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SHARP BOUNDS ON THE HEIGHT OF K-SEMISTABLE FANO VARIETIES II, THE LOG CASE

BY ROLF ANDREASSON & ROBERT J. BERMAN

ABSTRACT. — In our previous work we conjectured—inspired by an algebro-geometric result of Fujita—that the height of an arithmetic Fano variety $\mathcal{X}$ of relative dimension n is maximal when $\mathcal{X}$ is the projective space $\mathbb{P}^n_{\mathbb{Z}}$ over the integers, endowed with the Fubini-Study metric, if the corresponding complex Fano variety is K-semistable. In this work the conjecture is settled for diagonal hypersurfaces in $\mathbb{P}^{n+1}_{\mathbb{Z}}$. The proof is based on a logarithmic extension of our previous conjecture, of independent interest, which is established for toric log Fano varieties of relative dimension at most three, hyperplane arrangements on $\mathbb{P}^n_{\mathbb{Z}}$, as well as for general arithmetic orbifold Fano surfaces.

RÉSUMÉ (Bornes optimales pour la hauteur des variétés de Fano K-semi-stables II : le cas logarithmique)

Dans un travail antérieur, nous avons conjecturé — en nous inspirant d'un résultat algébro-géométrique de Fujita — que la hauteur d'une variété de Fano arithmétique $\mathcal{X}$ de dimension relative n est maximale lorsque $\mathcal{X}$ est l'espace projectif $\mathbb{P}^n_{\mathbb{Z}}$ sur les entiers, muni de la métrique de Fubini-Study, à condition que la variété de Fano complexe correspondante soit K-semi-stable. Dans ce travail, nous démontrons cette conjecture pour les hypersurfaces diagonales dans $\mathbb{P}^{n+1}_{\mathbb{Z}}$. La démonstration repose sur une extension logarithmique de notre conjecture précédente — d'un intérêt indépendant — que nous établissons pour les variétés de Fano logarithmiques toriques de dimension relative au plus 3, les arrangements d'hyperplans dans $\mathbb{P}^n_{\mathbb{Z}}$, ainsi que pour les surfaces de Fano arithmétiques orbifoldes générales.

CONTENTS

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1. Introduction

This is a sequel to [3], where a conjectural arithmetic analog of Fujita's sharp bound for the degree (volume) of K-semistable Fano varieties over $\mathbb{C}$ from [17] was proposed, concerning arithmetic Fano varieties $\mathcal{X}$. The case when $\mathcal{X}$ is the canonical integral model of a complex toric Fano variety X was settled in [3], when the relative dimension n is at most six (the extension to any n is conditioned on a conjectural gap hypothesis for the algebro-geometric degree). Here we will, in particular, show that the conjecture introduced in [3] holds for any diagonal Fano hypersurface $\mathcal{X}$ in $\mathbb{P}^{n+1}_{\mathbb{Z}}$ (see Section 1.1.3 below). The proof is based on the following extension of the conjecture in [3] to the logarithmic setting, which is the main focus of the present work:

Conjecture 1. — *Let $(\mathcal{X}, \mathcal{D})$ be an arithmetic log Fano variety and denote by (X, Δ) is complexification, which defines a complex log Fano variety. Then the following inequality of arithmetic intersection numbers holds for any volume-normalized continuous metric on $-(K_X + \Delta)$ with positive curvature current if (X, Δ) is K-semistable:*

$$(\overline{-\mathcal{K}_{(\mathcal{X},\mathcal{D})}})^{n+1} \leqslant (\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1},$$

where $-K_{\mathbb{P}^n_{\mathbb{C}}}$ is endowed with the volume-normalized Fubini-Study metric. Moreover, if $\mathcal{X}$ is normal equality holds if and only if $(\mathcal{X}, \mathcal{D}) = (\mathbb{P}^n_{\mathbb{Z}}, 0)$ and the metric is Kähler-Einstein, i.e., coincides with the Fubini-Study metric up to the action of an automorphism of $\mathbb{P}^n_{\mathbb{C}}$.

By definition, an *arithmetic log Fano variety* $(\mathcal{X}, \mathcal{D})$ is a projective flat scheme $\mathcal{X}$ over $\mathbb{Z}$, such that $\mathcal{X}$ is reduced and satisfies Serre's conditions S_2 (e.g., $\mathcal{X}$ normal), together with an effective $\mathbb{Q}$-divisor $\mathcal{D}$ on $\mathcal{X}$ such that

$$-\mathcal{K}_{(\mathcal{X},\mathcal{D})} := -(\mathcal{K}_{\mathcal{X}} + \mathcal{D})$$

defines a relatively ample $\mathbb{Q}$-line bundle, where $\mathcal{K}_{\mathcal{X}}$ denotes the relative canonical divisor on $\mathcal{X}$. We also assume that the corresponding complex variety X is normal. The complexification (X, Δ) of $(\mathcal{X}, \mathcal{D})$ thus defines a complex log Fano variety and we shall say that $(\mathcal{X}, \mathcal{D})$ is an *integral model* of (X, Δ) (see Section 2.2 for more background on this setup).

Following standard procedure, we denote by $\overline{\mathcal{L}}$ a metrized line bundle, i.e., a line bundle $\mathcal{L}$ on $\mathcal{X}$ endowed with an Hermitian metric over the complex points X of $\mathcal{X}$. Arithmetic intersection numbers of metrized line bundles were introduced by Gillet-Soulé in the context of Arakelov geometry [8]. The top arithmetic intersection number $\overline{\mathcal{L}}^{n+1}$ of $\overline{\mathcal{L}}$ is called the *height* of $\overline{\mathcal{L}}$. The height of $\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}}$ with respect to the volume-normalized Fubini-Study metric, appearing in the previous conjecture, is explicitly given by the following formula [1, Lem. 3.6], which, essentially, goes back to [20, §5.4]:

$$(\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1} = \frac{1}{2}(n+1)^{n+1}\bigg((n+1)\sum_{k=1}^{n} k^{-1} - n + \log(\pi^n/n!)\bigg). \tag{1.1}$$

As for the notion of K-stability it originally appeared in the context of the Yau-Tian-Donaldson conjecture for Fano manifolds X (see the survey [47] for recent developments, including connections to moduli spaces and the minimal model program in birational geometry). By [26] and [27, Th. 1.6] a log Fano variety (X,Δ) over $\mathbb{C}$ is *K-polystable* (which is a slightly stronger condition than K-semistability) if and only if it admits a *log Kähler-Einstein metric*, i.e., a locally bounded metric on $-(K_X+\Delta)$, whose curvature current ω induces a Kähler metric with constant Ricci curvature on the complement of Δ in the regular locus of X. After volume-normalization, any log Kähler-Einstein for (X,Δ) maximizes the height $(\overline{-\mathcal{K}_{(\mathcal{X},\mathcal{D})}})^{n+1}$ among all volume-normalized locally bounded metrics on $-(K_X+\Delta)$ with positive curvature (as shown precisely as in the case that $\mathcal{D}=0$ considered in [3, §2.3]). When (X,Δ) is log smooth any log Kähler-Einstein metric has conical singularities along Δ (see Section 2.1.2)

The K-semistability of (X,Δ) implies that (X,Δ) is Kawamata log terminal (klt) in the usual sense of birational geometry (see Remark 7). An important class of klt log Fano varieties (X,Δ) is provided by (smooth) *Fano orbifolds*, where the coefficients of Δ are of the form $(1-1/m_i)$ for positive integers m_i. Diophantine aspects of Fano orbifolds have recently been explored in a number of works, building on Campana's program [12] and its developments by Abramovich [1] (see [43] for a very recent survey). In particular, a logarithmic generalization of the Manin-Peyre conjecture for the density of rational points of bounded height on Fano varieties is proposed in [39], which, for example, is addressed for log Fano hyperplane arrangements and toric varieties in [10] and [38], respectively. See [4] for relations between height bounds, K-stability and the Manin-Peyre conjecture.

