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A HOLOGRAPHIC GLOBAL UNIQUENESS IN PASSIVE IMAGING

by Roman G. Novikov

Abstract. — We consider a radiation solution ψ for the Helmholtz equation in an exterior region in $\mathbb{R}^3$. We show that the restriction of ψ to any ray L in the exterior region is uniquely determined by its imaginary part $\operatorname{Im}\psi$ on an interval of this ray. As a corollary, the restriction of ψ to any plane X in the exterior region is uniquely determined by $\operatorname{Im}\psi$ on an open domain in this plane. These results have holographic prototypes in the recent work Novikov (2024, Proc. Steklov Inst. Math. 325, 218-223). In particular, these and known results imply a holographic type global uniqueness in passive imaging and for the Gelfand-Krein-Levitan inverse problem (from boundary values of the spectral measure in the whole space) in the monochromatic case. Some other surfaces for measurements instead of the planes X are also considered.

Résumé (Une unicité globale holographique en imagerie passive). — Nous considérons une solution de rayonnement ψ pour l'équation de Helmholtz dans une région extérieure de $\mathbb{R}^3$. Nous montrons que la restriction de ψ à tout rayon L de la région extérieure est déterminée de manière unique par sa partie imaginaire $\operatorname{Im}\psi$ sur un intervalle de ce rayon. En corollaire, la restriction de ψ à tout plan X de la région extérieure est déterminée de manière unique par $\operatorname{Im}\psi$ sur un domaine ouvert de ce plan. Ces résultats ont des prototypes holographiques dans l'article récent de Novikov (2024, Proc. Steklov Inst. Math. 325, 218-223). En particulier, ces résultats et des résultats connus impliquent une unicité globale de type holographique en imagerie passive et pour le problème inverse de Gelfand-Krein-Levitan (à partir des valeurs au bord de la mesure spectrale dans l'espace entier) dans le cas monochromatique. D'autres surfaces de mesure que les plans X sont également considérées.

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1. Introduction

We consider the Helmholtz equation

$$-\Delta\psi(x) = \kappa^2\psi(x), \quad x \in \mathcal{U},\ \kappa > 0, \tag{1}$$

where Δ is the Laplacian in x, and $\mathcal{U}$ is an exterior region in $\mathbb{R}^3$ that is $\mathcal{U}$ is an open unbounded connected set in $\mathbb{R}^3$ consisting of all points in exterior of a closed bounded regular surface S (as in [3], [29]).

For equation (1) we consider the radiation solutions ψ that is the solutions satisfying the Sommerfeld's radiation condition

$$|x|\Big(\frac{\partial}{\partial|x|} - i\kappa\Big)\psi(x) \longrightarrow 0 \quad \text{as } |x| \to +\infty, \tag{2}$$

uniformly in $x/|x|$. We assume that $\psi \in C^2$ in the closure of $\mathcal{U}$.

Let

$$L = L_{x_0,\theta} = \{x \in \mathbb{R}^3 :\ x = x(s) = x_0 + s\theta,\ 0 < s < +\infty\}, \quad x_0 \in \mathbb{R}^3,\ \theta \in \mathbb{S}^2, \tag{3}$$

where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$.

In the present work we show that, for any radiation solution ψ and any ray $L \subset \mathcal{U}$, the restriction of ψ to L is uniquely determined by its imaginary part $\operatorname{Im}\psi$ on an arbitrary interval of L; see Theorem 1 in Section 2.

As a corollary, we also obtain that, for any radiation solution ψ and any plane $X \subset \mathcal{U}$, the restriction of ψ to X is uniquely determined by $\operatorname{Im}\psi$ on an arbitrary open domain of X; see Theorem 2 in Section 2.

As a further corollary, we obtain that, for any radiation solution ψ and any plane $X \subset \mathcal{U}$, the solution ψ on the whole $\mathcal{U}$ is also uniquely determined by $\operatorname{Im}\psi$ on an arbitrary open domain of X; see Corollary 1 in Section 2.

These results have holographic prototypes in the recent work [23], where the aforementioned reconstructions of ψ are considered from the intensity $|e^{ikx} + \psi|^2$ in place of $\operatorname{Im}\psi$. Here, e^{ikx} is a plane wave solution of (1), i.e., $k \in \mathbb{R}^3,\ |k| = \kappa$. The results of [23] solve one of old mathematical questions of holography and admit straightforward applications to phaseless inverse scattering.

In the present work our studies are motivated by the Gelfand-Krein-Levitan inverse problem (from boundary values of the spectral measure in the whole space) and by passive imaging; see, e.g., [1], [2], [4], [6], [10], [11], [16], [28].

