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Subcritical contact surgeries and the topology of symplectic fillings

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SUBCRITICAL CONTACT SURGERIES AND  
THE TOPOLOGY OF SYMPLECTIC FILLINGS

BY PAOLO GHIGGINI, KLAUS NIEDERKRÜGER &amp; CHRIS WENDL

ABSTRACT. — By a result of Eliashberg, every symplectic filling of a three-dimensional contact connected sum is obtained by performing a boundary connected sum on another symplectic filling. We prove a partial generalization of this result for subcritical contact surgeries in higher dimensions: given any contact manifold that arises from another contact manifold by subcritical surgery, its belt sphere is zero in the oriented bordism group  $\Omega_*^{SO}(W)$  of any symplectically aspherical filling  $W$ , and in dimension five, it will even be nullhomotopic. More generally, if the filling is not aspherical but is semipositive, then the belt sphere will be trivial in  $H_*(W)$ . Using the same methods, we show that the contact connected sum decomposition for tight contact structures in dimension three does not extend to higher dimensions: in particular, we exhibit connected sums of manifolds of dimension at least five with Stein fillable contact structures that do not arise as contact connected sums. The proofs are based on holomorphic disk-filling techniques, with families of Legendrian open books (so-called “Lobs”) as boundary conditions.

RÉSUMÉ (Chirurgies de contact sous-critiques et topologie des remplissages symplectiques)

Un résultat d'Eliashberg affirme que tout remplissage symplectique d'une somme connexe de contact en dimension 3 est obtenu par somme connexe au bord d'un autre remplissage symplectique. Nous montrons une généralisation partielle de ce résultat pour les chirurgies de contact sous-critiques en dimension supérieure : étant donnée une variété de contact obtenue à partir d'une autre par chirurgie sous-critique, la cosphère de l'anse est nulle dans le groupe de bordisme orienté  $\Omega_*^{SO}(W)$  de tout remplissage symplectiquement asphérique  $W$ . En dimension 5, elle est même homotope à zéro. Plus généralement, si le remplissage n'est pas asphérique mais semi-positif, alors la cosphère de l'anse est triviale dans  $H_*(W)$ . Nous montrons aussi, en utilisant des méthodes similaires, que la décomposition en somme connexe de contact pour les structures de contact tendues en dimension 3 ne s'étend pas en dimension supérieure. Nous exhibons en particulier des sommes connexes de variétés de dimension au moins 5 qui portent une structure de contact Stein remplissable qui ne peut pas se mettre sous la forme d'une somme connexe de contact. Les démonstrations s'appuient sur les techniques de remplissage par disques holomorphes avec, pour conditions au bord, des familles de livres ouverts legendriens (que l'on abrège par « Lob »).

MATHEMATICAL SUBJECT CLASSIFICATION (2010). — 57R17, 53D10, 32Q65, 57R65.

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## 1. INTRODUCTION

1.1. THE MAIN RESULT AND COROLLARIES. — The idea of constructing contact manifolds as boundaries of symplectic  $2n$ -manifolds by attaching handles of index at most  $n$  goes back to Eliashberg [Eli90b] and Weinstein [Wei91]. In this context, a special role is played by *subcritical* handles, i.e., handles with index strictly less than  $n$ . One well-known result on this topic concerns subcritical Stein fillings, which are known to be *flexible* in the sense that their symplectic geometry is determined by homotopy theory, see [CE12]. There are also known restrictions on the topological types of subcritical fillings, e.g. by results of M.-L. Yau [Yau04] and Oancea-Viterbo [OV12, Prop. 5.7], the homology of a subcritical filling with vanishing first Chern class is uniquely determined by its contact boundary; this result can be viewed as a partial generalization of the Eliashberg-Floer-McDuff theorem [McD91] classifying symplectically aspherical fillings of standard contact spheres up to diffeomorphism. In dimension three, there is a much stronger result due to Eliashberg [Eli90a, CE12]: in this case every subcritical surgery is a connected sum, and the result states that if  $(M', \xi')$  is a closed contact 3-manifold obtained from another (possibly disconnected) contact manifold  $(M, \xi)$  by a connected sum, then every symplectic filling of  $(M', \xi')$  is obtained by attaching a Weinstein 1-handle to a symplectic filling of  $(M, \xi)$ . This implies that symplectic fillings of subcritically fillable contact 3-manifolds are unique up to symplectic deformation equivalence and blowup—in particular, their Stein fillings are unique up to symplectomorphism.

The present paper was motivated by the goal of generalizing Eliashberg's connected sum result to higher dimensions. The natural question in this setting is the following:

QUESTION 1.1. — *Given a closed contact manifold  $(M', \xi')$  that is obtained from another contact manifold  $(M, \xi)$  by subcritical contact surgery, is every (symplectically aspherical) filling of  $(M', \xi')$  obtained by attaching a subcritical Weinstein handle to a symplectic filling of  $(M, \xi)$ ?*

Classifying fillings up to symplectomorphism as suggested in this question would be far too ambitious in higher dimensions, e.g. the strongest result known so far,

the Eliashberg-Floer-McDuff theorem, is essentially a classification of fillings up to *homotopy type* (the *h*-cobordism theorem then improves it to a classification up to diffeomorphism). Our objective in this paper will therefore be to understand the main *homotopy-theoretic* obstruction to an affirmative answer to Question 1.1.

To state the main result, let us first recall some basic notions. Given an oriented  $(2n - 1)$ -dimensional manifold  $M$ , a (positive, co-oriented) *contact structure* on  $M$  is a hyperplane distribution of the form  $\xi = \ker \alpha$ , where the *contact form*  $\alpha$  is a smooth 1-form satisfying  $\alpha \wedge (d\alpha)^{n-1} > 0$ , and the co-orientation of  $\xi$  is determined by  $\alpha > 0$ . In this paper, contact structures will *always* be assumed to be both positive (with respect to a given orientation on  $M$ ) and co-oriented, and all contact forms will be assumed compatible with the co-orientation. A compact symplectic  $2n$ -manifold  $(W, \omega)$  with oriented boundary  $M = \partial W$  carrying a contact structure  $\xi$  is called a *strong symplectic filling* of  $(M, \xi)$  if  $\xi$  admits a contact form  $\lambda$  that extends to a primitive of  $\omega$  on a neighborhood of  $\partial W$ . It is equivalent to say that the boundary is *symplectically convex*, as the vector field  $\omega$ -dual to  $\lambda$  is then a Liouville vector field pointing transversely outward at  $\partial W$ . More generally, we say that  $(W, \omega)$  is a *weak symplectic filling* of  $(M, \xi)$  if  $\xi$  is the bundle of complex tangencies for some  $\omega$ -tame almost complex structure near  $\partial W$  that makes the boundary pseudoconvex (see [MNW13]).

Recall that if  $(M, \xi)$  contains a  $(k - 1)$ -dimensional isotropic sphere  $S_{\text{at}}^{k-1}$  with trivial normal bundle, then one can perform a *contact surgery of index  $k$*  on  $(M, \xi)$  by attaching to  $(-\varepsilon, 0] \times M$  a handle of the form  $D^k \times D^{2n-k}$  along a neighborhood of  $S_{\text{at}}^{k-1}$ . The new contact manifold  $(M', \xi')$  then contains the  $(2n - k - 1)$ -dimensional coisotropic sphere  $S_{\text{belt}}^{2n-k-1} = \{0\} \times \partial D^{2n-k}$ , which we call the *belt sphere* of the surgery. This surgery operation was first introduced by Weinstein [Wei91], and we will give a more precise description of it in Section 3. A Weinstein handle yields a symplectic cobordism that can be attached to any weak filling  $(W, \omega)$  of  $(M, \xi)$  for which  $\omega$  is exact along  $S_{\text{at}}^{k-1}$ ; the result is a weak filling of  $(M', \xi')$  in which the belt sphere is necessarily contractible (Figure 1.1).

Our main result is the following.

**THEOREM 1.2.** — *Suppose  $(M', \xi')$  is a closed contact manifold of dimension  $2n - 1 > 3$  that has been obtained from a manifold  $(M, \xi)$  by a contact surgery of index  $k \leq n - 1$ , with belt sphere  $S_{\text{belt}}^{2n-k-1} \subset M'$ . Assume  $(W', \omega')$  is a weak symplectic filling of  $(M', \xi')$ .*

(a) *If  $(W', \omega')$  is semipositive, then the belt sphere represents the trivial homology class in  $H_{2n-k-1}(W'; \mathbb{Z})$ .*

(b) *If  $(W', \omega')$  is symplectically aspherical, then the belt sphere represents the trivial element in the oriented bordism group  $\Omega_{2n-k-1}^{SO}(W')$ . If additionally either (1)  $M'$  is 5-dimensional, or (2)  $M'$  is 7-dimensional and  $k = 3$ , then  $S_{\text{belt}}^{2n-k-1}$  is contractible in  $W'$ , that is, it represents the trivial class in  $\pi_{2n-k-1}(W')$ .*

We now state two related results that follow via the same techniques.

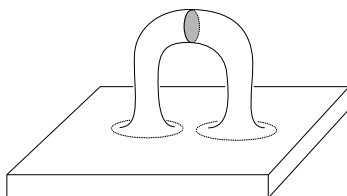


FIGURE 1.1. The belt sphere bounds an embedded disk inside the handle.

Recall that in dimension three, convex surface theory gives rise to a contact prime decomposition theorem, implying e.g. that every tight contact structure on a closed 3-manifold of the form  $M_0 \# M_1$  arises as a contact connected sum of tight contact structures on  $M_0$  and  $M_1$  (see [Col97], or [Gei08, §4.12] for more details). Some evidence against a higher-dimensional generalization of this result appeared in the recent work of Bowden, Crowley and Stipsicz [BCS15], providing also a negative answer to a topological version of Question 1.1: namely, there exist pairs of closed oriented manifolds  $M_0, M_1$  such that  $M_0 \# M_1$  admits a Stein fillable contact structure but  $M_0$  and  $M_1$  do not. This did not imply an actual answer to Question 1.1, however, as it was unclear whether the contact structures on  $M_0 \# M_1$  in the examples of [BCS15] could actually be *contact* connected sums, i.e., whether they arise from contact structures  $\xi_0$  on  $M_0$  and  $\xi_1$  on  $M_1$  by performing index 1 contact surgery. The following result gives a negative answer to the latter question, and shows that there is no hope of extending the contact prime decomposition theorem to higher dimensions. The theorem applies in particular whenever  $M$  is an almost contact  $S^{n-1}$ -bundle over  $S^n$  that is not a homotopy sphere, so for instance  $M$  could be  $S^{n-1} \times S^n$  or—as in [BCS15]—the unit cotangent bundle of a sphere.

**THEOREM 1.3.** — *Suppose  $M$  is a closed oriented manifold of dimension  $2n - 1 > 5$  that is not a homotopy sphere but admits an almost contact structure  $\Xi$  and a Morse function with unique local maxima and minima and otherwise critical points of index  $n - 1$  and  $n$  only. Then  $M \# (-M)$  admits a Stein fillable contact structure that is homotopic to the almost contact structure  $\Xi \# \bar{\Xi}$  but is not isotopic to  $\xi_1 \# \xi_2$  for any contact structures  $\xi_1$  and  $\xi_2$  on  $M$  and  $-M$  respectively.* <sup>(1)</sup>

Note that the contact structures in the above statement are necessarily tight in the sense of Borman-Eliashberg-Murphy [BEM15]; this follows from Stein fillability, using [Nie06] and the observation in [BEM15] that any overtwisted contact structure is also PS-overtwisted.

The holomorphic disk techniques developed in this article can also be used as in the work of Hofer [Hof93] to prove the Weinstein conjecture for a wide class of contact

<sup>(1)</sup>Given an oriented manifold  $M$  with almost contact structure  $\Xi$ , we denote by  $-M$  the same manifold with reversed orientation, and let  $\bar{\Xi}$  denote the almost contact structure on  $-M$  obtained by inverting the co-orientation of  $\Xi$ .

manifolds obtained by subcritical surgery. The following theorem, proved in Section 7, is related to the well-known result that every subcritically Stein fillable contact form admits a contractible Reeb orbit. (A similar result specifically for index 1 surgeries appeared recently in [GZ16].)

**THEOREM 1.4.** — *Assume  $(M', \xi')$  is the result of performing a contact surgery of index  $k \subset n - 1$  on a closed contact manifold  $(M, \xi)$  of dimension  $2n - 1 > 3$ , with belt sphere  $S_{\text{belt}}^{2n-k-1} \subset M'$ , and suppose at least one of the following conditions holds:*

- (1)  $[S_{\text{belt}}^{2n-k-1}] \neq 0$  in  $\Omega_{2n-k-1}^{SO}(M')$ ;
- (2)  $[S_{\text{belt}}^{2n-k-1}] \neq 0$  in  $\pi_{2n-k-1}(M')$  and either  $\dim M' = 5$  or  $\dim M' = 7$  with  $k = 3$ ;

(3)  $\dim M' = 5$  and  $(M', \xi')$  is a contact connected sum  $(M_0, \xi_0) \# (M_1, \xi_1)$  with the following two properties:

- (a) Neither  $M_0$  nor  $M_1$  is homeomorphic to a sphere;
- (b) If  $M_0$  and  $M_1$  are both rational homology spheres, then either both are not simply connected or at least one of them has infinite fundamental group.

Then every contact form for  $\xi$  admits a contractible Reeb orbit.

Before discussing the proofs, some further remarks about the main theorem are in order.

**REMARK 1.5.** — We do not know whether the dimensional restriction for the contractibility result in part (b) of Theorem 1.2 is essential, but given the wide range of known contact geometric phenomena that can happen *only* in sufficiently high dimensions, we consider it plausible that the contractibility statement could be false without some restriction of this type (thus implying a definitively negative answer to Question 1.1 in general). It is apparent in any case that our method will not work in all dimensions, as the improvement from “null-bordant” to “nullhomotopic” involves subtle topological difficulties that increase with the dimension; see the beginning of Section 5 for more discussion of this.

**REMARK 1.6.** — It is clear that nothing like Theorem 1.2 can be true for *critical surgeries* in general, i.e., the case  $k = n$ . There are obvious counterexamples already in dimension three, as any Legendrian knot  $L \subset (M', \xi')$  can be viewed as the belt sphere arising from a critical contact surgery—take  $(M, \xi)$  in this case to be the result of a Legendrian (+1)-surgery along  $L$ . It is certainly not true in general that arbitrary Legendrian knots are nullhomologous in every filling of  $(M, \xi)$ !

**REMARK 1.7.** — The semipositivity assumption in part (a) of Theorem 1.2 is there for technical reasons and could presumably be lifted using more advanced technology (e.g. polyfolds, see [HWZ11]). In contrast, symplectic asphericity in part (b) is a geometrically meaningful condition that, while not needed for Eliashberg’s three-dimensional version of this result, cannot generally be removed in higher dimensions;

see Example 1.8 below. The answer to Question 1.1 thus becomes negative without this assumption.

**EXAMPLE 1.8.** — The blowup of the total space of the rank 2 holomorphic vector bundle  $\mathcal{O}(-2) \oplus \mathcal{O}$  over  $CP^1$  at the zero section can be viewed as a (not symplectically aspherical) weak filling of a subcritically Stein fillable contact manifold  $(M', \xi')$  containing a belt sphere that is homotopically nontrivial in the filling. This is a special case of the following construction, which gives examples with subcritical handles of any even index  $k = 2m > 2$  in any dimension  $2n > 2k + 2 > 6$ .

Choose integers  $m, \ell > 1$  and set  $n = 2m + \ell$ , and suppose  $(\Sigma, \sigma)$  is a  $2m$ -dimensional closed symplectic manifold. Then consider the  $2n$ -dimensional Weinstein manifold  $T^*\Sigma \times C^\ell$ , i.e., the  $\ell$ -fold stabilization of  $T^*\Sigma$  with its standard Weinstein structure, and denote its ideal contact boundary by  $(M', \xi')$ . Any Morse function on  $\Sigma$  gives rise to a Weinstein handle decomposition of  $T^*\Sigma \times C^\ell$ , such that the function's maximum  $q \in \Sigma$  corresponds to an  $(n - \ell)$ -handle whose belt sphere  $S_{\text{belt}}^{n+\ell-1}$  is isotopic to the unit sphere in  $T_q^*\Sigma \times C^\ell$ . Let  $\Sigma \subset T^*\Sigma$  denote the zero section, so  $\Sigma \times \{0\}$  is an isotropic submanifold in  $T^*\Sigma \times C^\ell$ , and denote by  $\pi: T^*\Sigma \times C^\ell \rightarrow \Sigma \times \{0\}$  the obvious projection. Then for any  $\varepsilon > 0$  sufficiently small, adding  $\varepsilon\pi^*\sigma$  to the natural exact symplectic form on  $T^*\Sigma \times C^\ell$  gives a weak filling of  $(M', \xi')$  with  $\Sigma \times \{0\}$  as a symplectic submanifold. We can then blow up along this submanifold, as explained in [MS98, §7.1]. This produces a new weak filling  $(W', \omega')$  of  $(M', \xi')$ , in which the belt sphere  $S_{\text{belt}}^{n+\ell-1} \subset M'$  is nullhomologous but homotopically nontrivial: indeed, every fiber  $T_q^*\Sigma \times C^\ell$  has now been replaced by its blowup at the point  $(0, 0)$ , which can be viewed as the tautological line bundle over  $CP^{m+\ell-1}$ , so the bundle projection sends  $S_{\text{belt}}^{n+\ell-1}$  to a generator of  $\pi_{2(m+\ell)-1}(CP^{m+\ell-1}) \cong \mathbb{Z}$ .

The special case with  $\Sigma = S^2$  and  $\ell = 1$  gives the construction described at the beginning of this example, because the total space of  $\mathcal{O}(-2)$  is a deformation of  $T^*S^2$ .

The following represents another easy application of Theorem 1.2.

**EXAMPLE 1.9.** — Suppose  $(M_1, \xi_1)$  is a contact 5-manifold obtained by a subcritical surgery of index 2 on a sphere  $(S^5, \xi)$ , where  $\xi$  is any contact structure. Then  $M_1$  is diffeomorphic to either  $S^2 \times S^3$  or  $S^2 \tilde{\times} S^3$ , i.e., the trivial or nontrivial 3-sphere bundle over the 2-sphere. Indeed, closed loops in  $S^5$  are automatically unknotted, and the possible framings of the surgery are classified by the elements of  $\pi_1(\text{SO}(3)) \cong \mathbb{Z}_2$ . If  $(W, \omega)$  is any symplectically aspherical weak filling of  $(M_1, \xi_1)$ , then by Theorem 1.2, the fiber  $\{p\} \times S^3$  is a contractible 3-sphere in  $W$ .

Now take  $(M_2, \xi_2)$  to be the unit cotangent bundle of the 3-sphere or, more generally, any contact manifold supported by a contact open book with page  $T^*S^2$  and monodromy isotopic to a  $2k$ -fold product of Dehn twists for some integer  $k > 1$ . Then  $M_2$  will be diffeomorphic to  $S^2 \times S^3$ , but  $(M_2, \xi_2)$  admits a Stein filling that contracts to a bouquet of  $2k - 1$  three-dimensional spheres (see e.g. [vKN05]). We conclude that whenever  $(M_1, \xi_1)$  admits a symplectically aspherical weak filling, it is not contactomorphic to  $(M_2, \xi_2)$ . This implies for instance that the contact structures

induced on the ideal contact boundaries of  $T^*S^3$  and  $T^*S^2 \times \mathbb{C}$  (cf. Remark 1.7) are not isomorphic.

There are presumably other ways to distinguish  $\xi_1$  and  $\xi_2$  in many cases, e.g. using Symplectic Homology, but the technique described above is much more topological.

1.2. IDEA OF THE PROOF. — Our proof of Theorem 1.2 is based on a higher-dimensional analogue of the disk-filling methods underlying Eliashberg’s result in dimension three [Eli90a]. Such methods work whenever one can find a suitable submanifold to serve as a boundary condition for holomorphic disks, and the natural object to consider in this case is known as a *Legendrian open book* or “**Lob**”. Let us recall the definition, which is due originally to the second author and Rechtman [NR11].

DEFINITION 1.10. — A **Lob** in a  $(2n - 1)$ -dimensional contact manifold  $(M, \xi)$  is a closed  $n$ -dimensional submanifold  $L \subset M$  equipped with an open book decomposition  $\pi: L \setminus B \rightarrow S^1$  whose binding  $B \subset L$  is an  $(n - 2)$ -dimensional isotropic submanifold of  $(M, \xi)$ , and whose pages  $\pi^{-1}(*)$  are each Legendrian submanifolds of  $(M, \xi)$ .

