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POINCARÉ INEQUALITY AND QUANTITATIVE DE GIORGI METHOD FOR HYPOELLIPTIC OPERATORS

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ABSTRACT. — We propose a systematic approach based on trajectories to prove a Poincaré inequality for weak non-negative sub-solutions to hypoelliptic equations with an arbitrary number of Hörmander commutators, both in the local and in the non-local case. As a consequence, we deduce the weak Harnack inequality and Hölder regularity along the line of the De Giorgi method.

RÉSUMÉ (Inégalité de Poincaré et méthode quantitative de De Giorgi pour les opérateurs hypoelliptiques)

Nous proposons une approche systématique fondée sur les trajectoires pour démontrer une inégalité de Poincaré pour les sous-solutions faibles positives d'équations hypoelliptiques comportant un nombre arbitraire de commutateurs de Hörmander, tant dans le cas local que dans le cas non local. Nous en déduisons l'inégalité faible de Harnack et la régularité de Hölder, dans la lignée de la méthode de De Giorgi.

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1. INTRODUCTION

1.1. THE QUESTION AT HAND. — This paper is devoted to the regularity theory of hypoelliptic equations with rough coefficients. We consider equations with an arbitrary number of commutators in the sense of Hörmander [8], with a diffusive part that is either local, in which case the equation is of second order, or non-local, in which case

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we deal with an equation of fractional order. In either case though, the diffusion is *degenerate*, as it acts only along the direction of one vector field. Our aim is to recover regularity in all directions, by transferring regularity from the degenerate diffusivity onto all vector fields.

Concretely, we consider an unknown function $f = f(x, t)$ solving an equation of the form

$$(1.1) \quad \partial_t f + \mathcal{B}f + \mathcal{A}^* \mathcal{A}f = 0,$$

where $\mathcal{B}$ is a first-order differential operator and $\mathcal{A}$ is a, possibly fractional, differential operator. Here $t \in \mathbb{R}$ and $x = (x^{(\kappa)}, \dots, x^{(1)}, x^{(0)})$ with each $x^{(i)} \in \mathbb{R}^{d_i}$, $d_i \in \mathbb{N}^*$ and $i \in \{0, \dots, \kappa\}$ such that $N := d_0 + \dots + d_\kappa$. The differential operator $\mathcal{A}$ acts merely in one direction of x , namely $x^{(0)}$, which makes up the diffusive part of the equation. Even though we assume *no regularity* on the coefficients of $\mathcal{A}$, we still expect the diffusivity that stems from $\mathcal{A}$ to regularise the solution f of (1.1) at least in the variable $x^{(0)}$. To recover a regularisation in all remaining directions of x , we require a so-called *Hörmander's commutator condition*: the constant coefficient vector fields, and their commutators (defined as $[X, Y] = XY - YX$, where X and Y are any two smooth vector fields) appearing in (1.1) are supposed to span the whole space. Any equation with this property is called *hypoelliptic*. In our notation, κ denotes the number of commutators.

The aim of the current paper is to derive a Poincaré inequality for non-negative weak sub-solutions to (1.1), which exploits Hörmander's hypoellipticity to deal with the degeneracy in the diffusive part:

$$(1.2) \quad \int_{Q^+} \left(f - \frac{1}{|Q^-|} \int_{Q^-} f \right)_+ \lesssim \int_{\Omega} |\mathcal{A}f|,$$

where Q^-, Q^+ are two disjoint cylinders in the open domain $\Omega \subseteq \mathbb{R}^{N+1}$ so that Q^+ is a time-translation of Q^- into the future. The symbol $\lesssim$ denotes that the inequality holds up to a universal constant C . The striking feature of (1.2) is that on the right hand side there appears merely a differential in the diffusive direction: it is crucial that no differential in other directions appears in order to deduce the weak Harnack inequality or Hölder regularity of solutions to (1.1).

In this paper we prove a Poincaré inequality with a quantitative control on the constant hidden in the symbol $\lesssim$ in (1.2) for a large class of equations of type (1.1). The Poincaré inequality is a powerful integral way to measure the oscillation of the sub-solution, and its use in this context goes back to [15, 14]. We use it to quantitatively to deduce the De Giorgi lemmas (and thus the Hölder regularity for weak solutions), and the (weak) Harnack inequality for super-solutions following the methodology recently developed in [7, 11]. The main novelty is the method itself, but the result is also new when $\mathcal{A}$ is a fractional differential operator in the case that more than one commutator is involved (i.e., $\kappa > 1$).

The proof is based on a systematic construction of trajectories that encode the flow of the vector fields underlying (1.1), in such a way that they relate the future

cylinder Q^+ to any point in the past cylinder Q^- . The use of trajectories is inspired from [7], where they consider the local case (second order equation) with one commutator $\kappa = 1$. Their trajectories were combining the vector fields of the equation in a piecewise affine way; yet this was sub-optimal and they had an error term on the right hand side of (1.2). This approach was improved by L. Niebel and R. Zacher in [16], where they construct an ad-hoc smooth curve instead of a piecewise affine curve; however this was not sufficient to get rid of the error term that also appeared in [7]. Here we introduce an approach which picks up on ideas of both these papers: we construct a *smooth* trajectory as the solution of a control problem associated to the equation (1.1). We end up with a clean Poincaré inequality in the form (1.2) without any error terms on the right hand side.

1.2. EXAMPLES OF EQUATIONS. — When $\kappa = 1$, the prototypical local equation is the so-called *Kolmogorov equation*

$$(1.3) \quad \partial_t f + v \cdot \nabla_x f = \nabla_v \cdot [A \nabla_v f]$$

on the unknown $f = f(x, v, t)$ depending on $x, v \in \mathbb{R}^d$ with $d \in \mathbb{N}$, $t \in \mathbb{R}$, where the matrix $A = A(x, v, t)$ is measurable symmetric so that, for some $\Lambda \geq 1$,

$$(1.4) \quad \forall (x, v, t) \in \mathbb{R}^{2d+1}, \quad \Lambda^{-1} \leq A(x, v, t) \leq \Lambda.$$

When $\kappa \geq 2$, the underlying geometry becomes more involuted, and the prototypical local equation is

$$(1.5) \quad \partial_t f + x^{(\kappa-1)} \cdot \nabla_{x^{(\kappa)}} f + x^{(\kappa-2)} \cdot \nabla_{x^{(\kappa-1)}} f + \cdots + v \cdot \nabla_{x^{(1)}} f = \nabla_v \cdot [A \nabla_v f]$$

on the unknown

$$f = f(x^{(\kappa)}, x^{(\kappa-1)}, \dots, x^{(1)}, v, t),$$

depending on $x^{(\kappa)}, x^{(\kappa-1)}, \dots, x^{(1)}, v \in \mathbb{R}^d$, $t \in \mathbb{R}$, where the matrix A is measurable, symmetric and so that, for some $\Lambda \geq 1$,

$$(1.6) \quad \begin{aligned} \forall (x^{(\kappa)}, x^{(\kappa-1)}, \dots, x^{(1)}, v, t) \in \mathbb{R}^{(\kappa+1)d+1}, \\ \Lambda^{-1} \leq A(x^{(\kappa)}, x^{(\kappa-1)}, \dots, x^{(1)}, v, t) \leq \Lambda. \end{aligned}$$

From now on, we denote $v = x^{(0)}$ for notational consistency.

Our method can also handle fractional (non-local) operators. We demonstrate the application to the fractional setting by considering a fractional toy problem with rough coefficients. In the case $\kappa = 1$, it takes the form

$$(1.7) \quad \partial_t f + v \cdot \nabla_x f = (-\Delta_v)^{\beta/2} \cdot \left[a (-\Delta_v)^{\beta/2} f \right]$$

on the unknown $f = f(x, v, t)$ depending on $x, v \in \mathbb{R}^d$ and $t \in \mathbb{R}$, with $\beta \in (0, 1)$ and a scalar function a that is measurable and so that, for some $\Lambda \geq 1$,

$$(1.8) \quad a(x, v, t) \in [\Lambda^{-1}, \Lambda].$$

For a higher number of commutators, the corresponding version is

$$(1.9) \quad \partial_t f + x^{(\kappa-1)} \cdot \nabla_{x^{(\kappa)}} f + x^{(\kappa-2)} \cdot \nabla_{x^{(\kappa-1)}} f + \cdots + v \cdot \nabla_{x^{(1)}} f \\ = (-\Delta_v)^{\beta/2} \cdot \left[a (-\Delta_v)^{\beta/2} f \right]$$

on the unknown f which depends on $x^{(\kappa)}, x^{(\kappa-1)}, \dots, x^{(1)}, v \in \mathbb{R}^d, t \in \mathbb{R}$, and where the function a is measurable and so that, for some $\Lambda \geq 1$,

$$(1.10) \quad \forall (x^{(\kappa)}, x^{(\kappa-1)}, \dots, x^{(1)}, v, t) \in \mathbb{R}^{(\kappa+1)d+1}, \\ \Lambda^{-1} \leq a(x^{(\kappa)}, x^{(\kappa-1)}, \dots, x^{(1)}, v, t) \leq \Lambda.$$

The idea of this fractional setting is to have a divergence form equation of fractional order $2\beta \in (0, 2)$ with rough diffusion coefficients.

Such a setting is for example motivated by the Boltzmann equation (without cut-off), which is a kinetic equation with a degenerate fractional diffusion operator in velocity. In this application, the nonlocal operator $(-\Delta_v)^{\beta/2} \cdot [a (-\Delta_v)^{\beta/2} f]$ in (1.7) is replaced by

$$\mathcal{L}f(x^{(\kappa)}, \dots, x^{(1)}, v, t) \\ = \int_{\mathbb{R}^d} \left[f(x^{(\kappa)}, \dots, x^{(1)}, w, t) - f(x^{(\kappa)}, \dots, x^{(1)}, v, t) \right] K(x^{(\kappa)}, \dots, x^{(1)}, v, w, t) dw,$$

where the kernel K satisfies pointwise bounds

$$(1.11) \quad \forall (x^{(\kappa)}, \dots, x^{(1)}, v, w, t) \in \mathbb{R}^{(\kappa+2)d+1}, \\ \Lambda^{-1} |v - w|^{-(d+2\beta)} \leq K(x^{(\kappa)}, \dots, x^{(1)}, v, w, t) \leq \Lambda |v - w|^{-(d+2\beta)},$$

and the symmetry

$$(1.12) \quad \forall (x^{(\kappa)}, \dots, x^{(1)}, v, w, t) \in \mathbb{R}^{(\kappa+2)d+1}, \\ K(x^{(\kappa)}, \dots, x^{(1)}, v, w, t) = K(x^{(\kappa)}, \dots, x^{(1)}, w, v, t).$$

Such an operator $\mathcal{L}$ is of the usual form of nonlocal diffusion operators and can be related to Lévy processes. Our results can be adapted to this setting, even though the operator $\mathcal{L}$ is not of the form studied in (1.7). We will not specify this direction further in this work, since our focus is the development of an abstract method that is robust enough to treat a wide class of operators. The robustness of the method, however, is best demonstrated within a class of equations that can be described under a simple notation.

