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p-ADIC L-FUNCTIONS FOR $\mathrm{GSp}(4) \times \mathrm{GL}(2)$, II

by Zheng Liu

Abstract. — We construct a four-variable p-adic L-function for cuspidal Hida families on $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ and prove a complete interpolation formula for it. The Archimedean zeta integrals are computed by using a partial interpolation formula for the four-variable p-adic L-function, combined with Yoshida lifts and some previously constructed p-adic L-functions—specifically Kubota–Leopoldt p-adic L-functions, Rankin–Selberg p-adic L-functions, and p-adic (standard) L-functions for $\mathrm{Sp}(4)$.

Résumé (Fonctions L p-adiques pour $\mathrm{GSp}(4) \times \mathrm{GL}(2)$, II). — Nous construisons une fonction L p-adique à quatre variables pour les familles de Hida cuspidales sur $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ et en démontrons une formule d'interpolation complète. Les intégrales zêta archimédiennes sont calculées à l'aide d'une formule d'interpolation partielle pour la fonction L p-adique à quatre variables, combinée à des relèvements de Yoshida et à certaines fonctions L p-adiques déjà construites auparavant — notamment les fonctions L p-adiques de Kubota-Leopoldt, les fonctions L p-adiques de Rankin-Selberg et les fonctions L p-adiques (standard) pour $\mathrm{Sp}(4)$.

Contents

Mathematical subject classification (2020). — 11F67, 11F33, 11F55, 11F27, 11R23.
Keywords. — p-adic L-functions, p-adic families of automorphic forms, Archimedean zeta integrals.

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1. Introduction

In this paper, we generalize the construction of the cyclotomic-variable p-adic L-functions for $\mathrm{GSp}(4)\times\mathrm{GL}(2)$ in [Liu23] to construct four-variable p-adic L-functions for Hida families on $\mathrm{GSp}(4)$ and $\mathrm{GL}(2)$. (The weights included in the interpolation range satisfy the condition (1.1).) By exploiting the existence of the four-variable p-adic L-functions and their partial interpolation properties, we calculate the Archimedean zeta integrals in Furusawa's formula for holomorphic discrete series of general vector weights. This calculation yields a complete interpolation formula for the p-adic L-functions constructed in loc. cit. and this paper.

We describe the setting for stating the main theorem. Fix an odd prime p and an isomorphism $\overline{\mathbb{Q}}_p\cong\mathbb{C}$. Let F be a sufficiently large finite extension of $\mathbb{Q}_p$ and $\mathcal{O}$ be its ring of integers. For $G=\mathrm{GL}(2),\mathrm{GSp}(4)$, let Λ_G be the Iwasawa algebra for G over $\mathcal{O}$ defined in (2.2.1), and $\mathbb{T}_{G,\mathrm{ord}}$ be the Hecke algebra acting on the Λ_G-module of Hida families on G of tame level K_G^p (chosen as in Section 3.1). Given geometrically irreducible components

$$\mathscr{C}_1\subset\mathrm{Spec}(\mathbb{T}_{\mathrm{GL}(2),\mathrm{ord}}),\qquad \mathscr{C}_2\subset\mathrm{Spec}(\mathbb{T}_{\mathrm{GSp}(4),\mathrm{ord}}),$$

denote by $\mathbb{I}_{\mathscr{C}_1},\mathbb{I}_{\mathscr{C}_2}$ their coordinate rings and by $F_{\mathscr{C}_1},F_{\mathscr{C}_2}$ their function fields. We construct the (imprimitive) p-adic L-function for $\mathscr{C}_1,\mathscr{C}_2$ and verify its interpolation properties, without uncomputed local zeta integrals, as predicted by Coates and Perrin-Riou [CPR89, Coa91] when the weight ℓ of the specialization of $\mathscr{C}_1$ and the weight (ℓ_1,ℓ_2) of the specialization of $\mathscr{C}_2$ belong to the region

$$\min\{-\ell_1+\ell_2+\ell,\ \ell_1+\ell_2-\ell\}\geqslant 3, \tag{1.1}$$

(which is the region (D) in the convention of [LR25]). The constructed p-adic L-function is an element in

$$\mathscr{M}eas\big(\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q},f}^\times/U^p,\mathbb{I}_{\mathscr{C}_1}\widehat{\otimes}\mathbb{I}_{\mathscr{C}_2}\big)\otimes_{\mathbb{I}_{\mathscr{C}_1}\widehat{\otimes}\mathbb{I}_{\mathscr{C}_2}}(F_{\mathscr{C}_1}\widehat{\otimes}F_{\mathscr{C}_2}),$$

where U^p is a certain open compact subgroup of $(\mathbb{A}_{\mathbb{Q},f}^p)^\times$, and

$$\mathscr{M}eas\big(\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q},f}^\times/U^p,\mathbb{I}_{\mathscr{C}_1}\widehat{\otimes}\mathbb{I}_{\mathscr{C}_2}\big)$$

denotes the $\mathbb{I}_{\mathscr{C}_1}\widehat{\otimes}\mathbb{I}_{\mathscr{C}_2}$-module of $\mathbb{I}_{\mathscr{C}_1}\widehat{\otimes}\mathbb{I}_{\mathscr{C}_2}$-valued p-adic measures on $\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q},f}^\times/U^p$ (namely continuous $\mathbb{Z}_p$-linear maps from the space of all $\mathbb{Z}_p$-valued continuous functions on $\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q},f}^\times/U^p$, equipped with the topology of uniform convergence, to $\mathbb{I}_{\mathscr{C}_1}\widehat{\otimes}\mathbb{I}_{\mathscr{C}_2}$).

Theorem 1.1. — *Given Hida families $\mathscr{C}_1,\mathscr{C}_2$ on $\mathrm{GL}(2),\mathrm{GSp}(4)$ and the auxiliary data:*

– $\mathbb{S}=\begin{bmatrix}\mathbb{a} & \mathbb{b}/2\\ \mathbb{b}/2 & \mathbb{c}\end{bmatrix}\in\mathrm{Sym}_2(\mathbb{Q})_{>0}$,

– *a p-adically continuous Hecke character $\mathbf{\Lambda}:\mathcal{K}^\times\backslash\mathbb{A}_{\mathcal{K},f}^\times\to\Lambda_{\mathrm{GSp}(4)}^\times$ with $\mathcal{K}=\mathbb{Q}(\sqrt{-\det\mathbb{S}})$, such that $\mathbf{\Lambda}|_{\mathbb{A}_{\mathbb{Q}}^\times}=\omega_{\mathscr{C}_2}$, the central character associated to $\mathscr{C}_2$,*

– *a finite set S of places of $\mathbb{Q}$ containing p,∞ such that everything is unramified outside S, (see Section 3.1 for the precise condition on S),*

– *an open subgroup* U^p *of* $\widehat{\mathbb{Z}}^{p,\times}=\prod_{v\neq\infty,p}\mathbb{Z}_v^\times$ *containing* $\prod_{v\notin S}\mathbb{Z}_v^\times$,

taking $\beta_1\in\mathbb{Q}_{>0},\beta_2\in\mathrm{Sym}_2(\mathbb{Q})_{>0}$, *there exists a four-variable p-adic L-function*

$$\mathcal{L}^S_{\mathscr{C}_1,\mathscr{C}_2,\beta_1,\beta_2}\in\mathscr{M}eas\big(\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/U^p,\mathbb{I}_{\mathscr{C}_1}\widehat{\otimes}\mathbb{I}_{\mathscr{C}_2}\big)\otimes_{\mathbb{I}_{\mathscr{C}_1}\widehat{\otimes}\mathbb{I}_{\mathscr{C}_2}}(F_{\mathscr{C}_1}\widehat{\otimes}F_{\mathscr{C}_2})$$
$$\cong(\mathbb{I}_{\mathscr{C}_1}\widehat{\otimes}\mathbb{I}_{\mathscr{C}_2})[\![\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/U^p]\!]\otimes_{\mathbb{I}_{\mathscr{C}_1}\widehat{\otimes}\mathbb{I}_{\mathscr{C}_2}}(F_{\mathscr{C}_1}\widehat{\otimes}F_{\mathscr{C}_2})$$

satisfying the interpolation property:

$$\mathcal{L}^S_{\mathscr{C}_1,\mathscr{C}_2,\beta_1,\beta_2}(\kappa,x)=2^{-\ell-\ell_1-\ell_2}i^\ell\sum_{f\in\mathscr{S}_{\mathrm{GL}(2),x}}\frac{f_{\mathbb{c}}f_{\beta_1}}{\mathbf{P}(f,f)}\sum_{\varphi\in\mathscr{S}_{\mathrm{GSp}(4),x}}\frac{B^\dagger_{\mathbb{S},\Lambda}(\varphi)\,\varphi_{\beta_2}}{\mathbf{P}(\varphi,\varphi)}$$
$$\times E_\infty\Big(k+\frac{\ell+\ell_1+\ell_2}{2},\Pi_x\times\pi_x\times\chi\Big)E_p\Big(k+\frac{\ell+\ell_1+\ell_2}{2},\Pi_x\times\pi_x\times\chi\Big)$$
$$\times L^S\Big(k+\frac{\ell+\ell_1+\ell_2}{2},\Pi_x\times\pi_x\times\chi\Big),$$

where

– $x\in\mathscr{C}_1(\overline{\mathbb{Q}}_p)\times\mathscr{C}_2(\overline{\mathbb{Q}}_p)$ *is a point at which the weight projection map* (3.1.1) *is étale with arithmetic image*

$$(\tau,(\tau_1,\tau_2))=((\ell,\xi),(\ell_1,\ell_2,\xi_1,\xi_2))\in\mathrm{Hom}_{\mathrm{cont}}\Big(T^1_{\mathrm{GL}(2)}(\mathbb{Z}_p)\times T^1_{\mathrm{GSp}(4)}(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times\Big),$$

and $\kappa=(k,\chi)$ *is an arithmetic point in* $\mathrm{Hom}_{\mathrm{cont}}\big(\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/U^p,\overline{\mathbb{Q}}_p^\times\big)$, *(see Section* 3.1.1 *for some of the notations), such that*

$$(1.2)\qquad -\frac{\min\{-\ell_1+\ell_2+\ell,\ \ell_1+\ell_2-\ell\}}{2}+2\leqslant k+\frac{\ell_1+\ell_2+\ell}{2}\leqslant\frac{\min\{-\ell_1+\ell_2+\ell,\ \ell_1+\ell_2-\ell\}}{2}-1,$$

(when $\min\{-\ell_1+\ell_2+\ell,\ \ell_1+\ell_2-\ell\}\geqslant 3$, $s=k+(\ell+\ell_1+\ell_2)/2$ *for such k's are all the critical points for the L-function* $L(s,\Pi_x\times\pi_x\times\chi)$*)*,

– $\mathscr{S}_{\mathrm{GL}(2),x}$ *(resp.* $\mathscr{S}_{\mathrm{GSp}(4),x}$*) is an orthogonal basis (with respect to the modified Petersson inner product defined in* (3.3.5), (3.3.6)*) of the space spanned by ordinary cuspidal holomorphic forms on* GL(2) *of weight* ℓ *and tame level* $K^p_{\mathrm{GL}(2)}$ *(resp.* GSp(4) *of weight* (ℓ_1,ℓ_2) *and tame level* $K^p_{\mathrm{GSp}(4)}$*) with nebentypus at p given by* (3.1.3), *belonging to the Hecke eigenspace parameterized by x,*

– π_x *(resp.* Π_x*) is any unitary cuspidal irreducible automorphic representation of* $\mathrm{GL}(2,\mathbb{A}_\mathbb{Q})$ *(resp.* $\mathrm{GSp}(4,\mathbb{A}_\mathbb{Q})$*) inside the representation generated by* $\mathscr{S}_{\mathrm{GL}(2),x}$ *(resp.* $\mathscr{S}_{\mathrm{GSp}(4),x}$*) twisted by a real power of* $|\det|$,

– *the factors* E_p,E_∞ *are the modified Euler factors for p-adic interpolation (see* (3.3.2), (3.3.4) *for the precise formula),*

– $f_{\mathbb{c}},f_{\beta_1}$ *(resp.* φ_{β_2}*) denote the Fourier coefficient of f indexed by* $\mathbb{c},\beta_1$ *(resp. of* φ *indexed by* β_2*), the modified Petersson inner products* $\mathbf{P}(f,f),\mathbf{P}(\varphi,\varphi)$ *are defined as in* (3.3.5), (3.3.6),

– Λ *is the classical Hecke character corresponding to the specialization of* $\mathbf{\Lambda}$ *at* (τ_1,τ_2), *and* $B^\dagger_{\mathbb{S},\Lambda}(\varphi)$ *is the Bessel period with a modification at p. (See* (3.3.7) *and* [Liu23, §2.2.1] *for the precise definition.)*

Given a Hecke eigensystem corresponding to an x as in the above theorem, if $\Pi_{x,p}$ does not belong to IIa or IVa in the classification in [RS07] (in particular if $\Pi_{x,p}$ is unramified or isomorphic to a principal series inducing sufficiently ramified characters or $\Pi_{x,\infty}$ is not of scalar weight), then one can choose $(\mathbb{S},\Lambda),\mathbb{c},\beta_1,\beta_2$ and rearrange the tame level $K^p_{\mathrm{GL}(2)}$ if necessary such that the factor

$$\sum_{f\in\mathscr{S}_{\mathrm{GL}(2),x}}\frac{f_{\mathbb{c}}f_{\beta_1}}{\mathbf{P}(f,f)}\sum_{\varphi\in\mathscr{S}_{\mathrm{GSp}(4),x}}\frac{B^{\dagger}_{\mathbb{S},\Lambda}(\varphi)\,\varphi_{\beta_2}}{\mathbf{P}(\varphi,\varphi)}$$

in the interpolation formula is nonzero (cf. the end of [Liu23]). Then by Proposition 4.2.2, we also know that this factor does not vanish in a neighborhood of that given Hecke eigensystem in the Hida family. If $\Pi_{x,p}$ belongs to IIa or IVa, then more work is needed to check whether $(\mathbb{S},\Lambda),\mathbb{c},\beta_1,\beta_2$ can be chosen such that the factor is nonzero.

We make some remarks on comparisons between our results and other related works on p-adic L-functions for $\mathrm{GSp}(4)\times\mathrm{GL}(2)$. For weights in the region (1.1) considered in this paper, a one-variable p-adic L-function (with $\ell=\ell_\ell=\ell_2=-k-1$) is constructed in [Aga07], and a three-variable p-adic L-function with

$$k+\frac{\ell_1+\ell_2+\ell}{2}=\frac{\min\{-\ell_1+\ell_2+\ell,\,\ell_1+\ell_2-\ell\}}{2}-1,$$

is constructed in [LR25], and a one-variable cyclotomic p-adic L-function is constructed in [Liu23]. For weights in a different region where $-\ell_1+\ell_2+\ell\leqslant 1$ (which is the region (F) in the convention of [LR25]), a one-variable cyclotomic p-adic L-function is constructed in [LPSZ21], a three-variable p-adic L-function is constructed in [LZ21], and a four-variable p-adic L-function is constructed in [GR24]. (In all the previous constructions, the interpolations formulas are less complete than the one we prove in Theorem 1.1. They include unramified conditions at p or conditions at ramified places away from p or uncomputed local zeta integrals.) The constructions in [Aga07, Liu23] start with the same automorphic integral as we utilize in this paper, i.e., Furusawa's formula recalled in Section 2.3. The constructions in [LPSZ21, LZ21, LR25, GR24] start with a different automorphic integral involving globally generic (non-holomorphic) automorphic forms on GSp(4) and Eisenstein series on GL(2). The constructions in [LPSZ21, LZ21, LR25] are motivated by studying Euler systems for $\mathrm{GSp}(4)\times\mathrm{GL}(2)$ constructed from Siegel units for modular curves. One major motivation for our construction is studying congruences between Yoshida lifts and other cuspidal automorphic representations on GSp(4).

The main focus of [Liu23] is a complete computation of local zeta integrals at p in Furusawa's formula for p-adic interpolation under the ordinarity assumption. One of the main aspects here is the computation of Archimedean zeta integrals in Furusawa's formula for all holomorphic discrete series of weights satisfying (1.1). Such computation is crucial for studying algebraicity of critical values for $L(s,\Pi\times\pi)$ and have been extensively studied in [Fur93, PS09, Sah09, Pit11, Sah10, Mor14, Mor18]. Before [Mor18], the Archimedean zeta integrals have only been computed for holomorphic discrete series of scalar weights. For general vector weights, the computation achieved in [Mor18] is up to $\mathbb{Q}^\times$. Here, by utilizing the existence of the four-variable p-adic L-functions, we obtain a quantitative result (4.3.1).

The idea of the computation is to compare two four-variable p-adic L-functions: one for Hida families of Yoshida lifts (with uncomputed Archimedean zeta integrals contained in the interpolation formula), and the other one obtained as a product of Rankin–Selberg p-adic L-functions constructed in [Hid88], and deduce the full interpolation property of the former based on the later, and then obtain results on the Archimedean zeta integrals. The main difficulty in this comparison is handling the periods which are in terms of Petersson norms of Yoshida lifts and modular forms. One possible approach is to apply Rallis inner product formula. However, for our purpose, this approach requires precise computation of local zeta integrals at p in Rallis inner product formula for ramified principal series, which is not currently available. (The cases treated in [HN18] only include the unramified and Steinberg cases, and are not exactly the ones needed for p-adic interpolation.) Our strategy is to bypass this computation by making use of the p-adic standard L-functions for Hida families on Sp(4) constructed in [Liu20].

Remark 1.2. — At the end of this introduction, we include a few remarks. The first one is about computing local integrals in Rallis inner product formula. In the computation of local zeta integrals at p for p-adic interpolation, the case of (sufficiently) ramified principal series is typically easier to handle than the unramified and Steinberg cases. However, the Rallis inner product formula seems to follow the opposite pattern – in an ongoing joint work with Ming-Lun Hsieh, we manage to compute the unramified and Steinberg cases for p-adic interpolation furthering the computation in [HN18], but the case of ramified principal series seems more challenging. On the other hand, our computation here can be used to deduce results in that case.

Another remark is about the use of Hida families of Yoshida lifts. Yoshida lifts are a special case of functorial lifts, and it is well known that functorial lifts can be useful for calculating Archimedean zeta integrals. For example, the symmetric square lifts of modular forms are used in [LW21] for computing Archimedean zeta integrals for GL(3). Besides functorial lifts, a more crucial ingredient in our approach for computing Archimedean zeta integrals here is the multi-variable p-adic L-function for Hida families of functorial lifts. Because of this, comparing periods is the technical crux in our approach. In contrast, comparing periods are not needed in loc. cit. because what is essentially computed via functorial lifts there are ratios of Archimedean zeta integrals at $s=0$ and $s=j$ for the same representation and p-adic families of representations are not needed.

The last remark is about the condition (1.1). The Archimedean sections in our construction of the Eisenstein measure in Section 3.2 are chosen from a special type of sections, namely those obtained from applying differential operators to the standard holomorphic sections for Siegel Eisenstein series. The condition (1.1) ensures the existence of such sections with nonvanishing Archimedean zeta integrals. To consider weights in other regions listed in [LR25], one needs to consider more general sections.

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2. Notation and review of Hida theory and Furusawa's formula

2.1. Notation. — We fix an odd prime number p, an isomorphism $\overline{\mathbb{Q}}_p\cong\mathbb{C}$, and a sufficiently large finite extension F of $\mathbb{Q}_p$. Denote by $\mathcal{O}$ the ring of integers of F.

We use v to denote a place of $\mathbb{Q}$. We fix the additive character

$$\psi_{\mathbb{A}_\mathbb{Q}}=\bigotimes_v\psi_v:\mathbb{Q}\backslash\mathbb{A}\longrightarrow\mathbb{C}^\times,\qquad \psi_v(x)=\begin{cases}e^{-2\pi i\{x\}_v} & \text{if } v\neq\infty,\\ e^{2\pi i x} & \text{if } v=\infty,\end{cases}$$

where $\{x\}_v$ is the fractional part of x, i.e., $\{x\}_v\in\mathbb{Q}\cap[0,1)$ with $x-\{x\}_v\in\mathbb{Z}_v$.

Given a positive integer n, denote by $\mathbf{1}_n$ the identity matrix of size n, and define the algebraic group $\mathrm{GSp}(2n)$ over $\mathbb{Z}$ as

$$\mathrm{GSp}(2n,R)=\Big\{g\in\mathrm{GL}(2n,R):{}^{\mathrm{t}}g\left[\begin{smallmatrix}0&\mathbf{1}_n\\-\mathbf{1}_n&0\end{smallmatrix}\right]g=\nu_g\left[\begin{smallmatrix}0&\mathbf{1}_n\\-\mathbf{1}_n&0\end{smallmatrix}\right],\ \nu_g\in R^\times\Big\}\tag{2.1.1}$$

for all $\mathbb{Z}$-algebra R. Given an imaginary quadratic field $\mathcal{K}$, define the algebraic group $\mathrm{GU}(n,n)$ over $\mathbb{Z}$ as

$$\begin{aligned}&\mathrm{GU}(n,n)(R)\\ &\quad=\Big\{g\in\mathrm{GL}(2n,\mathcal{O}_\mathcal{K}\otimes R):{}^{\mathrm{t}}\overline{g}\left[\begin{smallmatrix}0&\mathbf{1}_n\\-\mathbf{1}_n&0\end{smallmatrix}\right]g=\nu_g\left[\begin{smallmatrix}0&\mathbf{1}_n\\-\mathbf{1}_n&0\end{smallmatrix}\right],\ \nu_g\in R^\times\Big\},\end{aligned}\tag{2.1.2}$$

where for $\alpha\in\mathcal{K}$, $\overline{\alpha}$ denotes its image under the nontrivial element in $\mathrm{Gal}(\mathcal{K}/\mathbb{Q})$. Given $g\in\mathrm{GSp}(2n)$ or $\mathrm{GU}(n,n)$, we call the ν_g in (2.1.1)(2.1.2) the similitude of g. In this paper, we will work with $\mathrm{GSp}(4),\mathrm{GSp}(2)=\mathrm{GL}(2),\mathrm{GU}(3,3),\mathrm{GU}(1,1)$.

