Journal de l'École polytechnique
Mathématiques

Dawei Chen, Quentin Gendron, Miguel Prado, & Guillaume Tahar

Isoresidual curves

Tome 13 (2026), p. 891-963.

https://doi.org/10.5802/jep.339

Publié avec le soutien
du Centre National de la Recherche Scientifique

Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 2270-518X

Journal de l'École polytechnique
Mathématiques
Tome 13, 2026, p. 891–963 DOI: 10.5802/jep.339

ISORESIDUAL CURVES

BY DAWEI CHEN, QUENTIN GENDRON, MIGUEL PRADO
& GUILLAUME TAHAR

ABSTRACT. — Given a partition μ of -2, the stratum $\mathcal{H}(\mu)$ parametrizes meromorphic differential one-forms on the Riemann sphere $\mathbb{CP}^1$ with n zeros and p poles of orders prescribed by μ. The isoresidual fibration is defined by assigning to each differential in $\mathcal{H}(\mu)$ its configuration of residues at the poles. In the case of differentials with $n = 2$ zeros, generic isoresidual fibers are complex curves endowed with a canonical translation structure, which we describe extensively in this paper. Quantitative characteristics of the translation structure on isoresidual fiber curves provide rich discrete invariants for these fibers. We determine the Euler characteristic of generic isoresidual fiber curves from intersection-theoretic computations, we describe a wall and chamber structure for the Euler characteristic of generic isoresidual fiber curves in terms of the partition μ, and we classify the connected components of generic isoresidual fibers for strata in genus zero with an arbitrary number of zeros.

RÉSUMÉ (Courbes isorésiduelles). — Étant donnée une partition μ de -2, la strate $\mathcal{H}(\mu)$ paramétrise les formes différentielles méromorphes de degré 1 sur la sphère de Riemann $\mathbb{CP}^1$, possédant n zéros et p pôles dont les ordres sont prescrits par μ. La fibration isorésiduelle est définie en associant à chaque différentielle de $\mathcal{H}(\mu)$ sa configuration de résidus aux pôles. Dans le cas des différentielles possédant $n = 2$ zéros, les fibres isorésiduelles génériques sont des courbes complexes munies d'une structure de translation canonique, que nous décrivons en détail dans cet article. Les caractéristiques quantitatives de la structure de translation sur les courbes isorésiduelles fournissent de riches invariants discrets de ces fibres. Nous déterminons la caractéristique d'Euler des courbes fibres isorésiduelles génériques à partir de calculs de nombres d'intersection, nous décrivons une structure en murs et chambres pour la caractéristique d'Euler des courbes isorésiduelles génériques en fonction de la partition μ, et nous classifions les composantes connexes des fibres isorésiduelles génériques pour les strates de genre 0 possédant un nombre arbitraire de zéros.

MATHEMATICAL SUBJECT CLASSIFICATION (2020). — 30F30, 32G15, 57M50.
KEYWORDS. — Isoresidual fibration, translation surfaces, multi-scale compactification, resonance arrangement, Gauss–Manin connection.

Research by D.C. is supported in part by NSF grant DMS-2301030, Simons Travel Support for Mathematicians, and a Simons Fellowship under Record ID SFI-MPS-SFM-00005694. D.C. thanks BIMSA and ICBS2024 for their invitation and hospitality, where part of the work was completed. Research by Q.G. is supported in part by the Grant PAAPIT UNAM-DFG DA100124 "Conectividad y conectividad simple de los estratos". Research by M.P. is supported by the DFG-UNAM project MO 1884/3-1 and the Collaborative Research Centre TRR 326 "Geometry and Arithmetic of Uniformized Structures." Research by G.T. is supported by the Beijing Natural Science Foundation (Grant IS23005) and the French National Research Agency under the project TIGerS (ANR-24-CE40-3604).

Contents

1. Introduction

For any family of positive integers $b_1, \dots, b_p$, we denote by $\Omega(b_1, \dots, b_p)$ the moduli space of meromorphic one-forms on the Riemann sphere $\mathbb{CP}^1$ with (labeled) poles of orders $b_1, \dots, b_p$ (up to biholomorphisms). The moduli space $\Omega(b_1, \dots, b_p)$ is stratified according to the orders $a_1, \dots, a_n$ of the (labeled) zeros of differential forms. Each stratum $\mathcal{H}(\mu)$ is characterized by a partition $\mu = (a_1, \dots, a_n, -b_1, \dots, -b_p)$ of -2 with $a_1, \dots, a_n \geqslant 1$. It is known that $\mathcal{H}(\mu)$ is a complex-analytic orbifold of dimension $n + p - 2$, whose underlying coarse space is a quasi-projective variety.

For each $p \geqslant 2$, we define the *residual space* $\mathcal{R}_p$ to be the complex vector space formed by the vectors $\lambda = (\lambda_1, \dots, \lambda_p)$ such that $\sum_{j=1}^{p} \lambda_j = 0$. Each moduli space $\Omega(b_1, \dots, b_p)$ is endowed with a *residual map*

$$\mathrm{res}\colon \Omega(b_1, \dots, b_p) \longrightarrow \mathcal{R}_p : \quad \omega \longmapsto (\mathrm{Res}_1\, \omega, \dots, \mathrm{Res}_p\, \omega)$$

that assigns to each differential ω the sequence of its residues at the poles. This map defines the *isoresidual fibration* for the entire moduli space and restricts to each stratum $\mathcal{H}(\mu)$.

The in-depth study of these maps began in [GT21], where the image of res is described for every stratum $\mathcal{H}(\mu)$ and in every genus. Then, in [GT22], the case of the *minimal strata* in genus zero, i.e., strata with a unique zero, was studied from a flat geometric perspective by two of the authors. The other two authors examined the same cases in [CP25] from an algebro-geometric point of view. In that case, these works show that the isoresidual fibration of such strata is a finite cover ramified over the *resonance arrangement* of the residual space.

Definition 1.1. — Let I be a nonempty proper subset of $\{1, \dots, p\}$. The *resonance hyperplane* A_I is the subspace defined by the equation $\sum_{i \in I} \lambda_i = 0$ in the residual space $\mathcal{R}_p$. The union of all the resonance hyperplanes is the *resonance arrangement* $\mathcal{A}_p \subset \mathcal{R}_p$.

In the case of the minimal stratum, the generic degree of the isoresidual cover has been computed in Theorem 1.2 of [GT22] as

$$f(a,p) := \frac{a!}{(a+2-p)!}, \tag{1.1}$$

where $a = -2+\sum_{j=1}^{p} b_j$ is the order of the unique zero. The cardinality of an arbitrary fiber over the resonance arrangement in $\mathcal{R}_p$ is given in [CP25, Th. 1.2].

It is worth noting that this isoresidual cover has recently appeared in various other contexts, such as the dynamics of polynomial maps in [Sug17], the topology of configuration spaces in [Sal25], and the Kadomtsev–Petviashvili hierarchy in [BR24]. In higher genus, the analog of the isoresidual fibration is the isoperiodic foliation which has been extensively described for the stratum $\mathcal{H}(1,1,-2)$ in [FTZ23] (see [KLS21, CD25, BG25] for related questions).

The goal of the present paper is to combine flat geometric and algebro-geometric approaches to study the isoresidual fiber in the strata of differentials on the Riemann sphere with $n=2$ zeros.

1.1. Main results. — We fix a stratum $\mathcal{H}(\mu)$ of differentials on $\mathbb{CP}^1$ with n zeros and p poles, where the orders of the singularities are prescribed by μ. For a given residue configuration $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathcal{R}_p$, we denote by $\mathcal{F}_\lambda$ the isoresidual fiber in the stratum $\mathcal{H}(\mu)$ parametrizing differentials with the residue configuration λ. Note that $\mathcal{F}_\lambda$ depends on μ, but we do not indicate it in order to keep the notation simple. We further denote by $\overline{\mathcal{F}}_\lambda$ the closure of $\mathcal{F}_\lambda$ in the multi-scale compactification of $\mathcal{H}(\mu)$ (see Section 3).

Deformation of a differential in the stratum $\mathcal{H}(\mu)$ means changing the periods of the differential along (relative) homology classes. Inside an isoresidual fiber, the absolute periods (i.e., the residues at the poles) are fixed, so the only degree of freedom is from the relative periods between the zeros. In the case of $n=2$ zeros, this relative period serves as a local coordinate defining the translation structure on the isoresidual fiber, and its differential gives rise to the canonical one-form ω_λ on $\mathcal{F}_\lambda$.

Our first main result describes the translation structure of the one-dimensional isoresidual fibers of the strata with $n=2$ zeros outside the resonance arrangement $\mathcal{A}_p$. We denote by $a = a_1 + a_2$ the total order of the two zeros.

Theorem 1.2. — *For a stratum $\mathcal{H}(a_1, a_2, -b_1, \dots, -b_p)$ and a configuration $\lambda \in \mathcal{R}_p \smallsetminus \mathcal{A}_p$, the closure $\overline{\mathcal{F}}_\lambda$ of the isoresidual fiber $\mathcal{F}_\lambda$ is a (possibly disconnected) compact Riemann surface endowed with a meromorphic one-form ω_λ. Additionally, ω_λ induces a translation structure on $\overline{\mathcal{F}}_\lambda$ that satisfies the following properties:*

(1) *$\mathcal{F}_\lambda$ coincides with the locus where ω_λ has neither a zero nor a pole;*

(2) *ω_λ has $a!/(a+2-p)!$ zeros, each of order a;*

(3) *ω_λ always has poles whose orders and residues are described in Sections 4.3 and 4.4;*

(4) *$(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ has finitely many saddle connections;*

(5) *each saddle connection of* $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ *has a period of the form* $\sum_{i\in I}\lambda_i$*, where* I *is a nonempty proper subset of* $\{1,\dots,p\}$*;*

(6) *if* $\mathcal{F}_\lambda$ *is disconnected, then all the components of* $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ *belong to the same stratum component;*

(7) *the stratum component containing* $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ *is independent of* λ*.*

Let us explain some parts of the theorem in the language of flat geometry. The zeros of ω_λ correspond to elements of the minimal stratum when the two zeros merge to form a single zero of order a. Hence, all the zeros of ω_λ have the same order, and there are $f(a,p)$ of them. The poles of ω_λ correspond to degenerations where the two zeros move infinitely far from each other in the flat metric of ω. The description of the number, orders, and residues of all the poles of a given isoresidual curve requires combinatorial work, which is done in Section 5.1. The list of possible poles of ω_λ crucially depends on the location of the partition μ in the following *singularity pattern space* $\mathcal{SP}_p$.

Definition 1.3. — Given $p\geqslant 1$, the *singularity pattern space* is the positive orthant
$$\mathcal{SP}_p=\Big\{(x_1,x_2,y_1,\dots,y_p)\in\mathbb{R}^{p+2}_{>0}\mid x_1+x_2=\textstyle\sum_{j=1}^p y_j\Big\}$$
of real dimension $p+1$, endowed with the family $\mathcal{W}_p$ of hyperplanes
$$\begin{aligned}W_1(I)&=\big\{\beta_1:=x_1-\textstyle\sum_{i\in I}y_i=0\big\},\\ W_2(I)&=\big\{\beta_2:=x_2-\textstyle\sum_{i\in I}y_i=0\big\},\end{aligned}\tag{1.2}$$
$$W_3(I,K,L)=\big\{\beta_3:=x_1-\textstyle\sum_{i\in I\cup L}y_i-|K\smallsetminus L|=0\big\},\tag{1.3}$$
$$W_4(J,K,M)=\big\{\beta_4:=x_2-\textstyle\sum_{i\in J\cup M}y_i-|K\smallsetminus M|=0\big\},\tag{1.4}$$
where $I\sqcup J\sqcup K$ is a partition of the index set of the poles $\{1,\dots,p\}$ into three disjoint subsets, and L and M are (possibly intersecting) arbitrary subsets of K.

Each partition μ determines a point $v_\mu=(a_1+1,a_2+1,b_1,\dots,b_p)\in\mathcal{SP}_p$. The sum of the orders of the poles of ω_λ is given by a complicated combinatorial formula that depends on the chamber of $(\mathcal{SP}_p,\mathcal{W}_p)$ containing v_μ. Nevertheless, the Euler characteristic of $\overline{\mathcal{F}}_\lambda$ possesses the following structure.

Theorem 1.4. — *For* $\lambda\in R_p\smallsetminus\mathcal{A}_p$ *and* $(a_1+1,a_2+1,b_1,\dots,b_p)$ *in each chamber of* $\mathcal{SP}_p$*, the Euler characteristic of* $\overline{\mathcal{F}}_\lambda$ *is a sum of homogeneous components of degree from* 0 *up to* $p-1$ *in terms of the variables* $a_1+1,a_2+1,b_1,\dots,b_p$*.*

In Proposition 5.1, we provide a more detailed description of the above theorem. Moreover, in Conjecture 5.3, we speculate that the main term in the formula of the Euler characteristic is always a homogeneous polynomial of degree exactly $p-1$ in the variables $a_1+1,a_2+1,b_1,\dots,b_p$. We check this conjecture in a specific chamber in Proposition 5.4.

In the case where all poles are simple, i.e., $b_i=1$ for all i, we denote by $\mu=(a_1,a_2,[-1]^{a+2})$ the partition $(a_1,a_2,-1,\dots,-1)$. The Euler characteristic of the generic isoresidual fiber is given as follows in that case.

Theorem 1.5. — *For $\mu=(a_1,a_2,[-1]^{a+2})$ with $a=a_1+a_2$ and $\lambda\in\mathcal{R}_p\smallsetminus\mathcal{A}_p$, we have*

$$2g\left(\overline{\mathcal{F}}_\lambda\right)-2=a!\Big(a-\frac{(a+2)(a+1)}{(a_1+1)(a_2+1)}\Big). \tag{1.5}$$

Note that, in general, we will show in part (2) of Theorem 1.7 that the generic isoresidual fiber of $\mathcal{H}(a_1,a_2,[-1]^{a+2})$ is disconnected if and only if a_1 and a_2 are both even. In that case, the generic isoresidual fiber has two connected components, and (1.5) gives the arithmetic genus of the disjoint union of these two components.

For general μ, although there is no simple formula to deduce the pole orders of ω_λ from μ for the corresponding generic isoresidual fiber, we can still describe some general properties.

Theorem 1.6. — *All the poles of $(\overline{\mathcal{F}}_\lambda,\omega_\lambda)$ are simple if and only if $\mu=(a_1,a_2,[-1]^{a+2})$.*

For $p\geqslant 3$, all the singularities of $(\overline{\mathcal{F}}_\lambda,\omega_\lambda)$ are of even orders if and only if $\mu=(2a_1,2a_2,-2b_1,\dots,-2b_p)$.

If all the orders of singularities of the differential ω_λ are even, then the stratum containing it is usually disconnected (as shown in [Boi15]). In such a case, it would be interesting to determine which connected component of the stratum contains the isoresidual fiber.

Every stratum $\mathcal{H}(\mu)$ of meromorphic differentials in genus zero is connected. However, the generic isoresidual fibers can be disconnected in certain cases. There are two families of such disconnected isoresidual fibers, related to the topological invariants of strata of translation and dilation surfaces of higher genus (see [Boi15, ABW23]).

Theorem 1.7. — *In strata of meromorphic one-forms on the Riemann sphere with $n\geqslant 2$ zeros, the generic isoresidual fibers are connected except for the following two families of strata:*

(1) $\mathcal{H}(ka_1,\dots,ka_n,-kb_1,\dots,-kb_{p-2},-1,-1)$ *for some $k\geqslant 2$ and $a_1,\dots,a_n$, $b_1,\dots,b_{p-2}$ being positive coprime integers, in which case the generic isoresidual fibers have k connected components;*

(2) $\mathcal{H}(2a_1,\dots,2a_n,-2b_1,\dots,-2b_{p-2g},[-1]^{2g})$ *with $g\geqslant 2$, where generic isoresidual fibers have two connected components.*

Besides the singularity pattern of $(\overline{\mathcal{F}}_\lambda,\omega_\lambda)$, the dependency of the periods of ω_λ on the underlying configuration of the residue tuple $\lambda=(\lambda_1,\dots,\lambda_p)$ provides an additional discrete invariant. It is shown in Sections 4.3 and 4.4 that the poles of ω_λ (whose orders are already discrete invariants) have residues that are linear combinations of $\lambda_1,\dots,\lambda_p$ with integer coefficients. These integer coefficients are also discrete invariants.

The same statement holds, in fact, for the period of every relative homology class in $H_1(X\smallsetminus P_\omega,Z_\omega)$, where Z_ω and P_ω are the sets of the zeros and poles of ω, respectively. It follows that the monodromy of the isoresidual fibration around resonance hyperplanes is encapsulated in a *Gauss–Manin system*, whose fiber is the relative homology group of the isoresidual fiber. The integer coefficients appearing in the formula of the periods form a *period central charge* that commutes with the monodromy of the fibration (see Section 7 for details).

Remark 1.8. — In the case of strata with $n \geqslant 3$ zeros, relative periods between the zeros still provide convenient local coordinates on isoresidual fibers and define a geometric structure that can serve as a reasonable higher-dimensional analog of a translation surface structure. Similarly to our computation of the Euler characteristic of one-dimensional isoresidual fibers, a geometric structure on higher-dimensional isoresidual fibers can provide a way to compute Chern and Hodge numbers of the (suitably compactified) fibers in terms of local invariants of the singular locus.

As far as we know, there is no universally accepted notion of translation manifolds yet. Polyhedral Kähler manifolds introduced in [Pan09] have good geometric properties, but they are usually defined inductively by gluing polyhedra. In contrast, isoresidual fibers directly arise from complex-analytic data. What is still missing is a natural category of objects that possess the two-faceted quality of translation surfaces: a complex-analytic definition (complex structure with a differential) and an equivalent geometric definition (translation atlas with local models for singularities).

1.2. Organization of the paper

– In Section 2, we review the standard background on translation structures, meromorphic differentials, period coordinates on strata of differentials, and the classification of connected components of strata. We also review known counting formulas for differentials with zero residues and recall the results of [GT22] and [CP25] regarding the isoresidual fibration in the case of strata of differentials in genus zero with a unique zero.

– In Section 3, we review the multi-scale compactification of strata and describe the closure of subspaces given by linear equations between the residues. In particular, we describe the boundary points of the generic isoresidual curves.

– In Section 4, we describe the canonical differential on the closure of the generic isoresidual curves whose translation structure is induced by the period atlas. In particular, we detail the local invariants of the singularities of the translation structure of isoresidual fibers in terms of the degeneration of the parametrized objects in the multi-scale compactification. We also provide a more precise description of their geometry in the case of isoresidual fibers over configurations of real residues. These results are summarized in Theorem 1.2, which is formally proved in Section 4.8.

– In Section 5, we analyze the wall and chamber structure for the expression of the Euler characteristic of generic isoresidual fibers. In particular, we prove Theorem 1.4 which shows that the Euler characteristic consists of homogeneous components with respect to the chamber structure described in Definition 1.3.

– In Section 6, we provide an alternative computation for the Euler characteristic of generic isoresidual fibers through intersection theory on the multi-scale compactification. We explicitly compute the Euler characteristic in the case where all poles are simple, proving Theorem 1.5.

– In Section 7, we construct a Gauss–Manin system associated with the isoresidual fibration. We discuss the linear dependence of the periods of the translation structure in terms of the underlying configuration of residues, encapsulated in a period central charge, which serves as an arithmetic invariant of the fibers and commutes with the monodromy of the fibration.

– In Section 8, we provide a classification of the connected components of one-dimensional generic isoresidual fibers, using rotation numbers and parity of spin structures. Drawing on the incidence relations between strata, we prove in Theorem 1.7 the complete classification of connected components of generic isoresidual fibers for strata in genus zero with an arbitrary number of zeros.

– In Section 9, we establish arithmetic relations between the singularity pattern of a stratum $\mathcal{H}(\mu)$ and the singularity pattern of its generic isoresidual fibers $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$, with a particular focus on proving Theorem 1.6.

Acknowledgements. — The authors thank Gianluca Faraco, Samuel Grushevsky, Myeongjae Lee, Martin Möller, and Scott Mullane for inspiring discussions on relevant topics. The authors are also grateful to the anonymous referee for their valuable comments, which have contributed to improving the rigor and clarity of the paper.

2. Meromorphic differentials and translation structures

2.1. Translation structures. — The details of the constructions presented in this section and the following one can be found in [Zor06, AM24, BCG+19a].

2.1.1. *The regular locus*. — On a compact Riemann surface X endowed with a meromorphic one-form ω, we denote by X^* the complement of the zeros and poles of ω in X. Local primitives of ω are injective around each point of X^*, providing an atlas of $\mathbb{C}$-valued local coordinates. Since local primitives of the same differential differ by a constant, the transition maps of this atlas are translations of the complex plane.

Conversely, such an atlas defines a complex structure on X^*, and the pullbacks of the one-form dz on $\mathbb{C}$ via local charts globalize to a holomorphic differential ω on X^*.

In a translation surface, for each slope θ in the circle $\mathbb{S}^1$ of directions, there is a foliation Fol_θ of the surface, where the leaves are locally conjugated to oriented lines of slope θ in the translation charts.

2.1.2. *Local models for singularities*. — At a zero of order a, a differential is locally represented by pulling back dz via a branched cover of degree $a+1$ of a disk, where the cover is totally ramified over the center of the disk. Consequently, the metric induced by the differential exhibits a conical singularity of angle $(a+1)2\pi$.

Before presenting local models for the poles of a meromorphic differential, we recall the following convention. The *residue* of a differential ω at a pole is defined as the period of ω over a positively oriented simple loop around the pole. This convention differs from the usual one by a factor of $2\pi\mathrm{i}$. Since our approach emphasizes periods rather than coefficients, this convention is more suitable. It is important to note that the residue is a local invariant of a pole. In particular, for a differential on $\mathbb{CP}^1$, if the residue at every pole is real, then the period of any closed loop is also real.

In a neighborhood of a simple pole, the differential ω is of the form rdz/z, where $2\pi ir$ is the residue at the pole. Its geometric interpretation is that of a semi-infinite cylinder, where the waist curves have a period equal to the residue $2\pi ir$ (see [Boi15, §2.2]).

A *flat cone of type* $a \geqslant 0$ is a flat surface associated with the differential ω on $\mathbb{P}^1$, which has a unique zero of order a and a unique pole of order $a+2$. In the case where $a = 0$, the flat cone has no conical singularity and features a unique pole of order 2, corresponding to the flat plane where the pole of order two is located at infinity.

The neighborhood of a pole of order $b > 1$ with trivial residue is the complement of a compact neighborhood around the conical singularity of the flat cone of type $a - 2$.

To construct local models for poles of order $b > 1$ with nontrivial residue, we start with a flat cone of type $a - 2$ and remove an ε-neighborhood of a semi-infinite line extending from the conical singularity, along with a neighborhood of the conical singularity itself. We then identify the resulting boundaries of the neighborhood using an isometry. By rotating and rescaling, we obtain a pole of order b with any nonzero residue. This construction is also explained in detail in [Boi15].

2.1.3. *Topological index of a loop in a translation surface*. — Directions are preserved by translations, so the usual winding number can be generalized from the flat plane to translation surfaces.

Definition 2.1. — Let γ be a smooth oriented loop in a translation surface punctured at the singularities. Assuming that γ is parametrized by arc length, $\gamma'(t)/|\gamma'(t)|$ defines a continuous map from $\mathbb{S}^1$ to $\mathbb{S}^1$. As such, it has a well-defined *topological index* Ind_γ, which depends only on the homotopy class of γ.

The topological index of a positively oriented loop around a singularity of order a is $a + 1$.

2.2. Period coordinates for strata

2.2.1. *The period atlas*. — Denote by Z_ω the set of zeros of ω and by P_ω the set of poles. Any smooth path γ joining two zeros of ω and avoiding the poles of ω represents a *relative homology class* $[\gamma]$ in $H_1(X \smallsetminus P_\omega, Z_\omega)$. The integrals of ω along an integral basis of $H_1(X \smallsetminus P_\omega, Z_\omega)$ provide local complex-analytic coordinates for the corresponding stratum of differentials with the same orders of singularities as ω, which are called *period coordinates*.

For a stratum $\mathcal{H}(a_1, \ldots, a_n, -b_1, \ldots, -b_p)$ with $\sum_{i=1}^n a_i - \sum_{j=1}^p b_j = 2g - 2$ and $n, p \geqslant 1$, the *period atlas* endows the stratum with the structure of a complex-analytic orbifold of complex dimension $2g + n + p - 2$.

2.2.2. *Saddle connections*. — In a translation surface (X, ω), a *saddle connection* is an arc joining two zeros, whose interior does not contain any singularities and is locally conjugated to a segment of constant slope in the translation charts. In the flat metric induced by ω, a saddle connection is a geodesic segment. A *closed saddle connection* is a saddle connection where the two ends coincide. Every saddle connection represents a class in the relative homology group $H_1(X \smallsetminus P_\omega, Z_\omega)$.

2.2.3. *Cylinders*. — It is well-known that translation surfaces of finite area (corresponding to holomorphic one-forms) have infinitely many saddle connections. However, meromorphic one-forms can have either finitely or infinitely many saddle connections, depending on the existence of invariant components of finite area in the directional foliation (see [Tah18, §5] for details).

Definition 2.2. — In a translation surface (X, ω), cutting along all the saddle connections sharing a given direction $\theta \in \mathbb{S}^1$ decomposes X into connected components that we refer to as *invariant components* because they are invariant under the directional flow in the direction θ.

Here, we are only interested in the case of meromorphic one-forms on $\mathbb{CP}^1$, whose characterization is simpler.

We recall that in a translation surface, closed geodesics (loops locally conjugated to straight segments of constant slope in the translation charts) that are disjoint from the singularities form one-parameter families known as *cylinders*. An end of a cylinder can be either a simple pole (in the case of an end at infinity) or consist of a chain of parallel saddle connections.

Proposition 2.3. — *For a translation surface $(\mathbb{CP}^1, \omega)$, where ω is a meromorphic one-form, one of the following statements holds:*

– *ω admits finitely many saddle connections;*

– *ω admits infinitely many saddle connections, and the accumulation points of the directions of these saddle connections in $\mathbb{S}^1$ coincide with the directions of closed geodesics in the cylinders of finite area in $(\mathbb{CP}^1, \omega)$.*

Proof. — It is proved in Proposition 5.10 of [Tah18] that the accumulation points of the directions of saddle connections are the directions of invariant components of the directional flow that have finite area. These components are either cylinders or minimal components. The latter only appears in genus at least one, so it cannot occur for $(\mathbb{CP}^1, \omega)$. Conversely, in any cylinder of finite area, we can find arbitrarily long saddle connections joining the two boundary components, and their slope approaches the slope of the closed geodesics of the cylinder.

Finally, having infinitely many saddle connections and infinity many slopes of saddle connections in $\mathbb{S}^1$ are equivalent, as there is a topological upper bound on the maximal number of saddle connections sharing a given direction (this is a special case of [Tah18, Prop. 7.6]). □

In this paper, we primarily focus on translation surfaces of genus zero with exactly $n=2$ conical singularities. In such surfaces, there is at most one cylinder of finite area.

Proposition 2.4. — *For a translation surface* $(\mathbb{CP}^1,\omega)$*, where* ω *is a meromorphic one-form with exactly two zeros, there is at most one cylinder of finite area in* $(\mathbb{CP}^1,\omega)$*.*

Proof. — In genus zero, each closed geodesic of a cylinder of finite area disconnects the surface into two components. Each component contains one boundary component of the cylinder and therefore exactly one conical singularity.

If $(\mathbb{CP}^1,\omega)$ contains two such cylinders, they cannot be disjoint because ω would then have at least three zeros. In the case where these two cylinders intersect, we have two closed geodesics γ and γ', that intersect each other. This situation is also impossible because γ decomposes $\mathbb{CP}^1$ into two connected components, X_1 and X_2. Since γ' is a periodic trajectory, it should cross γ from X_1 to X_2 and then from X_2 to X_1. In particular, if γ' enters X_2 by crossing γ positively (resp. negatively), then it has to leave X_2 by crossing γ negatively (positively). However, in a translation surface, the direction of a trajectory cannot change, so all intersections between the two trajectories have the same sign. This is a contradiction so there is no such pair of geodesics γ and γ' □

2.2.4. *The action of* $\mathrm{GL}_2^+(\mathbb{R})$. — Given a translation surface (X,ω), elements of $\mathrm{GL}_2^+(\mathbb{R})$ act by composition with the coordinate functions induced by ω. For a translation surface (X,ω) obtained by identifying parallel sides of a collection of (possibly infinite) polygons $P_1,\dots,P_t$ in the real plane $\mathbb{R}^2$, given $g\in\mathrm{GL}_2^+(\mathbb{R})$, the image $g\cdot(X,\omega)$ is obtained by identifying the corresponding sides of the polygons $g(P_i)$, where g acts as a linear transformation on $\mathbb{R}^2$.

Similarly, as $\mathbb{C}$ identifies with $\mathbb{R}^2$, the group $\mathrm{GL}_2^+(\mathbb{R})$ acts on the residual space $\mathcal{R}_p$. It is easy to check that the action of $\mathrm{GL}_2^+(\mathbb{R})$ preserves strata of meromorphic differentials and commutes with the residual map.

2.3. Core. — In a translation surface induced by a meromorphic differential, every neighborhood of a pole has infinite area. However, the essential information of the translation structure is contained within a domain of finite area called the *core*.

Definition 2.5. — A subset E of a translation surface (X,ω) is *convex* if and only if every geodesic segment joining two points of E lies entirely in E.

The *convex hull* of a subset F of a translation surface (X,ω) is the smallest closed convex subset of X that contains F.

The *core* of (X,ω) is the convex hull $\mathrm{core}(X)$ of the zeros of ω.