The simplest interesting example of a **Lob** occurs at the center of the “neck” in any 3-dimensional contact connected sum: here we find a 2-sphere  $S \subset M$  on which the characteristic foliation  $\xi \cap TS$  traces an  $S^1$ -family of longitudes connecting the north and south poles, so we can regard the longitudes as pages of an open book with the poles as binding. Such spheres were used as totally real boundary conditions for holomorphic disks in [BG83, Gro85, Eli90a], and similarly in Hofer’s proof of the Weinstein conjecture [Hof93] for contact 3-manifolds  $(M, \xi)$  with  $\pi_2(M) \neq 0$ .

In higher dimensions, a **Lob**  $L \subset (M, \xi)$  with binding  $B \subset L$  similarly defines a totally real submanifold  $\{0\} \times (L \setminus B)$  in the symplectization  $\mathbb{R} \times M$  of  $(M, \xi)$ , and thus serves as a natural boundary condition for pseudoholomorphic disks. Moreover,

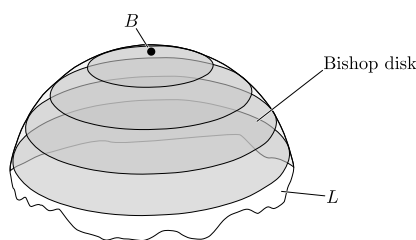


FIGURE 1.2. A schematic picture of the Bishop family around the binding of a **Lob**  $L$ .

for a suitably “standard” choice of almost complex structure near the binding, a **Lob** always gives rise to a canonical family of holomorphic disks near  $\{0\} \times B$  whose boundaries foliate a neighborhood of  $B$  in  $L \setminus B$  (see Figure 1.2). This is the so-called *Bishop family* of holomorphic disks, and it has the useful property that no other holomorphic curve can enter the region occupied by the Bishop disks from outside.



For a unified treatment of the essential analysis for Bishop disks with boundary on a **Lob**, see [Nie13].

As in the 3-dimensional case, we will see that the belt sphere of a surgery of index  $n - 1$  on a contact  $(2n - 1)$ -manifold is also naturally a **Lob**, so there is again a natural moduli space of holomorphic disks that fill the belt sphere, implying that it is nullhomologous. For surgeries of index  $k < n - 1$ , the belt sphere has dimension  $2n - k - 1 > n$ , and thus cannot be a **Lob**, but we will show that after a suitable deformation, the belt sphere can be viewed as a parametrized *family* of **Lobs**, giving rise to a well-behaved moduli space of disks with moving boundary condition. It should now be clear why this method cannot work for critical surgeries: the belt sphere in this case has dimension  $n - 1$ , so it is too small to define a totally real boundary condition.

The construction of the family of **Lobs** foliating a general subcritical belt sphere is somewhat less than straightforward: as we will see in Section 3, the standard model for a contact form after surgery does not lend itself well to this construction, but a natural family of **Lobs** can be found after deforming to a different model of the belt sphere as piecewise smooth boundary of a poly-disk.

Let us now discuss the topological reasons why the family of **Lobs** foliating  $S_{\text{belt}}^{2n-k-1} \subset (M', \xi')$  in Theorem 1.2 places constraints on the filling  $(W', \omega')$ . We focus for now on the case where  $(W', \omega')$  is symplectically aspherical, which rules out bubbling of holomorphic spheres. For a suitable choice of tame almost complex structure  $J$  on  $(W', \omega')$ , the Bishop families associated to  $S_{\text{belt}}^{2n-k-1}$  generate a compactified moduli space  $\overline{\mathcal{M}}$  of  $J$ -holomorphic disks in  $W'$  with one marked point, whose boundaries are mapped to  $S_{\text{belt}}^{2n-k-1}$ . In light of the marked point, this moduli space is necessarily diffeomorphic to a manifold with boundary and corners of the form

$$\overline{\Sigma} \times \mathbb{D}^2,$$

where  $\overline{\Sigma}$  is a smooth, compact, connected and oriented manifold with boundary and corners, whose boundary is a sphere. Furthermore, the natural evaluation map

$$\text{ev}: (\overline{\mathcal{M}}, \partial\overline{\mathcal{M}}) \longrightarrow (W', S_{\text{belt}}^{2n-k-1})$$

is smooth, and its restriction

$$\text{ev}|_{\partial\overline{\mathcal{M}}}: \partial\overline{\mathcal{M}} \longrightarrow S_{\text{belt}}^{2n-k-1}$$

is a continuous map of degree  $\pm 1$ . The latter follows readily from the properties of the Bishop family, which provide a nonempty open subset  $U \subset W'$  that intersects  $S_{\text{belt}}^{2n-k-1}$  and is the diffeomorphic image of  $\text{ev}^{-1}(U) \subset \overline{\mathcal{M}}$ .

This description of the evaluation map  $\text{ev}: \overline{\mathcal{M}} \rightarrow W'$  already implies the homological part of Theorem 1.2, i.e., that  $[S_{\text{belt}}^{2n-k-1}] = 0 \in H_{2n-k-1}(W')$ . To deduce stronger constraints, we will apply two further techniques. The first consists in performing surgery on the moduli space  $\overline{\mathcal{M}}$  to simplify its topology and suitably extending the evaluation map in order to prove  $[S_{\text{belt}}^{2n-k-1}] = 0 \in \pi_{2n-k-1}(W')$ . This method works when the dimension of  $\overline{\mathcal{M}}$  is not too large. The second method is relevant in particular

to the case  $k = 0$  of the main result, as well as to Theorem 1.3, and is based on the following topological lemma.

LEMMA 1.11. — *Let  $X, Y$  be smooth orientable compact manifolds with boundary and corners such that  $\partial Y$  is homeomorphic to a sphere and  $\dim X + 2 = \dim Y > 3$ . Write  $X' = X \times \mathbb{D}^2$ , and assume that*

$$f: (X', \partial X') \longrightarrow (Y, \partial Y)$$

*is a continuous map that is smooth on the interior of  $X'$ , and for which we find an open subset  $U \subset \mathring{Y}$  such that  $f|_{f^{-1}(U)}: f^{-1}(U) \rightarrow U$  is a diffeomorphism.*

*Then  $Y$  is contractible.*

While it will not be essential to most of our arguments, note that the  $h$ -cobordism theorem implies:

COROLLARY 1.12. — *If  $\dim Y > 5$  in the lemma, then  $Y$  is diffeomorphic to a ball.*

The  $k = 0$  case of Theorem 1.2 is the case where  $(M', \xi')$  is the standard contact sphere and the belt sphere is the entirety of  $M'$ . In this setting, applying the above lemma to the evaluation map  $\text{ev}: (\overline{\mathcal{M}} = \overline{\Sigma} \times \mathbb{D}^2, \partial \overline{\mathcal{M}}) \rightarrow (W', S^{2n-1})$  implies that  $W'$  must be diffeomorphic to a ball, hence this reproves the Eliashberg-Floer-McDuff theorem. We will explain this argument in more detail in Section 2, including the proof of the lemma (see Lemma 2.3). In another context, we will also apply the lemma in Section 6 to demonstrate that the contact structures arising on the boundaries of certain Stein domains which look topologically like connected sums cannot arise from index 1 contact surgery, thus proving Theorem 1.3.

Here is a brief outline of the paper. In Section 2, we provide a foretaste of the methods in the rest of the paper by using them to give an alternative proof of the Eliashberg-Floer-McDuff theorem. Section 3 then explains the general case of the family of Lobs associated to a subcritical belt sphere. In Section 4, we define the relevant moduli space of holomorphic disks and establish its basic properties, leading to the proof of the homological part of Theorem 1.2. Section 5 then improves this to a homotopical statement in cases where the moduli space has sufficiently low dimension. Finally, in Section 6 and Section 7 respectively we prove Theorems 1.3 and 1.4 on contact connected sums and contractible Reeb orbits. The paper concludes with a brief appendix addressing the technical question of orientability for our moduli space of holomorphic disks.

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## 2. THE ELIASHBERG-FLOER-McDUFF THEOREM REVISITED

In this section we modify slightly the proof of the Eliashberg-Floer-McDuff theorem [McD91, Th. 1.5] in order to illustrate the methods that will be applied in the rest of the article. The original argument worked by capping off the symplectic filling and then sweeping through it with a moduli space of holomorphic spheres. Our version will be the same in many respects, but has more in common with the 3-dimensional argument of Eliashberg in [Eli90a]: instead of spheres, we use holomorphic disks attached to a family of Lobs.

**THEOREM 2.1** (Eliashberg-Floer-McDuff). — *Let  $S^{2n-1} \subset \mathbb{C}^n$  be the unit sphere with its standard contact structure  $\xi_0$  given by the complex tangencies to the sphere, that is,*

$$\xi_0 = TS^{2n-1} \cap (i \cdot TS^{2n-1}).$$

*Every symplectically aspherical filling of  $(S^{2n-1}, \xi_0)$  is diffeomorphic to the  $(2n)$ -ball.*

Let  $\mathbf{z} = \mathbf{x} + i\mathbf{y} = (x_1 + iy_1, \dots, x_n + iy_n)$  be the coordinates of  $\mathbb{C}^n$ . The function  $f: \mathbb{C}^n \rightarrow [0, \infty)$  given by

$$f(\mathbf{z}) = \sum_{j=1}^n (x_j^2 + y_j^2)$$

is plurisubharmonic, and the unit sphere is the boundary of the ball

$$D^{2n} = \{\mathbf{z} \in \mathbb{C}^n \mid f(\mathbf{z}) \leq 1\}.$$

The main geometric ingredient needed for our proof is a foliation of  $S^{2n-1}$  (minus some singular subset) by a family of Lobs, but this idea does not seem to work when applied directly to the unit sphere  $f^{-1}(1)$ . Instead, we will deform the sphere to a different shape, which does contain a suitable family of Lobs that will suffice for our purposes. Define two functions  $g_A, g_B: \mathbb{C}^n \rightarrow [0, \infty)$  by

$$\begin{aligned} g_A(\mathbf{z}) &= y_1^2 + \dots + y_{n-1}^2 \\ g_B(\mathbf{z}) &= x_1^2 + \dots + x_{n-1}^2 + x_n^2 + y_n^2. \end{aligned}$$

Note that  $g_B$  is strictly plurisubharmonic and  $g_A$  is weakly plurisubharmonic as

$$\begin{aligned} -dd^c g_A &= 2 dx_1 \wedge dy_1 + \dots + 2 dx_{n-1} \wedge dy_{n-1} \\ -dd^c g_B &= 2 dx_1 \wedge dy_1 + \dots + 2 dx_{n-1} \wedge dy_{n-1} + 4 dx_n \wedge dy_n. \end{aligned}$$

We will now consider the subset (see Fig. 2.1)

$$\widehat{D}^{2n} = \{\mathbf{z} \in \mathbb{C}^n \mid g_B(\mathbf{z}) \leq 1\} \cap \{\mathbf{z} \in \mathbb{C}^n \mid g_A(\mathbf{z}) \leq 1\}.$$

Up to reordering the coordinates,  $\widehat{D}^{2n}$  is a bi-disk  $D^{n+1} \times D^{n-1} \subset \mathbb{R}^{2n}$ , which clearly contains the unit ball. Its boundary is not a smooth manifold, but is nonetheless homeomorphic to the unit sphere. It decomposes as

$$\partial \widehat{D}^{2n} \cong D^{n+1} \times (\partial D^{n-1}) \cup (\partial D^{n+1}) \times D^{n-1} = S_A \cup S_B,$$

where we have used the notation

$$S_A := \{g_A = 1\} \cap \partial \widehat{D}^{2n} \cong \mathbb{D}^{n+1} \times S^{n-2}$$

and

$$S_B := \{g_B = 1\} \cap \partial \widehat{D}^{2n} \cong S^n \times \mathbb{D}^{n-1}.$$

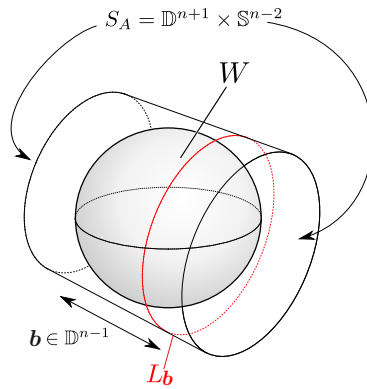


FIGURE 2.1. We find a family of Lobs by deforming the sphere to the boundary of a bi-disk. One of the two parts of the boundary, which we denote by  $S_B$ , will then be foliated by Lobs.

Let now  $(W, \omega)$  be a symplectically aspherical filling of  $(S^{2n-1}, \xi_0)$ . If it is only a weak filling, we can extend it by attaching a symplectic collar to obtain a strong symplectic filling of the sphere [MNW13, Rem. 2.11] because  $\omega|_{TS^{2n-1}}$  is exact. This filling is diffeomorphic to the initial one, and it is also still symplectically aspherical, because any 2-sphere can just be pushed by a homotopy entirely into the old symplectic filling. After rescaling the symplectic form, the extended symplectic manifold will be a strong symplectic filling of the unit sphere.

Remove now the interior  $\mathring{D}^{2n}$  of the unit ball from  $\widehat{D}^{2n}$ , and glue  $\widehat{D}^{2n} \setminus \mathring{D}^{2n}$  symplectically onto the filling  $W$ . Denote this new symplectic manifold by  $(\widehat{W}, \widehat{\omega})$ . Clearly  $\widehat{W}$  is homeomorphic to  $W$ . Using holomorphic disks, we will show as in the original paper by McDuff that  $\widehat{W}$  is contractible, so that the  $h$ -cobordism theorem [Sma62, Mil65] implies that  $\widehat{W}$  must be diffeomorphic to  $D^{2n}$  whenever  $2n - 1 > 5$ .

To study  $\widehat{W}$  using holomorphic curves, choose first an almost complex structure  $J$  on  $\widehat{W}$  that is tamed by  $\widehat{\omega}$  and that agrees on a small neighborhood of  $\partial \widehat{D}^{2n}$  in  $\widehat{W}$  with the standard complex structure  $i$  on  $C^n$ . The holomorphic curves we are interested in are attached to a family of Lobs, which we will introduce now. Let  $\Psi: S^n \times \mathbb{D}^{n-1} \rightarrow S_B \subset \partial \widehat{W}$  be the embedding into the boundary of  $\widehat{W}$  given by

$$((a_1, a_2, \dots, a_{n+1}); (b_1, \dots, b_{n-1})) \mapsto (a_1 + ib_1, \dots, a_{n-1} + ib_{n-1}, a_n + ia_{n+1}).$$

The image of  $\Psi$  lies in  $S_B \subset \widehat{\partial W}$ , and the  $J$ -complex tangencies on the corresponding part of  $\widehat{\partial W}$  are the kernel of the 1-form

$$-d^c g_B = 2x_1 dy_1 + \cdots + 2x_{n-1} dy_{n-1} + 2(x_n dy_n - y_n dx_n).$$

We obtain for the pull-back

$$\Psi^*(-d^c g_B) = 2a_1 db_1 + \cdots + 2a_{n-1} db_{n-1} + 2(a_n da_{n+1} - a_{n+1} da_n),$$

so that the restriction of  $\Psi^*(-d^c g_B)$  to each of the spheres

$$S^n \times \{(b_1, \dots, b_{n-1}) = \text{const}\}$$

gives

$$2(a_n da_{n+1} - a_{n+1} da_n).$$

This means that the projection

$$((a_1, a_2, \dots, a_{n+1}); (b_1, \dots, b_{n-1})) \mapsto \arg(a_n + ia_{n+1}) \in S^1$$

defines for each  $(b_1, \dots, b_{n-1})$  a **Lob** with the  $(n - 1)$ -ball as pages and trivial monodromy. From now on we denote the points in  $\mathbb{D}^{n-1}$  by  $\mathbf{b} = (b_1, \dots, b_{n-1})$ , and write for the **Lob**

$$L_{\mathbf{b}} = \Psi(S^n \times \{\mathbf{b}\}),$$

and  $B_{\mathbf{b}}$  for its binding.

For the technical details of the following part, we refer to Section 4.3. We will study the space

$$\widetilde{\mathcal{M}}_{\star} = \left\{ (\mathbf{b}, u, z_0) \mid \mathbf{b} \in \mathbb{D}^{n-1}, u: (\mathbb{D}^2, \partial\mathbb{D}^2) \rightarrow (\widehat{W}, L_{\mathbf{b}}), z_0 \in \mathbb{D}^2 \right\}$$

of nonconstant holomorphic maps from a disk, equipped with one marked point  $z_0$ , and with boundary sent into one of the **Lobs**  $L_{\mathbf{b}}$ . Additionally, we require that  $u$  is homotopic to a Bishop disk as an element in  $\pi_2(\widehat{W}, L_{\mathbf{b}} \cap B_{\mathbf{b}})$ , and we denote the corresponding subset by  $\widetilde{\mathcal{M}}$ . Next we divide  $\widetilde{\mathcal{M}}$  by the action of the group  $\text{Aut}(\mathbb{D}^2)$  of biholomorphic transformations on  $\mathbb{D}^2$ , where  $\varphi \in \text{Aut}(\mathbb{D}^2)$  acts on  $\widetilde{\mathcal{M}}$  via

$$\varphi \cdot (\mathbf{b}, u, z_0) = (\mathbf{b}, u \circ \varphi^{-1}, \varphi(z_0)).$$

We denote the moduli space  $\widetilde{\mathcal{M}}/\text{Aut}(\mathbb{D}^2)$  by  $\mathcal{M}$ . Note that for every class  $[\mathbf{b}, u, z]$  in  $\mathcal{M}$ , we can fix a unique representative  $(\mathbf{b}, u_0, z_0)$  by choosing a parametrization of  $u$  such that

$$(2.1) \quad u(z) \in \begin{cases} \text{the } 0 \text{ degree page of the } \mathbf{Lob}, & \text{if } z = 1, \\ \text{the } \pi/2 \text{ degree page of the } \mathbf{Lob}, & \text{if } z = i, \\ \text{the } \pi \text{ degree page of the } \mathbf{Lob}, & \text{if } z = -1. \end{cases}$$

A corollary of this is that the moduli space  $\mathcal{M}$  (before the compactification, see below) is a *trivial* disk bundle over the space of unmarked disks. This is the key fact that will allow us to “push” the topology of  $W$  into its boundary (which is the geometric analogue of the algebraic argument given in [McD91] and [OV12]).

Next, we need to understand the compactification of  $\mathcal{M}$ . Note first that typical holomorphic disks are surrounded by a neighborhood of other typical holomorphic

disks, that is, they represent smooth points of the interior of the moduli space  $\mathcal{M}$ . With “typical”, we mean smooth holomorphic disks whose interior points are mapped to the interior of  $\widehat{W}$ , and whose boundary sits on a Lob  $L_{\mathbf{b}}$  that is not a boundary Lob, i.e., for which  $\|\mathbf{b}\| < 1$ , and such that the disk does not touch the binding  $B_{\mathbf{b}}$  of the Lob.

Let us now consider the remaining cases. The boundary of  $\widehat{W}$  consists of  $S_A \subset \{g_A = 1\}$  and  $S_B \subset \{g_B = 1\}$ , which are weakly and strongly plurisubharmonic hypersurfaces respectively. A disk touching  $S_B$  with one of its interior points will automatically be constant. If the disk touches  $S_A$  instead, then it needs to be entirely contained in this hypersurface, and in particular its boundary will lie on a Lob with  $\|\mathbf{b}\| = 1$ ; below we will explain how to understand the disks in this second case explicitly.

For every Lob  $L_{\mathbf{b}}$ , there is a certain neighborhood of its binding  $B_{\mathbf{b}}$  that is only intersected by Bishop disks. Since there is exactly one disk meeting every point of this neighborhood, that is, the *evaluation map*

$$\text{ev}: \mathcal{M} \longrightarrow \widehat{W}, \quad [\mathbf{b}, u, z_0] \longmapsto u(z_0)$$

restricts close to  $B_{\mathbf{b}}$  to a diffeomorphism, it follows that the compactification  $\overline{\mathcal{M}}$  contains disks that collapse to a point in the binding. In [NR11] it was shown that adding these constant disks to  $\mathcal{M}$ , corresponds to adding points which lie on the smooth boundary of the compactification  $\overline{\mathcal{M}}$ .

Before understanding the bubbling, we will discuss disks whose boundary lies in a Lob  $L_{\mathbf{b}} \subset S_A$ .