The Gelfand-Krein-Levitan problem in question (in its fixed energy version in dimension $d = 3$) consists in determining the potential v in the Schrödinger equation

$$-\Delta\psi(x) + v(x)\psi(x) = \kappa^2\psi(x) + \delta(x - y),\ x, y \in \mathbb{R}^3,\ \kappa > 0, \tag{4}$$

from the imaginary part of the outgoing Green function $R_v^+(x, y, \kappa)$ for one κ and all x, y on some part of the boundary of a domain containing the support of v. Here, δ is the Dirac delta function. In this problem $\operatorname{Im} R_v^+$ is related to the spectral measure of the Schrödinger operator $H = -\Delta + v$. More precisely, H admits the following spectral decomposition in $L^2(\mathbb{R}^d)$, at least, for real-valued compactly supported $v \in L^\infty(\mathbb{R}^d)$:

$$H = \int_0^\infty \kappa^2 d\mu_\kappa + \sum_{j=1}^{N} E_j \pi_j, \quad d\mu_\kappa = \frac{2}{\pi} \operatorname{Im} R_v^+(\kappa)\kappa d\kappa, \tag{5}$$

where $d\mu_\kappa$ is the positive part of the spectral measure for H, E_j are nonpositive eigenvalues for H and π_j are orthogonal projectors on corresponding eigenspaces, $R_v^+(\kappa) = (H - \kappa^2 - i0)^{-1}$ is the limiting absorption resolvent for H, whose Schwartz kernel is given by $R_v^+(x, y, \kappa)$; see, e.g., [13, Lem. 14.6.1].

Note that the terminology "Gelfand-Krein-Levitan problem" is not conventional. Our motivation for this terminology is based on the Yu. M. Berezanskii's work [4], which is one of the very first mathematical works on the multidimensional inverse problems for differential equations. According to [4], the problem of finding potential v in the multidimensional Schrödinger equation from boundary values of the spectral measure for some fixed boundary condition was formulated originally by M. G. Krein, I. M. Gelfand and B. M. Levitan at a conference on differential equations in Moscow in 1952. In addition, [4] gives, in particular, uniqueness theorems on such problems including the case of the problem of determining v from boundary values (together with some normal derivatives) of the spectral measure arising in (5) for all real energies on a part of the boundary, at least, for piecewise real-analytic v.

In addition, $R_v^+(x, y, \kappa)$, for fixed y, can be defined as the solution ψ of equation (4) with the radiation condition (2). For more details, see [1], [4] and references therein.

Equation (4) at fixed κ can be also considered as the Helmholtz equation of acoustics or electrodynamics for monochromatic waves, where complex-valued $v(x) = v(x, \kappa)$ is related to the perturbation of the refraction index. In particular, the aforementioned mathematical problem of recovering v from boundary values of $\operatorname{Im} R_v^+$ arises in the framework of passive acoustic tomography (in ultrasonics, ocean acoustics, local helioseismology). In these framework $\operatorname{Im} R_v^+$ is related to cross correlations of wave fields generated by random sources; see, e.g., formula (49) in [11]. For more details, see [1], [2], [6], [11], [27], [28] and references therein.

Note that

$$R_0^+(x, y, \kappa) = \frac{e^{i\kappa|x-y|}}{4\pi|x-y|}, \quad x, y \in \mathbb{R}^3, \tag{6}$$

where R_0^+ is the outgoing Green function for equation (4) with $v \equiv 0$.

Let

$$R_{v,sc}^+(x, y, \kappa) = R_v^+(x, y, \kappa) - R_0^+(x, y, \kappa), \quad x, y \in \mathbb{R}^3. \tag{7}$$

Suppose that

$$\operatorname{supp} v \subset \mathbb{R}^3 \setminus \overline{\mathcal{U}}, \tag{8}$$

where $\overline{\mathcal{U}}$ is the closure of $\mathcal{U}$. Then, in view of the definitions of $R_{v,sc}^+$, R_v^+, and R_0^+, the function $\psi = R_{v,sc}^+(x, y, \kappa)$ is a radiation solution of equation (1) for each $y \in \mathbb{R}^3$.

Therefore, the aforementioned results on recovering a radiation solution ψ from $\operatorname{Im}\psi$ give a reduction of the Gelfand-Krein-Levitan problem (of inverse spectral theory and passive imaging in dimension $d = 3$) to the inverse scattering problem of finding v in (4) from boundary values of R_v^+. Note that studies on the latter problem also go back to [4].

This reduction and known results imply, in particular, that v in (4) is uniquely determined by $\operatorname{Im} R_v^+(x, y, \kappa)$ for one κ and all x, y on an arbitrary open domain $\mathcal{D}$ of X, where $\operatorname{supp} v \subset \Omega$, X is a plane in $\mathbb{R}^3 \smallsetminus \overline{\Omega}$, Ω is an open bounded connected domain in $\mathbb{R}^3$, $\overline{\Omega}$ is the closure of Ω; see Theorem 3 in Section 2. By this result we continue studies of [4] mentioned above and relatively recent studies of [1] and [2].

We also consider other surfaces for measurements instead of the planes X; see Example 1 and Theorems 4 and 5 in Section 2.

The main results of this work are presented in more detail and proved in Sections 2–7.

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2. Main results

In this work our key result is as follows.