1.1. Main results

1.1.1. *Toric log Fano varieties.* — We first consider the case when $(\mathcal{X},\mathcal{D})$ is the canonical integral model of a complex toric log Fano variety (X,Δ) (see [28, §2] and [11, Def. 3.5.6]). One advantage of the logarithmic setup is that on any given toric Fano variety X there exist an infinite number of toric $\mathbb{Q}-$divisors D such that $-(K_X+\Delta)$ is a K-semistable log Fano variety. Building on [1], where the case when $D=0$ was considered, we show:

Theorem 2. — *Let $(\mathcal{X},\mathcal{D})$ be the canonical integral model of a complex K-semistable toric log Fano variety (X,Δ). Conjecture 1 holds for $(\mathcal{X},\mathcal{D})$ under anyone of the following conditions:*

– *$n\leqslant 3$ and X is $\mathbb{Q}-$factorial (equivalently, X has at worst abelian quotient singularities);*
– *X is not Gorenstein or has some abelian quotient singularity.*

The starting point of the proof is the bound

$$\frac{(\overline{-\mathcal{K}_{(\mathcal{X},\mathcal{D})}})^{n+1}}{(n+1)!}\leqslant\frac{1}{2}\operatorname{vol}(X,\Delta)\log\Big(\frac{(2\pi^2)^n}{\operatorname{vol}(X,\Delta)}\Big),\quad \operatorname{vol}(X,\Delta):=\frac{-(K_X+\Delta)^n}{n!},\tag{1.2}$$

shown precisely as in the case when $\Delta = 0$, considered in [1]. For $X = \mathbb{P}^n$ the previous theorem is verified by an explicit calculation. In the remaining case, $X \neq \mathbb{P}^n$, the bound in Conjecture 1 follows, just as in [1], from combining the bound (1.2) with the following logarithmic analog of the "gap hypothesis" introduced in [1]:

$$\operatorname{vol}(X, \Delta) \leqslant \operatorname{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1) \tag{1.3}$$

for any K-semistable n-dimensional log Fano variety (X, Δ) such that $X \neq \mathbb{P}^n$.(1) In the case that X is singular the logarithmic gap hypothesis does hold in any dimension, just as in [1]. In the non-singular case there is, for any dimension, only a finite number of toric Fano varieties X. For $n \leqslant 6$ these appear in the database [31], which, as observed in [1], settles the gap hypothesis for $n \leqslant 6$, when $\Delta = 0$. However, in the present case there is for any given toric variety X an infinite number of toric divisors Δ on X such that (X, Δ) is a K-semistable Fano variety. In order to establish the logarithmic gap-hypothesis (1.3) we thus introduce the following invariant of a Fano manifold X:

$$S(X) := \sup_{\Delta} \{\operatorname{vol}(X, \Delta) : (X, \Delta)\, \text{K-semistable log Fano}\}$$

and show, by solving the corresponding optimization problem, that

$$S(X) \leqslant \operatorname{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)$$

when $X \neq \mathbb{P}^n$ and $n \leqslant 3$.

The invariant $S(X)$ and the corresponding maximizers Δ appear to be of independent interest in Kähler geometry. This is illustrated by some examples in Section 3.1, where we make contact with a rigidity property of the corresponding log Kähler-Einstein metric, first exhibited in [40].

1.1.2. *Hyperplane arrangements*. — We next turn to the case when $\mathcal{X}$ is the projective space over the integers, $\mathcal{X} = \mathbb{P}^n_{\mathbb{Z}}$ and $\mathcal{D}$ is a hyperplane arrangement, i.e., its irreducible components are hyperplanes.

Theorem 3. — *Conjecture 1 holds when $\mathcal{X} = \mathbb{P}^n_{\mathbb{Z}}$ and $\mathcal{D}$ is a hyperplane arrangement with simple normal crossings.*

The proof employs a convexity argument to reduce the problem to the case when $\mathcal{D}$ is toric, which is covered by Theorem 9. The argument leverages the explicit characterization of K-semistable hyperplane arrangements established in [18] and yields the following explicit bound:

$$\frac{(\overline{-\mathcal{K}_{(\mathcal{X},\mathcal{D})}})^{n+1}}{(n+1)!} \leqslant \frac{1}{2} \operatorname{vol}(X, \Delta) \log\Big(\frac{(n+1)^n e^{2a_n}}{(n+1)! \operatorname{vol}(X, \Delta)}\Big), \quad a_n = \frac{(\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1}}{(n+1)^{n+1}}, \tag{1.4}$$

with equality if and only if $\mathcal{D}$ is toric.

(1)See [3, §3.2.1] for a comparison between the gap hypothesis and the ODP-conjecture in [41] (very recently settled in the toric case [30]), which, however, yields a weaker inequality than the gap hypothesis in our setup.

1.1.3. *Application to diagonal hypersurfaces*. — Given a positive integer d and integers a_i, consider the diagonal hypersurface $\mathcal{X}_a$ of degree d in $\mathbb{P}^{n+1}_{\mathbb{Z}}$ cut out by the homogeneous polynomial $\sum_{i=0}^{n+1} a_i x_0^d$. The corresponding complex variety X_a is Fano if and only if $d \leqslant (n+1)$ and is always K-polystable (and, in particular, K-semistable); see, for example, [48] for an algebraic proof. Using the results stated in the previous two sections we will establish Conjecture 1 for $\mathcal{X}_a$, endowed with the trivial divisor 0:

Theorem 4. — *Conjecture* 1 *holds for any diagonal hypersurface $\mathcal{X}_a$ which is Fano (i.e., $d \leqslant n+1$) when the divisor $\mathcal{D}$ is trivial. More precisely,*

$$(\overline{-\mathcal{K}_{\mathcal{X}_a}})^{n+1} \leqslant (\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1} + (1-d)(n+2-d)^n \sum_{i=0}^{n+1} \log|a_i|,$$

and the inequality is strict if $d \geqslant 2$.

Note that the schemes $\mathcal{X}_a$ are mutually non-isomorphic over $\mathbb{Z}$, for any given degree d of at least two. In fact, in general, they are not even isomorphic over $\mathbb{Q}$. The proof of the previous theorem is first reduced to the case of a Fermat hypersurface, i.e., the case when $a_i = 1$. Expressing $\mathcal{X}_a$ as a Galois cover of $\mathbb{P}^n_{\mathbb{Z}}$ the estimate (1.4) can then be applied with $\mathcal{X} = \mathbb{P}^n_{\mathbb{Z}}$ and Δ the corresponding branching divisor, which reduces the problem to a simple toric case.

We recall that the Manin-Peyre conjecture has been settled for Fano hypersurfaces over $\mathbb{Q}$ of sufficiently small degree [35, 36]. In particular, there is an extensive literature on the diagonal case, in connection to Waring's classical problem, where the degree bound has been improved [37] (see also [7] and [46, Th. 15.6] for general number fields).