Fix the following maximal torus of $\mathrm{GSp}(2n)$, $\mathrm{GU}(n,n)$:

$$\begin{aligned}T_{\mathrm{GSp}(2n)}&=\big\{\mathrm{diag}(a_1,\dots,a_n,\nu a_1^{-1},\dots,\nu a_n^{-1})\in\mathrm{GSp}(2n)\big\},\\ T_{\mathrm{GU}(n,n)}&=\big\{\mathrm{diag}(\mathfrak{a}_1,\dots,\mathfrak{a}_n,\nu\overline{\mathfrak{a}}_1^{-1},\dots,\nu\overline{\mathfrak{a}}_n^{-1})\in\mathrm{GU}(n,n)\big\},\end{aligned}$$

and Siegel parabolic subgroups

$$\begin{aligned}Q_{\mathrm{GSp}(2n)}&=\Big\{\left[\begin{smallmatrix}A&B\\0&\nu{}^{\mathrm{t}}A^{-1}\end{smallmatrix}\right]\in\mathrm{GSp}(2n)\Big\},\\ Q_{\mathrm{GU}(n,n)}&=\Big\{\left[\begin{smallmatrix}\mathfrak{A}&\mathfrak{B}\\0&\nu{}^{\mathrm{t}}\overline{\mathfrak{A}}^{-1}\end{smallmatrix}\right]\in\mathrm{GU}(n,n)\Big\},\end{aligned}\tag{2.1.3}$$

where the matrices are written in $n\times n$ blocks. Denote by $M_{\mathrm{GSp}(2n)}\subset Q_{\mathrm{GSp}(2n)}$, $M_{\mathrm{GU}(n,n)}\subset Q_{\mathrm{GU}(n,n)}$ the Levi subgroups. Let

$$\begin{aligned}T^1_{\mathrm{GSp}(2n)}&=T_{\mathrm{GSp}(2n)}\cap\mathrm{Sp}(2n), & T^1_{\mathrm{GU}(n,n)}&=T_{\mathrm{GU}(n,n)}\cap\mathrm{U}(n,n),\\ M^1_{\mathrm{GSp}(2n)}&=M_{\mathrm{GSp}(2n)}\cap\mathrm{Sp}(2n), & M^1_{\mathrm{GU}(n,n)}&=M_{\mathrm{GU}(n,n)}\cap\mathrm{U}(n,n),\end{aligned}$$

where $\mathrm{Sp}(2n)$ (resp. $\mathrm{U}(n,n)$) is the subgroup of $\mathrm{GSp}(2n)$ (resp. $\mathrm{GU}(n,n)$) consisting of elements with similitude 1. We identify $M^1_{\mathrm{GSp}(2n)},M^1_{\mathrm{GU}(n,n)}$ with $\mathrm{GL}(n),\mathrm{GL}(n)_{/\mathcal{K}}$ via

$$A\longmapsto\begin{bmatrix}A&\\&{}^{\mathrm{t}}A^{-1}\end{bmatrix},\qquad\mathfrak{A}\longmapsto\begin{bmatrix}\mathfrak{A}&\\&{}^{\mathrm{t}}\overline{\mathfrak{A}}^{-1}\end{bmatrix}.$$

Denote by

$$U_{M^1_{\mathrm{GSp}(2n)}} \subset M^1_{\mathrm{GU}(n,n)}, \qquad U_{M^1_{\mathrm{GU}(n,n)}} \subset M^1_{\mathrm{GU}(n,n)}$$

the subgroup whose elements are upper triangular with diagonal entries being 1 under the above identification. Let

$$U_{\mathrm{GSp}(2n)} = \left\{ \begin{bmatrix} A & B \\ 0 & \nu {}^{\mathrm{t}}A^{-1} \end{bmatrix} \in Q_{\mathrm{GSp}(2n)} : A \in U_{M^1_{\mathrm{GSp}(2n)}} \right\},$$
$$U_{\mathrm{GU}(n,n)} = \left\{ \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ 0 & \nu {}^{\mathrm{t}}\overline{\mathfrak{A}}^{-1} \end{bmatrix} \in Q_{\mathrm{GU}(n,n)} : \mathfrak{A} \in U_{M^1_{\mathrm{GU}(n,n}} \right\}.$$

For all $\mathbb{Z}$-algebras R, we use $\mathrm{Her}_n(\mathcal{O}_{\mathcal{K}} \otimes R)$ (resp. $\mathrm{Sym}_n(R)$) to denote the set of all $n \times n$ matrices σ with entries in $\mathcal{O}_{\mathcal{K}} \otimes R$ (resp. R) satisfying ${}^{\mathrm{t}}\overline{\sigma} = \sigma$ (resp. ${}^{\mathrm{t}}\sigma = \sigma$).

Given a character $\theta : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ or a character $\Theta : \mathcal{K}_p^\times \to \mathbb{C}^\times$, we let

$$\theta^\circ = \mathbb{1}_{\mathbb{Z}_p^\times} \cdot \theta, \qquad \Theta^\circ = \mathbb{1}_{\mathcal{O}_{\mathcal{K},p}^\times} \cdot \Theta. \tag{2.1.4}$$

Given a Hecke character $\Theta : \mathcal{K}^\times \backslash \mathbb{A}_{\mathcal{K}}^\times \to \mathbb{C}^\times$, we let

$$\Theta_{\mathbb{Q}} = \Theta|_{\mathbb{A}_{\mathbb{Q}}^\times}.$$

We will use triv to denote the trivial character.

2.2. Review of Hida theory.

— We recall some constructions and results in Hida theory for symplectic groups. (We will use it for GSp(4) and GSp(2) = GL(2).) See [Hid02]or [Liu20, §6.1] for details.

For $G = \mathrm{GSp}(2n)$, define the Iwasawa algebra

$$\widetilde{\Lambda}_G = \mathcal{O}[\![T^1_G(\mathbb{Z}_p)]\!], \qquad \Lambda_G = \mathcal{O}[\![T^1_G(1+p\mathbb{Z}_p)]\!] \cong \mathcal{O}[\![T_1, T_2, \ldots, T_n]\!]. \tag{2.2.1}$$

Fix a neat open compact subgroup $K^p_G \subset G(\mathbb{A}^p_{\mathbb{Q},f})$. Let Y_G denote the Shimura variety for G of level $K^p_G G(\mathbb{Z}_p)$ defined over $\mathcal{O}$. Let $\mathscr{T}_{G,\ell,m}$ denote the ℓ-th layer of the Igusa tower over $\mathbb{Z}/p^m\mathbb{Z}$, which is an $M^1_G(\mathbb{Z}/p^\ell)$-étale cover of the ordinary locus $Y_{G,\mathrm{ord}}$, and $\mathscr{T}^{\mathrm{tor}}_{G,\ell,m}$ be a smooth partial toroidal compactification of $\mathscr{T}_{G,\ell,m}$ (with respect to a chosen polyhedral cone decomposition) with boundary C. (See [Hid02, §§2.1, 3.1, 3.2].) Put

$$V_{G,\ell,m} = H^0\left(\mathscr{T}^{\mathrm{tor}}_{G,\ell,m}, \mathcal{O}_{\mathscr{T}_{G,\ell,m}}(-C)\right)^{U_{M^1_G(\mathbb{Z}_p)}},$$

and

$$V_G = \varprojlim_m \varinjlim_\ell V_{G,\ell,m}, \qquad \mathcal{V}_G = \varinjlim_m \varinjlim_\ell V_{G,\ell,m}.$$

The group $T^1_G(\mathbb{Z}_p)$ naturally acts on these spaces, and they are all naturally modules over $\widetilde{\Lambda}_G$ and Λ_G.

For a tuple of integers $\underline{m} : m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n \geqslant 0$ and $m_0 \geqslant 0$, an operator $U^G_{p,\underline{m},m_0}$ (corresponding to $\mathrm{diag}(p^{m_1+m_0}, \ldots, p^{m_n+m_0}, p^{-m_1}, \ldots, p^{-m_n})$) on $V_{G,\ell,m}$ is defined. We call these operators $\mathbb{U}_p$-operators. Put $U^G_p = U^G_{p,\underline{m},0}$ with $\underline{m} = (n, n-1, \ldots, 1)$. As operators on $V_{G,\ell,m}$, the limit

$$e^G_{\mathrm{ord}} = \lim_{r\to\infty} \left(U^G_p\right)^{r!}$$

exists [Hid02, Th. 1.1 (3)][Liu20, §6.1.3], and is called the ordinary projector. Let

$$V_{G,\mathrm{ord}} = e^{G}_{\mathrm{ord}} V_G, \qquad \mathcal{V}_{G,\mathrm{ord}} = e^{G}_{\mathrm{ord}} \mathcal{V}_G, \qquad \mathcal{V}^*_{G,\mathrm{ord}} = \mathrm{Hom}(\mathcal{V}_{G,\mathrm{ord}}, F/\mathcal{O}).$$

The $\widetilde{\Lambda}_G$-module of Hida families of cuspidal p-adic automorphic forms on G of tame level K^p_G is defined to be

$$\mathcal{M}_{G,\mathrm{ord}} = \mathrm{Hom}_{\widetilde{\Lambda}_G}(\mathcal{V}^*_G, \widetilde{\Lambda}_G).$$

For each p-adic weight $\underline{\tau} \in \mathrm{Hom}_{\mathrm{cont}}\big(T^1_G(\mathbb{Z}_p), \overline{\mathbb{Q}}^{\times}_p\big)$, there is a natural map

$$\begin{aligned}(2.2.2)\quad V_{G,\mathrm{ord}}[\underline{\tau}] &\longrightarrow \mathrm{Hom}_{\mathcal{O}}\big(\mathrm{Hom}(V_{G,\mathrm{ord}}, \mathcal{O})/\mathcal{P}_{\underline{\tau}}, \widetilde{\Lambda}_G/\mathcal{P}_{\underline{\tau}}\big)\\ &\longrightarrow \mathrm{Hom}_{\mathcal{O}}\big(\mathrm{Hom}(\mathcal{V}_{G,\mathrm{ord}}, F/\mathcal{O})/\mathcal{P}_{\underline{\tau}}, \widetilde{\Lambda}_G/\mathcal{P}_{\underline{\tau}}\big) \longrightarrow \mathcal{M}_{G,\mathrm{ord}} \otimes_{\widetilde{\Lambda}_G} \widetilde{\Lambda}_G/\mathcal{P}_{\underline{\tau}},\end{aligned}$$

where $\mathcal{P}_{\underline{\tau}}$ is the prime ideal of $\widetilde{\Lambda}_G$ corresponding to $\underline{\tau}$.

Theorem 2.2.1 ([Hid02, Th. 1.1]). — *$\mathcal{M}_{G,\mathrm{ord}}$ is free over Λ_G of finite rank. The map (2.2.2) is an isomorphism. If $\underline{\tau}$ is algebraic and sufficiently regular, then*

$$V_{G,\mathrm{ord}}[\underline{\tau}] = e^{G}_{\mathrm{ord}} H^0\big(Y^{\mathrm{tor}}_G, \omega_{\underline{\tau}}(-C)\big),$$

where $\omega_{\underline{\tau}}$ is the automorphic vector bundle of weight $\underline{\tau}$ over a toroidal compactification of Y_G. (The right hand side can be identified with the space of classical holomorphic automorphic forms on G of weight $\underline{\tau}$.)

2.3. Review of Furusawa's formula. — We quickly recall a modification of Furusawa's formula for L-functions for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$. See [Liu23, §2.1] for details. Take

$$\mathbb{S} = \begin{bmatrix} \mathbb{a} & \mathbb{b}/2 \\ \mathbb{b}/2 & \mathbb{c} \end{bmatrix} \in \mathrm{Sym}_2(\mathbb{Q})_{>0},$$

and let $\mathcal{K} = \mathbb{Q}(\sqrt{-\det \mathbb{S}})$, and $\eta_{\mathcal{K}/\mathbb{Q}} : \mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q}} \to \mathbb{C}^\times$ be the quadratic character corresponding to $\mathcal{K}/\mathbb{Q}$. Let

$$\alpha_{\mathbb{S}} = \frac{\mathbb{b} + \sqrt{\mathbb{b}^2 - 4\mathbb{a}\mathbb{c}}}{2\mathbb{c}}, \qquad \imath_{\mathbb{S}}(\mathfrak{z}) = \begin{bmatrix} \alpha_{\mathbb{S}} & 1 \\ \overline{\alpha}_{\mathbb{S}} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathfrak{z} & \\ & \bar{\mathfrak{z}} \end{bmatrix} \begin{bmatrix} \alpha_{\mathbb{S}} & 1 \\ \overline{\alpha}_{\mathbb{S}} & 1 \end{bmatrix}$$

for $\mathfrak{z} \in \mathcal{K} \otimes_{\mathbb{Q}} R$ with R any $\mathbb{Q}$-algebra.

Given a Hecke character $\Xi : \mathcal{K}^\times\backslash\mathbb{A}^\times_{\mathcal{K}} \to \mathbb{C}^\times$, denote by $I_v(s, \chi, \Xi)$ the degenerate principal series on $\mathrm{GU}(3,3)(\mathbb{Q}_v)$ consisting of smooth functions $\mathtt{f}_v(s, \chi, \Xi) : \mathrm{GU}(3,3)(\mathbb{Q}_v) \to \mathbb{C}$ such that

$$\mathtt{f}_v(s, \chi, \Xi)\left(\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ 0 & \mathfrak{D} \end{smallmatrix}\right] g\right) = \Xi_v(\det \mathfrak{A})\chi_v(\det \mathfrak{A}\mathfrak{D}^{-1})|\det \mathfrak{A}\mathfrak{D}^{-1}|^{s+3/2}_v\, \mathtt{f}_v(s, \chi, \Xi)(g)$$

for all $g \in \mathrm{GU}(3,3)(\mathbb{Q}_v)$ and $\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ 0 & \mathfrak{D} \end{smallmatrix}\right] \in Q_{\mathrm{GU}(3,3)}(\mathbb{Q}_v)$, the Siegel parabolic group defined in (2.1.3). The Siegel Eisenstein series associated to a section $\mathtt{f}(s, \chi, \Xi) \in I(s, \chi, \Xi) = \bigotimes'_v I_v(s, \chi, \Xi)$ is defined as

$$E^{\mathrm{Sieg}}(g; \mathtt{f}(s, \chi, \Xi)) = \sum_{\gamma \in Q_{\mathrm{GU}(3,3)}(\mathbb{Q})\backslash \mathrm{GU}(3,3)(\mathbb{Q})} \mathtt{f}(s, \chi, \Xi)(\gamma g).$$

Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A}_{\mathbb{Q}})$. By taking a Hecke character $\Upsilon:\mathcal{K}^\times\backslash\mathbb{A}_{\mathcal{K}}^\times\to\mathbb{C}^\times$ with $\Upsilon_{\mathbb{Q}}$ equal to the central character of π, for every $f\in\pi$, we can extend it to an automorphic form f^Υ on $\mathrm{GU}(1,1)$ by

$$f^\Upsilon(\mathfrak{a}g)=\Upsilon(\mathfrak{a})f(g),\qquad \mathfrak{a}\in\mathbb{A}_{\mathcal{K}}^\times,\ g\in\mathrm{GL}(2,\mathbb{A}_{\mathbb{Q}}).\tag{2.3.1}$$

Then $\pi^\Upsilon=\{f^\Upsilon : f\in\pi\}$ is an irreducible cuspidal automorphic representation of $\mathrm{GU}(1,1)$. Denote the Whittaker period of $f\in\pi$ with respect to $\psi_{\mathbb{A}_{\mathbb{Q}},\mathbb{c}}$ (defined as $\psi_{\mathbb{A}_{\mathbb{Q}},\mathbb{c}}(x)=\psi_{\mathbb{A}_{\mathbb{Q}}}(\mathbb{c}x)$) by $W_{\mathbb{c}}(f)$, and define the function $\mathcal{W}_{\mathbb{c}}(f)$ on $\mathrm{GL}(2,\mathbb{A}_{\mathbb{Q}})$ as

$$\mathcal{W}_{\mathbb{c}}(f)(g)=W_{\mathbb{c}}(g\cdot f).$$

Let Π be an irreducible cuspidal automorphic representation of $\mathrm{GSp}(4,\mathbb{A}_{\mathbb{Q}})$, and $\Lambda:\mathcal{K}^\times\backslash\mathbb{A}_{\mathcal{K}}^\times\to\mathbb{C}^\times$ be a Hecke character such that $\Lambda_{\mathbb{Q}}$ equals the central character of Π. Then for $\varphi\in\Pi$, one can define its global Bessel period $B_{\mathbb{S},\Lambda}(\varphi)$ with respect to $\mathbb{S},\Lambda$ (and ψ) as recalled in [Liu23, §2.2.1]. We also define the function $\mathcal{B}_{\mathbb{S},\Lambda}(\varphi)$ on $\mathrm{GSp}(4,\mathbb{A}_{\mathbb{Q}})$ as

$$\mathcal{B}_{\mathbb{S},\Lambda}(\varphi)(g)=B_{\mathbb{S},\Lambda}(g\cdot\varphi).$$

Define $\mathrm{GSp}(4)\times_{\mathrm{GL}(1)}\mathrm{GU}(1,1)$ as $\{(g,h)\in\mathrm{GSp}(4)\times\mathrm{GU}(1,1):\nu_g=\nu_h\}$, and similarly define $\mathrm{GU}(2,2)\times_{\mathrm{GL}(1)}\mathrm{GU}(1,1)$. Denote by $\imath:\mathrm{GU}(2,2)\times_{\mathrm{GL}(1)}\mathrm{GU}(1,1)\hookrightarrow\mathrm{GU}(3,3)$ the embedding

$$\left(\begin{bmatrix}A&B\\C&D\end{bmatrix},\begin{bmatrix}\mathfrak{a}&\mathfrak{b}\\\mathfrak{c}&\mathfrak{d}\end{bmatrix}\right)\longmapsto\begin{bmatrix}A&&B&\\&\mathfrak{a}&&\mathfrak{b}\\C&&D&\\&\mathfrak{c}&&\mathfrak{d}\end{bmatrix}.$$

Let $R'_{\mathbb{S}}\subset\mathrm{GSp}(4)\times_{\mathrm{GL}(1)}\mathrm{GU}(1,1)$ be the subgroup

$$\left\{\left(\begin{bmatrix}\imath_{\mathbb{S}}(\mathfrak{z})&\\&{}^t\imath_{\mathbb{S}}(\bar{\mathfrak{z}})\end{bmatrix}\begin{bmatrix}\mathbf{1}_2&X\\&\mathbf{1}_2\end{bmatrix},\ \mathfrak{z}\cdot\mathbf{1}_2\right):\mathfrak{z}\in\mathrm{Res}_{\mathcal{K}/\mathbb{Q}}\mathrm{GL}(1),\ X\in\mathrm{Sym}_2\right\}.$$

Put

$$\eta_{\mathbb{S}}=\begin{bmatrix}1&&&\\\alpha_{\mathbb{S}}&1&&\\&&1&-\overline{\alpha}_{\mathbb{S}}\\&&&1\end{bmatrix}\in\mathrm{GU}(2,2)(\mathbb{Q}),\qquad \mathcal{S}=\begin{bmatrix}1&&&&&\\&1&&&&\\&&1&&&\\&&&1&&\\&&1&&1&\\&1&&&&1\end{bmatrix}\in\mathrm{GU}(3,3)(\mathbb{Q}).$$

The following formula is proved in [Liu23, Th. 2.1.1] by combining Furusawa's formula [Fur93] and Garrett's generalization of the doubling method [Gar89].

Theorem 2.3.1. — *Assume that $\Xi\Lambda^c\Upsilon^c=\mathrm{triv}$ and S is a subset of places of $\mathbb{Q}$ containing ∞ such that at all $v\notin S$,*

– $\pi,\Pi,\Upsilon,\Xi,\chi,\mathfrak{f}(s,\chi,\Xi)$ *are all unramified and φ,f are invariant under the action by* $\mathrm{GSp}(4,\mathbb{Z}_v),\mathrm{GU}(1,1)(\mathbb{Z}_v)$,

– $\mathbb{S}=\left[\begin{smallmatrix}\mathbb{a}&\mathbb{b}/2\\\mathbb{b}/2&\mathbb{c}\end{smallmatrix}\right]$ *belongs to $M_2(\mathbb{Z}_v)$ with $\mathbb{c}\in\mathbb{Z}_v^\times$ and* $\mathbb{b}^2-4\mathbb{a}\mathbb{c}=\mathrm{disc}(\mathcal{K}_v/\mathbb{Q}_v)$.