The core separates the poles from one another. The following result demonstrates that the complement of the core has as many connected components as the number of poles (see [Tah18, Prop. 4.4 & Lem. 4.5]).

Proposition 2.6. — *For a translation surface (X,ω), the boundary of the core $\partial\mathcal{C}(X)$ is a finite union of saddle connections. Moreover, each connected component of $X\smallsetminus \mathrm{core}(X)$ is a topological disk that contains a unique pole.*

We refer to these connected components as *polar domains.*

2.4. Graphs for translation surfaces with real periods. — In this section we introduce two graphs associated to a translation surface with real period. Both of them could be useful depending to the context. First we introduce a version of ribbon graphs suited for the description of the translation surfaces with real periods. Its definition is quite simple.

Definition 2.7. — Given a meromorphic differential ω with real periods on the Riemann sphere, its *associated graph* is the embedded graph in the sphere whose vertices are its zeros and edges are its saddle connections, oriented in the increasing real direction. Moreover, we draw an half edge for every horizontal half infinite ray starting from a zero of ω.

Note that each face of the graph corresponds to a pole. Moreover, by convention, we will draw the graph in such a way that the unbounded face corresponds to a pole with positive residue. The order of the zero is half of the number of ray starting at its corresponding vertex minus one. The order of the pole is minus half of the number of half edges contained in the corresponding face minus one.

In the case of a single zero, this graph corresponds to the dual of the graph introduced in [GT22].

To conclude, the merging zeros of ω by shrinking a saddle connection corresponds to the shrinking of the corresponding edge in the associated graph.

We now introduce the *decorated graphs* of a translation surface with real periods, which is the dual notion of the ribbon graphs. This generalizes the concept of *decorated trees* from [GT22], to provide a combinatorial description of these translation surfaces.

Definition 2.8. — A *decorated graph* is an embedded graph in the topological sphere such that:

– every face is labeled;
– every face is a topological disk;
– every vertex is labeled;
– edges are oriented;
– to every vertex is attached a nonnegative even number of unoriented half-edges;
– at a vertex, there is a nonnegative even number of half-edges between two adjacent edges with the same orientation and an odd number of half-edges between two edges of opposite orientation.

Now, we show how to construct a decorated graph corresponding to a translation surface of genus zero with real periods.

For any one-form ω on $\mathbb{CP}^1$ in a stratum $\mathcal{H}(a_1, \dots, a_n, -b_1, \dots, -b_p)$ with real periods, all the saddle connections are horizontal. Consequently, the *vertical directional foliation* induced by ω on $\mathbb{CP}^1$ is straightforward to describe (see [Tah18, Prop. 5.5] for more details). Vertical trajectories (oriented from bottom to top) can be:

– *Generic trajectories*: These are infinitely long and extend from one pole to another.
– One of the $\sum_{i=1}^{n}(a_i + 1)$ *critical trajectories* that go from a pole to a zero.
– One of the $\sum_{i=1}^{n}(a_i + 1)$ *critical trajectories* that go from a zero to a pole.

Generic trajectories assemble into one-parameter families that sweep through three kinds of subsurfaces:

– *Open left half-planes* with a unique conical singularity on the right vertical boundary.
– *Open right half-planes* with a unique conical singularity on the left vertical boundary.
– *Infinite vertical strips* with a unique conical singularity at the same height on each boundary line.

We construct the *decorated graph* $\mathfrak{gr}(\omega)$ in the following way:

– The vertices are the poles of ω.
– For each infinite vertical strip, we draw one of the vertical trajectories (oriented from bottom to top).
– For each open left or right half-plane, we draw a half-edge attached to the corresponding vertex.

We can immediately verify that:

– There is a complete correspondence between the oriented edges of $\mathfrak{gr}(\omega)$, infinite vertical strips, and the horizontal saddle connections joining their sides.
– Each face of $\mathfrak{gr}(\omega)$ is a topological disk containing exactly one zero of ω.
– To each pole of order b_j, there are $2b_j - 2$ half-edges attached (half of them correspond to left half-planes, while the other half correspond to right half-planes).
– The gluing of the boundaries of half-planes and strips is consistent with the orientation of the half-planes. Hence, there is a nonnegative even number of half-edges between two adjacent edges with the same orientation (at the vertex), and an odd number of half-edges between two edges of opposite orientation.

In a translation surface with only real periods, saddle connections are horizontal and can only meet at their endpoints. The union of the saddle connections cuts out p polar domains, each of which is a topological disk containing a unique pole (see, for example, [Tah18, Lem. 4.10]). The computation of the Euler characteristic formula shows that there are $n+p-2$ saddle connections, as well as the same number of infinite vertical strips. Each of these contains four sides of critical trajectories.

Since the local model of a pole of order $b_j > 1$ is the cyclic gluing of $b_j - 1$ planes, there are exactly $\sum_{j=1}^{p}(2b_j - 2)$ left or right half-planes, each containing two sides of critical trajectories. Therefore, the total number of critical trajectories is $2n-4+2\sum_{j=1}^{p} b_j$.

Since a zero of order a_i corresponds to a conical singularity with an angle of $(2a_i+2)\pi$, there are a_i+1 incoming and a_i+1 outgoing critical trajectories at this zero. Therefore, the total number of critical trajectories is equal to $2n+2\sum_{i=1}^{n} a_i$. This is consistent with the identity $\sum_{i=1}^{n} a_i - \sum_{j=1}^{p} b_j = -2$.

Example 2.9. — A differential in $\mathcal{H}(2,1,[-1]^5)$ is represented on the left side of Figure 1. The associated ribbon and decorated graphs are shown on the right side of the same figure, respectively on the top and on the bottom. Similarly, a differential in $\mathcal{H}(1,4,-1,-2,-4)$ and its associated graphs are presented in Figure 2.

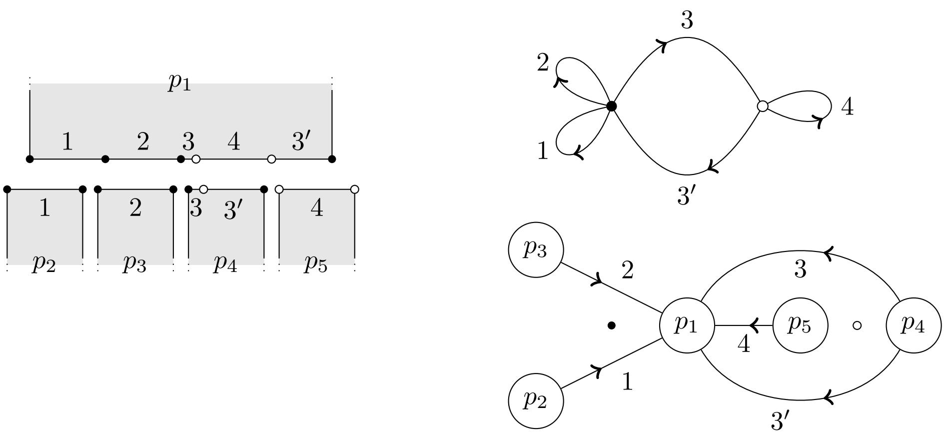

Figure 1. A differential ω and the associated ribbon and decorated graphs are shown, with extra labels on the edges given by the label of the corresponding saddle connection.

To conclude, we describe the effect of merging zeros of ω in terms of the associated graph $\mathfrak{gr}(\omega)$.

Lemma 2.10. — *Given a differential $\omega \in \mathcal{H}(a_1,a_2,-b_1,\dots,-b_p)$ with real periods and its associated graph $\mathfrak{gr}(\omega)$, if we shrink a saddle connection between two zeros, the associated graph of the limit is obtained by eliminating the corresponding edge of $\mathfrak{gr}(\omega)$.*

Proof. — The decorated graph $\mathfrak{gr}(\omega)$ has p vertices and p edges, and therefore a unique cycle that decomposes the underlying sphere into two connected components. The edges of the cycle correspond to the saddle connections joining the two zeros. Shrinking one of these saddle connections produces a translation surface ω_0 with one unique zero and the $p-1$ remaining saddle connections. Then, $\mathfrak{gr}(\omega_0)$ is a decorated tree where edges connect the same polar domains as in $\mathfrak{gr}(\omega)$. In other words, $\mathfrak{gr}(\omega_0)$

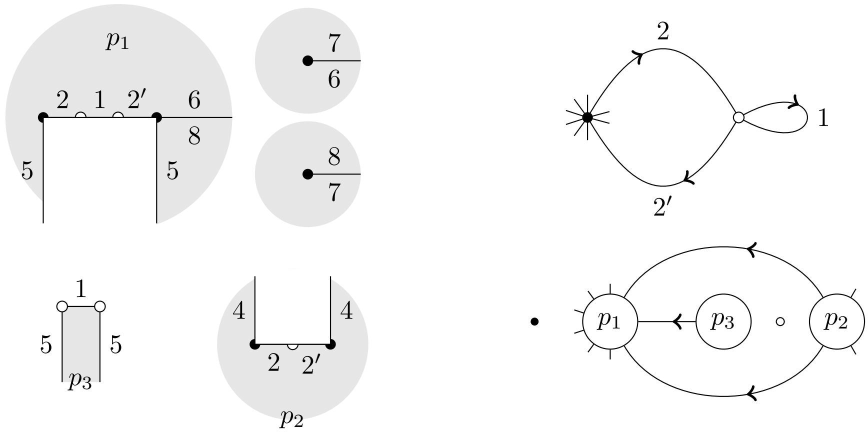

Figure 2. A differential in $\mathcal{H}(1,4,-1,-2,-4)$ and its associated graphs are presented.

is obtained from $\mathfrak{gr}(\omega)$ by removing the edge corresponding to the saddle connection we have shrunk. □

2.5. Classification of the connected components of strata. — A complete classification of connected components of strata of meromorphic differentials with prescribed orders of singularities has been provided in [Boi15]. Note that in the case of genus zero, every stratum of meromorphic differentials on $\mathbb{CP}^1$ is connected.

We first review the classification in genus one in terms of the rotation number of a translation surface (see Section 2.1.3 for the definition of rotation numbers).

Definition 2.11. — Let (X,ω) be a translation surface in a stratum

$$\mathcal{H}(a_1,\dots,a_n,-b_1,\dots,-b_p)$$

of differentials in genus one, with $n,p\geqslant 1$. Let (α,β) be two simple loops forming a symplectic basis of the homology of the underlying torus. The *rotation number* of (X,ω) is defined as $\gcd(a_1,\dots,a_n,b_1,\dots,b_p,\mathrm{Ind}_\alpha,\mathrm{Ind}_\beta)$.

Theorem 2.12. — *Let $\mathcal{H}=\mathcal{H}(a_1,\dots,a_n,-b_1,\dots,-b_p)$ be a stratum of translation surfaces of genus one. In each connected component of $\mathcal{H}$, the rotation numbers of the translation surfaces are the same. Conversely, there is exactly one connected component for each positive divisor of* $\gcd(a_1,\dots,a_n,b_1,\dots,b_p)$*, with one exception: in the stratum $\mathcal{H}(a,-a)$ (where $n=p=1$), there is no connected component corresponding to the rotation number a.*

For strata parameterizing translation surfaces of genus $g\geqslant 2$, their connected components are distinguished by *hyperellipticity* and *parity of spin structures*.

Definition 2.13. — For a stratum $\mathcal{H}(\mu)$ of translation surfaces of genus $g\geqslant 2$, the signature μ is said to be:

– *hyperelliptic type* if μ is of the form $(a,a,-b,-b)$, $(2a,-b,-b)$, $(a,a,-2b)$, or $(a,a,-b,-b)$, where a and b are positive integers;

– *even type* if μ is either of the form $(a_1,\dots,a_n,-b_1,\dots,-b_p)$ or $(a_1,\dots,a_n,-1,-1)$, where the orders $a_1,\dots,a_n,b_1,\dots,b_p$ are even.

Definition 2.14. — A stratum component $\mathcal{C}$ is said to be *hyperelliptic* if every translation surface (X,ω) in $\mathcal{C}$ admits a nontrivial involution τ of X such that $\tau^*\omega=-\omega$ and X/τ is isomorphic to $\mathbb{CP}^1$.

Definition 2.15. — In a stratum $\mathcal{H}(\mu)$ where μ is of even type, the *parity of spin structure* of a translation surface (X,ω) of genus $g\geqslant 2$ is defined as the parity of $\sum_{i=1}^{g}(\mathrm{Ind}_{\alpha_i}+1)(\mathrm{Ind}_{\beta_i}+1)$, where $(\alpha_1,\beta_1,\dots,\alpha_g,\beta_g)$ is a symplectic basis of the homology of the underlying topological surface of genus g. This parity is a topological invariant of each connected component of $\mathcal{H}(\mu)$.

Theorem 2.16. — *Let $\mathcal{H}(\mu)=\mathcal{H}(a_1,\dots,a_n,-b_1,\dots,-b_p)$ be a stratum of translation surfaces of genus $g\geqslant 2$. We distinguish several cases as follows:*

- *If $\sum_{j=1}^{p} b_j$ is odd, then $\mathcal{H}(\mu)$ is connected.*
- *If $\sum_{j=1}^{p} b_j=2$ and $g=2$, then:*

– *if μ is of hyperelliptic type, then there are two components: one is hyperelliptic while the other is not (in this case, these two components are also distinguished by the parity of the spin structure);*

– *if μ is not of hyperelliptic type, then $\mathcal{H}(\mu)$ is connected.*

- *If $\sum_{j=1}^{p} b_j>2$ or $g>2$, then:*

– *if μ is of hyperelliptic type, then $\mathcal{H}(\mu)$ has exactly one hyperelliptic component and either one or two non-hyperelliptic components, depending on whether μ is of even type or not;*

– *if μ is of even type, then $\mathcal{H}(\mu)$ has two non-hyperelliptic components distinguished by the parity of the spin structure;*

– *if μ is neither of hyperelliptic nor even type, then $\mathcal{H}(\mu)$ is connected.*

2.6. Differentials with zero residues. — For our later computations, we need to count the number of meromorphic differentials whose residues vanish.

Definition 2.17. — The number of meromorphic differentials in

$$\mathcal{H}(a_1,a_2,-b_1,\dots,-b_p)$$

whose residues at the poles are all zero is denoted by $\Xi(a_1,a_2;b_1,\dots,b_p)$.

This number was initially studied in [EMZ03], and an expression in terms of coefficient extraction from a generating series was given in Proposition 2.1 of [CMSZ20] as

$$(p-1)!\ \left[t^{a_1+1}\right]\prod_{i=1}^{p}\left(t+\cdots+t^{b_i-1}\right),$$

where $\left[t^j\right]P(t)$ denotes the coefficient of degree j of the polynomial P. Equivalently, up to the factor $(p-1)!$, the above formula counts the number of tuples $(c_1,\dots,c_p)$ such that $\sum_{i=1}^{p} c_i = a_1+1$ and $1\leqslant c_i\leqslant b_i-1$ for all i.

In the following, we prove that Ξ has a piecewise polynomial structure, where the walls that separate the polynomial chambers are defined by identities of the form $a_1+1-b_{i_1}-\cdots-b_{i_k}-(p-k)=0$.

Proposition 2.18. — *For any positive integers $a_1,a_2,b_1,\dots,b_p$ such that $\sum_{j=1}^{p} b_j = a_1+a_2+2$, the number $\Xi(a_1,a_2;b_1,\dots,b_p)$ is given by the following piecewise polynomial formula:*

$$(p-1)!\sum_{k=0}^{p-1}(-1)^k \sum_{a_1+1-b_{i_1}-\cdots-b_{i_k}-(p-k)\geqslant 0} \binom{a_1-b_{i_1}-\cdots-b_{i_k}+k}{p-1}. \tag{2.1}$$

In particular, Ξ is a polynomial of degree $p-1$ in every chamber.

Proof. — To obtain the coefficient of $\left[t^{a_1+1}\right]\prod_{i=1}^{p}\left(t+\cdots+t^{b_i-1}\right)$, note the inclusion-exclusion relation:

$$\begin{aligned}\left[t^{a_1+1}\right]\prod_{i=1}^{p}\left(t+\cdots+t^{b_i-1}\right)\\ &= \sum_{k=0}^{p-1}(-1)^k \sum_{a_1+1-b_{i_1}-\cdots-b_{i_k}-(p-k)\geqslant 0}\left[t^{a_1+1}\right]\left(\prod_{j=1}^{k}\frac{t^{b_{i_j}}}{1-t}\prod_{i\neq i_j}^{p}\frac{t}{1-t}\right).\end{aligned}$$

The right-hand side first counts the coefficient of $\left[t^{a_1+1}\right]\prod_{i=1}^{p} t/(1-t)$ whenever $a_1+1-p\geqslant 0$. However, it can occur that we took more than b_i-1 copies of t in the factor $(t+\cdots+t^{b_i-1}+\cdots)$ for k different factors corresponding to $b_{i_1},\dots,b_{i_k}$. This happens whenever $a_1+1-(b_{i_1}-1)-\cdots-(b_{i_k}-1)-p\geqslant 0$, and the corresponding extra coefficient is given by $\left[t^{a_1+1}\right]\left(\prod_{j=1}^{k} t^{b_{i_j}}/(1-t)\right)\left(\prod_{i\neq i_j}^{p} t/(1-t)\right)$. However, this case was also subtracted for every proper subset of $\{b_{i_1},\dots,b_{i_k}\}$, so the number of times we have already subtracted it is $\sum_{i=0}^{k-1}(-1)^i\binom{k}{i} = (-1)^{k-1}\binom{k}{k}$. Therefore, we need to add an extra term with coefficient $(-1)^k$ to ensure that this case is completely subtracted from the total sum.

Finally, observe that

$$\left[t^{a_1+1}\right]\left(\prod_{j=1}^{k}\frac{t^{b_{i_j}}}{1-t}\prod_{i\neq i_j}^{p}\frac{t}{1-t}\right) = \binom{a_1-b_{i_1}-\cdots-b_{i_k}+k}{p-1}.$$

This expression counts the number of ways we can distribute a_1+1 objects into p distinct boxes, where the boxes indexed by $b_{i_1},\dots,b_{i_k}$ must contain at least b_{i_j} objects respectively, and the $p-k$ remaining boxes must contain at least one object each. □

2.7. The isoresidual fibration of the minimal strata. — For the strata of meromorphic differentials on $\mathbb{CP}^1$ with a unique zero, the isoresidual fibration is a ramified cover. Its degree and ramification locus are described in the following statement, which is proved as [GT22, Th. 1.2].

Theorem 2.19. — *For every stratum $\mathcal{H}(a,-b_1,\dots,-b_p)$ of meromorphic differentials on $\mathbb{CP}^1$, the isoresidual fibration is an unramified cover of degree $f(a,p) := a!/(a+2-p)!$ over $\mathcal{R}_p \smallsetminus \mathcal{A}_p$.*

Elements of the generic isoresidual fibers of the strata $\mathcal{H}(a,-b_1,\dots,-b_p)$ over configurations of real residues are classified by the *decorated graphs* introduced in Section 2.4. Since the corresponding translation surfaces have exactly $p-1$ saddle connections, these graphs are decorated trees.

3. Multi-scale compactification of the isoresidual loci

In this section, we recall the basics of the multi-scale compactification introduced in [BCG+19a] and describe the closure of the loci of differentials with residues satisfying some linear relations. Moreover, we describe some properties for multi-scale differentials contained in the closure of the isoresidual loci within this compactification. Since the main Theorem 3.2 of this section applies to any genus, we denote by $\mathcal{H}_g(\mu)$ a stratum of differentials of genus $g \geqslant 0$ with signature μ when the results apply to such general cases.

3.1. The multi-scale compactification of strata. — The multi-scale compactification $\mathrm{MS}(\mu)$ of the projectivized strata $\mathbb{P}\mathcal{H}_g(\mu) := \mathcal{H}_g(\mu)/\mathbb{C}^*$ is constructed in [BCG+19b]. Briefly speaking, a *multi-scale differential* (X,ω) is defined on an underlying stable pointed nodal Riemann surface X, where ω consists of a non-identically-zero differential ω_i on each irreducible component X_i of X. Moreover, there is a total order that compares any two irreducible components of X, which encodes the information about vanishing rates when differentials from nearby smooth surfaces degenerate to these sub-surfaces X_i. In this sense, the total order induces a level structure on the dual graph of the nodal surface, which is called a *level graph*. Additionally, there are compatibility conditions that relate the zero and pole orders of the ω_i at the two branches of every node, as well as local and global residue conditions at the nodes, which we will review below.

If a node has two simple poles at its branches, the corresponding edge in the dual graph is called *horizontal*; otherwise, it is called *vertical*. At a vertical edge, the multi-scale differential has a zero of order $k \geqslant 0$ at the upper nodal point and a pole of order $-k-2 \leqslant -2$ at the lower nodal point. Moreover, the number $k+1$ is called the *prong number* of the edge. It encodes the number of ways to locally open up the node under the induced flat metric.

Next, we discuss the global residue condition. Consider a level L and a component Y of the part $\Gamma_{>L}$ of sub-surfaces lying strictly above L in the level graph Γ. We consider the edges that connect Y to the vertices of Γ at level L, and denote by $e_1,\dots,e_b$ the (lower) endpoints of these edges. If Y contains a marked pole, we do not impose any condition to Y. Now, suppose Y does *not* contain any marked poles. Then, we say that Y satisfies the (usual) *global residue condition* if the following condition holds:

GRC $$\sum_{i=1}^{b} \mathrm{Res}_{e_i}(\omega) = 0.$$

We remark that the residue theorem for each vertex (i.e., the total sum of residues on a component is equal to zero) still needs to hold, which we implicitly impose as a preexisting condition.

Finally, the multi-scale compactification has a system of local coordinates, which extends the period coordinates recalled in Section 2.2 to the boundary. These coordinates are called perturbed period coordinates and a variation of them corresponds to log period coordinates. These are defined and discussed in [BCG+19b, §§9.2, 13.3] and [Ben23, §5], respectively. These coordinates consist of two parts: one part is given by (usual) relative periods on the nodal curve and the other part is given by the parameters used to smooth the horizontal nodes and the nodes crossing a level of the multi-scale differential.

3.2. The closure of loci of differentials with linear residue conditions

One can consider a similar residual map from the multi-scale compactification $\mathrm{MS}(\mu)$ to the projective residue space $\mathbb{P}\mathcal{R}_p = (\mathcal{R}_p \smallsetminus \{0\})/\mathbb{C}^*$. Note that, in general, it is only a rational map; for example, it is undefined on the locus of residueless differentials. Indeed, in Corollary 3.4 below, we will show that in our case the closure of every isoresidual curve contains the zero-dimensional locus of residueless differentials.

Following [CMZ22, §4.1], we now describe the closure of the isoresidual fibers in the moduli space of multi-scale differentials. This discussion can generally be applied to arbitrary genera and partition μ. Additionally, we treat the more general setting where we fix a linear subspace of the residues (i.e., not just a point in the projectivized residue space by fixing all residues).

Consider a stratum $\mathbb{P}\mathcal{H}_g(\mu)$ of meromorphic differentials of genus g with signature μ. Let Λ be a linear subspace in the residual space $\mathcal{R}_p$, and consider the subspace $\mathcal{F}_\Lambda$ of differentials whose residues lie in Λ. We define $\Lambda^\vee$ in the dual vector space $\mathcal{R}_p^\vee$ of $\mathcal{R}_p$ to be the vector space of homogeneous linear equations satisfied by all residue tuples in Λ.

We want to characterize when a multi-scale differential (X,ω) is contained in the closure of $\mathcal{F}_\Lambda$. To this aim, we will state the *generalized global residue condition* imposed to (X,ω) by $\Lambda^\vee$, called $\mathcal{E}_\Lambda$-GRC. Denote by $q_1,\dots,q_p$ the marked poles in Γ. For each q_i, add a new vertex at level ∞ with a marked pole q_i' and turn the original q_i into an edge that connects to the new vertex. One can think of this new vertex as a semistable rational component with a pole at q_i' of order equal to that of q_i. Denote by Γ' the resulting level graph. For every level $L<\infty$ in Γ', let $Y_1,\dots,Y_s$ be the connected components of $\Gamma'_{>L}$. Then, the following conditions hold:

$\mathcal{E}_\Lambda$-GRC

(1) Each Y_i that does not contains any q_j' satisfies the usual GRC.
(2) For every equation $f\in\Lambda^\vee$ that can be written of the form

$$f=\sum_{i=1}^{s} a_i\Big(\sum_{q_j'\in Y_i}\mathrm{Res}_{q_j'}\Big),$$

we require that the equation $\sum_{i=1}^{s} a_i\big(\sum_{e_j\in Y_{i,L}}\mathrm{Res}_{e_j}(\omega)\big) = 0$ holds, where the inner summation ranges over the lower endpoints of the edges of Y_i that connect to vertices at level L.

The idea behind the $\mathcal{E}_\Lambda$-GRC is the following. Suppose a one-parameter family of differentials $\{\omega_t\}$ in $\mathcal{F}_\Lambda$ degenerates to the multi-scale differential ω as $t \to 0$, with the vanishing rate t^ℓ as they approach the limit components on level L for some $\ell \geqslant 0$. Then, given an equation f in $\Lambda^\vee$, if it satisfies the description in condition (2), it can descend to impose a relation for the residues at level L by applying Stokes' theorem to the limit differential $\lim_{t\to 0} t^{-\ell}\omega_t$ on the subsurfaces Y_i with boundary punctures at the poles. Note that those terms in f whose corresponding poles q_j connect to a vertex strictly below level L do not appear in the $\mathcal{E}_\Lambda$-GRC for level L, as their vanishing rates are faster than t^ℓ. Moreover, if a component Y_i contains a pole whose residue term does not appear in f, then condition (2) can be non-empty only if $a_i = 0$.

Additionally, we remark that up to a linear combination, there are indeed only finitely many non-trivial conditions imposed by the $\mathcal{E}_\Lambda$-GRC. To see this, if there are two independent relations in condition (2) that involve the same subset of Y_i with nonzero coefficients, we can use their combination to produce another relation that involves a smaller subset of Y_i. In the end, we can reduce to a finite set of relations that form an echelon form for each level.

We illustrate the $\mathcal{E}_\Lambda$-GRC through the following example.

Example 3.1. — Consider a stratum with four poles. We impose the residue relations $r_1 + r_2 + 2r_3 = 0$ and $2r_2 + r_3 = 0$ (together with the residue theorem relation $r_1 + r_2 + r_3 + r_4 = 0$). In other words, the given subspace of residue tuples Λ is spanned by $(3, 1, -2, -2)$.

Consider a multi-scale differential (X, ω) in the boundary of this isoresidual fiber whose level graph Γ is given on the left of Figure 3 and the associated level graph Γ' is given on the right, where the marked poles of Γ become the edges connecting to level ∞ in Γ'.

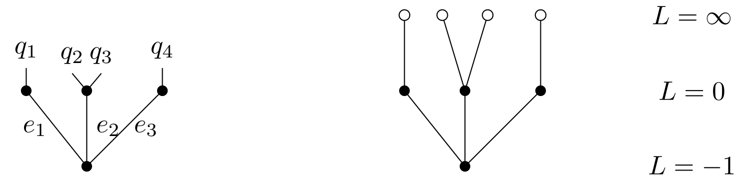

Figure 3. The level graphs Γ and Γ', illustrating the $\mathcal{E}_\Lambda$-GRC. The vertices at level infinity are pictured in white.

Note that all the poles q_i are at level zero. Hence, the $\mathcal{E}_\Lambda$-GRC for level zero implies that the imposed residue relations still hold at level zero. Combining with the residue theorem at each vertex, we conclude that $r_1 = r_2 = r_3 = r_4 = 0$ for the residues of ω at each q_i at level zero of Γ.

Next, denote by e_i for $i = 1, 2, 3$ the residues of the multi-scale differential ω at the lower endpoints of the edges joining the bottom component of Γ to level zero. Then, the $\mathcal{E}_\Lambda$-GRC at level -1 implies that there exists some $r \in \mathbb{C}$ such that $(e_1, e_2, e_3) = (r, -r/3, -2r/3)$. To see this, suppose that $e_1 = r$. Combining the two given equations gives $r_1 + 3(r_2 + r_3) + 0\ r_4 = 0$, which descends to the relation $e_1 + 3e_2 = 0$ by the $\mathcal{E}_\Lambda$-GRC. Therefore, $e_2 = -r/3$, which further implies $e_3 = -2r/3$ by the residue theorem on the bottom vertex.

Finally, note that the equations $2r_1 + 3r_4 = 0$ and $2(r_2 + r_3) - r_4 = 0$ are also in $\Lambda^\vee$. If we run the $\mathcal{E}_\Lambda$-GRC to them at level -1, we still obtain the same conclusion $2e_1 + 3e_3 = 0$ and $2e_2 - e_3 = 0$ as before.

After the above preparation, we can state the main result of this section as follows.

Theorem 3.2. — *A multi-scale differential is in the closure $\overline{\mathcal{F}}_\Lambda$ of $\mathcal{F}_\Lambda$ if and only if it satisfies the $\mathcal{E}_\Lambda$-GRC at every level. Moreover, $\overline{\mathcal{F}}_\Lambda$ is a smooth orbifold with normal crossing boundary. Finally, the codimension of a boundary stratum in $\overline{\mathcal{F}}_\Lambda$ is equal to the number of horizontal edges plus the number of levels below zero.*

We remark that the statement of Theorem 3.2 generalizes [CMZ22, Prop. 4.2], where partial sum residue conditions therein are replaced by arbitrary linear residue conditions in our setting.

Proof. — Note that the $\mathcal{E}_\Lambda$-GRC condition contains the usual GRC. In particular, such multi-scale differentials can be smoothed into the entire stratum interior. Therefore, we only need to verify the extra residue conditions imposed by $\mathcal{E}_\Lambda$. First, we remark that every horizontal node can be smoothed locally and independently. This is because the defining linear equations of $\mathcal{F}_\Lambda$ consists of loops around the marked poles, which do not cross any horizontal node (see [BDG22]). We can thus assume that the concerned multi-scale differential has only vertical nodes.