1.1.4. *Arithmetic log surfaces*. — Consider a polarized arithmetic log surface $(\mathcal{X}, \mathcal{D}; \mathcal{L})$, i.e., an arithmetic log surface $(\mathcal{X}, \mathcal{D})$ endowed with a relatively ample line bundle $\mathcal{L}$. Assume that the complexification $\mathcal{L} \otimes \mathbb{C}$ is isomorphic to $-K_{(X,\Delta)}$ (where (X, Δ) denotes, as before, the complexification of $(\mathcal{X}, \mathcal{D})$). Given a continuous metric on $\mathcal{L}$ we define the *arithmetic log Mabuchi functional* of $(\mathcal{X}, \mathcal{D}; \overline{\mathcal{L}})$ by

$$\mathcal{M}_{(\mathcal{X},\mathcal{D})}(\overline{\mathcal{L}}) := \frac{1}{2}\overline{\mathcal{L}}^2 + \overline{\mathcal{K}}_{(\mathcal{X},\mathcal{D})} \cdot \overline{\mathcal{L}}, \tag{1.5}$$

where $\overline{\mathcal{K}}_{(\mathcal{X},\mathcal{D})}$ is endowed with the volume-normalized metric induced by the curvature current ω of $\overline{\mathcal{L}}$, assuming that this metric on $\overline{\mathcal{K}}_{(\mathcal{X},\mathcal{D})}$ is continuous. When $\mathcal{D} = 0$ the functional $\mathcal{M}_{(\mathcal{X},\mathcal{D})}(\overline{\mathcal{L}})$ coincides with the modular height introduced in [33], up to normalization (see [3, §6.4] for a comparison between the different normalizations). For a given integral model $\mathcal{M}_{(\mathcal{X},\mathcal{D})}(\overline{\mathcal{L}})$ is minimized on a log Kähler-Einstein metric, if such a metric exists, and then

$$\mathcal{M}_{(\mathcal{X},\mathcal{D})}(\overline{\mathcal{L}}) = -(\overline{-\mathcal{K}_{(\mathcal{X},\mathcal{D})}})^2/2,$$

if the metric is volume-normalized and $\mathcal{L} = -\overline{\mathcal{K}}_{(\mathcal{X},\mathcal{D})}$.

Theorem 5. — *Let $(\mathcal{X}, \mathcal{D}; \mathcal{L})$ be a polarized arithmetic log surface $(\mathcal{X}, \mathcal{D}; \mathcal{L})$ with $\mathcal{X}$ normal, such that the complexification (X, Δ) of $(\mathcal{X}, \mathcal{D})$ is a K-semistable Fano variety and $\mathcal{L} \otimes \mathbb{C} = -K_{(X,\Delta)}$. If Δ is supported on (at most) three points, then*

$$\mathcal{M}_{(\mathcal{X},\mathcal{D})}(\overline{\mathcal{L}}) \geqslant \mathcal{M}_{(\mathbb{P}^1_{\mathbb{Z}},0)}(\overline{-\mathcal{K}_{\mathbb{P}^1_{\mathbb{Z}}}}) \quad (= -1 - \log \pi),$$

where $-\mathcal{K}_{\mathbb{P}^1_{\mathbb{Z}}}$ is endowed with the Fubini-Study metric. Moreover, equality holds if and only if $(\mathcal{X}, \mathcal{D})$ is isomorphic to $(\mathbb{P}^1_{\mathbb{Z}}, 0)$ and $\mathcal{L}$ is isomorphic to $-\mathcal{K}_{\mathbb{P}^1_{\mathbb{Z}}}$, endowed with a metric coinciding with the Fubini-Study metric, up to the application of an automorphism of $\mathbb{P}^1_{\mathbb{Z}}$ and a scaling of the metric.

It would be interesting to extend the previous theorem to the case when Δ is supported on any number of points. Anyhow, the assumption that Δ is supported on three points is always satisfied in the orbifold case. In particular, we get:

Corollary 6. — *Conjecture* 1 *holds for arithmetic normal log Fano surfaces, if Δ is supported on, at most, three points. In particular, the conjecture holds for all normal arithmetic log Fano orbifold surfaces (i.e., the case of coefficients of the form $1-1/m_i$ for $m_i \in \mathbb{N}$).*

In the setup of the previous theorem the corresponding complex variety X is always equal to $\mathbb{P}^1$ and thus (X, Δ) is a hyperplane arrangement. Accordingly, applying Theorem 3, the proof of Theorem 5 is reduced to showing that the canonical integral model $(\mathcal{X}_c, \mathcal{D}_c; -\mathcal{K}_{(\mathcal{X}_c,\mathcal{D}_c)})$ of $(X, \Delta; -K_{(X,\Delta)})$ obtained by setting $\mathcal{X}_c = \mathbb{P}^1_{\mathbb{Z}}$ and taking $\mathcal{D}_c$ to be the Zariski closure of $\{0, 1, \infty)\}$ in $\mathbb{P}^1_{\mathbb{Z}}$ minimizes $\mathcal{M}_{(\mathcal{X},\mathcal{D})}(\overline{\mathcal{L}})$ over all integral models $(\mathcal{X}, \mathcal{D}; \mathcal{L})$ of $(X, \Delta; -K_{(X,\Delta)})$, for any fixed metric on $-K_{(X,\Delta)}$. This minimization property can be viewed as logarithmic version of Odaka's minimization conjecture (proposed in any dimension in [33]). Our proof builds on [33], leveraging log canonical thresholds. We also show that the minimum is uniquely attained for $(\mathbb{P}^1_{\mathbb{Z}}, 0)$. See also [22] for very recent progress on Odaka's minimization conjecture in another direction.

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2. General setup

2.1. Log Fano varieties over $\mathbb{C}$ and volume-normalized metrics on $-(K_X + \Delta)$

A *log pair* (X, Δ) over $\mathbb{C}$ is a normal complex projective variety X together with an effective $\mathbb{Q}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, i.e., defines a $\mathbb{Q}$-line bundle, where K_X denotes the canonical divisor on X [24]. In the logarithmic setting this bundle plays the role of the canonical line bundle and is thus called the *log canonical line bundle* and is denoted by $K_{(X,\Delta)}$. A log pair (X, Δ) is said to be a *log Fano pair* if Δ is effective and $-(K_X + \Delta) > 0$. Any continuous metric $\|\cdot\|$ on $-(K_X + \Delta)$ induces a measure μ on X in a standard fashion. Indeed, when X is regular and $\Delta = 0$ this follows directly from the definition of metrics on $-K_X$ (see [3, §2.1.2]). In general, denoting by X_{reg} the regular locus of X, and using that the $\mathbb{Q}$-line bundles $-(K_X+\Delta)$ and $-K_X$ are isomorphic on the complement $X_{\mathrm{reg}} \smallsetminus \mathrm{supp}(\Delta)$ in X_{reg} of the support of Δ, the previous construction yields a measure on $X_{\mathrm{reg}} \smallsetminus \mathrm{supp}(\Delta)$. Its push-forward to X, under the inclusion map, thus yields a measure on X (see also [6, §3.1] for a slightly different representation of this measure). This measure has finite mass if and only if the log pair (X, Δ) is *klt* in the standard sense of birational algebraic geometry (see [6, §3.1] and Remark 7 below). A continuous metric on $-(K_X + \Delta)$ will be said to be *volume-normalized* if the corresponding measure is a probability measure.

2.1.1. *Local representations of metrics and measures*. — As in [3] we will use additive notation for metrics on holomorphic line bundles $L \to X$. This means that we identify a continuous Hermitian metric $\|\cdot\|$ on L with a collection of continuous local functions ϕ_U associated to a given covering of X by open subsets U and trivializing holomorphic sections e_U of $L \to U$:

$$\phi_U := -\log(\|e_U\|^2). \tag{2.1}$$

The curvature current of the metric may then, locally, be expressed as

$$dd^c\phi_U := \frac{i}{2\pi}\partial\overline{\partial}\phi_U$$

Accordingly, as is customary, we will symbolically denote by ϕ a given continuous Hermitian metric on L and by $dd^c\phi$ its curvature current. We will denote by $\mathcal{C}^0(L) \cap \mathrm{PSH}(L)$ the space of all continuous metrics on L whose curvature current is positive, $dd^c\phi \geqslant 0$ (which means that ϕ_U is plurisubharmonic, or psh, for short).

Given a log Fano pair (X, Δ) the measure corresponding to a given continuous metric ϕ on $-K_{(X,\Delta)}$ may be locally on X_{reg} be expressed as

$$\mu_\phi = e^{-\phi_U}\,|s_U|^{-2}\,(i/2)^{n^2} dz \wedge d\overline{z}, \quad dz := dz_1 \wedge \cdots \wedge dz_n$$

by taking $e_U = \partial/\partial z_1 \wedge \cdots \wedge \partial/\partial z_n \otimes e_\Delta$, where e_Δ is a local trivialization of the $\mathbb{Q}$-line bundle over X_{reg} corresponding to the divisor Δ and $s_U e_\Delta$ is the (multi-valued) holomorphic section cutting out Δ.