1.3. GENERAL ASSUMPTIONS. — We denote the variable $z := (x, t) \in \mathbb{R}^{N+1}$ with $t \in \mathbb{R}$ and $x = (x^{(\kappa)}, \dots, x^{(1)}, v)$ with each $x^{(i)} \in \mathbb{R}^{d_i}$, $d_i \in \mathbb{N}^*$ and $i \in \{1, \dots, \kappa\}$, and $v \in \mathbb{R}^{d_0}$. We assume that $d_0 \geq d_1 \geq \dots \geq d_\kappa \geq 1$ and denote $N := d_0 + \dots + d_\kappa$.

We consider (1.1) with operators $\mathcal{A}$ and $\mathcal{B}$ as follows. The operator $\mathcal{B}$ is given by

$$\mathcal{B} := (\mathcal{B}x) \cdot \nabla_x,$$

where the $N \times N$ matrix $\mathbf{B}$ has the form ($\mathcal{O}$ denotes a block matrix of zeros)

$$(1.13) \quad \mathbf{B} := \begin{pmatrix} \mathcal{O} & \mathbf{B}_\kappa & \mathcal{O} & \cdots & \cdots & \mathcal{O} \\ \vdots & \mathcal{O} & \mathbf{B}_{\kappa-1} & \mathcal{O} & \cdots & \mathcal{O} \\ \vdots & \vdots & \mathcal{O} & \ddots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \ddots & \mathbf{B}_2 & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \mathcal{O} & \mathbf{B}_1 \\ \mathcal{O} & \mathcal{O} & \cdots & \cdots & \cdots & \mathcal{O} \end{pmatrix},$$

with $d_i \times d_{i-1}$ blocks $\mathbf{B}_i$ which are constant matrices of rank d_i with $|\mathbf{B}| \leq \Lambda$.

The operator $\mathcal{A}$ is a linear operator acting only on the variable v and β -homogeneous in this variable with $\beta \in (0, 1]$ and it satisfies the following pointwise control from above and integral control from below, given $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that $\|\varphi\|_{L^\infty} \geq 1$:

$$(1.14) \quad \|\mathcal{A}\varphi\|_\infty \lesssim \|\nabla_v \varphi\|_{L^\infty}^{\beta+\varepsilon} \|\varphi\|_{L^\infty}^{1-\beta-\varepsilon}, \quad \int |\mathcal{A}\varphi| \, dz \gtrsim \int |(-\Delta_v)^{\beta/2} \varphi| \, dz,$$

where $\varepsilon = 0$ for $\beta = 1$, and $\varepsilon \in (0, 1 - \beta)$ for $\beta \in (0, 1)$. We point out that in case of $\beta = 1$, we identify with a slight abuse of notation $(-\Delta_v)^{\frac{1}{2}}$ with ∇_v , such that we are *not* violating the incorrect inequality $\int |\nabla \varphi| \, dz \gtrsim \int |(-\Delta_v)^{1/2} \varphi| \, dz$. In fact, in the local case where $\beta = 1$, there is no need to introduce fractional derivatives at all, but in order to treat $\beta \in (0, 1]$ we do so to ease the notation.

The transport operator $\mathcal{T} := \partial_t + \mathcal{B}$ naturally defines the non-commutative group of *hyper-Galilean transformations*, that generalises the Galilean transformations:

$$(1.15) \quad (\tilde{x}, \tilde{t}) \circ (x, t) = (x + \exp(t\mathbf{B})\tilde{x}, t + \tilde{t}), \quad (x, t), (\tilde{x}, \tilde{t}) \in \mathbb{R}^{N+1}.$$

This transformation leaves the operator $\mathcal{T}$ invariant: $(\mathcal{T}f)(\tilde{z} \circ z) = \mathcal{T}[f(\tilde{z} \circ \cdot)](z)$. When considering the principal part operator associated to (1.1), which is obtained by replacing A in (1.5) with the identity matrix of dimension d in the local case or by choosing a in (1.9) as the identity in the non-local case, the related equation naturally defines scaling properties encoded by the following *hypoelliptic dilations*, defined for $r > 0$

$$(1.16) \quad \delta_r := \text{diag}(r^{1+2\kappa\beta} \mathbb{I}_{d_\kappa}, \dots, r^{1+2\beta} \mathbb{I}_{d_1}, r \mathbb{I}_{d_0}, r^{2\beta}),$$

and we have the relation

$$(\mathcal{T} + \mathcal{A}^* \mathcal{A})[f \circ \delta_r] = r^{2\beta} [(\mathcal{T} + \mathcal{A}^* \mathcal{A}) f] \circ \delta_r.$$

These transformations allow to define the *hypoelliptic cylinders*

$$Q_r(\tilde{z}) := \tilde{z} \circ (\delta_r(Q_1)) \quad \text{with} \quad Q_1 := B_1 \times \cdots \times B_1 \times (-1, 0] \subset \mathbb{R}^{N+1}.$$

The (local) Kolmogorov equation (1.3) corresponds to $\kappa = 1$, $d_0 = d_1 = d$, $(x^{(1)}, x^{(0)}) = (x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, $\mathcal{A} := \sqrt{A} \nabla_v$ with a $d \times d$ -matrix $A = A(x, v, t)$ measurable symmetric so that $A \in [\Lambda^{-1}, \Lambda]$, and the matrix

$$(1.17) \quad \mathbf{B} := \begin{pmatrix} \mathcal{O} & \mathbb{I}_d \\ \mathcal{O} & \mathcal{O} \end{pmatrix}.$$

The (local) higher-order Kolmogorov equation (1.5) corresponds to $d_i = d$ for all i , $(x^{(\kappa)}, \dots, x^{(1)}, v) \in \mathbb{R}^{(\kappa+1)d}$, $\mathcal{A} := \sqrt{A} \nabla_v$ with a $d \times d$ -matrix $A = A(x^{(\kappa)}, \dots, x^{(1)}, v, t)$ measurable symmetric so that $A \in [\Lambda^{-1}, \Lambda]$, and the matrix

$$\mathbf{B} := \begin{pmatrix} \mathcal{O} & \mathbb{I}_d & \mathcal{O} & \dots & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathbb{I}_d & \mathcal{O} & \vdots \\ \mathcal{O} & \vdots & \ddots & \ddots & \mathcal{O} \\ \vdots & \vdots & \vdots & \ddots & \mathbb{I}_d \\ \mathcal{O} & \mathcal{O} & \dots & \dots & \mathcal{O} \end{pmatrix}.$$

The fractional Kolmogorov equations (1.7) and (1.9) are obtained along the same line with $\mathcal{A} := \sqrt{a}(-\Delta_v)^{\beta/2}$ and the same matrices $\mathbf{B}$.

1.4. MAIN RESULT. — Let us first give the definitions of weak sub-solutions and sub/super-solutions used in the results.

DEFINITION 1. — Let $\Omega = \Omega_x \times \Omega_t = \Omega_{(N-1)} \times \dots \times \Omega_{(1)} \times \Omega_{(0)} \times \Omega_t$ be an open subset of $\mathbb{R}^{N+1}$. A function f is a *locally integrable weak sub-solution* of (1.1) if f and $\mathcal{A}f$ are in $L^1_{\text{loc}}(\Omega)$ and satisfies

$$\partial_t f + \mathcal{B}f + \mathcal{A}^* \mathcal{A}f \leq 0,$$

in the distributional sense. A function f is a *sub-solution* of (1.1) if f is a weak sub-solution in $L^\infty(\Omega_t; L^2(\Omega_x)) \cap L^2(\Omega_{(N-1)} \times \dots \times \Omega_{(1)} \times \Omega_t; H^\beta(\Omega_{(0)}))$ and satisfies

$$(1.18) \quad \partial_t f + \mathcal{B}f \in L^2(\Omega_{(N-1)} \times \dots \times \Omega_{(1)} \times \Omega_t; H^{-\beta}(\Omega_{(0)})).$$

A function f is a *super-solution* of (1.1) if $-f$ is a *sub-solution*.

REMARK 2. — The condition (1.18) is used in [6] and it is weaker than the condition

$$\partial_t f + \mathcal{B}f \in L^2(\Omega),$$

used in [18]. In the definition of sub-solution, one could also consider a third condition, only valid for the local case, that reads as follows:

$\forall G: \mathbb{R} \rightarrow \mathbb{R}$ in C^2 with $G' \geq 0, G'' \geq 0$ both bounded, $G(f)$ is a weak sub-solution.

This condition was introduced in [7] and is weaker than the other two listed in the definition above. Indeed, it allows for instance to consider $f = f(t) = \mathbf{1}_{t \leq 0}$ as a sub-solution which explains the fact that the cylinders are disjoint in the following theorems.

Our first result is the Poincaré inequality:

THEOREM 3. — Consider operators $\mathcal{A}, \mathcal{B}$ satisfying the assumptions of Section 1.3. Let $R > 0$ be sufficiently large, depending on the number of commutators κ and Λ . Let $Q_R := \delta_R(Q_1)$ and suppose f is a weak non-negative sub-solution to (1.1) in $\tilde{\Omega}$, where

$\tilde{\Omega} = Q_R$ if $\beta = 1$, that is, in the local case, and $\tilde{\Omega} = B_R \times \dots \times B_R^{1+2(\kappa-1)\beta} \times \mathbb{R}^{d_0} \times (-5, 0)$, if $\beta \in (0, 1)$, that is, in the non-local case. Then

$$(1.19) \quad \int_{Q^+} \left(f - \frac{1}{|Q^-|} \int_{Q^-} f \right)_+ dz \lesssim \int_{\tilde{\Omega}} |Af| dz,$$

where $Q^+ := Q_1 \subset \tilde{\Omega}$ denotes the future cylinder, and $Q^- = B_1 \times \dots \times B_1 \times (-5, -4] \subset \tilde{\Omega}$ denotes the past cylinder (see Figure 1). The constant is universal and depends on κ , β , $d_0, \dots, d_\kappa$, B , R and Λ .

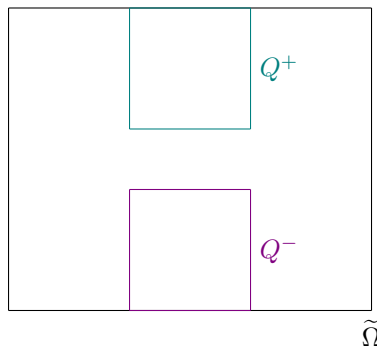


FIGURE 1. The different cylinders in the statement in Poincaré inequality (the time variable is represented vertically and upward).

REMARKS 4

(1) The radius R of the ambient space $\tilde{\Omega}$ is large enough such that the trajectories constructed in the proof (see (2.2)) do not exit $\tilde{\Omega}$. Since all maps involved in the construction of such trajectories are affine, we know that $R < +\infty$. We did not try to optimise the size of the ambient space.

(2) It is easy to include a source term S in the right hand side of (1.1). Then the L^1 norm of S on $\tilde{\Omega}$ should appear on the right hand side of (1.19).

(3) It is likely that the approach can be extended to the case where the matrix B has non-zero terms above the sub-diagonal, since they do not break the commutator structure. This, however, would require additional techniques, which we leave for further work.