LEMMA 2.2. — *Suppose  $u \in \overline{\mathcal{M}}$  maps  $\partial\mathbb{D}^2$  to a Lob  $L_{\mathbf{b}}$  such that  $\|\mathbf{b}\| = 1$ . Then the image of  $u$  is completely contained in  $S_A$ , and moreover, it is obtained by the intersection of a complex line parallel to the  $z_n$ -plane with  $S_A$ .*

*Proof.* — Parametrize the disk by polar coordinates  $re^{i\phi}$ . By acting on the coordinates  $z_1, \dots, z_{n-1}$  with a matrix in  $\text{SO}(n-1)$  (regarded as an element of  $\text{SU}(n-1)$  with real entries), we can assume without loss of generality that the Lob  $L_{\mathbf{b}}$  corresponds to the parameter  $\mathbf{b} = (1, 0, \dots, 0)$ , as the functions  $g_A$  and  $g_B$  are invariant under such an action. In particular it follows that the  $y_1$ -coordinate of  $u$  has its maximum on the boundary of  $u$ . The  $x_1$ -coordinate of  $u|_{\partial\mathbb{D}^2}$  is bounded, and hence there is an angle  $e^{i\phi_0}$  at which the derivative

$$\left. \frac{d}{d\phi} \right|_{\phi=\phi_0} x_1(u(e^{i\phi})) = 0$$

is zero. Complex multiplication gives  $i \cdot \partial_r = \partial_\phi$ , hence

$$dy_1(Du \cdot \partial_r) = dy_1(Du \cdot (-i \cdot \partial_\phi)) = -dy_1(i \cdot Du \cdot \partial_\phi) = -dx_1(Du \cdot \partial_\phi) = 0.$$

It follows that the outward derivative of the  $y_1$ -coordinate vanishes at the point  $e^{i\phi_0} \in \mathbb{D}^2$ , so that according to the boundary point lemma,  $y_1$  must equal the constant 1 on the whole disk, and as a consequence  $u$  lies entirely in  $S_A$ .

The  $y_2$ - to  $y_{n-1}$ -coordinates are all 0 on the boundary of the disk, and hence by the maximum principle, they need to be both maximal and minimal on all of the disk. With the Cauchy-Riemann equation we obtain that the  $x_1$ - up to  $x_{n-1}$ -coordinates of  $u$  need all to be constant on  $u$  (for more details read Section 4.2).

As explained in Section 4.3, no bubbling can occur under our assumptions, and hence  $\overline{\mathcal{M}}$  will be a compact manifold with boundary and corners (the boundary is smooth everywhere with the exception of the disks corresponding to the edges of  $\widehat{W}$ ). Moreover, the moduli space is orientable (see Appendix A) and the evaluation map

$$\text{ev}: (\overline{\mathcal{M}}, \partial\overline{\mathcal{M}}) \longrightarrow (\widehat{W}, \partial\widehat{W}), \quad [\mathbf{b}, u, z_0] \longmapsto u(z_0)$$

is a degree 1 map, that is, it maps the fundamental class  $[\overline{\mathcal{M}}] \in H_{2n}(\overline{\mathcal{M}}, \partial\overline{\mathcal{M}}; \mathbb{Z})$  onto the fundamental class  $[\widehat{W}] \in H_{2n}(\widehat{W}, \partial\widehat{W}; \mathbb{Z})$ . We are therefore in a position to apply the following topological result, which was stated as Lemma 1.11 in the introduction.

LEMMA 2.3. — *Let  $X, Y$  be smooth orientable compact manifolds with boundary and corners such that  $\partial Y$  is homeomorphic to a sphere and  $\dim X + 2 = \dim Y > 3$ . Write  $X' = X \times \mathbb{D}^2$ , and assume that*

$$f: (X', \partial X') \longrightarrow (Y, \partial Y)$$

*is a continuous map that is smooth on the interior of  $X'$ , and for which we find an open subset  $U \subset \mathring{Y}$  such that  $f|_{f^{-1}(U)}: f^{-1}(U) \rightarrow U$  is a diffeomorphism.*

*Then  $Y$  is contractible.*

*Proof.* — Note that by Whitehead’s theorem, it suffices to show that  $Y$  is weakly contractible, that is,  $\pi_j(Y) = 0$  for all  $j > 0$ . Using Hurewicz’s theorem we will show instead that  $Y$  is simply connected and satisfies  $H_j(Y; \mathbb{Z}) = 0$  for all  $j > 0$ .

(i) We will first consider the fundamental group of  $Y$ . Choose the base point  $p_0 \in U \subset \mathring{Y}$ . Let  $\gamma$  be a smooth, embedded loop representing a class in  $\pi_1(Y, p_0)$  that lies in the interior of  $Y$ . After a perturbation, we can assume that  $\gamma$  is transverse to the map  $f$ , so  $f^{-1}(\gamma)$  will be a finite collection of loops  $\Gamma_0, \dots, \Gamma_N$  in  $X'$ .

There is one loop, say  $\Gamma_0$ , that is mapped to  $\gamma$  with degree one. The reason for this is that  $\gamma$  runs through  $U$ , where  $f$  is a diffeomorphism. Then the loop  $f \circ \Gamma_0$  is homotopic to  $\gamma$ , and thus represents the same class in  $\pi_1(Y, p_0)$ .

Using the fact that  $X'$  is diffeomorphic to a trivial disk bundle, we may shift  $\Gamma_0$  into the boundary of  $X'$  (just by moving it inside the  $\mathbb{D}^2$ -factor). In particular, this shows that  $[\gamma] = [f \circ \Gamma_0]$  can be represented by a loop that lives in the boundary of  $Y$ , and is thus contractible.

(ii) Next we need to compute the homology of  $Y$ . It is easy to see that the image of

$$f_*: H_*(X'; \mathbb{Z}) \longrightarrow H_*(Y; \mathbb{Z})$$

is trivial. Indeed, all homology groups  $H_k(X'; \mathbb{Z})$  with  $k > \dim X$  are trivial, so that we only need to study  $k < \dim X < \dim \partial Y$ . Let  $A \in H_k(Y; \mathbb{Z})$  be a homology class that lies in the image of  $f_*$  so that there is a  $B \in H_k(X'; \mathbb{Z})$  with  $A = f_*B$ . Since

$X' = X \times \mathbb{D}^2$  where  $\mathbb{D}^2$  is contractible,  $B$  can be represented by a cycle in  $X \times \{p\}$  for any point  $p \in \mathbb{D}^2$ ; in particular we are free to choose  $p \in \partial\mathbb{D}^2$ , hence  $B$  is represented by a cycle in  $\partial X'$ . This implies that the class  $A$  is homologous to a cycle in the sphere  $\partial Y$ , which shows that  $A$  must be trivial.

We will now show that  $f_*: H_k(X'; Z) \rightarrow H_k(Y; Z)$  is surjective for every  $k \subset \frac{1}{2} \dim Y$ . Assume for now that  $k < \frac{1}{2} \dim Y$  and that we already have shown for every  $\ell < k$  that  $H_\ell(Y; Z) = 0$ . From (i), we see that  $k > 2$ . By Hurewicz's theorem we know that  $H_k(Y; Z)$  is isomorphic to  $\pi_k(Y, p_0)$  so that we can represent any homology class in  $H_k(Y; Z)$  by a map

$$s: S^k \longrightarrow Y.$$

We additionally assume again that the base point  $p_0$  lies in  $U$ . After a generic perturbation (see [Hir94, Th. II.2.12]),  $s$  will be an embedding that is transverse to  $f$ , and we find a closed (possibly disconnected) smooth submanifold  $f^{-1}(s) \subset X'$ . As in Step (i), there is a unique connected component  $S$  of  $f^{-1}(s)$  which passes through the base point  $p_0$ . Clearly we have a map  $S \rightarrow S^k$  of degree one that makes the following diagram commute:

$$\begin{array}{ccc} S & \xrightarrow{\quad} & S^k \\ & \searrow f & \swarrow s \\ & Y & \end{array}$$

As we wanted to show, it follows that  $f(S)$  represents the same class in  $H_k(Y; Z)$  as  $s$ , so that  $f_*: H_k(X'; Z) \rightarrow H_k(Y; Z)$  is surjective, and as a consequence  $H_k(Y; Z)$  is trivial.

Let us now study the case  $k = \frac{1}{2} \dim Y$ . Again we may represent any element in  $H_k(Y; Z)$  by a map

$$s: S^k \longrightarrow Y$$

that goes through  $p_0 \in U$ . Perturbing  $s$ , we may assume that it is an immersion with transverse self-intersections, and also that it is transverse to  $f$  (position the double-points at regular values of  $f$ ).

The map  $F := (f, s): X' \times S^k \rightarrow Y \times Y$  is transverse to the diagonal  $\Delta_Y \subset Y \times Y$ , and it follows that  $F^{-1}(\Delta_Y)$  is a closed smooth submanifold of  $X' \times S^k$ . Let  $S$  be the unique component of  $F^{-1}(\Delta_Y)$  that is mapped by  $F$  to  $(p_0, p_0)$ . By definition we have  $f \circ \Pi_1|_S = s \circ \Pi_2|_S$ , where  $\Pi_1: X' \times S^k \rightarrow X'$  and  $\Pi_2: X' \times S^k \rightarrow S^k$  are the canonical projections. Furthermore  $\Pi_2$  is a degree one map.

This ends the proof, because  $[f \circ \Pi_1|_S] = [s] \in H_k(Y; Z)$ .

(iii) Above we have seen that  $H_k(Y; Z)$  is trivial for all  $1 \subset k \subset \frac{1}{2} \dim Y$ . Using Poincaré-Lefschetz duality,  $H^{\dim Y - k}(Y, \partial Y; Z)$  is isomorphic to  $H_k(Y; Z)$  and thus it also vanishes (for  $1 \subset k \subset \frac{1}{2} \dim Y$ ). The long exact sequence of the pair then implies that  $H^{\dim Y - k}(Y; Z) = \{0\}$ , and finally the universal coefficient theorem tells us that  $H_{\dim Y - k}(Y; Z) = \{0\}$ , so that  $Y$  has trivial homology.



To conclude the proof of Theorem 2.1, simply apply the lemma and Corollary 1.12 to the moduli space and its evaluation map found above. It follows that  $\widehat{W}$  is diffeomorphic to a ball.

### 3. WEINSTEIN HANDLES AND CONTACT SURGERIES

In this section, we will give a precise description of Weinstein handle attachment, and then show how to deform the contact structure near the belt sphere in order to find a suitable family of Lobs.

Assume  $(W, \omega)$  is a weak symplectic filling of a  $(2n - 1)$ -dimensional contact manifold  $(M, \xi)$ , choose an isotropic sphere  $S_{\text{at}}^{k-1} \hookrightarrow (M, \xi)$  with trivial normal bundle  $\nu S_{\text{at}}^{k-1}$ , and suppose that  $\omega|_{TS_{\text{at}}^{k-1}}$  is exact (this is obviously always the case if  $k \neq 3$ ). After deforming the symplectic structure in a small neighborhood of the boundary using [MNW13, Rem. 2.11], we can find a Liouville vector field  $X_L$  on a neighborhood  $U \subset W$  close to  $S_{\text{at}}^{k-1}$  that points transversely out of  $W$ , and such that

$$\alpha|_{U \cap M} = (\iota_{X_L} \omega)|_{T(U \cap M)}$$

is a contact form for  $\xi$ .

Topologically, the handle attachment can be described as follows. Choose a trivialization of the normal bundle  $\nu S_{\text{at}}^{k-1}$  in  $M$ , identifying a tubular neighborhood with

$$T^*S^{k-1} \times \mathbb{R}^{2(n-k)+1} \cong S^{k-1} \times \mathbb{R}^{2n-k},$$

and let the  $k$ -handle  $H_k$  be the poly-disk

$$H_k := D^k \times D^{2n-k}.$$

The boundary of  $H_k$  can be written as the union

$$\partial H_k := S^{k-1} \times D^{2n-k} \cup D^k \times S^{2n-k-1}.$$

Using the obvious homeomorphism, we can “glue”  $H_k$  along the subset  $S^{k-1} \times D^{2n-k} \subset \partial H_k$  to the neighborhood of  $S_{\text{at}}^{k-1}$  using the trivialization chosen above. Denote the new manifold by

$$W' = W \cup_{\nu S_{\text{at}}^{k-1}} H_k.$$

Note that the gluing operation depends on the trivialization chosen for the normal bundle of  $S_{\text{at}}^{k-1}$ .

The boundary  $M' = \partial W'$  is obtained from the old contact manifold  $M$  by removing the neighborhood of  $S_{\text{at}}^{k-1}$ , and gluing in the free boundary component of the handle, that is,

$$M' = (M \setminus \nu S_{\text{at}}^{k-1}) \cup (D^k \times S^{2n-k-1}).$$

This operation changing  $M$  to  $M'$  is called a *contact surgery of index  $k$  along  $S_{\text{at}}^{k-1}$* . We will recall below how the natural symplectic structure on  $W'$  with weakly contact-type boundary  $M'$  is defined. The *belt sphere*  $S_{\text{belt}}^{2n-k-1}$  of the handle  $H_k$  is the “cosphere to the gluing sphere”,

$$S_{\text{belt}}^{2n-k-1} := \{\mathbf{0}\} \times S^{2n-k-1} \subset \partial H_k.$$

Note that contact surgery can also be defined as an operation on contact manifolds without assuming that they are symplectically fillable: one only need regard  $(M, \xi)$  as the contact boundary of a piece of its symplectization  $((-\varepsilon, 0] \times M, d(e^t \alpha))$ . Topologically, a surgery of index  $k$  is the operation of removing a small neighborhood of the isotropic sphere  $S_{\text{at}}^{k-1}$  from the contact manifold  $(M, \xi)$  and gluing into the cavity a standard patch that is diffeomorphic to  $D^k \times S^{2n-k-1}$ .

3.1. ATTACHING WEINSTEIN HANDLES. — We now define a symplectic model for a subcritical Weinstein handle of index  $k$  in dimension  $2n$ . For this, split  $C^n$  into  $C^k \times C^m \times C$  with  $n = k + m + 1$ , and write the coordinates on  $C^k$  as

$$\mathbf{z}^- = \mathbf{x}^- + i \mathbf{y}^- := (x_1^-, \dots, x_k^- + iy_k^-),$$

the ones on  $C^m$  as

$$\mathbf{z}^+ = \mathbf{x}^+ + i \mathbf{y}^+ := (x_1^+, \dots, x_m^+ + iy_m^+),$$

and the ones on  $C$  as

$$z^\circ = x^\circ + i y^\circ.$$

The coordinate  $z^\circ$  behaves like any other of the coordinates in  $\mathbf{z}^+$ , and in the usual descriptions of the handle attachment, it is not distinguished from  $\mathbf{z}^+$ . (For a critical handle attachment  $k = n$ , thus there are no  $\mathbf{z}^+$ - or  $z^\circ$ -coordinates.) The reason why we have introduced this more complicated notation is to prepare for the deformation we will perform in the next section, in which the  $z^\circ$ -coordinate will play a particular role.

As a model for the handle, take

$$H_r := D^k \times D_r^{2n-k} \subset C^n,$$

where the first disk corresponds to the  $\mathbf{y}^-$ -coordinates, the second disk to the  $(\mathbf{x}^-, \mathbf{z}^+, z^\circ)$ -coordinates, and  $r > 0$  is a constant, i.e.,  $H_r$  is the intersection of the two subsets

$$\{\|\mathbf{y}^-\|^2 \leq 1\} \cap \{\|\mathbf{x}^-\|^2 + \|\mathbf{z}^+\|^2 + |z^\circ|^2 \leq r^2\}.$$

We denote

$$\partial_- H_r = \partial D^k \times D_r^{2n-k}, \quad \partial_+ H_r = D^k \times \partial D_r^{2n-k}.$$

The core of  $\partial_- H_r$  is the  $(k - 1)$ -sphere

$$S_- := \{\|\mathbf{y}^-\|^2 = 1, \mathbf{x}^- = \mathbf{0}, \mathbf{z}^+ = \mathbf{0}, z^\circ = 0\},$$

which will be identified with the attachment sphere in a contact manifold; the core of  $\partial_+ H_r$  is the  $(n + m)$ -sphere (note that  $n + m = 2n - k - 1$ )

$$S_+ := \{\mathbf{y}^- = \mathbf{0}, \|\mathbf{x}^-\|^2 + \|\mathbf{z}^+\|^2 + |z^\circ|^2 = r^2\}.$$

Choose on  $\mathbb{C}^n$  the symplectic form

$$\omega = 2 \sum_{s=1}^k dx_s^- \wedge dy_s^- + 4 \sum_{t=1}^m dx_t^+ \wedge dy_t^+ + 4 dx^\circ \wedge dy^\circ.$$

It admits the Liouville form

$$\lambda = 2 \sum_{s=1}^k (2x_s^- dy_s^- + y_s^- dx_s^-) + 2 \sum_{t=1}^m (x_t^+ dy_t^+ - y_t^+ dx_t^+) + 2(x^\circ dy^\circ - y^\circ dx^\circ)$$

that is associated to the Liouville vector field

$$X_L = \sum_{s=1}^k \left( 2x_s^- \frac{\partial}{\partial x_s^-} - y_s^- \frac{\partial}{\partial y_s^-} \right) + \frac{1}{2} \sum_{t=1}^m \left( x_t^+ \frac{\partial}{\partial x_t^+} + y_t^+ \frac{\partial}{\partial y_t^+} \right) + \frac{1}{2} \left( x^\circ \frac{\partial}{\partial x^\circ} + y^\circ \frac{\partial}{\partial y^\circ} \right).$$

The field  $X_L$  points outward through  $\partial_+ H_r$  and inward at  $\partial_- H_r$ , so that both  $\partial_+ H_r$  and  $\partial_- H_r$  are contact type hypersurfaces with the corresponding coorientations. The core  $S_- \subset \partial_- H_r$  is an isotropic sphere with trivial conformal symplectic normal bundle.

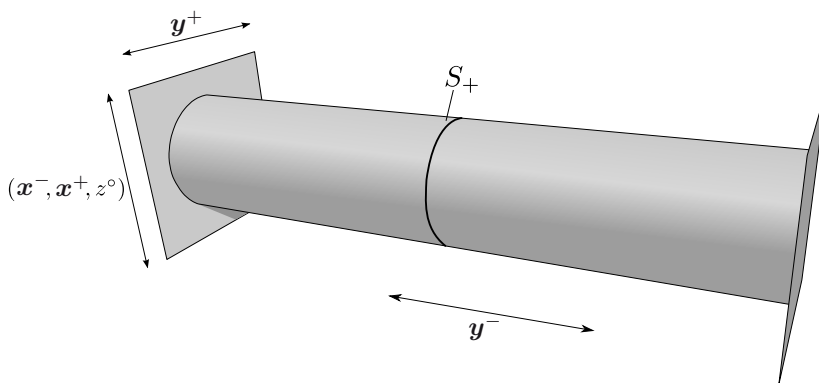


FIGURE 3.1. The handle can be glued onto a contact manifold.

Let now  $(M, \xi)$  be a given contact manifold and let  $S_{\text{at}}^{k-1} \subset M$  be a  $(k-1)$ -dimensional isotropic sphere with trivial conformal symplectic normal bundle that will serve as the *attaching sphere* of the  $k$ -handle  $H_r$ . Fixing  $r > 0$  small enough,  $\partial_- H_r$ , endowed with the contact structure induced by  $\lambda$ , is contactomorphic to a neighborhood  $\mathcal{N}(S_{\text{at}}^{k-1}) \subset (M, \xi)$  of  $S_{\text{at}}^{k-1}$ . We choose a contact form  $\alpha$  for  $\xi$  on  $M$  such that  $\alpha|_{\mathcal{N}(S_{\text{at}}^{k-1})}$  can be glued to  $\lambda|_{T(\partial_- H_r)}$  and define the Liouville manifold

$$(W_0, \lambda_0) := ((-\varepsilon, 0] \times M, e^t \alpha) \cup_{\partial_- H_r} (H_r, \lambda).$$

The positive boundary of  $W_0$  (denoted  $\partial_+ W_0$ ) has two smooth faces  $M \frown \mathcal{N}(S_{\text{at}}^{k-1})$  and  $\partial_+ H_r$ , meeting along a corner which is the image of the corner  $\partial D^k \times \partial D_r^{2n-k}$  in

$\partial H_r$ , see Fig. 3.1. Fix a small neighborhood  $\mathcal{U}$  of the corner, and choose a smooth hypersurface  $M'$  that matches  $\partial_+ W_0$  outside of  $\mathcal{U}$ , and is transverse to  $X_L$  in  $\mathcal{U}$ . Denote the induced contact structure on  $M'$  by  $\xi' = TM' \cap \ker \lambda_0$ .