2.1.2. *Log Kähler-Einstein metrics*. — Given a log Fano pair (X, Δ), a metric ϕ on $-K_{(X,\Delta)}$ is said to be a*log Kähler-Einstein metric*, if ϕ is locally bounded and its curvature current $dd^c\phi$ induces a Kähler metric with constant positive Ricci curvature on the complement of Δ in X_{reg} [6]. When (X, Δ) is log smooth, i.e., X is smooth and Δ has simple normal crossings, any log Kähler-Einstein for (X, Δ) has conical singularities along Δ. More precisely, ϕ is continuous and if Δ has coefficient $c_i \in \,]0,1[$ along a smooth prime divisor Δ_i, then $dd^c\phi$ has cone angle $2\pi(1 - c_i)$ along Δ_i [23, 21, 29].

2.1.3. *K-semistability*. — We next recall the definition of K-semistability in terms of intersection numbers (see the survey [47] for more background). A *test configuration* for a log Fano pair (X,Δ) is a $\mathbb{C}^*$-equivariant normal model $(\mathscr{X},\mathscr{L})$ for $(X,-K_{(X,\Delta)})$ over the complex affine line $\mathbb{C}$. More precisely, $\mathscr{X}$ is a normal complex variety, endowed with a $\mathbb{C}^*$-action ρ, a $\mathbb{C}^*$-equivariant holomorphic projection π to $\mathbb{C}$ and a relatively ample $\mathbb{C}^*$-equivariant $\mathbb{Q}$-line bundle $\mathscr{L}$ (endowed with a lift of ρ):

$$\pi:\mathscr{X}\longrightarrow\mathbb{C},\quad \mathscr{L}\longrightarrow\mathscr{X},\quad \rho:\mathscr{X}\times\mathbb{C}^*\longrightarrow\mathscr{X} \tag{2.2}$$

such that the fiber of $\mathscr{X}$ over $1\in\mathbb{C}$ is equal to $(X,-K_{(X,\Delta)})$. A log Fano pair (X,Δ) is said to be *K-semistable* if the Donaldson-Futaki invariants $\mathrm{DF}_\Delta(\mathscr{X},\mathscr{L})$ are non-negative for any test configuration $(\mathscr{X},\mathscr{L})$ of (X,Δ):

$$n!\mathrm{DF}_\Delta(\mathscr{X},\mathscr{L})=\frac{n}{(n+1)}\overline{\mathscr{L}}^{n+1}+\mathscr{K}_{(\overline{\mathscr{X}},\mathscr{D})/\mathbb{P}^1}\cdot\overline{\mathscr{L}}^n,$$

where $\overline{\mathscr{L}}$ denotes the $\mathbb{C}^*$-equivariant extension of $\mathscr{L}$ to the $\mathbb{C}^*$-equivariant compactification $\overline{\mathscr{X}}$ of $\mathscr{X}$ over $\mathbb{P}^1$ and $\mathscr{K}_{(\overline{\mathscr{X}},\mathscr{D})/\mathbb{P}^1}$ denotes the relative log canonical divisor of the pair $(\overline{\mathscr{X}},\mathscr{D})$ with $\mathscr{D}$ denoting the flat closure in $\overline{\mathscr{X}}$ of the $\mathbb{C}^*$-orbit of the divisor Δ.

Remark 7. — If a log Fano variety (X,Δ) is K-semistable, then (X,Δ) is klt (see [34, Th. 6.1] and [9, Cor. 9.6]). When X is non-singular and Δ has simple normal crossings this means that all the coefficients of Δ along its irreducible components are strictly smaller than 1.

2.2. Arithmetic log Fano varieties and integral models. — The notion of log pairs over $\mathbb{C}$ can be extended to schemes over excellent rings, as explained in the book [25]. Here we will consider the case when the ring in question is $\mathbb{Z}$. Henceforth, $\mathcal{X}$ will denote a projective flat scheme $\mathcal{X}\to\operatorname{Spec}\mathbb{Z}$ of relative dimension n such that $\mathcal{X}$ is reduced and satisfies Serre's conditions S_2 (this is, for example, the case if $\mathcal{X}$ is normal). Such a scheme $\mathcal{X}$ will be called an *arithmetic variety.* We will denote by π the corresponding structure morphism to $\operatorname{Spec}\mathbb{Z}$,

$$\pi:\mathcal{X}\longrightarrow\operatorname{Spec}\mathbb{Z}.$$

A *log pair* $(\mathcal{X},\mathcal{D})$ over $\mathbb{Z}$ of relative dimension n is an arithmetic variety $\mathcal{X}$ endowed with an effective $\mathbb{Q}$-divisor $\mathcal{D}$ on $\mathcal{X}$ such that $\mathcal{K}_{\mathcal{X}}+\mathcal{D}$ is $\mathbb{Q}$-Cartier, i.e., defines a $\mathbb{Q}$-line bundle, where $\mathcal{K}_{\mathcal{X}}$ denotes the relative canonical divisor on $\mathcal{X}$ (see [25, §1.1] and [3, §2.2.1]). We shall call such a pair $(\mathcal{X},\mathcal{D})$ an *arithmetic log variety.* The complexification of $(\mathcal{X},\mathcal{D})$ will be denoted by (X,Δ) and $(\mathcal{X},\mathcal{D})$ will be called an *integral model of* (X,Δ).

Remark 8. — We are using the notion of an integral model in a generalized sense – usually a scheme $\mathcal{X}$ over $\mathbb{Z}$ is said to be an integral model of a scheme $X_{\mathbb{Q}}$ defined over $\mathbb{Q}$ if $X_{\mathbb{Q}}$ is isomorphic to $\mathcal{X}\otimes_{\mathbb{Z}}\mathbb{Q}$ over $\mathbb{Q}$. Here X (without any subscript) will usually denote a variety over $\mathbb{C}$ and a scheme $\mathcal{X}$ over $\mathbb{Z}$ is thus called an integral model of X if X is isomorphic to $\mathcal{X}\otimes_{\mathbb{Z}}\mathbb{C}$.

A log pair $(\mathcal{X},\mathcal{D})$ over $\mathbb{Z}$ will be called an *arithmetic log Fano variety* if $-(\mathcal{K}_{\mathcal{X}}+\mathcal{D})$ is relatively ample and the corresponding complex variety X is normal. In particular, the complexification (X,Δ) of $(\mathcal{X},\mathcal{D})$ is a log Fano variety over $\mathbb{C}$. More generally, an arithmetic variety $\mathcal{X}$ endowed with a relatively ample line bundle $\mathcal{L}$ will be said to be *polarized* and $(\mathcal{X},\mathcal{L})$ will be called an *integral model* of a complex polarized variety (X,L) if (X,L) is isomorphic to the complexification of $(\mathcal{X},\mathcal{L})$.

