_w' does not coincide in D with the one induced by the symplectic form of T^*M , but if necessary, this can be remedied using Moser's trick. $\square$

Proof of the lemma. — Let $\sigma: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be a function satisfying $\sigma(t) = t$ for $t \leq 4\delta^2$ and $\sigma(t) \equiv 0$ for $t \geq 9\delta^2$. We define $\rho^\#: \mathbb{C}^n \rightarrow \mathbb{R}$ by

$$\rho^\#(z) := \rho_k(z) + \frac{c}{2}\sigma(|z|^2),$$

where c is a real parameter which we fix small enough that $\rho^\#$ is pseudoconvex.

On the one hand, the tangent space of F_w at any point $z = x + iy$ is the complex vector space defined by

$$\sum_1^n z_j dz_j = 0.$$

On the other hand,

$$d\rho^\# = d\rho_k + \frac{c}{2}\sigma'(|z|^2) d(|z|^2).$$

A routine calculation then shows that $T_z F_w$ is contained in the kernel of $\lambda^\# = d^c \rho^\#$ if and only if $z = (z_1, \dots, z_n)$, where $z_j = x_j + iy_j$, satisfies the following equations:

$$\begin{aligned} (1) \quad & y_\ell x_j = x_\ell y_j && \text{for } 1 \leq j < \ell \leq k, \\ (2) \quad & y_\ell x_j = x_\ell y_j && \text{for } k < j < \ell \leq n, \\ (3) \quad & x_\ell x_j = y_\ell y_j && \text{for } 1 \leq j \leq k < \ell \leq n, \\ (4) \quad & [1 + c\sigma'(|z|^2)] y_\ell x_j = c\sigma'(|z|^2) x_\ell y_j && \text{for } 1 \leq j \leq k < \ell \leq n. \end{aligned}$$

Let K be the set of those points $z \in E$ which are solutions of the above system. Since K is compact, it suffices to show that h does not vanish on K ; this can be checked by a case-by-case analysis that we sketch below.

For $z \in K$, we first note that $\sigma'(|z|^2) = 1$. Then we write $z = (z', z'') \in \mathbb{C}^k \times \mathbb{C}^{n-k}$, with $z' = x' + iy'$ and $z'' = x'' + iy''$. By (1), $x' = (x_1, \dots, x_k)$ and $y' = (y_1, \dots, y_k)$ are linearly dependent; by (2), $x'' = (x_{k+1}, \dots, x_n)$ and $y'' = (y_{k+1}, \dots, y_n)$ are also linearly dependent. Now assume for instance that $x' \neq 0$ and $y'' \neq 0$. Then $y' = \mu' x'$ and $x'' = \mu'' y''$ for some $\mu', \mu'' \in \mathbb{R}$. Next, (3) implies that $\mu'' = \mu' =: \mu$, and it follows from (4) that $\mu^2 = (1 + c)/c \neq 0$. As a result,

$$\operatorname{im} h(z) = 2\mu(|x'|^2 + |y''|^2) = \frac{2\mu}{1 + \mu^2} |z|^2 \neq 0.$$

The other cases are quite similar. $\square$

We now complete this paper with a few more remarks and comments about the topology and the geometry of the Lefschetz fiber we have constructed.

Fix $u \in \mathbb{R} \setminus \Delta_\varphi$. The Morse–Bott handle decomposition of F_u given by ρ_u can be viewed as follows. As we already said, the level set Q_u is the minimum of ρ_u . To detect the other critical submanifolds, we appeal to Proposition 13: by properties (2) and (3), for every point $a \in \Gamma_\varphi$, the value $f_r(C_a^+)$ (resp. $f_r(C_a^-)$) is an increasing (resp. decreasing) and unbounded function of r ; hence, for each $a \in \Gamma_\varphi$, there is a unique r such that $f_r(C_a^+) = u$ (resp. $f_r(C_a^-) = u$) if $\varphi(a) < u$ (resp. $\varphi(a) > u$), and none otherwise. By the Lagrange multiplier theorem (as seen in the proof above), the critical submanifolds of ρ_u , besides Q_u , are the spheres $C_a^\pm$ contained in F_u , and they lie in V_r where $f_r(C_a^\pm) = u$. Thus, ρ_u has exactly one critical sphere for each $a \in \Gamma_\varphi$, which is C_a^+ (of dimension $\operatorname{ind}(a) - 1$) if $\varphi(a) < u$ and C_a^- (of dimension $n - \operatorname{ind}(a) - 1$) if $\varphi(a) > u$.