The necessity of the $\mathcal{E}_\Lambda$-GRC condition has already been explained in the paragraph below the statement of the $\mathcal{E}_\Lambda$-GRC condition. Here we provide a more formal argument, following [BCG$^+$19a, Prop. 6.3]. Take a pointed topological surface Σ with $\mathcal{V} \subset \Sigma$ as a disjoint union of simple closed curves such that the degenerate surface with dual graph Γ is obtained by pinching the curves in $\mathcal{V}$ to form the corresponding nodes. The residue assignment ρ can be identified with an element of $H^1(\mathcal{V})$. It satisfies the usual GRC if and only if it factors through level quotients. Now, consider an imposed linear residue relation f as an element of $H^1(\Sigma \smallsetminus P, Z)$. In order to smooth the multi-scale differential while preserving f, we need its image under $H^1(\Sigma \smallsetminus P, Z) \to H^1(\mathcal{V})$ to contain ρ. This shows that the $\mathcal{E}_\Lambda$-GRC condition is necessary.

Conversely, we will show that the $\mathcal{E}_\Lambda$-GRC condition is also sufficient for smoothing the multi-scale differential while preserving the equations in $\Lambda^\vee$. The upshot is that since the imposed equations are linear, if we have two solutions, then their linear combination remains to be a solution.

As in [BCG$^+$18], the smoothing is level by level from the top to the bottom. Hence, we consider a level L and suppose by induction that all the edges whose lower endpoints are at a level strictly above L have been smoothed. Then, consider the connected components Y_i of $\Gamma'_{>L}$, and we will smooth their edges that connect to level L under the assumption that the $\mathcal{E}_\Lambda$-GRC is satisfied.

Consider the poles q'_j that lie in the components Y_i. In order to prove the result, we claim that it suffices to find a residue assignment r_j for each pole q'_j that satisfies the following conditions:

(a) The r_j satisfy the equations in $\Lambda^\vee$.
(b) For each Y_i, we have $\sum_{q'_j\in Y_i} r_j = \sum_{e_j\in Y_{i,L}} \mathrm{Res}_{e_j}(\omega)$.

Indeed, if we can find the assignments r_j as in the above, then we can find in each component Y_i a modifying differential with these residues at the q'_j such that the sum of the residues of these modifying differentials and the residues of ω at the nodes is zero. Moreover, note that if Y_i contains no marked poles at all, the usual GRC is satisfied. Consequently, we can smooth this level in such a way that the residues at the q'_j are proportional to the r_j. The linearity of $\Lambda^\vee$ implies that the corresponding equations are still satisfied. It follows that we can smooth the multi-scale differential in such a way that the equations of $\Lambda^\vee$ are satisfied by the residues after smoothing.

Therefore, it suffices to prove the existence of such residue assignments as in the above. We fix the residues of ω at the edges e_j and consider the set $\mathcal{R}$ of residue assignments satisfying part (b) of the condition. More precisely, let

$$w_i = \sum_{e_j\in Y_{i,L}} \mathrm{Res}_{e_j}(\omega) \in \mathbb{C}.$$

Then $\mathcal{R}$ is the solution set of the linear system

$$y_i(r_1,\dots,r_p) \coloneqq \sum_{q'_j\in Y_i} r_j = w_i$$

given by all connected components Y_i of $\Gamma'_{>L}$. Note that $\mathcal{R}$ is a nonempty affine subspace of the total residue space $\mathcal{R}_p$ (where $\mathcal{R}$ is not a linear subspace if some w_i is nonzero). We will show that if ω satisfies the $\mathcal{E}_\Lambda$-GRC, then for all linear subspaces V of dimension k in $\Lambda^\vee$, the locus of residue assignments in $\mathcal{R}\subset\mathcal{R}_p$ that are compatible with V (in the sense of satisfying the equations in V) remains to be a nonempty affine subspace. To this aim, we apply induction on k starting from $k=0$, and the final case of $k=\dim\Lambda^\vee$ thus provides the desired residue assignments to conclude the proof.

If $k=0$, then $V=0$. In this case, part (a) of the residue assignment holds trivially and thus $\mathcal{R}_V=\mathcal{R}$. Now, suppose by induction that given a k-dimensional subspace $V\subset\Lambda^\vee$, the locus of residue assignments $\mathcal{R}_V\subset\mathcal{R}$ is a nonempty affine subspace in $\mathcal{R}_p$. Next, we consider a subspace V' of dimension $k+1$, where $V\subset V'\subset\Lambda^\vee$.

Let $f\in V'\smallsetminus V$ and consider the system of equations given by $y_i(r_1,\dots,r_p)=w_i$ and $f_i(r_1,\dots,r_p)=0$, where the f_i form a basis of V. If adding the equation $f(r_1,\dots,r_p)=0$ increases the rank of the system, then the linear hyperplane defined by $f=0$ is not parallel to the affine subspace $\mathcal{R}_V$. Therefore, their intersection is a nonempty codimension-one affine subspace contained in $\mathcal{R}_V$. Otherwise, i.e., when the rank is preserved, since $f\notin V$, there exists a relation of the form

$$f+\sum\lambda_i f_i = \sum a_i y_i,$$

with at least one coefficient $a_i \neq 0$. Since ω satisfies the $\mathcal{E}_\Lambda$-GRC, its condition (2) implies that $\sum a_i w_i = 0$. Therefore, the linear hyperplane defined by $f = 0$ contains the affine subspace $\mathcal{R}_V$. It follows that the solution set $\mathcal{R}_{V'}$ of the linear system after adding the equation $f = 0$ is still given by $\mathcal{R}_V$, which is nonempty.

Finally, under the system of local coordinates along the boundary, recalled at the end of Section 3.1, the closure $\overline{\mathcal{F}}_\Lambda$ is still defined by linear equations at each boundary point. Hence, it remains smooth. The level-wise opening parameters and local horizontal-node smoothing parameters imply the desired normal crossing boundary structure as well as the codimension count. Alternatively, these claims follow from the general description of linear subvarieties (see e.g., [BDG22, Cor. 1.8]), as the defining equations of $\mathcal{F}_\Lambda$ consist of loops around each marked pole and thus do not cross any horizontal nodes. □

3.3. The closure of isoresidual fibers. — Since we are interested in the isoresidual fibers, we restrict to the case when $\Lambda = \mathbb{C}^* \cdot \lambda$, where $\lambda = (\lambda_1, \ldots, \lambda_p)$. In this case, we abuse notation and also write λ instead of Λ. Note that when $\lambda = (0, \ldots, 0)$, we have $\mathcal{E}_\lambda^\vee = \mathcal{R}_p^\vee$. If λ has at least one nonzero entry, then $\mathcal{E}_\lambda^\vee$ is the hyperplane generated by the equations of the form

$$f_{i,j}(q_1, \ldots, q_p) := \lambda_i \operatorname{Res}(q_j) - \lambda_j \operatorname{Res}(q_i) = 0. \tag{3.1}$$

Now, we describe some behavior of the residues in the closure of isoresidual fibers.

Corollary 3.3. — *Consider a multi-scale differential in the closure $\overline{\mathcal{F}}_\lambda$. If two poles are on different levels, then the pole on the higher level must have zero residue. Moreover, if two poles q_i and q_j are at the same level, then their residues satisfy $f_{i,j} = 0$ in* (3.1).

Proof. — For the first claim, if $\lambda_i = 0$ in λ, then the limit residue remains to be zero in the multi-scale differential. Next, consider two poles q_i and q_j such that λ_i and λ_j are both nonzero. Suppose that q_i lies strictly above q_j. Then, running the $\mathcal{E}_\lambda$-GRC for the level of q_i implies that the equation $f_{i,j} = 0$ holds with the residue at q_j set to be 0. This implies that $\lambda_j \operatorname{Res}(q_i) - \lambda_i \cdot 0 = 0$, and hence $\operatorname{Res}(q_i) = 0$.

For the other claim, suppose that the two poles q_i and q_j are at the same level. Then, running the $\mathcal{E}_\lambda$-GRC for that level implies that their residues satisfy Equation (3.1). □

Next, we show that if an isoresidual fiber has positive dimension, i.e., if $n \geqslant 2$ or $g \geqslant 1$, then the closure of an isoresidual fiber contains the locus of differentials with zero residues (up to a scalar multiple).

Corollary 3.4. — *Given $\lambda \in \mathcal{R}_p$, suppose (X, ω) is contained in the isoresidual fiber $\overline{\mathcal{F}}_\lambda$, where X is smooth. If there exists i such that $\lambda_i \neq 0$ and $\operatorname{Res}(p_i) = 0$ for ω, then every residue of ω is zero. Conversely, the locus of such residueless differentials (up to a scalar multiple) is contained in $\overline{\mathcal{F}}_\lambda$ for every $\lambda \in \mathcal{R}_p$.*

Proof. — For the first claim, note that all $f_{i,j}\colon \lambda_i\mathrm{Res}(q_j)-\lambda_j\mathrm{Res}(q_i)=0$ in Equation (3.1) are satisfied by differentials on smooth curves contained in $\mathcal{F}_\lambda$. If $\lambda_i\neq 0$ and $\mathrm{Res}(q_i)=0$, it follows from $f_{i,j}=0$ that $\mathrm{Res}(q_j)=0$ for all j.

For the other claim, residueless differentials on smooth curves satisfy $\mathcal{E}_\lambda$-GRC for all $\lambda\in\mathcal{R}_p$. Hence, they are contained in $\overline{\mathcal{F}}_\lambda$ for every λ. □

Finally, we show that the operation of splitting a zero can be performed within any isoresidual fiber.

Corollary 3.5. — *A zero of order a_1+a_2 in a translation surface can be locally split into two nearby zeros of respective orders a_1 and a_2, without affecting the translation structure outside a small flat geometric neighborhood containing the zeros.*

This operation is illustrated in detail in Section 8.1 of [EMZ03].

Proof. — We reinterpret this operation using multi-scale differentials. Let (X_1,ω_1) be a differential with a zero z_0 of order $a_0=a_1+a_2$ whose residues are given by λ. Take another differential $(\mathbb{CP}^1,\omega_2)$ in the stratum $\mathcal{H}(a_1,a_2,-a_0-2)$. Identifying z_0 with the pole q_0 of ω_2, we obtain a multi-scale differential (X,ω) by taking the unique equivalence class of prong-matchings σ at the node. The operation of breaking up the zero z_0 is the smoothing of the multi-scale differential (X,ω) into the respective stratum. By the residue theorem, the residue at the pole of ω_2 vanishes, and hence the $\mathcal{E}_\lambda$-GRC holds. Therefore, by Theorem 3.2, the multi-scale differential (X,ω) lies in the closure of the isoresidual fiber $\mathcal{F}_\lambda$. □

3.4. The multi-scale boundary of isoresidual curves. — In this section, we focus on the case of $g=0$ and μ with exactly two zero orders. We will analyze in detail how $\overline{\mathcal{F}}_\lambda$ intersects the boundary of the multi-scale compactification $\mathrm{MS}(\mu)$, especially when λ is generic.

First, we bound the types of level graphs that can appear in the boundary of $\overline{\mathcal{F}}_\lambda$.

Lemma 3.6. — *Suppose $g=0$ and μ has exactly two nonnegative entries. Given $\lambda\in\mathcal{R}_p$, suppose the level graph Γ of a multi-scale differential in $\overline{\mathcal{F}}_\lambda$ is not a single point (i.e., the underlying curve is not smooth). Then, either Γ has two levels without any horizontal edges, or Γ is a one-level graph with exactly one horizontal edge.*

Proof. — By assumption, $\mathcal{F}_\lambda$ is one-dimensional. Therefore, each of its boundary strata has codimension one. The claim thus follows from Theorem 3.2. □

Now, we can classify all possible degenerations in a generic isoresidual curve. By Lemma 3.6, for a level graph Γ in the boundary of $\overline{\mathcal{F}}_\lambda$, either Γ has two levels without horizontal edges, or Γ has one level with exactly one horizontal edge. We start by analyzing the case of Γ with two levels.

Proposition 3.7. — *Suppose $g=0$ and μ has exactly two nonnegative entries. For $\lambda\in\mathcal{R}_p\smallsetminus\mathcal{A}_p$, let (X,ω) be a multi-scale differential with more than one level in the boundary of $\overline{\mathcal{F}}_\lambda$. Then there are two cases:*

(1) *The level graph of X has two levels, where the bottom level has one vertex containing the zeros z_1 and z_2, while the top level has one vertex containing all the poles. Additionally, the residues at the top level poles are specified by λ.*

(2) *The level graph of X has two levels: the top level has one vertex containing a subset of poles K, while the bottom level has two vertices containing subsets of poles I (with z_1) and J (with z_2), respectively, where $I \sqcup J \sqcup K = \{1, \ldots, p\}$. Moreover, the top level differential has zero residue at every pole, while the residues of the bottom level multi-scale differential are specified by λ.*

We call the graphs of type (2) above "cherry graphs" because of their appearance as shown in Figure 4. Note that K cannot be empty (otherwise it would correspond to a horizontal degeneration). However, if either I or J is empty, then the corresponding graph is not stable. Its stabilization, in the case when $I = \varnothing$, is shown in (2') of Figure 4. Moreover, if both I and J are empty, then the stable model becomes a single $\mathbb{CP}^1$ with two zeros and residueless poles as explained in Corollary 3.4.

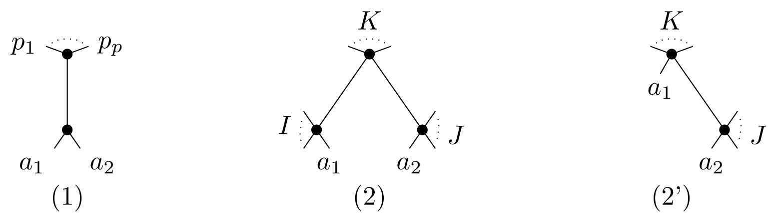

Figure 4. The types of level graphs in Proposition 3.7.

Proof. — By Lemma 3.6, in this case the level graph Γ has exactly two levels with no horizontal edges. Since every lower component has to contain a marked zero, there are at most two lower components. Moreover, for a top component, it either has at least two zeroes (including both marked zeros or zero edges), or its residues are not all equal to zero (see [GT21, Th. 1.2(i)]).

First, consider the case that the graph Γ has a unique lower component. Then, this component either contains one or both zeros. If the unique lower component contains both zeros, then Γ cannot have two or more top components. Otherwise, there would be a partial sum vanishing induced by the residue theorem on each top component, which is not possible since by hypothesis the residue tuple λ is generic. Hence, Γ has a unique top component. Moreover, the top component contains all the poles. Otherwise, there would still be a partial sum vanishing of the residues on top, which contradicts the generic choice of λ. Therefore, this case corresponds to merging the two zeros, as illustrated in type (1).

If the unique lower component contains only one zero, then again there must be a unique top component. By stability, the lower component contains at least one marked pole, and hence all the residues on the top component vanish by Corollary 3.3. This corresponds to the case where either I or J is empty, as illustrated in type (2') (for J nonempty).

Next, consider the case that Γ has exactly two lower components. Since each of the lower components contains a unique zero and they are not joined by any horizontal edges, it follows that the top level has a unique component. Then, Γ must be a cherry graph, and by stability, every component contains at least one marked pole. Therefore, this is the case when I, J, and K are all nonempty, as illustrated in type (2). Finally, the description of residues for each case follows from Corollary 3.3. □

Remark 3.8. — Note that in the above description, the assumption that λ is not contained in $\mathcal{A}_p$ is necessary. Otherwise, for non-generic λ, for example, if some residues are prescribed to be zero, when the corresponding poles move to the lower level, they do not constrain the top-level residues to be zero.

Remark 3.9. — We observe that in the cases of Proposition 3.7, once the level graph and the prescribed residues are given, the top-level differential (with exactly two zeros and residueless poles) and the bottom-level differential (with residues specified by λ) each have only finitely many choices (up to a scale multiple). This makes sense, because the intersection locus of the one-dimensional isoresidual fiber $\overline{\mathcal{F}}_\lambda$ with the boundary of $\mathrm{MS}(\mu)$ must be zero-dimensional.

Next, we describe how $\overline{\mathcal{F}}_\lambda$ intersects the horizontal boundary of $\mathrm{MS}(\mu)$.

Proposition 3.10. — *Suppose $g = 0$ and μ has exactly two nonnegative entries. For $\lambda \in \mathcal{R}_p$, suppose $\overline{\mathcal{F}}_\lambda$ contains a multi-scale differential (X, ω) with a horizontal edge. Then X consists of two irreducible components, X_1 and X_2, where X_1 contains the zero of order a_1 and the subset of poles labeled in I, while X_2 contains the zero of order a_2 and the subset of poles labeled in J. Furthermore, $I \sqcup J = \{1, \dots, p\}$, and $a_1 - \sum_{i\in I} b_i + 1 = 0$. Moreover, the residues of ω at the marked poles are determined by λ.*

Proof. — In this case, Lemma 3.6 implies that X has exactly two components joined by a unique horizontal node, which yields the desired configuration. Moreover, note that the residues at the marked poles do not all degenerate to zero (by the residue theorem and the fact that the horizontal node has non-zero residue). Hence, Corollary 3.3 implies that the residues at the marked poles are still determined by λ. □

4. The canonical translation structure of isoresidual curves

In this section we first introduce the translation structure on isoresidual fibers in Section 4.1 and study its singularities in Sections 4.2 to 4.4. The following sections give more details on this structure and in the last Section 4.8, we give the formal proof of Theorem 1.2.

4.1. The translation structure on isoresidual fibers. — Consider a stratum of meromorphic differentials $\mathcal{H}(a_1, a_2, -b_1, \ldots, -b_p)$ on $\mathbb{CP}^1$ with two zeros. The isoresidual fibers are defined by fixing the periods of differentials along close loops encircling the poles. In terms of period coordinates, the unique local parameter remaining to deform $(\mathbb{CP}^1, \omega)$ is the relative period of the differential along a path joining the two zeros. We recall that the zeros are labeled: the one of order a_i is denoted by z_i for $i = 1, 2$.

Any homotopy class of paths in $\mathbb{CP}^1$ punctured at the poles has at least one geodesic representative. Therefore, any homotopy class of paths from z_1 to z_2 has a representative that is a chain of saddle connections. Consequently, there is at least one saddle connection from z_1 to z_2.

Consider the isoresidual fiber $\mathcal{F}_\lambda$ determined by the configuration $\lambda = (\lambda_1, \ldots, \lambda_p)$. Let $\omega \in \mathcal{F}_\lambda$ and γ be a path joining the two zeros z_1 and z_2 of ω. In a neighborhood of ω in $\mathcal{F}_\lambda$, the period $z_\gamma = \int_\gamma \omega$ can serve as a local complex coordinate for $\mathcal{F}_\lambda$.

For any other path γ' joining the two zeros of ω, the difference $z_\gamma - z_{\gamma'}$ is the period of the differential along the closed loop $\gamma \cup -\gamma'$. This latter period is a linear combination of residues. Since the residues are fixed in $\mathcal{F}_\lambda$, the two local coordinates thus differ by a constant. Therefore, the differential dz_γ on $\mathcal{F}_\lambda$ does not depend on the choice of γ and hence defines a canonical *translation structure* on $\mathcal{F}_\lambda$.

The relative period z_γ is a locally injective complex function on an open subset of $\mathcal{F}_\lambda$. In particular, dz_γ (defined on the entire fiber $\mathcal{F}_\lambda$) has no zero or pole. On the other hand, zeros and poles of dz_γ appear on the boundary of the closure $\overline{\mathcal{F}}_\lambda$ in the multi-scale compactification of the stratum as given in Section 4.1. We will prove that the differential ω_λ extends to the closure $\overline{\mathcal{F}}_\lambda$ of $\mathcal{F}_\lambda$ as a meromorphic one-form.

4.2. Zeros of ω_λ. — First, we will describe the boundary points that are zeros of ω_λ. Recall that $a = a_1 + a_2$.

Proposition 4.1. — *Given a stratum $\mathcal{H}(a_1, a_2, -b_1, \ldots, -b_p)$, for any $\lambda \in \mathcal{R}_p \smallsetminus \mathcal{A}_p$, the boundary points of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ corresponding to Proposition* 3.7(1) *are the zeros of ω_λ.*

These zeros are of order a and there are $f(a,p) = a!/(a+2-p)!$ of them, corresponding to elements in the zero-dimensional generic isoresidual fiber of the minimal stratum $\mathcal{H}(a, -b_1, \ldots, -b_p)$.

Proof. — Let (X, ω) be the multi-scale differential in the boundary of $\overline{\mathcal{F}}_\lambda$ corresponding to the merging of the two zeros z_1 and z_2 into a zero z of order $a_1 + a_2$. The stable curve X is a union of two $\mathbb{CP}^1$ attached in such a way that z is glued to the other component containing z_1 and z_2 at a node. The breakup of z into two nearby zeros of order a_1 and a_2 is described in Corollary 3.5. The absolute period coordinates of the stratum at this point are given by the residues of the poles and by the opening parameter t of the node. Indeed, since we consider the projectivized period joining z_1 and z_2, this is zero-dimensional. Now, the differential ω_λ on the isoresidual fiber is given by the derivative of the period joining the two zeros. The lower level differential is multiplied by t^{a+1} when smoothed, as explained in [BCG+19b], especially in §§8.2 & 9.2.

Therefore, the differential of the period ct^{a+1}, where c is a constant, has the form $c(a+1)t^a dt$, which has a zero of order a. The number of such zeros corresponds to the cardinality of the zero-dimensional generic isoresidual fiber in $\mathcal{H}(a,-b_1,\ldots,-b_p)$. $\square$

Remark 4.2. — In terms of flat geometry, this proposition can be seen as follows. When splitting z into two zeros of orders a_1 and a_2, there are $a+1=a_1+a_2+1$ horizontal positive directions to open it up at z. Each of these yields a real positive saddle connection joining z_1 and z_2. Therefore, under the translation structure ω_λ induced by the primitive of such a saddle connection, (X,ω) is a conical singularity of angle $2\pi(a+1)$ in the isoresidual fiber.

Example 4.3. — A differential in $\mathcal{H}(2,3,-1,-2,-4)$ is shown in Figure 5. Its isoresidual deformations are given by the variation of γ, while fixing all the other saddle connections. Now, suppose that γ is very short and we start rotating it in the positive

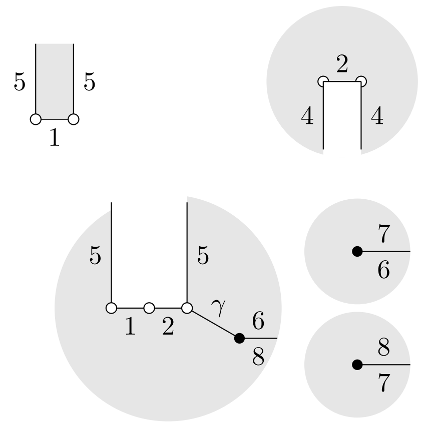

Figure 5. A differential in $\mathcal{H}(2,3,-1,-2,-4)$ such that the isoresidual deformations are given by varying γ only.

direction. It first meets the vertical ray labeled by 5, and then the saddle connection labeled by 1. In particular, after a rotation of angle 2π, it does not come back to its original position. Only after a rotation of angle 12π, it comes back to its original position. Therefore, this corresponds to a zero of order 5 under the canonical translation structure of the isoresidual fiber.

4.3. Simple poles of ω_λ. — The intersection of $\overline{\mathcal{F}}_\lambda$ with the horizontal boundary of $\mathrm{MS}(\mu)$ is described in Proposition 3.10. In the following, we will prove that each of these boundary points is a simple pole of the translation structure of $(\overline{\mathcal{F}}_\lambda,\omega_\lambda)$.

Proposition 4.4. — *In a stratum $\mathcal{H}(a_1,a_2,-b_1,\ldots,-b_p)$, for any $\lambda\in\mathcal{R}_p\smallsetminus\mathcal{A}_p$, the boundary points of $\overline{\mathcal{F}}_\lambda$ that correspond to multi-scale differentials with a horizontal edge are simple poles of ω_λ.*

For such a boundary point, the underlying nodal curve X consists of two irreducible components, X_1 and X_2. Here, X_1 contains the zero of order a_1 and the subset of poles labeled by I, while X_2 contains the zero of order a_2 and the subset of poles labeled by J. We have $I \sqcup J = \{1, \ldots, p\}$ and $a_1 - \sum_{i\in I} b_i + 1 = 0$. We denote by p_I and p_J the numbers of marked poles in X_1 and X_2, respectively.

For each such partition $I \sqcup J$, there are

$$\frac{a_1!}{(a_1 - p_I + 1)!} \cdot \frac{a_2!}{(a_2 - p_J + 1)!} \tag{4.1}$$

of these boundary points. The residue of ω_λ at each of these simple poles is given by $r_J := \sum_{j\in J} \lambda_j$.

Proof. — The counting of such simple poles of ω_λ follows from the formula in Theorem 2.19. Now let (X, ω) be a multi-scale differential described in Proposition 3.10. We use the log period coordinates as given in [BCG+19b, §§10.10 & 13.3]. In particular, from the last equation of that section, if the opening parameter of the node is t, then there exist constants c_1 and c_2 such that

$$\int_{z_1}^{z_2} \omega = r_J \log(t/c_1) + c_2.$$

Near (X, ω), the differential ω_λ is thus given by

$$\omega_\lambda = d\left(\int_{z_1}^{z_2} \omega\right) = \frac{r_J}{t}\, dt. \qquad \square$$

Remark 4.5. — The flat geometric way to understand the above result is the following. The multi-scale differential (X, ω) has an infinite cylinder C corresponding to the simple pole q, where the marked poles in the subsets I and J are separated on the two sides of q, respectively. Since the residues are determined by λ and do not vary in the isoresidual fiber, the residue of q, i.e., the period of the cylinder core curve, is fixed as $\sum_{j\in J} \lambda_j$ (up to orientation). Therefore, the deformation of (X, ω) into $\mathcal{F}_\lambda$ is parametrized by deforming the cylinder C back to a finite cylinder while preserving the period of the core curve. A local neighborhood of (X, ω) under the translation structure of ω_λ can be identified with a local neighborhood of q in (X, ω).

Example 4.6. — A flat geometric picture of an isoresidual deformation around a simple pole is given in Figure 6. The simple pole is formed by letting the height of γ go to infinity. Note that the action of the parabolic element $\left(\begin{smallmatrix}1 & s\\ 0 & 1\end{smallmatrix}\right)$ on the cylinder bounded by the saddle connections γ, 1, 2 and 9 is trivial, where s is the length of the saddle connection 9. This illustrates the fact that the corresponding pole has a residue equal to s as explained in Section 4.3.

4.4. Higher-order poles of ω_λ. — Recall that for generic residues the intersection points of $\overline{\mathcal{F}}_\lambda$ with the vertical boundary of $\mathrm{MS}(a_1, a_2, -b_1, \ldots, -b_p)$ are described in Proposition 3.7. Case (1) corresponds to the zeros of ω_λ as shown in Section 4.2. Now we treat case (2) and show that these are poles of order at least two in the translation structure of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$. These points are described by a partition $I \sqcup J \sqcup K$ of the set of

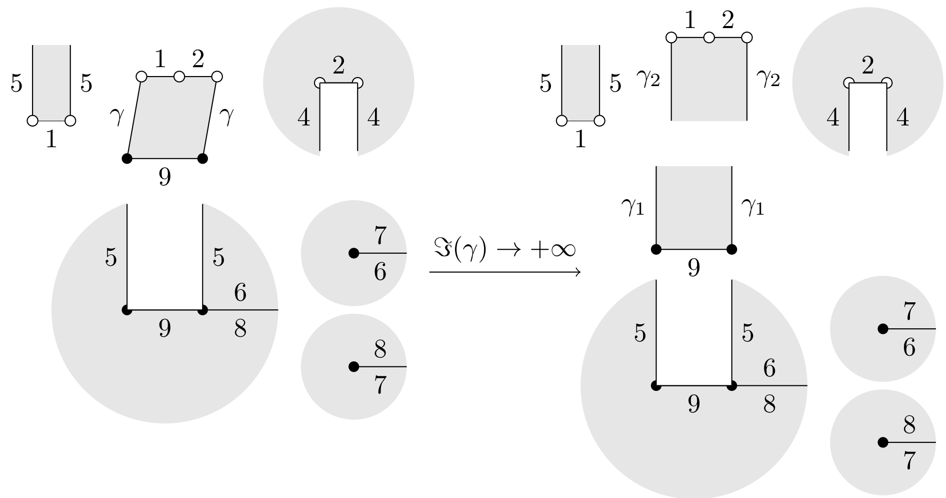

Figure 6. A isoresidual deformation in $\mathcal{H}(2,3,-1,-2,-4)$ leading to a simple pole of its flat structure.

the poles. Their level graphs are cherry graphs as pictured in Figure 4. The top level component X_{top} contains the subset K, while the bottom level has:

– one vertex X_1 that contains a subset of poles I with the zero z_1 of order a_1;
– one vertex X_2 that contains a subset of poles J with the zero z_2 of order a_2.

We denote by p_I, p_J, and p_K the numbers of marked poles in X_1, X_2, and X_{top}, respectively. We further define

$$c_1 = a_1 + 1 - \sum_{i\in I} b_i \quad \text{and} \quad c_2 = a_2 + 1 - \sum_{j\in J} b_j. \tag{4.2}$$

Finally, recall from Section 2.6 that $\Xi(\mu)$ is the number of residueless differentials in the stratum $\mathcal{H}(\mu)$.