Then

$$\int_{[\mathrm{GSp}(4)\times_{\mathrm{GL}(1)}\mathrm{GU}(1,1)]} E^{\mathrm{Sieg}}\big(\imath(g,h); f(s,\chi,\Xi)\big)\cdot\varphi(g)\cdot f^{\Upsilon}(h)\,\Xi^{-1}(\det h)\,dh\,dg$$
$$= W_{\mathfrak{c}}(f)\cdot B_{\mathbb{S},\Lambda}(\varphi)\cdot d_3^S\Big(s+\frac{1}{2},\Xi(\chi\circ\mathrm{Nm})\Big)^{-1}L^S\Big(s+\frac{1}{2},\widetilde{\Pi}\times\widetilde{\pi}\times\chi\Big)$$
$$\times\prod_{v\in S} Z_v\Big(f_v(s,\chi,\Xi),\mathcal{B}^{\Pi_v}_{\mathbb{S},\Lambda_v}(\varphi_v),\mathcal{W}^{\pi_v,\Upsilon_v}_{\mathfrak{c}}(f_v)\Big),$$

with

$$d_3^S\big(s,\Xi(\chi\circ\mathrm{Nm})\big)=\prod_{j=1}^{3}L_S\big(2s+j,\Xi_{\mathbb{Q}}\chi^2\eta^{n-j}_{\mathcal{K}/\mathbb{Q}}\big),$$

and

$$Z_v\Big(f_v(s,\chi,\Xi),,\mathcal{B}^{\Pi_v}_{\mathbb{S},\Lambda_v}(\varphi_v),\mathcal{W}^{\pi_v,\Upsilon_v}_{\mathfrak{c}}(f_v)\Big)=\mathcal{B}^{\Pi_v}_{\mathbb{S},\Lambda_v}(\varphi_v)(\mathbf{1}_4)^{-1}\mathcal{W}^{\pi_v,\Upsilon_v}_{\mathfrak{c}}(f_v)(\mathbf{1}_2)^{-1}$$
$$\times\int_{\big(R'_{\mathbb{S}}\backslash\,\mathrm{GSp}(4)\times_{\mathrm{GL}(1)}\mathrm{GU}(1,1)\big)(\mathbb{Q}_v)} f_v(s,\chi,\Xi)\,\big(\mathcal{S}^{-1}\imath(\eta_{\mathbb{S}}\,g,h)\big)$$
$$\times\,\mathcal{B}^{\Pi_v}_{\mathbb{S},\Lambda_v}(\varphi_v)(g)\mathcal{W}^{\pi_v,\Upsilon_v}_{\mathfrak{c}}(f_v)\,\big(\big[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\big]h\big)\,\Xi_v^{-1}(\det h)\,dhdg,$$

where Nm *denotes the norm map from* $\mathbb{A}^\times_{\mathcal{K}}$ *to* $\mathbb{A}^\times_{\mathbb{Q}}$, $\mathcal{B}^{\Pi_v}_{\mathbb{S},\Lambda_v}(\varphi_v)$ *is the element corresponding to* φ *in the local Bessel model of* Π_v, $\mathcal{W}^{\pi_v,\Upsilon_v}_{\mathfrak{c}}(f_v)$ *is the extension to* $\mathrm{GU}(1,1)(\mathbb{Q}_v)$ *via* Υ_v *of the element corresponding to* f *in the local Whittaker model of* π_v.

In [Liu23], a one-variable cyclotomic p-adic L-function for $\Pi\times\pi$ is constructed by using the above integral and interpolating Siegel Eisenstein series when s,χ vary. (Such a modification of Furusawa's integral is also used in [Mor14, Mor18].) In the next section, we let π,Π also vary in Hida families, and construct a four-variable Siegel Eisenstein family.

3. Four-variable p-adic L-function for $\mathrm{GSp}(4)\times\mathrm{GL}(2)$

3.1. The setup. — Let S be a finite set of places of $\mathbb{Q}$ containing p,∞ and U^p be an open subgroup of $\widehat{\mathbb{Z}}^{p,\times}$ containing $\prod_{v\notin S}\mathbb{Z}_v^\times$. We fix tame level groups

$$K^p_{\mathrm{GL}(2)}=\prod_{v\notin S}\mathrm{GL}(2,\mathbb{Z}_v)\times\prod_{v\in S-\{\infty\}}K_{\mathrm{GL}(2),v},\quad\text{with }\begin{bmatrix}1&\mathbb{Z}_v\\&1\end{bmatrix}\subset K_{\mathrm{GL}(2),v},$$
$$K^p_{\mathrm{GSp}(4)}=\prod_{v\notin S}\mathrm{GSp}(4,\mathbb{Z}_v)\times\prod_{v\in S-\{\infty\}}K_{\mathrm{GSp}(4),v},\quad\text{with }\begin{bmatrix}\mathbf{1}_2&\mathrm{Sym}_2(\mathbb{Z}_v)\\&\mathbf{1}_2\end{bmatrix}\subset K_{\mathrm{GSp}(4),v}.$$

Let $\mathbb{T}_{\mathrm{GL}(2),\mathrm{ord}}$ (resp. $\mathbb{T}_{\mathrm{GSp}(4),\mathrm{ord}}$) be the Hecke algebras acting on the $\widetilde{\Lambda}_{\mathrm{GL}(2)}$-module $\mathcal{M}_{\mathrm{GL}(2),\mathrm{ord}}$ of Hida families of tame level $K^p_{\mathrm{GL}(2)}$ (resp. $\widetilde{\Lambda}_{\mathrm{GSp}(4)}$-module $\mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}}$ of Hida families of tame level $K^p_{\mathrm{GSp}(4)}$) consisting of all the spherical Hecke operators away from S and the $\mathbb{U}_p$-operators at p. We have the natural weight projection map

$$\text{(3.1.1)}\qquad \mathrm{Spec}(\mathbb{T}_{\mathrm{GL}(2),\mathrm{ord}})\times\mathrm{Spec}(\mathbb{T}_{\mathrm{GSp}(4),\mathrm{ord}})\longrightarrow\mathrm{Spec}(\widetilde{\Lambda}_{\mathrm{GL}(2)})\times\mathrm{Spec}(\widetilde{\Lambda}_{\mathrm{GSp}(4)}).$$

Assume that we are given the following data:

– a geometrically irreducible component $\mathscr{C}_1\subset\mathrm{Spec}(\mathbb{T}_{\mathrm{GL}(2),\mathrm{ord}})$ with central character

$$\omega_{\mathscr{C}_1}:\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q},f}^\times\longrightarrow\Lambda_{\mathrm{GL}(2)}^\times,$$

– a geometrically irreducible component $\mathscr{C}_2\subset\mathrm{Spec}(\mathbb{T}_{\mathrm{GSp}(4),\mathrm{ord}})$ with central character

$$\omega_{\mathscr{C}_2}:\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q},f}^\times\longrightarrow\Lambda_{\mathrm{GSp}(4)}^\times,$$

plus the auxiliary data:

– an imaginary quadratic field $\mathcal{K}$ and a positive definite symmetric form

$$\mathbb{S}=\begin{bmatrix}\mathbb{a}&\mathbb{b}/2\\ \mathbb{b}/2&\mathbb{c}\end{bmatrix}\in\mathrm{Sym}_2(\mathbb{Q})_{>0}$$

with $\mathcal{K}=\mathbb{Q}(\sqrt{-\det\mathbb{S}})$ such that for all $v\notin S$,

$$\mathbb{S}\in\mathrm{Sym}_2(\mathbb{Z}_v),\qquad \mathbb{c}\in\mathbb{Z}_v^\times,\qquad 4\det\mathbb{S}=\mathrm{disc}(\mathcal{K}_v/\mathbb{Q}_v)$$

– a continuous character $\boldsymbol{\Upsilon}:\mathcal{K}^\times\backslash\mathbb{A}_{\mathcal{K},f}^\times\to\Lambda_{\mathrm{GL}(2)}^\times$ extending $\omega_{\mathscr{C}_1}$,

– a continuous character $\boldsymbol{\Lambda}:\mathcal{K}^\times\backslash\mathbb{A}_{\mathcal{K},f}^\times\to\Lambda_{\mathrm{GSp}(4)}^\times$ extending $\omega_{\mathscr{C}_2}$.

3.1.1. *Notation for some p-adic characters.* — We identify $T^1_{\mathrm{GL}(2)}(\mathbb{Z}_p),T^1_{\mathrm{GSp}(4)}(\mathbb{Z}_p)$ with $\mathbb{Z}_p^\times,\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times$ via

$$a\longmapsto\mathrm{diag}(a,a^{-1}),\qquad (a_1,a_2)\longmapsto\mathrm{diag}(a_1,a_2,a_1^{-1},a_2^{-1}).$$

We denote by τ (resp. (τ_1,τ_2)) a continuous character

$$T^1_{\mathrm{GL}(2)}(\mathbb{Z}_p)\longrightarrow\overline{\mathbb{Q}}_p^\times\qquad(\text{resp. } T^1_{\mathrm{GSp}(4)}(\mathbb{Z}_p)\longrightarrow\overline{\mathbb{Q}}_p^\times).$$

When the characters are arithmetic, i.e.,

$$\tau(x)=x^\ell\xi(x),\qquad\tau_i(x)=x^{\ell_i}\xi_i(x),\quad i=1,2$$

for some integers ℓ,ℓ_1,ℓ_2 and finite order characters ξ,ξ_1,ξ_2 of $\mathbb{Z}_p^\times$, we write

$$\tau=(\ell,\xi),\qquad(\tau_1,\tau_2)=(\ell_1,\ell_2,\xi_1,\xi_2),$$

and call ℓ (resp. (ℓ_1,ℓ_2)) the algebraic part of τ (resp. (τ_1,τ_2)), and ξ (resp. (ξ_1,ξ_2)) the finite order part. We view τ,τ_1,τ_2 also as a $\overline{\mathbb{Q}}_p$-valued character of $\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q},f}^\times/\widehat{\mathbb{Z}}^{p,\times}$ via the isomorphism $\mathbb{Z}_p^\times\xrightarrow{\sim}\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q},f}^\times/\widehat{\mathbb{Z}}^{p,\times}$ (induced by the embedding $\mathbb{Z}_p^\times\hookrightarrow\mathbb{A}_{\mathbb{A},f}^\times$).

We denote by κ a continuous character $\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q},f}^\times/U^p\to\overline{\mathbb{Q}}_p^\times$, and when it is arithmetic, we write

$$\kappa=(k,\chi)$$

with k the algebraic part and χ the finite order part.

In this paper, we call an arithmetic tuple $(\kappa,\tau,\tau_1,\tau_2)$ classical if their algebraic parts satisfy

$$-\min\{\ell_1+\ell_2,\ell+\ell_2\}+2\leqslant k\leqslant-\max\{\ell_1,\ell\}-1. \tag{3.1.2}$$

This condition corresponds to

$$-\frac{\min\{-\ell_1+\ell_2+\ell,\,\ell_1+\ell_2-\ell\}}{2}+2\leqslant k+\frac{\ell_1+\ell_2+\ell}{2}\leqslant\frac{\min\{-\ell_1+\ell_2+\ell,\,\ell_1+\ell_2-\ell\}}{2}-1$$

equivalent to $s=k+(\ell_1+\ell_2+\ell)/2$ being a critical point for the degree-8 L-function $L(s,\Pi\times\pi)$ with Π_∞ (resp. π_∞) isomorphic to the holomorphic discrete series of weight (ℓ_1,ℓ_2) (resp. ℓ).

3.1.2. *Convention on nebentypus at p and central characters.* — Given arithmetic $\tau=(\ell,\xi)$, $(\tau_1,\tau_2)=(\ell_1,\ell_2,\xi_1,\xi_2)$, we use the convention that the classical automorphic forms on $\mathrm{GL}(2),\mathrm{GSp}(4)$ contained in $V_{\mathrm{GL}(2)}[\tau],V_{\mathrm{GSp}(4)}[\tau_1,\tau_2]$ have

– weight $\ell,(\ell_1,\ell_2)$,

– nebentypus such that the right translation of $B_{\mathrm{GL}(2)}(\mathbb{Z}_p),B_{\mathrm{GSp}(4)}(\mathbb{Z}_p)$ acts by the character

$$\begin{bmatrix}a&*\\&d\end{bmatrix}\longmapsto\xi(a),\qquad\begin{bmatrix}a_1&*&*&*\\&a_2&*&*\\&&a_1^{-1}\nu&\\&&*&a_2^{-1}\nu\end{bmatrix}\longmapsto\xi_1(a_1)\xi_2(a_2),\tag{3.1.3}$$

– central character equal to the product of a finite order character unramified away from p and the classical character corresponding to

$$\tau:\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q},f}^\times/\widehat{\mathbb{Z}}^{p,\times}\longrightarrow\overline{\mathbb{Q}}_p^\times,\qquad\tau_1\tau_2:\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q},f}^\times/\widehat{\mathbb{Z}}^{p,\times}\longrightarrow\overline{\mathbb{Q}}_p^\times.$$

(Note that with this convention, the central characters of classical cuspidal automorphic forms in $V_{\mathrm{GL}(2)}[\tau],V_{\mathrm{GSp}(4)}[\tau_1,\tau_2]$ are not necessarily unitary.)

3.2. The Eisenstein measure

3.2.1. *The Siegel Eisenstein series and its Fourier coefficients.* — For a classical tuple $(\kappa,\tau,\tau_1,\tau_2)$ with κ fixed by U^p, put

$$E^{\mathrm{Sieg}}_{\kappa,\tau,\tau_1,\tau_2}(g)=|\nu_g|^{\frac{3}{2}(\ell_1+\ell_2+\ell)}\cdot\frac{\prod_{j=0}^{2}\Gamma\left(s+\left|k+\frac{\ell+\ell_1+\ell_2}{2}-\frac{1}{2}\right|+3-j\right)}{2^{-6}(-2i)^{6\left(\left|k+\frac{\ell+\ell_1+\ell_2}{2}-\frac{1}{2}\right|+\frac{3}{2}\right)}\pi^{3\left(s+\left|k+\frac{\ell+\ell_1+\ell_2}{2}-\frac{1}{2}\right|+2\right)}}$$
$$\times\,d_3^S\big(s,\Lambda_0\Upsilon_0(\chi\circ\mathrm{Nm})\big)\cdot E^{\mathrm{Sieg}}\left(g;\mathtt{f}_{\ell,\,\ell_1,\ell_2,\xi,\,\xi_1,\xi_2}(s,\chi,\Lambda_0\Upsilon_0)\right)\Big|_{s=k+\frac{\ell_1+\ell_2+\ell}{2}-\frac{1}{2}},$$

where $\Lambda_0=\Lambda|\cdot|^{-(\ell_1+\ell_2)/2}$, $\Upsilon_0=\Upsilon|\cdot|^{-\ell/2}$ with Λ,Υ the classical characters corresponding to the specializations at τ_1,τ_2,τ of $\boldsymbol{\Lambda},\boldsymbol{\Upsilon}$ fixed in Section 3.1, and the section

$$\mathtt{f}_{\ell,\,\ell_1,\ell_2,\xi,\,\xi_1,\xi_2}(s,\chi,\Lambda_0\Upsilon_0)\in\bigotimes_v I_v(s,\chi,\Lambda_0\Upsilon_0)$$

is chosen as in [Liu23, §3.3]. (For the place p, the $\eta^\circ_{\Pi_1,p},\eta^\circ_{\Pi_2,p},\eta^\circ_{\Pi_3,p}$ in loc. cit. correspond to $\xi_2^{-1},\xi_1^{-1},\xi_1^{-1}\xi_2^{-1}$ here, and $\eta^\circ_{\pi_p,1},\eta^\circ_{\pi_p,2}$ in loc. cit. correspond to ξ^{-1}, triv here. For the Archimedean place, $k+\epsilon/2$ in loc. cit. correspond to $k+(\ell_1+\ell_2+\ell)/2$ here.)

Given $\boldsymbol{\beta}\in\mathrm{Her}_3(\mathcal{K})$ and a nearly holomorphic automorphic form F on $\mathrm{GU}(3,3)$, we define the $\boldsymbol{\beta}$-th polynomial Fourier coefficient of F at $g_f\in\mathrm{GU}(3,3)(\mathbb{A}_{\mathbb{Q},f})$ as the polynomial $F_{\boldsymbol\beta}(g) \in \mathbb{C}[Y]$, where $\mathbb{C}[Y]$ denotes the ring of polynomials in entries of $Y = (Y_{ij})_{1\leqslant i,\ell\leqslant 3}$, such that for $z \in \bigl\{z \in M_{3,3}(\mathbb{C}) \mid i({}^{\mathrm t}\overline{z} - z) > 0\bigr\}$,

$$F_{\boldsymbol\beta}(g_f)\Bigl(Y = \Bigl(\frac{z-{}^{\mathrm t}\overline{z}}{2i}\Bigr)^{-1}\Bigr)\cdot e^{2\pi i \operatorname{Tr} z\boldsymbol\beta} = \int_{\operatorname{Her}_3(\mathcal{K})\backslash \operatorname{Her}_3(\mathbb{A}_{\mathcal{K}})} F\Biggl(\begin{bmatrix}\mathbf{1}_3 & \varsigma\\ 0 & \mathbf{1}_3\end{bmatrix} g_f \begin{bmatrix}\bigl(\frac{z-{}^{\mathrm t}\overline{z}}{2i}\bigr)^{1/2} & \frac{z+{}^{\mathrm t}\overline{z}}{2}\bigl(\frac{z-{}^{\mathrm t}\overline{z}}{2i}\bigr)^{-1/2}\\ 0 & \bigl(\frac{z-{}^{\mathrm t}\overline{z}}{2i}\bigr)^{-1/2}\end{bmatrix}_\infty\Biggr)\psi_{\mathbb{A}_{\mathbb{Q}}}\left(-\operatorname{Tr}\boldsymbol\beta\,\mathfrak{x}\right)d\mathfrak{x}.$$

(The existence of such a polynomial $F_{\boldsymbol\beta}(g)$ is implied by the definition of nearly holomorphic modular forms. See [Shi00, §13.11] or [Liu23, §3.2] for the definition of nearly holomorphic forms and [Liu19, §2.4] for their interpretations as global sections of automorphic sheaves over Shimura varieties.) Similarly, one defines polynomial Fourier coefficients of nearly holomorphic forms on $\mathrm{GU}(1,1), \mathrm{GL}(2)$ (resp. $\mathrm{GSp}(4)$) indexed by $\mathbb{Q} = \operatorname{Her}_1(\mathcal{K})$ (resp. $\operatorname{Sym}_2(\mathbb{Q})$).

By our choice of the Archimedean component of $\mathtt{f}_{\ell,\ell_1,\ell_2,\xi,\xi_1,\xi_2}(s,\chi,\Lambda_0\Upsilon_0)$, the Eisenstein series $E^{\mathrm{Sieg}}_{\kappa,\tau,\tau_1,\tau_2}$ is a nearly holomorphic automorphic form on $\mathrm{GU}(3,3)$. We have the following proposition on its polynomial Fourier coefficients.