(4) Under the same assumptions, it could be possible to prove L^p versions of the Poincaré inequality (1.19) by the same method, up to a suitable change in the choice of the trajectory, for $p \in (1, +\infty)$:

$$\int_{Q^+} (f - \langle f \rangle_{Q^-})_+^p dz \lesssim \int_{\tilde{\Omega}} |Af|^p dz.$$

Furthermore, it is even possible to adapt our proof to obtain a gain of integrability for f , see the related comments in Remark 8.

(5) The left hand side of (1.19) could be mollified as follows

$$(1.20) \quad \int_{\tilde{\Omega}} \left(f - \int_{\tilde{\Omega}} f \psi_- \right) \psi_+ \, dz \lesssim_{\psi_{\pm}} \int_{\tilde{\Omega}} |\mathcal{A}f| \, dz,$$

where $\psi_{\pm} \in C_c^{\infty}(Q^{\pm})$ are non-negative and integrate to one. This amounts to replacing respectively the indicator functions $\chi_{f(\cdot) \geq \langle f \rangle_{Q^-}} \chi_{Q^+}$ and χ_{Q^-} by some mollified versions ψ_+ and ψ_- in the proof.

(6) In such a mollified formulation, the Poincaré inequality in the local case would follow from the following “*wholespace hypoelliptic Bogovoskiĭ result*” (in the spirit of [4]): given $g = \psi_+ - \psi_-$ with $\psi_{\pm}$ as above, construct a vector field $\mathbf{F} := (F_T, F_v)$ with F_T one-dimensional and F_v d_0 -dimensional, so that

$$\mathcal{T}^* F_T - \nabla_v \cdot F_v = g \quad \text{and} \quad F_T \geq 0 \quad \text{and} \quad \|\nabla_v F_T\|_{\infty} + \|F_v\|_{\infty} \lesssim_g 1.$$

This is an open question, but we believe this vector field to exist.

(7) It is also possible to prove the local case of our Poincaré inequality through a “*parabolic wholespace Bogovoskiĭ inequality*” in $(x^{(\kappa)}, \dots, x^{(1)}, t)$ (excluding the $x^{(0)} = v$ variable, i.e., the velocity in case of a Kolmogorov equation), by following the arguments in [5]. This parabolic wholespace Bogovoskiĭ inequality is an easy generalisation of the Bogovoskiĭ inequality, where the explicit representation of Bogovoskiĭ is used to show that the first coordinate of the vector field solving the divergence problem is controlled below, due to the form of g given in the previous point (6).

It is possible to employ such a Poincaré inequality to prove a weak Harnack inequality for weak super-solutions to equation (1.1). We only sketch the proof of this here (see Section 4 for further details), since it follows from previous literature. Nevertheless, it is one of the main blocks to complete the De Giorgi-Nash-Moser theory for weak solutions, which consists in proving local quantitative regularity results (such as the a priori boundedness of weak solutions, the intermediate value lemma and the Harnack inequality) for weak solutions, and for this reason it is now stated as an independent result.

THEOREM 5 (Weak Harnack inequality). — *Let the operators $\mathcal{A}, \mathcal{B}$ be such that $\mathcal{B}$ is given by the assumptions of Section 1.3 and $\mathcal{A}$ is more specifically given by*

- (i) $\mathcal{A} := \sqrt{A} \nabla_v$ with $A \in [\Lambda^{-1}, \Lambda]$ measurable symmetric matrix (local case),
- (ii) $\mathcal{A} = \sqrt{a}(-\Delta_v)^{\beta/2}$ with $a \in [\Lambda^{-1}, \Lambda]$ scalar measurable (non-local case).

Let f be a super-solution to (1.1) on Q_1 , such that in the local case, $0 \leq f$ in Q_1 , and such that in the non-local case, $0 \leq f \leq 1$ on $B_1^{\otimes \kappa} \times \mathbb{R}^{d_0} \times [-1, 0]$. Then there is $C > 0$ and $\zeta > 0$ depending on $\beta, N, \kappa, \Lambda$ such that for $r_0 > 0$ sufficiently small the Weak Harnack inequality is satisfied:

$$(1.21) \quad \left(\int_{\tilde{Q}_{r_0/2}^-} f^{\zeta}(z) \, dz \right)^{1/\zeta} \leq C \inf_{Q_{r_0/2}} f,$$

where $\tilde{Q}_{r_0/2}^- := Q_{r_0/2}(0, \dots, 0, -\frac{5}{2}r_0^{2\beta} + \frac{1}{2}(r_0/2)^{2\beta}) \in \mathbb{R}^{N+1}$ (the local case is $\beta = 1$).

REMARK 6

(1) In the local case the strong Harnack follows immediately from the weak Harnack inequality above and the gain of integrability (first lemma of De Giorgi). By contrast, the strong Harnack inequality does not follow as immediately in the non-local case, since the function values outside the domain affect the solution inside the domain. It is however possible to adapt the recent work [1, 13, 12], which proves the strong Harnack inequality in the non-local case with $\kappa = 1$, to our setting.

(2) It is standard to deduce the Hölder regularity from the weak Harnack inequality. We do not repeat the argument, but it is identical for instance to those in [7, 11].

(3) The boundedness of f in the non-local case almost everywhere in the $x^{(0)} = v$ direction is required to make sense of the equation (1.1).

1.5. MOTIVATION. — Kolmogorov equations appear in the theory of stochastic processes: equation (1.3) was studied by Kolmogorov in 1934 [10], when A is the identity matrix, to understand the “time-integrated Brownian motion”. The equation is then the Kolmogorov forward equation for the process

$$dV_t = dW_t, \quad dX_t = V_t dt.$$

Kolmogorov obtained the explicit formula for its fundamental solution in [10] which inspired the seminal work [8]. Integrating twice in time the Brownian motion would lead to equation (1.5) with $\kappa = 2$ and A the identity matrix. Due to this connection to stochastic processes, several mathematical models involving linear and nonlinear Kolmogorov type equations have also appeared in finance, and in particular equations of type (1.5) appear in various models for pricing of path-dependent financial instruments. For example the equation for $P = P(S, A, t)$

$$(1.22) \quad \partial_t P + \frac{1}{2} \sigma^2 S^2 \partial_S^2 P + (\log S) \partial_A P + r(S \partial_S P - P) = 0, \quad S > 0, A, t \in \mathbb{R},$$

arises in the Black and Scholes option pricing problem

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t, \\ dA_t = S_t dt, \end{cases}$$

where σ is the volatility of the stock price S , μ is the interest rate of a riskless bond and $P = P(S, A, t)$ is the price of the Asian option depending on the price of the stock S , the geometric average A of the past price and the time to maturity t . For a more exhaustive treatment of the applications of Kolmogorov operators to finance and to stochastic theory, we refer to the monograph [17] by Pascucci.

Let us close this paragraph by mentioning that analogously, in the non-local case when $a = 1$, or equivalently when $\mathcal{A}$ corresponds to the fractional Laplacian, the stochastic process underlying this fractional diffusion is a Lévy process.

1.6. STRATEGY OF PROOF. — The starting point is the method developed in [7] for proving the Poincaré inequality using trajectories and deducing the second lemma of De Giorgi and the weak Harnack inequality.

The strategy to obtain a Poincaré inequality for the local case when $\kappa = 1$ in [7] can be summarised as follows. The first step is to mollify the characteristic function of Q^- , by adding an error term

$$\int_{Q^+} (f - \langle f \rangle_{Q^-})_+ dz \leq \int_{z_+ \in Q^+} \left(\int_{z_- \in Q^-} [f(z_+) - f(z_-)] \varphi_\varepsilon(y, w) dz_- \right)_+ dz_+ + \varepsilon^{2d} \|f\|_{L^2(Q_R)},$$

with $0 \leq \varphi_\varepsilon \leq 1$ smooth and only depending on (y, w) , $\varepsilon > 0$, $z_+ = (x, v, t) \in Q^+$, and $z_- = (y, w, s) \in Q^-$. The second step consists in constructing piecewise affine trajectories following the two vector fields $\mathcal{T} := \partial_t + v \cdot \nabla_x$ and ∇_v to connect any $z_+ \in Q^+$ with any $z_- \in Q^-$:

$$(x, v, t) \xrightarrow{\nabla_v} \left(x, \frac{x-y}{t-s}, t\right) \xrightarrow{\mathcal{T}} \left(y, \frac{x-y}{t-s}, s\right) \xrightarrow{\nabla_v} (y, w, s).$$

Note here that a positive time gap between the two cylinders Q^- and Q^+ is used to make sure the intermediate velocity $(x-y)/(t-s)$ remains bounded. We then write $f(z_+) - f(z_-)$ as an integral along the chosen trajectory and use $\mathcal{T}f + \mathcal{A}^* \mathcal{A}f \leq 0$:

$$\begin{aligned} \int_{Q^-} [f(z_+) - f(z_-)] \varphi_\varepsilon(y, w) dz_- &= \int_{Q^-} \int_{\text{trajectory}} \mathcal{T}f(\cdots) \varphi_\varepsilon(y, w) ds dz_- \\ &\quad + \int_{Q^-} \int_{\text{trajectory}} \nabla_v f(\cdots) \varphi_\varepsilon(y, w) ds dz_- \\ &\lesssim \int_{Q^-} \int_{\text{trajectory}} [\nabla_v \cdot (A \nabla_v f)](\cdots) \varphi_\varepsilon(y, w) ds dz_- \\ &\quad + \int_{Q^-} \int_{\text{trajectory}} \nabla_v f(\cdots) \varphi_\varepsilon(y, w) ds dz_-. \end{aligned}$$

We then want to integrate by parts the v -divergence in the past variable z_- . This integration by parts degenerates near the future point z_+ , since z_+ does not depend on z_- , and it produces a non-integrable singularity. In order to overcome this singularity, the paper [7] introduced an additional small fourth sub-trajectory along ∇_x :

$$\begin{aligned} (x, v, t) &\xrightarrow{\nabla_x} (x + \varepsilon w, v, t) \xrightarrow{\nabla_v} \left(x + \varepsilon w, \frac{x + \varepsilon w - y}{t-s}, t\right) \\ &\xrightarrow{\mathcal{T}} \left(y, \frac{x + \varepsilon w - y}{t-s}, s\right) \xrightarrow{\nabla_v} (y, w, s). \end{aligned}$$

This sub-trajectory “noises” the future position variable with the past velocity. This produces a second error term of the form $\varepsilon^\sigma \|f\|_{L_{t,v}^1 W_x^{\sigma,1}}$, which can, however, be controlled for a small ε by the integral regularity $L_{t,v}^1 W_x^{\sigma,1}$ (which can be established for sub-solutions to the Kolmogorov equation, for a small $\sigma > 0$). Finally it yields

$$\begin{aligned} \int_{Q^+} (f - \langle f \rangle_{Q^-})_+ dz &\lesssim C_\varepsilon \int_{Q_R} |S| dz + \int_{Q_R} |\nabla_v f| dz + \varepsilon^\sigma \|f\|_{L_{t,v}^1 W_x^{\sigma,1}(Q_R)} + \varepsilon^{2d} \|f\|_{L^2(Q_R)} \end{aligned}$$

for a constant $C_\varepsilon > 0$ depending on $\varepsilon > 0$. The two additional errors weaken the Poincaré inequality, but the latter is shown to be sufficient for implementing the next steps of the De Giorgi theory in [7].