Proposition 4.7. — *Given a stratum $\mathcal{H}(a_1,a_2,-b_1,\dots,-b_p)$, $\lambda \in \mathcal{R}_p \smallsetminus \mathcal{A}_p$ and a boundary point of $(\overline{\mathcal{F}}_\lambda,\omega_\lambda)$ corresponding to the partition of poles $I\sqcup J\sqcup K$. Up to reordering the indices, we assume that the poles in K are the poles of order $b_1,\dots,b_{p_K}$. We have the following cases:*

– Both I and J are nonempty. *There are*

$$\frac{a_1!}{(a_1-p_I+1)!}\cdot\frac{a_2!}{(a_2-p_J+1)!}\cdot \gcd(c_1,c_2)\cdot \Xi(c_1-1,c_2-1;b_1,\dots,b_{p_K}) \tag{4.3}$$

of these boundary points. Each of them is a pole of order $1+\operatorname{lcm}(c_1,c_2)$ for ω_λ, with residue

$$-\frac{c_2}{\gcd(c_1,c_2)}\sum_{i\in I}\lambda_i + \frac{c_1}{\gcd(c_1,c_2)}\sum_{j\in J}\lambda_j. \tag{4.4}$$

– I is empty and J is nonempty. *There are*

$$\frac{a_2!}{(a_2 - p_J + 1)!} \cdot \Xi(a_1, c_2 - 1; b_1, \ldots, b_{p_K}) \tag{4.5}$$

of these boundary points. Each of them is a pole of order $1 + c_2$ *with residue*

$$\sum_{j \in J} \lambda_j. \tag{4.6}$$

– I is nonempty and J is empty. *There are*

$$\frac{a_1!}{(a_1 - p_I + 1)!} \cdot \Xi(c_1 - 1, a_2; b_1, \ldots, b_{p_K}) \tag{4.7}$$

of these boundary points. Each of them is a pole of order $1 + c_1$ *with residue*

$$-\sum_{i \in I} \lambda_i. \tag{4.8}$$

– Both I and J are empty. *There are* $\Xi(a_1, a_2; b_1, \ldots, b_p)$ *of these boundary points. Each of them is a double pole with zero residue.*

As a general remark, if we transpose the zeros z_1 and z_2, we can multiply ω_λ by -1 and transpose the subsets I and J. This symmetry is reflected in the local invariants of the poles of ω_λ.

Proof. — We first focus on the case when I, J, and K are all nonempty. We start by counting the number of multi-scale differentials associated to this partition in the closure of the isoresidual fiber. The data of a multi-scale differential has various pieces. First, it has two differentials in the lower level strata with a unique zero respectively of order a_i and residues of poles prescribed. This gives the first two terms of Equation (4.3). On the top level, we have a differential with two zeros and poles of orders $b_1, \ldots, b_{p_K}$ and zero residues. Their number gives the last term of Equation (4.3). Additionally, a multi-scale differential contains the data of an equivalence class of *prong-matchings* (see [BCG⁺19a, §5.4]). In the case of a cherry graph with the prong numbers c_1 and c_2 at the two vertical edges, there are $\gcd(c_1, c_2)$ of such classes, giving the remaining term of Equation (4.3).

Next, we determine the flat geometric structure at such points. In order to avoid heavy notations, we write only the relevant part of the log period coordinates. The isoresidual curve is parametrized by the opening parameter t of both nodes. The period of a path γ joining z_1 and z_2 is a multi-valued function (since the residue corresponding to such pole is not zero in general). Nevertheless, since we consider a one-parameter family, [Ben23, Cor. 6.3] shows that its log period $\psi_\gamma(t)$ is a holomorphic function of t. Moreover, it gives a formula for the value of this log period that we can compute in our context (following the notation of [Ben23, Def. 5.1]). We denote by $m_i = \operatorname{lcm}(c_1, c_2)/c_i$ for $i = 1, 2$. The rescaling parameter of the top component is equal to 1, so $t_{\lceil \top(\gamma) \rceil} = 1$. Now consider the edge e_1 between the top component and the bottom component containing z_1 and the corresponding vanishing cycle λ_{e_1}. The intersection pairing $\langle \gamma, \lambda_{e_1} \rangle = 1$. The residue $r_{e_1}(t)$ is given by $t^a \sum_{i \in I} \lambda_i$, where $a = \operatorname{lcm}(c_1, c_2)$. Finally, σ_e is given by m_i based on the formula at the beginning of [Ben23, §7.4]. Doing the same for e_2 and noting that $\langle \gamma, \lambda_{e_2} \rangle = -1$, we obtain that the log period of γ on the isoresidual curve is

$$\psi_\gamma(t) = \int_\gamma \omega(t) - \lambda_I t^a m_1 \log(t) + \lambda_J t^a m_2 \log(t),$$

where $\lambda_I = \sum_{i\in I} \lambda_i$ and $\lambda_J = \sum_{j\in J} \lambda_j$. Since $\psi_\gamma(0) \neq 0$ (by the second part of [Ben23, Cor. 6.3]), the period of γ is given by

$$t^{-a}\psi_\gamma(t) - \lambda_I m_1 \log(t) + \lambda_J m_2 \log(t).$$

Therefore, its derivative

$$\Big(t^{-a-1}(-a\psi_\gamma(t) + t\psi_\gamma'(t)) + \frac{-\lambda_I m_1 + \lambda_J m_2}{t}\Big)dt$$

has a pole of order $a+1$ with residue as given by Equation (4.4).

If I is empty and J is nonempty, the stable model of the level graph has exactly one top vertex, one bottom vertex, and one vertical edge, where the top vertex contains z_1 and the poles in K, the bottom vertex contains z_2 and the poles in J, and the prong number of the edge is c_2. In this case, there is a unique prong-matching equivalence class. The same argument as in the preceding case justifies the desired claim. The case when I is nonempty and J is empty follows by symmetry.

If I and J are both empty, the stable model of the level graph consists of a single vertex only, where the residue of every pole becomes zero. The number of such differentials with zero residues is computed in Section 2.6 as $\Xi(a_1, a_2; b_1, \dots, b_p)$. Consider such a residueless differential $(\mathbb{CP}^1, \omega_0)$. At this point, the stratum has the usual period coordinates $(r_1, \dots, r_{p-1}, t)$, where r_i is the residue at the pole q_i and t is a period between z_1 and z_2. Hence, the differential ω_0 corresponds to $(0, \dots, 0, c)$ under these coordinates, where c is nonzero, and nearby points in the isoresidual fiber correspond to the coordinates $(s\lambda_1, \dots, s\lambda_{p-1}, f(s))$, where $f(s)$ is a holomorphic function such that $f(0) = c$. Therefore, the differential ω_λ is locally given by $d\,(f(s)/s) = [(f'(s)s - f(s))/s^2]ds$, where the Taylor series of the numerator at 0 is of the form $f(0) + o(s^2)$. Hence, ω_λ has a pole of order 2 whose residue vanishes. $\square$

Example 4.8. — The flat geometric intuition behind Proposition 4.7 is that the two zeros can move arbitrarily far away from each other under the flat metric. An example of an isoresidual deformation in the stratum $\mathcal{H}(2, 3, -1, -2, -4)$ illustrating the second case is given in Figure 7. In the picture, the zero z_1 of order 2 is labeled in white while the zero z_2 of order 3 is in black. In the deformation, both saddle connections corresponding to α and β go to infinity, in such a way that the difference of their periods $\int_\alpha \omega - \int_\beta \omega$ remains constant. In the picture, we only illustrate the bottom components of the resulting multi-scale differential. Note that the component containing z_1 is unstable. Hence, its stabilization is of the form (2') in Figure 4.

To compute the orders of the poles, we can proceed in the following way. Consider the pair of saddle connections (α, β) with very large periods as in the picture. Then start rotating them such that their difference remains constant. In the upper right

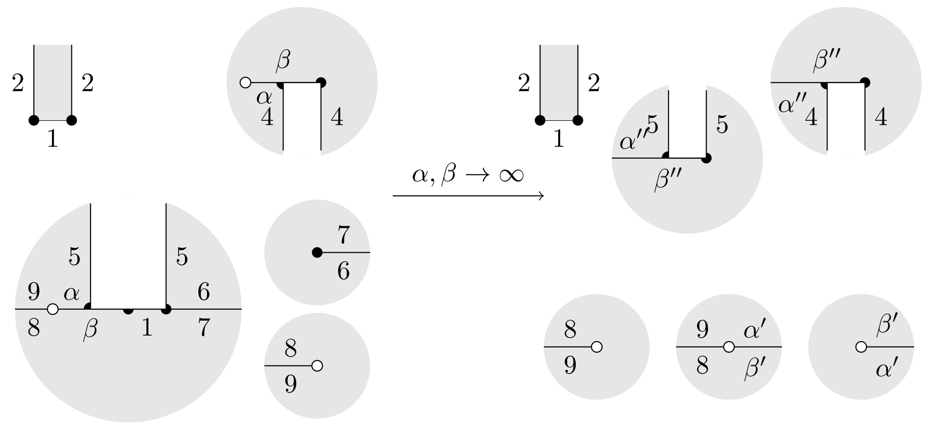

Figure 7. An isoresidual deformation in $\mathcal{H}(2,3,-1,-2,-4)$ converging to a multi-scale differential of cherry type, where $I=\varnothing$, J contains the simple pole, and K contains the other two poles.

of the first picture, this pair comes back to its original position after a rotation of angle 2π. In the lower part of the picture, the saddle connections come back to their initial positions only after a rotation of angle 4π. Therefore, the whole picture comes back to its initial position after a rotation of (α,β) of angle 4π. This implies that the corresponding pole has order -3.

Moreover, we can determine the residue of the pole as follows. Note that after a rotation of angle 4π, the pair (α,β) does not come to its original position. Indeed, they are both shortened by the difference of the original periods of β and α. It implies that the difference $\int_\beta \omega - \int_\alpha \omega$ is the residue of the corresponding pole in the isoresidual curve, which is equal to the prescribed residue of the simple pole of the differentials parameterized in the isoresidual curve. Therefore, this confirms Equation (4.6) of Proposition 4.7.

Finally, in Figure 5, we can produce another example by letting γ go to infinity. In that case, we have $I=\varnothing$, J contains the poles of orders -1 and -2, and K contains the pole of order -4.

We remark that Propositions 4.1, 4.4, and 4.7 fully describe the zeros, simple poles, higher-order poles, and their orders for the translation structure ω_λ on $\overline{\mathcal{F}}_\lambda$. In particular, they determine the Euler characteristic $2-2g(\overline{\mathcal{F}}_\lambda)$. Later, in Proposition 6.3, we will recover the Euler characteristic using the alternative method of intersection theory on the multi-scale compactification of strata of differentials.

4.5. Cylinders in isoresidual fibers. — We observe that every cylinder belonging to a generic isoresidual fiber must be infinite.

Proposition 4.9. — *For a stratum $\mathcal{H}(\mu)$ with $n=2$ zeros and every configuration of residues $\lambda\in\mathcal{R}_p\smallsetminus\mathcal{A}_p$, any cylinder in the closure $\overline{\mathcal{F}}_\lambda$ of the isoresidual fiber $\mathcal{F}_\lambda$ bounds a simple pole.*

Proof. — We assume that $(\overline{\mathcal{F}}_\lambda,\omega_\lambda)$ admits a closed geodesic γ. Up to scaling, we can assume that γ is horizontal and has unit length. We will show that the closed geodesic γ belongs to an infinite cylinder.

In $(\overline{\mathcal{F}}_\lambda,\omega_\lambda)$, we introduce the flow $(A_t)_{t\in\mathbb{R}}$ that acts on the parametrized differentials by keeping the residues unchanged while the relative period $z_{1,2}$ joining the two zeros z_1 and z_2 becomes $z_{1,2}+t$. Any differential ω parametrized in γ is periodic with a period of 1 under the action of $(A_t)_{t\in\mathbb{R}}$.

We first consider the case where, for some differential ω of $\mathcal{F}_\lambda$ contained in the closed geodesic γ, $(\mathbb{CP}^1,\omega)$ contains a cylinder $\mathcal{C}$ of finite area foliated by horizontal closed geodesics. Since $(\mathbb{CP}^1,\omega)$ is of genus zero, the cylinder $\mathcal{C}$ decomposes the surface into two connected components, each containing exactly one zero of ω. These two components are unchanged by the action of $(A_t)_{t\in\mathbb{R}}$, while A_1 induces a Dehn twist on the cylinder $\mathcal{C}$ (recall that $A_1\omega=\omega$). Taking the period of a diagonal of the cylinder as a local coordinate, we can make its imaginary part arbitrarily large in absolute value. This proves that γ belongs to an infinite cylinder bounded by some simple pole of ω_λ.

In the following, we will assume that for any differential ω of $\mathcal{F}_\lambda$ that is contained in the closed geodesic γ, there is no horizontal cylinder of finite area in $(\mathbb{CP}^1,\omega)$. We then introduce $\Phi_\omega\subset\mathbb{R}$ as follows: $s\in\Phi_\omega$ if s is the imaginary part of the period of a saddle connection in $(\mathbb{CP}^1,\omega)$ joining z_1 and z_2 (with this orientation). The subset Φ_ω is nonempty because any length-minimizing path between the two zeros (for the flat metric induced by ω) contains a saddle connection between them.

We first observe that if $0\in\Phi_\omega$, then $(\mathbb{CP}^1,\omega)$ has a horizontal saddle connection, and the two zeros of ω collide in finite time under the action of $(A_t)_{t\in\mathbb{R}}$. This is impossible because the differential ω belongs to the closed geodesic γ and is supposed to be periodic under the action of $(A_t)_{t\in\mathbb{R}}$.

Since $(\mathbb{CP}^1,\omega)$ has no horizontal cylinder of finite area, Proposition 2.3 implies that the slopes of saddle connections of $(\mathbb{CP}^1,\omega)$ cannot accumulate in the horizontal direction. It follows that for any $s>0$, $\Phi_\omega\cap[-s,s]$ is a finite set (the number of saddle connections of length smaller than some bound is always finite). We deduce that the infimum δ_ω of the absolute values of elements of Φ_ω is realized by some saddle connection of $(\mathbb{CP}^1,\omega)$.

Along the action of the flow, a saddle connection α_t of $A_t\omega$ joining the two distinct zeros is destroyed when its interior is crossed by a conical singularity (two conical singularities cannot collide because γ remains in the regular locus of $\mathcal{F}_\lambda$). Consequently, our initial saddle connection is cut into two saddle connections, one of which still joins the two zeros. Observe that this latter saddle connections persists under small perturbations of t, and the imaginary part of its period is (in absolute value) strictly smaller than that of α_t. It follows that $\delta_{A_t\omega}$ is constant.

The same argument shows that a saddle connection α of $(\mathbb{CP}^1,\omega)$ that realizes δ_ω persists under the action of $(A_t)_{t\in\mathbb{R}}$. It follows that for a saddle connection α in $(\mathbb{CP}^1,\omega)$ of period z, there is another saddle connection in $(\mathbb{CP}^1,\omega)$ of period $z+1$. Consequently, there is an infinite sequence of saddle connections in $(\mathbb{CP}^1,\omega)$ whose directions approach the horizontal direction. Proposition 2.3 proves that there is a horizontal cylinder of finite area in $(\mathbb{CP}^1,\omega)$. This case has already been settled. This concludes the proof. □

Translation surfaces with infinitely many saddle connections have been characterized in [Tah18, Cor. 5.13] as those that contain a cylinder of finite area.

Corollary 4.10. — *For a stratum $\mathcal{H}(\mu)$ with $n=2$ zeros and any configuration of residues $\lambda\in\mathcal{R}_p\smallsetminus\mathcal{A}_p$, the closure $\overline{\mathcal{F}}_\lambda$ of the isoresidual fiber $\mathcal{F}_\lambda$ contains finitely many saddle connections.*

4.6. Isoresidual fibers over configurations of real residues. — In a translation surface (X,ω) of genus zero with real residues, the group generated by the absolute periods of ω is contained in $\mathbb{R}$. Consequently, even though the relative periods between two zeros are defined up to the addition of an absolute period, the imaginary part of the relative period between any two fixed zeros does not depend on the integration path.

Lemma 4.11. — *For a stratum $\mathcal{H}(a_1,a_2,-b_1,\dots,-b_p)$ and a configuration λ of real residues, there exists a harmonic function $\mathcal{I}\colon\mathcal{F}_\lambda\to\mathbb{R}$ such that for any differential $\omega\in\mathcal{F}_\lambda$ and any path γ joining the two zeros in $(\mathbb{CP}^1,\omega)$, we have*

$$\Im\left(\int_\gamma\omega\right)=\mathcal{I}(\omega).$$

Proof. — In $(\mathbb{CP}^1,\omega)$, the integral of ω along two distinct paths between the two zeros z_1 and z_2 differs by an absolute period of ω. Given that the latter is real, $\Im\big(\int_\gamma\omega\big)$ is a real function $\mathcal{I}$ of ω, which is globally defined on $\mathcal{F}_\lambda$. We observe that for any local period coordinate of the translation structure of $\mathcal{F}_\lambda$, $\mathcal{I}$ coincides with its imaginary part. In other words, $\mathcal{I}$ is the imaginary part of a holomorphic function on a Riemann surface, and it is therefore a harmonic function. □

For an isoresidual fiber $\mathcal{F}_\lambda$, where λ consists of real residues, we define the *real locus* $\mathbb{R}\mathcal{F}_\lambda\subset\mathcal{F}_\lambda$ as the locus formed by differentials for which the relative period between the two zeros is also real. Equivalently, $\mathbb{R}\mathcal{F}_\lambda=\mathcal{I}^{-1}(0)$.

Lemma 4.12. — *In a stratum $\mathcal{H}(a_1,a_2,-b_1,\dots,-b_p)$, for any configuration of real residues $\lambda\in\mathcal{R}_p\smallsetminus\mathcal{A}_p$, every zero of $(\overline{\mathcal{F}}_\lambda,\omega_\lambda)$ belongs to the closure of the real locus $\mathbb{R}\mathcal{F}_\lambda$. Moreover, every saddle connection of $(\overline{\mathcal{F}}_\lambda,\omega_\lambda)$ is horizontal.*

Proof. — When the two zeros of the parametrized translation surfaces collide, the relative period computed by integrating the differential on the shrinking saddle connection tends to zero. In particular, its imaginary part (well-defined since the absolute periods of the differential are real) also tends to zero. Additionally, a zero of order a_1+a_2 can be locally split via a horizontal opening direction to two zeros of order a_1 and a_2. Thus, every zero of ω_λ belongs to the closure of the real locus.

Following Lemma 4.11, the imaginary part of any local period coordinate of the translation atlas in $\mathcal{F}_\lambda$ is given by the harmonic function $\mathcal{I}$. Saddle connections of $(\overline{\mathcal{F}}_\lambda,\omega_\lambda)$ have constant slopes in the translation atlas, and their endpoints belong to the zero set of $\mathcal{I}$. It follows that every saddle connection of $(\overline{\mathcal{F}}_\lambda,\omega_\lambda)$ is completely contained in the real locus $\mathbb{R}\mathcal{F}_\lambda=\mathcal{I}^{-1}(0)$ and is therefore horizontal. □

Any differential ω in $\mathbb{R}\mathcal{F}_\lambda$ can be described by the following data:

– the decorated graph $\mathfrak{gr}(\omega)$ formed by p vertices and p edges (see Section 2.4);

– the lengths of the p saddle connections of ω.

Since $\mathfrak{gr}(\omega)$ has p vertices and p edges, it contains a uniquely defined loop. This loop has:

– *coherent orientation* if all the edges of the loop have the same orientation;

– *incoherent orientation* if two consecutive edges in the loop have opposite orientations.

Because the distinction between coherent or incoherent orientation depends only on the graph, this notion induces a dichotomy on the connected components of $\mathbb{R}\mathcal{F}_\lambda$.

Proposition 4.13. — *We consider a stratum $\mathcal{H}(\mu)$ with $n=2$ zeros and any configuration of real residues λ lying outside $\mathcal{A}_p$. Let $\mathcal{C}$ be a connected component of the real locus $\mathbb{R}\mathcal{F}_\lambda$ of the isoresidual fiber $\mathcal{F}_\lambda$. Then one of the following statements holds:*

– If the decorated graph of the differentials in $\mathcal{C}$ has coherent orientation, then $\mathcal{C}$ is an infinite horizontal trajectory between a zero and a pole of $(\overline{\mathcal{F}}_\lambda,\omega_\lambda)$.

– If the decorated graph of the differentials in $\mathcal{C}$ has incoherent orientation, then $\mathcal{C}$ is a horizontal saddle connection of $(\overline{\mathcal{F}}_\lambda,\omega_\lambda)$ joining two zeros of ω_λ. Moreover, its length is $|\sum_{j\in J}\lambda_j|$ for some subset J of $\{1,\dots,p\}$.

Proof. — We first consider the case where the decorated graph of the differentials in $\mathcal{C}$ has coherent orientation. The fact that the residues at the poles are fixed means that the difference between the lengths of the saddle connections corresponding to two adjacent oriented edges of the loop is fixed. This is the only constraint on them. It follows that the lengths of the saddle connections corresponding to the edges of the loop (in other words, the saddle connections joining two distinct zeros) can be made arbitrarily large. Thus, $\mathcal{C}$ is an infinite arc in $(\overline{\mathcal{F}}_\lambda,\omega_\lambda)$.

As the lengths of these edges become infinite, the parametrized differential degenerates in the way described in Sections 3.4 and 4.4. This corresponds to a pole of ω_λ. Conversely, if we decrease the lengths of these saddle connections, at some point one of them shrinks to zero (only one at a time; otherwise, a resonance equation would hold between the residues). In other words, at that moment, the two zeros collide, leading to a degeneration corresponding to a zero of ω_λ.

Next, we discuss the case of a decorated graph with incoherent orientation. The edges of the loop form two subsets E_1 and E_2 according to the orientation of the edges. Again, the only degree of freedom is the length L of the saddle connection corresponding to an arbitrary edge in E_1. Indeed, the lengths of the closed saddle connections are determined by the sums of the residues of the poles they enclose. As L increases, the lengths of the other saddle connections in E_1 also increase, while the lengths of the saddle connections in E_2 decrease. Since all these lengths must remain positive, L is constrained to an interval. The connected component $\mathcal{C}$ is thus a horizontal segment whose endpoints correspond to the collision of the two zeros. Therefore, $\mathcal{C}$ is a horizontal saddle connection of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$.

It remains to compute the length of this saddle connection. At the left (resp. right) endpoint of $\mathcal{C}$, exactly one saddle connection from E_1 (resp. E_2) shrinks to zero. It is impossible for two saddle connections to shrink at the same time because their lengths are related by equations involving the residues, which would lead to a resonance equation. We denote by e_1 and e_2 the edges of the loop corresponding to the saddle connections that shrink at the left and right endpoints of $\mathcal{C}$, respectively. By removing e_1 and e_2 from the decorated graph, we decompose it into exactly two connected components, G_1 and G_2. Denote by $I \sqcup J$ the corresponding bipartition of the set of poles $\{1, \ldots, p\}$. We find that the lengths of the two saddle connections corresponding to the edges e_1 and e_2 sum to $|\sum_{i\in I} \lambda_i| = |\sum_{j\in J} \lambda_j|$. Since these represent the only saddle connections that can shrink as the parametrized differential is deformed along $\mathcal{C}$, this gives the length of $\mathcal{C}$. □

4.7. Periods of saddle connections. — Deformations of translation surfaces are governed by the deformation of the periods of their saddle connections. Since one-dimensional isoresidual fibers are translation surfaces, it is essential to understand how their saddle connections change in relation to the underlying configuration of residues. In the following, we will prove the linear dependence of the periods of an isoresidual fiber in terms of λ.

Proposition 4.14. — *For a stratum $\mathcal{H}(\mu)$ of differentials on $\mathbb{CP}^1$ with two zeros, for any configuration of residues $\lambda \in \mathcal{R}_p \smallsetminus \mathcal{A}_p$, and for any relative homology class $[\gamma]$ of $H_1(\overline{\mathcal{F}}_\lambda \smallsetminus P_{\omega_\lambda}, Z_{\omega_\lambda})$, we have*

$$\int_{[\gamma]} \omega_\lambda = \sum_{j=1}^{p} w_j \lambda_j,$$

where $w_j \in \mathbb{Z}$ for all $1 \leqslant j \leqslant p$.

Proof. — We first consider the case of an isoresidual fiber $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ where λ is a configuration of real residues. Following Lemma 4.12, all the saddle connections of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ belong to the real locus $\mathbb{R}\mathcal{F}_\lambda$. Proposition 4.13 shows that their periods are given by partial sums of residues. As with any translation surface, saddle connections generate the relative homology group $H_1(\overline{\mathcal{F}}_\lambda \smallsetminus P_{\omega_\lambda}, Z_{\omega_\lambda})$. It follows that the integral of ω_λ along any relative homology class is a linear combination of the residues $\lambda_1, \ldots, \lambda_p$

with integer coefficients. Using analytic continuation, we deduce that the same claim holds for any $\lambda \in \mathcal{R}_p \smallsetminus \mathcal{A}_p$. $\square$

Actually, the fact that the period of any saddle connection of the isoresidual fiber is a partial sum of residues holds for any fiber, not just those corresponding to configurations of real residues.

Proposition 4.15. — *Consider a stratum $\mathcal{H}(a_1, a_2, -b_1, \dots, -b_p)$ and a configuration of residues $\lambda \in \mathcal{R}_p \smallsetminus \mathcal{A}_p$. For any saddle connection γ in the closure $\overline{\mathcal{F}}_\lambda$ of the isoresidual fiber $\mathcal{F}_\lambda$, we have*

$$\int_\gamma \omega_\lambda = \sum_{j \in J} \lambda_j$$

for some subset J of $\{1, \dots, p\}$.

Proof. — Following Corollary 4.10, the isoresidual fiber $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$, as a translation surface, has finitely many saddle connections. Each of these saddle connections has a period that is a linear combination of $\lambda_1, \dots, \lambda_p$ with integer coefficients. We will prove that this linear combination is, in fact, a partial sum.

Up to a small perturbation of λ, we assume that saddle connections of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ have the same slope if and only if their periods are given by the same coefficients (up to multiplication by a constant real factor). Any saddle connection γ_1 belongs to a maximal family $\gamma_1, \dots, \gamma_t$ of saddle connections of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ that have the same slope $\theta \in \mathbb{RP}^1$.

Since $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ has finitely many saddle connections, there is an open interval $U \in \mathbb{RP}^1$ that contains θ but no other slopes of saddle connections. Up to the action of $\mathrm{GL}_2^+(\mathbb{R})$ (recall that this action commutes with the isoresidual map; see Section 2.2.4), we can assume that θ is the vertical direction, while the other saddle connections have slopes arbitrarily close to the horizontal direction.

By using the subgroup of $\mathrm{GL}_2^+(\mathbb{R})$ that preserves the horizontal direction while contracting exponentially fast a direction arbitrarily close to the vertical direction, we can conjugate $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ to the fiber corresponding to a configuration λ' that is arbitrarily close to a configuration of real residues, where the saddle connections conjugate to $\gamma_1, \dots, \gamma_t$ are the smallest saddle connections of the surface.

It follows that these saddle connections are obtained by deformations of the saddle connections of a fiber corresponding to a configuration with real residues. Since such saddle connections have periods given by partial sums of residues (see Proposition 4.13), the same holds for $\gamma_1, \dots, \gamma_t$. $\square$

Remark 4.16. — Observe that Proposition 4.15 imposes highly nontrivial constraints on the geometry of isoresidual fibers. Indeed, when three saddle connections $\gamma_1, \gamma_2, \gamma_3$ form the sides of a triangle, if the periods of the homology classes $[\gamma_1]$ and $[\gamma_2]$ are partial sums of the underlying residues, there is no reason for the period $[\gamma_3] = -[\gamma_1] - [\gamma_2]$ to also be a partial sum of residues.

4.8. Formal proof of Theorem 1.2. — We summarize all the results of Section 4 in Theorem 1.2.

Proof of Theorem 1.2. — The translation structure of $(\mathcal{F}_\lambda, \omega_\lambda)$ is described in Section 4.1, while its compactification is given in Section 3.4. It is proved in Sections 4.2, 4.3, and 4.4 that ω_λ extends to $\overline{\mathcal{F}}_\lambda$ as a meromorphic differential. The periods of saddle connections are computed in Proposition 4.15, and we know from Corollary 4.10 that there are finitely many of them.

We deduce that for a configuration of real residues in $\mathcal{R}_p \setminus \mathcal{A}_p$, all the saddle connections of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ are horizontal. Such a translation surface cannot correspond to a holomorphic differential, as this would imply that the area of the holomorphic translation surface is zero. Therefore, ω_λ must have at least one pole.

The fact that the isoresidual fibration is nonsingular outside $\mathcal{A}_p$ follows from Proposition 4.15. Since the periods of saddle connections are given by partial sums of residues, the translation structure of an isoresidual fiber $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ cannot degenerate as long as λ stays away from the resonance hyperplanes.

Finally, if $\mathcal{F}_\lambda$ is not connected, then the monodromy of the fibration acts transitively on the set of connected components; otherwise, the whole stratum would not be connected. The singularity pattern of the translation structure of each connected component of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ is invariant under deformation. It follows that these patterns are the same for each connected component of $\mathcal{F}_\lambda$. □

5. The Euler characteristic of generic isoresidual fibers

In this section, we study the combinatorial structure for the Euler characteristic of generic isoresidual fibers. In particular, we will prove Theorem 1.4.

5.1. The wall and chamber structures. — We first recall the wall and chamber structures described in Definition 1.3. Given $p \geqslant 1$, the singularity pattern space $\mathcal{SP}_p$ is $\{(x_1, x_2, y_1, \ldots, y_p) \in \mathbb{R}_{>0}^{p+2} \mid x_1 + x_2 = \sum y_j\}$ endowed with the family $\mathcal{W}_p$ of hyperplanes $W_1(I), W_2(I)$ of Equation (1.2) and $W_3(I, K, L), W_4(J, K, M)$ of Equations (1.3) and (1.4), where $I \sqcup J \sqcup K$ is a partition of the index set of the poles $\{1, \ldots, p\}$ into three disjoint subsets, and L and M are (possibly intersecting) arbitrary subsets of K.