Proposition 3.2.1. — *For $\mathfrak{A} \in \mathrm{GL}(2,\mathbb{A}^p_{\mathcal{K},f})$, $\nu \in \mathbb{A}^{\times,p}_f$ and $\boldsymbol\beta \in \operatorname{Her}_3(\mathcal{K})$, we have $\bigl(E^{\mathrm{Sieg}}_{\kappa,\tau,\tau_1,\tau_2}\bigr)_{\boldsymbol\beta} = 0$ unless $\boldsymbol\beta > 0$, and for $\boldsymbol\beta > 0$,*

$$\bigl(E^{\mathrm{Sieg}}_{\kappa,\tau,\tau_1,\tau_2}\bigr)_{\boldsymbol\beta}\Bigl(\begin{bmatrix}\mathfrak{A} & \\ & \nu{}^{\mathrm t}\overline{\mathfrak{A}}^{-1}\end{bmatrix}^p_f\Bigr)(Y=0) = A(\boldsymbol\beta;\kappa,\tau,\tau_1,\tau_2)\Bigl(\begin{bmatrix}\mathfrak{A} & \\ & \nu{}^{\mathrm t}\overline{\mathfrak{A}}^{-1}\end{bmatrix}^p_f\Bigr)\cdot C_{k,\ell,\ell_1,\ell_2}(\boldsymbol\beta)$$

with

$$(3.2.1)\quad A(\boldsymbol\beta;\kappa,\tau,\tau_1,\tau_2)\Bigl(\begin{bmatrix}\mathfrak{A} & \\ & \nu{}^{\mathrm t}\overline{\mathfrak{A}}^{-1}\end{bmatrix}^p_f\Bigr) = \Lambda\Upsilon(\det(\nu{}^{\mathrm t}\overline{\mathfrak{A}}^{-1})\cdot(\chi|\cdot|^k)(\det(\nu\mathfrak{A}^{-1}{}^{\mathrm t}\overline{\mathfrak{A}}^{-1}))$$
$$\times\prod_{v\nmid Np\infty} h_{v,\nu^{-1}{}^{\mathrm t}\overline{\mathfrak{A}}\boldsymbol\beta\mathfrak{A}}\left(\Lambda_{\mathbb{Q},v}\Upsilon_{\mathbb{Q},v}\chi_v^2(\varpi_v)|\varpi_v|^{2k+2}\right)\prod_{v\mid N}\mathcal{F}\Phi^{\mathrm{vol}}_{v,\mathrm{val}_v(N)}\bigl(\nu_v^{-1}{}^{\mathrm t}\overline{\mathfrak{A}}_v\boldsymbol\beta\mathfrak{A}_v\bigr)$$
$$\times\Lambda^{\circ-1}_{\overline{\mathfrak{p}}}\xi_1(\varrho_{\mathfrak{p}}(\beta_{13}))\varrho_{\mathfrak{p}}(\beta_{13})^{-r_{\Lambda,2}+\ell_1}\cdot\Lambda^{\circ-1}_{\mathfrak{p}}\xi_1(\varrho_{\overline{\mathfrak{p}}}(\beta_{13}))\varrho_{\overline{\mathfrak{p}}}(\beta_{13})^{-r_{\Lambda,1}+\ell_1}$$
$$\times\xi_2\xi\chi_p^\circ\Bigl(\frac{\overline{\beta}_{13}\beta_{23}-\beta_{13}\overline{\beta}_{23}}{\alpha_{\mathbb{S}}-\overline{\alpha}_{\mathbb{S}}}\Bigr)\Bigl(\frac{\overline{\beta}_{13}\beta_{23}-\beta_{13}\overline{\beta}_{23}}{\alpha_{\mathbb{S}}-\overline{\alpha}_{\mathbb{S}}}\Bigr)^{\ell_2+\ell+k-2}$$
$$\times\xi_1\xi_2\chi_p^\circ\Bigl(\frac{(\alpha_{\mathbb{S}}-\overline{\alpha}_{\mathbb{S}})(\beta_{12}-\overline{\beta}_{12})}{2}\Bigr)\Bigl(\frac{(\alpha_{\mathbb{S}}-\overline{\alpha}_{\mathbb{S}})(\beta_{12}-\overline{\beta}_{12})}{2}\Bigr)^{\ell_1+\ell_2+k-2},$$

and

$$(3.2.2)\quad C_{k,\ell,\ell_1,\ell_2}(\boldsymbol\beta) = \begin{cases}\Bigl(\Bigl(\frac{\beta_{12}-\overline{\beta}_{12}}{2}\det\begin{bmatrix}\overline{\beta}_{13} & \overline{\beta}_{23}\\ \beta_{13} & \beta_{23}\end{bmatrix}\Bigr)^{-1}\det\boldsymbol\beta\Bigr)^{2k+\ell_1+\ell_2+\ell-1} & \text{if } k+(\ell_1+\ell_2+\ell)/2 \geqslant 1/2,\\ 1 & \text{if } k+(\ell_1+\ell_2+\ell)/2 \leqslant 1/2.\end{cases}$$

Here, Λ,Υ are the classical characters associated to the specialization of $\boldsymbol\Lambda$ at (τ_1,τ_2), $\boldsymbol\Upsilon$ at τ, and $(r_{\Lambda,1},r_{\Lambda,2})$ is the ∞-type of Λ, $\varrho_p,\varrho_{\overline{\mathfrak{p}}}$ are embedding of $\mathcal{K}$ into $\overline{\mathbb{Q}}_p$ inducing $\mathfrak{p},\overline{\mathfrak{p}}$. (See (2.1.4) *for the notation superscript $^\circ$ on characters.) $\mathcal{F}\Phi^{\mathrm{vol}}_{v,\mathrm{val}_v(N)}$ is the Fourier transform of the Schwartz function defined in* [Liu23, §3.3.3]*, which does not depend on $\kappa,\tau,\tau_1,\tau_2$.*

Proof. — This proposition is the same as [Liu23, Prop. 3.5.1], aside from a notational difference. The characters $\eta^\circ_{\Pi_1,p},\eta^\circ_{\Pi_2,p},\eta^\circ_{\Pi_3,p}$ in loc. cit. correspond to our $\xi_2^{-1},\xi_1^{-1},\xi_1^{-1}\xi_2^{-1}$ here, $\eta^\circ_{\pi_p,1},\eta^\circ_{\pi_p,2}$ in loc. cit. correspond to our ξ^{-1}, triv here, and $k+\epsilon/2,\Lambda,\Xi$ in loc. cit. correspond to our $k+(\ell+\ell_1+\ell_2)/2$, $\Lambda_0^{-c},\Lambda_0\Upsilon_0$ here. □

3.2.2. *The p-adic measure interpolating restrictions of Siegel Eisenstein series*. — Given a compact and totally disconnected topological space Y and a p-adically complete $\mathbb{Z}_p$-module M, we denote by $\mathcal{M}eas(Y,M)$ the space of all continuous $\mathbb{Z}_p$-linear maps from $\mathscr{C}(Y,\mathbb{Z}_p)$ to M, where $\mathscr{C}(Y,\mathbb{Z}_p)$ is the space of all continuous $\mathbb{Z}_p$-valued functions on Y equipped with the topology of uniform convergence.

Theorem 3.2.2. — *There exists a p-adic measure*

$$\mu_{\mathcal{E}}\in \mathcal{M}eas\big(\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/U^p,\mathcal{M}_{\mathrm{GL}(2),\mathrm{ord}}\widehat{\otimes}_{\mathcal{O}}\mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}}\big)$$

such that for all classical $(\kappa,\tau,\tau_1,\tau_2)=((k,\chi),(\ell,\xi),(\ell_1,\ell_2,\xi_1,\xi_2))$ *with* k,ℓ,ℓ_1,ℓ_2 *satisfying* (3.1.2), *denoting by* $\mathrm{sp}_{\tau,(\tau_1,\tau_2)}$ *the specialization map at* $\tau,(\tau_1,\tau_2)$, *we have*

$$\mathrm{sp}_{\tau,(\tau_1,\tau_2)}(\mu_{\mathcal{E}}(\kappa))=e^{\mathrm{GL}(2)}_{\mathrm{ord}}e^{\mathrm{GSp}(4)}_{\mathrm{ord}}\,\mathrm{Proj}_{K^p_{\mathrm{GL}(2)},K^p_{\mathrm{GSp}(4)}}\big(E^{\mathrm{Sieg}}_{\kappa,\tau,\tau_1,\tau_2}|_{\mathrm{GL}(2)\times\mathrm{GSp}(4)}\big)\\ \times\mathrm{sp}_{\tau,(\tau_1,\tau_2)}(\omega^{-1}_{\mathscr{C}_1}\omega^{-1}_{\mathscr{C}_2}\circ\det{}_{\mathrm{GL}(2)}).$$

Here, $-|_{\mathrm{GL}(2)\times\mathrm{GSp}(4)}$ *means restriction to* $\mathrm{GL}(2)\times_{\mathrm{GL}(1)}\mathrm{GSp}(4)$ *followed by extension by zero to* $\mathrm{GL}(2)\times\mathrm{GSp}(4)$, *and the projection* $\mathrm{Proj}_{K^p_{\mathrm{GL}(2)},K^p_{\mathrm{GSp}(4)}}$ *is defined as* $\int_{K^p_{\mathrm{GL}(2)}}\int_{K^p_{\mathrm{GSp}(4)}}$ *translation by* $(h,g)\,dhdg$.

Proof. — For $G=\mathrm{GSp}(4),\mathrm{GL}(2)$, denote by $\mathcal{M}eas\left(T^1_G(\mathbb{Z}_p),V_{G,\mathrm{ord}}\right)$ the set of p-adic measures on $T^1_G(\mathbb{Z}_p)$ valued in $V_{G,\mathrm{ord}}$. The group $T^1_G(\mathbb{Z}_p)$ acts on it in two ways: via its action by translation on $T^1_G(\mathbb{Z}_p)$ and its action on $V_{G,\mathrm{ord}}$. Let

$$\mathcal{M}eas\left(T^1_G(\mathbb{Z}_p),V_{G,\mathrm{ord}}\right)^\natural\subset\mathcal{M}eas\left(T^1_G(\mathbb{Z}_p),V_{G,\mathrm{ord}}\right)$$

be the subset on which the two action of $T^1_G(\mathbb{Z}_p)$ agree and make it a $\widetilde{\Lambda}_G$-module. Unfolding the definition of $\mathcal{M}_{G,\mathrm{ord}}$ gives a natural map from $\mathcal{M}eas\big(T^1_G(\mathbb{Z}_p),V_{G,\mathrm{ord}}\big)^\natural$ to $\mathcal{M}_{G,\mathrm{ord}}$ such that for each p-adic weight $\underline{\tau}\in\mathrm{Hom}_{\mathrm{cont}}\big(T^1_G(\mathbb{Z}_p),\overline{\mathbb{Q}}_p^\times\big)$, we have the commutative diagram

$$\begin{array}{ccc}\mathcal{M}eas\left(T^1_G(\mathbb{Z}_p),V_{G,\mathrm{ord}}\right)^\natural & \longrightarrow & \mathcal{M}_{G,\mathrm{ord}}\\ \text{evaluate at }\underline{\tau}\downarrow & & \downarrow\\ V_{G,\mathrm{ord}}[\underline{\tau}] & \xrightarrow{(2.2.2)} & \mathcal{M}_{G,\mathrm{ord}}\otimes_{\widetilde{\Lambda}_G}\widetilde{\Lambda}_G/\mathcal{P}_{\underline{\tau}}\end{array}\tag{3.2.3}$$

(cf. [Liu20, §6.1.4]). Therefore, it suffices to show that inside the space,

$$\mathcal{M}eas\Big((\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/U^p)\times T^1_{\mathrm{GL}(2)}\times T^1_{\mathrm{GSp}(4)},V_{\mathrm{GL}(2),\mathrm{ord}}\otimes_{\mathcal{O}}V_{\mathrm{GSp}(4),\mathrm{ord}}\Big)^\natural,\tag{3.2.4}$$

there exists a p-adic measure $\mu_{\mathcal{E}}$ such that

$$\mu_{\mathcal{E}}(\kappa,\tau,(\tau_1,\tau_2)) = e_{\mathrm{ord}}^{\mathrm{GL}(2)} e_{\mathrm{ord}}^{\mathrm{GSp}(4)} \operatorname{Proj}_{K^p_{\mathrm{GL}(2)},K^p_{\mathrm{GSp}(4)}}\big(E^{\mathrm{Sieg}}_{\kappa,\tau,\tau_1,\tau_2}\big|_{\mathrm{GL}(2)\times\mathrm{GSp}(4)}\big) \times \mathrm{sp}_{\tau,(\tau_1,\tau_2)}(\omega_{\mathscr{C}_1}^{-1}\omega_{\mathscr{C}_2}^{-1}\circ\det{}_{\mathrm{GL}(2)}) \tag{3.2.5}$$

for all classical $(\kappa,\tau,\tau_1,\tau_2)=((k,\chi),(\ell,\xi),(\ell_1,\ell_2,\xi_1,\xi_2))$. The existence of $\mu_{\mathcal{E}}$ can be shown by p-adic interpolation of Fourier coefficients.

Take sufficiently small tame level group $K^{p,\prime}_{\mathrm{GL}(2)}\subset K^p_{\mathrm{GL}(2)}, K^{p,\prime}_{\mathrm{GSp}(4)}\subset K^p_{\mathrm{GSp}(4)}$ such that $E^{\mathrm{Sieg}}_{\kappa,\tau,\tau_1,\tau_2}\big|_{\mathrm{GL}(2)\times\mathrm{GSp}(4)}$ is fixed by $K^{p,\prime}_{\mathrm{GL}(2)}, K^{p,\prime}_{\mathrm{GSp}(4)}$ for all classical $(\kappa,\tau,\tau_1,\tau_2)$. Denote by $V'_{\mathrm{GL}(2)}, V'_{\mathrm{GSp}(4)}$ the space of p-adic forms of tame levels $K^{p,\prime}_{\mathrm{GL}(2)}, K^{p,\prime}_{\mathrm{GSp}(4)}$. Thanks to the strong approximation theorem for $\mathrm{Sp}(4)$ and $\mathrm{SL}(2)$, we can pick $\nu_i\in\mathbb{A}^{\times,p}_{\mathbb{Q},f}$, $i=1,\dots,c_1$, and $\nu'_j\in\mathbb{A}^{\times,p}_{\mathbb{Q},f}$, $j=1,\dots,c_2$, such that each connected component of the Shimura variety for $\mathrm{GL}(2)\times\mathrm{GSp}(4)$ of level $K^{p,\prime}_{\mathrm{GL}(2)}\mathrm{GL}(2)(\mathbb{Z}_p)\times K^{p,\prime}_{\mathrm{GSp}(4)}\mathrm{GSp}(4,\mathbb{Z}_p)$ contains a cusp corresponding to $\big(\big[\begin{smallmatrix}\mathbf{1}_2&\\&\nu_i\cdot\mathbf{1}_2\end{smallmatrix}\big],\big[\begin{smallmatrix}1&\\&\nu'_j\end{smallmatrix}\big]\big)$ for some i,j. Taking the q-expansions at these cusps gives an injection

$$\varepsilon_{q\text{-exp}}: V'_{\mathrm{GL}(2)}\widehat{\otimes}_{\mathcal{O}}V'_{\mathrm{GSp}(4)}\longrightarrow \mathcal{O}[\![\mathbb{Q}_{>0}\times\mathrm{Sym}_2(\mathbb{Q})_{>0}]\!]^{\oplus c_1c_2}, \tag{3.2.6}$$

and the image of $\varepsilon_{q\text{-exp}}$ is closed in $\mathcal{O}[\![\mathrm{Sym}_2(\mathbb{Q})_{>0}\times\mathbb{Q}_{>0}]\!]^{\oplus c_1c_2}$ (for the p-adic topology). (The q-expansion maps are defined via evaluations at Mumford objects [FC90, Ch. III], cf. [Hid02, §3.4], [Eis12, §4.2]. The injectivity and the closedness of the image follows from the irreducibility of Igusa towers [Hid04, Cor. 8.17].) We view $V'_{\mathrm{GL}(2)}\widehat{\otimes}_{\mathcal{O}}V'_{\mathrm{GSp}(4)}$ as a subspace of $\mathcal{O}[\![\mathrm{Sym}_2(\mathbb{Q})_{>0}\times\mathbb{Q}_{>0}]\!]^{\oplus c_1c_2}$. The $\mathbb{U}_p$-operators on $V'_{\mathrm{GL}(2)}\widehat{\otimes}_{\mathcal{O}}V'_{\mathrm{GSp}(4)}$ extend to operators on $\mathcal{O}[\![\mathrm{Sym}_2(\mathbb{Q})_{>0}\times\mathbb{Q}_{>0}]\!]^{\oplus c_1c_2}$. Writing elements in $\mathcal{O}[\![\mathrm{Sym}_2(\mathbb{Q})_{>0}\times\mathbb{Q}_{>0}]\!]^{\oplus c_1c_2}$ as $\sum_{i,j,\beta_1,\beta_2}a_{(i,j)}(\beta_1,\beta_2)\,q^{(\beta_1,\beta_2)}_{(i,j)}$, with summation over $1\leqslant i\leqslant c_1$, $1\leqslant j\leqslant c_2$, $\beta_1,\beta_2\in\mathrm{Sym}_2(\mathbb{Q})_{>0}\times\mathbb{Q}_{>0}$, the extension has the formula

$$\begin{aligned}U^{\mathrm{GL}(2)}_{p,m_3,0}U^{\mathrm{GSp}(4)}_{p,m_1,m_2,0}&\Big(\sum_{i,j,\beta_1,\beta_2}a_{i,j}(\beta_1,\beta_2)\,q^{(\beta_1,\beta_2)}\Big)\\ &=\sum_{x\in\mathbb{Z}/p^{m_1-m_2}\mathbb{Z}}\ \sum_{i,j,\beta_1,\beta_2}a_{(i,j)}\Big(\big[\begin{smallmatrix}p^{m_1-m_2}&\\x&1\end{smallmatrix}\big]p^{m_2}\beta_1\big[\begin{smallmatrix}p^{m_1-m_2}&x\\&1\end{smallmatrix}\big],p^{2m_3}\beta_2\Big)\,q^{(\beta_1,\beta_2)}.\end{aligned}$$

On the other hand, letting $\mathcal{N}_G$ denote the space of classical nearly holomorphic forms on $G=\mathrm{GSp}(4),\mathrm{GL}(2)$ over $\mathcal{O}$ of all weights, invariant under the right translation by $K^{p,\prime}_G U_G(\mathbb{Z}_p)$, and vanishing along all p-adic cusps, then the unit root splitting [Kat73, Th. 4.1] gives rise to a map

$$\imath_{p\text{-adic}}:\mathcal{N}_{\mathrm{GL}(2)}\otimes_{\mathcal{O}}\mathcal{N}_{\mathrm{GSp}(4)}\longrightarrow V'_{\mathrm{GL}(2)}\widehat{\otimes}_{\mathcal{O}}V'_{\mathrm{GSp}(4)},$$

which is actually an embedding by [Liu19, Prop. 3.12.1]. Moreover, for a nearly holomorphic form F on $\mathrm{GL}(2)\times\mathrm{GSp}(4)$, we have

$$\begin{aligned}&(\beta_1,\beta_2)\text{-coefficient of }(i,j)\text{-component of }\varepsilon_{q\text{-exp}}(\imath_{p\text{-adic}}(F))\\ &\qquad= F_{\beta_1,\beta_2}\Big(\big[\begin{smallmatrix}\mathbf{1}_2&\\&\nu_i\cdot\mathbf{1}_2\end{smallmatrix}\big],\big[\begin{smallmatrix}1&\\&\nu'_j\end{smallmatrix}\big]\Big)(Y=0).\end{aligned}$$

It follows that

$$\varepsilon_{q\text{-exp}}\left(\imath_{p\text{-adic}}\left(E^{\mathrm{Sieg}}_{\kappa,\tau,\tau_1,\tau_2}\big|_{\mathrm{GL}(2)\times\mathrm{GSp}(4)}\right)\right) = \sum_{\substack{(i,j),(\beta_1,\beta_2)\\ \nu_i=\nu'_j}}\left(\sum_{\substack{\boldsymbol{\beta}\in\mathrm{Her}_3(\mathcal{K})_{>0}\\ \frac{\boldsymbol{\beta}+\overline{\boldsymbol{\beta}}}{2}=\left[\begin{smallmatrix}\beta_1 & *\\ * & \beta_2\end{smallmatrix}\right]}} A(\boldsymbol{\beta};\kappa,\tau,\tau_1,\tau_2)\left(\begin{bmatrix}\mathbf{1}_3 & \\ & \nu_i\mathbf{1}_3\end{bmatrix}^p_f\right) C_{k,\ell,\ell_1,\ell_2}(\boldsymbol{\beta})\right) q^{(\beta_1,\beta_2)}_{(i,j)}$$

with $A(\boldsymbol{\beta};\kappa,\tau,\tau_1,\tau_2), C_{k,\ell,\ell_1,\ell_2}(\boldsymbol{\beta})$ given by the formulas (3.2.1) and (3.2.2). From (3.2.1), it is easy to see that for each $\boldsymbol{\beta}$ and i,j such that $\nu_i=\nu'_j$, there exists a p-adic measure

$$\mu_{(i,j),\boldsymbol{\beta}}\in\mathcal{M}eas\Big(\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/U^p\times T^1_{\mathrm{GL}(2)}(\mathbb{Z}_p)\times T^1_{\mathrm{GSp}(4)}(\mathbb{Z}_p),\mathcal{O}\Big)$$

such that for call classical tuples $(\kappa,\tau,\tau_1,\tau_2)$,

$$\mu_{(i,j),\boldsymbol{\beta}}(\kappa,\tau,\tau_1,\tau_2)=A(\boldsymbol{\beta};\kappa,\tau,\tau_1,\tau_2)\left(\begin{bmatrix}\mathbf{1}_3 & \\ & \nu_i\mathbf{1}_3\end{bmatrix}^p_f\right).$$

Define

$$\mu_{\mathcal{E},q\text{-exp}}\in\mathcal{M}eas\Big(\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/U^p\times T^1_{\mathrm{GL}(2)}(\mathbb{Z}_p)\times T^1_{\mathrm{GSp}(4)}(\mathbb{Z}_p),\mathcal{O}[\![\mathbb{Q}_{>0}\times\mathrm{Sym}_2(\mathbb{Q})_{>0}]\!]^{\oplus c_1c_2}\Big)$$

as

$$\mu_{\mathcal{E},q\text{-exp}}=\sum_{\substack{(i,j),(\beta_1,\beta_2)\\ \nu_i=\nu'_j}}\left(\sum_{\boldsymbol{\beta}\in\mathrm{Her}_3(\mathcal{K})_{>0},\ \frac{\boldsymbol{\beta}+\overline{\boldsymbol{\beta}}}{2}=\left[\begin{smallmatrix}\beta_1 & *\\ * & \beta_2\end{smallmatrix}\right]}\mu_{(i,j),\boldsymbol{\beta}}\right) q^{(\beta_1,\beta_2)}_{(i,j)}.$$

Observe that $C_{k,\ell,\ell_1,\ell_2}(\beta)\equiv 1 \bmod p^m$ for all $\boldsymbol{\beta}$ such that $\boldsymbol{\beta}+\overline{\boldsymbol{\beta}}\equiv 0\bmod p^m$. Thus, for all classical $(\kappa,\tau,\tau_1,\tau_2)$,

$$\begin{aligned}(3.2.7)\quad &\left(U^{\mathrm{GL}(2)}_{p,1,0}U^{\mathrm{GSp}(4)}_{p,2,1,0}\right)^m\mu_{\mathcal{E},q\text{-exp}}(\kappa,\tau,\tau_1,\tau_2)\\ &\equiv\varepsilon_{q\text{-exp}}\left(\left(U^{\mathrm{GL}(2)}_{p,1,0}U^{\mathrm{GSp}(4)}_{p,2,1,0}\right)^m\imath_{p\text{-adic}}\left(E^{\mathrm{Sieg}}_{\kappa,\tau,\tau_1,\tau_2}\big|_{\mathrm{GL}(2)\times\mathrm{GSp}(4)}\right)\right)\mod p^m.\end{aligned}$$