Note that the constant  $r > 0$  can be made arbitrarily small, without changing the isotopy class of the contact structure on  $M'$ ; we can shrink the size of the handle continuously (including the smoothing) which allows us to apply Gray stability to obtain an isotopy with support in the model neighborhood.

The belt sphere  $S_{\text{belt}}^{2n-k-1} = S_{\text{belt}}^{n+m}$  of the  $k$ -handle is the core  $S_+$  of  $\partial_+ H_r$ .

**3.2. FAMILIES OF LOBS ON A DEFORMED SUBCRITICAL HANDLE.** — To find the desired family of Lobs, we will now modify the contact structure in a neighborhood of the belt sphere in two steps. The first deformation is borrowed from the recent article [GZ16]; it replaces a technically more complicated method that was used in an earlier version of this paper.

Consider again a “thin” handle  $H_r = \mathbb{D}^k \times \mathbb{D}_r^{2n-k} \subset \mathbb{C}^n$  with  $r \ll 1$  as used above. Suppose that the rounding of the corners has been performed for values of  $\|\mathbf{y}^-\|$  in the interval  $[1 - \varepsilon, 1]$ . The part of  $\partial_+ H_r$  outside the smoothing region lies in the level set  $\{f = r^2\}$  of the function

$$f(\mathbf{z}^-, \mathbf{z}^+, z^\circ) = \|\mathbf{x}^-\|^2 + \|\mathbf{z}^+\|^2 + |z^\circ|^2.$$

We would like to modify the Liouville field on a neighborhood of the belt sphere so that the induced contact structure coincides with the field of complex hyperplanes on the boundary. For this, add the Hamiltonian vector field  $X_H$  of a function  $H: \mathbb{C}^n \rightarrow \mathbb{R}$  to  $X_L$ , since then  $\widehat{X}_L := X_L + X_H$  will still be a Liouville field.

Let  $\rho: [0, \infty) \rightarrow [-1, 0]$  be a smooth function that is equal to  $-1$  on the interval  $[0, \sqrt{1-2\varepsilon}]$ , equal to  $0$  on the interval  $[\sqrt{1-\varepsilon}, 1]$  and that increases monotonically in between. Define the Hamiltonian function

$$H(\mathbf{z}^-, \mathbf{z}^+, z^\circ) := 2 \langle \mathbf{x}^-, \mathbf{y}^- \rangle \rho(\|\mathbf{y}^-\|^2).$$

The Hamiltonian vector field corresponding to  $H$  is

$$X_H = - \sum_{s=1}^k y_s^- \rho(\|\mathbf{y}^-\|^2) \frac{\partial}{\partial y_s^-} + \sum_{s=1}^k \left( 2 \langle \mathbf{x}^-, \mathbf{y}^- \rangle y_s^- \rho'(\|\mathbf{y}^-\|^2) + x_s^- \rho(\|\mathbf{y}^-\|^2) \right) \frac{\partial}{\partial x_s^-}.$$

The vector field  $\widehat{X}_L$  agrees outside the support of  $\rho$  with  $X_L$ , and it is everywhere transverse to  $\partial_+ H_r$  as can be seen from

$$\begin{aligned} \mathcal{L}_{\widehat{X}_L} f &= \mathcal{L}_{X_L} f + \mathcal{L}_{X_H} f \\ &= 4 \|\mathbf{x}^-\|^2 + \|\mathbf{z}^+\|^2 + |z^\circ|^2 + \mathcal{L}_{X_H} \|\mathbf{x}^-\|^2 \\ &= (4 + 2\rho) \|\mathbf{x}^-\|^2 + \|\mathbf{z}^+\|^2 + |z^\circ|^2 + 4 \langle \mathbf{x}^-, \mathbf{y}^- \rangle^2 \rho' > 0, \end{aligned}$$

because  $\rho > -1$ , and  $\rho' > 0$ .

It follows that  $\lambda$  and  $\widehat{\lambda} := \iota_{\widehat{X}_L} \omega$  induce isotopic contact structures on  $M'$ .

The contact structure on the domain  $\partial_+ H_r \cap \{\|\mathbf{y}^-\|^2 \leq 1 - 2\varepsilon\}$  is the kernel of the Liouville form

$$\widehat{\lambda} = \lambda + dH = 2 \sum_{s=1}^k x_s^- dy_s^- + 2 \sum_{t=1}^m (x_t^+ dy_t^+ - y_t^+ dx_t^+) + 2(x^\circ dy^\circ - y^\circ dx^\circ).$$

REMARK 3.1. — This first deformation shows that the surgered manifold contains a neighborhood of the belt sphere that is contactomorphic to a cylinder

$$\{f = r^2\} \cap \{\|\mathbf{y}^-\|^2 \leq 1/2\} \subset \mathbb{C}^n,$$

with  $r$  arbitrarily small and a contact structure given as kernel of  $\widehat{\lambda}$ . Note that  $\widehat{\lambda}$  on the domain under consideration is equal to the differential  $-d^c f$ , i.e., the contact structure on our domain coincides with the complex tangencies. This is the key fact that we will exploit in the second deformation below.

To continue, we consider the setting of Theorem 1.2, in which  $(M', \xi')$  was a fillable contact manifold obtained by subcritical surgery. Since this will be the main object of study from now on, it will be convenient to simplify the notation, hence we assume (unlike in the statement of Theorem 1.2) that  $(M, \xi)$  is a closed contact manifold of dimension  $2n - 1$  that has been obtained by a surgery of index  $k \in \{1, \dots, n - 1\}$  from another contact manifold, and let  $(W, \omega)$  be a weak symplectic filling of  $(M, \xi)$ . Since the theorem in dimension three already follows from the much stronger result of Eliashberg [Eli90a], we are free to assume  $n > 3$ . The belt sphere then has dimension  $2n - k - 1 > n > 3$ , hence the restriction of  $\omega$  to  $S_{\text{belt}}^{2n-k-1}$  is automatically exact. It follows (using [MNW13, Rem. 2.11]) that  $\omega$  can be deformed in a collar neighborhood of  $\partial W$  so that an outwardly transverse Liouville vector field exists near  $S_{\text{belt}}^{2n-k-1}$ , and we are therefore free to pretend in the following discussion that  $(W, \omega)$  is a *strong* filling of  $(M, \xi)$ . In particular, we may assume that the symplectic structure on a collar neighborhood close to the belt sphere looks like the symplectic structure on the boundary of the model of the handle, and we may identify both.

Let  $f: \mathbb{C}^n \rightarrow [0, \infty)$  be again the plurisubharmonic function

$$f(\mathbf{z}^-, \mathbf{z}^+, z^\circ) = \|\mathbf{x}^-\|^2 + \|\mathbf{z}^+\|^2 + |z^\circ|^2.$$

By the explanations above (see Remark 3.1), the belt sphere  $S_{\text{belt}}^{n+m}$  has a neighborhood  $\mathcal{U}_M \subset M$  that is contactomorphic to the cylinder

$$C_r := \{(\mathbf{z}^-, \mathbf{z}^+, z^\circ) \in \mathbb{C}^n \mid f(\mathbf{z}^-, \mathbf{z}^+, z^\circ) = r^2, \|\mathbf{y}^-\|^2 < 1/2\}$$

for arbitrarily small  $r \ll 1$  with contact structure  $\widehat{\xi}$  given as the kernel of the Liouville form  $\widehat{\lambda} = -d^c f|_{C_r}$ , and since the cylinder is a level set of  $f$ , this also means that  $\widehat{\xi}$  are the complex tangencies of  $C_r$ . We denote by  $\mathcal{U}_W$  a small neighborhood of  $\mathcal{U}_M$  in  $W$  that is symplectomorphic to the subset

$$\{(\mathbf{z}^-, \mathbf{z}^+, z^\circ) \in \mathbb{C}^n \mid f(\mathbf{z}^-, \mathbf{z}^+, z^\circ) \in (r^2 - \delta, r^2], \|\mathbf{y}^-\|^2 < 1/2\}$$

with symplectic form  $\omega = -dd^c f$ .

Using the embedding of  $\mathcal{U}_W$  into our model, we can extend the symplectic filling  $(W, \omega)$  by attaching the following compact symplectic subdomain of  $\mathbb{C}^n$ : replace  $f$  by  $G := \max\{g_A, g_B\}$  which is obtained as the maximum of the two functions:

$$g_A(\mathbf{z}^-, \mathbf{z}^+, z^\circ) := \|\mathbf{y}^+\|^2$$

and 
$$g_B(\mathbf{z}^-, \mathbf{z}^+, z^\circ) := \|\mathbf{x}^-\|^2 + \|\mathbf{x}^+\|^2 + \psi(\|\mathbf{y}^-\|) \cdot \|\mathbf{y}^+\|^2 + |z^\circ|^2,$$

where  $\psi$  is a cut-off function that vanishes close to 0, and increases until it reaches 1 close to  $\|\mathbf{y}^-\| = 1/\sqrt{2}$ .

Clearly  $g_A$  is a weakly plurisubharmonic function. The function  $g_B$  is strictly plurisubharmonic on a neighborhood of  $\{\mathbf{y}^+ = \mathbf{0}\}$  because the last term of

$$-dd^c g_B = 2 \sum_{s=1}^k dx_s^- \wedge dy_s^- + 2 \sum_{t=1}^m dx_t^+ \wedge dy_t^+ + 4 dx^\circ \wedge dy^\circ - dd^c(\psi(\|\mathbf{y}^-\|) \cdot \|\mathbf{y}^+\|^2)$$

simplifies along this subset to

$$\begin{aligned} -dd^c(\psi(\|\mathbf{y}^-\|) \cdot \|\mathbf{y}^+\|^2) &= -\psi(\|\mathbf{y}^-\|) dd^c \|\mathbf{y}^+\|^2 - d\|\mathbf{y}^+\|^2 \wedge d^c \psi(\|\mathbf{y}^-\|) \\ &\quad - d\psi(\|\mathbf{y}^-\|) \wedge d^c \|\mathbf{y}^+\|^2 - \|\mathbf{y}^+\|^2 dd^c \psi(\|\mathbf{y}^-\|) \\ &= 2\psi(\|\mathbf{y}^-\|) \sum_{t=1}^m dx_t^+ \wedge dy_t^+, \end{aligned}$$

which is weakly plurisubharmonic. This implies that if the chosen handle  $C_r$  is thin enough, that is if  $r > 0$  has been chosen sufficiently small, then  $g_B$  will be strictly plurisubharmonic on its neighborhood.

For large values of  $\|\mathbf{y}^-\|^2$ , the cut-off function is equal to 1 and  $g_B$  agrees with  $f$ . Since it also dominates  $g_A$ , the level set  $\{G = r^2\}$  glues smoothly to the given contact manifold, and it bounds a symplectic manifold

$$\widehat{W} = W \cup \{(\mathbf{z}^-, \mathbf{z}^+, z^\circ) \in \mathbb{C}^n \mid f(\mathbf{z}^-, \mathbf{z}^+, z^\circ) > r^2 \text{ and } G(\mathbf{z}^-, \mathbf{z}^+, z^\circ) \leq r^2\}$$

obtained from the given symplectic filling  $W$  by attaching to it the symplectic domain lying in our model between the level sets  $\{f = r^2\}$  and  $\{G = r^2\}$ , see Figures 3.2 and 3.3. Note that the boundaries of  $W$  and  $\widehat{W}$  are continuously isotopic. We also write  $W_{\text{model}}$  for the subdomain  $\mathcal{U}_W \cup (\widehat{W} \cap W)$  that lies entirely in  $\mathbb{C}^n$ .

We decompose the boundary of  $\widehat{W}$  into three domains, which we denote by  $M_{\text{reg}}, M_A$  and  $M_B$ . Here  $M_A$  and  $M_B$  are the parts of  $\partial \widehat{W}$  that lie in the level set  $\{g_A = r^2\}$  or  $\{g_B = r^2\}$  respectively, and satisfy  $\|\mathbf{y}^-\|^2 < 1/2$ ;  $M_{\text{reg}}$  is the remaining part of the boundary of  $\widehat{W}$ , i.e., the part that is disjoint from the boundary of the deformed handle.

The boundary of the handle contains a deformation of the belt sphere, which we will still write as  $S_{\text{belt}}^{n+m} = \{\mathbf{y}^- = \mathbf{0}\}$  even though it has edges. The cut-off function  $\psi$  vanishes on a neighborhood of  $S_{\text{belt}}^{n+m}$ , so that  $G$  simplifies to

$$G(\mathbf{z}^-, \mathbf{z}^+, z^\circ) := \max\{\|\mathbf{y}^+\|^2, \|\mathbf{x}^-\|^2 + \|\mathbf{x}^+\|^2 + |z^\circ|^2\}.$$

It follows that  $S_{\text{belt}}^{n+m}$  is the boundary of the poly-disk

$$\left\{ \|\mathbf{y}^+\|^2 \leq r^2, \mathbf{y}^- = \mathbf{0} \right\} \cap \left\{ \|\mathbf{x}^-\|^2 + \|\mathbf{x}^+\|^2 + |z^0|^2 \leq r^2, \mathbf{y}^- = \mathbf{0} \right\} \cong \mathbb{D}^m \times \mathbb{D}^{n+1}.$$

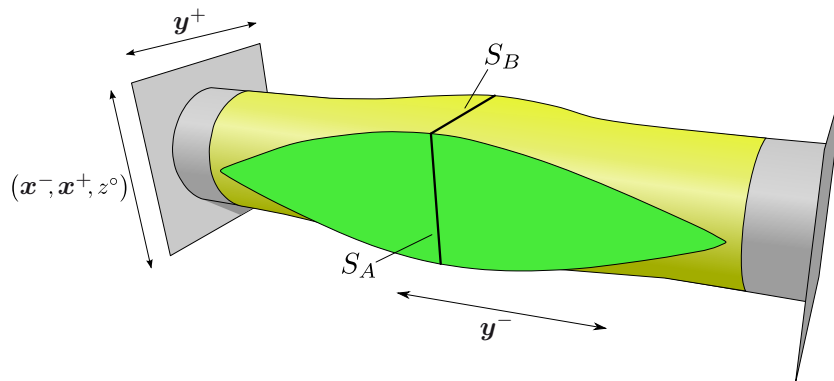


FIGURE 3.2. We need to deform the handle to find a suitable family of Lobs in the belt sphere. The new handle will have edges. The green area in the picture represents the boundary  $M_A$ , the yellow one is the boundary  $M_B$ , and the grey part is  $M_{\text{reg}}$ . The belt sphere  $S_A \cup S_B$  corresponding to the deformed handle also has edges. The part  $S_B$  is foliated by Lobs.

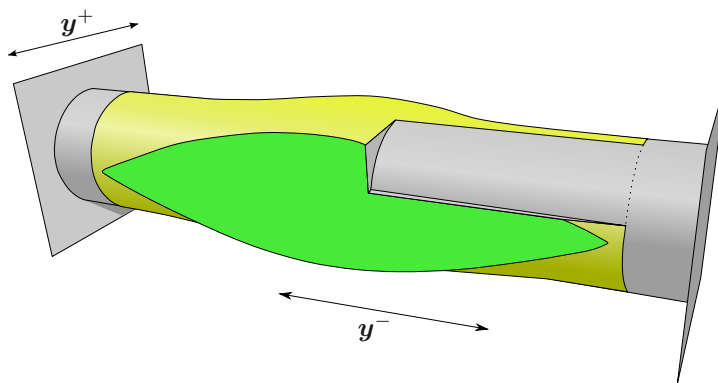


FIGURE 3.3. The deformed handle differs from the original one by the attachment of a symplectic cobordism. This cobordism can also be added to any other symplectic filling of the surgered contact manifold. The figure shows a cut through this cobordism.

We can decompose the boundary of the poly-disk  $\mathbb{D}^m \times \mathbb{D}^{n+1}$  as a union of two smooth parts

$$\partial(\mathbb{D}^m \times \mathbb{D}^{n+1}) = S^{m-1} \times \mathbb{D}^{n+1} \cup \mathbb{D}^m \times S^n.$$

We will denote the first part of the belt sphere by

$$S_A := \left\{ \mathbf{y}^- = \mathbf{0}, \|\mathbf{y}^+\|^2 = r^2, \|\mathbf{x}^-\|^2 + \|\mathbf{x}^+\|^2 + |z^\circ|^2 \subset r^2 \right\} \cong S^{m-1} \times \mathbb{D}^{n+1},$$

but for now, we will be mostly interested in the second part

$$S_B := \left\{ \mathbf{y}^- = \mathbf{0}, \|\mathbf{y}^+\|^2 \subset r^2, \|\mathbf{x}^-\|^2 + \|\mathbf{x}^+\|^2 + |z^\circ|^2 = r^2 \right\} \cong \mathbb{D}^m \times S^n.$$

It lies in the  $i$ -convex hypersurface  $M_B := \{g_B = r^2\}$ , whose complex tangencies are the kernel of the 1-form  $-(d^c g_B)|_{TM_B}$ . Close to  $\{\mathbf{y}^- = \mathbf{0}\}$ , we compute

$$-d^c g_B = 2 \sum_{s=1}^k x_s^- dy_s^- + 2 \sum_{t=1}^m x_t^+ dy_t^+ + 2(x^\circ dy^\circ - y^\circ dx^\circ),$$

which simplifies on  $S_B \subset \{\mathbf{y}^- = \mathbf{0}\}$  further to

$$-d^c g_B = 2 \sum_{t=1}^m x_t^+ dy_t^+ + 2(x^\circ dy^\circ - y^\circ dx^\circ).$$

The submanifold  $S_B \cong \mathbb{D}^m \times S^n$  can be foliated by the  $n$ -spheres with constant  $\mathbf{y}^+$ -value. For every fixed value of  $\mathbf{y}^+ = \mathbf{b}^+ \in \mathbb{D}_r^m$  we write the corresponding leaf as

$$L_{\mathbf{b}^+} = \left\{ (\mathbf{y}^-, \mathbf{y}^+) = (\mathbf{0}, \mathbf{b}^+), \|\mathbf{x}^-\|^2 + \|\mathbf{x}^+\|^2 + |z^\circ|^2 = r^2 \right\}.$$

The restriction of  $-d^c g_B$  to each of these spheres is

$$2(x^\circ dy^\circ - y^\circ dx^\circ),$$

so that  $L_{\mathbf{b}^+}$  is actually a spherical Lob in the (strictly)  $i$ -convex level set  $M_B$ .

The binding  $B_{\mathbf{b}^+}$  of the Lob  $L_{\mathbf{b}^+}$  is given by the set of points where  $z^\circ$  vanishes, i.e.,  $B_{\mathbf{b}^+} \cong S^{n-2}$ ; the pages of the open book are the fibers of the map

$$\vartheta: L_{\mathbf{b}^+} \cap B_{\mathbf{b}^+} \longrightarrow S^1, (z^-, z^+, z^\circ) \longmapsto \frac{z^\circ}{|z^\circ|}.$$

In the following sections we will study holomorphic disks that each have boundary on one of the Lobs  $L_{\mathbf{b}^+}$ .

#### 4. THE SPACE OF HOLOMORPHIC DISKS ATTACHED TO THE BELT SPHERE

We will now construct the moduli space of pseudoholomorphic disks needed for the proof of Theorem 1.2 and show that it is a smooth manifold with boundary. We continue with the setup and notation used in Section 3.2 above.

**ASSUMPTIONS 4.1.** — Choose an almost complex structure  $J$  on  $\widehat{W}$  with the following properties:

- $J$  is tamed by  $\omega$ ;
- $J$  agrees on  $W_{\text{model}}$  with the standard complex structure  $i$ ;
- the unmodified domain  $M_{\text{reg}}$  in  $\partial W$  is  $J$ -convex, and its  $J$ -complex tangencies agree with  $\xi$ .



4.1. THE TOP STRATUM OF THE MODULI SPACE. — We define  $\widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$  as the moduli space of “parametrized” curves  $(\mathbf{b}^+, u, z_0)$ , where:

- (i)  $\mathbf{b}^+ \in \mathbb{D}_r^m$  is a point in the  $m$ -disk parametrizing the Lobs  $L_{\mathbf{b}^+} \subset S_B$  described in the previous section;
- (ii)  $u: (\mathbb{D}^2, \partial\mathbb{D}^2) \rightarrow (\widehat{W}, L_{\mathbf{b}^+} \cap B_{\mathbf{b}^+})$  is a  $J$ -holomorphic map which is trivial in  $\pi_2(\widehat{W}, L_{\mathbf{b}^+})$ ; and
- (iii)  $z_0$  is a marked point in the closed unit disk  $\mathbb{D}^2$ .

Additionally we require that

$$(\vartheta \circ u)|_{\partial\mathbb{D}^2}: S^1 \longrightarrow S^1$$

is a degree 1 map, that is, the boundary of each disk makes one turn around the binding of the open book.