2.3. Arithmetic intersection numbers and heights. — We recall some well-known facts about heights (see [3] for more background and references). A *metrized line bundle* $\overline{\mathcal{L}}$ is a line bundle $\mathcal{L}\to\mathcal{X}$ such that the corresponding line bundle $L\to X$ is endowed with a metric, that we shall denote by ϕ (as in Section 2.1.1); $\overline{\mathcal{L}}:=(\mathcal{L},\phi)$. The χ-*arithmetic volume* of a polarized arithmetic variety $(\mathcal{X},\mathcal{L})$ is defined by

$$\widehat{\mathrm{vol}}_{\chi}\left(\overline{\mathcal{L}}\right):=\lim_{k\to\infty}k^{-(n+1)}\log\mathrm{Vol}\bigl\{s_k\in H^0(\mathcal{X},k\mathcal{L})\otimes\mathbb{R}:\sup_X\|s_k\|_{\phi}\leqslant 1\bigr\},\tag{2.3}$$

where $H^0(\mathcal{X},k\mathcal{L})\otimes\mathbb{R}$ may be identified with the subspace of real sections in $H^0(X,kL)$. More generally, $\widehat{\mathrm{vol}}_{\chi}\left(\overline{\mathcal{L}}\right)$ is naturally defined for $\mathbb{Q}-$line bundles, since it is homogeneous with respect to tensor products of $\overline{\mathcal{L}}$:

$$\widehat{\mathrm{vol}}_{\chi}\left(m\overline{\mathcal{L}}\right)=m^{n+1}\widehat{\mathrm{vol}}_{\chi}\left(\overline{\mathcal{L}}\right),\quad\text{if }m\in\mathbb{Z}_+.\tag{2.4}$$

Moreover, $\widehat{\mathrm{vol}}_{\chi}\left(\overline{\mathcal{L}}\right)$ is additively equivariant with respect to scalings of the metric:

$$\widehat{\mathrm{vol}}_{\chi}\left(\mathcal{L},\phi+\lambda\right)=\widehat{\mathrm{vol}}_{\chi}\left(\overline{\mathcal{L}}\right)+\frac{\lambda}{2}\mathrm{Vol}(L),\quad\text{if }\lambda\in\mathbb{R}.\tag{2.5}$$

If the metric on L has positive curvature current (i.e., if ϕ is psh), then, by the arithmetic Hilbert-Samuel theorem,

$$\widehat{\mathrm{vol}}_{\chi}(\overline{\mathcal{L}})=\frac{\overline{\mathcal{L}}^{n+1}}{(n+1)!},\tag{2.6}$$

where $\overline{\mathcal{L}}^{n+1}$ denotes the top arithmetic intersection number in the sense of Gillet-Soulé [20], which, defines the *height* of $\mathcal{X}$ with respect to $\overline{\mathcal{L}}$ [16, 8]. For the purpose of the present paper formula (2.6) may be taken as the definition of $\overline{\mathcal{L}}^{n+1}$ (arithmetic intersections between general $n+1$ metrized line bundles could then be defined by polarization, i.e., using multilinearity). Following standard practice we will use the shorthand $h_{\phi}(\mathcal{X},\mathcal{L})$ for the height $(\mathcal{L},\phi)^{n+1}$ and $\widehat{h}_{\phi}(\mathcal{X},\mathcal{L})$ for the normalized height:

$$h_{\phi}(\mathcal{X},\mathcal{L}):=(\mathcal{L},\phi)^{n+1},\quad\widehat{h}_{\phi}(\mathcal{X},\mathcal{L}):=\frac{(\mathcal{L},\phi)^{n+1}}{(n+1)L^n}.$$

The definition of $\widehat{h}_{\phi}(\mathcal{X},\mathcal{L})$ is made so that

$$\widehat{h}_{\phi+\lambda}(\mathcal{X},\mathcal{L})=\widehat{h}_{\phi}(\mathcal{X},\mathcal{L})+\lambda/2\quad\text{if }\lambda\in\mathbb{R}.$$

We also recall that, given two continuous psh metrics ϕ and ϕ_0 on the complexification $L \to X$ of $\mathcal{L} \to \mathcal{X}$, we have that

$$
\begin{aligned}
2h\left(\mathcal{L},\phi\right) - 2h\left(\mathcal{L},\phi_0\right) &= \mathcal{E}(\phi,\phi_0) \\
&:= \frac{1}{(n+1)!}\int_X (\phi-\phi_0)\sum_{j=0}^{n}(dd^c\phi)^j \wedge (dd^c\phi_0)^{n-j}.
\end{aligned}
\tag{2.7}
$$

2.4. The canonical height of an arithmetic log Fano variety. — We define the *canonical height* $h_{\mathrm{can}}(\mathcal{X},\mathcal{D})$ of an arithmetic log Fano variety $(\mathcal{X},\mathcal{D})$ by

$$
h_{\mathrm{can}}(\mathcal{X},\mathcal{D}) := \sup\Big\{h_\phi\left(-\mathcal{K}_{(\mathcal{X},\mathcal{D})}\right) : \phi \,\text{cont. psh}, \int_X \mu_\phi = 1\Big\}
$$

(when $\mathcal{D}=0$ we shall use the short hand $h_{\mathrm{can}}(\mathcal{X})$ for $h_{\mathrm{can}}(\mathcal{X},0)$). As shown precisely as in the case $\mathcal{D}=0$, considered in [3], $h_{\mathrm{can}}(\mathcal{X},\mathcal{D})<\infty$ if and only if the corresponding log Fano variety (X,Δ) over $\mathbb{C}$ is K-semistable. By [2, Prop. 3.3], the sup defining $h_{\mathrm{can}}(\mathcal{X},\mathcal{D})$ may as well be taken over all locally bounded metrics ϕ on $-K_{(X,\Delta)}$ (or, more generally, over all finite energy psh metrics ϕ). Moreover, (X,Δ) is K-polystable if and only if the sup defining $h_{\mathrm{can}}(\mathcal{X},\mathcal{D})$ is attained at some locally bounded metric ϕ, namely a log Kähler-Einstein metric. Hence, if (X,Δ) is K-polystable, then the canonical height $h_{\mathrm{can}}(\mathcal{X},\mathcal{D})$ is computed by any volume-normalized log Kähler-Einstein metric.

Finally, we note that, by scaling the metric,

$$
\begin{aligned}
-\widehat{h}_{\mathrm{can}}(\mathcal{X},\mathcal{D}) &:= -\frac{h_{\mathrm{can}}(\mathcal{X})}{(n+1)(-K_{(X,\Delta)})^n} = \inf_\phi \frac{1}{2}\widehat{D}_{\mathbb{Z}}(\phi), \\
\widehat{D}_{\mathbb{Z}}(\phi) &:= -2\widehat{h}_\phi\left(-\mathcal{K}_{(\mathcal{X},\mathcal{D})}\right) - \log\int_X \mu_\phi,
\end{aligned}
\tag{2.8}
$$

where the inf ranges over all continuous psh metrics ϕ on $-K_{(X,\Delta)}$. Additionally, by [2, Prop. 3.3], $-h_{\mathrm{can}}(\mathcal{X},\mathcal{D})$ coincides with the infimum of the arithmetic Mabuchi functional attached to $(\mathcal{X},\mathcal{D};-\mathcal{K}_{(\mathcal{X},\mathcal{D})},\phi)$. The functional $\widehat{D}_{\mathbb{Z}}(\phi)$ in (2.8) is an arithmetic analog of the *Ding functional* $\mathcal{D}_{\phi_0}$ in Kähler geometry (that appears in the proof of [3, Th. 2.5] in the case when $\mathcal{D}=0$), defined by

$$
\mathcal{D}_{\phi_0}(\phi) := -\mathcal{E}(\phi,\phi_0)/\operatorname{vol}(-K_{(X,\Delta)}) - \log\int_X \mu_\phi \tag{2.9}
$$

with respect to a reference metric ϕ_0 on $-K_{(X,\Delta)}$.

3. Toric log Fano varieties

Recall that an n-dimensional complex projective variety X is said to be *toric* if the complex torus $T_{\mathbb{C}} := (\mathbb{C}^*)^n$ acts on X with an open dense orbit [15]. We can thus view X is a $T_{\mathbb{C}}$-equivariant compactification of $(\mathbb{C}^*)^n$. A log pair (X,Δ) over $\mathbb{C}$ is said to be toric if X and Δ are toric, i.e., if X is toric and the $\mathbb{Q}$-divisor Δ is invariant under the torus action on X. A line bundle L over a complex toric variety X is called toric if is endowed with a $T_{\mathbb{C}}$-action covering the $T_{\mathbb{C}}$-action on X. An ample toric line bundle $L \to X$ corresponds to a convex rational polytope $P \subset \mathbb{R}^n$, called the *moment* *polytope*, determined by the following property: for any given positive integer k, the complex vector space $H^0(X, kL$ decomposes as follows under the $T_{\mathbb{C}}$-action,

$$H^0(X,kL)\cong\bigoplus_{m\in\mathbb{Z}^n\cap kP}\mathbb{C}\chi^m,\quad \chi^m(z)=z^m:=z_1^{m_1}\dots z_1^{m_1}, z\in(\mathbb{C}^*)^n, \tag{3.1}$$

where we have identified $L\to X$ with the trivial line bundle over the dense open orbit of $T_{\mathbb{C}}$ in X (from an invariant point of view, the real vector space $\mathbb{R}^n$ above arises as $M\otimes_{\mathbb{Z}}\mathbb{R}$, where M is the lattice $\mathrm{Hom}(T_c,\mathbb{C}^*)$ of characters of the group T_c [15]).