At a classical point, because $E^{\mathrm{Sieg}}_{\kappa,\tau,\tau_1,\tau_2}\big|_{\mathrm{GL}(2)\times\mathrm{GSp}(4)}$ belongs to a finite-rank $\mathcal{O}$-module of classical nearly holomorphic forms of a certain level on which $U^{\mathrm{GL}(2)}_{p,1,0}U^{\mathrm{GSp}(4)}_{p,2,1,0}$ acts, the limit

$$\lim_{n\to\infty}\left(U^{\mathrm{GL}(2)}_{p,1,0}U^{\mathrm{GSp}(4)}_{p,2,1,0}\right)^{n!}E^{\mathrm{Sieg}}_{\kappa,\tau,\tau_1,\tau_2}\big|_{\mathrm{GL}(2)\times\mathrm{GSp}(4)}$$

exists. It follows that when $m=n!$ with $n\to\infty$, the limit of the right hand side of (3.2.7) exists. Combining this with the Zariski density of the classical points inside the weight space $\mathrm{Hom}_{\mathrm{cont}}\big((\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/U^p)\times T^1_{\mathrm{GL}(2)}\times T^1_{\mathrm{GSp}(4)},\overline{\mathbb{Q}}_p^\times\big)$, we deduce from (3.2.7) that the limit

$$\lim_{n\to\infty}\left(U^{\mathrm{GL}(2)}_{p,1,0}U^{\mathrm{GSp}(4)}_{p,2,1,0}\right)^{n!}\mu_{\mathcal{E},q\text{-exp}}$$

exists and interpolates $\varepsilon_{q\text{-exp}}\big(e_{\mathrm{ord}}^{\mathrm{GL}(2)}e_{\mathrm{ord}}^{\mathrm{GSp}(4)}\imath_{p\text{-adic}}\big(E^{\mathrm{Sieg}}_{\kappa,\tau,\tau_1,\tau_2}|_{\mathrm{GL}(2)\times\mathrm{GSp}(4)}\big)\big)$ at all classical $(\kappa,\tau,\tau_1,\tau_2)$. Denote this limit by

$$e_{\mathrm{ord}}^{\mathrm{GL}(2)}e_{\mathrm{ord}}^{\mathrm{GSp}(4)}(\mu_{\mathcal{E},q\text{-exp}}) \in \mathcal{M}eas\Big(\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/U^p\times T^1_{\mathrm{GL}(2)}(\mathbb{Z}_p)\times T^1_{\mathrm{GSp}(4)}(\mathbb{Z}_p),\mathcal{O}[\![\mathbb{Q}_{>0}\times\mathrm{Sym}_2(\mathbb{Q})_{>0}]\!]^{\oplus c_1c_2}\Big).$$

Since the classical points are dense in the weight space and the image of (3.2.6) is closed, this limit must come from the q-expansion of a p-adic measure valued in p-adic forms, i.e., there exists

$$\mu'_{\mathcal{E}}\in\mathcal{M}eas\Big((\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/U^p)\times T^1_{\mathrm{GL}(2)}\times T^1_{\mathrm{GSp}(4)},V'_{\mathrm{GL}(2),\mathrm{ord}}\otimes_{\mathcal{O}}V'_{\mathrm{GSp}(4),\mathrm{ord}}\Big),$$

such that

$$\varepsilon_{q\text{-exp}}(\mu'_{\mathcal{E}})=e_{\mathrm{ord}}^{\mathrm{GL}(2)}e_{\mathrm{ord}}^{\mathrm{GSp}(4)}(\mu_{\mathcal{E},q\text{-exp}}).$$

Moreover, by knowing that the evaluation of $\mu'_{\mathcal{E}}$ at all classical points $(\kappa,\tau,\tau_1,\tau_2)$ has weight $\tau,(\tau_1,\tau_2)$, we deduce that the two natural actions of $T^1_{\mathrm{GL}(2)}\times T^1_{\mathrm{GSp}(4)}$ on $\mu'_{\mathcal{E}}$ agree with each other, and

$$\mu'_{\mathcal{E}}\in\mathcal{M}eas\Big((\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/U^p)\times T^1_{\mathrm{GL}(2)}\times T^1_{\mathrm{GSp}(4)},V'_{\mathrm{GL}(2),\mathrm{ord}}\otimes_{\mathcal{O}}V'_{\mathrm{GSp}(4),\mathrm{ord}}\Big)^{\natural}.$$

Then $\mu_{\mathcal{E}}=\mathrm{Proj}_{K^p_{\mathrm{GL}(2)},K^p_{\mathrm{GSp}(4)}}(\mu'_{\mathcal{E}})\cdot\omega^{-1}_{\mathscr{C}_1}\omega^{-1}_{\mathscr{C}_2}\circ\det_{\mathrm{GL}(2)}$ is the desired measure satisfying (3.2.5). □

3.3. The four-variable p-adic L-function and its interpolation formula I

3.3.1. *Hida families and idempotent operators.* — Let $F_{\mathscr{C}_1},F_{\mathscr{C}_2}$ be the function fields of the irreducible components $\mathscr{C}_1,\mathscr{C}_2$ fixed in Section 3.1. Then the maps $\widetilde{\Lambda}_{\mathrm{GL}(2)}\to F_{\mathscr{C}_1}$, $\widetilde{\Lambda}_{\mathrm{GSp}(4)}\to F_{\mathscr{C}_2}$ factors through projections $\widetilde{\Lambda}_{\mathrm{GL}(2)}\to\Lambda_{\mathrm{GL}(2)}$, $\widetilde{\Lambda}_{\mathrm{GSp}(4)}\to\Lambda_{\mathrm{GSp}(4)}$ induced by characters of $T^1_{\mathrm{GL}(2)}(\mathbb{Z}/p\mathbb{Z})$ and $T^1_{\mathrm{GSp}(4)}(\mathbb{Z}/p\mathbb{Z})$. We view $F_{\mathscr{C}_1}$ as an algebra over $\Lambda_{\mathrm{GL}(2)}$ and $F_{\mathscr{C}_2}$ as an $\Lambda_{\mathrm{GSp}(4)}$-algebra through these factorizations.

Denote by $\mathbb{I}_{\mathscr{C}_1},\mathbb{I}_{\mathscr{C}_2}$ the integral closures of $\Lambda_{\mathrm{GL}(2)},\Lambda_{\mathrm{GSp}(4)}$ inside $F_{\mathscr{C}_1},F_{\mathscr{C}_2}$. The universal ordinary Hecke algebras $\mathbb{T}_{\mathrm{GL}(2),\mathrm{ord}},\mathbb{T}_{\mathrm{GSp}(4),\mathrm{ord}}$ are known to be reduced. Therefore, we have

$$\mathbb{T}_{\mathrm{GL}(2),\mathrm{ord}}\otimes F_{\mathscr{C}_1}=F_{\mathscr{C}_1}\oplus R_{\mathscr{C}_1},\qquad \mathbb{T}_{\mathrm{GSp}(4),\mathrm{ord}}\otimes F_{\mathscr{C}_2}=F_{\mathscr{C}_2}\oplus R_{\mathscr{C}_2}$$

as $F_{\mathscr{C}_1}$-algebras and $F_{\mathscr{C}_2}$-algebras such that the projection onto the first factor agrees with the natural maps $\mathbb{T}_{\mathrm{GL}(2),\mathrm{ord}}\to\mathbb{I}_{\mathscr{C}_1}$, $\mathbb{T}_{\mathrm{GSp}(4),\mathrm{ord}}\to\mathbb{I}_{\mathscr{C}_2}$. Let

$$\mathbb{1}_{\mathscr{C}_1}\in\mathbb{T}_{\mathrm{GL}(2),\mathrm{ord}}\otimes F_{\mathscr{C}_1},\qquad \mathbb{1}_{\mathscr{C}_2}\in\mathbb{T}_{\mathrm{GSp}(4),\mathrm{ord}}\otimes F_{\mathscr{C}_2}\tag{3.3.1}$$

be the idempotent associated to the first factor in the above decomposition.

3.3.2. *The modified Euler factors at p and ∞.* — We let

$$\lambda_{\mathrm{GL}(2),0,1}\in\mathbb{I}^\times_{\mathscr{C}_1},\qquad \lambda_{\mathrm{GSp}(4),0,0,1},\lambda_{\mathrm{GSp}(4),1,0,0}\in\mathbb{I}^\times_{\mathscr{C}_2}$$

denote the eigenvalues of the $\mathbb{U}_p$-operators associated to $\left[\begin{smallmatrix}p&\\&1\end{smallmatrix}\right]$, $\begin{bmatrix}p&&&\\&p&&\\&&1&\\&&&1\end{bmatrix}$, $\begin{bmatrix}p&&&\\&1&&\\&&p^{-1}&\\&&&1\end{bmatrix}$.

Given a point $x \in \mathscr{C}_1(\overline{\mathbb{Q}}_p) \times \mathscr{C}_2(\overline{\mathbb{Q}}_p)$ where the weight projection map (3.1.1) is étale and the image of x is an arithmetic tuple $(\tau, \tau_1, \tau_2) = (\ell, \xi, \ell_1, \xi_1, \ell_2, \xi_2)$, we let

$$\mathscr{S}_{\mathrm{GL}(2),x} \qquad (\text{resp. } \mathscr{S}_{\mathrm{GSp}(4),x})$$

be an orthogonal basis (with respect to the modified Petersson inner product defined in (3.3.5), (3.3.6)) of the space spanned by ordinary cuspidal holomorphic forms on $\mathrm{GL}(2)$ of weight ℓ, tame level $K^p_{\mathrm{GL}(2)}$ (resp. $\mathrm{GSp}(4)$ of weight (ℓ_1, ℓ_2), tame level $K^p_{\mathrm{GSp}(4)}$) and nebentypus at p given by (3.1.3), belonging to the Hecke eigenspace parameterized by x. Let π_x be the unitary irreducible automorphic representation of $\mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$ generated by forms $\mathscr{S}_{\mathrm{GL}(2),x}$ twisted by a real power of $|\det|$, and Π_x be a unitary irreducible automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ inside the representation generated by forms in $\mathscr{S}_{\mathrm{GSp}(4),x}$ twisted by a real power of $|\det|$. (There can be more than one choices of Π_x, but the partial L-function and modified Euler factors at p, ∞ do not depend on the choice of Π_x.) Let $L^S(s, \Pi_x \times \pi_x \times \chi)$ be the degree 8 partial L-function, and

$$\begin{aligned}(3.3.2)\quad & E_p(s, \Pi_x \times \pi_x \times \chi) \\ &= \gamma_p(s, \chi_p \eta_{x,1} \eta'_{x,1})^{-1} \gamma_p(s, \chi_p \eta_{x,1} \eta'_{x,2})^{-1} \gamma_p(s, \pi_{x,p} \times \chi_p \eta'_{x,3})^{-1},\end{aligned}$$

where the characters $\eta_{x,1}, \eta_{x,2}, \eta'_{x,1}, \eta'_{x,2}, \eta'_{x,3}$ are:

$$\begin{aligned}
\eta_{x,1}(a) &= \xi(a|a|_p)\Big(p^{-(\ell-1)/2}\lambda_{\mathrm{GL}(2),0,1}(x)\Big)^{\mathrm{val}_p(a)}, \\
\eta_{x,2}(a) &= \Big(p^{(\ell-1)/2}(\omega_{\mathscr{C}_1,p}(p)\lambda^{-1}_{\mathrm{GL}(2),0,1})(x)\Big)^{\mathrm{val}_p(a)}, \\
(3.3.3)\qquad \eta'_{x,1}(a) &= \xi_1(a|a|_p)\Big(p^{(-\ell_1+\ell_2-1)/2}\omega_{\mathscr{C}_2,p}(p)\lambda_{\mathrm{GSp}(4),1,0}\lambda^{-1}_{\mathrm{GSp}(4),0,1}(x)\Big)^{\mathrm{val}_p(a)}, \\
\eta'_{x,2}(a) &= \xi_2(a|a|_p)\Big(p^{(\ell_1-\ell_2+1)/2}(\lambda_{\mathrm{GSp}(4),0,1}\lambda^{-1}_{\mathrm{GSp}(4),1,0})(x)\Big)^{\mathrm{val}_p(a)}, \\
\eta'_{x,3}(a) &= \xi_1\xi_2(a|a|_p)\Big(p^{-(\ell_1+\ell_2-3)/2}\lambda_{\mathrm{GSp}(4),0,1}(x)\Big)^{\mathrm{val}_p(a)},
\end{aligned}$$

and

$$\begin{aligned}(3.3.4)\quad & E_\infty(s, \Pi_x \times \pi_x \times \chi) \\ &= e^{-(4s+\ell_1+\ell_2+\ell)\cdot \pi i/2}\,\Gamma_\mathbb{C}\Big(s + \frac{\ell_1+\ell_2+\ell}{2} - 2\Big)\Gamma_\mathbb{C}\Big(s + \frac{\ell_1+\ell_2-\ell}{2} - 1\Big) \\ &\qquad \times \Gamma_\mathbb{C}\Big(s + \frac{-\ell_1+\ell_2+\ell}{2} - 1\Big)\Gamma_\mathbb{C}\Big(s + \frac{\ell_1-\ell_2+\ell}{2}\Big),\end{aligned}$$

where $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$.

The factors E_p, E_∞ are obtained by unfolding the definitions in [CPR89, Coa91]. In [Coa91, p. 163], for a homogeneous motive M over $\mathbb{Q}$ with coefficients in K and σ a fixed embedding $K \hookrightarrow \mathbb{C}$, the modified Euler factor at p for its L-function $L(\sigma, M, s)$ is defined as $\prod_U E_p(\sigma, U, \rho, s)$, where U runs over irreducible components of the semi-simplification of the Weil-Deligne representation at p associated to the λ-adic realization of M with λ a prime in K coprime to p, and ρ is a choice of $\pm i$, for which we choose $\rho = i$. Interpreting this definition in our automorphic context here with Π_x (resp. π_x) a subquotient of the induced representation associated to

$$\begin{bmatrix} a_1 & * & * & * \\ & a_2 & * & * \\ & & \delta a_2 & \\ & & * & \delta a_1 \end{bmatrix} \longmapsto \eta'_{x,1}(a_1)\,\eta'_{x,2}(a_2)\,\eta'_{x,3}(\delta)$$

$$\Bigl(\text{resp. } \begin{bmatrix} a & * \\ & d \end{bmatrix} \longmapsto \eta_{x,1}(d)\,\eta_{x,2}(a)\Bigr),$$

the irreducible component U corresponds to $\eta = \eta_{x,i}\eta'_{x,j}, \eta_{x,i}\eta'_{x,1}\eta'_{x,2}\eta'^{,-1}_{x,3}$, $i = 1, 2$, $j = 1, 2, 3$. Denoting by Φ an element in $G_{\mathbb{Q}_p}$ whose image in $G_{\mathbb{Q}_p}/I_p$ is the geometric Frobenius, the factor $E_p(\sigma, U, \rho, s)$ is defined to be 1 or the inverse of the γ-factor of U depending on whether the inverse roots of $\det(1-\Phi.X|U)$ have p-adic valuations larger or smaller than $-1/2$, namely in our context here whether $\mathrm{val}_p(\eta(p)p^{-s})$ is larger or smaller than $-1/2$. Note that the λ's in (3.3.3) are all p-adic units. Thus, when

$$-\frac{\min\{-\ell_1+\ell_2+\ell,\, \ell_1+\ell_2-\ell\}}{2} + 2 \leqslant s \leqslant \frac{\min\{-\ell_1+\ell_2+\ell,\, \ell_1+\ell_2-\ell\}}{2} - 1,$$

among all the η's, the ones with $\mathrm{val}_p(\eta(p)p^{-s}) < -1/2$ are exactly

$$\eta_{x,1}\eta'_{x,1},\ \eta_{x,1}\eta'_{x,2},\ \eta_{x,1}\eta'_{x,3},\ \eta_{x,2}\eta'_{x,3}.$$

Therefore, the modified Euler factor at p is given as (3.3.2). In [Coa91, (38) on p. 157], the modified Euler factor at ∞ for M is defined as $\prod_U E_\infty(U, \rho, s)$ where U runs over the direct summands of the Hodge decomposition of $H_B(M) \otimes_{K,\sigma} \mathbb{C}$. In our case here, we only have U's of type (a) in loc. cit. where $E_\infty(U, \rho, s)$ can be obtained from $L_\infty(U, s)$ by replacing $\Gamma_{\mathbb{C}}(s)$ with $\Gamma_{\mathbb{C},\rho}(s) = \rho^{-s}\Gamma_{\mathbb{C}}(s)$. It follows that $E_\infty(s, \Pi_x \times \pi_x \times \chi)$ equals the product of

$$\begin{aligned} L_\infty(s, \Pi_x \times \pi_x \times \chi) = \Gamma_{\mathbb{C}}\Bigl(s + \frac{\ell_1+\ell_2+\ell}{2} - 2\Bigr)\Gamma_{\mathbb{C}}\Bigl(s + \frac{\ell_1+\ell_2-\ell}{2} - 1\Bigr) \\ \times \Gamma_{\mathbb{C}}\Bigl(s + \frac{-\ell_1+\ell_2+\ell}{2} - 1\Bigr)\Gamma_{\mathbb{C}}\Bigl(s + \frac{\ell_1-\ell_2+\ell}{2}\Bigr), \end{aligned}$$

and $\rho^{-(s+(\ell_1+\ell_2+\ell)/2-2)-(s+(\ell_1+\ell_2-\ell)/2-1)-(s+(-\ell_1+\ell_2+\ell)/2-1)-(s+(\ell_1-\ell_2+\ell)/2)}$.

3.3.3. *The modified Petersson inner product and Bessel period.* — For ordinary holomorphic automorphic forms f_1, f_2 on $\mathrm{GL}(2)$ and φ_1, φ_2 on $\mathrm{GSp}(4)$, we define the modified Petersson inner product $\mathbf{P}(f_1, f_2), \mathbf{P}(\varphi_1, \varphi_2)$ and the modified Bessel period $B^{\dagger}_{\mathbb{S},\Lambda}(\varphi_1)$ as follows:

$$\begin{aligned}(3.3.5)\quad &\mathbf{P}(f_1,f_2)\\ &= \lambda_p(f_1)^{-m}\int_{[\mathrm{GL}(2)]} f_1(g)\, f_2\big(g\left[\begin{smallmatrix} & 1\\ 1 & \end{smallmatrix}\right]_{p\infty}\left[\begin{smallmatrix} p^m & \\ & p^{-m}\end{smallmatrix}\right]_p\big)\cdot\omega_\pi(\det g)^{-1}\,dg,\end{aligned}$$

$$\begin{aligned}(3.3.6)\quad &\mathbf{P}(\varphi_1,\varphi_2) = \lambda_{p,1}(\varphi_1)^{-m_1}\lambda_{p,2}(\varphi_1)^{-m_2}\\ &\times\int_{[\mathrm{GSp}(4)]}\varphi_1(g)\,\varphi_2\left(g\left[\begin{smallmatrix} & \mathbf{1}_2\\ \mathbf{1}_2 & \end{smallmatrix}\right]_{p\infty}\begin{bmatrix} p^{m_1} &&&\\ & p^{m_2} &&\\ && p^{-m_1} &\\ &&& p^{-m_2}\end{bmatrix}_p\right)\cdot\omega_\Pi(\nu_g)^{-1}\,dg,\end{aligned}$$

$$(3.3.7)\quad B^{\dagger}_{\mathbb{S},\Lambda}(\varphi_1) = \lambda_{p,1}(\varphi_1)^{-m_1}\lambda_{p,2}(\varphi_1)^{-m_2}\cdot B_{\mathbb{S},\Lambda}\left(\begin{bmatrix} p^{m_1} &&&\\ & p^{m_2} &&\\ && p^{-m_1} &\\ &&& p^{-m_2}\end{bmatrix}_p\varphi_1\right),$$

with $m_1\gg m_2\gg 0$, $m\gg 0$, where $\lambda_p(f_1),\lambda_{p,1}(\varphi_1),\lambda_{p,2}(\varphi_1)$ are defined by

$$\lambda_p(f_1)^m f_1 = \int_{U_{\mathrm{GL}(2)}(\mathbb{Z}_p)}\pi_p\left(u\left[\begin{smallmatrix} p^m & \\ & p^{-m}\end{smallmatrix}\right]\right) f_1\,du,$$

$$\lambda_{p,1}(\varphi_1)^{m_1}\lambda_{p,2}(\varphi_1)^{m_2}\varphi_1 = \int_{U_{\mathrm{GSp}(4)}(\mathbb{Z}_p)}\Pi_p\left(u\begin{bmatrix} p^{m_1} &&&\\ & p^{m_2} &&\\ && p^{-m_1} &\\ &&& p^{-m_2}\end{bmatrix}\right)\varphi_1\,du,$$

and $B_{\mathbb{S},\Lambda}$ is the Bessel period as defined in [Liu23, §2.2.1]. (One can check that the right hand sides of (3.3.5), (3.3.6), (3.3.7) do not depend on m_1,m_2,m as long as m_1-m_2,m_2,m are sufficiently large. See [Liu23, Prop. 2.7.1] for a proof of this for (3.3.7).)

3.3.4. *The four-variable p-adic L-function.* — With the various objects defined in Sections 3.3.2, 3.3.3, we apply the idempotent in (3.3.1) to the p-adic measure $\mu_{\mathcal{E}}$ constructed in Theorem 3.2.2 to obtain the p-adic measure for $\mathscr{C}_1,\mathscr{C}_2$. Before stating the theorem, we introduce some more notation.

With $\mathbb{S}=\left[\begin{smallmatrix}\mathbb{a} & \mathbb{b}/2\\ \mathbb{b}/2 & \mathbb{c}\end{smallmatrix}\right]\in\mathrm{Sym}_2(\mathbb{Q})_{>0}$ and $\mathbf{\Lambda}:\mathcal{K}^\times\backslash\mathbb{A}^\times_{\mathcal{K},f}\to\Lambda^\times_{\mathrm{GSp}(4)}$ as fixed in Section 3.1, for $f\in\mathscr{S}_{\mathrm{GL}(2),x},\varphi\in\mathscr{S}_{\mathrm{GSp}(4),x}$, we let $f_{\mathbb{c}}$ denote the Fourier coefficients of f indexed by $\mathbb{c}$, and $B^{\dagger}_{\mathbb{S},\Lambda}(\varphi)$ denote the modified Bessel period in (3.3.7) with Λ the classical Hecke character corresponding to the specialization of $\mathbf{\Lambda}$ at $(\ell_1,\ell_2,\xi_1,\xi_2)$. We denote by $(r_{\Lambda,1},r_{\Lambda,2})$ the ∞-type of Λ.