Theorem 1. — *Let ψ be a radiation solution of equation* (1) *as in* (2)*. Let L be a ray as in* (3) *such that $L \subset \mathcal{U}$, where $\mathcal{U}$ is the region in* (1)*. Then ψ on L is uniquely determined by $\operatorname{Im}\psi$ on Λ, where Λ is an arbitrary non-empty open interval of L.*

As a corollary, we also get, in particular, the following result.

Theorem 2. — *Let ψ be a radiation solution of equation* (1) *as in* (2)*. Let X be a two-dimensional plane in $\mathbb{R}^3$ such that $X \subset \mathcal{U}$. Then ψ on X is uniquely determined by $\operatorname{Im}\psi$ on $\mathcal{D}$, where $\mathcal{D}$ is an arbitrary non-empty open domain of X.*

Theorem 1 is proved in Section 4 using the Atkinson-Wilcox expansion of [3], [29] for the radiation solutions ψ of equation (1), and a modified version of holographic techniques of [21], [23]. In particular, in this proof we use Proposition 1 of Section 3, which yields a two-point approximation for ψ in terms of $\operatorname{Im}\psi$. This two-point approximation is also of independent interest.

Note that ψ and $\operatorname{Im}\psi$ are real-analytic on $\mathcal{U}$, and, therefore, on L in Theorem 1 and on X in Theorem 2. Because of this analyticity, Theorem 1 reduces to the case when $\Lambda = L$ and Theorem 2 reduces to the case when $\mathcal{D} = X$.

Using this reduction, Theorem 2 is proved as follows. We assume that $\mathcal{D} = X$. Then to determine ψ at an arbitrary $x \in X$, we consider a ray $L = L_{x_0,\theta} \subset X$ such that $x \in L$ and use Theorem 1 for this L.

Corollary 1. — *Under the assumptions of Theorem* 2*, the imaginary part* $\operatorname{Im}\psi$ *on* $\mathcal{D}$*, uniquely determines ψ in the entire region $\mathcal{U}$.*

Corollary 1 follows from Theorem 2, formula (18) recalled in Section 3.3, and analyticity of ψ in $\mathcal{U}$.

Remark 1. — In the one dimensional case, an analog of Theorem 1 follows from a very simple form of one dimensional radiation solutions. In particular, in this case an analog of the two-point approximation of Proposition 1 is exact. The two dimensional case is considered in [17] using results of [14], [18] in place of the three dimensional Atkinson-Wilcox expansion (12) used in the present work. We expect that the case of dimension $d > 3$ is similar to the three-dimensional case if d is odd and is similar to the two-dimensional case if d is even.

Theorems 1 and 2, and Corollary 1 have holographic prototypes in [23]; see Introduction for some comments in this connection.

Using Theorem 2 and known results on direct and inverse scattering we obtain the following global uniqueness theorem for the Gelfand-Krein-Levitan inverse problem mentioned in Introduction.

Theorem 3. — *Let $v \in L^\infty(\mathbb{R}^3)$, $\operatorname{supp} v \subset \Omega$, and $X \subset \mathbb{R}^3 \setminus \overline{\Omega}$, where Ω is an open bounded connected domain in $\mathbb{R}^3$, $\overline{\Omega}$ is the closure of Ω, and X is a two-dimensional plane. Let property (23) hold for fixed $\kappa > 0$ and $R_v^+(x, y, \kappa)$ be the outgoing Green function for equation (4). Then v is uniquely determined by $\operatorname{Im} R_v^+(\cdot, \cdot, \kappa)$ on $\mathcal{D} \times \mathcal{D}$, where $\mathcal{D}$ is an arbitrary non-empty open domain of X.*

In Theorem 3 we do not assume that v is real-valued, but we assume that property (23) formulated in Section 3.4 holds.

Remark 2. — Under the conditions that v is real-valued, $v \in L^\infty(\mathbb{R}^d)$, $\operatorname{supp} v \subseteq \overline{\Omega}$, where Ω is an open bounded domain in $\mathbb{R}^d$ with $\partial\Omega \in C^{2,1}$, $\overline{\Omega} = \Omega \cup \partial\Omega$, $d \geqslant 2$, a local uniqueness for determining v from $\operatorname{Im} R_v^+(\cdot, \cdot, \kappa)$ on $\partial\Omega \times \partial\Omega$ for one κ is proved in [1]. The assumption that v is real-valued is very essential in this proof.

Remark 3. — In connection with applications to helioseismology, it is natural to assume that v is complex-valued, $v \in L^\infty(\mathbb{R}^3)$, $v(x) = \widetilde{v}(|x|)$, $\widetilde{v}(r) = \alpha/r$, $r \geqslant r_0$, for some constants $\alpha \in \mathbb{R}$ and $r_0 > 0$. Let $M_r = \mathbb{S}_r^2 \times \mathbb{S}_r^2$ and $r_2 > r_1 > r_0$, where $\mathbb{S}_r^2$ is defined by (9). Under these assumptions, a global uniqueness for determining v from $\operatorname{Im} R_v^+(\cdot, \cdot, \kappa)$ on $M_{r_1} \cup M_{r_2}$, for fixed κ and nonsingular pair r_1, r_2, is proved in [2].