Any toric ample line bundle L over a complex projective variety X admits a canonical integral model $\mathcal{L}\to\mathcal{X}$ with $\mathcal{X}$ normal (see [28, §2] and [11, Def. 3.5.6]). It is determined by the property that the $\mathbb{Z}-$module $H^0(\mathcal{X},k\mathcal{L})$ is generated by the sections corresponding to the characters χ^m, appearing in formula (3.1) (see [28, Prop. 2.3.10]). Likewise, a toric log pair (X,Δ) admits a canonical integral model $(\mathcal{X},\mathcal{D})$. Moreover, if (X,Δ) is a log Fano variety, then $(\mathcal{X},\mathcal{D})$ is an arithmetic log Fano variety (as shown precisely as in the case $\mathcal{D}=0$, considered in [3, §3.1.6]).

In this section we will prove the following result, by building on the proof of [3, Th. 1.2].

Theorem 9. — *Let $(\mathcal{X},\mathcal{D})$ be the canonical integral model of a complex K-semistable toric log Fano variety (X,Δ). Conjecture 1 holds for $(\mathcal{X},\mathcal{D})$ under anyone of the following conditions:*

– *$n\leqslant 3$ and X is $\mathbb{Q}$-factorial (equivalently, X has at worst abelian quotient singularities);*

– *X is not Gorenstein or has some abelian quotient singularity.*

We start by introducing some notation, following [5]. Given a complex toric log Fano variety (X,Δ) set $L=-(K_X+\Delta)$ and denote by P the corresponding moment polytope in $\mathbb{R}^n$. Then

$$P=\{p\in\mathbb{R}^n:\langle l_F,p\rangle\geqslant -a_F,\ \forall F\}, \tag{3.2}$$

where $a_F\in\,]0,1]$ (generalizing the Fano case when $a_F=1$ for all F; see [5]) and l_F is a primitive integer vector. As shown in [5] (X,Δ) is K-semistable if and only if 0 is the barycenter of P if and only if the log Ding functional $\mathcal{D}_{\phi_0}$ is bounded from below. Moreover, the infimum of $\mathcal{D}_{\phi_0}$ is attained at a T-invariant psh metric ϕ on L. We will identify the metric ϕ with a continuous convex function on $\mathbb{R}^n$, as in [3]. More precisely, on $(\mathbb{C}^*)^n\hookrightarrow X$, let $x_i=\log(|z_i|^2)$. Trivializing $-(K_X+\Delta)$ with $\frac{dz_1}{z_1}\wedge\dots\wedge\frac{dz_n}{z_n}\otimes s_Ue_\Delta$ over $U=(\mathbb{C}^*)^n$, and abusing notation slightly, we let $\phi(x):=\phi_U(z)$ in the chosen trivialization over $U=(\mathbb{C}^*)^n$. Then ϕ as a function of x is a continuous convex function on $\mathbb{R}^n$. Via this recipe we also define a reference metric from the convex function $\psi_P(x):=\sup_{p\in P}\langle p,x\rangle$. As in [3, Eq. (3.8)], we still have that

$$\mathcal{D}_{\psi_P}(\phi)=\int_P\phi^*dy/V-\log\int_{\mathbb{R}^n}e^{-\phi(x)}dx-n\log\pi,\quad V:=\mathrm{vol}(P)$$

(since the support of Δ is contained in the complement of $(\mathbb{C}^*)^n$ in X). Thus the inequality in [3, Prop. 3.7] generalizes to the canonical toric model $\mathcal{L}$ of L (which coincides with $-\mathcal{K}_{(\mathcal{X},\mathcal{D})}$):

$$2\widehat{\mathrm{vol}}_{\mathcal{X}}\left(-\mathcal{K}_{(\mathcal{X},\mathcal{D})},\phi\right) \leqslant -\operatorname{vol}(X,\Delta)\log\Big(\frac{\operatorname{vol}(X,\Delta)}{(2\pi^2)^n}\Big), \tag{3.3}$$

where $\operatorname{vol}(X,\Delta) := \operatorname{vol}(-K_{(X,\Delta)})$.

We will first prove Theorem 9 in the case that $X = \mathbb{P}^n$, using the following lemma, formulated in terms of the divisor D_0 cut out by the T_c-invariant element of $H^0(X,-K_X)$ (given by $\frac{\mathrm{d}z_1}{z_1}\wedge\cdots\wedge\frac{\mathrm{d}z_n}{z_n}$ over $(\mathbb{C}^*)^n$). In other words,

$$D_0 = \sum_F D_F,$$

where D_F is the irreducible divisor corresponding to the facet F of the moment polytope corresponding to X (see [5]). The lemma is a special case of [3, Prop. 3.12].

Lemma 10. — *Let $\mathcal{X}$ be the canonical integral model of an n-dimensional complex K-semistable toric Fano variety X and denote by D_0 the standard anti-canonical divisor on X. Then*

$$\frac{(\overline{-\mathcal{K}_{(\mathcal{X},(1-t)\mathcal{D}_0)}})^{n+1}/(n+1)!}{\left(-(K_X+(1-t)D_0)\right)^n/n!} = \frac{(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}/(n+1)!}{(-K_X)^n/n!} - \frac{1}{2}\log(t^n), \tag{3.4}$$

$$t^n = \Big(\frac{\left(-(K_X+(1-t)D_0)\right)^n}{(-K_X)^n}\Big), \tag{3.5}$$

with respect to the volume-normalized Kähler-Einstein metrics.

We next deduce the following

Lemma 11. — *Let $(\mathcal{X},\mathcal{D})$ be a toric K-semistable log Fano variety such that $\mathcal{X}=\mathbb{P}^n_{\mathbb{Z}}$. Then $(\overline{-\mathcal{K}_{(\mathcal{X},\mathcal{D})}})^{n+1} \leqslant (\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1}$ with equality if and only if $\mathcal{D}=0$.*

Proof. — First observe that there exists $t\in[0,1]$ such that $\mathcal{D}=(1-t)\mathcal{D}_0 =: \mathcal{D}_t$. This is a special case of [18, Cor. 1.6], which applies to $\mathbb{P}^n$, in any dimension n, using that toric log Fano varieties are never uniformly K-stable. It will thus be enough to show that $t\mapsto(\overline{-\mathcal{K}_{(\mathbb{P}^n_{\mathbb{Z}},\mathcal{D}_t)}})^{n+1}$ is increasing on $[0,1]$ (and thus its maximum is attained at $t=1$). By the previous lemma

$$\frac{2(\overline{-\mathcal{K}_{(\mathbb{P}^n_{\mathbb{Z}},\mathcal{D}_t)}})^{n+1}/(n+1)!}{(-K_{\mathbb{P}^n})^n/n!} = t^n 2\frac{(\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1}/(n+1)!}{(-K_{\mathbb{P}^n})^n/n!} - t^n\log(t^n).$$

Differentiating with respect to (t^n) reveals that the right hand side above is increasing with respect to t if and only if

$$2\,\frac{(\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1}/(n+1)!}{(-K_{\mathbb{P}^n})^n/n!} \geqslant 1.$$

The latter inequality is indeed satisfied, as follows from the explicit formula (1.1). □

Combining the universal bound (3.3) with [3, Lem. 3.8], all that remains to prove Theorem 9 is to establish the "logarithmic gap hypothesis"

$$\operatorname{vol}(X,\Delta) \leqslant \operatorname{vol}(\mathbb{P}^{n-1}\times\mathbb{P}^1), \tag{3.6}$$

assuming that $X\neq\mathbb{P}^n$ and that (X,Δ) satisfies the assumptions of Theorem 9.