Since  $L_{\mathbf{b}^+}$  lies in the strictly convex hypersurface  $M_B$ , the map  $(\vartheta \circ u)|_{\partial\mathbb{D}^2}$  is a diffeomorphism, i.e., the disk intersects every page of the Lob  $L_{\mathbf{b}^+}$  precisely once; see [Nie13, Cor. II.1.11]. It is very easy to deduce from this that the disks in  $\widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$  are somewhere injective near the boundary. However, since we are not really free to choose the almost complex structure near the boundary, we need more to achieve transversality. We say that a  $J$ -holomorphic disk  $u$  is *simple* if it is somewhere injective in an open dense set of  $\mathbb{D}^2$ . For closed holomorphic curves, simple means not multiply covered, but for disks the situation is more complicated; see [Laz00]. However our situation is not the most general one, and it is possible to adapt the arguments of [MS12, Prop. 2.5.1] to prove that all disks in  $\widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$  are simple.

LEMMA 4.2. — *Every disk  $u \in \widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$  is simple.*

*Proof.* — We know  $u: \mathbb{D}^2 \rightarrow W$  is embedded near the boundary. Let  $X$  denote the set of points  $z \in \mathbb{D}^2$  such that either of the following is true:

- (i)  $D_z u = 0$ , or
- (ii) There exists a different point  $z' \in \mathbb{D}^2$  such that  $u$  restricted to disjoint neighborhoods of  $z$  and  $z'$  has an isolated intersection  $u(z) = u(z')$ .

(Recall that either  $u(z) = u(z')$  is an isolated intersection, or there exists neighbourhoods of  $z$  and  $z'$  with the same image.)

Standard local results plus the fact that  $u$  is embedded at  $\partial\mathbb{D}^2$  tell us that  $X$  is a finite set of interior points. The image  $u(\mathbb{D}^2 \cap X)$  is then a smoothly embedded  $J$ -holomorphic submanifold of  $W$ ; in particular, it is a Riemann surface  $\dot{\Sigma}$  with connected boundary and finitely many punctures. The inclusion of  $\dot{\Sigma}$  into  $W$  is then a  $J$ -holomorphic embedding, and it extends over the punctures to a  $J$ -holomorphic map  $v: \Sigma \rightarrow W$ , which is not necessarily an embedding but has only finitely many critical points and self-intersections.

At this point we don't know the topology of  $\Sigma$ , except that it has connected boundary. But the original map  $u$ , restricted to  $\mathbb{D}^2 \cap X$ , defines a holomorphic map to  $\dot{\Sigma}$ , which then extends by removal of singularities to a holomorphic map  $\varphi: \mathbb{D}^2 \rightarrow \Sigma$

such that  $u = v \circ \varphi$ . Given the properties of  $u$  at the boundary,  $\varphi$  must restrict to a diffeomorphism  $\partial D^2 \rightarrow \partial \Sigma$ , and it maps interior to interior. So it has degree one, and is therefore biholomorphic.

Lemma 4.2 will allow us to use the following transversality result.

**PROPOSITION 4.3.** — *Let  $(W, J)$  be an almost complex manifold, and let  $D_\varepsilon^m \times L \subset W$  be a submanifold for which every slice  $L_{\mathbf{x}} := \{\mathbf{x}\} \times L$  is a totally real submanifold. For generic choices of  $J$  satisfying Assumptions 4.1, the following holds. Suppose  $u_0: (D^2, \partial D^2) \rightarrow (W, L_0)$  is any  $J$ -holomorphic map such that*

- *the interior points of  $u_0$  do not touch the boundary of  $W$ ;*
- *the boundary of  $u_0$  lies in the interior of  $L_0$ ;*
- *the disk  $u_0$  is simple.*

Let  $\widetilde{\mathcal{M}}$  be the space of all  $J$ -holomorphic maps

$$u: (D^2, \partial D^2) \longrightarrow (W, L_{\mathbf{x}})$$

for all  $\mathbf{x} \in D_\varepsilon^m$ .

Then the space of solutions in  $\widetilde{\mathcal{M}}$  close to  $u_0$  forms a smooth ball that has  $u_0$  as its center and whose dimension is

$$\dim \widetilde{\mathcal{M}} = \frac{1}{2} \dim W + \mu(u_0^*TW, u_0^*TL_0) + m,$$

where  $\mu(u_0^*TW, u_0^*TL_0)$  denotes the Maslov index of the disk  $u_0$ .

*Proof.* — The result is standard if  $m = 0$ , in which case  $\frac{1}{2} \dim W + \mu(u_0^*TW, u_0^*TL_0)$  is the Fredholm index of the linearized Cauchy-Riemann operator on a suitable Banach space of sections of  $u_0^*TW$  with totally real boundary condition; see [MS12, §3.2]. For  $m > 0$ , the linearized problem is the same as that of the  $m = 0$  case, but with an extra  $m$ -dimensional space of smooth sections added to the domain in order to allow for the moving boundary condition, cf. [Wen05, §4.5]. Thus the Fredholm index becomes larger by  $m$ . Given the corresponding enlargement of the nonlinear configuration space, the proof of transversality for generic  $J$  works as in the standard case by defining a suitable universal moduli space and applying the Sard-Smale theorem, see e.g. [MS12, Chap. 3].

Recall from Section 3.2 that the family of Lobs is parametrized by a disk  $D_r^m$  of some fixed radius  $r \ll 1$ . We define

$$\widetilde{\mathcal{M}}_{\text{int}}(\widehat{W}, S_B; J) = \{(\mathbf{b}^+, u, z_0) \in \widetilde{\mathcal{M}}(\widehat{W}, S_B; J) \mid \|\mathbf{b}^+\| < r\}.$$

Since  $g_B$  is plurisubharmonic and  $g_A$  is weakly plurisubharmonic, this subspace consists of  $J$ -holomorphic disks that map the interior of  $D^2$  to the interior of  $\widehat{W}$ ; see also Proposition 4.8.

**COROLLARY 4.4.** — *The subspace  $\widetilde{\mathcal{M}}_{\text{int}}(\widehat{W}, S_B; J)$  of the parametrized moduli space is a smooth manifold with boundary, and its dimension is  $2n - k + 3$ , where  $k$  is the index of the surgery. Its boundary consists of triples  $(\mathbf{b}^+, u, z_0)$  with  $z_0 \in \partial D^2$ .*

Note that triples  $(\mathbf{b}^+, u, z_0)$  with  $\|\mathbf{b}^+\| = r$  do not belong to  $\widetilde{\mathcal{M}}_{\text{int}}(\widehat{W}, S_B; J)$ , and therefore are not points of its boundary.

*Proof.* — By definition the elements of  $\widetilde{\mathcal{M}}_{\text{int}}(\widehat{W}, S_B; J)$  satisfy the hypotheses of Proposition 4.3, and therefore every  $J$ -holomorphic disk  $u \in \widetilde{\mathcal{M}}_{\text{int}}(\widehat{W}, S_B; J)$  has an open neighbourhood which is diffeomorphic to a ball of dimension  $\frac{1}{2} \dim \widehat{W} + \mu(u^*T\widehat{W}, u^*TL_{\mathbf{b}^+}) + m + 2$  — the presence of the marked point adds 2 to the index. A simple computation shows that  $\mu(u^*T\widehat{W}, u^*TL_{\mathbf{b}^+}) = 2$  for all  $(\mathbf{b}^+, u, z_0)$ , so that  $\widetilde{\mathcal{M}}_{\text{int}}(\widehat{W}, S_B; J)$  is a smooth manifold with boundary of dimension

$$\dim \widetilde{\mathcal{M}}_{\text{int}}(\widehat{W}, S_B; J) = n + m + 4.$$

(The boundary points are those with the mark point in  $\partial D^2$ .) Since  $m = n - k - 1$ , we obtain the desired formula for the dimension.

In the next subsection we will analyze what happens when  $\|\mathbf{b}^+\| = r$ , and we will also show that  $\widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$  is non-empty. To consider geometric disks instead of parametrized ones, we divide  $\widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$  by the group of biholomorphic reparametrizations of  $D^2 \subset \mathbb{C}$ . We define the moduli space of “unparametrized” curves:

$$\mathcal{M}(\widehat{W}, S_B; J) = \widetilde{\mathcal{M}}(\widehat{W}, S_B; J) / \sim,$$

where  $(\mathbf{b}^+, u, z_0) \sim (\widetilde{\mathbf{b}}^+, \widetilde{u}, \widetilde{z}_0)$  if and only if  $\mathbf{b}^+ = \widetilde{\mathbf{b}}^+$  and there exists a transformation  $\varphi \in \text{Aut}(D^2)$  with  $u = \widetilde{u} \circ \varphi^{-1}$  and  $z_0 = \varphi(\widetilde{z}_0)$ . The action of the reparametrization group  $\text{Aut}(D^2)$  preserves  $\widetilde{\mathcal{M}}_{\text{int}}(\widehat{W}, S_B; J)$ ; we denote its quotient by  $\mathcal{M}_{\text{int}}(\widehat{W}, S_B; J)$ .

**PROPOSITION 4.5.** — *The subspace  $\mathcal{M}_{\text{int}}(\widehat{W}, S_B; J)$  of the moduli space is a smooth manifold with boundary of dimension  $2n - k$ .*

*Proof.* — The map  $\vartheta: S_B \cap \{z^\circ = 0\} \rightarrow S^1$  is globally defined for all Lobs in the family. Therefore we can define the subset

$$\widetilde{\mathcal{M}}_0(\widehat{W}, S_B; J) \subset \widetilde{\mathcal{M}}_{\text{int}}(\widehat{W}, S_B; J)$$

consisting of triples  $(\mathbf{b}^+, u, z_0)$  such that

$$(4.1) \quad \vartheta(u(z)) = \begin{cases} 1 & \text{if } z = 1, \\ i & \text{if } z = i, \\ -1 & \text{if } z = -1. \end{cases}$$

We know that  $\widetilde{\mathcal{M}}_0(\widehat{W}, S_B; J)$  is a submanifold of  $\widetilde{\mathcal{M}}_{\text{int}}(\widehat{W}, S_B; J)$  because  $\vartheta \circ u|_{\partial D^2}$  is a diffeomorphism and the biholomorphism group of the disk is triply transitive on  $\partial D^2$ . Then the subset  $\widetilde{\mathcal{M}}_0(\widehat{W}, S_B; J)$  provides a global slice for the action of  $\text{Aut}(D^2)$  on  $\widetilde{\mathcal{M}}_{\text{int}}(\widehat{W}, S_B; J)$ .

4.2. THE BISHOP DISKS. — In this section, we want to study a certain class of disks in  $\mathcal{M}(\widehat{W}, S_B; J)$  that lie entirely in the model neighborhood  $W_{\text{model}}$  and that can be described explicitly.

A *Bishop disk* is a disk that we obtain by intersecting a  $z^\circ$ -plane in  $\mathbb{C}^n$  with constant  $(z^-, z^+)$ -coordinates with the model neighborhood  $W_{\text{model}}$ . A possible way to parametrize it is as a map

$$u: (\mathbb{D}^2, \partial\mathbb{D}^2) \longrightarrow (W, L_{\mathbf{b}^+}),$$

with constant coordinates  $(\mathbf{y}^-, \mathbf{y}^+) = (\mathbf{0}, \mathbf{b}^+)$ , constant  $\mathbf{x}^-$  and  $\mathbf{x}^+$ -coordinates, so we write

$$u(z) = (\mathbf{x}^-; \mathbf{x}^+ + i\mathbf{b}^+; Cz),$$

where  $C = \sqrt{r^2 - \|\mathbf{x}^-\|^2 - \|\mathbf{x}^+\|^2}$ . The Bishop disks are the buds from which the moduli space will grow, and it is therefore important to establish that they are Fredholm regular, meaning that their linearized Cauchy-Riemann operators are surjective. This is ensured by the following “automatic” transversality lemma (see [Nie13, §III.1.3]).

LEMMA 4.6. — *Let  $u: \mathbb{D}^2 \rightarrow \widehat{W}$  be a Bishop disk with image in  $W_{\text{model}}$  and boundary mapped to the Lob  $L_{\mathbf{b}^+}$ . Then its linearized Cauchy-Riemann operator  $D_{(\mathbf{b}^+, u)}$ , defined on suitable Banach space completions with totally real boundary condition determined by the Lob  $L_{\mathbf{b}^+}$ , is surjective.*

COROLLARY 4.7. — *The triples  $(\mathbf{b}^+, u, z_0)$  where  $u$  is a Bishop disk with image in  $W_{\text{model}}$  are regular points of the moduli space  $\mathcal{M}(\widehat{W}, S_B; J)$ .*

*Proof.* — The relevant linearized operator is the same as  $D_{(\mathbf{b}^+, u)}$  in Lemma 4.6, except that the moving boundary condition satisfied by  $J$ -holomorphic maps in  $\mathcal{M}(\widehat{W}, S_B; J)$  means that this domain must be enlarged by some finite-dimensional space of smooth sections, allowing the boundary to move to different Lobs in the family (see the appendix for more details). The target of the operator remains the same, so surjectivity of  $D_{(\mathbf{b}^+, u)}$  in Lemma 4.6 immediately implies surjectivity on the enlarged domain.

The rest of this subsection will be concerned with the proof that the Bishop disks are the only holomorphic disks in  $W_{\text{model}}$ .

PROPOSITION 4.8. — *If a holomorphic disk*

$$u: (\mathbb{D}^2, \partial\mathbb{D}^2) \longrightarrow (\widehat{W}, L_{\mathbf{b}^+})$$

*touches the boundary of  $\widehat{W}$  at an interior point of  $\mathbb{D}^2$ , then either it is constant or it is a multiple cover of a Bishop disk that is completely contained in  $S_A \subset M_A \cap S_{\text{belt}}^{n+m}$ .*

*Proof.* — Let  $z_0 \in \mathring{\mathbb{D}}^2$  be a point in the interior of the disk at which  $u$  touches  $M_A$ ,  $M_B$ , or  $M_{\text{reg}}$ . We will obtain the desired statement by using the maximum principle; we only need to be a bit more careful compared with the standard situation,

because the boundary of  $\widehat{W}$  is defined piecewise as a union of level sets of different plurisubharmonic functions.

Assume first that  $u(z_0)$  touches  $M_B$ . The function  $g_B$  is not defined on the whole symplectic filling, but we may nonetheless assume that  $g_B$  exists on a small neighborhood of  $u(z_0)$ , hence we find an open subset  $U \subset \mathring{D}^2$  containing  $z_0$  such that

$$(g_B \circ u)|_U : U \longrightarrow \mathbb{R}$$

is a plurisubharmonic function having a maximum at  $z_0$ . It follows from the maximum principle that  $g_B \circ u|_U$  is constant, and due to strong convexity it even follows that the holomorphic map  $u|_U$  itself must be constant. This implies that the open set  $U$  chosen above can in fact be extended to the whole disk, and  $u$  will be a constant disk.

Note that this argument also remains valid if  $u(z_0)$  lies in the edge where  $M_A$  and  $M_B$  meet. The disk lies in the model locally in the domain with  $g_B \subset r^2$ , and thus  $g_B \circ u|_U$  still has a local maximum at  $z_0 \in \mathring{D}^2$ , as used previously. Similarly, the argument can be used verbatim for disks that touch  $M_{\text{reg}}$ , and this implies in fact that there are no disks at all touching  $M_{\text{reg}}$  at interior points, because a constant disk must lie in  $L_{\mathbf{b}^+} \subset S_{\text{belt}}^{n+m}$ , which is disjoint from  $M_{\text{reg}}$ .

Let us now assume that the disk  $u$  touches the hypersurface  $M_A$  at  $z_0$ . Again, we find an open subset  $U \subset \mathring{D}^2$  containing  $z_0$  for which

$$(g_A \circ u)|_U : U \longrightarrow \mathbb{R}$$

is defined and has a maximum at  $z_0$ . By weak plurisubharmonicity, this function must be constant.

Now it is easy to see that we can choose  $U$  to be the whole disk  $D^2$ , because by continuity, the image of every point  $z \in \overline{U}$  lies in  $\partial\widehat{W}$ . If  $z$  is an interior point of the disk, and if  $u(z)$  is an interior point of  $M_A$ , i.e., it does not lie in  $M_A \cap M_B$ , then we can extend  $U$  to a larger open domain that contains  $z$  in its interior. If  $z$  is an interior point but  $u(z)$  *does* lie in  $M_A \cap M_B$ , then we know by the first part of the proof that  $u$  must be a constant map. In both cases the whole disk lies in  $M_A$ .

It remains to see that a nonconstant holomorphic disk lying in  $M_A$  must be a Bishop disk (or a multiple cover). We know that all coordinate functions are harmonic, and hence each of them must attain both its maximum and its minimum at a point on the boundary of the disk. The boundary of  $u$  lies in  $L_{\mathbf{b}^+} \subset \{\mathbf{y}^- = \mathbf{0}\}$ , and hence it follows that all of the  $\mathbf{y}^-$ -coordinates vanish on the disk. From the Cauchy-Riemann equation, we then see that the  $\mathbf{x}^-$ -coordinates of the disk will be constant.

Similarly, the  $\mathbf{y}^+$ -coordinates of the disk must all be equal to  $\mathbf{b}^+$ , because  $L_{\mathbf{b}^+} \subset \{\mathbf{y}^+ = \mathbf{b}^+\}$ , and again by the Cauchy-Riemann equation also the  $\mathbf{x}^+$ -coordinates will be constant.

The only nonconstant coordinate functions of the disk are the  $z^\circ$ -coordinate, and they span a round disk.

Recall that

$$B_{\mathbf{b}^+} = L_{\mathbf{b}^+} \cap \{z^\circ = 0\}$$

is the binding of the Lob  $L_{\mathbf{b}^+}$ .

**PROPOSITION 4.9.** — *There exists an open subset  $V \subset W_{\text{model}}$ , containing  $B_{\mathbf{b}^+}$  for every  $\mathbf{b}^+ \in \mathbb{D}_r^m$ , such that every holomorphic disk*

$$u: (\mathbb{D}^2, \partial\mathbb{D}^2) \longrightarrow (\widehat{W}, L_{\mathbf{b}^+})$$

*in  $\widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$  intersecting  $V$  must be a Bishop disk up to reparametrization.*

*Proof.* — Note that

$$h(\mathbf{z}^-, \mathbf{z}^+, z^\circ) = \|\mathbf{x}^-\|^2 - \frac{1}{2} \|\mathbf{y}^-\|^2 + \|\mathbf{x}^+\|^2$$

is a weakly plurisubharmonic function on  $W_{\text{model}}$ . Its value on the binding  $B_{\mathbf{b}^+}$  is  $r^2$ , and it decreases along the  $\text{Lob}$ . If we choose a sufficiently small  $\varepsilon > 0$ , we can make sure that  $V := h^{-1}((r^2 - \varepsilon, +\infty)) \cap W_{\text{model}}$  is an open neighborhood of  $B_{\mathbf{b}^+}$  with  $\overline{V} \subset W_{\text{model}}$ . It follows in fact from  $g_B \triangleleft r^2$  and  $h > r^2 - \varepsilon$  that

$$g_B(\mathbf{z}^-, \mathbf{z}^+, z^\circ) - h(\mathbf{z}^-, \mathbf{z}^+, z^\circ) = \frac{1}{2} \|\mathbf{y}^-\|^2 + \psi(\|\mathbf{y}^-\|) \cdot \|\mathbf{y}^+\|^2 + |z^\circ|^2 < \varepsilon,$$

so that both the  $\mathbf{y}^-$  and the  $z^\circ$ -coordinates are small in  $V$ , and in particular we can assume that  $\psi = 0$  on  $V$ . On the other hand,

$$\|\mathbf{x}^-\|^2 + \|\mathbf{x}^+\|^2 > r^2 - \varepsilon + \frac{1}{2} \|\mathbf{y}^-\|^2 > r^2 - \varepsilon$$

implies that every point in  $V$  lies in an arbitrarily small neighborhood of  $S_B$ .

Let now  $u: (\mathbb{D}^2, \partial\mathbb{D}^2) \rightarrow (\widehat{W}, L_{\mathbf{b}^+})$  be a holomorphic disk whose image intersects  $V$ . Assume that  $h \circ u$  is not constant: then we can choose by Sard's theorem a slightly smaller number  $\varepsilon' < \varepsilon$  for which  $r^2 - \varepsilon'$  will be a regular value of  $h \circ u$ , so that the subdomain

$$G := \{z \in \mathbb{D}^2 \mid (h \circ u)(z) > r^2 - \varepsilon'\}$$

is compact and has piecewise smooth boundary, which we denote by

$$\partial G = \partial_+ G \cup \partial_- G,$$

where  $\partial_+ G = G \cap \partial\mathbb{D}^2$  lies in the boundary of the unit disk, and  $\partial_- G$  lies in the interior of the unit disk. Denote the restriction

$$u|_G: G \longrightarrow W_{\text{model}}$$

by  $u_G$ .