Now take two critical points a, b with consecutive values in Δ_φ . If $u < \varphi(a) < \varphi(b)$, then $\rho_u(C_a^-) < \rho_u(C_b^-)$ (for the copies of C_a^-, C_b^- sitting in F_u); but if $\varphi(a) < u < \varphi(b)$, the ordering of $\rho_u(C_a^+)$ and $\rho_u(C_b^-)$ depends on the position of u between $\varphi(a)$ and $\varphi(b)$ (and for some u , the two spheres lie at the same level of ρ_u).

In the situation where $\varphi(a) < u < \varphi(b)$, we can also locate Q_u with respect to the vanishing cycles $Z_u(a), Z_u(b) \subset F_u$ associated with a, b . The invariant manifolds $E^+(a)$ and $E^-(b)$ being transverse to each other, their sphere conormal bundles are disjoint. Moreover, it follows from the discussion in Section A-3 that Q_u intersects cleanly $Z_u(a)$ and $Z_u(b)$ along the attaching spheres $K_u(a) = Q_u \cap E^+(a)$ and

$K_u(b) = Q_u \cap E^-(b)$, respectively. The vanishing cycle $Z_u(a)$ is the union of $K_u(a)$ and the stable manifold of C_a^+ for the gradientlike field $\vec{\lambda}_u$, and similarly for $Z_u(b)$. Finally, Lemma 5 implies that, for any real regular value u_- (resp. u_+) in the component of $\mathbb{R} \setminus \Delta_\varphi$ immediately below $\varphi(a)$ (resp. above $\varphi(b)$), the parallel transport $F_{u_\pm} \rightarrow F_u$ takes $Q_{u_\pm}$ to an exact Lagrangian submanifold isotopic to the Morse–Bott Lagrangian surgery of Q_u with $Z_u(a)$ and $Z_u(b)$, respectively. All these incidence relations, of course, are preserved by parallel transport.

EXAMPLE 2 (Lefschetz fiber of a Heegaard splitting). — Let M be a closed oriented 3-manifold, $\varphi: M \rightarrow \mathbb{R}$ an ordered Morse function with only one minimum and one maximum, ν an adapted gradient satisfying the Morse–Smale condition, and Q a regular level set of φ separating the critical points of index 1 from those of index 2. The above discussion shows that the Lefschetz fiber of (φ, ν) is the Weinstein 4-manifold F obtained as follows from the Heegaard splitting given by Q .

Let g denote the genus of Q . Then the unstable (resp. stable) manifolds of the critical points of index 1 (resp. 2) intersect Q along g disjoint embedded curves $\alpha_1, \dots, \alpha_g$ (resp. $\beta_1, \dots, \beta_g$), and since ν satisfies the Morse–Smale condition, each α_j is transverse to each β_ℓ . This transversality implies that the sphere conormal bundles

$$\mathrm{SN}^* \alpha_1, \dots, \mathrm{SN}^* \alpha_g, \mathrm{SN}^* \beta_1, \dots, \mathrm{SN}^* \beta_g \subset \mathrm{ST}^* Q$$

are disjoint, and hence provide $4g$ disjoint embedded framed curves in the boundary of the disk cotangent bundle $\mathrm{DT}^* Q$. Then F is the (completion of) the Weinstein domain obtained by attaching a Weinstein handle to $\mathrm{DT}^* Q$ along each of these $4g$ curves. The vanishing cycle associated to any critical point of index 1 (resp. 2) is the union of the corresponding $\mathrm{N}^* \alpha_j$ (resp. $\mathrm{N}^* \beta_\ell$) and the two Weinstein handles attached to it. The vanishing cycle associated with the minimum (resp. the maximum), or equivalently the level set just above the minimum (resp. below the maximum), appears as the result of the g Morse–Bott Lagrangian surgeries of Q with the vanishing cycles including the curves α_j (resp. β_ℓ). We refer to [Sre24] for details.

A more global picture of the Lefschetz fiber is provided by the hypersurface $\mathrm{SN}^*(\nu)$ of $\mathrm{ST}^* M$ which, as already mentioned, is its double. To see this, pick a real value $u \in \mathbb{R} \setminus \varphi(M)$ which is so large that F_u lies entirely in the region of $\mathrm{T}^* M$ where h is homogeneous — that is, where $f_r = r f_\infty$ in the notations of Proposition 13. Then the projection $F_u \subset \mathrm{T}^* M \setminus M \rightarrow \mathrm{ST}^* M$ maps F_u diffeomorphically to one half of $\mathrm{SN}^*(\nu)$ (determined by the position of u with respect to $\varphi(M)$) limited by the level set $f_\infty = 0$. The proof is roughly as follows: first of all, everything takes place in $\mathrm{N}^*(\nu) = \{g = 0\}$, and $\vec{\lambda}$ is tangent to $\mathrm{N}^*(\nu)$; next, $r f_\infty = u$ is equivalent to $f_\infty = u/r$; this shows that $\vec{\lambda}$ is transverse to F_u inside $\mathrm{N}^*(\nu)$, and so it projects diffeomorphically to its image in $\mathrm{SN}^*(\nu)$. Moreover, this diffeomorphism maps the Liouville field $\vec{\lambda}_u$ to a vector field which is proportional to $\vec{\nu}$ (because at each point, $\vec{\lambda}$, $\vec{\nu}$ and $\vec{\nu}$ lie in the same tangent plane).