One can see that Theorem 3 contains a principal progress on the Gelfand-Krein-Levitan inverse problem in comparison with the results of [1], [2] mentioned in Remarks 2 and 3. In comparison with the aforementioned result of [1], Theorem 3 is global and does not assume that v is real-valued. In comparison with the aforementioned result [2], Theorem 3 does not assume that v is spherically symmetric.

Theorem 3 is proved in Section 5. In particular, in this proof we use Proposition 2 of Section 5, which stays that v is uniquely determined by $R_v^+(\cdot, \cdot, \kappa)$ on $X \times X$ for fixed κ. To our knowledge, Proposition 2 is slightly different from the results reported in the literature. Therefore, just in case, for completeness of presentation this proposition is also proved in Section 5.

In connection with other surfaces of measurements instead of the planes X our results are as follows.

The results of Theorems 1 and 2 don't hold for some other curves in place of L and surfaces in place of X. An example is as follows.

Let

$$\mathbb{S}^2_r = \{x \in \mathbb{R}^3 : \ |x| = r\}, \quad r > 0. \tag{9}$$

Example 1. — Let $\psi(x) = G^+(x, \kappa)$, where G^+ is defined by (19). Then ψ is a non-zero radiation solution of equation (1) if $0 \in \mathbb{R}^3 \setminus \overline{\mathcal{U}}$, but $\operatorname{Im} \psi \equiv 0$ on the spheres $\mathbb{S}^2_r$ for $r = n\pi/\kappa$, $n \in \mathbb{N}$.

Nevertheless, the uniqueness results of Theorem 2, Corollary 1, and Theorem 3 remain valid for many interesting surfaces instead of the planes X. Suppose that

$$\begin{aligned} &\mathcal{B} \text{ is an open bounded domain in } \mathbb{R}^3, \\ &Y = \partial\mathcal{B} \text{ is real-analytic and connected.} \end{aligned} \tag{10}$$

In particular, we have the following uniqueness theorems.

Theorem 4. — *Let ψ be a radiation solution of equation* (1) *as in* (2)*. Let Y be a surface as in* (10)*, where κ is not a Dirichlet eigenvalue for $\mathcal{B}$, and $\overline{\mathcal{B}} = \mathcal{B} \cup Y \subset \mathcal{U}$. Then ψ on $\mathcal{U}$ is uniquely determined by $\operatorname{Im} \psi$ on $\mathcal{D}$, where $\mathcal{D}$ is an arbitrary non-empty open domain of Y.*

Theorem 5. — *Let $v \in L^\infty(\mathbb{R}^3)$, $\operatorname{supp} v \subset \mathbb{R}^3 \setminus \overline{\mathcal{U}}$, where $\mathcal{U}$ is as in* (1)*, and property* (23) *holds for fixed $\kappa > 0$. Let Y be a surface as in* (10)*, where κ is not a Dirichlet eigenvalue for $\mathcal{B}$, and $\overline{\mathcal{B}} = \mathcal{B} \cup Y \subset \mathcal{U}$. Then v is uniquely determined by $\operatorname{Im} R^+_v(\cdot, \cdot, \kappa)$ on $\mathcal{D} \times \mathcal{D}$, where $\mathcal{D}$ is an arbitrary non-empty open domain of Y.*

Theorems 4 and 5 are proved in Sections 6 and 7. Note that the determinations in Theorems 1, 2, 3 and especially in Theorems 4 and 5 include analytic continuations. Studies on more stable reconstructions appropriate for numerical implementation will be continued elsewhere. In this respect one can use, in particular, results of Section 3.2.

3. Preliminaries

3.1. The Atkinson-Wilcox expansion. — Let

$$B_r = \{x \in \mathbb{R}^3 : \ |x| < r\}, \quad r > 0. \tag{11}$$

If ψ is a radiation solution of equation (1) and $\mathbb{R}^3 \setminus B_r \subset \mathcal{U}$, then the following Atkinson-Wilcox expansion holds:

$$\psi(x) = \frac{e^{i\kappa|x|}}{|x|} \sum_{j=1}^{\infty} \frac{f_j(\theta)}{|x|^{j-1}} \quad \text{for } x \in \mathbb{R}^3 \setminus B_r, \ \theta = \frac{x}{|x|}, \tag{12}$$

where the series converges absolutely and uniformly; see [3], [29].

3.2. A two-point approximation for ψ. — Let

$$I(x) = |x| \operatorname{Im} \psi(x), \quad x \in \mathcal{U}, \tag{13}$$

where ψ is a radiation solution of equation (1).

We have that

$$2iI(x) = e^{i\kappa|x|} f_1(\theta) - e^{-i\kappa|x|}\overline{f_1(\theta)} + O(|x|^{-1}) \quad \text{as } |x| \to +\infty, \tag{14}$$

uniformly in $\theta = x/|x|$, where f_1 is the leading coefficient in (12).