In the above, x_i corresponds to the zero order plus one, i.e., $a_i + 1$ for $i = 1, 2$, while y_j corresponds to the pole order b_j for $j = 1, \ldots, p$. Now, we can show that in each chamber of the singularity pattern space $\mathcal{SP}_p$, separated by the walls in $\mathcal{W}_p$, the Euler characteristic of the isoresidual curve $\overline{\mathcal{F}}_\lambda$ has the desired piecewise homogeneous structure.

We say that a function is *homogeneous of degree k* if scaling all variables by λ scales the function value by λ^k. For example, $\gcd(x_1 - y_2, x_2 - y_3)$ is homogeneous of degree one for $x_i, y_j \in \mathbb{Z}$. Now, we can prove Theorem 1.4 through the following more detailed statement.

Proposition 5.1. — *For $\lambda \in \mathcal{R}_p \smallsetminus \mathcal{A}_p$ and $(a_1+1, a_2+1, b_1, \ldots, b_p)$ in any given chamber of $\mathcal{SP}_p$, the Euler characteristic of $\overline{\mathcal{F}}_\lambda$ is a sum of homogeneous components of degree from 0 up to $p-1$ in terms of the variables $a_1+1, a_2+1, b_1, \ldots, b_p$. Moreover, the terms containing the* gcd *have degree at most $p-2$.*

Proof. — According to Propositions 4.1, 4.4, and 4.7 (see also Proposition 6.3 below), we can express the Euler characteristic of $\overline{\mathcal{F}}_\lambda$ in terms of its boundary points and the residueless locus. It then suffices to verify that the contributions from these loci to the Euler characteristic satisfy the desired claim in each chamber of $\mathcal{SP}_p$. Recall that the boundary of $\overline{\mathcal{F}}_\lambda$ consists of three types of graphs. The first type is given by two-level graphs Γ_{z_1,z_2}, where both zeros are in the unique bottom component and all the poles are in the unique top component, with the residues of the poles determined by λ. By Proposition 4.1, their total contribution to the Euler characteristic of $\overline{\mathcal{F}}_\lambda$ is a polynomial of degree $p-1$ in the variables a_1+1 and a_2+1, whose expression is independent of the chambers.

The second type is given by horizontal graphs satisfying the equations $(a_1+1) - \sum_{i\in I} b_i = 0 = (a_2+1) - \sum_{j\in I^c} b_j$ of type β_1 and β_2 in $\mathcal{W}_p$, where $I \subsetneq \{1, \ldots, p\}$ parametrizes the poles in the same component that contains the zero z_1, and the residues of the poles are determined by λ. By Proposition 4.4, their contribution to the Euler characteristic of $\overline{\mathcal{F}}_\lambda$ is a polynomial of degree $p-2$.

The third type, which we call cherry graphs as described in Proposition 4.7, is given by a partition of the poles $I \sqcup J \sqcup K = \{1, \ldots, p\}$, where the zero z_1 belongs to a bottom component with the poles indexed by I, the zero z_2 belongs to the other bottom component with the poles indexed by J, and the remaining poles are in the top component, indexed by K. These cherry graphs satisfy the conditions $(a_1+1) - \sum_{i\in I} b_i \geqslant 1$ and $(a_2+1) - \sum_{j\in J} b_j \geqslant 1$ as a consequence of the level-graph definition at the edges. Moreover, the multi-scale differential restricted to the top component is residueless, and, according to Proposition 2.18, the enumeration of such residueless differentials is given by the function

$$\begin{aligned}\Xi\left(c_{1,I}-1, c_{2,J}-1, \{-b_i\}_{i\in K}\right) &= \sum_{c_{1,I\sqcup L}-|K\smallsetminus L|\geqslant 0} (-1)^{|L|} f(c_{1,I\sqcup L}-1+|L|, |K|+1)\\ &= \sum_{c_{2,J\sqcup M}-|K\smallsetminus M|\geqslant 0} (-1)^{|M|} f(c_{2,J\sqcup M}-1+|M|, |K|+1),\end{aligned}$$

where $c_{j,I} = (a_j+1) - \sum_{i\in I} b_i$ and $f(a,n) = a!/(a-(n-2))!$. Note that the conditions in the above summation correspond to the walls of type β_3 and β_4 in $\mathcal{W}_p$. Moreover, this description also includes the residueless locus in the interior of $\overline{\mathcal{F}}_\lambda$, corresponding to the case $I = J = \varnothing$. By examining the cases in Proposition 4.7, we see that in each case, the contribution to the Euler characteristic of $\overline{\mathcal{F}}_\lambda$ consists of homogeneous components of degree at most $p-1$.

Finally, note that the term $\gcd(c_1, c_2)$ appears in Proposition 4.7 when I and J are both nonempty in the cherry graph. In the contribution to the Euler characteristic, it is multiplied by $1+\operatorname{lcm}(c_1,c_2)$. Since $\gcd(c_1,c_2)\operatorname{lcm}(c_1,c_2)=c_1c_2$, the remaining terms that contain the gcd only have degree strictly smaller than the main terms. ☐

In what follows, we provide an example to illustrate the description in Proposition 5.1.

Example 5.2. — Consider the stratum $\mathcal{H}(x_1-1,x_2-1,-y_1,-y_2,-y_3)$, where $y_2+y_3<x_1,x_2<y_1$. The singularities of ω_λ on $\overline{\mathcal{F}}_\lambda$ are:

- x_1+x_2-2 zeros of order x_1+x_2-2;
- y_2-1 poles of order x_2-y_3+1 (where $K=\{1,2\}$, $I=\varnothing$, and $J=\{3\}$);
- y_3-1 poles of order x_2-y_2+1 (where $K=\{1,3\}$, $I=\varnothing$, and $J=\{2\}$);
- y_2-1 poles of order x_1-y_3+1 (where $K=\{1,2\}$, $I=\{3\}$, and $J=\varnothing$);
- y_3-1 poles of order x_1-y_2+1 (where $K=\{1,3\}$, $I=\{2\}$, and $J=\varnothing$);
- $\gcd(x_1-y_2,x_2-y_3)$ poles of order $\operatorname{lcm}(x_1-y_2,x_2-y_3)+1$ (where $K=\{1\}$, $I=\{2\}$, and $J=\{3\}$);
- $\gcd(x_1-y_3,x_2-y_2)$ poles of order $\operatorname{lcm}(x_1-y_3,x_2-y_2)+1$ (where $K=\{1\}$, $I=\{3\}$, and $J=\{2\}$);
- x_1-1 poles of order $x_1-y_2-y_3+1$ (where $K=\{1\}$, $I=\{2,3\}$, and $J=\varnothing$);
- x_2-1 poles of order $x_2-y_2-y_3+1$ (where $K=\{1\}$, $I=\varnothing$, and $J=\{2,3\}$);
- $2(y_2-1)(y_3-1)$ doubles poles (where $I=J=\varnothing$).

Then, the total sum of the orders of the singularities is:

$$(x_1+x_2)(y_2+y_3)-2y_2y_3-2x_1-2x_2-2y_2-2y_3 \\ -\gcd(x_1-y_2,x_2-y_3)-\gcd(x_1-y_3,x_2-y_2)+6.$$

Note that the principal term is a homogeneous polynomial of degree 2, which equals $p-1$ for $p=3$, as predicted by Proposition 5.1.

Moreover, for the special case $y_2=y_3=1$ and $x_1=x_2=a+1$, the above formula simplifies to $-2\gcd(x_1-1,x_2-1)=-2a$. By Theorem 1.7, a generic isoresidual fiber of the stratum $\mathcal{H}(a,a,-2a,-1,-1)$ consists of a connected components. Since these connected components have the same topology, each of them has Euler characteristic -2. Therefore, a generic isoresidual fiber of the stratum $\mathcal{H}(a,a,-2a,-1,-1)$ is a disjoint union of a spheres.

5.2. The top term of the Euler characteristic of $\overline{\mathcal{F}}_\lambda$. — Following Proposition 5.1, the Euler characteristic of the generic isoresidual fiber for a given stratum is a sum of homogeneous components of degree from 0 up to $p-1$ in terms of the variables $a_1+1,a_2+1,b_1,\dots,b_p$. We conjecture that the term of degree $p-1$ does not vanish in any chamber and therefore provides the asymptotics for the Euler characteristic of the generic isoresidual fiber when the orders of singularities become large (for a fixed number of poles).

Conjecture 5.3. — *For $\lambda\in\mathcal{R}_p\smallsetminus\mathcal{A}_p$, the top term of the Euler characteristic of $\overline{\mathcal{F}}_\lambda$ is a piecewise homogeneous polynomial of degree $p-1$.*

In the chambers of $\mathcal{SP}_p$ where the order of the smallest zero is smaller than the order of every pole, we confirm Conjecture 5.3 by providing an explicit formula for the term of degree $p-1$ in the Euler characteristic.

Proposition 5.4. — *For any $p \geqslant 2$, the top term of the Euler characteristic of the generic isoresidual fiber in the chamber of $\mathcal{SP}_p$ defined by the inequalities $x_1 > p$ and $x_1 < y_j + 1 - p$ for any $j \in \{1, \dots, p\}$, is the homogeneous polynomial*

$$x_1^{p-1} + x_2^{p-1} - (x_1 + x_2)^{p-1}.$$

Proof. — Using the results of Sections 4.2 and 4.4, we compute the term of degree $p-1$ in the Euler characteristic of $\overline{\mathcal{F}}_\lambda$ by adding the terms of degree $p-1$ in the total order of the zeros and poles of ω_λ.

Since $x_1 < y_j - p$ for any $j \in \{1, \dots, p\}$, the poles of ω_λ correspond to degenerations where $I = \varnothing$ while J and K form an arbitrary partition of $\{1, \dots, p\}$ where $K \neq \varnothing$.

For each such partition $J \cup K$ where $J, K \neq \varnothing$, the total order of the poles (computed using the explicit formula for Ξ given in Proposition 2.18) is $x_2^{|J|-1} x_1^{|K|-1} \cdot (x_2 - \sum_{j \in J} y_j)$ (plus terms of lower order). Summing on the set of partitions where $|J| = j$ and $|K| = k$, we obtain

$$\binom{p}{j} x_2^{|J|-1} x_1^{|K|-1} \left(x_2 - \frac{j}{p} \sum_{j=1}^{p} y_j \right) = \binom{p}{j} x_2^{|J|-1} x_1^{|K|-1} \Big(\frac{k}{p} x_2 - \frac{j}{p} x_1 \Big).$$

For the (unique) partition where J is empty, the total order is $2x_1^{p-1}$ (plus terms of lower order). Summing these contributions on the possible values of j, we obtain a telescopic sum equal to $x_1^{p-1} + x_2^{p-1}$.

Finally, the term of degree $p-1$ in the total order of the zeros is $(x_1 + x_2)^{p-1}$. Therefore, the term of degree $p-1$ in the Euler characteristic of $\overline{\mathcal{F}}_\lambda$ is $x_1^{p-1} + x_2^{p-1} - (x_1 + x_2)^{p-1}$. □

6. Intersection calculations on the multi-scale compactification

In this section, we interpret and recover some of the previous results by using degeneration and intersection theory on the multi-scale compactification of strata of differentials.

6.1. The genus of a complete intersection curve. — First, we review how to use the adjunction formula successively to calculate the arithmetic genus p_a of a complete intersection curve. Note that the arithmetic genus and the geometric genus coincide for smooth curves, although they can differ for singular curves.

Let $C = H_1 \cap \cdots \cap H_{n-1}$ be a complete intersection curve of arithmetic genus p_a in an n-dimensional variety X, where each H_i is a hypersurface (i.e., an effective Cartier divisor) in X. Then

$$K_{H_1 \cap \cdots \cap H_k} = \big(K_{H_1 \cap \cdots \cap H_{k-1}} + H_k \big) \,|_{H_1 \cap \cdots \cap H_k},$$

where K stands for the canonical divisor class. It follows that

$$2p_a(C) - 2 = \left(K_X + \sum_{i=1}^{n-1} H_i\right) \prod_{i=1}^{n-1} H_i.$$

Recall that $\mu = (a_1, a_2, -b_1, \ldots, -b_p)$ is a partition of -2 with two positive entries and p negative entries. The projectivized moduli space of differentials of type μ on $\mathbb{CP}^1$ is isomorphic to $\mathcal{M}_{0,p+2}$, and let $\mathrm{MS}(\mu)$ be the multi-scale compactification. We denote by z_1 and z_2 the two zeros, and by $q_1, \ldots, q_p$ the poles.

We use Γ to denote the level graph of a multi-scale differential. If Γ has exactly two levels, then the locus of multi-scale differentials with the level graph Γ is a boundary divisor class denoted by D_Γ. We use $\Gamma^\top$ and $\Gamma^\bot$ to denote the top and bottom levels of Γ, and we denote by $N_\Gamma^\top$ and $N_\Gamma^\bot$ the (unprojectivized) dimensions of the top and bottom level strata, respectively. We also denote by ℓ_Γ the lcm of the prong numbers of the vertical edges in Γ. The set of two-level graphs in $\mathrm{MS}(\mu)$ is denoted by $\mathrm{LG}_1(\mu)$.

If Γ has only one horizontal edge (i.e., with two simple poles at the branches of the corresponding node), we denote this horizontal boundary divisor class by D_h.

Given a generic configuration of residues $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathcal{R}_p \smallsetminus \mathcal{A}_p$, let D_i be the divisorial locus in $\mathrm{MS}(\mu)$ where $\lambda_i \operatorname{Res}_{q_p} \omega - \lambda_p \operatorname{Res}_{q_i} \omega = 0$ for $i = 1, \ldots, p-1$. From this description, the divisor class of D_i is given by

$$D_i = -\eta,$$

where η is the first Chern class of the tautological line bundle $\mathcal{O}(-1)$, whose fibers are generated by the top level differentials on $\mathrm{MS}(\mu)$.

We remark that D_i can contain extraneous boundary divisor components that arise in the following two ways. First, q_i and q_p can both go to the lower levels of a level graph, resulting in their residues becoming zero (before twisting to obtain the multi-scale differential). Alternatively, suppose one of them is on the top level. If the top level vertex containing this pole has it as the unique pole, then the residue is still automatically zero.

Recall that $\overline{\mathcal{F}}_\lambda$ is the closure of the isoresidual fiber $\mathcal{F}_\lambda$ in $\mathrm{MS}(\mu)$ parametrizing differentials with λ as the prescribed configuration of residues at the poles (up to simultaneous scaling). Let

$$\mathcal{D}_\lambda = D_1 \cap \cdots \cap D_{p-2}.$$

From the above construction, it is clear that $\overline{\mathcal{F}}_\lambda \subset \mathcal{D}_\lambda$. In what follows, we will study the differences between them. Recall also that in Section 3.4, we described the multi-scale differentials contained in the boundary of $\overline{\mathcal{F}}_\lambda$. Next, we study multi-scale differentials that are contained in the boundary of $\mathcal{D}_\lambda$.

Proposition 6.1. — *Let λ be a generic tuple of residues. Then a multi-scale differential (X, ω) contained in the boundary of $\mathcal{D}_\lambda$ satisfies the following conditions:*

– *If the level graph of (X, ω) has only one level with a horizontal edge, then the horizontal edge separates two components X_1 and X_2, where X_1 contains z_1 and the subset of poles labeled in I, and X_2 contains z_2 and the subset of poles labeled in J (with $I \sqcup J = \{1, \ldots, p\}$), and $a_1 - \sum_{i \in I} b_i + 1 = 0$. Moreover, the residues of the marked poles are determined by λ.*

– *If the level graph has multiple levels and if the lower levels contain a marked pole, then the residue of every top-level marked pole is zero.*

Proof. — In the first case, since there are only two zeros and each vertex (of genus zero) must contain a zero, it follows that X has two components joined by the horizontal edge. Since there is no lower level, the residues do not degenerate to zero; hence, they are still determined by λ.

In the second case, suppose a marked pole q_i is on the top level for $i \leqslant p-2$. If q_p is in a lower level, then since the residue at q_p is zero (before twisting to obtain the multi-scale differential), it follows that the residue at q_i is zero. If q_p is also on top, then using some q_j in a lower level implies that the residue at q_p is zero, and hence the residue of each top-level pole is zero. Finally, if q_{p-1} is the only top-level pole, then by the residue theorem, its residue is zero. □

Note that a simple pole cannot have zero residue. Therefore, the case when all the poles are simple is easier to treat using Proposition 6.1. For the case of higher-order poles, we need to subtract extraneous boundary components from each D_i before taking the complete intersection. In the following two sections, we will deal with the two cases separately.

6.2. Strata with only simple poles.

— In this section, we will prove Theorem 1.5 using intersection theory on $\mathrm{MS}(\mu)$.

Corollary 6.2. — *Let λ be a generic tuple of residues. If all poles are simple, i.e., $b_i = 1$ for all i, then $\mathcal{D}_\lambda$ does not contain any multi-scale differential whose level graph has more than one level with a pole below the top level. In this case, $\mathcal{D}_\lambda = \overline{\mathcal{F}}_\lambda$ and*

$$2g(\overline{\mathcal{F}}_\lambda) - 2 = \Big(2\eta - D_h + \sum_{\Gamma \in \mathrm{LG}_1(\mu)} (N_\Gamma^\perp \ell_\Gamma - 1) D_\Gamma\Big)(-\eta)^{p-2}. \tag{6.1}$$

Proof. — By Proposition 6.1, for a multi-scale differential contained in $\mathcal{D}_\lambda$ with multiple levels, if there is a pole below the top level, then every top-level pole has zero residue. Note that every vertex (of genus zero) on top level must contain at least one pole. Additionally, a simple pole on the top level cannot have zero residue. This thus justifies the first claim. We remark that D_λ can contain multi-scale differentials with a two-level, two-vertex graph, where all simple poles are on the top-level vertex with prescribed generic residues. Note that the locus of such multi-scale differentials is 0-dimensional and thus does not produce any extraneous boundary component in $\mathcal{D}_\lambda$ but not in $\overline{\mathcal{F}}_\lambda$.

Since $\overline{\mathcal{F}}_\lambda$ is smooth by Theorem 3.2, we have

$$\begin{aligned} 2g(\overline{\mathcal{F}}_\lambda) - 2 = 2p_a(\mathcal{D}_\lambda) - 2 &= \Big(K_{\mathrm{MS}(\mu)} + \sum_{i=1}^{p-2} D_i\Big) \prod_{i=1}^{p-2} D_i \\ &= \Big(2\eta - D_h + \sum_{\Gamma \in \mathrm{LG}_1(\mu)} (N_\Gamma^\perp \ell_\Gamma - 1) D_\Gamma\Big)(-\eta)^{p-2}, \end{aligned} \tag{6.2}$$

where we used the canonical divisor class formula of $\mathrm{MS}(\mu)$ from [CMZ22, Th. 1.1]. □

Consider $\mu = (a_1, a_2, [-1]^{a_1+a_2+2})$ where $a_1, a_2 > 0$ and all the poles are simple. By Corollary 6.2, in this case, the generic isoresidual fiber $\overline{\mathcal{F}}_\lambda = \mathcal{D}_\lambda = D_1 \cap \cdots \cap D_{p-2}$; hence, we can apply Formula (6.1) to compute the genus of $\overline{\mathcal{F}}_\lambda$.

Proof of Theorem 1.5. — In Formula (6.1), we need to evaluate

$$D_h(-\eta)^{a_1+a_2}, \quad (-\eta)^{a_1+a_2+1}, \quad D_\Gamma(-\eta)^{a_1+a_2}$$

where Γ is a two-level graph with $N_\Gamma^\perp = 1$ (otherwise, the intersection of D_Γ with $\eta^{a_1+a_2}$ is zero since the top level does not have enough dimension). For simplicity of notation, we denote

$$N(a_1, a_2) = \int_{\mathrm{MS}(\mu)} (-\eta)^{a_1+a_2+1}.$$

We also denote and recall

$$N(a) = \int_{\mathrm{MS}(a,[-1]^{a+2})} (-\eta)^a = a!$$

from [CP25].

First, a general differential in D_h is of type $(a_1, [-1]^{a_1+1}; -1) \times (a_2, [-1]^{a_2 1+1}; -1)$ with matching residues at the horizontal node. Let π_1 and π_2 be the projections from the product of the strata of the two horizontal components to each of them, respectively. Then D_h can be identified with the zero locus of a section of $-\eta$ over $\mathbb{P}(\pi_1^*\mathcal{O}(-1) \oplus \pi_2^*\mathcal{O}(-1))$, using the matching residue condition at the horizontal node. We also need to choose $a_1 + 1$ simple poles to be in the component that contains z_1. It follows that

$$\int_{\mathrm{MS}(\mu)} D_h(-\eta)^{a_1+a_2} = \binom{a_1+a_2+2}{a_1+1} a_1! a_2! = \frac{(a_1+a_2+2)!}{(a_1+1)(a_2+1)}.$$

Next, we recall the divisor class relation

$$\eta = (-1+1)\psi_{q_1} - \sum_{q_1 \in \Gamma^\perp} \ell_\Gamma D_\Gamma$$

from [CMZ22, Prop. 8.2]. Then we have

$$N(a_1, a_2) = \Bigg(\sum_{\substack{q_1 \in \Gamma^\perp \\ N_\Gamma^\perp = 1}} \ell_\Gamma D_\Gamma\Bigg)(-\eta)^{a_1+a_2} = a_1 N(a_1-1, a_2) + a_2 N(a_1, a_2-1).$$

In the above, the two terms come from the two cases of the bottom vertex of Γ containing q_1 and z_1, or containing q_1 and z_2, respectively. Note that $N(0,0) = 0$. By induction, we conclude that $N(a_1, a_2) = 0$ for all a_1 and a_2.

Finally, we compute $\sum_{N_\Gamma^\perp = 1} \ell_\Gamma D_\Gamma(-\eta)^{a_1+a_2}$, where the sum runs over the two-level graphs with (unprojectivized) bottom dimension $N_\Gamma^\perp = 1$. Since every top vertex must carry a marked pole, the global residue condition (GRC) is void. There are two cases, depending on whether a simple pole (say q_1) is in the lower level or not. First, suppose $q_1 \in \Gamma^\perp$. Then the top vertex contains one marked zero and one zero edge. In this case, we obtain $D_\Gamma(-\eta)^{a_1+a_2} = 0$ as before. Next, suppose all simple poles are on the top level. Then there are two sub-cases. If both z_1 and z_2 are on the bottom, then we obtain $D_\Gamma(-\eta)^{a_1+a_2} = N(a_1+a_2)$ with $\ell_\Gamma = a_1+a_2+1$. If z_1 is on the bottom and there are two top vertices—one containing k many simple poles and the other containing z_2 with the remaining simple poles—then the corresponding contribution is

$$\sum_{k=2}^{a_1} \binom{a_1+a_2+2}{k} N(k-2)N(a_1-k,a_2) = 0.$$

Similarly, if z_2 is on the bottom and there are two top vertices, the contribution is

$$\sum_{k=2}^{a_2} \binom{a_1+a_2+2}{k} N(k-2)N(a_1,a_2-k) = 0.$$

In summary, we conclude that

$$\begin{aligned}2g(\overline{\mathcal{F}}_\lambda)-2 &= -D_h(-\eta)^{a_1+a_2} - 2(-\eta)^{a_1+a_2+1} + \sum_{N_\Gamma^\perp=1}(N_\Gamma^\perp \ell_\Gamma - 1)D_\Gamma(-\eta)^{a_1+a_2}\\ &= -\frac{(a_1+a_2+2)!}{(a_1+1)(a_2+1)} - 2\cdot 0 + (a_1+a_2+1-1)\cdot(a_1+a_2)!\\ &= (a_1+a_2)!\left(a_1+a_2-\frac{(a_1+a_2+2)(a_1+a_2+1)}{(a_1+1)(a_2+1)}\right).\end{aligned}$$

This completes the proof of Theorem 1.5. □

6.3. Strata with arbitrary poles. — Now, we consider a partition

$$\mu = (a_1, a_2, -b_1, \dots, -b_p)$$

of -2, where the b_i's can be arbitrary positive integers.

We denote $T_0 = \mathrm{MS}(\mu)$. For $i = 1,\dots,p-2$, we define the spaces T_i successively, each as the closure inside T_{i-1} of the locus of differentials $(\mathbb{CP}^1,\omega)$ satisfying the *residue equation*

$$\lambda_p \operatorname{Res}_{q_i}\omega - \lambda_i \operatorname{Res}_{q_p}\omega = 0$$

for q_i.

We denote and recall

$$f(a,n) = \frac{a!}{(a-n+2)!}.$$

Moreover, we denote and recall

$$c_{1,I} = a_1 + 1 - \sum_{i\in I} b_i \quad \text{and} \quad c_{2,J} = a_2 + 1 - \sum_{j\in J} b_j$$

for subsets I and J of $\{1,\dots,p\}$, as used in Proposition 4.7. We also recall the quantity $\Xi(a_1,a_2;b_1,\dots,b_n)$, which is described in Proposition 2.18 and counts the number of residueless differentials of type μ.

Proposition 6.3. — *For any generic configuration of residues $\lambda \in \mathcal{R}_p \setminus \mathcal{A}_p$, we have $T_{p-2} = \overline{\mathcal{F}}_\lambda$. Moreover, the genus g_λ of $\overline{\mathcal{F}}_\lambda$ is determined by*

$$\begin{aligned}2g_\lambda - 2 = (a_1 + a_2) f(a_1 + a_2, p) - \sum_{c_{1,I}=0} f(a_1, |I| + 1) f(a_2, |I^c| + 1) \\ - \sum_{c_{1,I}>0} (c_{1,I} + 1) f(a_1, |I| + 1) \Xi(c_{1,I} - 1, a_2; \{b_i\}_{i\in I}) \\ - \sum_{c_{2,J}>0} (c_{2,J} + 1) f(a_2, |J| + 1) \Xi(a_1, c_{2,J} - 1; \{b_j\}_{j\in J}) \\ - \sum_{\substack{c_{1,I}>0,\\ c_{2,J}>0}} [(1 + \operatorname{lcm}(c_{1,I}, c_{2,J})) f(a_1, |I| + 1) f(a_2, |J| + 1) \\ \cdot \gcd(c_{1,I}, c_{2,J}) \Xi(c_{1,I} - 1, c_{2,J} - 1; \{b_i\}_{i\in K})] \\ - 2\Xi(a_1, a_2; b_1, \ldots, b_n).\end{aligned}$$

We note that the terms in the above formula correspond precisely to the description of the zeros, poles, and their orders for the translation structure ω_λ on $\overline{\mathcal{F}}_\lambda$, as given in Propositions 4.1, 4.4, and 4.7. Therefore, this provides an alternative cross-check for the computation of the Euler characteristic of $\overline{\mathcal{F}}_\lambda$.

Proof. — Every element $(X, \omega) \in T_i$ satisfies the residue equations $\lambda_p \operatorname{Res}_{q_j} \omega - \lambda_j \operatorname{Res}_{q_p} \omega = 0$ for every $1 \leqslant j \leqslant i$. In particular, every element in the interior of T_{p-2} satisfies the residue equations for the first $p-2$ poles; hence, by the residue theorem, all the residues are determined by λ up to scaling. Therefore, the interior of T_{p-2} coincides with the open isoresidual fiber curve $\mathcal{F}_\lambda$. Since at each step we remove the extraneous boundary components, the closure T_{p-2} of its interior inside the closed space T_{p-3} is exactly the closure of $\mathcal{F}_\lambda$ inside $\mathrm{MS}(\mu)$. It follows that $T_{p-2} = \overline{\mathcal{F}}_\lambda$.

To compute the genus of T_{p-2}, we observe that the divisor class of T_i inside T_{i-1} satisfies

$$[T_i] = -\eta - \sum_{\Gamma \in \mathcal{G}_i} \ell_\Gamma D_\Gamma$$

where $\mathcal{G}_i$ is the collection of two-level graphs Γ that automatically guarantee the residue equation for the pole q_i by using the residue equations of the previous poles $q_1, \ldots, q_{i-1}$ as well as the associated global residue condition for Γ.

Let K_{T_i} be the canonical divisor class of T_i. By the adjunction formula, we obtain that

$$K_{T_i} = \left(K_{T_{i-1}} + [T_i]\right)[T_i].$$

This implies that

$$\begin{aligned}2g(T_{p-2}) - 2 &= \Bigg(K_{\mathrm{MS}(\mu)} + \sum_{i=1}^{p-2} \Big(-\eta - \sum_{\Gamma \in \mathcal{G}_i} \ell_\Gamma D_\Gamma\Big)\Bigg)[T_{p-2}] \\ &= \Bigg(p\eta - D_h + \sum_{\Gamma \in \mathrm{LG}_1} (N_\Gamma^\perp \ell_\Gamma - 1) D_\Gamma - (p-2)\eta - \sum_{i=1}^{p-2} \sum_{\Gamma \in \mathcal{G}_i} \ell_\Gamma D_\Gamma\Bigg)\left[\overline{\mathcal{F}}_\lambda\right].\end{aligned}$$

Therefore, we only need to consider those two-level graphs Γ such that D_Γ intersects $\overline{\mathcal{F}}_\lambda$ as described in Propositions 3.7 and 3.10.

We denote by Γ_{z_1,z_2} the two-level graph with both zeros in the unique bottom component and all the poles in the unique top component. We denote by $\Gamma_{I,J}$ the two-level "cherry" graphs with the zero z_1 in one bottom component, along with some marked poles indexed by I, the zero z_2 in the other bottom component with some marked poles indexed by J, and the rest of the marked poles in the unique top component indexed by K. Note that either I or J can be empty, but not both. Moreover, neither Γ_{z_1,z_2} nor the horizontal graphs corresponding to the components of D_h belong to any $\mathcal{G}_i$.