In the present article, we improve the method by simplifying the proof and removing the two error terms in the inequality. Moreover, the gain of integrability can directly be extracted from the proof of the Poincaré inequality. The first novel idea is to use three cylinders Q^- , Q^0 and Q^+ , and mollify the characteristic function only in the intermediate cylinder Q^0 , see Figure 2.

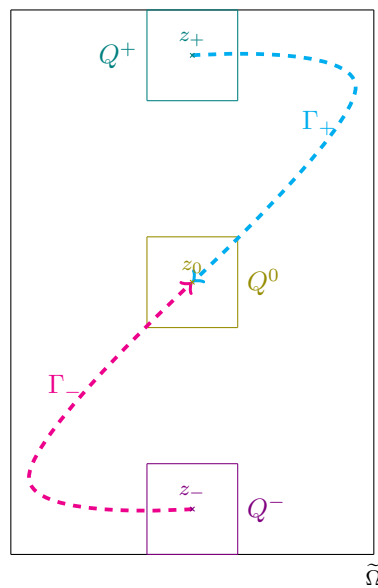


FIGURE 2. Construction of the trajectories. The curve Γ_+ connects any point $z_+ \in Q^+$ to some intermediate point $z_0 \in Q^0$, whereas the curve Γ_- connects any point $z_- \in Q^-$ to some intermediate point $z_0 \in Q^0$.

It is easy to deduce from the triangle inequality

$$\begin{aligned}
 (1.23) \quad & \int_{Q^+} [f(z_+) - \langle f \rangle_{Q^-}]_+ \, dz_+ \\
 & \leq \int_{Q^+} \left(\int_{Q^0} [f(z_+) - f(z_0)] \varphi_\varepsilon(x_0, v_0) \, dz_0 \right)_+ \, dz_+ \\
 & \quad + \int_{Q^-} \left(\int_{Q^0} [f(z_0) - f(z_-)] \varphi_\varepsilon(x_0, v_0) \, dz_0 \right)_+ \, dz_-
 \end{aligned}$$

for $\varphi_\varepsilon \geq 0$ smooth, with mass 1, where $z_0 = (x_0, v_0, t_0) \in Q^0$, $z_+ \in Q^+$, $z_- \in Q^-$, and where we have denoted by $f_Q := |Q|^{-1} \int_Q$ the normalised integral. We then construct trajectories to connect z_+ to z_0 , and z_- to z_0 . This removes the first error term we had introduced in relation to the mollification φ_ε .

The second novel idea is to connect the points by a curved trajectory, rather than a piecewise affine one, by solving a control problem with a well-chosen forcing (i.e., control function), in order to reduce the order of the singularity of the integration by parts near $z_{\pm}$. More precisely, the first idea is to consider trajectories whose speed diverges at $z_{\pm}$ but with an integrable divergence so that the trajectories remain bounded. This idea is inspired from the reading of [16], but we propose simpler and more systematic trajectories. The second important idea is to use $\kappa + 1$ linearly independent control power functions with such well-balanced diverging behaviour at $z_{\pm}$ in order to connect all points between the cylinders, thanks to the Hörmander commutator condition.

1.7. OUTLINE. — In Section 2 we provide the explicit construction of the trajectories connecting points in the future to points in the past. These trajectories are then used in Section 3 to prove the Poincaré inequality; they allow to estimate the L^1 norm in the future of the difference between f and the past average of f , or in other words, the left hand side of (1.19). In Section 4, we prove the Harnack inequalities on the basis of this Poincaré inequality.

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2. CONSTRUCTION OF THE TRAJECTORIES

2.1. THE BASE CASE. — Let us first consider the case with one commutator ($\kappa = 1$), $d_1 = d_0 = d$ and $B_1 = \text{Id}$. Given three points $z_+ \in Q^+$, $z_0 \in Q^0$ and $z_- \in Q^-$, we want to construct two paths $s \rightarrow \Gamma_+(s)$ and $s \rightarrow \Gamma_-(s)$ for $s \in [0, 1]$ such that (see Figures 2 and 3)

$$\begin{aligned} \Gamma_+(s) &= (X_+^{(1)}(s), X_+^{(0)}(s), T_+(s))^T, & \Gamma_+(0) &= z_+, & \Gamma_+(1) &= z_0, \\ \Gamma_-(s) &= (X_-^{(1)}(s), X_-^{(0)}(s), T_-(s))^T, & \Gamma_-(0) &= z_-, & \Gamma_-(1) &= z_0, \end{aligned}$$

where

$$(2.1) \quad \begin{cases} \frac{d}{ds} X_{\pm}^{(0)}(s) = \mathbf{m}_{\pm}^{(0)} g_0''(s) + \mathbf{m}_{\pm}^{(1)} g_1''(s), \\ \frac{d}{ds} X_{\pm}^{(1)}(s) = \delta_{\pm} X_{\pm}^{(0)}(s), \\ \frac{d}{ds} T_{\pm}(s) = \delta_{\pm}, \end{cases}$$

where $\delta_{\pm} := t_0 - t_{\pm}$ and the control functions $g_i \in C^2((0, 1])$ with $g_i(0) = g_i'(0) = 0$ for $i = 0, 1$, and $\mathbf{m}_{\pm}^{(i)} \in \mathbb{R}^d$ for $i = 0, 1$. Solving the differential equations yields

$$(2.2) \quad \begin{cases} X_{\pm}^{(0)}(s) = x_{\pm}^{(0)} + \mathbf{m}_{\pm}^{(0)} g_0'(s) + \mathbf{m}_{\pm}^{(1)} g_1'(s), \\ X_{\pm}^{(1)}(s) = x_{\pm}^{(1)} + s \delta_{\pm} x_{\pm}^{(0)} + \delta_{\pm} [\mathbf{m}_{\pm}^{(0)} g_0(s) + \mathbf{m}_{\pm}^{(1)} g_1(s)], \\ T_{\pm}(s) = s t_0 + (1 - s) t_{\pm}. \end{cases}$$

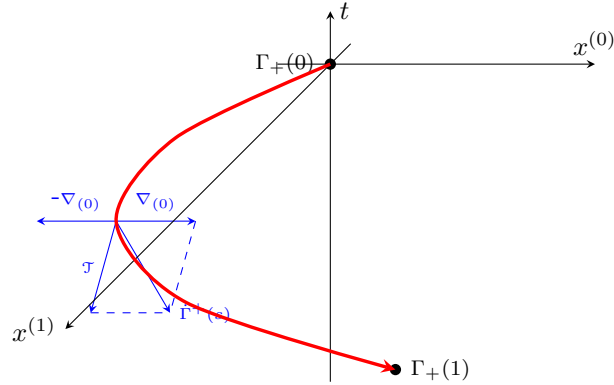


FIGURE 3. Construction of the trajectories. The curve Γ_+ connects any point $\Gamma_+(0) = z_+ \in Q^+$ to some intermediate point $\Gamma_+(1) = z_0 \in Q^0$ along the vector fields $\mathcal{J}$ and $\nabla_{(0)}$.

Let us denote $\mathbf{1}_d$ the d -vector of 1's, $\mathbf{0}_d$ the d -vector of 0's, Id_d the $d \times d$ identity matrix, $\mathcal{O}_d$ the $d \times d$ zero block, and

$$\begin{aligned} \mathbf{M}_\pm &:= \begin{pmatrix} \mathbf{m}_\pm^{(0)} \\ \mathbf{m}_\pm^{(1)} \end{pmatrix}, & \mathbf{W}(s) &:= \begin{pmatrix} g'_0(s)\text{Id}_d & g'_1(s)\text{Id}_d \\ g_0(s)\text{Id}_d & g_1(s)\text{Id}_d \end{pmatrix}, \\ \mathbf{W}_\pm^\delta(s) &:= \begin{pmatrix} \text{Id}_d & \mathcal{O}_d \\ \mathcal{O}_d & \delta_\pm \text{Id}_d \end{pmatrix} \mathbf{W}(s), & \mathbf{Y}_\pm &:= \begin{pmatrix} y_\pm^{(0)} \\ y_\pm^{(1)} \end{pmatrix} := \begin{pmatrix} x_0^{(0)} - x_\pm^{(0)} \\ x_0^{(1)} - x_\pm^{(1)} - \delta_\pm x_\pm^{(0)} \end{pmatrix}. \end{aligned}$$

The boundary conditions $\Gamma_\pm(1) = z_0$ impose

$$(2.3) \quad \mathbf{W}_\pm^\delta(1)\mathbf{M}_\pm = \mathbf{Y}_\pm \iff \mathbf{M}_\pm = \mathbf{W}_\pm^\delta(1)^{-1}\mathbf{Y}_\pm,$$

provided that the Wronskian matrix $\mathbf{W}(1)$ is invertible at $s = 1$. We deduce

$$\begin{pmatrix} X_\pm^{(0)}(s) \\ X_\pm^{(1)}(s) \end{pmatrix} = \mathbf{W}_\pm^\delta(s)\mathbf{M}_\pm + \begin{pmatrix} \text{Id}_d & \mathcal{O}_d \\ \delta_\pm s \text{Id}_d & \text{Id}_d \end{pmatrix} \begin{pmatrix} x_\pm^{(0)} \\ x_\pm^{(1)} \end{pmatrix}.$$

Using (2.3), we obtain that

$$\begin{aligned} \begin{pmatrix} X_\pm^{(0)}(s) \\ X_\pm^{(1)}(s) \end{pmatrix} &= \mathbf{W}_\pm^\delta(s)\mathbf{W}_\pm^\delta(1)^{-1}\mathbf{Y}_\pm + \begin{pmatrix} \text{Id}_d & \mathcal{O}_d \\ \delta_\pm s \text{Id}_d & \text{Id}_d \end{pmatrix} \begin{pmatrix} x_\pm^{(0)} \\ x_\pm^{(1)} \end{pmatrix} \\ &= \mathbf{W}_\pm^\delta(s)[\mathbf{W}_\pm^\delta(1)]^{-1} \begin{pmatrix} x_0^{(0)} \\ x_0^{(1)} \end{pmatrix} \\ &\quad - \mathbf{W}_\pm^\delta(s)[\mathbf{W}_\pm^\delta(1)]^{-1} \begin{pmatrix} x_\pm^{(0)} \\ x_\pm^{(1)} + \delta_\pm x_\pm^{(0)} \end{pmatrix} + \begin{pmatrix} \text{Id}_d & \mathcal{O}_d \\ \delta_\pm s \text{Id}_d & \text{Id}_d \end{pmatrix} \begin{pmatrix} x_\pm^{(0)} \\ x_\pm^{(1)} \end{pmatrix} \\ &= \mathfrak{A}^s \begin{pmatrix} x_0^{(0)} \\ x_0^{(1)} \\ x_0 \end{pmatrix} + \mathfrak{B}^s = \mathfrak{A}^s x_0 + \mathfrak{B}^s =: \Phi_\pm^s(x_0) \end{aligned}$$

is an affine function with matrix $\mathfrak{A}^s := W_{\pm}^{\delta}(s)[W_{\pm}^{\delta}(1)]^{-1}$ and a vector $\mathfrak{B}^s$ that depend only on s and $x_{\pm}$. We still have to prove that the matrix $\mathfrak{A}^s$ is invertible for $s \in (0, 1]$. If so, given $s \in (0, 1]$, the derivative along the first variable $X_{\pm}^{(0)}(s)$ of the inverse is

$$\nabla_{(0)} (\Phi_{\pm}^s)^{-1} = W_{\pm}^{\delta}(1)[W_{\pm}^{\delta}(s)]^{-1} \begin{pmatrix} \mathbf{1}_d \\ \mathbf{0}_d \end{pmatrix}.$$