Theorem 3.3.1. — *Given the data in Section* 3.1*, there exists*

$$\mu^S_{\mathscr{C}_1,\mathscr{C}_2}\in\mathcal{M}eas\Big(\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/U^p,\mathcal{M}_{\mathrm{GL}(2),\mathrm{ord}}\widehat{\otimes}_{\mathcal{O}}\mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}}\Big)\otimes_{\widetilde{\Lambda}_{\mathrm{GL}(2)}\widehat{\otimes}_{\mathcal{O}}\widetilde{\Lambda}_{\mathrm{GSp}(4)}}(F_{\mathscr{C}_1}\widehat{\otimes}_{\mathcal{O}}F_{\mathscr{C}_2})$$

satisfying the interpolation properties: Suppose that $x\in\mathscr{C}_1(\overline{\mathbb{Q}}_p)\times\mathscr{C}_2(\overline{\mathbb{Q}}_p)$ is a point at which the weight projection map (3.1.1) *is étale. Then $\mu^S_{\mathscr{C}_1,\mathscr{C}_2}$ has no poles along x. Let $\tau\in\mathrm{Hom}_{\mathrm{cont}}\big(T^1_{\mathrm{GL}(2)}(\mathbb{Z}_p),\overline{\mathbb{Q}}_p\big)$, $(\tau_1,\tau_2)\in\mathrm{Hom}_{\mathrm{cont}}\big(T^1_{\mathrm{GSp}(4)}(\mathbb{Z}_p),\overline{\mathbb{Q}}_p\big)$ be the projection of x to the weight space.*

For a character $\kappa\in\mathrm{Hom}_{\mathrm{cont}}\big(\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q},f}^\times/U^p,\overline{\mathbb{Q}}_p\big)$ such that $(\kappa,\tau,\tau_1,\tau_2)$ is classical (as defined at the end of Section 3.1.1*),*

$$\mu^S_{\mathscr{C}_1,\mathscr{C}_2}(\kappa,x) \tag{3.3.8}$$
$$=\sum_{f\in\mathscr{S}_{\mathrm{GL}(2),x}}\frac{f_{\mathfrak{c}}f}{\mathbf{P}(f,f)}\sum_{\varphi\in\mathscr{S}_{\mathrm{GSp}(4),x}}\frac{B^\dagger_{\mathbb{S},\Lambda}(\varphi)\varphi}{\mathbf{P}(\varphi,\varphi)}\cdot i^{r_{\Lambda,1}-r_{\Lambda,2}}I_\infty(k,\mathcal{D}_{\ell_1,\ell_2},\mathcal{D}_\ell,\Lambda_\infty)$$
$$\times E_p\Big(k+\frac{\ell+\ell_1+\ell_2}{2},\Pi_x\times\pi_x\times\chi\Big)\cdot L^S\Big(k+\frac{\ell+\ell_1+\ell_2}{2},\Pi_x\times\pi_x\times\chi\Big),$$

with $I_\infty(k,\mathcal{D}_{\ell_1,\ell_2},\mathcal{D}_\ell,\Lambda_\infty)$ the Archimedean zeta integral given in [Liu23, (4.2.5)] *with $k+\epsilon/2$, t_k in loc. cit. equal to $k+(\ell+\ell_1+\ell_2)/2$, $|2k+\ell+\ell_1+\ell_2-1|+3$ here and $\mathcal{D}_{\ell_1,\ell_2},\mathcal{D}_\ell$ holomorphic discrete series of* $\mathrm{GSp}(4),\mathrm{GL}(2)$ *of weights $(\ell_1,\ell_2),\ell$. If further assuming that $\ell=\ell_1=\ell_2$, then*

$$\mu^S_{\mathscr{C}_1,\mathscr{C}_2}(\kappa,x)=\mathfrak{c}\sqrt{\det\mathbb{S}}\,2^{-3\ell+2}i^\ell\sum_{f\in\mathscr{S}_{\mathrm{GL}(2),x}}\frac{f_{\mathfrak{c}}f}{\mathbf{P}(f,f)}\sum_{\varphi\in\mathscr{S}_{\mathrm{GSp}(4),x}}\frac{B^\dagger_{\mathbb{S},\Lambda}(\varphi)\,\varphi}{\mathbf{P}(\varphi,\varphi)}$$
$$\times E_\infty\Big(k+\frac{3\ell}{2},\Pi_x\times\pi_x\times\chi\Big)E_p\Big(k+\frac{3\ell}{2},\Pi_x\times\pi_x\times\chi\Big)L^S\Big(k+\frac{3\ell}{2},\Pi_x\times\pi_x\times\chi\Big).$$

(See Sections 3.3.2*,* 3.3.3 *for the definitions of the terms appearing in the above interpolation formulas.)*

Proof. — We first examine the evaluations of

$$(\mathbb{1}_{\mathscr{C}_1}\otimes\mathbb{1}_{\mathscr{C}_2})\cdot\mu_{\mathcal{E}}\in\mathcal{M}eas\Big(\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/U^p,\mathcal{M}_{\mathrm{GL}(2),\mathrm{ord}}\widehat{\otimes}_{\mathcal{O}}\mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}}\Big)$$
$$\otimes_{\widetilde{\Lambda}_{\mathrm{GL}(2)}\widehat{\otimes}_{\mathcal{O}}\widetilde{\Lambda}_{\mathrm{GSp}(4)}}(F_{\mathscr{C}_1}\widehat{\otimes}_{\mathcal{O}}F_{\mathscr{C}_2}).$$

By the construction of $\mu_{\mathcal{E}}$, we know that at (κ,x) with $(\kappa,\tau,\tau_1,\tau_2)$ classical,

$$((\mathbb{1}_{\mathscr{C}_1}\otimes\mathbb{1}_{\mathscr{C}_2})\cdot\mu_{\mathcal{E}})(\kappa,x)=\sum_{f\in\mathscr{S}_{\mathrm{GL}(2),x}}\sum_{\varphi\in\mathscr{S}_{\mathrm{GSp}(4),x}}\frac{\mathbf{P}\left(E^{\mathrm{Sieg}}_{\kappa,\tau,\tau_1,\tau_2}|_{\mathrm{GL}(2)\times\mathrm{GSp}(4)},f\otimes\varphi\right)}{\mathbf{P}(f,f)\mathbf{P}(\varphi,\varphi)}f\otimes\varphi.$$

(Here we use that the definition of $\mathbf{P}(-,-)$ plus the tame level and ordinarity of f,φ implies that applying $\mathbf{P}(-,f\otimes\varphi)$ to $E^{\mathrm{Sieg}}_{\kappa,\tau,\tau_1,\tau_2}|_{\mathrm{GL}(2)\times\mathrm{GSp}(4)}$ and

$$\mu_{\mathcal{E}}(\kappa,x)=e^{\mathrm{GL}(2)}_{\mathrm{ord}}e^{\mathrm{GSp}(4)}_{\mathrm{ord}}\,\mathrm{Proj}_{K^p_{\mathrm{GL}(2)},K^p_{\mathrm{GSp}(4)}}\Big(E^{\mathrm{Sieg}}_{\kappa,\tau,\tau_1,\tau_2}|_{\mathrm{GL}(2)\times\mathrm{GSp}(4)}\Big)$$

produces the same value.) The computation in the proof of [Liu23, Th. 4.2.1] (or more precisely the formula for $I_p(s)$ and $C_{k,\chi,\Pi,\pi}(s)$ in loc. cit. gives

$$\mathbf{P}\big(E^{\mathrm{Sieg}}_{\kappa,\tau,\tau_1,\tau_2},f\otimes\varphi|_{\mathrm{GL}(2)\times\mathrm{GSp}(4)}\big)=C(k,\chi,x)\cdot f_{\mathfrak{c}}B^\dagger_{\mathbb{S},\Lambda}(\varphi)\cdot I_\infty(k,\Pi_{x,\infty},\pi_{x,\infty},\Lambda_\infty)$$
$$\times E_p\Big(k+\frac{\ell+\ell_1+\ell_2}{2},\Pi_x\times\pi_x\times\chi\Big)\cdot L^S\Big(k+\frac{\ell+\ell_1+\ell_2}{2},\Pi_x\times\pi_x\times,\chi\Big)$$

and

$$
\begin{aligned}
(3.3.9)\quad & C(k,\chi,x)\\
&= \operatorname*{vol}_S \frac{1-(\mathcal{K}/p)\,p^{-1}}{1+p^{-1}}\left|\frac{\alpha_{\mathbb{S}}-\overline{\alpha}_{\mathbb{S}}}{\sqrt{\operatorname{disc}(\mathcal{K}/\mathbb{Q})}}\right|_p\cdot|\mathbb{c}|_p^2|(\alpha_{\mathbb{S}}-\overline{\alpha}_{\mathbb{S}})^2|_p^2\cdot\chi_p(-1)(-1)^k\\
&\times(\chi_p^{-1}|\cdot|_p^{-k})(\mathbb{c})\mathbb{c}^{-k}\cdot\xi^{-1}\xi_1^{-1}\xi_2^{-1}(\mathbb{c}|\mathbb{c}|_p)(\mathbb{c}|\mathbb{c}|_p)^{-\ell-\ell_1-\ell_2}\cdot\lambda_{\mathrm{GL}(2)}\lambda_{\mathrm{GSp}(4)}(x)^{-\operatorname{val}_p(\mathbb{c})}\\
&\times(\chi_p^{-1}|\cdot|_p^{-k})(-(\alpha_{\mathbb{S}}-\overline{\alpha}_{\mathbb{S}})^2)(-(\alpha_{\mathbb{S}}-\overline{\alpha}_{\mathbb{S}})^2)^{-k}\\
&\times\xi^{-1}(-(\alpha_{\mathbb{S}}-\overline{\alpha}_{\mathbb{S}})^2|\alpha_{\mathbb{S}}-\overline{\alpha}_{\mathbb{S}}|_p^2)(-(\alpha_{\mathbb{S}}-\overline{\alpha}_{\mathbb{S}})^2|\alpha_{\mathbb{S}}-\overline{\alpha}_{\mathbb{S}}|_p^2)^{-\ell}\\
&\times\Lambda_p^{-c}(-(\alpha_{\mathbb{S}}-\overline{\alpha}_{\mathbb{S}}))(\alpha_{\mathbb{S}}-\overline{\alpha}_{\mathbb{S}})^{-r_{\Lambda_1}}(-\alpha_{\mathbb{S}}+\overline{\alpha}_{\mathbb{S}})^{-r_{\Lambda_2}}\\
&\times\lambda_{\mathrm{GL}(2)}(x)^{-\operatorname{val}_p((\alpha_{\mathbb{S}}-\overline{\alpha}_{\mathbb{S}})^2)}\cdot i^{r_{\Lambda,1}-r_{\Lambda,2}},
\end{aligned}
$$

where vol_S is a nonzero constant independent of k,χ,x (and can be expressed in terms of the volumes of some open compact subgroups of $\mathrm{GL}_2(\mathbb{Q}_v),\mathrm{GSp}(4,\mathbb{Q}_v)$ at finite $v\in S$), $\lambda_{\mathrm{GL}(2)}\in\mathbb{I}_{\mathscr{C}_1}^\times,\lambda_{\mathrm{GSp}(4)}\in\mathbb{I}_{\mathscr{C}_2}^\times$ denote the eigenvalues of the $\mathbb{U}_p$-operators corresponding to $\left[\begin{smallmatrix}p&\\&1\end{smallmatrix}\right],\left[\begin{smallmatrix}p\cdot\mathbf{1}_2&\\&\mathbf{1}_2\end{smallmatrix}\right]$ along $\mathscr{C}_1,\mathscr{C}_2$. (To plug in the formulas in [Liu23], s, $k+\epsilon/2$, $\Lambda,\eta_{\pi_p,1}(a)$, $\eta_{\Pi_p,3}(a)$ for $a\in\mathbb{Q}_p$ in loc. cit. correspond to

$$
\begin{gathered}
k+\frac{\ell+\ell_1+\ell_2-1}{2},\quad k+\frac{\ell+\ell_1+\ell_2}{2},\\
\Lambda^{-c}|\cdot|_{\mathbb{A}_{\mathcal{K}}}^{(\ell_1+\ell_2)/2},\ \xi^{-1}(a|a|_p)\cdot(p^{(\ell-1)/2}\lambda_{\mathrm{GL}(2)}^{-1}(x))^{\operatorname{val}_p(a)},\\
\xi_1^{-1}\xi_2^{-1}(a|a|_p)\cdot(p^{(\ell_1+\ell_2-3)/2}\lambda_{\mathrm{GSp}(4)}^{-1}(x))^{\operatorname{val}_p(a)}
\end{gathered}
$$

here. Also, note that the integral in loc. cit. is over $\mathrm{GU}(1,1)\times\mathrm{GSp}(4)$ with f extended to $\mathrm{GU}(1,1)$ by Υ equals our integral over $\mathrm{GL}(2)\times\mathrm{GSp}(4)$ here because the central characters match.)

From the above formula for $C(k,\chi,x)$, it is easy to see that there exists

$$\mathcal{C}\in\Big(\mathcal{O}[\![\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q},f}^\times/U^p]\!]\widehat{\otimes}_{\mathcal{O}}\mathbb{I}_{\mathscr{C}_1}\widehat{\otimes}_{\mathcal{O}}\mathbb{I}_{\mathscr{C}_2}\otimes_{\mathcal{O}}F\Big)^\times$$

such that for all (κ,x) with $(\kappa,\tau,\tau_1,\tau_2)$ classical,

$$\mathcal{C}(\kappa,x)=i^{-r_{\Lambda,1}+r_{\Lambda,2}}\cdot C(k,\chi,x).$$

Let

$$\mu^S_{\mathscr{C}_1,\mathscr{C}_2}=\mathcal{C}^{-1}\left((\mathbb{1}_{\mathscr{C}_1}\otimes\mathbb{1}_{\mathscr{C}_2})\cdot\mu_{\mathcal{E}}\right).$$

Then for (κ,x) as in the statement of the theorem,

$$
\begin{aligned}
&\mu^S_{\mathscr{C}_1,\mathscr{C}_2}(\kappa,x)\\
&\quad=\sum_{f\in\mathscr{S}_{\mathrm{GL}(2),x}}\ \sum_{\varphi\in\mathscr{S}_{\mathrm{GSp}(4),x}}\frac{f_{\mathbb{c}}B^\dagger_{\mathbb{S},\Lambda}(\varphi)}{\mathbf{P}(f,f)\mathbf{P}(\varphi,\varphi)}\cdot i^{r_{\Lambda,1}-r_{\Lambda,2}}I_\infty(k,\mathcal{D}_{\ell_1,\ell_2},\mathcal{D}_\ell,\Lambda_\infty)\\
&\times E_p\Big(k+\frac{\ell+\ell_1+\ell_2}{2},\Pi_x\times\pi_x\times\chi\Big)\cdot L^S\Big(k+\frac{\ell+\ell_1+\ell_2}{2},\Pi_x\times\pi_x\times\chi\Big)\cdot f\otimes\varphi.
\end{aligned}
$$

When $\ell_1=\ell_2=\ell$, the integral $I_\infty(k,\mathcal{D}_{\ell_1,\ell_2},\mathcal{D}_\ell,\Lambda_\infty)$ is evaluated in [Liu23, Prop. 4.2]. Like in loc. cit., plugging the result into the right hand side of the above formula gives the complete interpolation formula in the case $\ell_1=\ell_2=\ell$. □

4. Specialization to Hida families of Yoshida lifts

The interpolation formula in Theorem 3.3.1 is incomplete in the sense that it contains an uncomputed Archimedean zeta integral $I_\infty(k,\mathcal{D}_{\ell_1,\ell_2},\mathcal{D}_\ell,\Lambda_\infty)$ when ℓ_1,ℓ_2,ℓ are not all equal. In order to calculate this integral, we let $\mathscr{C}_2=\theta(\mathscr{B},\mathscr{B}')$, the Hecke eigensystem associated to the Yoshida lifts of Hida families $\mathscr{B},\mathscr{B}'$ on $\mathrm{GL}(2)$, and compare $\mu^S_{\mathscr{C}_1,\theta(\mathscr{B},\mathscr{B}')}$ with the product of Rankin–Selberg p-adic L-functions for $\mathscr{B},\mathscr{C}_1$ and $\mathscr{C}_1,\mathscr{B}'$. The main difficulty in carrying out this strategy lies in comparing the periods, which boils down to comparing the (modified) Petersson norms of a pair of modular forms and that of their Yoshida lift. To circumvent the explicit calculation of the corresponding Rallis inner product formula in our specific case, we make use of the standard p-adic L-function for $\theta(\mathscr{B},\mathscr{B}')$, which has the (modified) Petersson norms of Yoshida lifts as periods.

4.1. Some previous results on p-adic L-functions.

— For simplicity, in this section and Section 4.2, we assume that S contains some finite place $v\neq p$, and for all such $v\in S$,

$$K^p_{\mathrm{GL}(2),v}=\left\{g\in\mathrm{GL}(2,\mathbb{Z}_v):g\equiv\left[\begin{smallmatrix}*&*\\0&*\end{smallmatrix}\right]\bmod\varpi_v\right\}.$$

We recall some previous results on constructions of Kubota–Leopold p-adic L-functions, Rankin–Selberg p-adic L-functions, and p-adic (degree 5) standard L-functions for $\mathrm{Sp}(4)$. Complete interpolation properties have been established for these p-adic L-functions, and they will be used to formulate the right hand side in the key identity (4.2.1). To simplify the writing of the interpolation properties, we use the following convention: Given an automorphic representation σ with σ_∞ isomorphic to holomorphic discrete series and ordinary at p, we let

$$D^S(s,\sigma)=E_\infty(s,\sigma)\,E_p(s,\sigma)\,L^S(s,\sigma)$$

with $E_\infty(s,\sigma),E_p(s,\sigma)$ the modified Euler factor at ∞ and p for p-adic interpolation as defined in [CPR89, Coa91].

4.1.1. *Kubota–Leopoldt p-adic L-function*

Theorem 4.1.1. — *Given a Dirichlet character ϕ unramified outside $S\smallsetminus\{p\}$, there exists*

$$\mathcal{L}^S_{\mathrm{KL},\phi}\in\mathcal{M}eas\big(\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/\widehat{\mathbb{Z}}^{p,\times},\mathcal{O}\big)$$

such that for all arithmetic $\kappa=(k,\chi)\in\mathrm{Hom}_{\mathrm{cont}}\big(\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/\widehat{\mathbb{Z}}^{p,\times},\overline{\mathbb{Q}}_p\big)$ with $\phi\chi(-1)=(-1)^k$ and $k\geqslant1$ or $\phi\chi(-1)=(-1)^{k+1}$ and $k\leqslant0$,

$$\mathcal{L}^S_{\mathrm{KL},\phi}(\kappa)=D^S(k,\phi\chi).$$

(Here, S is assumed to contain finite places other than p. Hence, the imprimitive Kubota–Leopoldt p-adic L-function does not have poles.)

4.1.2. *Rankin–Selberg p-adic L-function*

Theorem 4.1.2. — *Let $\mathscr{B}_1, \mathscr{B}_2 \subset \operatorname{Spec}(\mathbb{T}_{\mathrm{GL}(2),\mathrm{ord}})$ be two geometrically irreducible components. We assume that $\mathscr{B}_1$ is primitive, i.e., the newforms in the automorphic representations corresponding to classical points of $\mathscr{B}_1$ has tame level equal to $K^p_{\mathrm{GL}(2)}$). Denoting by $F_{\mathscr{B}_1}, F_{\mathscr{B}_2}$ the function fields of $\mathscr{B}_1, \mathscr{B}_2$, there exists*

$$\mathcal{L}^S_{\mathscr{B}_1,\mathscr{B}_2} \in \mathscr{M}eas\left(\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{A}}/U^p, \Lambda_{\mathrm{GL}(2)}\widehat{\otimes}_{\mathcal{O}}\Lambda_{\mathrm{GL}(2)}\right) \otimes_{\Lambda_{\mathrm{GL}(2)}\widehat{\otimes}_{\mathcal{O}}\Lambda_{\mathrm{GL}(2)}} \left(F_{\mathscr{B}_1}\widehat{\otimes}_{\mathcal{O}} F_{\mathscr{B}_2}\right)$$

satisfying the interpolation property: Suppose that $(x_1, x_2) \in \mathscr{B}_1(\overline{\mathbb{Q}}_p) \times \mathscr{B}_2(\overline{\mathbb{Q}}_p)$ is a classical point of weights $t_1 > t_2 \geqslant 2$. Then for an arithmetic character $\kappa = (k, \chi) \in \operatorname{Hom}_{\mathrm{cont}}\left(\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/U^p, \overline{\mathbb{Q}}_p^\times\right)$ such that

$$-t_1 + 1 \leqslant k \leqslant -t_2$$

i.e., $s = k + (t_1 + t_2)/2$ is a critical point for the Rankin–Selberg L-function $L(s, \sigma_{x_1} \times \sigma_{x_2})$, we have

$$\mathcal{L}^S_{\mathscr{B}_1,\mathscr{B}_2}(\kappa, x) = \frac{D^S\left(k + (t_1 + t_2)/2, \sigma_{x_1} \times \sigma_{x_2} \times \chi\right)}{(-2i)^{t_1+1}\mathbf{P}(f_{x_1}, f_{x_1})}, \tag{4.1.1}$$

where σ_{x_j}, $j = 1, 2$, is the (unique) unitary automorphic representation of GL(2) *(with unitary central character) giving rise to the Hecke eigensystem parameterized by x_j (up to a twist by a real power of $|\det|$), and $f_{x_1} \in \sigma_{x_1}$ is the normalized eigenform for the Hecke eigensystem parameterized by x_1. (The modified Petersson inner product $\mathbf{P}(f_{x_1}, f_{x_1})$ is defined as in* (3.3.5)*.)*

This theorem is proved in [Hid88, Th. 5.1d], see also [CH20, Th. A]. Our $\mathcal{L}^S_{\mathscr{B}_1,\mathscr{B}_2}$ are obtained from the p-adic L-functions in loc. cit. by removing L-factors at $v \in S \smallsetminus \{p, \infty\}$ which are p-adically interpolatable and by a change of variable. Note that the classical Petersson norm for the new form inside σ_{x_1} is used in the interpolation formula in loc. cit., while in the above theorem, the modified Petersson norm $\mathbf{P}(f_{x_1}, f_{x_1})$ is used. The relation between the two Petersson norms are proved in [Hsi21, Lem. 3.6], via which one can see that the interpolation formula (4.1.1) follows from that in [Hid88, CH20]. (We also note that there is a slight difference between our convention of nebentypus here and that in loc. cit. If σ_j has central character ω_j, then in our convention the p-nebentypus is $\omega_j|_{\mathbb{Z}_p^\times}$ and in the conventions in loc. cit., the nebentypus sends q_v to $\omega_v(q_v)$, so essentially is $\omega_j^{-1}|_{\mathbb{Z}_p^\times}$.)