Proposition 1. — *Let ψ be a radiation solution of equation* (1)*. Then*

$$\begin{aligned} &f_1(\theta) = \frac{1}{\sin(\kappa\tau)}\big(-e^{-i\kappa|y|}I(x) + e^{-i\kappa|x|}I(y) + O(|x|^{-1})\big), \\ &x, y \in L_{x_0,\theta}, \quad x_0 = 0,\ y = x + \tau\theta, \quad \theta \in \mathbb{S}^2, \quad \tau > 0, \end{aligned} \tag{15}$$

uniformly in θ, where f_1 is the coefficient in (14)*, I is defined by* (13)*, $L_{x_0,\theta}$ is the ray defined by* (3)*, and $\sin(\kappa\tau) \neq 0$ for fixed τ.*

Formula (15) is a two-point approximation for f_1 and together with (12) also gives a two-point approximation for ψ in terms of I. For phaseless inverse scattering and holography formulas of such a type go back to [21] (see also [25], [24], [23]).

Remark 4. — If an arbitrary function I on $L_{0,\theta}$ satisfies (14), then formula (15) holds, for fixed $\theta \in \mathbb{S}^2$.

We obtain (15) from the system of equations for f_1 and $\overline{f_1}$:

$$\begin{aligned} e^{i\kappa|x|} f_1(\theta) - e^{-i\kappa|x|}\overline{f_1(\theta)} &= 2iI(x) + O(|x|^{-1}), \\ e^{i\kappa|y|} f_1(\theta) - e^{-i\kappa|y|}\overline{f_1(\theta)} &= 2iI(y) + O(|y|^{-1}), \end{aligned} \tag{16}$$

where x, y are as in (15). In particular, we use that $|y| = |x| + \tau$ and that

$$D = 2i\sin(\kappa\tau), \tag{17}$$

where D is the determinant of system (16).

In turn, (16) follows from (14).

3.3. A Green type formula. — The following formula holds:

$$\psi(x) = 2\int_X \frac{\partial G^+(x-y,\kappa)}{\partial \nu_y}\psi(y)dy, \quad x \in V_X, \tag{18}$$

$$G^+(x,\kappa) = -\frac{e^{i\kappa|x|}}{4\pi|x|}, \quad x \in \mathbb{R}^3, \tag{19}$$

where ψ is a radiation solution of equation (1), X and V_X are plane and open half-space in $\mathcal{U}$, where X is the boundary of V_X, ν is the outward normal to X relative to V_X; see, for example, [5, Eq. (5.84)].

Recall that $R_0^+(x,y,\kappa) = -G^+(x-y,\kappa)$, where R_0^+ is the outgoing Green function for equation (4) with $v \equiv 0$.

For completeness of presentation note that formula (18) follows from the formula

$$\begin{aligned}(20)\qquad &\psi(x)=\int_X \frac{\partial G_X^+(x,y,\kappa)}{\partial\nu_y}\psi(y)dy,\quad x\in V_X,\\ &G_X^+(x,y,\kappa)=G^+(x-y,\kappa)-G^+(x-y^*,\kappa),\quad x,y\in V_X\cup X,\end{aligned}$$

where y^* is symmetric to y with respect to X. The point is that G_X^+ is the Green function for the Helmholtz operator $\Delta+\kappa^2$ in V_X with Dirichlet boundary condition on X and Sommerfeld radiation condition at infinity.

3.4. Some facts of direct scattering. — We consider equation (4) assuming for simplicity that

$$\begin{aligned}(21)\qquad &v\in L^\infty(\Omega),\quad v\equiv 0 \text{ on } \mathbb{R}^3\smallsetminus\overline{\Omega}\\ &\Omega \text{ is an open bounded connected domain in } \mathbb{R}^3.\end{aligned}$$

The outgoing Green function R_v^+ for equation (4) satisfies the integral equation

$$(22)\qquad R_v^+(x,y,\kappa)=-G^+(x-y,\kappa)+\int_\Omega G^+(x-z,\kappa)v(z)R_v^+(z,y,\kappa)dz,$$

where $x,y\in\mathbb{R}^3$, G^+ is given by (19).

Actually, in addition to (21), we assume that, for fixed $\kappa>0$,

$$(23)\qquad \text{equation (22) is uniquely solvable for } R_v^+(\cdot,y,\kappa)\in L^2(\Omega).$$

In particular, it is known that if v satisfies (21) and is real-valued (or $\operatorname{Im} v\leqslant 0$), then (23) is fulfilled automatically; see, for example, [7].

We also consider the scattering wave functions ψ^+ for the homogeneous equation (4) (i.e., without δ):

$$(24)\qquad \psi^+=\psi^+(x,\theta,\kappa)=e^{i\kappa\theta x}+\psi_{sc}^+(x,\theta,\kappa),\quad x\in\mathbb{R}^3,\ \theta\in\mathbb{S}^2,$$

where ψ_{sc}^+ satisfies the radiation condition (2) at fixed θ.