We conclude this discussion with a few words about the relationship between our Lefschetz fibration and the Weinstein structure of the cotangent bundle $\mathrm{T}^* M$.

Symplectic Lefschetz fibrations on a Weinstein manifold W are often requested to satisfy more conditions than those given in Definition 3 (see for instance the notion of “abstract Weinstein Lefschetz fibration” in [GP17]). Mainly, the Lefschetz thimbles associated to a complete system of vanishing paths (see [Sei08]) are required to appear also as the top-dimensional handles in a Weinstein presentation of W . Our Lefschetz fibration $h: W = T^*M \rightarrow \mathbb{C}$ has this property (for any Morse function $\varphi: M \rightarrow \mathbb{R}$), so it may be worth briefly explaining why.

Take a smooth arc I in $\mathbb{C}$ which avoids Δ_φ and intersects each bounded component of $\mathbb{R} \setminus \Delta_\varphi$ transversely in a single point. Then a sufficiently small metric neighborhood of I is a smooth disk $D \subset \mathbb{C} \setminus \Delta_\varphi$ which intersects each bounded component of $\mathbb{R} \setminus \Delta_\varphi$ in a segment, and we can find disjoint segments $J_u \subset \mathbb{R}$, $u \in \Delta_\varphi$, such that each J_u avoids D in its interior but connects u to a point in ∂D . By Proposition 15, we can endow each fiber $F_w = h^{-1}(w)$, $w \in D$, with a Weinstein structure. Then $h^{-1}(D)$, equipped with the vanishing cycles provided in its boundary by the segments J_u , $u \in \Delta_\varphi$, is roughly what is called an abstract Weinstein Lefschetz fibration in [GP17]. In other words, h can be regarded as a Weinstein Lefschetz fibration, that is hopefully somewhat more concrete at this point.

REFERENCES

- [Cie24] K. CIELIEBAK – “A note on gradient-like vector fields”, 2024, [arXiv:2406.02985](#).
- [Don99] S. K. DONALDSON – “Lefschetz pencils on symplectic manifolds”, *J. Differential Geometry* **53** (1999), no. 2, p. 205–236.
- [EG91] Y. ELIASBERG & M. GROMOV – “Convex symplectic manifolds”, in *Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989)*, Proc. Sympos. Pure Math., vol. 52, Part 2, American Mathematical Society, Providence, RI, 1991, p. 135–162.
- [Gir91] E. GIROUX – “Convexité en topologie de contact”, *Comment. Math. Helv.* **66** (1991), no. 4, p. 637–677.
- [GP17] E. GIROUX & J. PARDON – “Existence of Lefschetz fibrations on Stein and Weinstein domains”, *Geom. Topol.* **21** (2017), no. 2, p. 963–997.
- [Joh09] J. JOHNS – “The Picard–Lefschetz theory of complexified Morse functions”, 2009, [arXiv:0906.1218](#).
- [Joh10] ———, “Complexifications of Morse functions and the directed Donaldson–Fukaya category”, *J. Symplectic Geom.* **8** (2010), no. 4, p. 403–500.
- [Lee25] S. LEE – “Lefschetz fibrations on cotangent bundles and some plumbings”, *Selecta Math. (N.S.)* **31** (2025), no. 5, article no. 91 (86 pages), [arXiv:2105.11200v2](#).
- [Sei08] P. SEIDEL – *Fukaya categories and Picard–Lefschetz theory*, Zurich Lectures in Advanced Math., European Mathematical Society, Zürich, 2008.
- [Sma61] S. SMALE – “On gradient dynamical systems”, *Ann. of Math. (2)* **74** (1961), p. 199–206.
- [Sre24] M. SRECKOVIĆ – “On the Floer theory of Lefschetz fibrations on the cotangent bundle extending Morse functions”, PhD Thesis, ENS–Paris, 2024, available at <https://theses.fr/2024UPSLE003>.

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