4.1.3. *Standard p-adic L-function for Yoshida lifts.* — Let $\mathscr{B}, \mathscr{B}' \subset \operatorname{Spec}(\mathbb{T}_{\mathrm{GL}(2),\mathrm{ord}})$ be two geometrically irreducible components. Let $F_{\mathscr{B}}, F_{\mathscr{B}'}$ be the function fields of $\mathscr{B}, \mathscr{B}'$ and $\mathbb{I}_{\mathscr{B}}, \mathbb{I}_{\mathscr{B}'}$ be the integral closures of $\Lambda_{\mathrm{GL}(2)}$ in $F_{\mathscr{B}}, F_{\mathscr{B}'}$. Denote by

$$\lambda_{\mathscr{B}} : \mathbb{T}_{\mathrm{GL}(2),\mathrm{ord}} \longrightarrow \mathbb{I}_{\mathscr{B}}, \qquad \lambda_{\mathscr{B}'} : \mathbb{T}_{\mathrm{GL}(2),\mathrm{ord}} \longrightarrow \mathbb{I}_{\mathscr{B}'}$$

the corresponding Hecke eigensystems, and

$$\omega_{\mathscr{B}}, \omega_{\mathscr{B}'} : \mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f} \longrightarrow \Lambda^\times_{\mathrm{GL}(2)}$$

the central characters. We fix a square root for each of them
$$\omega_{\mathscr{B}}^{1/2},\omega_{\mathscr{B}'}^{1/2}:\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q},f}^\times\longrightarrow\Lambda_{\mathrm{GL}(2)}^\times.$$
(The equality proved in Proposition 4.2.3 below does not depend on the choice of the square roots.)

We have the group homomorphism
$$\begin{gathered}T^1_{\mathrm{GL}(2)}\times T^1_{\mathrm{GL}(2)}\longrightarrow T^1_{\mathrm{GSp}(4)},\\ \big(\mathrm{diag}(a_1,a_1^{-1}),\mathrm{diag}(a_2,a_2^{-1})\big)\longmapsto\mathrm{diag}(a_1a_2,a_1a_2^{-1},a_1^{-1}a_2^{-1},a_1^{-1}a_2)\end{gathered}$$
which induces
$$\widetilde{\Lambda}_{\mathrm{GL}(2)}\widehat{\otimes}\widetilde{\Lambda}_{\mathrm{GL}(2)}\longrightarrow\widetilde{\Lambda}_{\mathrm{GSp}(4)}.$$
Let
$$\mathbb{I}_{\theta(\mathscr{B},\mathscr{B}')}=\widetilde{\Lambda}_{\mathrm{GSp}(4)}\otimes_{\widetilde{\Lambda}_{\mathrm{GL}(2)}\widehat{\otimes}\widetilde{\Lambda}_{\mathrm{GL}(2)}}(\mathbb{I}_{\mathscr{B}}\widehat{\otimes}_{\mathcal{O}}\mathbb{I}_{\mathscr{B}'}).$$
It follows from the theory of theta lifts [Rob01] that if there exists a finite place $v\neq p$ such that the classical specializations of $\mathscr{B},\mathscr{B}'$ are discrete series at v, then for suitable $K^p_{\mathrm{GSp}(4)}$, there exists a geometrically irreducible component $\theta(\mathscr{B},\mathscr{B}')\subset\mathrm{Spec}(\mathbb{T}_{\mathrm{GSp}(4),\mathrm{ord}})$ with the $\widetilde{\Lambda}_{\mathrm{GSp}(4)}$-algebra homomorphism
$$\lambda_{\theta(\mathscr{B},\mathscr{B}')}:\mathbb{T}_{\mathrm{GSp}(4),\mathrm{ord}}\longrightarrow\mathbb{I}_{\theta(\mathscr{B},\mathscr{B}')}$$
such that the central character equals $\mathbb{Q}^\times\backslash\mathbb{A}_f^\times\xrightarrow{\omega_{\mathscr{B}}}\Lambda^\times_{\mathrm{GL}(2)}\to\Lambda^\times_{\mathrm{GSp}(4)}$ where the second map is induced by $T^1_{\mathrm{GL}(2)}\to T^1_{\mathrm{GSp}(4)}$, $\mathrm{diag}(a,a^{-1})\mapsto\mathrm{diag}(a,a,a^{-1},a^{-1})$, and
$$\begin{aligned}&\lambda_{\theta(\mathscr{B},\mathscr{B}')}\Big(\mathrm{GSp}(4,\mathbb{Z}_v)\Big[\begin{smallmatrix}\varpi_v&&&\\&\varpi_v&&\\&&1&\\&&&1\end{smallmatrix}\Big]\mathrm{GSp}(4,\mathbb{Z}_v)\Big)=\lambda_{\mathscr{B}}\big(\mathrm{GL}(2,\mathbb{Z}_v)\big[\begin{smallmatrix}\varpi_v&\\&1\end{smallmatrix}\big]\mathrm{GL}(2,\mathbb{Z}_v)\big),\\&\lambda_{\theta(\mathscr{B},\mathscr{B}')}\Big(\mathrm{GSp}(4,\mathbb{Z}_v)\Big[\begin{smallmatrix}\varpi_v&&&\\&1&&\\&&1&\\&&&\varpi_v\end{smallmatrix}\Big]\mathrm{GSp}(4,\mathbb{Z}_v)\Big)\\&\qquad\qquad=\omega_{\mathscr{B}}^{1/2}\,\omega_{\mathscr{B}'}^{-1/2}(\varpi_v)\lambda_{\mathscr{B}'}\big(\mathrm{GL}(2,\mathbb{Z}_v)\big[\begin{smallmatrix}\varpi_v&\\&1\end{smallmatrix}\big]\mathrm{GL}(2,\mathbb{Z}_v)\big).\end{aligned}$$
(It follows from the results in [Rob01] that for all classical specializations $\sigma\not\cong\sigma'$ of $\mathscr{B},\mathscr{B}'$ of weights $t>t'\geqslant 2$ such that they are both discrete series at a finite place v and the product of the central characters is a square, there is a nonzero Yoshida lift of $\sigma\boxtimes\sigma'\otimes(\omega_\sigma,\omega_{\sigma'})^{1/2}\circ\det$ to $\mathrm{GSp}(4)$ with Archimedean component isomorphic to a holomorphic discrete series. This implies the existence of the geometrically irreducible component $\theta(\mathscr{B},\mathscr{B}')\subset\mathrm{Spec}(\mathbb{T}_{\mathrm{GSp}(4),\mathrm{ord}})$.)

Theorem 4.1.3. — *Let ϕ be a Dirichlet character unramified outside $S\smallsetminus\{p\}$ with $\phi^2\neq\mathrm{triv}$ and $F_{\theta(\mathscr{B},\mathscr{B}')}=\mathrm{Frac}\big(\mathbb{I}_{\theta(\mathscr{B},\mathscr{B}')}\big)$. There exists*
$$\mu^S_{\theta(\mathscr{B},\mathscr{B}'),\phi}\in\mathscr{M}eas\big(\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/\widehat{\mathbb{Z}}^{p,\times},\mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}}\otimes_{\widetilde{\Lambda}_{\mathrm{GSp}(4)}}\mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}}\big)\otimes_{\widetilde{\Lambda}_{\mathrm{GSp}(4)}}F_{\theta(\mathscr{B},\mathscr{B}')}$$
satisfying the interpolation property: Suppose that $x\in\theta(\mathscr{B},\mathscr{B}')(\overline{\mathbb{Q}}_p)$ is point where the weight projection map $\widetilde{\Lambda}_{\mathrm{GSp}(4)}\to\mathbb{T}_{\mathrm{GSp}(4),\mathrm{ord}}$ is étale and has arithmetic image

$(\ell_1,\ell_2,\xi_1,\xi_2)$ *with* $\ell_1 \geqslant \ell_2 \geqslant 3$. *If* $\kappa = (k,\xi) \in \mathrm{Hom}_{\mathrm{cont}}\big(\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q},f}^\times/U^p, \overline{\mathbb{Q}}_p^\times\big)$ *is an arithmetic point with* $\ell_2 \geqslant k \geqslant 3$ *and* $\phi\chi(-1) = (-1)^k$, *then*

$$\begin{aligned}\mu^S_{\theta(\mathscr{B},\mathscr{B}'),\phi}&(\kappa,x)\\ &= 2^{-\ell_1-\ell_2}\cdot D^S(k-2,\theta(\mathscr{B},\mathscr{B}')_x\times\phi\chi)\sum_{\varphi\in\mathscr{S}_{\mathrm{GSp}(4),x}}\frac{\varphi\boxtimes\varphi}{\mathbf{P}(\varphi,\varphi)}\\ &= 2^{-\ell_1-\ell_2}\cdot D^S(k-2,\phi\chi)\,D^S\big(k-2,\sigma_x\times\sigma'_x\times\omega_{\sigma_x}^{-1/2}\omega_{\sigma'_x}^{-1/2}\phi\chi\big)\sum_{\varphi\in\mathscr{S}_{\mathrm{GSp}(4),x}}\frac{\varphi\boxtimes\varphi}{\mathbf{P}(\varphi,\varphi)},\end{aligned}$$

where σ_x,σ'_x *are the (unique) automorphic representations of* $\mathrm{GL}(2)$ *giving rise to the Hecke eigensystems parameterized by the point in* $\mathscr{B}(\overline{\mathbb{Q}}_p)\times\mathscr{B}'(\overline{\mathbb{Q}}_p)$ *induced by* x *and the natural map* $\mathbb{I}_{\mathscr{B}}\widehat{\otimes}_{\mathcal{O}}\mathbb{I}_{\mathscr{B}'}\to\mathbb{I}_{\theta(\mathscr{B},\mathscr{B}')}$, *the set* $\mathscr{S}_{\mathrm{GSp}(4),x}$ *is an orthogonal basis of the space spanned by ordinary cuspidal holomorphic Siegel modular forms on* $\mathrm{GSp}(4)$ *of weight* (ℓ_1,ℓ_2), *tame level* $K^p_{\mathrm{GSp}(4)}$ *belonging to the Hecke eigenspace parameterized by* x, *or equivalently of* $\theta(\sigma_x,\sigma'_x)$.

Proof. — In [Liu20], the p-adic standard L-functions for ordinary families on symplectic groups (over $\mathbb{Q}$) are constructed by using the doubling method formula of Garrett [Gar84] and Piatetski-Shapiro–Rallis [PSR87]. We apply that construction to the special case of $\theta(\mathscr{B},\mathscr{B}')$ on GSp(4). (The Archimedean zeta integral is left as an uncomputed factor in loc. cit.. It has been calculated in [Liu21] and verified to agree with what is expected according to the conjecture of Coates and Perrin–Riou on p-adic L-functions.) The last factor in the formula for $\mu^S_{\theta(\mathscr{B},\mathscr{B}'),\phi}$ here is slightly different from the formula [Liu20, (7.0.1)] because the interpolation formula is computed by applying $\langle\;,\overline{\varphi}\rangle$ to the specialization. If we apply instead $\mathbf{P}(\;,\varphi)$ to the specialization, we get the above formula. ☐

4.2. Comparison of p-adic L-functions. — Let $\mathscr{C}_1,\mathscr{B},\mathscr{B}'$ be primitive geometrically irreducible components of $\mathrm{Spec}(\mathbb{T}_{\mathrm{GL}(2),\mathrm{ord}})$ such that $\mathrm{Spec}(\mathbb{T}_{\mathrm{GSp}(4),\mathrm{ord}})$ has a geometrically irreducible component $\theta(\mathscr{B},\mathscr{B}')$. By using the results on p-adic L-functions in Section 4.1, we can deduce the following proposition.

Proposition 4.2.1. — *There exists*

$$\begin{aligned}\mathcal{L}^{S,*}_{\mathscr{B},\mathscr{C}_1},&\mathcal{L}^{S,*}_{\mathscr{C}_1,\mathscr{B}'}\\ &\in \mathcal{M}eas\big(\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/U^p,\Lambda_{\mathrm{GL}(2)}\widehat{\otimes}_{\mathcal{O}}\Lambda_{\mathrm{GSp}(4)}\big)\otimes_{\widetilde{\Lambda}_{\mathrm{GL}(2)}\widehat{\otimes}_{\mathcal{O}}\widetilde{\Lambda}_{\mathrm{GSp}(4)}}\big(F_{\mathscr{C}_1}\widehat{\otimes}_{\mathcal{O}}F_{\theta(\mathscr{B},\mathscr{B}')}\big)\end{aligned}$$

and

$$\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}')}\in\big(\mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}}\otimes_{\widetilde{\Lambda}_{\mathrm{GSp}(4)}}\mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}}\big)\otimes_{\widetilde{\Lambda}_{\mathrm{GSp}(4)}}F_{\theta(\mathscr{B},\mathscr{B}')}$$

satisfying the following interpolation properties: In the setting of Theorem 3.3.1 $\mathscr{C}_2=\theta(\mathscr{B},\mathscr{B}')$,

$$\mathcal{L}^{S,*}_{\mathscr{B},\mathscr{C}_1}(\kappa,x)=\frac{D^S\big(k+(\ell+\ell_1+\ell_2)/2,\sigma_x\times\pi_x\times\chi\big)}{(-2i)^{\ell_1+\ell_2-1}\mathbf{P}(h_x,h_x)}$$

$$\mathcal{L}^{S,*}_{\mathscr{C}_1,\mathscr{B}'}(\kappa,x)=\frac{D^S\big(k+(\ell+\ell_1+\ell_2)/2,\pi_x\times\sigma'_x\times\omega_{\sigma_x}^{1/2}\omega_{\sigma'_x}^{-1/2}\chi\big)}{(-2i)^{\ell+1}\mathbf{P}(f_x,f_x)}$$

and

$$\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}')}(x) = 2^{-1} i^{-\ell_1-\ell_2+1}\,\mathbf{P}(h_x,h_x) \sum_{\varphi\in\mathscr{S}_{\mathrm{GSp}(4),x}} \frac{\varphi\boxtimes\varphi}{\mathbf{P}(\varphi,\varphi)}$$

with $\pi_x, f_x \in \pi_x$ *as in Theorem* 3.3.1, σ_x, σ'_x *as in Theorem* 4.1.3, *and* $h_x \in \sigma_x$ *the unique normalized ordinary form fixed by* $K^p_{\mathrm{GL}(2)}$. *(Note that the weights of the Archimedean components of* σ_x, σ'_x *are* $\ell_1+\ell_2-2, \ell_1-\ell_2+2$.*)*

Proof. — Applying pullback and change of variable to the Rankin–Selberg p-adic L-function in Theorem 4.1.2 shows the existence of $\mathcal{L}^{S,*}_{\mathscr{B},\mathscr{C}_1}, \mathcal{L}^{S,*}_{\mathscr{C}_1,\mathscr{B}'}$. In the same way, for a tame Dirichlet character ϕ as in Theorem 4.1.3, we get

$$\mathcal{L}^{S,*}_{\mathscr{B},\mathscr{B}',\phi} \in \mathcal{M}eas\big(\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/\widehat{\mathbb{Z}}^{p,\times}, \mathbb{I}_{\theta(\mathscr{B},\mathscr{B}')} \otimes_{\mathbb{I}_{\theta(\mathscr{B},\mathscr{B}')}} F_{\theta(\mathscr{B},\mathscr{B}')}$$

satisfying the interpolation property: For all points

$$(\kappa, x_2) \in \mathrm{Hom}_{\mathrm{cont}}\big(\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/\widehat{\mathbb{Z}}^{p,\times}, \overline{\mathbb{Q}}_p^\times\big) \times \theta(\mathscr{B},\mathscr{B}')(\overline{\mathbb{Q}}_p)$$

satisfying the conditions in Theorem 4.1.3 (where x_2 denotes the projection of x to $\theta(\mathscr{B},\mathscr{B}')(\overline{\mathbb{Q}}_p)$)

$$\mathcal{L}^{S,*}_{\mathscr{B},\mathscr{B}',\phi}(\kappa,x) = \frac{D^S\big(k-2, \sigma_x\times\sigma'_x\times\omega_{\sigma_x}^{-1/2}\omega_{\sigma'_x}^{-1/2}\chi\phi\big)}{(-2i)^{\ell_1+\ell_2-1}\mathbf{P}(h_x,h_x)}.$$

A change of variable (i.e., μ to $f \mapsto \int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q}}/\widehat{\mathbb{Z}}^{p,\times}} f(x)x^{-2}\,d\mu(x)$) to the Kubota–Leopoldt p-adic L-function recalled in Theorem 4.1.1 gives

$$\mathcal{L}^{S,*}_{\mathrm{KL},\phi} \in \mathcal{M}eas\big(\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{A}}/\widehat{\mathbb{Z}}^{p,\times}, \mathcal{O}\big)$$

such that for κ as as in Theorem 4.1.3

$$\mathcal{L}^{S,*}_{\mathrm{KL},\phi}(\kappa) = D^S(k-2,\chi\phi).$$

The desired $\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}')}$ can be obtained as $(\mathcal{L}^{S,*}_{\mathscr{B},\mathscr{B}',\phi}\mathcal{L}^{S,*}_{\mathrm{KL},\phi})^{-1}\mu^S_{\theta(\mathscr{B},\mathscr{B}'),\phi}$ with $\mu^S_{\theta(\mathscr{B},\mathscr{B}'),\phi}$ as in Theorem 4.1.3. □

Next, we show that we can take the $(\mathbb{S},\mathbf{\Lambda})$-Bessel coefficient on the second factor of $\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}')}$.