We choose $g_i(s) := (1 + \alpha_i)^{-1}(2 + \alpha_i)^{-1}s^{2+\alpha_i}$, $i = 0, 1$, with $\alpha_0, \alpha_1 \in (-1, 0)$ and $\alpha_0 \neq \alpha_1$. Then the Wronskian matrix is invertible for all $s \neq 0$:

$$(2.4) \quad \begin{aligned} \det W(s) &= (g'_0(s)g_1(s) - g_0(s)g'_1(s))^d \\ &= \frac{(\alpha_0 - \alpha_1)^d}{(1 + \alpha_0)^d(1 + \alpha_1)^d(2 + \alpha_0)^d(2 + \alpha_1)^d} s^{(3+\alpha_0+\alpha_1)d} \neq 0. \end{aligned}$$

Our choice of two linearly independent control functions g_0 and g_1 is guided by ensuring the Wronskian matrix to be invertible. Moreover,

$$\begin{aligned} W_{\pm}^{\delta}(1) &= \begin{pmatrix} \frac{\text{Id}_d}{(1+\alpha_0)} & \frac{\text{Id}_d}{(1+\alpha_1)} \\ \frac{\delta_{\pm}\text{Id}_d}{(1+\alpha_0)(2+\alpha_0)} & \frac{\delta_{\pm}\text{Id}_d}{(1+\alpha_1)(2+\alpha_1)} \end{pmatrix}, \\ W_{\pm}^{\delta}(s)^{-1} &= \frac{1}{\delta_{\pm} \det W(s)} \begin{pmatrix} \frac{\delta_{\pm}s^{2+\alpha_1}\text{Id}_d}{(1+\alpha_1)(2+\alpha_1)} & \frac{-s^{1+\alpha_1}\text{Id}_d}{(1+\alpha_1)} \\ \frac{-\delta_{\pm}s^{(2+\alpha_0)}\text{Id}_d}{(1+\alpha_0)(2+\alpha_0)} & \frac{s^{1+\alpha_0}\text{Id}_d}{(1+\alpha_0)} \end{pmatrix}, \end{aligned}$$

which finally yields

$$\begin{aligned} \nabla_{(0)} (\Phi_{\pm}^s)^{-1} &= W_{\pm}^{\delta}(1)[W_{\pm}^{\delta}(s)]^{-1} \begin{pmatrix} \mathbf{1}_d \\ \mathbf{0}_d \end{pmatrix} \\ &= \frac{1}{(\alpha_0 - \alpha_1)} \begin{pmatrix} (2 + \alpha_0)s^{-1-\alpha_0} - (2 + \alpha_1)s^{-1-\alpha_1} \mathbf{1}_d \\ \delta_{\pm} (s^{-1-\alpha_0} - s^{-1-\alpha_1}) \mathbf{1}_d \end{pmatrix} \\ &= O(s^{-1-\alpha_0}) + O(s^{-1-\alpha_1}), \end{aligned}$$

which remains integrable for $\alpha_0, \alpha_1 \in (-1, 0)$ with $\alpha_0 \neq \alpha_1$. Observe also that our choice of control functions g_0 and g_1 implies that all their derivatives up to order two are integrable on $s \in [0, 1]$, which implies that the trajectories are bounded (with integrable tangent vector field).

2.2. THE GENERAL CASE. — Let us now consider the general case with $\kappa \geq 2$ commutators, general dimensions $d_0 \geq d_1 \geq \dots \geq d_{\kappa} \geq 1$ and surjective $d_i \times d_{i-1}$ matrices B_i . Consider for $s \in [0, 1]$ the paths

$$\begin{aligned} \Gamma_+(s) &= (X_+^{(\kappa)}(s), \dots, X_+^{(0)}(s), T_+(s))^T, & \Gamma_+(0) &= z_+, & \Gamma_+(1) &= z_0, \\ \Gamma_-(s) &= (X_-^{(\kappa)}(s), \dots, X_-^{(0)}(s), T_-(s))^T, & \Gamma_-(0) &= z_-, & \Gamma_-(1) &= z_0, \end{aligned}$$

and the following control problem

$$(2.5) \quad \begin{cases} \frac{d}{ds} X_{\pm}^{(0)}(s) = \sum_{i=0}^{\kappa} \mathbf{m}_{\pm}^{(i)} g_i^{(\kappa+1)}(s), \\ \frac{d}{ds} X_{\pm}^{(1)}(s) = \delta_{\pm} \mathbf{B}_1 X_{\pm}^{(0)}(s), \\ \vdots \\ \frac{d}{ds} X_{\pm}^{(\kappa)}(s) = \delta_{\pm} \mathbf{B}_{\kappa} X_{\pm}^{(\kappa-1)}(s), \\ \frac{d}{ds} T_{\pm}(s) = \delta_{\pm} \end{cases}$$

with $\delta_{\pm} := t_0 - t_{\pm}$ and the control functions $g_i \in C^{\kappa+1}((0, 1])$ so that $g_i(0) = g'_i(0) = \dots = g_i^{(\kappa)}(0) = 0$ for $i = 0, 1, \dots, \kappa$, and $\mathbf{m}_{\pm}^{(i)} \in \mathbb{R}^{d_0}$ for $i = 0, \dots, \kappa$. Then

$$(2.6) \quad \begin{cases} X_{\pm}^{(0)}(s) = x_{\pm}^{(0)} + \sum_{i=0}^{\kappa} \mathbf{m}_{\pm}^{(i)} g_i^{(\kappa)}(s), \\ X_{\pm}^{(1)}(s) = x_{\pm}^{(1)} + s\delta_{\pm} \tilde{\mathbf{B}}_{1,1} x_{\pm}^{(0)} + \delta_{\pm} \left[\sum_{i=0}^{\kappa} \tilde{\mathbf{B}}_{1,1} \mathbf{m}_{\pm}^{(i)} g_i^{(\kappa-1)}(s) \right], \\ X_{\pm}^{(2)}(s) = x_{\pm}^{(2)} + s\delta_{\pm} \tilde{\mathbf{B}}_{2,2} x_{\pm}^{(1)} + \frac{(s\delta_{\pm})^2}{2} \tilde{\mathbf{B}}_{2,1} x_{\pm}^{(0)} + \delta_{\pm}^2 \left[\sum_{i=0}^{\kappa} \tilde{\mathbf{B}}_{2,1} \mathbf{m}_{\pm}^{(i)} g_i^{(\kappa-2)}(s) \right], \\ \vdots \\ X_{\pm}^{(\kappa)}(s) = x_{\pm}^{(\kappa)} + \sum_{i=1}^{\kappa} \frac{(s\delta_{\pm})^i}{i!} \tilde{\mathbf{B}}_{\kappa, \kappa-i+1} x_{\pm}^{(\kappa-i)} + \delta_{\pm}^{\kappa} \left[\sum_{i=0}^{\kappa} \tilde{\mathbf{B}}_{\kappa,1} \mathbf{m}_{\pm}^{(i)} g_i(s) \right], \\ T_{\pm}(s) = st_0 + (1-s)t_{\pm}, \end{cases}$$

where $\tilde{\mathbf{B}}_{i,j} := \mathbf{B}_i \mathbf{B}_{i-1} \dots \mathbf{B}_j$ for $1 \leq j \leq i$, which is a $d_i \times d_{j-1}$ block matrix.

We denote by $\mathcal{O}$ a zero block matrix with arbitrary size and

$$(2.7) \quad \mathbf{M}_{\pm} := \begin{pmatrix} \mathbf{m}_{\pm}^{(0)} \\ \vdots \\ \mathbf{m}_{\pm}^{(\kappa)} \end{pmatrix}, \quad \mathbf{W}(s) := \begin{pmatrix} g_0^{(\kappa)}(s) \text{Id}_{d_0} & \dots & g_{\kappa}^{(\kappa)}(s) \text{Id}_{d_0} \\ \vdots & \vdots & \vdots \\ g_0(s) \text{Id}_{d_0} & \dots & g_{\kappa}(s) \text{Id}_{d_0} \end{pmatrix},$$

$$(2.8) \quad \mathbf{W}_{\pm}^{\delta}(s) := \begin{pmatrix} \text{Id}_{d_0} & \mathcal{O} & \dots & \dots & \mathcal{O} \\ \mathcal{O} & \tilde{\mathbf{B}}_{1,1} \delta_{\pm} & \mathcal{O} & \dots & \mathcal{O} \\ \vdots & \mathcal{O} & \tilde{\mathbf{B}}_{2,1} \delta_{\pm}^2 & \ddots & \mathcal{O} \\ \vdots & \vdots & \ddots & \ddots & \mathcal{O} \\ \mathcal{O} & \dots & \dots & \mathcal{O} & \tilde{\mathbf{B}}_{\kappa,1} \delta_{\pm}^{\kappa} \end{pmatrix} \mathbf{W}(s) =: \mathbf{R} \mathbf{W}(s),$$

$$(2.9) \quad \mathbf{Y}_{\pm} = \begin{pmatrix} y_{\pm}^{(0)} \\ \vdots \\ y_{\pm}^{(\kappa)} \end{pmatrix} := \begin{pmatrix} x_0^{(0)} \\ \vdots \\ x_0^{(\kappa)} \end{pmatrix} - \mathbf{T}_{\pm}(1) \begin{pmatrix} x_{\pm}^{(0)} \\ \vdots \\ x_{\pm}^{(\kappa)} \end{pmatrix},$$

$$(2.10) \quad T_{\pm}(s) := \begin{pmatrix} \text{Id}_{d_0} & \mathcal{O} & \cdots & \cdots & \mathcal{O} \\ (s\delta_{\pm})\tilde{B}_{1,1} & \text{Id}_{d_1} & \mathcal{O} & \cdots & \mathcal{O} \\ \vdots & (s\delta_{\pm})\tilde{B}_{2,2} & \text{Id}_{d_2} & \ddots & \mathcal{O} \\ \vdots & \vdots & \ddots & \ddots & \mathcal{O} \\ \frac{(s\delta_{\pm})^{\kappa}}{\kappa!}\tilde{B}_{\kappa,1} & \cdots & \cdots & (s\delta_{\pm})\tilde{B}_{\kappa,\kappa} & \text{Id}_{d_{\kappa}} \end{pmatrix}.$$