The following formulas hold:

$$(25)\qquad R_v^+(x,y,\kappa)=R_v^+(y,x,\kappa),\quad x,y\in\mathbb{R}^3;$$

$$(26)\qquad R_v^+(x,y,\kappa)=\frac{e^{i\kappa|x|}}{4\pi|x|}\,\psi^+\Big(y,-\frac{x}{|x|},\kappa\Big)+O\Big(\frac{1}{|x|^2}\Big)\quad\text{as } |x|\to+\infty \text{ at fixed } y;$$

$$(27)\qquad \psi_{sc}^+(x,\theta,\kappa)=\frac{e^{i\kappa|x|}}{|x|}\,A\Big(\theta,\frac{x}{|x|},\kappa\Big)+O\Big(\frac{1}{|x|^2}\Big)\quad\text{as } |x|\to+\infty \text{ at fixed } \theta,$$

where A arising in (27) is the scattering amplitude for the homogeneous equation (4) and is defined on $\mathbb{S}^2\times\mathbb{S}^2$ at fixed κ.

In view of (6), (7), (24)–(26), we also have that

$$(28)\qquad R_{v,sc}^+(x,y,\kappa)=R_{v,sc}^+(y,x,\kappa),\ x,y\in\mathbb{R}^3;$$

$$(29)\qquad R_{v,sc}^+(x,y,\kappa)=\frac{e^{i\kappa|x|}}{4\pi|x|}\,\psi_{sc}^+\Big(y,-\frac{x}{|x|},\kappa\Big)+O\Big(\frac{1}{|x|^2}\Big)\text{ as } |x|\to+\infty \text{ at fixed } y.$$

In connection with aforementioned facts concerning R_v^+ and ψ^+ see, e.g., [9, Chap. IV, §1].

REMARK 5. — It is well known that, under assumptions (21), (23), the scattering amplitude $A = A(\cdot,\cdot,\kappa)$ is real analytic on $\mathbb{S}^2 \times \mathbb{S}^2$.

4. Proof of Theorem 1

4.1. CASE $L \subseteq L_{0,\theta}$. — First, we give the proof for the case when $L \subseteq L_{0,\theta}$. In this case it is essentially sufficient to prove that $\operatorname{Im}\psi$ on L uniquely determines $f_j(\theta)$ in (12) for all j. Such a determination is presented below in this subsection. The rest follows from the convergence of the series in (12) and analyticity of ψ and $\operatorname{Im}\psi$ on L.

The determination of f_1 follows from (15). Suppose that $f_1,\dots,f_n$ are determined, then the determination of f_{n+1} is as follows. Let

$$\psi_n(x) = \frac{e^{i\kappa|x|}}{|x|}\sum_{j=1}^{n}\frac{f_j(\theta)}{|x|^{j-1}}, \quad \text{where } \theta = \frac{x}{|x|}, \tag{30}$$

$$I_n(x) = |x|\operatorname{Im}\psi_n, \tag{31}$$

$$J_n(x) = |x|^n(I(x) - I_n(x)), \tag{32}$$

where x is as in (12), $I(x)$ is defined by (13).

We have that

$$2iI(x) = 2iI_n(x) + \frac{e^{i\kappa|x|}}{|x|^n}f_{n+1}(\theta) - \frac{e^{-i\kappa|x|}}{|x|^n}\overline{f_{n+1}(\theta)} + O(|x|^{-n-1}), \tag{33}$$

$$2iJ_n(x) = e^{i\kappa|x|}f_{n+1}(\theta) - e^{-i\kappa|x|}\overline{f_{n+1}(\theta)} + O(|x|^{-1}), \tag{34}$$

as $|x| \to +\infty$, where I is defined by (13).

Due to (34) and Remark 4, we get

$$\begin{aligned} &f_{n+1}(\theta) = \frac{1}{\sin(\kappa\tau)}\big(-e^{-i\kappa|y|}J_n(x) + e^{-i\kappa|x|}J_n(y) + O(|x|^{-1})\big),\\ &x, y \in L_{x_0,\theta}, \quad x_0 = 0, \quad y = x + \tau\theta, \quad \theta \in \mathbb{S}^2, \quad \tau > 0, \end{aligned} \tag{35}$$

assuming that $\sin(\kappa\tau) \neq 0$ for fixed τ (where the parameter τ can be always fixed in such a way for fixed $\kappa > 0$).

Formulas (13), (30)–(32) and (35) determine f_{n+1}, give the step of induction for finding all f_j, and complete the proof of Theorem 1 for the case $L \subseteq L_{0,\theta}$.

4.2. GENERAL CASE. — The general case reduces to the case of Section 4.1 by the change of variables

$$x' = x - q \quad \text{for some fixed } q \in \mathbb{R}^3 \text{ such that } L \subseteq L_{q,\theta}. \tag{36}$$

In the new variables $x' \in \mathcal{U}' = \mathcal{U} - q$, we have that:

$$\psi = \frac{e^{i\kappa|x'|}}{|x'|}\sum_{j=1}^{\infty}\frac{f'_j(\theta)}{|x'|^{j-1}} \quad \text{for } x' \in \mathbb{R}^3 \smallsetminus B_{r'},\ \theta = \frac{x'}{|x'|}, \tag{37}$$

for some new f'_j, where r' is such that $\mathbb{R}^3 \smallsetminus B_{r'} \subset \mathcal{U}'$;

$$L \subseteq L_{q,\theta} = L_{0,\theta}. \tag{38}$$

In addition, the series in (37) converges absolutely and uniformly.