To describe the set $\left\{\Gamma_{I,J} \in \bigsqcup_{i=1}^{p-2} \mathcal{G}_i\right\}$, depending on the positions of q_{p-1} and q_p, we analyze the following four cases:

Case 1. — Suppose $q_{p-1}, q_p \in \Gamma_{I,J}^\top$. First, we cannot determine the residue equation for any of the poles in the top component using only the residue equations for the previous ones, due to the presence of q_{p-1}. Next, let q_{i_0} be the pole with the smallest index at the bottom level. Imposing its residue equation forces the residue at q_p to be zero because the differential ω (before twisting) vanishes at the bottom level. This guarantees the residue equations for all the other poles at the bottom level, other than q_{i_0}. Consequently, $\Gamma_{I,J} \in \mathcal{G}_i$ for each i such that $i_0 < i \leqslant p-2$ and $q_i \in \Gamma_{I,J}^\perp$.

Case 2. — Suppose $q_{p-1} \in \Gamma_{I,J}^\top$ and $q_p \in \Gamma_{I,J}^\perp$. Similarly, we cannot determine the residue equation for any of the poles in the top component using only the residue equations for the previous ones. However, all the residue equations for the other poles at the bottom level are guaranteed, due to the vanishing of ω at the bottom level. This means that $\Gamma_{I,J} \in \mathcal{G}_i$ for each $i \leqslant p-2$ such that $q_i \in \Gamma_{I,J}^\perp$.

Case 3. — Suppose $q_{p-1} \in \Gamma_{I,J}^\perp$ and $q_p \in \Gamma_{I,J}^\top$. First, consider the case where the other poles in the top component are the first i poles, $q_1, \ldots, q_i$. We cannot determine the residue equation for any of these poles using the residue equations of the previous ones. However, if we impose the residue equations for all of them, the residue theorem for the top component and the generality of λ imply that all the poles in the top component are residueless, including in particular q_p. This guarantees the residue equations for the remaining poles at the bottom level, except for q_{p-1}. In this case, $\Gamma_{I,J} \in \mathcal{G}_i$ for each $i \leqslant p-2$ such that $q_i \in \Gamma_{I,J}^\perp$.

Next, consider the case where q_{i_0} is the pole with the smallest index at the bottom level and there exists at least one pole q_i at the top level such that $i_0 < i < p-1$. Imposing the residue equation for q_{i_0} forces the residue at q_p to be zero. This ensures that the residue equations for the remaining poles at the bottom level are satisfied, except for q_{p-1}. Additionally, let q_{i_1} be the pole with the largest index in the top component. By imposing the residue equations for the poles with indices smaller than i_1, and since the residue at q_p is zero, all the poles in the top component become residueless, and the residue equation for q_{i_1} is guaranteed. In this case, we conclude that $\Gamma_{I,J} \in \mathcal{G}_i$ for $i = i_1$, and for all other $i \leqslant p-2$ such that $i \neq i_0$ and $q_i \in \Gamma_{I,J}^\perp$.

Case 4. — Suppose $q_{p-1}, q_p \in \Gamma^{\perp}_{I,J}$. The residue equations for all the poles at the bottom level, except for q_{p-1} and q_p, are guaranteed. For the top level, only the residue equation for the pole with the largest index i_1 is guaranteed by imposing the residue equations for the previous ones. In this case, $\Gamma_{I,J} \in \mathcal{G}_i$ for $i = i_1$, and for all other $i \leqslant p-2$ such that $q_i \in \Gamma^{\perp}_{I,J}$.

Note that for any of these four cases, the graph $\Gamma_{I,J}$ appears $|I|+|J|-1$ times in $\bigsqcup_{i=1}^{p-2} \mathcal{G}_i$, with $N^{\perp}_{\Gamma_{I,J}} = |I|+|J|$. Therefore,

$$\begin{aligned} 2g(\overline{F}_\lambda) - 2 &= \Big(2\eta - D_h + (\ell_{\Gamma_{z_1,z_2}} - 1)D_{\Gamma_{z_1,z_2}} + \sum_{\Gamma_{I,J}} (\ell_{\Gamma_{I,J}} - 1)D_{\Gamma_{I,J}}\Big)\,[\mathcal{F}_\lambda] \\ &= (\ell_{\Gamma_{z_1,z_2}} - 1)D_{\Gamma_{z_1,z_2}}\,[\mathcal{F}_\lambda] - D_h\,[\mathcal{F}_\lambda] - 2\Big(-\eta - \sum_{\Gamma_{I,J}} \ell_{\Gamma_{I,J}} D_{\Gamma_{I,J}}\Big)\,[\mathcal{F}_\lambda] \\ &\qquad - \sum_{\Gamma_{I,J}} (\ell_{\Gamma_{I,J}} + 1)D_{\Gamma_{I,J}}[\mathcal{F}_\lambda], \end{aligned}$$

where

$$\begin{aligned} (\ell_{\Gamma_{z_1,z_2}} - 1)D_{\Gamma_{z_1,z_2}}\,[\mathcal{F}_\lambda] &= (a_1+a_2)f(a_1+a_2,p), \\ D_h\,[\mathcal{F}_\lambda] &= \sum_{c_{1,I}=0} f(a_1,|I|+1)f(a_2,|I^c|+1), \\ -2\Big(-\eta - \sum_{\Gamma_{I,J}} \ell_{\Gamma_{I,J}} D_{\Gamma_{I,J}}\Big)\,[\mathcal{F}_\lambda] &= -2\Xi(a_1,a_2;b_1,\dots,b_p) \ \text{ by Proposition 6.4 below,} \end{aligned}$$

and, denoting $d_{I,J} := \gcd(c_{1,I}, c_{2,J})$,

$$D_{\Gamma_{I,J}}[\mathcal{F}_\lambda] = \begin{cases} f(a_1,|I|+1)\Xi(c_{1,I}-1,a_2;\{b_i\}_{i\in K}) & \text{if } J=\varnothing, \\ f(a_2,|J|+1)\Xi(a_1,c_{2,J}-1;\{b_i\}_{i\in K}) & \text{if } I=\varnothing, \\ f(a_1,|I|+1)f(a_1,|J|+1)d_{I,J}\Xi(c_{1,I}-1,c_{2,J}-1;\{b_i\}_{i\in K})) & \text{otherwise.} \end{cases}$$

□

It remains to prove the following identity:

Proposition 6.4. — *The number of residueless differentials $\Xi(a_1,a_2;b_1,\dots,b_p)$ in stratum $\mathcal{H}(a_1,a_2,-b_1,\dots,-b_p)$ is*

$$\Xi(a_1,a_2;b_1,\dots,b_p) = \Big(-\eta - \sum_{\Gamma_{I,J}} \ell_{\Gamma_{I,J}} D_{\Gamma_{I,J}}\Big)[\mathcal{F}_\lambda]. \tag{6.3}$$

Proof. — We rewrite the right-hand side of Equation (6.3) with respect to the pole q_p as

$$\Big(-\eta - \sum_{\Gamma_{I,J}} \ell_{\Gamma_{I,J}} D_{\Gamma_{I,J}}\Big)\,[\mathcal{F}_\lambda] = \Big((b_p-1)\psi_{q_p} - \sum_{q_p\in\Gamma^{\top}_{I,J}} \ell_{\Gamma_{I,J}} D_{\Gamma_{I,J}}\Big)[\mathcal{F}_\lambda].$$

Consider the forgetful morphism $\pi\colon \mathrm{MS}(\mu) \to \overline{\mathcal{M}}_{0,p+2}$, which remembers only the $p+2$ marked zeros and poles. We have

$$\pi_*(\psi_{q_p}) = \sum_{\Delta\in B^{q_p}_{z_1,z_2}} \Delta,$$

where $B^{q_p}_{z_1,z_2}$ is the set of boundary divisors in $\overline{\mathcal{M}}_{0,p+2}$ whose dual graphs have the pole q_p in one component and the zeros z_1 and z_2 in the other component.

Let $\pi^*(B^{q_p}_{z_1,z_2}) = \{D_\Gamma \mid \pi_*(D_\Gamma) \in B^{q_p}_{z_1,z_2}\}$. Then, the only boundary divisor $D_\Gamma \in \pi^*(B^{q_p}_{z_1,z_2})$ that intersects $\overline{\mathcal{F}}_\lambda$ is the one for which z_1 and z_2 are in the bottom component of Γ and all the poles are in the unique top component. We conclude that

$$\text{(6.4)}\quad \Big((b_p-1)\psi_{q_p} - \sum_{q_p\in\Gamma^\top} \ell_\Gamma D_\Gamma\Big)\,[\overline{\mathcal{F}}_\lambda] = (b_p-1)f(a_1+a_2,p) \\ - \sum_{\substack{c_{1,I'}>0\\ c_{2,J'}>0\\ |I'|+|J'|\geqslant 1\\ p\in K'}} c_{1,I'}c_{2,J'}f(a_1,|I'|+1)f(a_2,|J'|+1)\Xi(c_{1,I'}-1,c_{2,J'}-1;\{b_i\}_{i\in K'}).$$

Recall that the number of residueless differentials is given by Proposition 2.18, where Equation (2.1) can be rewritten in two different ways with respect to the zeros z_1 and z_2, as

$$\Xi(a_1,a_2;b_1,\dots,b_p) = \sum_{c_{1,I}+|I|-p\geqslant 0} (-1)^{|I|} f(c_{1,I}-1+|I|,p+1) \\ = \sum_{c_{2,J}+|J|-p\geqslant 0} (-1)^{|J|} f(c_{2,J}-1+|J|,p+1).$$

If we set $b_p=1$, every term $(-1)^{|I|}f(c_{1,I}-1+|I|,p+1)$ such that $p\in I$ cancels out with the term $(-1)^{|I\smallsetminus\{p\}|}f(c_{1,I\smallsetminus\{p\}}-1+|I\smallsetminus\{p\}|,p+1)$, and the same holds for z_2 and J. Therefore,

$$\Xi(a_1,a_2;b_1,\dots,b_p)|_{b_p=1} = \sum_{\substack{b_p-1>c_{1,I}+|I|-p\geqslant 0\\ p\notin I}} (-1)^{|I|} f(c_{1,I}-1+|I|,p+1) \\ = \sum_{\substack{b_p-1>c_{2,J}+|I|-p\geqslant 0\\ p\notin J}} (-1)^{|J|} f(c_{2,J}-1+|J|,p+1).$$

Note that

$$(-1)^{|I|}f(c_{1,I}-1+|I|,p+1)|_{b_p=1} = (-1)^{|J|}f(c_{2,J}-1+|J|,p+1)|_{b_p=1}$$

for $I\sqcup J=\{1,\dots,p-1\}$. Moreover, if $b_p-1>c_{1,I}+|I|-p\geqslant 0$ and $b_p-1> c_{2,J}+|J|-p\geqslant 0$ for $I\sqcup J=\{1,\dots,p-1\}$, then

$$c_{1,I'},c_{2,J'}>0, \\ b_p-1>c_{1,I}+|I|-(p-|J'|)\geqslant 0, \\ b_p-1>c_{2,J}+|J|-(p-|I'|)\geqslant 0,$$

for every $I' \subset I$ and $J' \subset J$. This implies that

$$\Big((b_p-1)\psi_{q_p} - \sum_{q_p\in\Gamma^\top} \ell_{\Gamma_{I,J}} D_{\Gamma_{I,J}}\Big)\,[\overline{\mathcal{F}}_\lambda]\,\Big|_{b_p=1}$$

$$= -\sum_{\substack{c_{1,I'}>0\\ c_{2,J'}>0\\ |I'|+|J'|\geqslant 1\\ p\in K'}} c_{1,I'}c_{2,J'} f(a_1,|I'|+1)f(a_2,|J'|+1)\Xi(c_{1,I'}-1,c_{2,J'}-1,\{b_i\}_{i\in K'})|_{b_p=1}$$

$$= \sum_{\substack{b_p-1>c_{1,I}+|I|-|I'|-|K'|\geqslant 0\\ I'\subset I,\ J'\subset\{1,\dots,p-1\}-I\\ |I'|+|J'|\geqslant 1\\ p\in K'}} \Big[(-1)^{|I|-|I'|-1}c_{1,I'}c_{2,J'} \cdot f(a_1,|I'|+1)f(a_2,|J'|+1)f(c_{1,I}-1+|I|-|I'|,|K'|+1)\Big]$$

$$= \sum_{\substack{b_p-1>c_{1,I}+|I|-p\geqslant 0\\ p\notin I}} (-1)^{|I|-1}\sum_{i=0}^{|I|}\sum_{\substack{j=0\\ i+j\geqslant 1}}^{p-1-|I|}\Big[(-1)^i f(a_1,i+1)f(a_2,j+1) \cdot f(c_{1,I}-1+|I|-i,p+1-i-j)\cdot\Big(\tbinom{|I|-1}{i-1}c_{1,I}+\tbinom{|I-1|}{i}(a_1+1)\Big) \cdot\Big(\tbinom{p-2-|I|}{j-1}(1-c_{1,I})+\tbinom{p-2-|I|}{j}(a_2+1)\Big)\Big]$$

$$= \sum_{\substack{b_p-1>c_{1,I}+|I|-p\geqslant 0\\ p\notin I}} (-1)^{|I|}f(c_{1,I}-1+|I|,p+1) + (-1)^{|I|-1}\sum_{i=0}^{|I|}\Big[(-1)^i f(a_1,i+1)\Big(\tbinom{|I|-1}{i-1}c_{1,I}+\tbinom{|I|-1}{i}(a_1+1)\Big) \cdot\Big(-\sum_{j=1}^{p-1-|I|} f(a_2,j+1)f(c_{1,I}-1+|I|-i,p+2-i-j) + \sum_{j=0}^{p-2-|I|} f(a_2,j+2)f(c_{1,I}-1+|I|-i,p+1-i-j)\Big)\Big]$$

$$= \sum_{\substack{b_p-1>c_{1,I}+|I|-p)\geqslant 0\\ p\notin I}} (-1)^{|I|}f(c_{1,I}+|I|,p+1).$$

Finally, since $a_2=\sum_{i=1}^p b_i-a_1-2$, the left-hand side and right-hand side of Equation (6.3) are polynomials $L,R\in\mathbb{Z}[a_1,b_1,\dots,b_p]$ of degree $p-1$. The computations above imply that $(b_p-1)\mid(L-R)$. Furthermore, since we can use any other pole instead of q_p, it follows that $\prod_{i=1}^p(b_i-1)\mid(L-R)$. Since $L-R$ has degree at most $p-1$, we conclude that $L-R$ is the zero polynomial. □

7. Monodromy of the isoresidual fibration

7.1. Gauss–Manin connection. — For a stratum $\mathcal{H}(a_1,a_2,-b_1,\dots,-b_p)$, we introduce the vector bundle $\pi\colon\mathcal{E}\to\mathcal{B}$, where:

– $\mathcal{B}$ is the complement $\mathcal{R}_p \setminus \mathcal{A}_p$ of the resonance arrangement in the residual space;
– for each $\lambda \in \mathcal{B}$, the fiber $\mathcal{E}_\lambda$ is $\mathbb{C} \otimes H_1(\overline{\mathcal{F}}_\lambda \setminus P_{\omega_\lambda}, Z_{\omega_\lambda}, \mathbb{Z})$.

The vector bundle $\pi\colon \mathcal{E} \to \mathcal{B}$ is endowed with a connection ∇_{GM} that is locally trivial on the elements of $\mathcal{H}_\lambda := H_1(\overline{\mathcal{F}}_\lambda \setminus P_{\omega_\lambda}, Z_{\omega_\lambda}, \mathbb{Z})$ (which is a lattice in the fiber). This connection is called the *Gauss–Manin connection* of $\mathcal{E}$.

7.2. Period central charge

Definition 7.1. — In the dual space $(\mathcal{R}_p)^*$ of the residual space $\mathcal{R}_p$, we denote by $L^\vee$ the lattice generated by the partial sums $\lambda \mapsto \sum_{j\in J} \lambda_j$ for subsets $J \subset \{1, \ldots, p\}$.

It follows from Proposition 4.14 that for each relative homology class $[\gamma]$ of $\mathcal{H}_\lambda$, its period depends on λ as an element of $L^\vee$. Since the saddle connections generate the relative homology group of any translation surface, the period of any relative homology class $[\gamma]$ in $\mathcal{H}_\lambda$ is given by an element $w_{[\gamma]} \in L^\vee$. Moreover, the map $[\gamma] \mapsto w_{[\gamma]}$ is locally constant under small variations of λ. We describe this structure as a morphism from $\mathcal{E}$ to the trivial bundle $(\mathcal{R}_p)^* \times \mathcal{B}$.

Definition 7.2. — The *period central charge* is the bundle morphism $\Psi\colon \mathcal{E} \to \mathcal{B} \times (\mathcal{R}_p)^*$ such that for any configuration $\lambda \in \mathcal{B}$ and any relative homology class $[\gamma] \in \mathcal{H}_\lambda$, we have $\Psi(\lambda, [\gamma]) = (\lambda, w)$, where w is the element of $L^\vee$ prescribing the deformation of $\int_{[\gamma]} \omega_\lambda$ as λ changes. The morphism Ψ extends linearly to arbitrary elements of $\mathcal{E}_\lambda \coloneqq \mathbb{C} \otimes H_1(\overline{\mathcal{F}}_\lambda \setminus P_{\omega_\lambda}, Z_{\omega_\lambda}, \mathbb{Z})$.

Isoresidual fibers are translation surfaces, and deformations of translation surfaces correspond to deformations of the periods of their relative homology classes. The period central charge describes how the periods of the relative homology classes of $\mathcal{H}_\lambda$ change in terms of λ. It follows that the geometry of the isoresidual fibration is entirely described by the Gauss–Manin connection of $\mathcal{E}$, obtained by replacing the fiber with its homology group (up to a change of coefficients).

7.3. Monodromy. — For any based loop α in $\mathcal{B}$, the Gauss–Manin connection along α induces an automorphism T_α of $\mathcal{E}_\lambda = \mathbb{C} \otimes H_1(\overline{\mathcal{F}}_\lambda \setminus P_{\omega_\lambda}, Z_{\omega_\lambda})$. A priori, computing the monodromy of the Gauss–Manin connection on $\mathcal{E}_\lambda$ is a difficult problem. Nevertheless, the isoresidual fibration enjoys additional structures that impose significant constraints on the possible form of this monodromy. In this section, we explain the following three results:

(1) The period central charge Ψ commutes with the monodromy automorphisms.

(2) The action of the monodromy on the singular points of the isoresidual fiber can be deduced from the study of the monodromy in the case $n = 1$ investigated in [GT22, §5].

(3) Saddle connections that shrink as λ approaches a resonance hyperplane survive as saddle connections along the monodromy around this hyperplane.

7.3.1. *The period central charge commutes with monodromy*

Proposition 7.3. — *For any $\lambda \in \mathcal{B}$, any based loop α in $\mathcal{B}$, and any homology class $[\gamma] \in \mathcal{H}_\lambda$, we have:*

$$\Psi(\lambda, T_\alpha[\gamma]) = \Psi(\lambda, [\gamma]).$$

Proof. — The Gauss–Manin connection identifies nearby relative homology classes (with integer coefficients) and linear dependence of their periods in terms of λ, which can be shown to be the same through analytic continuation. □

Proposition 7.3 provides the construction of many invariant subbundles of $\mathcal{E}$.

Corollary 7.4. — *For any linear subspace V of $(\mathcal{R}_p)^*$, $\mathcal{E}^V = \Psi^{-1}(\mathcal{B} \times V)$ is an invariant subbundle of $\mathcal{E}$.*

In particular, the subbundle $\mathcal{E}^{\{0\}}$ generated by the relative homology classes with vanishing periods is invariant under the monodromy of the Gauss–Manin connection.

7.3.2. *Action of the monodromy on the zeros and poles of ω_λ*. — The relative homology group $\mathcal{H}_\lambda$ is part of the exact sequence

$$0 \longrightarrow H_1(\overline{\mathcal{F}}_\lambda \smallsetminus P_{\omega_\lambda}) \longrightarrow H_1(\overline{\mathcal{F}}_\lambda \smallsetminus P_{\omega_\lambda}, Z_{\omega_\lambda}) \longrightarrow H_0(Z_{\omega_\lambda}) \longrightarrow H_0(\overline{\mathcal{F}}_\lambda \smallsetminus P_{\omega_\lambda}) \longrightarrow 0.$$

Following Section 4.2, the zeros of ω_λ are the elements of the isoresidual fiber of the minimal stratum $\mathcal{H}(a_1 + a_2, -b_1, \ldots, -b_p)$. The action of the monodromy on $H_0(Z_\omega)$ partially determines the action of the monodromy on $\mathcal{H}_\lambda$. This monodromy on $H_0(Z_\omega)$ is fully described in Section 5 of [GT22] for the case $p = 3$ and partially for some other cases.

We can now describe part of the action of the monodromy on $\mathcal{H}_\lambda$ by the action on the poles of ω_λ. As in the case of the zeros, it follows from the study of the action of the monodromy on some discrete isoresidual fibers.

Following the results of Sections 4.3 and 4.4, in the multi-scale compactification, the poles of ω_λ correspond to nodal curves with at most three irreducible components. The top-level component corresponds to a differential with zero residues, making it invariant under the monodromy. In contrast, the monodromy usually acts nontrivially on the other components. Since each of the latter components contains exactly one zero, they are described by the isoresidual fibration for the case $n = 1$ as in [GT22].

This action of the monodromy, via permutations on P_{ω_λ}, determines the action of the monodromy on the subspace of $H_1(\overline{\mathcal{F}}_\lambda \smallsetminus P_{\omega_\lambda})$ generated by the homology classes of the loops around the poles.

7.3.3. *Action of the monodromy on saddle connections*. — In $\mathcal{B}$, we consider a positively oriented simple loop α around the resonance hyperplane A_J corresponding to the resonance equation $\sum_{j \in J} \lambda_j = 0$. We will use the geometry of the saddle connections of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ as λ approaches A_J to construct a block decomposition of the monodromy automorphism T_α on $H_1(\overline{\mathcal{F}}_\lambda \smallsetminus P_{\omega_\lambda})$.

Denoting by V the complex line in $(\mathcal{R}_p)^*$ generated by $\lambda \mapsto \sum_{j\in J} \lambda_j$, Corollary 7.4 shows that $\mathcal{E}^V$ is an invariant subbundle. In particular, $\mathcal{E}^V_\lambda$ is invariant under the action of T_α.

For λ close enough to A_J, we introduce the bundle $\mathcal{F}$, where $\mathcal{G}_\lambda$ is the subspace of $\mathcal{E}^V_\lambda$ generated by the saddle connections whose period central charges coincide with $\sum_{j\in J} \lambda_j$. These saddle connections become arbitrarily small (in contrast to the other saddle connections in the fiber) as λ approaches A_J. Therefore, they cannot cross any singularity and persist as saddle connections as λ moves along α.

Proposition 7.5. — *For λ close enough to A_J, the automorphism T_α acts by permutation on a basis of the subbundle $\mathcal{G}_\lambda \subset \mathcal{E}_\lambda$. Similarly, T_α acts by permutation on a basis of the quotient bundle $\mathcal{E}_\lambda/\mathcal{G}_\lambda$.*

Proof. — In an arbitrarily small neighborhood of a point in the regular locus of A_J, the Gauss-Manin connection around A_J preserves saddle connections whose period is given by $\sum_{j\in J} \lambda_j$. Indeed, since these saddle connections are arbitrarily small compared to the other saddle connections of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$, they cannot be crossed by them. It follows that they are permuted by T_α.

Since the relative homology group $\mathcal{H}_\lambda$ is generated by the homology classes of saddle connections, the fiber $\mathcal{E}_\lambda/\mathcal{G}_\lambda$ of the quotient bundle is generated by saddle connections whose periods are not given by $\sum_{j\in J} \lambda_j$.

Along a loop in $\mathcal{B}$ that is close enough to the resonance hyperplane, the saddle connections that do not shrink can only be crossed by the vanishing saddle connections. It follows that their classes in the quotient fiber $\mathcal{E}_\lambda/\mathcal{G}_\lambda$ are the same, up to permutation. □

7.4. Example: the case of $\mathcal{H}(1,1,-2,-1,-1)$. — Note first that when $p = 2$, the monodromy around the unique resonance hyperplane is trivial because the generic isoresidual fibers are related to each other by the scaling action of $\mathbb{C}^*$. Hence, the stratum $\mathcal{H}(1,1,-2,-1,-1)$ is the first nontrivial example.

Here, $\mathcal{B} = \mathcal{R}_3 \setminus \mathcal{A}_3$ is the complement of three resonance hyperplanes A_1, A_2, and A_3, corresponding respectively to the vanishing of λ_1 (the residue at the double pole), and λ_2 and λ_3 (the residues at the simple poles).

The zeroes of ω_λ are described in Theorem 1.2 while the poles are described in Sections 4.3 and 4.4. We obtain that the generic isoresidual fiber is a punctured sphere endowed with a translation structure belonging to the stratum $\mathcal{H}(2,2,-2,-2,-1,-1)$. In particular, it is of genus zero. The residues at the double poles are $\pm(\lambda_2 - \lambda_3)$, while the residues at the simple poles are $\pm\lambda_1$. The vector bundle $\mathcal{E}$ has rank 4.

We deduce from Section 7.3.2 that the monodromy of the isoresidual fibration preserves each pole of ω_λ. Therefore, $\mathcal{E}$ admits a subbundle $\mathcal{E}_p$ of rank 3, on which the monodromy acts trivially. In particular, $\mathcal{E}_p$ contains the subbundle $\mathcal{E}^{\{0\}}$ of rank 2, generated by the classes whose period charge vanishes (see Corollary 7.4).

The relative homology class of any arc joining the two zeros of ω_λ generates the quotient bundle $\mathcal{E}/\mathcal{E}_p$ of rank 1. Following Section 7.3.2, we can verify that the monodromy of loops around A_2 and A_3 permutes the two zeros and therefore acts as $-\mathrm{Id}$ on $\mathcal{E}/\mathcal{E}_p$. In contrast, the monodromy of loops around A_1 preserves the two zeros individually and therefore acts trivially on $\mathcal{E}/\mathcal{E}_p$. The eigenvalues of the monodromy operators are then:

– $(1,1,1,1)$ for the loops around A_1;
– $(1,1,1,-1)$ for the loops around A_2 and A_3).

In order to compute these monodromy operators, we will describe explicitly the translation structure of $\mathcal{F}_\lambda$ for a generic configuration of real residues λ (see Figure 8). This can be accomplished by exhaustively writing down the ribbon graphs associated to ω_λ as defined in Section 2.4. We observe that as λ approaches any of the three resonance hyperplanes, exactly two of the four saddle connections of $(\mathcal{F}_\lambda,\omega_\lambda)$ shrink. According to Proposition 7.5 there exists of a subbundle $\mathcal{G}_{\lambda,A_i}$ of rank 2 (defined in a neighborhood of the resonance hyperplane A_i) on which the monodromy around that hyperplane acts as a permutation, while it also acts by permuting the quotient bundle.

7.4.1. *Computation of the monodromy around* A_1. — The eigenvalues of the monodromy operator are $(1,1,1,1)$ so these two permutations are trivial. Looking at the translation structure for a generic configuration of real residues close to A_1, we observe that each of the two shrinking saddle connections is a closed saddle connection enclosing a simple pole of ω_λ. We deduce that the subbundle $\mathcal{G}_{\lambda,A_1}$ of rank 2 (defined in a neighborhood of A_1) is contained in the rank-3 subbundle $\mathcal{E}_p$ where the monodromy acts trivially.

Using the block decomposition of the matrix together with its commutation with the period charge, we obtain a matrix of the following form:

$$\begin{pmatrix}1&0&1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

7.4.2. *Computation of the monodromy around* A_2 *and* A_3. — The eigenvalues for the monodromy around A_2 and A_3 are $(1,1,1,-1)$. Again, by looking at the translation structure for a generic configuration of real residues close to A_2 (or A_3), we observe that the two shrinking saddle connections join to the two zeros. Since the monodromy around these hyperplanes transposes the two zeros, it follows that the monodromy acts as a transposition on $\mathcal{G}_{\lambda,A_i}$ and trivially on the quotient bundle $\mathcal{E}_p/\mathcal{G}_{\lambda,A_i}$ for $i=2$ and 3.

In order to write down explicitly the monodromy action corresponding to A_2 (or equivalently A_3) up to conjugacy, we factorize the matrix into two *half-monodromy transformations* corresponding to the two half-loops around the resonance hyperplane connecting the two chambers of the space of configurations of real residues, where we have $\lambda_2 > 0$ and $\lambda_2 < 0$, respectively. We obtain that the monodromy matrix is given by

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

in a basis (u_1, u_2, u_3, u_4) of $\mathcal{H}_\lambda$, formed by the four saddle connections of $(\mathcal{F}_\lambda, \omega_\lambda)$. Here, $\lambda_2, \lambda_3 > 0$ while $\lambda_1 < 0$, u_3 and u_4 generate $\mathcal{G}_{\lambda, A_2}$, and $\Psi(u_1) = \Psi(u_2) = \lambda_3$ while $\Psi(u_3) = \Psi(u_4) = \lambda_2$ (see Figure 8 for a picture of the translation structure). We can check that the monodromy commutes with the period central charge.

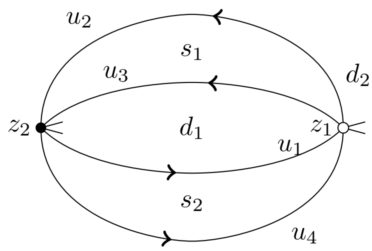

Figure 8. The graph associated to the differential ω_λ where z_1, z_2 are the zeros, s_1, s_2 are the simple poles and d_1, d_2 are the double poles.