By the maximum principle, it follows that the maximum of  $h \circ u_G$  on each component of  $G$  must lie on the boundary of that component. Clearly then the boundary of every component of  $G$  must intersect  $\partial_+ G$ , because otherwise  $h \circ u_G$  would have an interior maximum on that component, so it would be equal to  $r^2 - \varepsilon'$ , but this contradicts the assumption that  $r^2 - \varepsilon'$  is a regular value.

It follows then that every component of  $G$  must intersect  $\partial\mathbb{D}^2$ , and since  $h \circ u_G$  is minimal along  $\partial_- G$ , the maximum of  $h \circ u_G$  must lie at a point  $z_0 \in \partial_+ G \subset \partial\mathbb{D}^2$ . By the boundary point lemma, a version of the maximum principle at the boundary (see for example [Nie13, Th. II.1.3]), the derivative of  $h \circ u_G$  at  $z_0$  in the outward radial

direction must be strictly positive. We choose polar coordinates  $(r, \varphi)$  on  $\mathbb{D}^2$ . Using the fact that  $u$  is  $J$ -holomorphic, we can write

$$\begin{aligned}\partial_r(h \circ u) &= dh(Du \cdot \partial_r) = dh(Du \cdot (-i \cdot \partial_\varphi)) \\ &= -dh(i \cdot Du \cdot \partial_\varphi) = -d^c h(Du \cdot \partial_\varphi),\end{aligned}$$

but note that

$$-d^c h = \sum_{s=1}^k (2x_s^- dy_s^- + y_r^- dx_r^-) + 2 \sum_{t=1}^m x_t^+ dy_t^+.$$

We obtain  $-d^c h \cdot Du \cdot \partial_\varphi = 0$  along the whole boundary of the disk, because the boundary of  $u$  lies in the  $\mathbf{Lob} L_{\mathbf{b}^+}$ , which is a subset of  $\{\mathbf{y}^- = \mathbf{0}, \mathbf{y}^+ = \mathbf{b}^+\}$ . It follows that  $\partial_r(h \circ u) = -d^c h(Du \cdot \partial_\varphi)$  vanishes at  $z_0$ , and by the boundary point lemma, the disk must be contained in one of the level sets of  $h$ , so in particular it lies in  $V \subset W_{\text{model}}$ .

The rest of the statement follows from standard arguments. All of the coordinate functions on  $W_{\text{model}}$  are harmonic, hence they must attain their maxima and minima on the boundary of the disk. Since  $\mathbf{y}^- = \mathbf{0}$  along  $\partial\mathbb{D}^2$ , the  $\mathbf{y}^-$ -coordinates of  $u$  are zero on the whole disk, and using the Cauchy-Riemann equation, we see that the  $\mathbf{x}^-$ -coordinates must be constant on the disk. Similar arguments work for  $\mathbf{y}^+$  and  $\mathbf{x}^+$ , and we finally conclude that  $u$  must be a Bishop disk.

**PROPOSITION 4.10.** — *Let  $L_{\mathbf{b}^+}$  be a  $\mathbf{Lob}$  that lies in the hypersurface  $M_A$ , i.e.,  $\mathbf{b}^+ \in \mathbb{D}_r^m$  has been chosen such that  $\|\mathbf{b}^+\| = r$ . Then up to parametrization, every holomorphic disk  $u$  in  $\mathcal{M}(\widehat{W}, S_B; J)$  whose boundary lies in  $L_{\mathbf{b}^+}$  is a Bishop disk.*

*Proof.* — Note that  $L_{\mathbf{b}^+}$  lies in the level set of the weakly plurisubharmonic function  $g_A: \widehat{W} \rightarrow \mathbb{R}$ . It suffices to prove that the image of  $u$  has to lie entirely in  $M_A \subset W_{\text{model}}$ , as this already implies the desired statement by Proposition 4.8.

Since the whole boundary  $u(\partial\mathbb{D}^2)$  lies in  $M_A$ , we can find a closed annulus  $G \subset \mathbb{D}^2$  having  $\partial\mathbb{D}^2$  as one of its boundary components such that

$$(g_A \circ u)|_G: G \longrightarrow \mathbb{R}$$

is defined and everywhere weakly plurisubharmonic, and it takes its maximum along  $\partial\mathbb{D}^2 \subset G$ .

Assume first that the disk  $u$  is tangent to  $M_A$  at one of its boundary points. We can apply the boundary point lemma around this point (see again [Nie13, Th. II.1.3]) to deduce that  $(g_A \circ u)|_G$  has to be constant on all of  $G$ . In particular this implies that  $u(G)$  lies in  $M_A$ , and  $u$  touches  $M_A$  also with one of its interior points. Proposition 4.8 then implies that  $u$  is either constant or one of the Bishop disks.

Conversely suppose that  $u$  is everywhere transverse to  $M_A$ , meaning that the function  $\partial_r(g_A \circ u)(z)$  is *strictly* positive for every  $z \in \partial\mathbb{D}^2$ . The restriction  $u|_G$  is a  $J$ -holomorphic map whose image lies in  $W_{\text{model}}$ ; moreover,  $g_A \circ u \equiv r^2$  on  $\partial\mathbb{D}^2$  and  $g_A \circ u < r^2$  on the inner boundary of  $G$ . Introduce on  $G$  the polar coordinates  $z = \rho e^{i\varphi}$ .

Note that along  $\partial D^2$ , all of the  $\mathbf{y}^+$ -coordinates are constant in the  $\varphi$ -direction, because the boundary of the disk lies in the Lob  $L_{\mathbf{b}^+}$ . Multiplying the complex coordinates  $\mathbf{z}^+$  by a suitable  $SO(m)$ -matrix (the standard complex structure  $i$  and the functions  $g_A, g_B$  are invariant under such a multiplication), we may assume that  $\mathbf{b}^+ = (r, 0, \dots, 0)$ . It follows that the  $y_1^+$ -coordinate of  $u|_G$  has its maximum on  $\partial D^2$ . Note now that the  $x_1^+$ -coordinate of  $u|_{S^1}$  is bounded, and hence it necessarily must take a maximum at some point  $e^{i\varphi_0} \in S^1 = \partial D^2$ , so that

$$\frac{d}{d\varphi} \Big|_{\varphi=\varphi_0} x_1^+(u(e^{i\varphi})) = 0.$$

Again, we can use complex multiplication to see  $i \cdot \partial_\rho = \partial_\varphi$ , hence

$$dy_1^+(Du \cdot \partial_\rho) = dy_1^+(Du \cdot (-i \cdot \partial_\varphi)) = -dy_1^+(i \cdot Du \cdot \partial_\varphi) = -dx_1^+(Du \cdot \partial_\varphi),$$

and in particular the radial derivative of  $y_1^+$  vanishes at  $e^{i\varphi_0}$ , so that by the boundary point lemma,  $y_1^+$  must be constant on all of  $G$ .

Using the fact that  $r^2 = |y_1^+|^2 \circlearrowleft g_A(\mathbf{z}^-, \mathbf{z}^+, z^\circ) \circlearrowright r^2$  everywhere on  $G$ , we deduce that all of  $u(G)$  lies in  $M_A$ . In particular,  $u$  touches  $M_A$  at an interior point, which allows us to conclude the proof by applying Proposition 4.8.

We end this subsection with a description of the global topology of the moduli spaces  $\widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$  and  $\mathcal{M}(\widehat{W}, S_B; J)$ .

**PROPOSITION 4.11.** — *The parametrized moduli space  $\widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$  is a smooth  $(2n - k + 3)$ -dimensional manifold with boundary and corners. Its boundary has two smooth strata, one corresponding to elements  $(\mathbf{b}^+, u, z_0)$  with  $\|\mathbf{b}^+\| = r$ , and the other corresponding to elements  $(\mathbf{b}^+, u, z_0)$  with  $|z_0| = 1$ .*

*The moduli space  $\mathcal{M}(\widehat{W}, S_B; J)$  is a smooth  $(2n - k)$ -dimensional manifold with boundary and corners, which decomposes as a product*

$$\mathcal{M}(\widehat{W}, S_B; J) = \Sigma \times D^2,$$

where  $\Sigma$  is a (non-compact) manifold with boundary.

*Proof.* — Let  $(\mathbf{b}^+, u, z_0)$  be an element of  $\mathcal{M}_{\text{int}}(\widehat{W}, S_B; J)$ . Since  $\widetilde{\mathcal{M}}_{\text{int}}(\widehat{W}, S_B; J)$  is open in  $\widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$ , it follows from Proposition 4.8 that the image of  $u$  does not touch  $\partial \widehat{W}$  with any interior point if  $\|\mathbf{b}^+\| < r$ , and that  $(\mathbf{b}^+, u, z_0)$  has a neighborhood in  $\widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$  which is diffeomorphic to an open ball  $\overset{\circ}{D}^{2n-k+3}$  if  $|z_0| < 1$ , or to a half-ball  $\overset{\circ}{D}^{2n-k+1} \times \{z \in \mathbb{C} \mid \text{Im } z > 0\}$  if  $|z_0| = 1$ .

Now we consider the elements  $(\mathbf{b}^+, u, z_0)$  of  $\widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$  such that the image of  $u$  touches  $\partial \widehat{W}$  at an interior point. Again by Proposition 4.8 this implies  $\|\mathbf{b}^+\| = r$ . We know by Proposition 4.10 that  $u$  will be a Bishop disk up to reparametrization. Since Bishop disks are regular by Corollary 4.7, a neighborhood of  $(\mathbf{b}^+, u, z)$  in  $\widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$  looks like the neighborhood of a boundary point of an  $(n + m + 2)$ -manifold. A priori the boundary of the unparametrized moduli space may contain one more stratum consisting of triples  $(\mathbf{b}^+, u, z_0)$  such that the image of  $u$  touches the



binding  $B_{\mathbf{b}^+}$  of the Lob  $L_{\mathbf{b}^+}$  at a boundary point. However, in this case, the image of  $u$  would have to intersect the neighborhood  $V$  from Proposition 4.9, and therefore  $u$  would be a Bishop disk. Since Bishop disks which intersect the binding are constant and, by definition, do not belong to  $\widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$ , the possibility that the image of  $u$  touches  $B_{\mathbf{b}^+}$  cannot occur. (We will see later that those constant disks must be added to the compactification of the moduli space.)

By definition, the space  $\widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$  of parametrized disks with a marked point is a trivial disk bundle. The moduli space  $\mathcal{M}(\widehat{W}, S_B; J)$  is then a disk bundle over the moduli space of the same holomorphic curves without the marked point, and the projection map is simply the map that forgets the marked point.

In our case though, it is even true that  $\mathcal{M}(\widehat{W}, S_B; J)$  is a *trivial* disk bundle. Recall that the map  $\vartheta: S_B \setminus \{z^\circ = 0\} \rightarrow S^1$  is globally defined for all LobS in the family. Hence every equivalence class  $[\mathbf{b}^+, u, z]$  in  $\mathcal{M}(\widehat{W}, S_B; J)$  has a unique representative  $(\mathbf{b}^+, u_0, z_0)$ , defined by fixing a parametrization of  $u$  such that

$$\vartheta(u(z)) = \begin{cases} 1 & \text{if } z = 1, \\ i & \text{if } z = i, \\ -1 & \text{if } z = -1, \end{cases}$$

as we did in the proof of Proposition 4.5. This choice of parametrization gives a global slice for the action of  $\text{Aut}(\mathbb{D}^2)$  on  $\widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$ , which identifies  $\mathcal{M}(\widehat{W}, S_B; J)$  with a subset of  $\widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$ . Then  $\mathcal{M}(\widehat{W}, S_B; J)$  is a trivial disk bundle because the same is true of  $\widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$ .

**4.3. TOPOLOGY OF THE COMPACTIFIED MODULI SPACE.** — In the previous sections, we introduced the moduli space we want to use, and we showed that all the disks intersecting certain domains of the model neighborhood  $W_{\text{model}}$  must be Bishop disks. Our aim in this section is to study the topology of the natural compactification of that moduli space. The compactification of  $\mathcal{M}(\widehat{W}, S_B; J)$  involves two phenomena: (1) Gromov convergence to stable nodal holomorphic disks (see e.g. [Fra08, FZ15]), and (2) degeneration to constant maps in the binding of a Lob. In order to accommodate the latter without losing the extra disk-bundle structure provided by the marked point, we shall (as in Proposition 4.5) replace  $\mathcal{M}(\widehat{W}, S_B; J)$  by the space  $\widetilde{\mathcal{M}}_0(\widehat{W}, S_B; J)$  of parametrized curves that satisfy the condition (4.1). This introduces a hint of extra book-keeping into the following statement, but the reader should keep in mind that the space we are actually interested in is always  $\mathcal{M}(\widehat{W}, S_B; J)$ .

**PROPOSITION 4.12.** — *Any sequence  $(\mathbf{b}_j^+, u_j, z_j) \in \widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$  satisfying the condition (4.1) has a subsequence that converges to a unique configuration of one of the following types:*

(1) *An element of the moduli space  $(\mathbf{b}_\infty^+, u_\infty, z_\infty) \in \widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$ , still satisfying (4.1);*

(2) A triple  $(\mathbf{b}_\infty^+, p_\infty, z_\infty)$ , where  $\mathbf{b}_\infty^+ \in \mathbb{D}^m$ ,  $p_\infty$  represents the constant map at the point  $p_\infty \in B_{\mathbf{b}_\infty^+}$  and  $z_\infty \in \mathbb{D}^2$ ; or

(3) A triple  $(\mathbf{b}_\infty^+, \mathbf{t}_\infty, z_\infty)$ , where  $\mathbf{b}_\infty^+ \in \mathbb{D}^m$ ,  $\mathbf{t}_\infty$  is a stable nodal holomorphic disk with boundary on  $L_{\mathbf{b}_\infty^+} \subset S_{\text{belt}}^{n+m}$ , consisting of a single nonconstant disk with a tree of sphere bubbles attached, and  $z_\infty$  is a marked point on the domain of  $\mathbf{t}_\infty$ .

Convergence in cases (1) and (2) is in the  $C^\infty$ -topology, and in case (3) it is in the sense of Gromov. If  $W$  is symplectically aspherical, then the third case does not occur.

*Proof.* — Since the parameters  $\mathbf{b}_j^+$  belong to the closed ball  $\mathbb{D}_r^m$ , we can extract a first subsequence from  $(\mathbf{b}_j^+, u_j, z_j)$  for which the parameters converge to some limit  $\mathbf{b}_\infty^+ \in \mathbb{D}_r^m$ . For simplicity, we still denote this subsequence by  $(\mathbf{b}_j^+, u_j, z_j)$ .

The usual statement of Gromov’s compactness theorem for holomorphic disks (see [Fra08, §4]) applies to sequences of *unparametrized* curves with fixed numbers of interior marked points and/or boundary marked points. Thus in order to apply the theorem to  $(\mathbf{b}_j^+, u_j, z_j) \in \mathcal{M}(\widehat{W}, S_B; J)$ , it will be convenient to observe that parametrized curves satisfying (4.1) can be identified in a canonical way with unparametrized stable nodal  $J$ -holomorphic disks carrying one interior marked point (corresponding to  $z_j$ ) and three extra boundary marked points (corresponding to the points  $1, i, -1$ ), where the latter are required to satisfy incidence conditions under the evaluation map. In this picture, *smooth* (i.e., non-nodal) unparametrized curves with extra boundary marked points correspond to triples  $(\mathbf{b}_j^+, u_j, z_j)$  with  $z_j \in \mathbb{D}^2 \setminus \partial\mathbb{D}^2$ , and triples with  $z_j \in \partial\mathbb{D}^2$  are identified with *nodal* curves that consist of a nonconstant disk  $u_j$  attached by a node at  $z_j$  to a single constant (“ghost”) disk on which the interior marked point lives.

With this identification understood, suppose the maps  $u_j$  have images bounded away from the binding  $B_{\mathbf{b}_\infty^+}$ . Then after taking a subsequence, we can assume by Gromov compactness that the corresponding sequence of unparametrized stable curves with extra boundary marked points converges in the Gromov topology to a smooth or nodal  $J$ -holomorphic disk. Note that each of the unparametrized curves has a unique parametrization for which the (ordered) set of boundary marked points is  $(1, i, -1)$ , thus if the Gromov limit is smooth, then this means  $z_j$  converges to an interior point of  $\mathbb{D}^2$  and  $u_j$  converges in  $C^\infty$  to a smooth  $J$ -holomorphic disk  $u_\infty$ . Similarly, if the nodal limit consists only of one nonconstant  $J$ -holomorphic disk  $u_\infty$  and one ghost disk containing the interior marked point, then this means that  $u_j$  converges in  $C^\infty$  to  $u_\infty$  while  $z_j$  converges to a point in  $\partial\mathbb{D}^2$ . In all other cases,  $u_j$  can be viewed as converging to a bubble tree which may include both spheres and disks, while  $z_j$  converges to an interior or boundary point on one of the components.

Suppose now that the sequence  $(u_j, z_j)$  converges to a bubble tree  $(\mathbf{t}_\infty, z_\infty)$ . We will show that  $\mathbf{t}_\infty$  does not contain any nonconstant disk bubble. (Since the boundary marked points are always mapped to distinct points in the image, stability then implies that with the exception of the cases interpreted above as smooth limits,  $\mathbf{t}_\infty$  contains no disk bubbles at all.) Suppose on the contrary that the sequence  $u_j$  bubbles a

nonconstant disk  $v$  with boundary on the Lob  $L_{\mathbf{b}_\infty^+}$ . The points  $1, -1, i$  divide  $\partial\mathbb{D}^2$  into three segments, one of which is necessarily disjoint from the bubbling region. The fact that for each  $j$  the function  $\vartheta \circ u_j|_{\partial\mathbb{D}^2}$  is a diffeomorphism then implies that  $\vartheta \circ v|_{\partial\mathbb{D}^2}$  is not surjective. Thus the boundary of  $v$  is somewhere tangent a page of the Lob  $L_{\mathbf{b}_\infty^+}$ , but it follows from a standard argument using the boundary point lemma [Nie13, Th. II.1.3] that the disk  $v$  cannot exist. We conclude that  $\mathfrak{t}_\infty$  is a bubble tree containing only holomorphic spheres. This is case (3).

If there is a sequence  $w_j \in \mathbb{D}^2$  such that  $u_j(w_j)$  approaches the binding, then the maps  $u_j$  are Bishop disks for  $j$  large enough because, sooner or later, the images of the  $u_j$  will intersect the domain  $V$  as in Proposition 4.9 nontrivially. This implies that the limit  $u_\infty$  is the constant map at a point  $p_\infty \in B_{\mathbf{b}_\infty^+}$ , and we have case (2).

Using the natural identification of  $\mathcal{M}(\widehat{W}, S_B; J)$  with the space of parametrized curves satisfying (4.1), we can now compactify  $\mathcal{M}(\widehat{W}, S_B; J)$  by adding the limiting configurations described in Proposition 4.12. We will denote this compactified moduli space by  $\overline{\mathcal{M}}(\widehat{W}, S_B; J)$ . Its “boundary”

$$\partial\overline{\mathcal{M}}(\widehat{W}, S_B; J) \subset \overline{\mathcal{M}}(\widehat{W}, S_B; J)$$

can be defined naturally as the set of equivalence classes  $[(\mathbf{b}^+, u, z)]$  for which either  $\mathbf{b}^+ \in \partial\mathbb{D}_r^2$ ,  $z \in \partial\mathbb{D}^2$  (including cases where the domain of  $u$  contains sphere bubbles), or  $u$  is a constant map into the binding of a Lob. The compactification can also be decomposed naturally into two disjoint pieces,

$$\overline{\mathcal{M}}(\widehat{W}, S_B; J) = \mathcal{M}_{\text{smooth}} \cup \mathcal{M}_{\text{bubble}},$$

defined as the subsets consisting of non-nodal and nodal curves respectively. We define

$$\partial\mathcal{M}_{\text{smooth}} := \partial\overline{\mathcal{M}}(\widehat{W}, S_B; J) \cap \mathcal{M}_{\text{smooth}}.$$

The next proposition describes the topology of the compactified moduli space. We refer to [MS12, §6.5] for general facts about pseudocycles, and [Sch99, Zin08] for the fact that pseudocycles up to bordism can be identified with integral homology classes.