In view of (37), (38), we complete the proof of Theorem 1 by repeating the proof of Section 4.1.

Remark 6. — Our proof of Theorem 1 has a holographic prototype in [23]. Additional formulas for finding f_j from $\operatorname{Im}\psi$ on L can be obtained proceeding also from the approaches of [22], [25], [26].

5. Proof of Theorem 3

Under our assumptions on v, Ω, X, and κ, we have, in particular, that

$$R^+_{v,sc}(\cdot, y, \kappa) \text{ is real-analytic on } X \text{ for fixed } y \in X, \tag{39}$$

$$R^+_{v,sc}(x, \cdot, \kappa) \text{ is real-analytic on } X \text{ for fixed } x \in X, \tag{40}$$

where $R^+_{v,sc}$ is defined by (6), (7). Here, we use that $R^+_{v,sc}(x, y, \kappa)$ satisfies the homogeneous equation (4) and, therefore, is real-analytic outside of $\operatorname{supp} v$, and that symmetry (28) holds.

In view of (6), (7), (39), (40), Theorem 3 reduces to the case when $\mathcal{D} = X$. In turn, Theorem 3 with $\mathcal{D} = X$ follows from formulas (6), (7) and from Lemma 1 and Proposition 2 given below.

Lemma 1. — *Under the conditions of Theorem* 3, $\operatorname{Im} R^+_{v,sc}(\cdot, y, \kappa)$ *on* X *uniquely determines* $R^+_{v,sc}(\cdot, y, \kappa)$ *on* X, *where* $y \in \mathbb{R}^3$, *and* $\operatorname{Im} R^+_{v,sc}(\cdot, \cdot, \kappa)$ *on* $X \times X$ *uniquely determines* $R^+_{v,sc}(\cdot, \cdot, \kappa)$ *on* $X \times X$.

Lemma 1 follows from Theorem 2 with $\mathcal{U}$ such that $\overline{\Omega} \subset \mathbb{R}^3 \smallsetminus \overline{\mathcal{U}}$, $X \subset \mathcal{U}$, and the property that $\psi = R^+_{v,sc}(x, y, \kappa)$ is a radiation solution of equation (1) for each $y \in \mathbb{R}^3$.

Proposition 2. — *Under the conditions of Theorem* 3, $R^+_{v,sc}(\cdot, \cdot, \kappa)$ *on* $X \times X$ *uniquely determines* v *(for fixed* κ*).*

Proposition 2 is proved as follows (for example). First,

$$\begin{aligned} &\psi = R^+_{v,sc}(\cdot, x', \kappa) \text{ on } X \text{ uniquely determines } \psi = R^+_{v,sc}(\cdot, x', \kappa) \text{ on } V_X\\ &\text{via formula (18), for each fixed } x' \in X. \end{aligned} \tag{41}$$

Second,

$$\begin{aligned} &R^+_{v,sc}(\cdot, x', \kappa) \text{ on } V_X \cup X \text{ uniquely determines } \psi^+_{sc}(x', \theta, \kappa) \text{ for } \theta \in \Theta^+_X\\ &\text{via formula (29), for each fixed } x' \in X, \end{aligned} \tag{42}$$

where

$$\Theta^\pm_X = \{\theta \in \mathbb{S}^2 : \ \pm\theta\nu \geqslant 0\}, \tag{43}$$

where ν is the outward normal to X relative to V_X considered as the interior of X.

Third,

$$\begin{aligned} &\psi = \psi^+_{sc}(\cdot, \theta, \kappa) \text{ on } X \text{ uniquely determines } \psi = \psi^+_{sc}(\cdot, \theta, \kappa) \text{ on } V_X\\ &\text{via formula (18).} \end{aligned} \tag{44}$$

Fourth,

$$\psi^+_{sc}(\cdot,\theta,\kappa) \text{ on } V_X\cup X \text{ uniquely determines } A(\theta,\theta',\kappa) \text{ for } \theta'\in\Theta^-_X \text{ via formula (27), for each fixed } \theta\in\mathbb{S}^2, \tag{45}$$

where Θ^-_X is defined in (43).

Therefore, we get that

$$R^+_{v,sc}(\cdot,\cdot,\kappa) \text{ on } X\times X \text{ uniquely determines } A(\cdot,\cdot,\kappa) \text{ on } \Theta^+_X\times\Theta^-_X \text{ via (41), (42), (44), (45).} \tag{46}$$

In addition,

$$A(\cdot,\cdot,\kappa) \text{ on } \Theta^+_X\times\Theta^-_X \text{ uniquely determines } A(\cdot,\cdot,\kappa) \text{ on } \mathbb{S}^2\times\mathbb{S}^2 \text{ by analyticity,} \tag{47}$$

in view of Remark 5.