Remark 7.6. — The lattice of absolute periods of ω_λ is $(\lambda_2 + \lambda_3)\mathbb{Z} + (\lambda_2 - \lambda_3)\mathbb{Z}$. Indeed, the isoresidual fiber has genus zero, and we know the residues at the poles. In contrast, we can find saddle connections with periods λ_2 or λ_3 between the two zeros of ω_λ. The relative period lattice of ω_λ is therefore $\lambda_2\mathbb{Z} + \lambda_3\mathbb{Z}$. These two lattices do not coincide.

8. Connected components of generic isoresidual fibers

The goal of this section is to prove the classification of the connected components of generic isoresidual fibers in genus zero, as stated in Theorem 1.7. To achieve this, we introduce topological invariants in Section 8.1, prove that they are complete for one-dimensional isoresidual fibers in Section 8.3, and then use the incidence relations between strata to deduce the result in full generality in Section 8.2.

8.1. Topological invariants. — Recall that the two exceptional families of strata where generic isoresidual fibers are disconnected are as follows:

- $\mathcal{H}(kc_1, \ldots, kc_m, -1, -1)$ with $k \geqslant 2$ and $\gcd(c_1, \ldots, c_m) = 1$;
- $\mathcal{H}(c_1, \ldots, c_m, [-1]^p)$ with $p, c_1, \ldots, c_m$ even.

In each case, we introduce topological invariants of the connected components in an indirect way as follows:

(1) We consider isoresidual fibers corresponding to configurations of residues where, for each pair of simple poles, one pole has a real positive residue while the other has a real negative residue. Notice that this condition can be satisfied by some residue configurations that do not belong to any resonance hyperplane (see Definition 1.1).

(2) For each translation surface (X,ω) belonging to any such generic isoresidual fiber $\mathcal{F}_\lambda$, we glue the cylinders corresponding to the pairs of simple poles using a homothety. The resulting object is not a translation surface, but rather a dilation surface of genus $g \geqslant 1$.

(3) The moduli space of dilation surfaces is stratified in exactly the same way as that of translation surfaces. The charts of the complex affine structure around singular points are given by branches of $\int z^m g(z)dz$, where g is a nonzero holomorphic function and $m \in \mathbb{C}$, with m representing the order of the singularities. The Gauss–Bonnet identity $\sum m_i = 2g-2$ also holds for complex affine structures. When the complex affine structure reduces to a dilation structure, its singularities satisfy $\Re(m) \in \mathbb{Z}$, and the moduli space is stratified according to these integer values (see [ABW23] for details).

(4) Any stratum $\mathcal{H}(a_1,\dots,a_n,-b_1,\dots,-b_p)$ of meromorphic differentials naturally embeds into the corresponding stratum $\mathcal{D}(a_1,\dots,a_n,-b_1,\dots,-b_p)$ of dilation surfaces. The topological invariants deduced from the rotation numbers (which still make sense in a dilation surface, since slopes are globally defined) that we described in Section 2.5 distinguish the connected components of $\mathcal{D}(a_1,\dots,a_n,-b_1,\dots,-b_p)$.

8.1.1. *The first exceptional family of strata*. — For a stratum of the form

$$\mathcal{H}(kc_1,\dots,kc_m,-1,-1),$$

where $k \geqslant 2$ and $c_1,\dots,c_m$ are positive or negative coprime integers satisfying $\sum_{i=1}^{m} c_i = 0$, we consider a generic isoresidual fiber $\mathcal{F}_\lambda$ corresponding to a configuration $\lambda = (\lambda_1,\dots,\lambda_p)$ of residues, where λ_{p-1} is a positive real number and λ_p is a negative real number.

We introduce the following surgery: for any translation surface (X,ω) in $\mathcal{F}_\lambda$, we glue the two cylinders corresponding to the two simple poles together. We choose one regular closed geodesic on each cylinder and identify them with each other using a homothety. As a result, we obtain a dilation surface X' of genus one with a distinguished closed geodesic γ.

There is a well-defined notion of topological index (see Section 2.1.3) in the dilation surface X'. For any pair of simple loops α,β in X' crossing γ exactly once (with a positive orientation), the topological indices of α and β differ by a combination of the orders $kc_1,\dots,kc_m$. We thus define the *surgical rotation number* of (X,ω) as the class of Ind_α in $\mathbb{Z}/k\mathbb{Z}$.

We can easily check that:

– the surgical rotation number of (X,ω) does not depend on the choice of closed geodesics in the surgery performed on (X,ω);

– deformations of (X,ω) in the isoresidual fiber $\mathcal{F}_\lambda$ preserve the surgical rotation number.

Therefore, for each connected component $\mathcal{C}$ of $\mathcal{F}_\lambda$, there is a well-defined *surgical rotation number* in $\mathbb{Z}/k\mathbb{Z}$. We will see in Proposition 8.2 that the number of connected components in a generic isoresidual fiber is indeed k, proving in particular that the *surgical rotation number* is a complete invariant of the connected components for the fibers where it is defined (where λ_{p-1} and λ_p are respectively a positive and a negative real number).

Algebro-geometrically speaking, gluing the two simple poles yields a rational nodal curve as a degeneration of elliptic curves. The torsion number for the connected components of the stratum $\mathcal{H}(kc_1,\dots,kc_m)$ in genus one (see [CG22, §3.4]) can also be used to induce the surgical rotation number.

8.1.2. *The second exceptional family of strata*. — For a stratum of the form

$$\mathcal{H}(c_1,\dots,c_m,[-1]^{2g}),$$

where $g \geqslant 1$ and $c_1,\dots,c_m$ are positive or negative even integers satisfying $\sum_{i=1}^m c_i = 2g-2$, we consider a generic isoresidual fiber $\mathcal{F}_\lambda$ corresponding to a configuration of residues, where g simple poles have pairwise distinct real positive residues while the other g simple poles have pairwise distinct negative real residues.

The surgery we introduce is as follows: We choose a regular closed geodesic on each cylinder corresponding to a simple pole. We denote by $\gamma_1,\dots,\gamma_g$ (resp. $\delta_1,\dots,\delta_g$) the closed geodesics chosen on the cylinders corresponding to the simple poles with negative (resp. positive) residues. Without loss of generality, we assume that $\gamma_1,\dots,\gamma_g$ (resp. $\delta_1,\dots,\delta_g$) are ordered by increasing length. Using homotheties to identify γ_i with δ_i for each $i \in \{1,\dots,g\}$, we obtain a dilation surface X' of genus $g \geqslant 1$.

We know from [ABW23, Kaw18] that the parity of the spin structure (or equivalently, the Arf invariant) generalizes from translation surfaces to dilation surfaces. We can then check that the parity of

$$\sum_{i=1}^{g}(\mathrm{Ind}_{\alpha_i}+1)(\mathrm{Ind}_{\beta_i}+1)$$

does not depend on the choice of geodesics $\gamma_1,\dots,\gamma_g$ and $\delta_1,\dots,\delta_g$. It is also constant along deformations of (X,ω) inside the isoresidual fiber, since the closed geodesics of the cylinders retain their directions.

Then, for each connected component $\mathcal{C}$ of $\mathcal{F}_\lambda$, there is a well-defined *surgical Arf invariant* in $\mathbb{Z}/2\mathbb{Z}$. We will see in Proposition 8.2 that the number of connected components in a generic fiber is indeed 2, proving in particular that the *surgical Arf invariant* is a complete invariant of the connected components for the fibers where it is defined (where the g residues are positive real numbers while the other g residues are negative real numbers).

8.2. Incidence relations. — In this section, we consider strata with the same pattern of poles $b_1, \dots, b_p$, but the pattern of zeros can vary. To avoid confusion between isoresidual fibers of distinct strata, we introduce the notation $\mathcal{F}_\lambda(a_1, \dots, a_n)$ for the isoresidual fiber of $\mathcal{H}(a_1, \dots, a_n) = \mathcal{H}(a_1, \dots, a_n, -b_1, \dots, -b_p)$ corresponding to a configuration of residues $\lambda = (\lambda_1, \dots, \lambda_p)$.

We deduce from the surgery of breaking up zeros (see Corollary 3.5) the following lemma about incidence relations between these isoresidual fibers.

Lemma 8.1. — *Consider a stratum $\mathcal{H}(a_1, \dots, a_n)$ with $n \geqslant 2$, any configuration of residues λ, and a subset $I \subset \{1, \dots, n\}$. For any connected component of $\mathcal{F}_\lambda(a_1, \dots, a_n)$, there exists a connected component of $\mathcal{F}_\lambda(\sum_{i\in I} a_i, \{a_j\}_{j\notin I})$ contained in its closure. Moreover, each connected component of $\mathcal{F}_\lambda(\sum_{i\in I} a_i, \{a_j\}_{j\notin I})$ is adjacent to a unique connected component of $\mathcal{F}_\lambda(a_1, \dots, a_n)$.*

The proof of this result is a refinement of [Boi15, Prop. 7.1 & 7.2].

Proof. — We first treat the case where $I = \{1, \dots, n\}$ and show that, given $\mathcal{F}_\lambda(a_1, \dots, a_n)$, there is an element in its closure that belongs to $\mathcal{F}_\lambda(\sum_{i\in I} a_i)$.

We associate to $\omega \in \mathcal{F}_\lambda(a_1, \dots, a_n)$ an oriented graph in the following way. Consider a generic direction and the associated decomposition into basic domains (see [Boi15, §3.3]). Each pole corresponds to a vertex in the graph. For each saddle connection, we draw an arrow between the poles corresponding to the basic domains it bounds, such that the orientation goes from the lower domain to the upper domain.

The dimension of $\mathcal{H}(a_1, \dots, a_n)$ is $p + n - 2$, so the graph has at least p edges, since $n \geqslant 2$. Therefore, it contains at least one simple closed path. Let γ be such a path. If γ is a loop (i.e., it contains only one vertex), then we are done by letting the corresponding saddle connection degenerate.

If γ is a sum of edges, consider one of the shortest edges. Shrink it by a quantity d. Then, consider the next edge in the loop. If both edges point in the same direction, we change the latter saddle connection by $-d$. If one edge points toward the vertex and the other points away from it, we add d to the corresponding saddle connection. More generally, if the change at an edge is $\pm d$, then the change at the next edge is $\pm d$ if both edges point toward or away from the vertex, and $\mp d$ if the edges point in opposite directions.

Now, this can be done in a coherent way. Indeed, if γ is a directed graph, then we add d to all the corresponding saddle connections. If we reverse the orientation of exactly one edge in γ, then this only changes its contribution to $-d$ and makes no other change. Since any orientation of the edges can be obtained from the directed one by such changes, this proves the result.

Now we can show uniqueness and the case $I \neq \{1, \dots, n\}$ at the same time. The multi-scale compactification is smooth at the points corresponding to $\mathcal{F}_\lambda(a_1, \dots, a_n)$ on the boundary. Hence, we can break the zeros not in I, and then those in I. Since breaking the zeros does not change the residues, this shows (by reversing this construction) that we can merge only the zeros in I. Finally, the smoothness implies the uniqueness statement, since if two components were to meet, this would create a singularity. □

8.3. Connected components of generic one-dimensional isoresidual fibers

In this section, we prove that the surgical Arf invariant and the surgical rotation number are complete invariants for the connected components of generic isoresidual fibers in strata with $n = 2$ zeros.

Proposition 8.2. — *In strata of meromorphic differentials of genus zero, generic isoresidual fibers are connected, except for the following two families of strata:*

(1) $\mathcal{H}(ka_1, ka_2, -kb_1, \dots, -kb_s, -1, -1)$ *for some* $k \geqslant 2$ *and* $a_1, a_2, b_1, \dots, b_s$ *positive integers such that* $\gcd(a_1, a_2, b_1, \dots, b_s) = 1$*, where generic isoresidual fibers have* k *connected components;*

(2) $\mathcal{H}(2a_1, 2a_2, -2b_1, \dots, -2b_s, [-1]^{2g})$ *with* $a_1, a_2, b_1, \dots, b_s \geqslant 1$ *and* $g \geqslant 2$*, where generic isoresidual fibers have two connected components.*

This entire Section 8.3 is dedicated to the proof of this proposition. We begin with some general considerations.

We know from Theorem 1.2 that the type of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ as a translation structure is the same for any $\lambda \in \mathcal{R}_p \smallsetminus \mathcal{A}_p$. Therefore, it suffices to determine the number of connected components of generic isoresidual fibers in the special case of an isoresidual fiber $\mathcal{F}_\lambda$, where $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathcal{R}_p \smallsetminus \mathcal{A}_p$ satisfies $\lambda_1 \in \mathbb{R}_{>0}$ and $\lambda_j \in \mathbb{R}_{<0}$ for $2 \leqslant j \leqslant p$ as shown in Figure 1.

Following Section 4.2, the zeros of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ correspond to elements of stratum $\mathcal{H}(a_1 + a_2, -b_1, \dots, -b_s, [-1]^t)$ that realize the configuration λ. We know from Lemma 8.1 that every connected component of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ contains a zero of ω_λ. To classify the connected components of $\mathcal{F}_\lambda$, we determine which pairs of zeros of ω_λ can be joined by a chain of saddle connections of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$.

It is proved in Lemma 4.12 that when λ is a configuration of real residues, every saddle connection of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ is horizontal. Differentials that belong to such a saddle connection define the same decorated graph (see Section 2.4), which we describe in the following lemma.

Lemma 8.3. — *Consider a stratum* $\mathcal{H}(a_1, a_2, -b_1, \dots, -b_p)$ *and a configuration of real residues* $\lambda \in \mathcal{R}_p \smallsetminus \mathcal{A}_p$ *that satisfies* $\lambda_1 \in \mathbb{R}_{>0}$ *and* $\lambda_j \in \mathbb{R}_{<0}$ *for* $2 \leqslant j \leqslant p$*. Saddle connections of* $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ *correspond to decorated graphs that satisfy the following two sets of properties.*

The first set of properties ensures that the decorated graph is compatible with the data of $\mathcal{H}(a_1, a_2, -b_1, \dots, -b_p)$*:*

(1) *faces are labeled according to the zeros (of orders* a_1 *and* a_2*);*

(2) *vertices are labeled according to the poles (with orders and residues* $b_1, \dots, b_p$ *and* $\lambda_1, \dots, \lambda_p$*);*

(3) *each face corresponding to a zero of order* a_i *contains* $2a_i + 2$ *corners;*

(4) *for each vertex corresponding to a pole of order b_j, there are $b_j - 1$ incoming half-edges and $b_j - 1$ outgoing half-edges attached to the vertex.*

The second set of properties ensures that the orientations of the edges are compatible with the signs of the partial sums of residues (which are positive if and only if they include λ_1):

(1) *the graph contains a unique loop formed by $r+s+2$ vertices called*

$$T, V_1, \ldots, V_r, B, U_1, \ldots, U_s$$

(in this cyclic order) for $r, s \geqslant 0$;

(2) *the edges of the loop are oriented from B (as "bottom") to T (as "top") in the two branches of the loop (through $U_1, \ldots, U_s$ or $V_r, \ldots, V_1$);*

(3) *the vertex P, corresponding to the pole of order b_1 with the only positive residue λ_1, either coincides with T or belongs to a tree attached to T;*

(4) *in a tree attached to a vertex of the loop (provided that the tree does not contain P), the edges are oriented towards the loop;*

(5) *in a tree attached to T that contains P as a vertex, the edges are oriented towards P.*

Denote by λ_B the total (negative) residue of the poles corresponding to the vertex B and the vertices of the trees attached to B. Then, the length of the corresponding saddle connection is $|\lambda_B|$. Moreover, each end of the saddle connection corresponds to shrinking one of the two edges of the loop incident to B.

Proof. — We first verify the properties of a decorated graph associated with a differential parameterized by a saddle connection γ of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$. Note that the first set of properties follows directly from Sections 2.4 and 4.6.

For the second set of properties, the number of edges in the decorated graph is $n+p-2 = p$, while the number of vertices is p. It follows that the graph has only one loop. As proved in Proposition 4.13, the orientation of the edges in the loop is incoherent. In other words, there are some vertices of the loop where the two incident edges of the loop are incoming, and the same number of vertices where the two incident edges (of the loop) are outgoing. For a translation surface associated with γ, cutting along all the saddle connections corresponding to the edges of the loop in the decorated graph decomposes the translation surface into connected components, each having two boundary saddle connections. It is clear that the connected component corresponding to the vertex of the loop with two incoming edges has a positive total residue. It follows that:

– there is only one vertex, called T, in the loop with two incoming edges;

– the vertex P of the graph, corresponding to the unique pole with a positive residue, either coincides with T or belongs to a tree attached to the loop at T;

– there is only one vertex, called B, in the loop with two outgoing edges.

The orientations of the edges in the loop are now fixed. The other edges correspond to closed saddle connections that decompose $\mathbb{CP}^1$ into two connected components.

The orientations of the corresponding edges are fully determined by the signs of the partial sums of the residues.

Since there is only one positive residue, we observe that the lengths of the saddle connections corresponding to the edges of the loop are decreasing. Specifically, the lengths of the saddle connections corresponding to the edges between $T, V_1, \dots, V_r, B$ (or similarly $T, U_s, \dots, U_1, B$) decrease and differ by constant partial sums of negative residues. The only degree of freedom in the saddle connection γ is the length of the saddle connection corresponding to the edges of the loop, as it represents the relative period of the differential between the two zeros. Thus, the two saddle connections that can shrink correspond to the edges of the loop incident to B. The parametrization of γ is given by the lengths of these two saddle connections, which sum to λ_B (up to sign), and this is the length of γ. In particular, the two ends of γ correspond to the shrinking of one of these saddle connections and the removal of the corresponding edges from the decorated graph (see Lemma 2.10).

Conversely, for any such configuration of real residues with a unique positive residue, the orientation of the edges we prescribed to the graph ensures that we can construct the translation surface corresponding to the decorated graph in such a way that the lengths of the saddle connections corresponding to the edges are positive. ☐

We describe the decorated graph of a differential ω parameterized by a saddle connection of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ in the case of strata with only simple poles.

Example 8.4. — Consider a stratum $\mathcal{H}(a_1, a_2, [-1]^t)$ in genus zero with only simple poles. A decorated graph parameterized in a saddle connection of ω_λ has t vertices $V_1, \dots, V_t$ (corresponding to the poles $p_1, \dots, p_t$) and t edges pointing towards the vertex V_1. It follows that there is a unique pole of residue $\lambda_j < 0$ whose corresponding vertex V_j has valency 2. The loop is formed by the two edges from V_j to V_1 and decomposes the underlying sphere into two connected components. The other $t-2$ vertices are attached to V_1 each through a unique edge. They are split into two families having respectively a_1 and a_2 vertices according to which face they belong to. This example is illustrated in Figure 1 for the case $t=5$ and $j=4$.

In what follows, we first prove Proposition 8.2 in the case of strata without simple poles.

Lemma 8.5. — *In strata $\mathcal{H}(a_1, a_2, -b_1, \dots, -b_p)$ of meromorphic differentials of genus zero with $b_i \geqslant 2$ for all i, the generic isoresidual fibers are connected.*

Proof. — We consider the case of a unique positive residue at a pole of lowest order and negative residues at all the other poles. The case with two poles is trivial since there is only one decorated graph. Now, suppose there are $p \geqslant 3$ poles, and let p_i denote the poles such that their orders satisfy $-b_1 \geqslant -b_2 \geqslant \dots \geqslant -b_p$. We will assume that the residue of p_1 is positive, and the residue of the other poles are negative.

Now consider the graph associated with a differential in the generic isoresidual fiber of the stratum $\mathcal{H}(a_1+a_2, -b_1, \dots, -b_p)$. This graph has a special vertex V_1, such that all the other vertices point towards it. First, we deform the graph so that all vertices are directly adjacent to V_1. Consider a leaf of the graph. Move it by connecting its first quadrant to the rest of the graph, such that the bounded region corresponds to the zero of order a_1, and then delete the original edge (see the top of Figure 9). Repeat this operation until we return to the original place. If the vertex is attached to V_1 at some step, we are done. Otherwise, we start from the second quadrant as pictured at the bottom of Figure 9 and repeat the operation until we return to the original place. Note there is a forbidden quadrant between the first and the second one due to the compatibility condition of the directions of the edges: this will always be the case and we will forget these quadrant in our counts. With this operation, we can visit all the places of the previous case, shifted by one quadrant. We encounter the same dichotomy, and if we do not meet V_1, we can repeat the procedure to eventually reach this vertex.

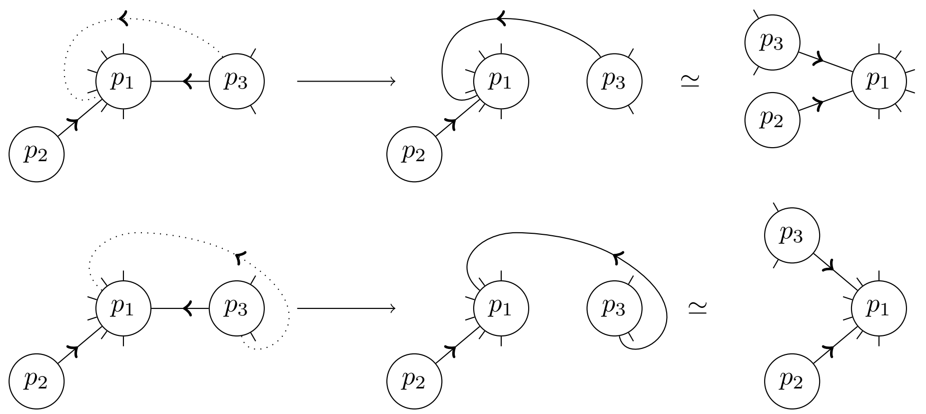

Figure 9. A move of a leaf from the first quadrant on the top and from the second quadrant on the bottom.

So, the graph we consider is given by a root vertex V_1 and $p-1$ leaves $V_2, \dots, V_p$ pointing towards it. We show that this graph can be put into the following standard form: the vertices point towards the same quadrant of the vertex V_1 and are arranged in a cyclic increasing order. Take V_3 and move it along the graph by adding an edge starting from its first quadrant, as described in the preceding paragraph (and shown at the top of Figure 9). Repeating this operation, either it reaches the first quadrant of V_1 after V_2, or it returns to its original position. In the latter case, we proceed as in the preceding paragraph, and eventually, it reaches the desired position. We apply the same procedure to every edge: this is possible since all the vertices have half-edges and thus at least two quadrants. □

Next, we prove Proposition 8.2 for strata with only simple poles.

Lemma 8.6. — *In a stratum $\mathcal{H}(a_1, a_2, [-1]^t)$ of differentials in genus zero with only simple poles, the generic isoresidual fibers have:*

– *two connected components, classified by the surgical Arf invariant, if* a_1 *and* a_2 *are even;*
– *only one connected component, otherwise.*

Proof. — Since λ is a configuration of real residues with a unique positive residue, the associated decorated trees are easy to describe. The $(t-2)!$ zeros of the translation structure ω_λ (see Proposition 4.1) correspond to trees with one vertex V_1 of valency $t-1$ and $t-1$ vertices attached to it according to some cyclic order. Each zero of ω_λ thus corresponds to a permutation σ on $\{2,\dots,t\}$ such that $\sigma(j)$ follows j in the cyclic order.

Suppose there exists a saddle connection γ on $(\overline{\mathcal{F}}_\lambda,\omega_\lambda)$ joining two zeros of ω_λ, corresponding respectively to permutations σ and τ of $\{2,\dots,t\}$. Then, σ and τ differ only by the location of j_0 in the cyclic order, depending on which edge incident to V_{j_0} is deleted, as explained in Lemma 2.10 and Figure 10. The location of j_0 in the cyclic order changes either by a_1 or by a_2. In other words, $\sigma\circ\tau^{-1}$ is a cycle of order a_1+1 or a_2+1. Since each such cycle corresponds to a saddle connection of ω_λ, these cycles generate the whole alternate group $\mathfrak{A}_{t-1}$ if a_1, a_2 are even, and the whole symmetric group $\mathfrak{S}_{t-1}$ otherwise.

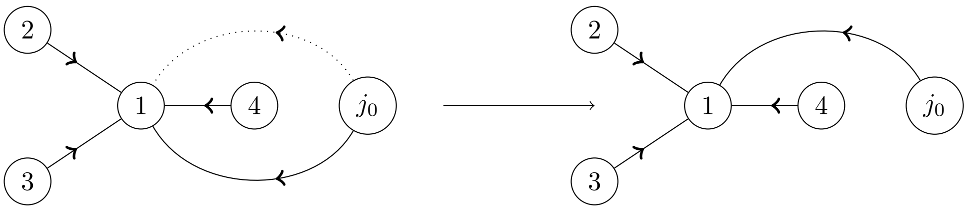

Figure 10. Going from one tree to another.

It follows that, provided a_1 and a_2 are even, the two zeros corresponding to the permutations σ and τ belong to the same connected component of $(\overline{\mathcal{F}}_\lambda,\omega_\lambda)$ if and only if σ and τ have the same signature. Indeed, the decomposition of $\sigma\circ\tau^{-1}$ into a product of cycles of length a_1+1 provides the chain of saddle connections joining the corresponding zeros. If either a_1 or a_2 is odd, the same argument shows that every pair of zeros can be joined by a chain of saddle connections. Therefore, the isoresidual fiber $\overline{\mathcal{F}}_\lambda$ is connected. □

We now prove the case in which there is a unique non-simple pole.

Lemma 8.7. — *In a stratum* $\mathcal{H}(a_1,a_2,-b,[-1]^t)$ *of differentials in genus zero with* $b\geqslant 2$*, the generic isoresidual fibers have:*
– k *connected components, where* $k=\gcd(a_1,a_2,b)$*, if* $t=2$*;*
– *two connected components, if* a_1,a_2,b *are even and* $t\geqslant 4$*;*
– *only one connected component, otherwise.*

Proof. — We consider an isoresidual fiber $\mathcal{F}_\lambda$, where the residues at the simple poles are negative real numbers and the residue at the pole of order b is positive. The decorated graphs corresponding to the zeros of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ have a vertex V_1 with $2(b-1)$ half-edges and $t-1$ vertices pointing towards it. We analyze which zeros can be connected by a saddle connection.

Since the case with a unique simple pole is trivial (as there is a unique graph), we begin with the case of two simple poles. We fix the vertex V_2 corresponding to the first simple pole and move V_3 along the graph. The move consists of drawing a new edge that partitions the vertices and half-edges into two sets of cardinality a_1 and a_2, and then deleting the original edge (see Figure 10). There are b possible positions for V_3, and the distance between consecutive positions is either a_1 or a_2. Therefore, there are $\gcd(a_1, a_2, b)$ possible orbits for the position of V_3.

Now, suppose there are $t \geqslant 3$ simple poles. Again, we fix the position of V_2 and want to place the other vertices V_i for $3 \leqslant i \leqslant t-1$ in the same quadrant and in increasing order. Consider V_3 and move it along the graph using the operation shown in Figure 10. If it moves next to V_2, we are done. Otherwise, continue moving it until it jumps above V_t. Then, we make one move of V_t so that it jumps over V_3 and let V_3 jump over it again. In this way, the position that V_3 will visit will be shifted by one quadrant relative to the previous moves. Either it will reach the correct position, or we can repeat the operation until it does. Thus, we can place all the V_i with $i \leqslant t-1$ in their correct positions.

Next, consider the last vertex V_t. We first place it after the vertices V_{t-2} and V_{t-1}, possibly switching the order of these two vertices. After moving V_t along the graph, if it reaches the correct quadrant, we are done. Otherwise, either we jump over both V_{t-2} and V_{t-1}, or we end up between them. In both cases, first move V_{t-2}, then V_t, and finally place V_{t-2} back. This will switch the position of V_t by one quadrant and swap the relative positions of V_{t-2} and V_{t-1}. In the second case, we are done, and in the first case, we can repeat this operation after rotating V_t along the graph. Eventually, the vertex v_t can be placed in the correct quadrant.

Finally, it remains to show that if one of a_1, a_2 or b is odd, then we can switch the positions of V_{t-1} and V_{t-2}. In the case with three simple poles, this is achieved by first moving V_{t-1}, then V_{t-2}, and finally V_t in the negative direction. If there are more simple poles, we first perform this move to switch the positions of V_{t-1} and V_{t-2}. This, however, changes the position of the triple of vertices by $-a_1+1$ or $-a_2+1$ quadrants. Then, by going back, we see that this triple is shifted by one quadrant. Thus, by repeating this operation, we obtain the triple in the desired order in any quadrant with an odd distance from the original one. Since the total number of quadrants where this triple can be is $b+t-3$, if $b+t$ is odd, then we are done. If either a_1 or a_2 is odd, say a_1, we can first move the triple by a_1 quadrants and then perform the same operation as in the previous case to place it in the desired position. Now, suppose both b and t are odd, and a_1, a_2 are even. In that case, we first move all the remaining $t-3$ simple poles by $-a_1$. Then, the triple lands in a new quadrant an odd distance from the desired quadrant. Thus, we can repeat the previous steps from this new position to reach the desired final configuration. □

Now, we can prove the general case.

Proof of Proposition 8.2. — By the previous results, we can assume that there are at least two poles of higher orders and at least one simple pole. If there is a unique simple pole, we can assume that it has the unique positive real residue. Then, all the other poles can be moved freely, as shown in the proof of Lemma 8.5. Hence, the generic isoresidual fiber is connected.