Remark 12. — This is a logarithmic generalization of the "gap hypothesis", discussed in [3, §3.2.1] and it seems natural to ask if it holds for any K-semistable log Fano variety (X,Δ) such that $X\neq\mathbb{P}^n$? For example, by the proof of [3, Lem. 3.1], the logarithmic gap hypothesis holds for K-semistable products. More generally, as pointed out by the referee, the logarithmic gap hypothesis holds when (X,Δ) admits a morphism to $\mathbb{P}^1$ with K-semistable general fibers, by [13, Cor. 1.17] (even without assuming that the total space (X,Δ) is K-semistable).

Proposition 13. — *The logarithmic gap hypothesis holds for all toric K-semistable log Fano varieties (manifolds) (X,Δ) such that $X\neq\mathbb{P}^n$ if and only if the following bound holds for all Fano varieties (manifolds) $X\neq\mathbb{P}^n$*

$$S(X)\leqslant \operatorname{vol}(\mathbb{P}^{n-1}\times\mathbb{P}^1),\quad S(X):=\sup\{\operatorname{vol}(-(K_X+\Delta)):(X,\Delta)\ \textit{K-semistable}\}. \tag{3.7}$$

The "logarithmic gap hypothesis" holds for all log Fano varieties (X,Δ) such that X is $\mathbb{Q}$-factorial and of dimension $n\leqslant 3$ and for any dimensions n if X has some abelian quotient singularity or if X is not Gorenstein.

Proof. — Since, trivially, $\operatorname{vol}(X,\Delta)\leqslant S(X)$ the first equivalence follows directly from the definitions. Next, let us show the last statement of the proposition, first assuming that X is singular, which means that the moment polytope P of (X,Δ) is "singular" in the sense that there exists a vertex of ∂P such that the corresponding primitive vectors $l_{F_1},\ldots,l_{F_n}$ do not generate $\mathbb{Z}^n$. It follows from the proof of [3, Lem. 3.9] that

$$\operatorname{vol}(P)\leqslant\frac{1}{2}(n+1)^n/n!\leqslant\operatorname{vol}(\mathbb{P}^{n-1}\times\mathbb{P}^1)$$

Indeed, since $a_F\leqslant 1$ the first inequality follows from the inequality [3, (3.13)], using that $\delta\geqslant 2$, according to the singularity assumption on P (for the second inequality see formula [3, (3.14)]). All that remains is thus to show the bound (3.7) for $S(X)$ when $n\leqslant 3$ and X is non-singular. First assume that $n=2$. This means, by classical classification results, that X is either $\mathbb{P}^1\times\mathbb{P}^1$ or the blow-up $X^{(m)}$ of $\mathbb{P}^2$ in m points for $m\leqslant 3$. But $(-K_{X^{(m)}})^2=(-K_{\mathbb{P}^2})^2-m$ and thus $\operatorname{vol}(X_1)\leqslant 4=\operatorname{vol}(\mathbb{P}^{n-1}\times\mathbb{P}^1)$, proving the bound (3.7). Finally, consider the case when $n=3$. Starting with the trivial bound $\operatorname{vol}(X,\Delta)\leqslant\operatorname{vol}(X)$ it follows the classification [31] of all non-singular toric Fano varieties of dimension 3 that it is enough to show that the bound (3.7) holds when X is $\mathbb{P}^3$ blown-up in one point or $\mathbb{P}(\mathcal{O}(1)\oplus\mathcal{O}(2))$ (whose degrees are 56 and 62, respectively). According to the following proposition the corresponding invariants $S(X)n!$ are, approximately, given by 41.8 and 30.3, respectively, which are well below the degree 54 of $\mathbb{P}^2\times\mathbb{P}^1$, as desired. □

3.1. The invariant $S(X)$ for $n\leqslant 3$.

— In the proof above we used the following result.

Proposition 14. — *After rounding to the nearest decimal place the invariant $n!S(X)$ (formula (3.7)) is given by 41.8 and 30.3 when X equals $\mathbb{P}^3$ blown up in one point and $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(2))$, respectively.*

Proof. — Given a convex subset P of $\mathbb{R}^n$ let

$$s(P) := \sup\{\mathrm{vol}(P_0) : P_0 \subset P, \, b_{P_0} = 0\},$$

where P_0 is a closed subset of P with barycenter b_{P_0} at the origin. We will compute $s(P)$ when P is the moment polytope of the manifolds X appearing in the proposition, showing at the same time that $s(P) = S(X)$. The moment polytopes P of both $\mathbb{P}^3$ blown up in one point and $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(2))$ are of the form a simplex, with a simplex subset removed, by chopping off a vertex (see ID 20 and ID7 in the database [31])). After a general linear transformation, they are of the form $(a\Delta_3 - \mathbf{1}) \smallsetminus (b\Delta_3 - \mathbf{1})$, where Δ_3 is the standard unit simplex in dimension three, $\mathbf{1}$ is the vector with all entries equal to 1, and a and b are positive real numbers. For $\mathbb{P}^3$ blown up in one point we can transform the moment polytope to $(4\Delta_3 - \mathbf{1}) \smallsetminus (2\Delta_3 - \mathbf{1})$ and for $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$ we get $(5\Delta_3 - \mathbf{1}) \smallsetminus (\Delta_3 - \mathbf{1})$. In the first case, the linear transformation is unimodular, but in the second case the transformation has determinant 2. This will not matter when computing $s(P)$ as long as we correct for the non-unit determinant. Next we compute the barycenter b_P of these polytopes, a simple task using the explicit barycenter of the standard unit simplex, $b_{\Delta_n} = \mathbf{1}/(n+1)$, and then scaling and linearity properties of the volume times the barycenter. The barycenter of $(a\Delta_3 - \mathbf{1}) \smallsetminus (b\Delta_3 - \mathbf{1})$ is given by

$$\frac{a^3/3!(a/4-1) - b^3/3!(b/4-1)}{a^3/3! - b^3/3!}\mathbf{1}.$$

Next we use a general fact, to be proved in the lemma below, stating that the closed subset P' of P which maximizes volume, with the relaxed constraint

$$b_{P'} \cdot \mathbf{1} = 0 \tag{3.8}$$

is the one given by $P \cap H$ where H is a half-space with normal $\mathbf{1}$. In our case, by symmetry, this P' automatically satisfies the stronger constraint $b_{P'} = 0$. Moreover, since the boundary of $P \cap H$ is parallel to a facet of P it corresponds to a divisor Δ on X defining a log Fano pair (X, Δ). Thus (X, Δ) is also the K-semistable log Fano pair realizing the sup in the definition of $S(X)$, showing that $s(P) = S(X)$. We can find H by imposing the constraint. We introduce the weight w such that

$$P \cap H = ((a-w)\Delta_3 - \mathbf{1}) \smallsetminus (b\Delta_3 - \mathbf{1}).$$

From here it is clear that if $b_{P'} \cdot \mathbf{1} = 0$, then, in fact, the entire barycenter will vanish and the condition $b_{P'} \cdot \mathbf{1} = 0$ turns into the following fourth order polynomial equation for w:

$$(a-w)^3/3!((a-w)/4-1) - b^3/3!(b/4-1) = 0.$$

The solution w and the corresponding value $s(P)$ for $\mathbb{P}^3$ blown up in one point, is given by

$$w = \frac{2}{3}\Bigl(5 - \frac{4}{\sqrt[3]{19-3\sqrt{33}}} - \sqrt[3]{19-3\sqrt{33}}\Bigr)$$

and
$$n!s(P) = n!\,\mathrm{vol}(P') = ((4-w)^3 - 2^3) \approx 41.8$$

and for $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$,

$$w = \left(4 - \sqrt[3]{\frac{4}{2-\sqrt{2}}} - \sqrt[3]{2(2-\sqrt{2})}\right)$$

and
$$n!S(P) = \frac{1}{2} n! \operatorname{vol}(P') = \frac{1}{2}((5-w)^3 - 1^3) \approx 30.3,$$

where we have corrected for the non-unimodular transformation used in the second case. □

In the above proof we used the following

Lemma 15. — *Let P be a closed subset of $\mathbb{R}^n$ with the origin as an interior point. Given $v \in \mathbb{R}^n$ assume that $\int_P x \cdot v > 0$. Then the maximum*

$$\max_{Q \subset P : \int_Q v \cdot x \mathrm{d}\lambda(\mathrm{x}) = 0} \int_Q \mathrm{d}\lambda$$

is attained at $Q = P \cap H$ with H a closed half-space with outward pointing normal v. Here $\mathrm{d}\lambda$ is Lebesgue measure.