Finally, Proposition 2 follows from (46), (47) and the result of [19] (see also [20], [12], [8]) that, under assumptions (21), (23), the scattering amplitude A at fixed κ uniquely determines v. In this result the assumption that v is real-valued or $\operatorname{Im} v\leqslant 0$ is not essential and can be replaced by assumption (23).

This completes the proofs of Proposition 2 and Theorem 3.

6. Proof of Theorem 4

Under the assumptions of Theorem 4, the functions ψ and $\operatorname{Im}\psi$ are real-analytic in $\mathcal{U}$, in general, and on Y, in particular. Therefore, $\operatorname{Im}\psi$ on $\mathcal{D}$ uniquely determines $\operatorname{Im}\psi$ on Y by analytic continuation, taking also into account that Y is real-analytic and connected. In turn, $\operatorname{Im}\psi$ on Y uniquely determines $\operatorname{Im}\psi$ in $\mathcal{B}$ by solving the Dirichlet problem for the Helmholtz equation. In turn, $\operatorname{Im}\psi$ in $\mathcal{B}$ uniquely determines $\operatorname{Im}\psi$ in $\mathcal{U}$ by analytic continuation. Finally, $\operatorname{Im}\psi$ in $\mathcal{U}$ uniquely determines ψ in $\mathcal{U}$, in view of Corollary 1 in Section 2.

This completes the proof of Theorem 4.

7. Proof of Theorem 5

Recall that $\operatorname{Im} R^+_v(\cdot,\cdot,\kappa)$ on $\mathcal{D}\times\mathcal{D}$ uniquely determines $\operatorname{Im} R^+_{v,sc}(\cdot,\cdot,\kappa)$ on $\mathcal{D}\times\mathcal{D}$, in view of (6), (7). Under the assumptions of Theorem 5, we have, in particular, that

$$\psi = R^+_{v,sc}(\cdot,y,\kappa) \text{ is a radiation solution of equation (1)},\ y\in\mathbb{R}^3. \tag{48}$$

Therefore, due to Theorem 4, $\operatorname{Im} R^+_{v,sc}(\cdot,y,\kappa)$ on $\mathcal{D}$ uniquely determines $R^+_{v,sc}(\cdot,y,\kappa)$ in $\mathcal{U}$.

Similarly, $\operatorname{Im} R^+_{v,sc}(x,\cdot,\kappa)$ on $\mathcal{D}$ uniquely determines $R^+_{v,sc}(x,\cdot,\kappa)$ in $\mathcal{U}$, due to (28), where $x\in\mathbb{R}^3$. Therefore, $\operatorname{Im} R^+_{v,sc}(\cdot,\cdot,\kappa)$ on $\mathcal{D}\times\mathcal{D}$ uniquely determines $R^+_{v,sc}(\cdot,\cdot,\kappa)$ on $\mathcal{U}\times\mathcal{U}$. Finally, $R^+_{v,sc}(\cdot,\cdot,\kappa)$ on $\mathcal{U}\times\mathcal{U}$ uniquely determines v by different ways.

For example: $R^+_{v,sc}(\cdot,\cdot,\kappa)$ on $\mathcal{U}\times\mathcal{U}$ uniquely determines $A(\cdot,\cdot,\kappa)$ on $\mathbb{S}^2\times\mathbb{S}^2$, in view of (27), (29); A at fixed κ uniquely determines v as recalled at the end of proof of Theorem 3.

This completes the proof of Theorem 5.

Remark 7. — Formulas relating R_v^+ on $\mathbb{S}_r^2 \times \mathbb{S}_r^2$ and A on $\mathbb{S}^2 \times \mathbb{S}^2$ at fixed κ, where $v(x) = 0$ for $|x| > r$, $\mathbb{S}_r^2$ is defined by (9), were given for the first time in [4].

Remark 8. — The case is also of interest when the boundary Y in (10) is not connected but consists of two disjoint connected components Y_1 and Y_2, where $Y_1 = \partial\mathcal{B}_1$, $Y_2 = \partial\mathcal{B}_2$ and $\mathcal{B}_1$, $\mathcal{B}_2$ are open bounded domains such that $\mathbb{R}^3 \smallsetminus \mathcal{U} \subset \mathcal{B}_1 \subset \mathcal{B}_2$. In this case Theorem 4 is valid with $\mathcal{D}$ replaced by $\mathcal{D}_1 \cup \mathcal{D}_2$, whereas Theorem 5 is valid with $\mathcal{D} \times \mathcal{D}$ replaced by $(\mathcal{D}_1 \cup \mathcal{D}_2) \times \mathcal{D}_1$ (as well as with $\mathcal{D} \times \mathcal{D}$ replaced by $(\mathcal{D}_1 \cup \mathcal{D}_2) \times \mathcal{D}_2$, where $\mathcal{D}_1$, $\mathcal{D}_2$ are arbitrary non-empty open domains of Y_1 and Y_2, respectively. It is of interest to compare the later result with Theorem 4.3 in the recent thesis [15], which gives uniqueness for monochromatic passive imaging from cross correlations on $(Y_1 \cup Y_2) \times (Y_1 \cup Y_2)$. These comparisons may lead to further important results.

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