Note that the matrix $W(s)$ is the Wronskian of the family of functions $(g_i)_{i=0}^{\kappa}$ and is invertible when they are the linearly independent solutions to a $(\kappa + 1)$ -order linear ODE. The boundary conditions $\Gamma_{\pm}(1) = z_0$ implies $W_{\pm}^{\delta}(1)M_{\pm} = Y_{\pm}$ which has at least the following solution

$$M_{\pm} := W_{\pm}^{\delta}(1)^{-1}Y_{\pm}$$

thanks to Moore-Penrose theory, see for example [3], where $(W_{\pm}^{\delta}(1))^{-1}$ is the right pseudo-inverse of $W_{\pm}^{\delta}(1)$. Indeed, the right pseudo-inverse of the matrix $W_{\pm}^{\delta}(1)$ always exists, and is given by the product of the inverse of $W(1)$ times the right pseudo-inverse of R that we denote here by R^{-1} . Thus we deduce that

$$(2.11) \quad \begin{pmatrix} X_{\pm}^{(0)}(s) \\ \vdots \\ X_{\pm}^{(\kappa)}(s) \end{pmatrix} = W_{\pm}^{\delta}(s)[W_{\pm}^{\delta}(1)]^{-1}x_0 + \left(T_{\pm}(s) - W_{\pm}^{\delta}(s)[W_{\pm}^{\delta}(1)]^{-1}T_{\pm}(1)\right)x_{\pm} \\ = \mathfrak{A}^s x_0 + \mathfrak{B}^s =: \Phi_{\pm}^s(x_0)$$

is an affine function with matrix $\mathfrak{A}^s := W_{\pm}^{\delta}(s)[W_{\pm}^{\delta}(1)]^{-1} = RW(s)W(1)^{-1}R^{-1}$ and vector $\mathfrak{B}^s$ that depend only on s and $x_{\pm}$. We still have to prove that the matrix $\mathfrak{A}^s$ is invertible for $s \in (0, 1]$. If so, given $s \in (0, 1]$, the derivative along the first variable $X_{\pm}^{(0)}(s)$ of the inverse is

$$(2.12) \quad \nabla_{(0)}(\Phi_{\pm}^s)^{-1} = W_{\pm}^{\delta}(1)[W_{\pm}^{\delta}(s)]^{-1} \begin{pmatrix} \mathbf{1}_{d_0} \\ \mathbf{0}_{d_1} \\ \vdots \\ \mathbf{0}_{d_{\kappa}} \end{pmatrix}.$$

We finally choose the control functions

$$(2.13) \quad g_i(s) := \frac{s^{1+\kappa+\alpha_i}}{(1+\alpha_i)(2+\alpha_i)\cdots(1+\kappa+\alpha_i)}$$

with $\alpha_i \in (-1, 0)$ pairwise distinct for $i = 0, \dots, \kappa$. Thus, we are left to prove that $W(1)$ and $\mathfrak{A}_s$ are invertible for $s \in (0, 1]$, but this boils down to prove the matrix $W(s)$ is invertible for $s \in (0, 1]$. Indeed, when $s \neq 0$ the Wronskian matrix is invertible with the precise s -behaviour given by the following lemma:

LEMMA 7. — Consider the $(\kappa + 1) \times (\kappa + 1)$ matrix $P(s) := (s^{1+\alpha_j+i}/p_{i,j})_{i,j=0}^\kappa$ with $p_{i,j} := (1+\alpha_j) \cdots (1+i+\alpha_j)$ and the matrix⁽¹⁾ $W := P \otimes \text{Id}_{d_0}$. Then their determinants are given by

$$(2.14) \quad \begin{aligned} \det P(s) &= \left(\frac{\prod_{i,j=0}^\kappa (\alpha_i - \alpha_j)}{\prod_{i,j=0}^\kappa (1+i+\alpha_j)} \right) s^{\frac{(\kappa+1)(\kappa+2)}{2} + \sum_{i=0}^\kappa \alpha_i} \\ \det W(s) &= \left(\frac{\prod_{i,j=0}^\kappa (\alpha_i - \alpha_j)^{d_0}}{\prod_{i,j=0}^\kappa (1+i+\alpha_j)^{d_0}} \right) s^{d_0 \frac{(\kappa+1)(\kappa+2)}{2} + d_0 \sum_{i=0}^\kappa \alpha_i}. \end{aligned}$$

Proof of Lemma 7. — The factorisation in s gives the exponents. It remains to compute the determinant of matrix $P(1) := (1/p_{i,j})_{i,j=0}^\kappa$. From now on, for every $n \geq 1$, we define $Q_n(X) = \prod_{j=0}^{n-1} (X + \kappa + 1 - j)$ a polynomial of degree n . Then, we observe:

$$\prod_{i,j=0}^\kappa (1+i+\alpha_j) \det P(1) = \det \begin{pmatrix} Q_\kappa(\alpha_0) & Q_\kappa(\alpha_1) & \cdots & \cdots & Q_\kappa(\alpha_\kappa) \\ Q_{\kappa-1}(\alpha_0) & Q_{\kappa-1}(\alpha_1) & \cdots & \cdots & Q_{\kappa-1}(\alpha_\kappa) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ Q_1(\alpha_0) & Q_1(\alpha_1) & \cdots & \ddots & Q_1(\alpha_\kappa) \\ 1 & 1 & \cdots & \cdots & 1 \end{pmatrix},$$

where the equivalence follows by multiplying the j^{th} -column by $\prod_{i=0}^\kappa (1+i+\alpha_j)$ according to classical laws of multiplication by scalars for determinants of matrices. Then the determinant we are interested in is equal to the determinant of the equivalent matrix that comes from performing linear combinations of rows to get $\alpha_{j-1}^{\kappa+1-i}$ on the i^{th} row and j^{th} column which gives a Vandermonde determinant of value $\prod_{i,j=0}^\kappa (\alpha_i - \alpha_j)$. The determinant of W directly follows by tensor calculus. $\square$

Then, by Lemma 7, we thus find

$$W(s)^{-1} := \frac{1}{\det W(s)} \text{Comatrix } W(s).$$

Going back to the calculation of (2.12), and combining (2.14) with the s -scaling of the entries of the cofactor matrix and our choice (2.13), we get

$$(2.15) \quad \nabla_{(0)} (\Phi_\pm^s)^{-1} = W_\pm^\delta(1) [W_\pm^\delta(s)]^{-1} \begin{pmatrix} \text{Id}_{d_0} \\ \mathcal{O}_{d_1} \\ \vdots \\ \mathcal{O}_{d_\kappa} \end{pmatrix} = \sum_{i=0}^\kappa O(s^{-1-\alpha_i}),$$

which is integrable for $\alpha_i \in (-1, 0)$. Observe also that our choice of control functions g_i , $i = 1, \dots, \kappa$, implies that all their derivatives up to order $\kappa + 1$ are integrable on $s \in [0, 1]$, which implies that the trajectories are bounded (with integrable tangent vector field).

⁽¹⁾The notation $P \otimes \text{Id}_{d_0}$ is the classical tensor product between the matrices P and Id_{d_0} .

3. PROOF OF THE POINCARÉ INEQUALITY

We consider the three cylinders $Q^+, Q^-, Q^0 \subset \tilde{\Omega}$ as in Figure 2, operators $\mathcal{A}$ and $\mathcal{B}$ that satisfy the assumptions of Section 1.3, and a weak sub-solution f of (1.1). Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be a non-negative function in the first N variables $(x^{(\kappa)}, \dots, x^{(0)})$ (excluding the time component) with compact support in any time-slice of Q^0 and such that $\int_{Q^0} \varphi \, dx = 1$. Then

$$\begin{aligned}
 \int_{Q^+} \left(f(z_+) - \langle f \rangle_{Q^-} \right)_+ \, dz_+ &\leq \int_{Q^+} \int_{Q^-} \left(f(z_+) - f(z_-) \right)_+ \, dz_- \, dz_+ \\
 &\leq \int_{Q^+} \int_{Q^-} \left(f(z_+) - \langle f \varphi \rangle_{Q^0} \right)_+ \, dz_- \, dz_+ \\
 &\quad + \int_{Q^+} \int_{Q^-} \left(\langle f \varphi \rangle_{Q^0} - f(z_-) \right)_+ \, dz_- \, dz_+ \\
 (3.1) \qquad &\leq \int_{Q^+} \left\{ \underbrace{\int_{Q^0} (f(z_+) - f(z_0)) \varphi(x_0) \, dz_0}_{=: \mathcal{J}^+} \right\}_+ \, dz_+ \\
 &\quad + \int_{Q^-} \left\{ \underbrace{\int_{Q^0} (f(z_0) - f(z_-)) \varphi(x_0) \, dz_0}_{=: \mathcal{J}^-} \right\}_+ \, dz_-,
 \end{aligned}$$

where $z_+ = (x_+, t_+)$, $z_- = (x_-, t_-)$ and $z_0 = (x_0, t_0)$. Note that $t_- < t_0 < t_+$. We now use the trajectories constructed in Section 2 to estimate the right hand side.

Using the chain rule, (1.1), (2.5) with the choice (2.13) we get

$$\begin{aligned}
 \mathcal{J}^\pm &= \pm \int_{Q^0} \left(f(z_\pm) - f(z_0) \right) \varphi(x_0) \, dz_0 = \mp \int_{Q^0} \int_0^1 \frac{d}{ds} f(\Gamma_\pm(s)) \varphi(x_0) \, ds \, dz_0 \\
 &= \mp \delta_\pm \int_{Q^0} \int_0^1 (\mathcal{T}f)(\Gamma_\pm(s)) \varphi(x_0) \, ds \, dz_0 \\
 &\quad \mp \int_{Q^0} \int_0^1 \left(\frac{d}{ds} X^{(0)}(s) \right) \cdot (\nabla_{(0)} f)(\Gamma_\pm(s)) \varphi(x_0) \, ds \, dz_0 \\
 (3.2) \qquad &\leq \underbrace{\pm \delta_\pm \int_{Q^0} \int_0^1 [(\mathcal{A}^* \mathcal{A})f](\Gamma_\pm(s)) \varphi(x_0) \, ds \, dz_0}_{=: \mathcal{J}_1^\pm} \\
 &\quad \underbrace{\mp \int_{Q^0} \int_0^1 \sum_{i=0}^\kappa s^{\alpha_i} \mathbf{m}_\pm^{(i)} \cdot (\nabla_{(0)} f)(\Gamma_\pm(s)) \varphi(x_0) \, ds \, dz_0}_{=: \mathcal{J}_2^\pm},
 \end{aligned}$$

where the above integrals are interpreted in the duality sense. The only difference between the two terms $\mathcal{J}^+$ and $\mathcal{J}^-$ is the role of z_0 : in the former case it is the past variable, in the latter it is the future variable.