Proposition 4.2.2. — *Take nonzero* $\mathcal{H} \in \mathbb{I}_{\theta(\mathscr{B},\mathscr{B}')}$ *such that*

$$\mathcal{H}\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}')} \in \mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}} \otimes_{\widetilde{\Lambda}_{\mathrm{GSp}(4)}} \mathbb{I}_{\theta(\mathscr{B},\mathscr{B}')}.$$

Given $\mathbb{S} = \left[\begin{smallmatrix} \mathbb{a} & \mathbb{b}/2 \\ \mathbb{b}/2 & \mathbb{c}\end{smallmatrix}\right] \in \mathrm{Sym}_2(\mathbb{Q})_{>0}$ *and* $\mathbf{\Lambda} \in \mathrm{Hom}_{\mathrm{cont}}(\mathcal{K}^\times\backslash\mathbb{A}^\times_{\mathcal{K},f}, \Lambda^\times_{\mathrm{GSp}(4)})$ *extending* $\omega_{\theta(\mathscr{B},\mathscr{B}')} = \omega_{\mathscr{B}}$, *where* $\mathcal{K} = \mathbb{Q}(\sqrt{-\det\mathbb{S}})$, *there exists*

$$\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}'),\mathbb{S},\mathbf{\Lambda}} \in \mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}} \otimes_{\widetilde{\Lambda}_{\mathrm{GSp}(4)}} F_{\theta(\mathscr{B},\mathscr{B}')}$$

such that for $x \in \theta(\mathscr{B},\mathscr{B}')(\overline{\mathbb{Q}}_p)$ *as in Proposition* 4.2.1 *and not a pole of* $\mathcal{H}$,

$$\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}'),\mathbb{S},\mathbf{\Lambda}}(x) = 2^{-1} i^{-\ell_1-\ell_2+1}\mathbf{P}(h_x,h_x) \sum_{\varphi\in\mathscr{S}_{\mathrm{GSp}(4),x}} \frac{B^\dagger_{\mathbb{S},\Lambda}(\varphi)\varphi}{\mathbf{P}(\varphi,\varphi)}.$$

Proof. — We follow the method in [HY24, §10.2] to construct the desired $\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}'),\mathbb{S},\mathbf{\Lambda}}$ from the $\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}')}$ in Proposition 4.2.1. Fix $c \geqslant 0$ such that $p^c\alpha_{\mathbb{S}} \in \mathcal{O}_{\mathcal{K}}$ and an open compact subgroup $U_{\mathcal{K}}^p \subset \mathbb{A}_{\mathcal{K}}^{\times,p}$ such that $\mathbf{\Lambda}$ factors through the quotient by $U_{\mathcal{K}}^p$. Given a positive integer n, we let

$$U_{\mathcal{K},p,n} = \mathbb{Z}_p^\times(1+p^{n+c}\mathbb{Z}_p\alpha_{\mathbb{S}}), \qquad U_{\mathcal{K},n} = U_{\mathcal{K}}^p U_{\mathcal{K},p,n},$$

and

$$\rho_n : \widetilde{\Lambda}_{\mathrm{GSp}(4)} \longrightarrow \mathcal{O}\big[\,T^1_{\mathrm{GSp}(4)}(\mathbb{Z}/p^n\mathbb{Z})\,\big]$$

be the natural projection induced by $T^1_{\mathrm{GSp}(4)}(\mathbb{Z}_p) \to T^1_{\mathrm{GSp}(4)}(\mathbb{Z}/p^n\mathbb{Z})$. Put

$$\mathbb{I}_{\theta(\mathscr{B},\mathscr{B}'),n} = \mathbb{I}_{\theta(\mathscr{B},\mathscr{B}')} \otimes_{\widetilde{\Lambda}_{\mathrm{GSp}(4)},\rho_n} \mathcal{O}\big[\,T^1_{\mathrm{GSp}(4)}(\mathbb{Z}/p^n\mathbb{Z})\,\big].$$

Then ρ_n naturally induces $\rho_n : \mathbb{I}_{\theta(\mathscr{B},\mathscr{B}')} \to \mathbb{I}_{\theta(\mathscr{B},\mathscr{B}'),n}$. Taking the q-expansion of the second factor at $\left[\begin{smallmatrix} A & \\ & D\end{smallmatrix}\right] \in \mathrm{GSp}(4,\mathbb{A}_{\mathbb{Q},f})$ and taking the coefficient indexed by $\mathbb{S}$ gives a map

$$\varepsilon_{q\text{-exp},\mathbb{S}}\left(\,\cdot\,,\left[\begin{smallmatrix} A & \\ & D\end{smallmatrix}\right]\right) : \mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}} \otimes_{\widetilde{\Lambda}_{\mathrm{GSp}(4)}} \mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}} \longrightarrow \mathcal{M}_{\mathrm{GSp}(4)}.$$

Define $\Theta_n \in \mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}} \otimes_{\widetilde{\Lambda}_{\mathrm{GSp}(4)}} \mathbb{I}_{\theta(\mathscr{B},\mathscr{B}'),n}$ as

$$\sum_{\mathfrak{z}\in\mathcal{K}^\times\mathbb{A}^\times_{\mathbb{Q},f}\backslash\mathbb{A}^\times_{\mathcal{K},f}/U_{\mathcal{K},n}} \rho_n\left(\lambda^{-n-c}\cdot\mathbf{\Lambda}(\mathfrak{z})^{-1}\,\varepsilon_{q\text{-exp},\mathbb{S}}\left(\mathcal{H}\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}')},\begin{bmatrix}\imath_{\mathbb{S}}(\mathfrak{z}) & \\ & {}^{\mathrm{t}}\imath_{\mathbb{S}}(\bar{\mathfrak{z}})\end{bmatrix}\begin{bmatrix}p^2 &&& \\ & p && \\ && 1 & \\ &&& p\end{bmatrix}_p^{n+c}\right)\right)[\mathfrak{z}],$$

with λ equal to the product of $\omega_{\mathscr{B},p}(p)$ and the eigenvalue of the $\mathbb{U}_p$-operator associated to $\left[\begin{smallmatrix} p &&& \\ & 1 && \\ && p^{-1} & \\ &&& 1\end{smallmatrix}\right]$ corresponding to $\theta(\mathscr{B},\mathscr{B}')$. For $y \in \mathbb{Z}_p$, we have

$$\begin{aligned}&\imath_{\mathbb{S}}(1+p^{n+c}y\alpha_{\mathbb{S}})\\ &\qquad = \begin{bmatrix}1 & p^{n+c}y && \\ & 1 && \\ && 1 & \\ && -p^{n+c}y & 1\end{bmatrix}\begin{bmatrix}(1+p^{n+c}y\alpha_{\mathbb{S}})(1+p^{n+c}y\bar{\alpha}_{\mathbb{S}}) &&& \\ -p^{n+c}y\alpha_{\mathbb{S}}\bar{\alpha}_{\mathbb{S}} & 1 && \\ && 1 & p^{n+c}y\alpha_{\mathbb{S}}\bar{\alpha}_{\mathbb{S}} \\ &&& (1+p^{n+c}y\alpha_{\mathbb{S}})(1+p^{n+c}y\bar{\alpha}_{\mathbb{S}})\end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}&\rho_n\left(\varepsilon_{q\text{-exp},\mathbb{S}}\left(\sum_{y\in\mathbb{Z}/p}\mathcal{H}\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}')},\begin{bmatrix}\imath_{\mathbb{S}}(\mathfrak{z}(1+p^{n+c}\alpha_{\mathbb{S}}y)) & \\ & {}^{\mathrm{t}}\imath_{\mathbb{S}}(\bar{\mathfrak{z}}(1+p^{n+c}\bar{\alpha}_{\mathbb{S}}y))\end{bmatrix}\begin{bmatrix}p^2 &&& \\ & p && \\ && 1 & \\ &&& p\end{bmatrix}_p^{n+1+c}\right)\right)\\
&\quad = \rho_n\left(\varepsilon_{q\text{-exp},\mathbb{S}}\left(\sum_{y\in\mathbb{Z}/p}\mathcal{H}\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}')},\begin{bmatrix}\imath_{\mathbb{S}}(\mathfrak{z}) & \\ & {}^{\mathrm{t}}\imath_{\mathbb{S}}(\bar{\mathfrak{z}})\end{bmatrix}\begin{bmatrix}1 & p^{n+c}y && \\ & 1 && \\ && 1 & \\ && -p^{n+c}y & 1\end{bmatrix}\begin{bmatrix}p^2 &&& \\ & p && \\ && 1 & \\ &&& p\end{bmatrix}_p^{n+1+c}\right)\right)\\
&\quad = \rho_n\left(\varepsilon_{q\text{-exp},\mathbb{S}}\left(\sum_{y\in\mathbb{Z}/p}\mathcal{H}\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}')},\begin{bmatrix}\imath_{\mathbb{S}}(\mathfrak{z}) & \\ & {}^{\mathrm{t}}\imath_{\mathbb{S}}(\bar{\mathfrak{z}})\end{bmatrix}\begin{bmatrix}p^2 &&& \\ & p && \\ && 1 & \\ &&& p\end{bmatrix}_p^{n+c}\begin{bmatrix}1 & y && \\ & 1 && \\ && 1 & \\ && -y & 1\end{bmatrix}\begin{bmatrix}p^2 &&& \\ & p && \\ && 1 & \\ &&& p\end{bmatrix}_p\right)\right)\\
&\quad = \rho_n\left(\lambda_{\mathrm{GSp}(4),2,1}\cdot\varepsilon_{q\text{-exp},\mathbb{S}}\left(\mathcal{H}\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}')},\begin{bmatrix}\imath_{\mathbb{S}}(\mathfrak{z}) & \\ & {}^{\mathrm{t}}\imath_{\mathbb{S}}(\bar{\mathfrak{z}})\end{bmatrix}\begin{bmatrix}p^2 &&& \\ & p && \\ && 1 & \\ &&& p\end{bmatrix}_p^{n+c}\right)\right),\end{aligned}$$

from which it follows that

$$\rho_n(\Theta_{n+1}) = \Theta_n.$$

Therefore, the Θ_n's define an element $\Theta \in \mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}} \otimes_{\widetilde{\Lambda}_{\mathrm{GSp}(4)}} \mathbb{I}_{\theta(\mathscr{B},\mathscr{B}')}$. Notice that for ordinary φ invariant under $\{g \in \mathrm{GSp}(4,\mathbb{Z}_p) : g \bmod p^n \in U_{\mathrm{GSp}(4)}(\mathbb{Z}/p^n)\}$, $B^{\dagger}_{\mathbb{S},\Lambda}(\varphi)$ can be computed with $m_1 = n + c, m_2 = 0$ in (3.3.7). Then from the definition of Θ_n's, we see that up to a scalar, $\mathcal{H}^{-1}\Theta$ gives the desired $\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}'),\mathbb{S},\mathbf{\Lambda}}$. □

Proposition 4.2.3. — *Suppose that $\mathscr{C}_1$ is primitive and $\boldsymbol{f} \in \mathcal{M}_{\mathrm{GL}(2),\mathrm{ord}}$ is the Hida family corresponding to $\mathscr{C}_1$ normalized such that the first Fourier coefficient is* 1. *Then*

$$\mu^S_{\mathscr{C}_1,\theta(\mathscr{B},\mathscr{B}')} = \mathbb{c}\sqrt{\det \mathbb{S}}\, 2^3 i^{-1} \boldsymbol{f}_{\mathbb{c}}\boldsymbol{f} \cdot \mathcal{L}^{S,*}_{\mathscr{B},\mathscr{C}_1} \mathcal{L}^{S,*}_{\mathscr{C}_1,\mathscr{B}'} \cdot \mathcal{F}_{\theta(\mathscr{B},\mathscr{B}'),\mathbb{S},\mathbf{\Lambda}} \tag{4.2.1}$$

Proof. — It suffices to check that in the setting of Theorem 3.3.1 with $\mathscr{C}_2 = \theta(\mathscr{B},\mathscr{B}')$, the evaluations of both sides agree at all (κ, x) with $\ell_1 = \ell_2 = \ell$. It follows from Propositions 4.2.1, 4.2.2 that

$$\begin{aligned}\mathrm{RHS}(\kappa,x) &= \mathbb{c}\sqrt{\det \mathbb{S}}\, 2^3 i^{-1} f_{x,\mathbb{c}} f_x \cdot 2^{-3\ell-1} i^{\ell+1} \\ &\times \frac{D^S(k+3\ell/2, \sigma_x \times \pi_x \times \chi)\, D^S\Big(k+3\ell/2, \pi_x \times \sigma'_x \times \omega^{1/2}_{\sigma_x}\omega^{-1/2}_{\sigma'_x} \times \chi\Big)}{\mathbf{P}(f_x,f_x)} \\ &\qquad\times \sum_{\varphi \in \mathscr{S}_{\mathrm{GSp}(4),x}} \frac{B^{\dagger}_{\mathbb{S},\Lambda}(\varphi)\varphi}{\mathbf{P}(\varphi,\varphi)},\end{aligned}$$

where f_x denotes the specialization of $\boldsymbol{f}$ at x. Note that when $\Pi_x = \theta(\mathscr{B},\mathscr{B}')_x$, the L-function for Π_x decomposes as the product of L-functions for σ_x and $\sigma'_x \otimes \omega^{1/2}_{\sigma_x}\omega^{-1/2}_{\sigma'_x} \circ \det$, and we have

$$D^S(s, \Pi_x \times \pi_x \times \chi) = D^S(s, \sigma_x \times \pi_x \times \chi)\, D^S\big(s, \pi_x \times \sigma'_x \times \omega^{1/2}_{\sigma_x}\omega^{-1/2}_{\sigma'_x} \times \chi\big).$$

Hence,

$$\begin{aligned}&\mathrm{RHS}(\kappa,x) \\ &= \mathbb{c}\sqrt{\det \mathbb{S}}\, 2^{-3\ell+2} i^{\ell} \cdot D^S\Big(k + \frac{3\ell}{2}, \Pi_x \times \pi_x \times \chi\Big) \cdot \frac{f_{x,\mathbb{c}} f_x}{\mathbf{P}(f_x,f_x)} \sum_{\varphi \in \mathscr{S}_{\mathrm{GSp}(4),x}} \frac{B^{\dagger}_{\mathbb{S},\Lambda}(\varphi)\varphi}{\mathbf{P}(\varphi,\varphi)}\end{aligned}$$

which equals exactly $\mu^S_{\mathscr{C}_1,\theta(\mathscr{B},\mathscr{B}')}$ by the formula in Theorem 3.3.1. □

4.3. The four-variable p-adic L-function and its interpolation formula II

With Proposition 4.2.3, we can deduce a formula for the Archimedean zeta $I_\infty(k, \mathcal{D}_{\ell_1,\ell_2}, \mathcal{D}_\ell, \Lambda_\infty)$ appearing in the interpolation formula in Theorem 3.3.1 and finish the proof of Theorem 1.1.

Proof of Theorem 1.1.. — We choose $K^p_{\mathrm{GL}(2)}, K^p_{\mathrm{GSp}(4)}$ and primitive Hida families $\mathscr{B}, \mathscr{B}'$ of tame level $K^p_{\mathrm{GL}(2)}$ such that $\mathrm{Spec}(\mathbb{T}_{\mathrm{GSp}(4),\mathrm{ord}})$ has an irreducible component $\theta(\mathscr{B},\mathscr{B}')$. With such a choice, we can further choose $\mathbb{S}$ and $\mathbf{\Lambda}$ such that $\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}'),\mathbb{S},\mathbf{\Lambda}} \neq 0$. (By the interpolation property of $\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}'),\mathbb{S},\mathbf{\Lambda}}$ in Proposition 4.2.2, to show the existence of such a $\mathbb{S}$ and $\mathbf{\Lambda}$, it suffices to show that there exists x satisfying the conditions there for which $B^{\dagger}_{\mathbb{S},\Lambda}(\varphi) \neq 0$ for some $\varphi \in \mathscr{S}_{\mathrm{GSp}(4),x}$. Take an x with corresponding weight (ℓ_1,ℓ_2), $\ell_1 \gg \ell_2 \gg 0$ and $\varphi \in \mathscr{S}_{\mathrm{GSp}(4),x}$. One can choose $\mathbb{S},\Lambda$ such that the usual Bessel period $B_{\mathbb{S},\Lambda}(\varphi)\neq 0$. Then by [Liu23, Prop. 2.7.1], we know that $B^{\dagger}_{\mathbb{S},\Lambda}(\varphi)\neq 0$.) Then we can choose the finite set S such that the conditions in Section 3.1 is satisfied. By the primitivity of $\mathscr{C}_1$ and our assumption on $K^p_{\mathrm{GL}(2)}$ made at the beginning of Section 4.1, $\boldsymbol{f}_{\mathbb{c}}\neq 0$. Hence, both sides of the identity in Proposition 4.2.3 are nonzero elements in $\mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}}\otimes_{\widetilde{\Lambda}_{\mathrm{GSp}(4)}} F_{\theta(\mathscr{B},\mathscr{B}')}$. (There are many interpolation points corresponding to s belonging to the absolute convergence range, at which one can check that the evaluations are nonzero.)

Let (κ,x) be a point of $\mathrm{Hom}_{\mathrm{cont}}\big(\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q},f}/U^p,\overline{\mathbb{Q}}_p^\times\big)\times\mathscr{C}_1(\overline{\mathbb{Q}}_p)\times\theta(\mathscr{B},\mathscr{B}')(\overline{\mathbb{Q}}_p)$ as in Theorem 3.3.1 with $\mathscr{C}_2=\theta(\mathscr{B},\mathscr{B}')$. Then, for the evaluations of both sides of (4.2.1) at (κ,x), we have

$$\mathrm{RHS}(\kappa,x)=\mathbb{c}\sqrt{\det\mathbb{S}}\,2^3 i^{-1}f_{x,\mathbb{c}}f_x\cdot 2^{-\ell-\ell_1-\ell_2-1}i^{\ell+1}\sum_{\varphi\in\mathscr{S}_{\mathrm{GSp}(4),x}}\frac{B^{\dagger}_{\mathbb{S},\Lambda}(\varphi)\varphi}{\mathbf{P}(\varphi,\varphi)}$$
$$\times\frac{D^S\left(k+\frac{\ell+\ell_2+\ell_2}{2},\sigma_x\times\pi_x\times\chi\right)D^S\Big(k+\frac{\ell+\ell_1+\ell_2}{2},\pi_x\times\sigma'_x\times\omega^{1/2}_{\sigma_x}\omega^{-1/2}_{\sigma'_x}\times\chi\Big)}{\mathbf{P}(f_x,f_x)}$$

by the interpolation properties of $\mathcal{L}^{S,*}_{\mathscr{B},\mathscr{C}_1},\mathcal{L}^{S,*}_{\mathscr{C}_1,\mathscr{B}'},\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}'),\mathbb{S},\boldsymbol{\Lambda}}$, and

$$\mathrm{LHS}(\kappa,x)=i^{r_{\Lambda,1}-r_{\Lambda,2}}f_{x,\mathbb{c}}f_x\cdot\frac{I_\infty(k,\mathcal{D}_{\ell_1,\ell_2},\mathcal{D}_\ell,\Lambda_\infty)}{E_\infty\Big(k+\frac{\ell+\ell_1+\ell_2}{2},\mathcal{D}_{\ell_1,\ell_2}\times\mathcal{D}_\ell\times\chi\Big)}\sum_{\varphi\in\mathscr{S}_{\mathrm{GSp}(4),x}}\frac{B^{\dagger}_{\mathbb{S},\Lambda}(\varphi)\varphi}{\mathbf{P}(\varphi,\varphi)}$$
$$\times\frac{D^S\left(k+\frac{\ell+\ell_2+\ell_2}{2},\sigma_x\times\pi_x\times\chi\right)D^S\Big(k+\frac{\ell+\ell_1+\ell_2}{2},\pi_x\times\sigma'_x\times\omega^{1/2}_{\sigma_x}\omega^{-1/2}_{\sigma'_x}\times\chi\Big)}{\mathbf{P}(f_x,f_x)}$$

by Theorem 3.3.1. It follows that

$$\begin{aligned}(4.3.1)\quad & I_\infty(k,\mathcal{D}_{\ell_1,\ell_2},\mathcal{D}_\ell,\Lambda_\infty)\\ &=\mathbb{c}\sqrt{\det\mathbb{S}}\,2^{-\ell-\ell_1-\ell_2+2}i^{\ell-r_{\Lambda,1}+r_{\Lambda,2}}\cdot E_\infty\Big(k+(\ell+\ell_1+\ell_2)/2,\mathcal{D}_{\ell_1,\ell_2}\times\mathcal{D}_\ell\times\chi\Big),\end{aligned}$$

for all (k,ℓ,ℓ_1,ℓ_2) which equals the algebraic part of the projection of an arithmetic (κ,x) to the weight space such that x is not a pole of either side of(4.2.1) and $\mathrm{RHS}(\kappa,x)$ and $\mathrm{LHS}(\kappa,x)$ are nonzero.

Since we have made the choices such that both sides of (4.2.1) are nonzero, the points for which the weight projection map (3.1.1) is not étale at x or $\mathrm{LHS}(\kappa,x)=\mathrm{LHS}(\kappa,x)=0$ are not Zariski dense. For any (k,ℓ,ℓ_1,ℓ_2) satisfying (1.2), the classical points (κ,x) whose projections to the weight space has algebraic part equal to (k,ℓ,ℓ_1,ℓ_2) are Zariski dense, so there exist (κ,x) which satisfies the conditions for the above comparison to deduce (4.3.1) for the given (k,ℓ,ℓ_1,ℓ_2). Thus, (4.3.1) is true for all (k,ℓ,ℓ_1,ℓ_2) satisfying (1.2). Plugging it into the interpolation formula in Theorem 3.3.1 shows that

$$\mathcal{L}^S_{\mathscr{C}_1,\mathscr{C}_2,\beta_1,\beta_2}=\big(\mathbb{c}\sqrt{\det\mathbb{S}}\big)^{-1}2^{-2}\cdot\varepsilon_{q\text{-exp},\beta_1,\beta_2}\left(\mu^S_{\mathscr{C}_1,\mathscr{C}_2}\right)$$

is the desired p-adic L-function. ☐

REMARK 4.3.1. — With (1.2), the interpolation formula for the one-variable cyclotomic p-adic L-function $\mathcal{L}^S_{\Pi,\pi}$ in [Liu23, Th. 1.0.1] with an uncomputed Archimedean zeta integral becomes

$$\mathcal{L}^S_{\Pi,\pi}\big((\chi|\cdot|^k)_{p\text{-adic}}\big) = \mathbb{c}\sqrt{\det\mathbb{S}}\,2^{-\ell-\ell_1-\ell_2+2}i^{\ell-r_{\Lambda,1}+r_{\Lambda,2}}\cdot\frac{B^{\dagger}_{\mathbb{S},\Lambda}\big(\varphi_{\mathrm{ord}}\big)\,W_{\mathbb{c}}(f_{\mathrm{ord}})}{\mathbf{P}(\varphi_{\mathrm{ord}},\varphi_{\mathrm{ord}})\,\mathbf{P}(f_{\mathrm{ord}},f_{\mathrm{ord}})}$$
$$\times\begin{cases} E_\infty\big(k,\widetilde{\Pi}\times\widetilde{\pi}\times\chi\big)E_p\big(k,\widetilde{\Pi}\times\widetilde{\pi}\times\chi\big)\cdot L^S\big(k,\widetilde{\Pi}\times\widetilde{\pi}\times\chi\big), & \ell_1+\ell_2+\ell\text{ even},\\ E_\infty\big(k+1/2,\widetilde{\Pi}\times\widetilde{\pi}\times\chi\big)E_p\big(k+1/2,\widetilde{\Pi}\times\widetilde{\pi}\times\chi\big)\cdot L^S\big(k+1/2,\widetilde{\Pi}\times\widetilde{\pi}\times\chi\big), & \ell_1+\ell_2+\ell\text{ odd},\end{cases}$$

with (k,χ) satisfying the conditions in loc. cit.

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