**PROPOSITION 4.13.** — *Let  $(W, \omega)$  be a symplectic filling of a contact  $(2n - 1)$ -manifold  $(M, \xi)$ . Suppose that  $(M, \xi)$  has been obtained by a surgery of index  $k \subset n - 1$  from another contact manifold, and that  $S_{\text{belt}}^{n+m}$  is the corresponding belt sphere (with  $n = k + m + 1$ ). Deform  $W$  as described in Section 3.2 to  $(\widehat{W}, \omega)$ , and let  $\overline{\mathcal{M}}(\widehat{W}, S_B; J)$  be the compactification of the moduli space  $\mathcal{M}(\widehat{W}, S_B; J)$  of disks attached to  $S_B \subset S_{\text{belt}}^{n+m}$ . Then  $\text{ev}: \mathcal{M}(\widehat{W}, S_B; J) \rightarrow \widehat{W}$  extends to a continuous map*

$$\overline{\text{ev}}: \left(\overline{\mathcal{M}}(\widehat{W}, S_B; J), \partial\overline{\mathcal{M}}(\widehat{W}, S_B; J)\right) \longrightarrow \left(\widehat{W}, S_{\text{belt}}^{n+m}\right).$$

Moreover:

- (a) *If  $(W, \omega)$  is semipositive, then*

$$\text{ev}|_{\partial\mathcal{M}_{\text{smooth}}}: \partial\mathcal{M}_{\text{smooth}} \longrightarrow \widehat{W}$$

defines an  $(n + m)$ -dimensional pseudocycle in  $\widehat{W}$  representing the homology class  $\pm[S_{\text{belt}}^{n+m}] \in H_{n+m}(\widehat{W}; \mathbb{Z})$ , and

$$\text{ev}|_{\mathcal{M}_{\text{smooth}}} : \mathcal{M}_{\text{smooth}} \longrightarrow \widehat{W}$$

defines a bordism of the above pseudocycle to the empty  $(n + m)$ -dimensional pseudocycle in  $\widehat{W}$ .

(b) If  $(W, \omega)$  is symplectically aspherical, then  $\overline{\mathcal{M}}(\widehat{W}, S_B; J)$  is homeomorphic to a manifold with boundary and corners of the form

$$\overline{\Sigma} \times \mathbb{D}^2,$$

where  $\overline{\Sigma}$  is a smooth, compact, connected and oriented  $(n + m - 1)$ -manifold with boundary and corners, whose boundary is homeomorphic to  $S^{n+m-2}$ . Furthermore,

$$\text{ev}|_{\partial \overline{\mathcal{M}}(\widehat{W}, S_B; J)} : \partial \overline{\mathcal{M}}(\widehat{W}, S_B; J) \longrightarrow S_{\text{belt}}^{n+m}$$

is a map of degree  $\pm 1$ .

*Proof.* — Let us describe the natural topology on  $\overline{\mathcal{M}}(\widehat{W}, S_B; J)$ , using again the identification of  $\mathcal{M}(\widehat{W}, S_B; J)$  with the slice in  $\widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$  defined via the conditions (4.1). The boundary of the uncompactified space  $\mathcal{M}(\widehat{W}, S_B; J)$  consists of holomorphic disks whose marked points lie in  $\partial \mathbb{D}^2$ , together with Bishop disks with boundary on a  $\text{Lob } L_{\mathbf{b}^+}$  with  $\mathbf{b}^+ \in \partial \mathbb{D}_r^2$ . Proposition 4.12, provides a description of the two additional limit objects we need to consider. If  $(\mathbf{b}^+, p_\infty, z)$  is one of the constant disks appearing in case (2) of Proposition 4.12, then it follows from Proposition 4.9 that it is surrounded only by Bishop disks. Using the parametrization of the Bishop disks given at the beginning of Section 4.2—in this description the constant disks are obtained by choosing  $C = 0$ —we can add the constant disks  $(\mathbf{b}^+, p_\infty, z)$  to the chosen slice in  $\widetilde{\mathcal{M}}(\widehat{W}, S_B; J)$ , and give it a smooth structure that agrees with the one induced by  $C^\infty$ -convergence of maps. Attaching the constant disks in this way corresponds to adding boundary points to the global slice. Defining a smooth structure on  $\overline{\mathcal{M}}(\widehat{W}, S_B; J)$  in this way, it is straightforward to see that the evaluation map extends smoothly to the constant disks in  $\overline{\mathcal{M}}(\widehat{W}, S_B; J)$ .

The other singular points we need to consider in  $\overline{\mathcal{M}}(\widehat{W}, S_B; J)$  are bubble trees, each consisting of one holomorphic disk and several holomorphic spheres. If  $(W, \omega)$  is symplectically aspherical as in case (b), it does not contain any holomorphic spheres, and hence no bubbles can appear. In this case, the compactified moduli space  $\overline{\mathcal{M}}(\widehat{W}, S_B; J)$  will be diffeomorphic to

$$\overline{\Sigma} \times \mathbb{D}^2,$$

according to Proposition 4.11, where  $\overline{\Sigma}$  is a smooth compact manifold with boundary and corners. If we are in case (a), then bubbling of spheres may occur, but standard index counting arguments using the semipositivity assumption imply that such bubbling is a “codimension 2 phenomenon”. The restriction of  $\text{ev}$  to  $\partial \mathcal{M}_{\text{smooth}}$  is then a

pseudocycle, and the restriction to  $\mathcal{M}_{\text{smooth}}$  is a bordism of this pseudocycle to the trivial one.

In the absence of bubbling,  $\overline{\mathcal{M}}(\widehat{W}, S_B; J)$  is a trivial disk bundle over the compact base manifold  $\overline{\Sigma}$ , whose boundary consists of Bishop disks sitting on boundary Lobs and/or collapsing into the binding. The boundary Lobs are parametrized by  $\mathbf{b}^+ \in \partial D_r^m \cong S^{m-1}$ , and there is precisely one (unparametrized) Bishop disk going through every point of the page of the Legendrian open book of  $L_{\mathbf{b}^+}$ , hence we conclude that the first disks can be parametrized by  $S^{m-1} \times D^{n-1}$ . The binding of a Lob  $L_{\mathbf{b}^+}$ , on the other hand, is diffeomorphic to  $S^{n-2}$ , and since there is a  $D_r^m$ -worth of Lobs, we conclude that  $\overline{\mathcal{M}}(\widehat{W}, S_B; J)$  contains a family of constant disks that is parametrized by  $D_r^m \times S^{n-2}$ . These two parts meet at their boundaries and form a (topological) manifold homeomorphic to  $S^{n+m-2}$ , as claimed.

Finally, we observe that the restriction of  $\overline{\text{ev}}$  to  $\partial \overline{\mathcal{M}}(\widehat{W}, S_B; J)$  is always bijective on some subset consisting of Bishop disks, so it is a map of degree  $\pm 1$  onto  $S_{\text{belt}}^{n+m}$  whenever  $\partial \overline{\mathcal{M}}(\widehat{W}, S_B; J)$  is a topological manifold. More generally, this implies that the pseudocycle  $\text{ev}|_{\partial \mathcal{M}_{\text{smooth}}} : \partial \mathcal{M}_{\text{smooth}} \rightarrow S_{\text{belt}}^{n+m}$  represents a generator of  $H_{n+m}(S_{\text{belt}}^{n+m}; \mathbb{Z})$  whenever it is well defined.

The orientability of  $\mathcal{M}(\widehat{W}, S_B; J)$  is shown in the appendix.

*Proof of Theorem 1.2.* — The proof of statement (a) follows directly from part (a) of Proposition 4.13, using the natural identification between singular homology classes and bordism classes of pseudocycles, see [Sch99, Zin08].

Statement (b) can be obtained by using part (b) of Proposition 4.13. Since  $\overline{\mathcal{M}}(\widehat{W}, S_B; J)$  is diffeomorphic to a trivial disk bundle, we can apply Proposition 5.1 in the general situation, or Propositions 5.2 and 5.3 respectively, when  $n + m = 3$  or  $n + m = 4$ . This implies that  $S_{\text{belt}}^{n+m}$  is the trivial element in the oriented bordism group  $\Omega_{n+m}^{SO}(W)$ , and it is even contractible in  $W$ , if  $n + m = 3$  or  $4$  as claimed.

## 5. SURGERY ON MODULI SPACES

If bubbling can be ruled out in the proof of Theorem 1.2 given in the previous section, we can use topological results to conclude that the belt sphere is not only null-homologous but null-bordant in  $\Omega_*^{SO}(\widehat{W})$ , and in cases where  $\dim \mathcal{M}(\widehat{W}, S_B; J) \leq 5$ , it is even null-homotopic. The idea in both cases is to attach handles to the moduli space and extend the evaluation map accordingly so that we obtain a new space together with a map into  $\widehat{W}$  which will be topologically simpler than the original moduli space. Our argument for this will make essential use of the fact that the moduli space is naturally a trivial disk bundle.

Note that if  $\Sigma$  is a compact oriented  $k$ -manifold with boundary, then using handle attachments to turn  $\Sigma \times D^2$  into a ball cannot succeed unless  $\Sigma \times D^2$  admits an embedding into  $\mathbb{R}^{k+2}$ , which cannot always be guaranteed, i.e., in general there are topological obstructions to the applicability of this technique to obtain contractibility of the belt sphere. We will show that these can be overcome if  $\dim \Sigma \leq 3$ .

**PROPOSITION 5.1.** — *Let  $W$  be a compact manifold possibly with boundary, and let  $S \subset W$  be an embedded  $(k + 1)$ -sphere. Assume that  $\Sigma$  is a compact connected orientable  $k$ -manifold with non-empty boundary. Let  $X$  be  $\Sigma \times \mathbb{D}^2$ , and let*

$$f: (X, \partial X) \longrightarrow (W, S)$$

*be a continuous map, whose restriction to the boundary*

$$f|_{\partial X}: \partial X \longrightarrow S$$

*is of degree 1. Then it follows that  $S$  is null-bordant in  $\Omega_{k+1}^{SO}(W)$ .*

*Proof.* — We will assume  $k > 2$ , since for  $k \leq 1$ ,  $\Sigma$  is either a point or a closed interval. The manifold  $\Sigma$  has a handle decomposition that consists of a  $k$ -disk with finitely many handles of index  $< k$  attached to it (attached successively in order of their indices). Since the product of a  $k$ -dimensional handle with  $\mathbb{D}^2$  is a  $(k + 2)$ -dimensional handle of the same index,  $X = \Sigma \times \mathbb{D}^2$  is built up by attaching  $(k + 2)$ -dimensional handles of index  $< k$ . For each  $q = 0, \dots, k - 1$ , let  $X^{(q)} \subset X$  denote the union of all the handles up to index  $q$ , so  $X = X^{(k-1)}$ .

For every 1-handle in  $X^{(1)}$  we can find a closed curve in the boundary of  $X^{(1)}$  by pushing the core of the handle into  $\partial X^{(1)}$ , and connecting the end points with a path in  $\partial X^{(1)}$  that does not intersect any other 1-handle. This is possible, because  $X^{(0)}$  consists of a unique 0-handle that in particular is connected. These curves intersect the belt sphere of the corresponding handle exactly once. Moreover since  $\dim \partial X^{(1)} = k + 1$ , there is enough space to assume that the loops corresponding to different 1-handles are disjoint from each other and also disjoint from any of the attaching circles of the 2-handles needed to obtain  $X^{(2)}$ . The loops thus embed into  $\partial X^{(2)}$ , and we can repeat this reasoning to see that they also embed into  $\partial X^{(3)}$  and so on up to  $\partial X$ .

Standard Morse theory implies that a  $q$ -handle can be canceled out by attaching a  $(q + 1)$ -handle along an embedded  $q$ -sphere that intersects the belt sphere of the  $q$ -handle exactly once. It is thus possible to convert  $X^{(1)}$  into a ball by attaching 2-handles, each corresponding to one of the 1-handles. Since the loops also embed into  $\partial X$ , we may equally well attach the 2-handles to  $X$ , obtaining in this way a compact connected orientable  $(k + 2)$ -manifold  $X'$  that admits a handle decomposition with exactly one 0-handle, and no handles of index 1 or of index  $> k$ .

The map  $f: X \rightarrow W$  can be extended to the newly added 2-handles because  $\partial X$ , and hence also the attaching curves, are mapped to  $S$ , which is simply connected. We can thus construct a continuous map

$$f': (X', \partial X') \longrightarrow (W, S).$$

The restriction of  $f'$  to the boundary is still of degree 1, because the image of the 2-handles can be assumed to be a thin set in the  $(k + 1)$ -sphere  $S$ .

To finish the proof, consider the double of  $X'$ , obtained by gluing a copy of  $X'$  with reversed orientation to itself along its boundary. One can decompose the double into handles such that for each  $q$ -handle in the original  $X'$ , there is a corresponding  $(k + 2 - q)$ -handle in the second copy (think of the two copies of  $X'$  as carrying Morse

functions  $f$  and  $c - f$  for some constant  $c \in \mathbb{R}$ ). Our handle decomposition of the double therefore has exactly one handle of index  $k + 2$  and none of index  $k + 1$ . Let  $\widehat{X}$  denote the result of removing the  $(k + 2)$ -handle. Then  $\widehat{X}$  is a compact connected orientable  $(k+2)$ -manifold obtained from  $X'$  by attaching additional handles of various indices  $3, \dots, k$ , and  $\partial\widehat{X} \cong S^{k+1}$ . Since  $\pi_q(S) = 0$  for all  $q = 2, \dots, k$ , the map  $f'$  extends from  $X'$  to a continuous map

$$\widehat{f}: (\widehat{X}, \partial\widehat{X}) \longrightarrow (W, S),$$

mapping all of the additional handles into  $S$ . Note in particular that the restriction of  $\widehat{f}$  to  $\partial\widehat{X}$  is a degree 1 map to  $S$ , because all the handles added to  $X'$  are of index lower than  $\dim S$ .

The restriction  $\widehat{f}|_{\partial\widehat{X}}$  is homotopic to a diffeomorphism between the two  $(k + 1)$ -spheres  $\partial\widehat{X}$  and  $S$ , and it follows that  $S$  is null-bordant in  $\Omega_{k+1}^{SO}(W)$ .

**PROPOSITION 5.2.** — *Let  $W$  be a compact manifold possibly with boundary, and let  $S \subset W$  be an embedded 3-sphere. Assume that  $\Sigma$  is a compact connected orientable surface with non-empty boundary. Let  $X$  be  $\Sigma \times \mathbb{D}^2$ , and let*

$$f: (X, \partial X) \longrightarrow (W, S)$$

*be a continuous map, whose restriction to the boundary*

$$f|_{\partial X}: \partial X \longrightarrow S$$

*is of degree 1. Then it follows that  $S$  is contractible in  $W$ .*

*Proof.* — The proof is a special case of the argument used for Proposition 5.1. The difference is that  $X$  consists only of a 4-disk and 1-handles, thus after passing to  $X'$  by attaching 2-handles, it already follows that  $X'$  is diffeomorphic to a 4-disk  $\mathbb{D}^4$ . We then obtain a continuous map

$$f': (\mathbb{D}^4, \partial\mathbb{D}^4) \longrightarrow (W, S)$$

whose restriction to the boundary is still of degree 1, and is therefore homotopic to a homeomorphism  $\partial\mathbb{D}^4 \rightarrow S$ .

**PROPOSITION 5.3.** — *Let  $W$  be a compact manifold, possibly with boundary, and let  $S \subset W$  be an embedded 4-sphere. Let  $\Sigma$  be a compact connected orientable 3-manifold with non-empty boundary  $\partial\Sigma \cong S^2$ . Assume that  $X$  is  $\Sigma \times \mathbb{D}^2$ , and that*

$$f: (X, \partial X) \longrightarrow (W, S)$$

*is a continuous map, whose restriction to the boundary*

$$f|_{\partial X}: \partial X \longrightarrow S$$

*is of degree 1. Then it follows that  $S$  is contractible in  $W$ .*

*Proof.* — The manifold  $\Sigma$  is an orientable 3-manifold minus a ball. It admits a handle decomposition given by a 3-ball with 1- and 2-handles attached, and we may assume that  $\Sigma$  has been obtained by attaching first the 1-handles and then the 2-handles. As in the proof of Proposition 5.1, it follows that the manifold  $X$  is built up by first attaching 1- and then 2-handles to a 5-ball  $D^5$ . Denote by  $X^{(1)}$  the intermediate space consisting only of the 5-ball and the 1-handles.

It is easy to find for every 1-handle an embedded loop in  $\partial X^{(1)}$  that intersects the belt sphere of the handle exactly once. For dimensional reasons, these loops will be generically disjoint from each other, but they will also generically not intersect any of the attaching circles of the 2-handles. We can cancel all 1-handles of  $X^{(1)}$  by attaching 2-handles along the chosen loops. The chosen loops also embed into  $\partial X$ , hence we can also kill all 1-handles by attaching 2-handles to  $X$ . Note that we could get rid of the 1-handles without choosing a particular framing when attaching the 2-handles; however  $X$  is parallelizable (as is any oriented 3-manifold), and using Lemma 5.4 below, we attach the 2-handles in such a way that the resulting manifold is also parallelizable. Since the image  $f(\partial X)$  lies in the sphere  $S$ , we can extend  $f$  to the additional 2-handles without changing the degree of  $f|_{\partial X}$ .

After the previous step, we will assume that  $X$  is a 5-manifold with trivial tangent bundle that has been obtained by gluing 2-handles to the 5-ball. Every embedding of  $S^1$  into  $S^4 = \partial D^5$  is isotopic to a standard one for dimensional reasons (see [Hae63]), and it follows that the 2-handles are all attached along unknots. Note also that these unknots are unlinked since we may shrink the first unknot into an arbitrarily small ball, so that the other loops will bound embedded disks that are disjoint from this ball.

We may therefore assume that  $X$  is the boundary sum of a finite collection of 5-manifolds, each consisting of a 5-ball with a single 2-handle attached along an unknot. The only invariant of each such manifold is the framing of the 2-handle. It is given by a loop in  $SO(3)$ , which means there are only two choices, corresponding to the two elements of  $\pi_1(SO(3))$ . In fact, each of these manifolds is diffeomorphic to either the trivial rank 3 bundle over  $S^2$  or the twisted one,  $S^2 \tilde{\times} D^3$ . The total space of the twisted one is not parallelizable: it suffices to study  $T(S^2 \tilde{\times} D^3)|_{S^2 \tilde{\times} \{0\}}$  which is obtained by clutching two copies of  $\mathbb{C} \oplus \mathbb{R}^3$  over two disks together. The gluing map is  $e^{2i\phi} \oplus \psi$ , where  $\psi$  is the nontrivial loop in  $\pi_1(SO(3))$ , but since this is the nontrivial element of  $\pi_1(SO(5))$ , the bundle is not trivial. It follows that  $X$  is the boundary connected sum of copies of  $S^2 \times D^3$ .

We can then also kill the 2-handles by attaching 3-handles, and the map  $f$  extends to this new manifold. This proves that  $S$  is homotopically trivial in the filling.

The following lemma was used above in the proof of Proposition 5.3.

LEMMA 5.4. — *Let  $X$  be a compact parallelizable  $n$ -manifold with boundary, and let  $\gamma$  be an embedded loop in  $\partial X$ . Assume  $n > 5$ . Then one can choose a framing of  $\gamma$  such*



that the manifold

$$X \cup_{\gamma} H_2$$

obtained by attaching a 2-handle  $H_2$  along  $\gamma$  is also parallelizable.

*Proof.* — A framing of  $\gamma$  is an oriented trivialization of the normal bundle  $\nu(\gamma)$  of  $\gamma$  in  $\partial X$ . Given one framing, any other one can be obtained by multiplying the first one in each fiber with a matrix in  $\mathrm{GL}^+(n-2)$ , i.e. the second framing can be represented with respect to the first one by a map  $S^1 \rightarrow \mathrm{GL}^+(n-2)$ . We are only interested in framings up to homotopy, hence it follows that all framings are classified by  $\pi_1(\mathrm{GL}^+(n-2))$ , and since  $\mathrm{GL}^+(n-2) \simeq \mathrm{SO}(n-2)$ , there are only two possible choices.