In the following computations to estimate $\mathcal{J}_1^\pm$ and $\mathcal{J}_2^\pm$, the local and the non-local case can be treated similarly, *upon replacing the integration domain over Q^0 by $Q_v^0 \times \mathbb{R}^{d_0}$ in the non-local case, where $Q_v^0 \subset \mathbb{R}^{N+1-d_0}$ is the domain of the cylinder Q^0 for the variables $(x^{(1)}, \dots, x^{(\kappa)}, t)$, that is we slice out the $x^{(0)}$ variable.*

We now integrate by parts the terms $\mathcal{J}_1^\pm$ after a change of variables $x_0 \mapsto y := \Phi_\pm^s(x_0)$ for s, t_0 fixed $\Phi_\pm^s$ is the affine map defined in (2.11), which is invertible for $s \neq 0$):

$$\begin{aligned} \mathcal{J}_1^\pm &= \pm \frac{\delta_\pm}{|Q^0|} \int_{Q^0} \int_0^1 [(\mathcal{A}^* \mathcal{A}) f](\Gamma_\pm(s)) \varphi(x_0) \, ds \, dz_0 \\ &= \pm \frac{\delta_\pm}{|Q^0|} \int_{Q^0} \int_0^1 [(\mathcal{A}^* \mathcal{A}) f](\Phi_\pm^s(x_0), st_0 + (1-s)t_\pm) \varphi(x_0) \, ds \, dx_0 \, dt_0 \\ &= \pm \frac{\delta_\pm}{|Q^0|} \int_{(\Phi_\pm^s \otimes \text{Id})(Q^0)} \int_0^1 [(\mathcal{A}^* \mathcal{A}) f](y, st_0 + (1-s)t_\pm) \varphi\left((\Phi_\pm^s)^{-1}(y)\right) \frac{ds \, dy \, dt_0}{|\det \mathfrak{A}^s|} \\ &= \pm \frac{\delta_\pm}{|Q^0|} \int_{(\Phi_\pm^s \otimes \text{Id})(Q^0)} \int_0^1 [\mathcal{A} f](y, st_0 + (1-s)t_\pm) \mathcal{A} \left[\varphi\left((\Phi_\pm^s)^{-1}(y)\right) \right] \frac{ds \, dy \, dt_0}{|\det \mathfrak{A}^s|} \\ &= \pm \frac{\delta_\pm}{|Q^0|} \int_{Q^0} \int_0^1 [\mathcal{A} f](\Gamma_\pm(s)) \left\{ \mathcal{A} \left[\varphi\left((\Phi_\pm^s)^{-1}(y)\right) \right] \right\}_{|y=\Phi_\pm^s(x_0)} \, ds \, dx_0 \, dt_0. \end{aligned}$$

We then use the first bound in (1.14) on the operator $\mathcal{A}$ of Section 1.3: for $\varepsilon \in (0, 1-\beta)$

$$\begin{aligned} \left| \left\{ \mathcal{A} \left[\varphi\left((\Phi_\pm^s)^{-1}(y)\right) \right] \right\}_{|y=\Phi_\pm^s(x_0)} \right| &\lesssim_\varphi \left\| \nabla_{(0)} (\Phi_\pm^s)^{-1} \right\|_\infty^{\beta+\varepsilon} \\ &\lesssim_\varphi \left(\sum_{i=0}^\kappa s^{-1-\alpha_i} \right)^{\beta+\varepsilon} \lesssim_\varphi \left(\sum_{i=0}^\kappa s^{-(\beta+\varepsilon)(1+\alpha_i)} \right) \end{aligned}$$

and deduce finally

$$\mathcal{J}_1^\pm \lesssim \frac{\delta_\pm}{|Q^0|} \int_{Q^0} \int_0^1 |[\mathcal{A} f](\Gamma_\pm(s))| \left(\sum_{i=0}^\kappa s^{-(\beta+\varepsilon)(1+\alpha_i)} \right) \, ds \, dx_0 \, dt_0.$$

Now let us turn to $\mathcal{J}_2^\pm$ in (3.2). We find, using the same change of variables,

$$\begin{aligned} \mathcal{J}_2^\pm &= \mp \int_0^1 \int_{Q^0} \sum_{i=0}^\kappa s^{\alpha_i} \mathbf{m}_\pm^{(i)} \cdot \nabla_{(0)} f(\Gamma_\pm(s)) \varphi(x_0) \, ds \, dx_0 \, dt_0 \\ &= \mp \frac{1}{|Q^0|} \sum_{i=0}^\kappa \int_0^1 \int_{(\Phi_\pm^s \otimes \text{Id})(Q^0)} s^{\alpha_i} \mathbf{m}_\pm^{(i)} \cdot \nabla_{(0)} f(y, st_0 + (1-s)t_\pm) \\ &\quad \times \varphi\left((\Phi_\pm^s)^{-1}(y)\right) \frac{ds \, dy \, dt_0}{|\det \mathfrak{A}^s|} \\ &= \pm \frac{1}{|Q^0|} \sum_{i=0}^\kappa \int_0^1 \int_{Q^0} s^{\alpha_i} \left((-\Delta_{(0)})^{\beta/2} f \right)(\Gamma_\pm(s)) \\ &\quad \times \mathbf{m}_\pm^{(i)} \cdot \left\{ (-\Delta_{(0)})^{-\beta/2} \nabla_{(0)} \left\{ \varphi\left((\Phi_\pm^s)^{-1}(y)\right) \right\} \right\}_{|y=\Phi_\pm^s(x_0)} \, ds \, dx_0 \, dt_0. \end{aligned}$$

For $0 < \beta < 1$, we claim that

$$(3.3) \quad \left| (-\Delta_{(0)})^{-\beta/2} \nabla_{(0)} \left\{ \varphi \left[(\Phi_{\pm}^s)^{-1}(y) \right] \right\} \right| \lesssim \left\| \nabla_{(0)} (\Phi_{\pm}^s)^{-1} \right\|_{\infty}^{1-\beta} \\ \lesssim \left(\sum_{j=0}^{\kappa} s^{-1-\alpha_j} \right)^{1-\beta} \leq \sum_{j=0}^{\kappa} s^{-(1+\alpha_j)(1-\beta)}.$$

This can be seen as follows. For $0 < \beta < 1$, the operator

$$T_{\beta} := (-\Delta_{(0)})^{-\beta/2} \nabla_{(0)}$$

has order $1 - \beta$ and kernel

$$K_{\beta}(v - w) = c_{d_0, \beta} \frac{v - w}{|v - w|^{d_0+2-\beta}}.$$

Writing $y = (y', v)$, where $y' = (y^{(\kappa)}, \dots, y^{(1)})$ is fixed, we set

$$g(v) := \varphi((\Phi_{\pm}^s)^{-1}(y', v)),$$

so that, using the cancellation of this odd kernel,

$$|T_{\beta} g(v)| \lesssim \int_{\mathbb{R}^{d_0}} \frac{|g(v) - g(w)|}{|v - w|^{d_0+1-\beta}} dw.$$

By splitting the integral at radius ρ ,

$$|T_{\beta} g(v)| \lesssim \|\nabla_{(0)} g\|_{\infty} \rho^{\beta} + \|g\|_{\infty} \rho^{\beta-1}.$$

Choosing $\rho = \|g\|_{\infty} / \|\nabla_{(0)} g\|_{\infty}$ gives

$$\|T_{\beta} g\|_{\infty} \lesssim_{\beta, d_0} \|g\|_{\infty}^{\beta} \|\nabla_{(0)} g\|_{\infty}^{1-\beta}.$$

By the chain rule,

$$\|\varphi((\Phi_{\pm}^s)^{-1}(y))\|_{\infty} \leq \|\varphi\|_{\infty}, \quad \|\nabla_{(0)} (\varphi((\Phi_{\pm}^s)^{-1}(y)))\|_{\infty} \lesssim_{\varphi} \|\nabla_{(0)} (\Phi_{\pm}^s)^{-1}\|_{\infty}.$$

Thus, combined with (2.15), this implies (3.3). We deduce

$$\mathcal{J}_{\pm}^{\pm} \leq \frac{1}{|Q^0|} \int_0^1 \int_{Q^0} \left(\sum_{i,j=0}^{\kappa} s^{\alpha_i + \beta(1+\alpha_j) - \alpha_j - 1} \right) \left| \left((-\Delta_{(0)})^{\beta/2} f \right) (\Gamma_{\pm}(s)) \right| ds dx_0 dt_0.$$

It follows from (3.1) and (3.2) that we are left with estimating

$$\mathcal{J} := \mathcal{J}^+ + \mathcal{J}^-, \quad \text{where} \quad \mathcal{J}^{\pm} := \oint_{Q^{\pm}} \{\mathcal{J}_1^{\pm}\}_+ dz_{\pm} + \oint_{Q^{\pm}} \{\mathcal{J}_2^{\pm}\}_+ dz_{\pm}.$$

The previous estimates imply

$$(3.4) \quad \mathcal{J}^{\pm} \lesssim \int_{Q^{\pm}} \int_0^1 \int_{Q^0} \left(\sum_{i=0}^{\kappa} \frac{1}{s^{(\alpha_i+1)(\beta+\varepsilon)}} \right) |(\mathcal{A}f)(\Gamma_{\pm}(s))| dz_{\pm} ds dz_0 \\ + \int_{Q^{\pm}} \int_0^1 \int_{Q^0} \left(\sum_{i,j=0}^{\kappa} s^{\alpha_i + \beta(1+\alpha_j) - \alpha_j - 1} \right) \left| \left((-\Delta_{(0)})^{\beta/2} f \right) (\Gamma_{\pm}(s)) \right| dz_{\pm} ds dz_0.$$

Given some $\varepsilon > 0$, we first choose $\varepsilon > 0$ sufficiently small and then choose the α_i 's pairwise distinct and sufficiently close to $(2 + \beta + \varepsilon)^{-1} - 1$.

REMARK 8. — Note also that, by adopting the mollified formulation (1.20) and by keeping the test functions $\psi_{\pm}$ until the integral $\mathcal{J}^{\pm}$ above, one could use the variable z_0 to integrate the test function in the non-singular region $s \in [s_0, 1]$, keep an L^{∞} control of the test function in the singular region, and optimise s_0 in order to deduce a gain of integrability.

Then (3.4) simplifies into, for some ε as small as wanted,

$$\begin{aligned} \mathcal{J}^{\pm} &\lesssim \int_{Q^{\pm}} \int_0^1 \int_{Q^0} s^{-\frac{\beta}{2+\beta}-\varepsilon} |(\mathcal{A}f)(\Gamma_{\pm}(s))| \, dz_{\pm} \, ds \, dz_0 \\ &\quad + \int_{Q^{\pm}} \int_0^1 \int_{Q^0} s^{-1+\frac{\beta}{2+\beta}-\varepsilon} \left| \left((-\Delta_{(0)})^{\beta/2} f \right) (\Gamma_{\pm}(s)) \right| \, dz_{\pm} \, ds \, dz_0. \end{aligned}$$

We are now in a position to use not only the intermediate variable z_0 , but also the future/past variables $z_{\pm}$ for a change of variables, since we are now integrating in both Q^0 and $Q_{\pm}$. Note that it was not possible to use the integration in $Q_{\pm}$ before because of the positive value around the Q^0 integral.

We split $\mathcal{J}^{\pm}$ as follows, given $s_0 \in (0, 1)$,

$$\begin{aligned} \int_{Q^{\pm}} \int_{Q^0} \int_0^1 (\dots) \, ds \, dz_0 \, dz_{\pm} \\ = \underbrace{\int_{Q^{\pm}} \int_{Q^0} \int_0^{s_0} (\dots) \, ds \, dz_0 \, dz_{\pm}}_{=: \mathcal{J}_1^{\pm}} + \underbrace{\int_{Q^{\pm}} \int_{Q^0} \int_{s_0}^1 (\dots) \, ds \, dz_0 \, dz_{\pm}}_{=: \mathcal{J}_2^{\pm}}. \end{aligned}$$

The two changes of variables on each part are represented in Figure 4.