Now, suppose there are $s = 2$ simple poles. In this case, we fix one simple pole to be the unique pole with a positive residue. Then, the non-simple poles can be placed freely, as in the proof of Lemma 8.5. We place them such that they all point towards the simple pole with the positive residue. Now, consider the second simple pole, called q. Note that there are are $\sum b_i$ possible positions, so by performing the operation of Figure 10, we can visit all the quadrants at a distance of a_1 or a_2 from the previous one. Hence, there are $d := \gcd(a_1, a_2, \sum b_i)$ distinct classes of positions. If $d = k$ then we are done. Otherwise, take b_j such that $\gcd(d, b_j) < d$. Move the corresponding vertex V_j parallel to q such that they do not interact. After q passes the quadrant where V_j was attached, stop it and place b_j back in its original position. Then q is switched by b_j, and we obtain all the quadrants with a distance of $\gcd(a_1, a_2, b_j, \sum b_i)$. We complete the result by considering all the b_i's.

Finally, we consider the case where there are $s \geqslant 3$ simple poles. We first place the graph without the three simple poles in the desired form, as in the proof of Lemma 8.7. Then, the three simple poles can be placed in any position, up to order, if all the other orders are divisible by 2, as in the proof of Lemma 8.5. Hence, we obtain the desired upper bound on the number of connected components. □

8.4. Proof of Theorem 1.7. — By combining Proposition 8.2 with Lemma 8.1, we can extend the classification of the connected components of the generic isoresidual fibers to strata in genus zero with an arbitrary number of zeros.

Proof of Theorem 1.7. — We first consider the case of an isoresidual fiber
$$\mathcal{F}_\lambda(a_1, \ldots, a_n)$$
for which there is a permutation σ such that $\mathcal{F}_\lambda(a_{\sigma(1)} + \cdots + a_{\sigma(a_{n-1})}, a_{\sigma(n)})$ is connected. Then, any connected component $\mathcal{C}$ of $\mathcal{F}_\lambda(a_1, \ldots, a_n)$ contains the isoresidual fiber $\mathcal{F}_\lambda(a_{\sigma(1)} + \cdots + a_{\sigma(a_{n-1})}, a_{\sigma(n)})$ in its closure. Lemma 8.1 then shows that $\mathcal{C}$ is the only connected component of $\mathcal{F}_\lambda(a_1, \ldots, a_n)$.

In the remaining cases, we consider isoresidual fibers $\mathcal{F}_\lambda(a_1, \ldots, a_n)$ with $n \geqslant 3$, such that for any permutation σ, the stratum
$$\mathcal{H}(a_{\sigma(1)} + \cdots + a_{\sigma(a_{n-1})}, a_{\sigma(n)}, -b_1, \ldots, -b_p)$$
belongs to one of the two exceptional families described in Proposition 8.2. In particular, the orders of the poles are either $(-1, \ldots, -1)$, or of the form
$$(-kd_1, \ldots, -kd_{p-2}, -1, -1) \quad \text{with } k \geqslant 2.$$

We first consider the case where $b_1 = \cdots = b_p = 1$. If, for every permutation σ, the isoresidual fiber $\mathcal{F}_\lambda(a_{\sigma(1)}+\cdots+a_{\sigma(a_{n-1})}, a_{\sigma(n)})$ fails to be connected, then, by Proposition 8.2, it follows that all the $a_1, \ldots, a_n$ are even. The surgical Arf invariant, described in Section 8.1.2, is a well-defined topological invariant for the connected components of $\mathcal{F}_\lambda(a_1, \ldots, a_n)$. The argument that works in the general case then proves that the subset of $\mathcal{F}_\lambda(a_1, \ldots, a_n)$ consisting of differentials with an even (resp. odd) surgical Arf invariant (see Section 8.1.2) is connected. Since every fiber of the form $\mathcal{F}_\lambda(a_{\sigma(1)} + \cdots + a_{\sigma(a_{n-1})}, a_{\sigma(n)})$ has exactly one (non-empty) connected component for each parity of the invariant, we deduce that $\mathcal{F}_\lambda(a_1, \ldots, a_n)$ has exactly two connected components.

Finally, we consider the case of a stratum $\mathcal{H}(a_1, \ldots, a_n, -Ld_1, \ldots, -Ld_{p-2}, -1, -1)$, where two poles are simple and the others have orders divisible by $L \geqslant 2$. Since for every permutation σ and any t, the isoresidual fiber $\mathcal{F}_\lambda(\sum_{1\leqslant i\leqslant t} a_i, \sum_{t<i\leqslant n} a_i)$ fails to be connected, it follows from Proposition 8.2 that for any i, we have $L_{\sigma,t} \neq 1$, where $L_{\sigma,t} = \gcd(\sum_{1\leqslant i\leqslant t} a_i, L)$.

For each stratum $\mathcal{H}(\sum_{1\leqslant i\leqslant t} a_i, \sum_{t<i\leqslant n} a_i)$, the generic isoresidual fibers have $L_{\sigma,t}$ connected components, which are classified by the surgical rotation number in $\mathbb{Z}/L_{\sigma,t}\mathbb{Z}$ (see Section 8.1.1). We observe that the monodromy of the isoresidual fibration around the resonance hyperplane given by $\lambda_p = 0$ changes the surgical rotation number by one. Since every connected component of $\mathcal{F}_\lambda(\sum_{1\leqslant i\leqslant t} a_i, \sum_{t<i\leqslant n} a_i)$ belongs to the closure of exactly one connected component of $\mathcal{F}_\lambda(a_1, \ldots, a_n)$, it follows that the number of connected components of $\mathcal{F}_\lambda(a_1, \ldots, a_n)$ is at most the greatest common divisor of the $L_{\sigma,t}$. In particular, $\mathcal{F}_\lambda(a_1, \ldots, a_n)$ has at most k connected components, where $k = \gcd(a_1, \ldots, a_n, L)$.

Conversely, we know that $\mathcal{F}_\lambda(a_1, \ldots, a_n)$ has at least k connected components because the surgical rotation number in $\mathbb{Z}/k\mathbb{Z}$ is a topological invariant (see Section 8.1.1), and every number in $\mathbb{Z}/k\mathbb{Z}$ can be realized by some connected component of $\mathcal{F}_\lambda(a_1, a_2+\cdots+a_n)$. For example, the connected components may be classified by a surgical rotation number in a larger cyclic group, but the class modulo k remains unchanged by the splitting of the zero. Therefore, $\mathcal{F}_\lambda(a_1, \ldots, a_n)$ has exactly k connected components, and they are classified by their surgical rotation number in $\mathbb{Z}/k\mathbb{Z}$. □

9. The orders of singularities in strata and isoresidual fibers

We know from Section 2.5 that some strata of meromorphic differentials can be disconnected if the singularity pattern satisfies some arithmetic properties. The formulas in Section 3.4 provide a way to compute the number and order of the zeros and poles of the translation structure of generic isoresidual fibers in terms of the singularity pattern of the stratum. In this section, we use these formulas to derive general combinatorial relations between these orders.

In our notation, we distinguish between the singularities of a generic isoresidual fiber $\mathcal{F}$ and the orders of singularities in the pattern that define the stratum $\mathcal{H}$. By a slight abuse of notation, we will refer to the singularities of $\mathcal{H}$ (which, in fact, are the singularities of the translation surfaces parametrized by $\mathcal{H}$).

9.1. Simple poles in strata and isoresidual fibers

We will prove that every pole of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ is a simple pole if and only if the stratum is of the form $\mathcal{H}(a_1, a_2, -1, \ldots, -1)$.

Proof of the first part of Theorem 1.6. — We first consider a stratum $\mathcal{H}$ parametrizing differentials with only simple poles. Then, every pole of a generic isoresidual fiber of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ corresponds to a degeneration where K is empty (see Section 4.3). It follows that every pole of ω_λ is a simple pole.

Conversely, we assume that the partition μ contains a number $c \leqslant -2$. Suppose, by contradiction, that every pole of ω_λ is a simple pole. For such a simple pole of ω_λ, there is a partition $I \sqcup J$ (where both I and J are nonempty) of the set of poles of the parametrized differentials. Without loss of generality, we assume that the pole of order c belongs to J.

If I contains a simple pole, then we can transfer this simple pole from I to J, and the pole of order c from J to K. This gives a degeneration corresponding to a non simple pole of ω_λ.

If I does not contain any simple pole, then it contains a non simple pole (since we know by hypothesis that $I \neq \varnothing$). We then transfer this pole from I to K, while transferring the pole of order c to K at the same time. We obtain a degeneration corresponding to a non simple pole of ω_λ. Thus, ω_λ must have a non simple pole.

Notice that the case where ω_λ has no poles at all cannot occur because, when the configuration λ is real, all the periods of ω_λ are real. This would be impossible for a translation surface of finite area (which corresponds to a holomorphic one-form). ☐

9.2. Even singularities in strata and isoresidual fibers. — The goal of this section is to prove the second part of Theorem 1.6. We first show that when the singularity pattern of $\mathcal{H}(\mu)$ contains only singularities of even order, the translation structure ω_λ of generic isoresidual fibers also have only zeros and poles of even order.

Proposition 9.1. — *For a stratum $\mathcal{H}(a_1, a_2, -b_1, \ldots, -b_p)$ where $a_1, a_2, b_1, \ldots, b_p$ are even, every zero or pole of a generic isoresidual fiber $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ has even order.*

Proof. — For the zeros of ω_λ, the claim follows from Proposition 4.1. Any pole of ω_λ corresponds to a tripartition $I \sqcup J \sqcup K$ of $b_1, \ldots, b_p$. The order of the pole can be $a_1 - b_I + 2$ (if J is empty), $a_2 - b_J + 2$ (if I is empty), 2 (if both I and J are empty) or $1 + \operatorname{lcm}(a_1 - b_I + 1, a_2 - b_J + 1)$ otherwise. By hypothesis, $a_1 - b_I + 1$ and $a_2 - b_J + 1$ are odd, so a pole of ω_λ is of even order in all these cases. ☐

The converse of Proposition 9.1 (which forms the second half of Theorem 1.6) does not hold for $p = 2$. Using the results from Section 3.4, we compute the following example.

Example 9.2. — Generic isoresidual fibers $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ of the stratum $\mathcal{H}(3,3,-1,-7)$ are translation surfaces parametrized by $\mathcal{H}(6,-4,-4)$.

In the following, we will focus on strata of differentials with at least three poles (i.e., $p \geqslant 3$).

Lemma 9.3. — *Consider a stratum $\mathcal{H}(a_1, a_2, [-1]^s, -b_1, \dots, -b_t)$ with $p = s + t \geqslant 3$, $b_1, \dots, b_t \geqslant 2$, and $a_1 \leqslant a_2$. If every singularity of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ has even order, then $s = 0$, and additionally, a_1 and a_2 must have the same parity.*

Proof. — The zeros of ω_λ are of even order, so $a_1 + a_2$ is even. It follows that a_1 and a_2 must have the same parity. We also have $t \geqslant 1$, because otherwise, Theorem 1.6 would imply that every pole of ω_λ is simple.

We first assume $s \geqslant 1$ and consider a degeneration corresponding to a non simple pole of ω_λ. This degeneration corresponds to a tripartition $I \sqcup J \sqcup K$. Without loss of generality, we assume that I contains a simple pole (which cannot belong to K). The pole of ω_λ corresponding to this degeneration is of order $a_1 - b_I + 2$ or $1 + \mathrm{lcm}(a_1 - b_I + 1, a_2 - b_J + 1)$, depending on whether J is empty or not. Since every pole of ω_λ is of even order, it follows that $a_1 - b_I + 1$ is odd.

By moving a simple pole from I to J, we obtain a tripartition $I' \sqcup J' \sqcup K$ corresponding to another pole of ω_λ. If I' is not empty, the pole is of order $1 + \mathrm{lcm}(a_1 - b_I + 2, a_2 - b_J)$, which is odd (since $a_1 - b_I + 1$ is odd), leading to a contradiction. Therefore, I' must be empty, and the pole of ω_λ corresponding to this degeneration is of order $a_2 - b_J + 1$, which is thus even.

Now we consider two cases depending on whether J is empty or not. If J is nonempty, then the pole corresponding to the tripartition $I \sqcup J \sqcup K$ is of order $1 + \mathrm{lcm}(a_1 - b_I + 1, a_2 - b_J + 1)$, and thus both $a_1 - b_I + 1$ and $a_2 - b_J + 1$ are odd. However, we already know that $a_2 - b_J + 1$ is even, which leads to a contradiction. Consequently, the partitions corresponding to degenerations in the fiber $\mathcal{F}_\lambda$ are very specific. One subset of I and J contains a unique simple pole, while the other subset is empty, and every other pole belongs to the subset K. In particular, there is only one simple pole. However, since $p \geqslant 3$, at least one pole satisfies $b_j \leqslant a_2 + 1$. We can then find a tripartition $I \sqcup J \sqcup K$ where this pole belongs to J, which again leads to a contradiction. □

We treat separately the case in which one zero has a very small order.

Proposition 9.4. — *Consider a stratum $\mathcal{H}(a_1, a_2, -b_1, \dots, -b_p)$ with $a_1 \leqslant p - 2$. If every singularity of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ is of even order, then $a_1, a_2, b_1, \dots, b_p$ are even.*

Proof. — Following Lemma 9.3, we have $b_1, \dots, b_p \geqslant 2$. We consider partitions where K contains exactly $a_1 + 1$ poles, while J contains the remaining $p - a_1 - 1$ poles (with I being empty). We can freely choose the orders of the poles in J and K. Such a partition corresponds to a pole of ω_λ of order $2 + a_2 - b_J$ (which is even by hypothesis). Thus, b_J must have the same parity as a_2. Unless all the $b_1, \dots, b_p$

have the same parity, this is not possible for every partition corresponding to a degeneration of $\mathcal{F}_\lambda$. Therefore, all the $b_1, \dots, b_p$ must have the same parity.

If all the $b_1, \dots, b_p$ are odd, then a_2 must have the same parity as $p - a_1 - 1$. Since a_1 and a_2 have the same parity (see Lemma 9.3), we conclude that p is odd. Thus, $a_1 + a_2 - \sum_{j=1}^{p} b_j$ is odd, which is a contradiction.

If all the $b_1, \dots, b_p$ are even, then a_2 and $2 + a_2 - b_J$ must have the same parity. Since $2 + a_2 - b_J$ is even (it is the order of the corresponding pole of ω_λ), we deduce that a_2, and therefore a_1, are even. Consequently, all singularities of ω_λ are of even order. □

We will now prove the general case.

Proof of the second part of Theorem 1.6. — Following Propositions 9.1, 9.4, and Lemma 9.3, it remains to prove that in the case of a stratum $\mathcal{H}(a_1, a_2, -b_1, \dots, -b_p)$ with $p - 1 \leqslant a_1 \leqslant a_2$ and $b_1, \dots, b_p \geqslant 2$, if every singularity of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ is of even order, then $a_1, a_2, b_1, \dots, b_p$ are even.

We know from Lemma 9.3 that a_1 and a_2 have the same parity. We first assume that both a_1 and a_2 are odd. We then split the case based on whether all the $b_1, \dots, b_p$ are odd.

First, we assume that one of the pole orders b_j is even. If $b_j \leqslant a_2 + 1$, then there is a degeneration where J is a singleton formed by this pole. In this case, $a_2 - b_j + 1$ is even, so the corresponding pole of ω_λ would be of odd order. This is a contradiction, so we assume that every even order b_j among $b_1, \dots, b_p$ satisfies $b_j \geqslant a_2 + 3$. Consequently, there can be at most one such pole, b_1 (if there were two, their sum would exceed $a_1 + a_2 + 2$). Thus, the sum of the orders of the poles of odd order, $b_2, \dots, b_p$, is at most $a_1 - 1$. Therefore, there exists a tripartition $I \sqcup J \sqcup K$ such that I contains all the poles of odd order, while K contains at least the pole of order b_1. Since the number $p - 1$ of poles of odd order is even, the pole corresponding to this tripartition is automatically of odd order (because $a_1 - b_I + 2$ is odd).

Now, we consider the case where all the $b_1, \dots, b_p$ are odd. Since $a_1 + a_2$ is even, p is also even and satisfies $p \geqslant 4$. The two poles with the smallest orders have a total order of at most $1 + a_2$. There is a partition where these two poles are the only poles in J. This partition also corresponds to a pole of odd order for ω_λ, which is impossible. Thus, we have eliminated all cases where the two zeros are of odd order.

Now, we consider the case where both a_1 and a_2 are even. The number α of odd numbers among $b_1, \dots, b_p$ is even. We need to prove that $\alpha = 0$. Since $p \geqslant 3$, if there are poles of odd order among $b_1, \dots, b_p$, the smallest of them has order at most a_2 (because we would have $\alpha \geqslant 2$). There is a partition where this pole is the only one in J. Such a partition corresponds to a pole of odd order. Therefore, all the $b_1, \dots, b_p$ must be even when the singularities of ω_λ are of even order. □

9.3. Isoresidual fibers lying in disconnected strata of translation surfaces

It is possible for the connected components of isoresidual fibers to belong to strata that are not connected. In such cases, the connected components of $\mathcal{F}_\lambda$ are classified using two types of topological invariants: hyperellipticity and invariants derived from the winding numbers of loops in the translation structure (see Section 2.5).

Recall that strata with hyperelliptic components have at most two zeros and two poles. We will prove the following: if an isoresidual fiber has so few singularities, then it is automatically of genus zero (and therefore belongs to a connected stratum).

Proposition 9.5. — *The only strata $\mathcal{H}(a_1, a_2, -b_1, \ldots, -b_p)$ such that every connected component of the generic isoresidual fiber $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ has at most two zeros are:*

– *strata for which $p = 2$;*
– *stratum $\mathcal{H}(1, 1, [-1]^4)$, for which $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ belongs to $\mathcal{H}(2, 2, [-1]^6)$;*
– *strata $\mathcal{H}(k, k, -2k, -1, -1)$, where $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ consists of k connected components, each belonging to $\mathcal{H}(2k, 2k, -k, -k, -1-k, -1-k)$.*

In each of these cases, every connected component of $\mathcal{F}_\lambda$ has genus zero.

Proof. — In the case where $\mathcal{F}_\lambda$ is connected, the claim follows from Proposition 4.1 and the enumeration of poles provided in Sections 4.3 and 4.4.

If $\mathcal{F}_\lambda$ is disconnected, then Theorem 1.2 shows that all of them belong to the same stratum in the moduli space of translation surfaces. Theorem 1.7 precisely characterizes the strata $\mathcal{H}(a_1, a_2, -b_1, \ldots, -b_p)$ where this can occur.

In the first family of exceptional strata, a_1 and a_2 are even so we have $a_1+a_2 \geqslant 4$. The number of simple poles is even and positive so we have $p \geqslant 3$. It follows that the generic isoresidual fiber contains at least four zeros. The only case where it contains exactly four zeros implies $a_1 + a_2 = 4$ and $p = 3$. This characterizes stratum $\mathcal{H}(2, 2, -4, -1, -1)$ which also belongs to the second exceptional family.

In the second exceptional family, there exist $k \geqslant 2$ and positive coprime integers $c_1, c_2, d_1, \ldots, d_{p-2}$ such that strata are of the form

$$\mathcal{H}(kc_1, kc_2, -kd_1, \ldots, -kd_{p-2}, -1, -1).$$

In this case, the generic isoresidual fiber has k connected components and therefore contains at most $2k$ zeros. Since $p \geqslant 3$, we must have $c_1 = c_2 = 1$. Such a stratum is either $\mathcal{H}(k, k, -k, -k, -1, -1)$ or $\mathcal{H}(k, k, -2k, -1, -1)$. In the first case, Proposition 4.1 proves that the isoresidual fiber contains $2k(2k-1)$ zeros, which is impossible. In the second case, an explicit enumeration of the possible degenerations of the isoresidual fiber $\mathcal{F}_\lambda$ shows that it consists of k spheres. □

Recall that for strata of translation surfaces of genus one, the number of connected components is determined by the greatest common divisor of the orders of the singularities. In the following, we will prove that for a generic isoresidual fiber, this common divisor is either 1 or 2.

Proposition 9.6. — *Consider a stratum $\mathcal{H}(a_1, a_2, -b_1, \ldots, -b_p)$ with $p \geqslant 3$. If the orders of the singularities of $(\overline{\mathcal{F}}_\lambda, \omega_\lambda)$ are integer multiples of a positive integer k, then $k \in \{1, 2\}$.*

Proof. — In the following, we assume that $k \neq 1$, and we will then prove that $k = 2$. In particular, we have $b_1, \ldots, b_p \geqslant k$. By convention, we assume that $a_1 \leqslant a_2$. If $a_1 \geqslant p-1$, then there is a tripartition $I \sqcup J \sqcup K$, where both I and J are trivial. In this case, formulas from Section 4.4 show that the corresponding pole of ω_λ is a double pole, and thus $k = 2$. In the rest of the proof, we will assume that $a_1 \leqslant p - 2$.

We first prove that every pole order b_j is of the form $r + km_j$, where r is a constant number satisfying $rp \equiv 2 \pmod{k}$. Since $b_1, \ldots, b_p \geqslant k$, there are degenerations in $\mathcal{F}_\lambda$ such that $a_1 + 1$ poles belong to K, while the others belong to J (with I being empty). Since $a_1 \leqslant p-2$, J is nonempty, and the order $a_2 - b_J + 2$ of the pole of ω_λ corresponding to this degeneration changes when we change the choice of the poles of J. More precisely, $a_1 - b_J + 1$ changes by a multiple of k if we permute one pole of J with a pole of K. Consequently, there exists a number r such that every pole order b_j is of the form $r + kb_j$. The sum $a_1 + a_2$ is also an integer multiple of k. Since $a_1 + a_2 - \sum_{j=1}^{p} = -2$, we obtain that $rp \equiv 2 \pmod{k}$.

We first assume that some b_j among $b_1, \ldots, b_p$ satisfies $b_p \geqslant 3$. We consider a tripartition where K contains $a_1 + 1$ poles (including the pole of order b_p), while I is empty. The inequality $a_1 \leqslant p-2$ then implies that J is nonempty, and the corresponding pole of ω_λ has order $a_2 - b_J + 2$. We construct a new tripartition by moving a pole of order b_i (with $i \neq p$) from K to J. This results in a degeneration corresponding to a pole of order $a_2 - b_J - b_i + 2$. The fact that the order of every pole of ω_λ is a multiple of k implies that b_i is a multiple of k. We already know that every order b_j is of the form $r + km_j$, where r is a constant, so $r = 0$ and all the $b_1, \ldots, b_p$ are integer multiples of k. Since $a_1 + a_2$ is also an integer multiple of k, and -2 is a multiple of k, we conclude that $k = 2$.

In the last case, we have $b_1 = \cdots = b_p = 2$, and therefore $a_1 + a_2 = 2p - 2$. If a_1 and a_2 are odd, then, because $a_1 \leqslant p-2$, a bipartition $I \sqcup J$, where I contains $(a_1+1)/2$ double poles, corresponds to a simple pole of ω_λ, which is impossible.

We will assume that a_1 and a_2 are even. We consider the tripartition where I, J, and K contain $a_1/2$, $a_2/2$, and one double poles, respectively. We have $a_1 - b_I + 1 = 1$ and $a_2 - b_J + 1 = 1$, so the corresponding pole of ω_λ is a double pole. It follows that $k = 2$ in this case. □

It follows from Propositions 9.5, 9.6, and the results of Section 2.5 that if the generic isoresidual fibers of some stratum belong to a disconnected stratum, then the latter has exactly two connected components, classified by the parity of the spin structure. Following the second part of Theorem 1.6, this occurs for a generic isoresidual fiber $\mathcal{F}_\lambda$ of a stratum $\mathcal{H}(a_1, a_2, -b_1, \ldots, -b_p)$ if and only if the following three conditions are satisfied:

(1) The number p of poles satisfies $p \geqslant 3$;

(2) The orders $a_1, a_2, -b_1, \ldots, -b_p$ of the singularities of the stratum are even;

(3) The genus g of the generic isoresidual fiber $\mathcal{F}_\lambda$ satisfies $g \geqslant 1$ (Theorem 1.7 proves that for such μ, $\mathcal{F}_\lambda$ is always connected).

Open Problem 9.7. — *Show that the third condition regarding the genus of the fiber is redundant. Indeed, the simplest example of strata satisfying the first two conditions is* $\mathcal{H}(2,2,-2,-2,-2)$*, and we can check directly that its generic isoresidual fibers belong to* $\mathcal{H}(4^4,(-2)^8)$ *and are therefore elliptic curves.*

Open Problem 9.8. — *Provide a systematic computation of the parity of the spin structure for isoresidual fibers of strata satisfying the aforementioned three conditions.*

References

[ABW23] P. Apisa, M. Bainbridge & J. Wang – "Moduli spaces of complex affine and dilation surfaces", *J. reine angew. Math.* **2023** (2023), no. 796, p. 229–243.

[AM24] J. Athreya & H. Masur – *Translation surfaces*, Graduate Studies in Math., vol. 242, American Mathematical Society, Providence, RI, 2024.

[BCG+18] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky & M. Möller – "Compactification of strata of abelian differentials", *Duke Math. J.* **167** (2018), no. 12, p. 2347–2416.

[BCG+19a] ———, "Strata of k-differentials", *Algebraic Geom.* **6** (2019), no. 2, p. 196–233.

[BCG+19b] ———, "The moduli space of multi-scale differentials", 2019, `arXiv:1910.13492`.

[Ben23] F. Benirschke – "The boundary of linear subvarieties", *J. Eur. Math. Soc. (JEMS)* **25** (2023), no. 11, p. 4521–4582.

[BDG22] F. Benirschke, B. Dozier & S. Grushevsky – "Equations of linear subvarieties of strata of differentials", *Geom. Topol.* **26** (2022), no. 6, p. 2773–2830.

[BG25] A. Bogatyrëv & Q. Gendron – "The space of solvable Pell–Abel equations", *Compositio Math.* **161** (2025), no. 7, p. 1483–1511.

[Boi15] C. Boissy – "Connected components of the strata of the moduli space of meromorphic differentials", *Comment. Math. Helv.* **90** (2015), no. 2, p. 255–286.

[BR24] A. Buryak & P. Rossi – "Counting meromorphic differentials on $\mathbb{CP}^1$", *Lett. Math. Phys.* **114** (2024), no. 4, article no. 97 (27 pages).

[CD25] G. Calsamiglia & B. Deroin – "Isoperiodic meromorphic forms: two simple poles", *Groups Geom. Dyn.* **20** (2025), no. 1, p. 107–168.

[CG22] D. Chen & Q. Gendron – "Towards a classification of connected components of the strata of k-differentials", *Documents Math.* **27** (2022), p. 1031–1100.

[CMSZ20] D. Chen, M. Möller, A. Sauvaget & D. Zagier – "Masur–Veech volumes and intersection theory on moduli spaces of Abelian differentials", *Invent. Math.* **222** (2020), p. 283–373.

[CP25] D. Chen & M. Prado – "Counting differentials with fixed residues", *Lett. Math. Phys.* **115** (2025), no. 3, article no. 53 (26 pages).

[CMZ22] M. Costantini, M. Möller & J. Zachhuber – "The Chern classes and Euler characteristic of the moduli spaces of Abelian differentials", *Forum Math. Pi* **10** (2022), article no. e16 (55 pages).

[EMZ03] A. Eskin, H. Masur & A. Zorich – "Moduli spaces of Abelian differentials: the principal boundary, counting problems, and the Siegel-Veech constants", *Publ. Math. Inst. Hautes Études Sci.* **97** (2003), p. 61–179.

[FTZ23] G. Faraco, G. Tahar & Y. Zhang – "Isoperiodic foliation of the stratum $\mathcal{H}(1,1,-2)$", 2023, `arXiv:2305.06761`.

[GT21] Q. Gendron & G. Tahar – "Différentielles abéliennes à singularités prescrites", *J. Éc. polytech. Math.* **8** (2021), p. 1397–1428.

[GT22] ———, "Isoresidual fibration and resonance arrangements", *Lett. Math. Phys.* **112** (2022), no. 2, article no. 33 (36 pages).

[Kaw18] N. Kawazumi – "The mapping class group orbits in the framings of compact surfaces", *Q. J. Math.* **69** (2018), no. 4, p. 1287–1302.

[KLS21] I. Krichever, S. Lando & A. Skripchenko – "Real-normalized differentials with a single order 2 pole", *Lett. Math. Phys.* **111** (2021), no. 2, article no. 36 (19 pages).

[Pan09] D. Panov – "Polyhedral Kähler manifolds", *Geom. Topol.* **13** (2009), no. 4, p. 2205 – 2252.

[Sal25] N. Salter – "Stratified braid groups: monodromy", *Math. Proc. Cambridge Philos. Soc.* **178** (2025), no. 2, p. 259–292.

[Sug17] T. Sugiyama – "The moduli space of polynomial maps and their fixed-point multipliers", *Adv. Math.* **322** (2017), p. 132–185.

[Tah18] G. Tahar – "Counting saddle connections in flat surfaces with poles of higher order", *Geom. Dedicata* **196** (2018), no. 1, p. 145–186.

[Zor06] A. Zorich – "Flat surfaces.", in *Frontiers in number theory, physics, and geometry I. On random matrices, zeta functions, and dynamical systems (Les Houches, 2003)*, Springer, 2006, p. 437–583.

Manuscript received 19th June 2025
accepted 8th May 2026

Dawei Chen, Department of Mathematics, Boston College,
Chestnut Hill, MA 02467, USA
E-mail : dawei.chen@bc.edu
Url : https://sites.google.com/bc.edu/dawei-chen

Quentin Gendron, Instituto de Matemáticas de la UNAM,
Ciudad Universitaria, Tenochtitlán, 04510, México
E-mail : quentin.gendron@im.unam.mx
Url : https://qgendron-math.bee.wf/

Miguel Prado, Goethe University,
Frankfurt am Main, Hessen 60325, Germany
E-mail : prado@math.uni-frankfurt.de
Url : https://sites.google.com/bc.edu/miguelprado

Guillaume Tahar, Beijing Institute of Mathematical Sciences and Applications,
Huairou District, Beijing, China
E-mail : guillaume.tahar@bimsa.cn
Url : https://bimsa.net/people/tahar/