Choose now a trivialization of  $TX$ . Such a trivialization allows us to identify  $TX|_{\gamma}$  with  $S^1 \times \mathbb{R}^n$ . Any other trivialization of  $TX|_{\gamma}$  can be represented with respect to the first one by a map  $S^1 \rightarrow \mathrm{GL}^+(n)$ , that is, up to homotopy there are also only two trivializations of  $TX|_{\gamma}$ , corresponding to the elements of  $\pi_1(\mathrm{SO}(n))$ . In particular, any framing  $(e_1, \dots, e_{n-2})$  of  $\nu(\gamma)$  extends to a basis  $(f_1, f_2, e_1, \dots, e_{n-2})$  of  $TX|_{\gamma}$ , where the vector fields  $f_1$  and  $f_2$  are given by

$$\begin{aligned} f_1 &= \vec{n} \cos \phi - \dot{\gamma} \sin \phi \\ f_2 &= \vec{n} \sin \phi + \dot{\gamma} \cos \phi, \end{aligned}$$

where  $\phi$  parametrizes  $\gamma$ . Here  $\vec{n}$  denotes the outward normal vector field to the boundary  $\partial X$ , and  $\dot{\gamma}$  is the tangent vector field to the loop  $\gamma$ . If this basis is not homotopic to the given trivialization of  $TX|_{\gamma}$ , it suffices to choose instead

$$(f_1, f_2, e_1 \cos \phi - e_2 \sin \phi, e_1 \sin \phi + e_2 \cos \phi, e_3, \dots, e_{n-2}),$$

which corresponds to the second framing of  $\gamma$ , but also to the other homotopy class of possible trivializations of  $TX|_{\gamma}$ . It is thus possible to homotope the trivialization of  $TX$  into one that coincides close to  $\gamma$  with  $(f_1, f_2, e_1, \dots, e_{n-2})$ , where  $(e_1, \dots, e_{n-2})$  is a framing of  $\gamma$ .

On  $H_2 = \mathbb{D}^2 \times \mathbb{D}^{n-2}$  with coordinates  $(x, y; \mathbf{z}) \in \mathbb{D}^2 \times \mathbb{D}^{n-2}$ , the attaching circle  $\{x^2 + y^2 = 1, \mathbf{z} = \mathbf{0}\}$  has the obvious framing  $(\partial_u, \partial_v, \partial_{z_1}, \dots, \partial_{z_{n-2}})$ . If we glue  $H_2$  to  $X$  with the chosen framing, then the trivialization  $(f_1, f_2, e_1, \dots, e_{n-2})$  extends to  $(\partial_u, \partial_v, \partial_{z_1}, \dots, \partial_{z_{n-2}})$ , so the manifold  $X \cup_{\gamma} H_2$  has trivial tangent bundle, as desired.

## 6. CONTACT STRUCTURES THAT ARE NOT CONTACT CONNECTED SUMS

In this section we prove Theorem 1.3. The construction we are going to use is inspired by a similar one in [BCS15], though we do not need the full strength of that paper.

Let  $M$  be a closed  $(2n-1)$ -dimensional manifold that admits an almost contact structure and that has a handle decomposition with a single handle of index 0, a single one of index  $2n-1$ , and otherwise only handles of indices  $n-1$  and  $n$ . We assume also that  $M$  is not a homotopy sphere, which by the Hurewicz theorem implies that it must

have nontrivial homology in dimension  $n - 1$  or  $n$ . Possible examples include the unit cotangent bundle of  $S^n$ , and  $S^{n-1} \times S^n$ ; the first carries a canonical contact structure, and the second is easily seen to be almost contact since it is stably parallelizable.

Remove a small open disk  $D$  from  $M$  and denote the resulting manifold by  $M^*$ . The product manifold  $W = M^* \times [-1, 1]$  is compact and has boundary and corners, and after smoothing, its boundary

$$\partial W = M^* \times \{-1\} \cup (\partial M^* \times [-1, 1]) \cup M^* \times \{+1\}$$

is diffeomorphic to  $M \# (-M)$ .

Now we can proceed with the proof of Theorem 1.3.

*Proof of Theorem 1.3.* — By assumption, the manifold  $M^*$  admits a Morse function with outward pointing gradient at the boundary and critical points of index at most  $n$ , and the same is therefore true of  $W$ . Moreover, any almost contact structure  $\Xi$  on  $M$  induces an almost complex structure on  $W$ , thus by a well-known theorem of Eliashberg [Eli90b], there is a Stein structure whose complex structure is homotopic to the given almost complex structure. The boundary  $\partial W \cong M \# (-M)$  inherits from this Stein structure a contact structure  $\xi$  which is homotopic to the almost contact structure  $\Xi \# \bar{\Xi}$ . Note that the belt sphere of the connected sum (i.e., the center of the “neck” in  $M \# (-M)$ ) is

$$S := \partial M^* \times \{0\} \subset \partial W.$$

Arguing by contradiction, suppose now that  $\xi_1$  and  $\xi_2$  are positive contact structures on  $M$  and  $-M$  respectively such that  $\xi_1 \# \xi_2$  is isotopic to  $\xi$ . Then after a deformation of the Stein structure near  $\partial W$  and hence an isotopy of  $\xi$ , we can assume  $\xi$  in a neighborhood of  $S$  is contactomorphic to the contact structure on a neighborhood of the belt sphere of an index one Weinstein handle. According to Proposition 4.13 there is a compact  $(2n - 1)$ -dimensional moduli space  $\overline{\mathcal{M}} \cong \overline{\Sigma} \times \mathbb{D}^2$  and an evaluation map  $\text{ev}: (\overline{\mathcal{M}}, \partial \overline{\mathcal{M}}) \rightarrow (W, S)$  such that

- (1)  $\text{ev}|_{\partial \overline{\mathcal{M}}}: \partial \overline{\mathcal{M}} \rightarrow S$  has degree one, and
- (2)  $\overline{\Sigma}$  is a compact orientable  $(2n - 3)$ -manifold with non-empty boundary.

Moreover,  $\text{ev}$  is a diffeomorphism on some open subset.

Consider the projection  $p: W = M^* \times [-1, 1] \rightarrow M^*$ , which maps  $S$  to  $\partial M^*$ , and denote by  $f: \overline{\mathcal{M}} \rightarrow M^*$  the composition  $f = p \circ \text{ev}$ . It is easy to check that  $f: \overline{\mathcal{M}} \rightarrow M^*$  now satisfies the conditions of Lemma 1.11. This implies that  $M^*$  has vanishing homology in positive degrees, and is thus a contradiction.

On the other hand, note that there is no homotopical obstruction to decomposing  $(M \# (-M), \xi)$ , because  $\xi$  is, by construction, homotopic to  $\Xi \# \bar{\Xi}$ .

**REMARK 6.1.** — A similar argument can be used to find examples of Stein fillable contact structures that are homotopic (through almost contact structures) but not isotopic to contact structures obtained via subcritical surgery of arbitrary index  $k = 1, \dots, n - 1$ . The above is the  $k = 1$  case of this result.

## 7. THE WEINSTEIN CONJECTURE FOR SUBCRITICAL SURGERIES

We will now prove Theorem 1.4, the existence of contractible Reeb orbits for certain contact manifolds  $(M', \xi')$  obtained by subcritical surgery.

Under either of the first two conditions stated in the theorem, the proof is a trivial modification of the proof of Theorem 1.2, following [Hof93]. Suppose  $\alpha$  is the contact form for which we'd like to find a contractible Reeb orbit, and let  $\alpha'$  denote a second contact form that matches the one given in our Weinstein surgery model near the belt sphere  $S_{\text{belt}}^{2n-k-1}$ . After rescaling  $\alpha$ , we can find an exact symplectic structure on  $\mathbb{R} \times M'$  that matches  $d(e^t\alpha)$  on  $(-\infty, -1] \times M'$  and  $d(e^t\alpha')$  on  $[-1/2, \infty) \times M'$ . We then choose a compatible almost complex structure and, as in Theorem 1.2, study the moduli space of holomorphic disks in  $\mathbb{R} \times M'$  with boundary in the Lobs obtained by deformation from  $\{0\} \times S_{\text{belt}}^{2n-k-1} \subset \mathbb{R} \times M'$ . If  $\alpha$  admits no contractible Reeb orbits, then bubbling is impossible, so the proof of Theorem 1.2 shows that  $S_{\text{belt}}^{2n-k-1}$  will be null-bordant in  $\mathbb{R} \times M'$ , and thus also in  $M'$ . If  $n = 3$ , or  $n = 4$  with  $k = 3$ , it also shows that  $S_{\text{belt}}^{2n-k-1}$  is trivial in  $\pi_{2n-k-1}(\mathbb{R} \times M') = \pi_{2n-k-1}(M')$ .

It remains to handle the third condition in Theorem 1.4, which specifically concerns contact connected sums in dimension five. The above argument shows that in this situation, if there is no contractible Reeb orbit, then the belt sphere must be nullhomotopic. But the following theorem of Ruberman [Rub97] says that this can only happen in the cases excluded by the third condition.

**THEOREM 7.1** (Ruberman). — *Let  $M$  be a closed oriented manifold, and suppose  $S$  is an embedded codimension 1 sphere that is nullhomotopic. Then either  $S$  is the boundary of a homotopy-ball embedded in  $M$ , or  $M$  is the connected sum*

$$M = N_0 \# N_1$$

*of two rational homology spheres  $N_0$  and  $N_1$ , one of which is simply connected, while the other has finite fundamental group.*

The proof of Theorem 1.4 is thus complete.

### APPENDIX. ORIENTABILITY OF THE MODULI SPACES

In this appendix we prove that the moduli spaces used in this paper are orientable. Let us fix some notation which will be used in the proof. Fix a real number  $p > 2$ . Let  $\mathcal{B}$  denote the space of pairs  $(\mathbf{b}^+, u)$  where:

- $\mathbf{b}^+ \in \mathbb{D}_r^m$  with  $\|\mathbf{b}^+\| < r$ ;
- $u: \mathbb{D}^2 \rightarrow W$  is a map of class  $W^{1,p}$  such that  $u(\partial\mathbb{D}^2) \subset L_{\mathbf{b}^+} \cap B_{\mathbf{b}^+}$  where  $L_{\mathbf{b}^+}$  denotes the Lob indexed by  $\mathbf{b}^+$  and  $B_{\mathbf{b}^+}$  its binding; and
- $\vartheta \circ u|_{\partial\mathbb{D}^2}$  has degree one, where  $\vartheta: L_{\mathbf{b}^+} \cap B_{\mathbf{b}^+} \rightarrow S^1$  is the fibration of the Lob.

Of course the information about  $\mathbf{b}^+$  is already contained in  $u$ , and  $\mathbf{b}^+$  only serves for book-keeping. We denote by  $\mathcal{B}_{\mathbf{b}^+}$  the fibers of the projection  $\mathbf{p}: \mathcal{B} \rightarrow \mathbb{D}_r^m$ , i.e.  $\mathbf{p}(\mathbf{b}^+, u) = \mathbf{b}^+$ . Then  $\mathcal{B}_{\mathbf{b}^+}$  consists of the maps  $u \in \mathcal{B}$  such that  $u(\partial\mathbb{D}^2) \subset L_{\mathbf{b}^+} \cap B_{\mathbf{b}^+}$ .

The linearized Cauchy-Riemann operator at  $(\mathbf{b}^+, u) \in \mathcal{B}$  (or to be more precise, the *vertical differential* of the nonlinear Cauchy-Riemann operator, as defined in [MS12]), will be denoted  $\tilde{D}_{(\mathbf{b}^+, u)}$ . Recall that this depends on a choice of connection on  $W$ , though it is independent of this choice whenever  $u$  is  $J$ -holomorphic. We define

$$\det \tilde{D}_{(\mathbf{b}^+, u)} = \Lambda^{\text{top}} \ker \tilde{D}_{(\mathbf{b}^+, u)} \otimes \Lambda^{\text{top}}(\text{coker } \tilde{D}_{(\mathbf{b}^+, u)})^*.$$

The *determinant bundle*  $\mathcal{D} \rightarrow \mathcal{B}$  is the real rank-one bundle whose fiber at  $(\mathbf{b}^+, u)$  is  $\det \tilde{D}_{(\mathbf{b}^+, u)}$ . In order to prove that the moduli space  $\widetilde{\mathcal{M}}_{\text{int}}(\widehat{W}, S_B; J)$  is orientable, it suffices to show that  $\mathcal{D} \rightarrow \mathcal{B}$  is trivial.

To better understand the determinant bundle we take a closer look at the linearized Cauchy-Riemann operator. The tangent space  $T_{(\mathbf{b}^+, u)}\mathcal{B}$  consists of sections  $\xi \in W^{1,p}(u^*T\widehat{W})$  such that, for all  $z \in \partial\mathbb{D}^2$ , they satisfy  $\xi(z) \in T_{u(z)}S_{\text{belt}}^{n+m}$  and moreover the projection of  $\xi(z)$  to  $T_{\mathbf{b}^+}D_r^m$  is independent of  $z \in \partial\mathbb{D}^2$ . It contains the subspace  $T_u\mathcal{B}_{\mathbf{b}^+} \subset T_{(\mathbf{b}^+, u)}\mathcal{B}$  which is defined as

$$T_u\mathcal{B}_{\mathbf{b}^+} = \{ \xi \in W^{1,p}(u^*T\widehat{W}) \mid \xi(z) \in T_{u(z)}L_{\mathbf{b}^+} \forall z \in \partial\mathbb{D}^2 \},$$

and therefore we can identify

$$(A.1) \quad T_{(\mathbf{b}^+, u)}\mathcal{B} \cong T_u\mathcal{B}_{\mathbf{b}^+} \oplus T_{\mathbf{b}^+}D_r^m.$$

The tangent spaces  $T_u\mathcal{B}_{\mathbf{b}^+}$  are the fibers of a vector bundles over  $\mathcal{B}$  which we will denote  $T^{\text{vert}}\mathcal{B}$  and the decomposition (A.1) globalizes to a bundle isomorphism

$$T\mathcal{B} \cong T^{\text{vert}}\mathcal{B} \oplus \mathfrak{p}^*TD_r^m.$$

Although the above isomorphism is not canonical, its homotopy class is.

We denote the restriction of  $\tilde{D}_{(\mathbf{b}^+, u)}$  to  $T_u\mathcal{B}_{\mathbf{b}^+}$  by  $D_{(\mathbf{b}^+, u)}$ . If we write the elements of  $T_{(\mathbf{b}^+, u)}\mathcal{B}$  as pairs  $(\xi, v) \in T_u\mathcal{B}_{\mathbf{b}^+} \oplus T_{\mathbf{b}^+}D_r^m$  using the identification in Equation (A.1), we can decompose  $\tilde{D}_{(\mathbf{b}^+, u)}$  as

$$\tilde{D}_{(\mathbf{b}^+, u)}(\xi, v) = D_{(\mathbf{b}^+, u)}(\xi) + K_{(\mathbf{b}^+, u)}(v).$$

The operator  $D_{(\mathbf{b}^+, u)}$  is the linearization at  $u$  of the nonlinear Cauchy-Riemann operator defined on  $\mathcal{B}_{\mathbf{b}^+}$ , and therefore it is a linear Cauchy-Riemann type operator.

Let  $\mathcal{D}' \rightarrow \mathcal{B}$  the real line bundle whose fiber at  $(\mathbf{b}^+, u)$  is  $\det D_{(\mathbf{b}^+, u)}$ . Since the determinant line bundles of homotopic families of Fredholm operators are isomorphic, we obtain an isomorphism

$$(A.2) \quad \mathcal{D} \cong \mathcal{D}' \otimes \mathfrak{p}^*\Lambda^m TD_r^m$$

by homotoping the operators  $K_{(\mathbf{b}^+, u)}$  to the zero operator via a linear homotopy. Note that this defines a homotopy of families of Fredholm operators because the operators  $K_{(\mathbf{b}^+, u)}$  are defined on a finite dimensional space. By the isomorphism (A.2), the triviality of  $\mathcal{D}$  is equivalent to the triviality of  $\mathcal{D}'$ , so from now on we will concentrate on this second bundle.

Triviality of a rank-one real line bundle can be checked on loops. Thus let  $(\mathbf{b}_\bullet^+, u_\bullet): S^1 \rightarrow \mathcal{B}$  be a loop in  $\mathcal{B}$ , i.e.  $\theta \mapsto (\mathbf{b}_\theta^+, u_\theta)$ . From a different point of view we have a map  $\tilde{u}: S^1 \times \mathbb{D}^2 \rightarrow W$  defined as  $\tilde{u}(\theta, z) = u_\theta(z)$ . We define the vector bundle

$T^{\text{vert}}S_{\text{belt}}^{n+m}$  such that  $T_p^{\text{vert}}S_{\text{belt}}^{n+m} = T_pL_{\mathbf{b}^+(p)}$ , where  $L_{\mathbf{b}^+(p)}$  denotes the Lob containing  $p$ . We define a complex vector bundle  $E \rightarrow S^1 \times D^2$  by  $E = \tilde{u}^*TW \oplus \underline{\mathbb{C}}$  and a real vector subbundle  $F \rightarrow S^1 \times \partial D^2$  of  $E|_{S^1 \times \partial D^2}$  by  $F = (\tilde{u}|_{S^1 \times \partial D^2})^*T^{\text{vert}}S_{\text{belt}}^{n+m} \oplus \underline{\mathbb{R}}$ . Here  $\underline{\mathbb{C}}$  and  $\underline{\mathbb{R}}$  denote the trivial complex and real line bundle, respectively.

We denote by  $E_\theta$  the restriction of  $E$  to  $\{\theta\} \times D^2$ , by  $F_\theta$  the restriction of  $F$  to  $\{\theta\} \times D^2$  and by  $\Gamma(E_\theta, F_\theta)$  the sections of  $E_\theta$  which take values in  $F_\theta$  along  $\partial D^2$ . Similarly, let  $\Gamma(\underline{\mathbb{C}}, \underline{\mathbb{R}})$  denote the sections of the trivial line bundle  $\underline{\mathbb{C}}$  over  $D^2$  with real values at  $\partial D^2$ , and  $D_0$  the standard Cauchy-Riemann operator acting on  $\Gamma(\underline{\mathbb{C}}, \underline{\mathbb{R}})$ . We consider the family of linear Cauchy-Riemann type operators  $D_\theta^+ = D'_{u_\theta} \oplus D_0$  acting on  $\Gamma(E_\theta, F_\theta)$ . This family gives rise to a determinant line bundle  $\mathcal{D}^+ \rightarrow S^1$ . Since  $\det D_0 = \mathbb{R}$ , we have  $\mathcal{D}^+ \cong u_*\mathcal{D}'$ . Therefore studying the orientability of  $\mathcal{D}^+$  is equivalent to studying the orientability of the moduli space.

Being spheres, the Lobes are stably parallelizable and the parameter space  $D_r^m$  is contractible, so  $T^{\text{vert}}S_{\text{belt}}^{n+m}$  is stably trivial. An orthonormal trivialization of  $T^{\text{vert}}S_{\text{belt}}^{n+m} \oplus \underline{\mathbb{R}}$  can be pulled back to an orthonormal trivialization  $\nu_0$  of  $F$ . We can also regard  $\nu_0$  as a unitary trivialization of  $E|_{S^1 \times \partial D^2}$ , because  $F$  is a Lagrangian subbundle of  $E|_{S^1 \times \partial D^2}$ . However  $\nu_0$  does not extend to a unitary trivialization of  $E$ . In fact it does not extend to the meridian disks of  $S^1 \times D^2$ , because  $(E_\theta, F_\theta)$  has Maslov index two. We choose a map  $A: \partial D^2 \rightarrow U(n)$  such that the trivialization  $\nu$  defined as  $\nu(\theta, z) = A(z)^{-1}\nu_0(\theta, z)$  extends to a trivialization of  $E$  over any meridian disk. (Of course the new trivialization is no longer an orthonormal trivialization of  $F$ .) We extend  $\nu$  to a trivialization of  $E$  on a regular neighborhood of  $(S^1 \times \partial D^2) \cup (\{\theta_0\} \times D^2)$  for a fixed  $\theta_0 \in S^1$ . The complement of this neighborhood in  $S^2 \times D^2$  is a ball. We can extend  $\nu$  inside this ball because  $\pi_2(U(n)) = 0$ .

Then  $\nu$  defines an isomorphism  $(E, F) \cong (\underline{\mathbb{C}}^n, F')$ , where  $F'_{(\theta, z)} = A(z)R^n$ . The operators  $D_\theta^+$  become  $D_0^+ + a_\theta$ , where  $D_0^+$  is the standard Cauchy-Riemann operator on  $\Gamma(\underline{\mathbb{C}}^n, F')$  and  $a_\theta \in \Omega^{0,1}(TD^2, E)$ . Since  $a_\theta$  belongs to a contractible space, the loop  $\theta \mapsto D_\theta^+$  can be continuously deformed to a constant loop. Then  $\mathcal{D}'$  is a trivial line bundle. This ends the proof of the orientability of the moduli space  $\widetilde{\mathcal{M}}_{\text{int}}(\widehat{W}, S_B; J)$ .

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