To control the part $\mathcal{J}_1^{\pm}$ we parametrise $\Gamma_{\pm}(s)$ by the $z_{\pm}$ coordinates, for $s \in [0, s_0]$ and $t_0, t_{\pm}$ all fixed:

$$(x_{\pm}, t_{\pm}) \mapsto \Gamma_{\pm}(s) = (\bar{\Gamma}_{\pm}^s, t_0 s + (1-s)t_{\pm}) =: \Psi_{\pm}^s(x_{\pm}, t_{\pm})$$

Since s does not approach 1, we can prove that this change of variables is not singular. The mappings $\Psi_{\pm}^s$ are determined by solving (2.11) for $x_{\pm}$ instead of x_0 . It yields

$$\Psi_{\pm}^s(x_{\pm}, t_{\pm}) = (\mathbf{a}_{\pm}^s x_{\pm} + \mathbf{b}_{\pm}^s, t_0 s + (1-s)t_{\pm})$$

with the matrix $\mathbf{a}_{\pm}^s$ and vector $\mathbf{b}_{\pm}^s$ given by

$$\mathbf{a}_{\pm}^s := (\mathbf{T}_{\pm}(s) - \mathbf{W}_{\pm}^{\delta}(s) [\mathbf{W}_{\pm}^{\delta}(1)]^{-1} \mathbf{T}_{\pm}(1)), \quad \mathbf{b}_{\pm}^s := \mathbf{W}_{\pm}^{\delta}(s) [\mathbf{W}_{\pm}^{\delta}(1)]^{-1} x_0,$$

which depend only on s and x_0 . Since we have proved in Lemma 7 that

$$\mathbf{W}(s) = \mathcal{O}\left(s^{d_0 \frac{(\kappa+1)(\kappa+2)}{2} + d_0 \sum_{i=0}^{\kappa} \alpha_i}\right),$$

it follows that $\mathbf{W}_{\pm}^{\delta}(s) = \mathcal{O}\left(s^{d_0 \frac{(\kappa+1)(\kappa+2)}{2} + d_0 \sum_{i=0}^{\kappa} \alpha_i}\right)$ and thus $\mathbf{W}_{\pm}^{\delta}(s)$ goes to zero as $s \rightarrow 0$. Since $\mathbf{T}_{\pm}^{\delta}$ is a lower triangular matrix with unitary diagonal it is invertible

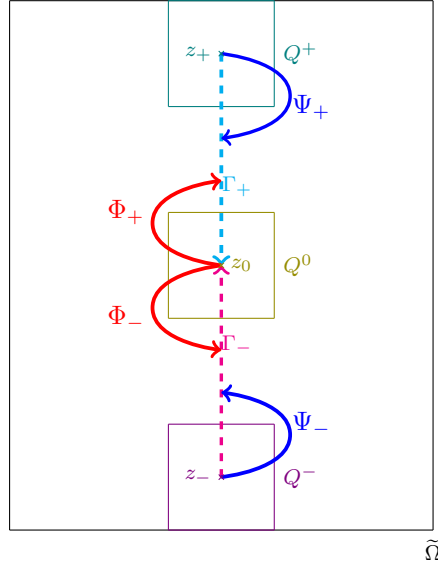


FIGURE 4. The change of variables that we use in the proof for some fixed $s \in (0, 1)$. For $s \in (0, s_0)$ we use $\Psi_{\pm}$ that map $z_{\pm}$ onto $\Gamma_{\pm}$, whereas for $s \in (s_0, 1)$ we use $\Phi_{\pm}$ that map z_0 onto $\Gamma_{\pm}$.

for all s , and finally $\mathfrak{a}_{\pm}^s$ is invertible with uniformly bounded inverse on $s \in [0, s_0]$ for $s_0 > 0$ small enough. We apply this change of variables to estimate $\mathcal{J}_1^{\pm}$:

$$\begin{aligned} \mathcal{J}_1^{\pm} &= \int_{Q^0} \int_0^{s_0} \int_{\Psi^{-1}(Q^{\pm})} s^{-\frac{\beta}{2+\beta}-\varepsilon} |(\mathcal{A}f)(y, t)| \frac{dy dt ds dz_0}{|(1-s) \det \mathfrak{a}^s|} \\ &\quad + \int_{Q^0} \int_0^{s_0} \int_{\Psi^{-1}(Q^{\pm})} s^{-1+\frac{\beta}{2+\beta}-\varepsilon} \left| \left((-\Delta_{(0)})^{\beta/2} f \right)(y, t) \right| \frac{dy dt ds dz_0}{|(1-s) \det \mathfrak{a}^s|} \\ &\lesssim_{s_0} \int_{\tilde{\Omega}} |\mathcal{A}f(z)| dz, \end{aligned}$$

where we also applied Fubini-Tonelli's theorem and integrated over s . The part $\mathcal{J}_2^{\pm}$ is controlled by using the same change of variables $\Phi_{\pm}^s$ as before, this time also involving the time component, and we call it

$$\tilde{\Phi}_{\pm}^s(x_0, t_0) := (\Phi_{\pm}^s(x_0), t_0 s + (1-s)t_{\pm})$$

for this reason:

$$\begin{aligned} \mathcal{J}_2^{\pm} &= \int_{Q^{\pm}} \int_{s_0}^1 \int_{\tilde{\Phi}_{\pm}^{s-1}(Q^0)} s^{-1+\frac{\beta}{2+\beta}-\varepsilon} \left| \left((-\Delta_{(0)})^{\beta/2} f \right)(y, t) \right| \frac{dy dt ds dz_{\pm}}{|s \det \mathfrak{A}^s|} \\ &\quad + \int_{Q^{\pm}} \int_{s_0}^1 \int_{\tilde{\Phi}_{\pm}^{s-1}(Q^0)} s^{-\frac{\beta}{2+\beta}-\varepsilon} |\mathcal{A}f(y, t)| \frac{dy dt ds dz_{\pm}}{|s \det \mathfrak{A}^s|} \\ &\lesssim \int_{\tilde{\Omega}} |\mathcal{A}f(z)| dz, \end{aligned}$$

where we have used first that the integral in s avoids the singularity at 0, and second assumption (1.14) of Section 1.3. This concludes the proof.

4. PROOF OF THE WEAK HARNACK INEQUALITY

In this section we briefly explain how to adapt the approach of [7] for deducing the intermediate value of lemma of De Giorgi from the Poincaré inequality we have proved. This approach was developed in the local case with $\kappa = 1$ and with a weaker version of the Poincaré inequality. A non-local extension, when $\kappa = 1$, was developed in [11]. This approach can be described as a “trajectory viewpoint” on the De Giorgi theory, and it is fully quantitative.

In order to carry out this analysis, one needs to have an a priori local quantitative boundedness result for weak solutions to (1.1), also known as the first De Giorgi lemma, which writes in its simplest form as follows.

LEMMA 9 (First lemma of De Giorgi). — *Let the operators $\mathcal{A}, \mathcal{B}$ be such that $\mathcal{B}$ is given by the assumptions of Section 1.3 and $\mathcal{A}$ is more specifically given by*

- (i) $\mathcal{A} := \sqrt{A}\nabla_{(0)}$ with $A \in [\Lambda^{-1}, \Lambda]$ measurable symmetric matrix (local case),
- (ii) $\mathcal{A} = \sqrt{a}(-\Delta_v)^{\beta/2}$ with $a \in [\Lambda^{-1}, \Lambda]$ scalar measurable (non-local case).

Let f be a sub-solution to (1.1) in Q_1 , such that in the local case $0 \leq f$ in Q_1 , and such that in the non-local case, $0 \leq f \leq 1$ in $B_1^{\otimes \kappa} \times \mathbb{R}^d \times [-1, 0]$. Then there exist $\varepsilon > 0$ and $r \in (0, 1)$ so that $\int_{Q_1} f^2 \leq \varepsilon$ implies $f \leq 1/2$ on Q_r .

In the local case, a stronger statement holds where the L^∞ norm of f on Q_r is controlled by a constant (depending on $\mathcal{A}, \mathcal{B}$ and r) times the L^2 norm of f on Q_1 . In the non-local case, variants of this stronger statement exist if f is not only a sub-solution but also a super-solution; we refer to [13, 12]. The proof is based on an energy estimate and the gain of integrability of the fundamental solution of the Kolmogorov equation, local or non-local. This is classical and we refer to [7, 11, 2, 18, 9, 1]. The non-local case with $\kappa \geq 2$ is obtained by scaling arguments, or by calculating the fundamental solutions in Fourier.

As a next step, one needs to prove the second De Giorgi lemma, also known as the intermediate value theorem, which controls the gradient of a sub-solution by getting an explicit bound on the measures of the sets where the solution is in between two values. It reads as follows:

LEMMA 10 (Second lemma of De Giorgi). — *Let the operators $\mathcal{A}, \mathcal{B}$ be such that $\mathcal{B}$ is given by the assumptions of Section 1.3 and $\mathcal{A}$ is more specifically given by*

- (i) $\mathcal{A} := \sqrt{A}\nabla_{(0)}$ with $A \in [\Lambda^{-1}, \Lambda]$ measurable symmetric matrix (local case),
- (ii) $\mathcal{A} = \sqrt{a}(-\Delta_v)^{\beta/2}$ with $a \in [\Lambda^{-1}, \Lambda]$ scalar measurable (non-local case).

Let $0 \leq f \leq 1$ be a sub-solution to (1.1) in Q_1 . Let $\delta_1, \delta_2 \in (0, 1)$. Then there are constants $r_0 > 0$ and $\nu \in (0, 1)$ and $\theta \in (0, 1)$ such that whenever

$$|\{f \leq 0\} \cap Q_{r_0}^-| \geq \delta_1 |Q_{r_0}^-| \quad \text{and} \quad |\{f \geq 1 - \theta\} \cap Q_{r_0}| \geq \delta_2 |Q_{r_0}|,$$

there holds

$$|\{0 < f < 1 - \theta\} \cap Q_{1/2}| \geq \nu |Q_{1/2}|$$

in the local case, or

$$|\{0 < f < 1 - \theta\} \cap B_{(1/2)^{1+\kappa\beta}} \times \cdots \times B_{(1/2)^{1+\beta}} \times [-3, 0]| \geq \nu |Q_{1/2}|$$

in the non-local case.

The proof of this lemma follows from our Poincaré inequality and the first lemma of De Giorgi by using the argument of [7, §3.2] in the local case (in fact the proof is simpler since we do not have the error terms in the Poincaré inequality that were present in [7]), and it follows from [11, §5.2] in the non-local case (note that we have removed the mollifications F_i of the cutoff functions in the statement of [11] since they can be used in the proof but removed from the statement by making the border region small enough). These proofs are done in the case of one commutator but they extend straightforwardly to the case of several commutators because they do not depend on the transport part $\mathcal{T} = \partial_t + \mathcal{B}$ of the equation, but only on $\mathcal{A}$ and the Poincaré inequality and first lemma of De Giorgi.

With the first and second De Giorgi lemma at hand, one can deduce the “measure-to-pointwise estimate” and the weak Harnack inequality by following [7] in the local case, and [11] in the non-local case, see [1] also. These proofs are independent of the equation and only depend on the previously established functional inequalities. The only minor change is that the kinetic cylinders have a slightly more complicated scaling on the variables $(x^{(\kappa)}, \dots, x^{(0)}, t)$ but the technical changes needed are, although slightly tedious, clear.

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