Proof. — Without loss of generality we can assume that $v = (0, \ldots, 0, 1)$. Denote by $(x_1, x_2, \ldots, x_{n-1}, y)$ the coordinates on $\mathbb{R}^n$. Since the origin is an interior point of P and $\int_P x \cdot v > 0$ there is a closed half-space H as in the lemma satisfying $\int_{P \cap H} y \mathrm{d}\lambda = 0$. Hence, any candidate Q for the maximum in question satisfies $\int_{P \cap H} y \mathrm{d}\lambda = \int_Q y \mathrm{d}\lambda$. Subtracting the left hand side from the right hand side and vice versa yields $\int_{P \cap H \smallsetminus Q} y \mathrm{d}\lambda = \int_{Q \smallsetminus P \cap H} y \mathrm{d}\lambda$. Since $\sup_{P \cap H \smallsetminus Q} y \leqslant \inf_{Q \smallsetminus (P \cap H)} y$ it follows that $\operatorname{vol}(P \cap H \smallsetminus Q) \geqslant \operatorname{vol}(Q \smallsetminus P \cap H)$, so that $\operatorname{vol}(P \cap H) \geqslant \operatorname{vol}(Q)$, as desired. □

In fact, with just a slight variation of the argument above, any maximizer must be of the special form above and, in addition, assuming connectedness of P, the maximizer is unique. The proof of the previous proposition thus reveals that the unique toric divisor Δ on X realizing the sup defining the invariant $S(X)$ is a multiple of the prime divisor D_F defined by the zero-section of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2)) \to \mathbb{P}^2$ and hyperplane "at infinity" in $\mathbb{P}^3$ blown up at the origin in $\mathbb{C}^3 \subset \mathbb{P}^3$, respectively (i.e., the zero-section of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{P}^2$). A similar argument also applies when X is the blow-up of $\mathbb{P}^2$ at the origin in $\mathbb{C}^2$ (i.e., the first Hirzebruch surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{P}^2$). The unique maximizer for the invariant $S(X)$ is then a log Fano pair (X, Δ) for a multiple of the hyperplane D "at infinity" (i.e., the zero-section of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{P}^2$). Interestingly, this K-polystable log pair (X, Δ) was also singled out in [40, Cor. 1.5] by the following rigidity property (answering a question of Cheltsov): it admits a rigid Kähler-Einstein metric in the sense that for any other multiple cD the log pair (X, cD) does not admit a Kähler-Einstein metric. The same rigidity property holds for the two three-dimensional log pairs discussed above (since there is a unique half-space H satisfying the constraint in formula (3.8)).

3.2. Estimates on the canonical height. — Theorem 1.3 from [3] (and its corollary) generalizes directly to the case of log Fano pairs and their Kähler-Einstein metrics in any relative dimension n (with the same proof, by letting P be the moment polytope corresponding to (X,Δ)):

$$\frac{1}{2}\operatorname{vol}(X,\Delta)\log\Big(\frac{n!m_n\pi^n}{\operatorname{vol}(X,\Delta)}\Big)\leqslant\frac{h_{\mathrm{can}}(\mathcal{X},\mathcal{D})}{(n+1)!}\leqslant\frac{1}{2}\operatorname{vol}(X,\Delta)\log\Big(\frac{(2\pi)^n\pi^n}{\operatorname{vol}(X,\Delta)}\Big).$$

Interestingly, Lemma 10 reveals that the family of log Fano pairs $(\mathcal{X},\mathcal{D})$ appearing in the lemma may be explicitly expressed in terms of the algebro-geometric volume $\operatorname{vol}(X,\Delta)$ in the same functional form as the one appearing in the previous upper and lower bounds:

$$\frac{(\overline{-\mathcal{K}_{(\mathcal{X},\mathcal{D})}})^{n+1}}{(n+1)!}=\frac{1}{2}\operatorname{vol}(X,\Delta)\log\Big(\frac{be^{2a}}{\operatorname{vol}(X,\Delta)}\Big)$$

with
$$a:=\frac{(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}/(n+1)!}{(-K_X)^n/n!}\quad\text{and}\quad b=\operatorname{vol}(X).$$

4. Hyperplane arrangements

In this section we prove Theorem 3 concerning hyperplane arrangements. Recall that a log Fano pair (X,Δ) is called a *log Fano hyperplane arrangement* if $X=\mathbb{P}^n$ and $\Delta=\sum_{i=1}^m w_iH_i$ where $w_i\in\mathbb{Q}_{>0}$ and the H_i are distinct hyperplanes. Furthermore we will call (X,Δ) *simple normal crossing*, abbreviated snc, if the support of Δ has simple normal crossings.

For an snc log Fano hyperplane arrangement, if $m=n+1$ and all the weights w_i are equal, then (X,Δ) is a toric log-pair (see Lemma 10). The following lemma shows that for given hyperplanes $H_1,\dots,H_m$ and a fixed volume $\operatorname{vol}(X,\Delta)$, the "toric" weights form the vertices of the convex polytope of all weights w_i corresponding to K-semistable (X,Δ).

Lemma 16. — *Fix $m\geqslant 1$ and a real number $0<D\leqslant(-K_{\mathbb{P}^n})^n=(n+1)^n$. Let as before for a real m-tuple w, $\Delta=\sum_{i=1}^m w_iH_i$ for distinct hyperplanes H_i. Then the set of weights*

$$S=\{w\in\mathbb{R}^n:(-(K_{\mathbb{P}^n}+\Delta))^n=D\text{ and }(\mathbb{P}^n,\Delta)\text{ is K-semistable}\}$$

is either empty or $m\geqslant n+1$ and S is a polytope with $\binom{n}{m}$ vertices given by any reordering of the tuple $w_1=w_2=\cdots=w_{n+1}=\frac{1}{m}(n+1-D^{1/n})$, $w_\ell=0\ \forall\ell>n+1$.

Proof. — By [18], for $w\in\mathbb{R}^n$ and $\Delta=\sum_{i=1}^m w_iH_i$, $(\mathbb{P}^n,\Delta)$ is a K-semistable log Fano pair if and only if w is in the convex set C defined by the following inequalities:

$$0\leqslant w_i<1\quad\forall i=1,\dots,m$$

$$k\sum_{i=1}^m w_i\geqslant(n+1)\sum_{j=1}^k w_{i_j}\qquad\begin{cases}\forall k=1,\dots,n,\\ \forall i_1,\cdots,i_k\text{ with }1\leqslant i_1<\cdots<i_k\leqslant m.\end{cases}\tag{4.1}$$

Here it should be noted that in fact, it suffices to consider the second inequality for index combinations i_j of length 1. The other inequalities for larger index combinations follows. Hence, K-semistability of $(\mathbb{P}^n, \Delta)$ is equivalent to

$$\begin{aligned} 0 \leqslant w_i < 1 \quad \forall i = 1, \dots, m, \\ w_i \leqslant \frac{1}{n+1} \sum_{j=1}^{m} w_j \quad \forall i = 1, \dots, m. \end{aligned} \tag{4.2}$$

Fix m and D as in the statement of the theorem. The goal is to understand the intersection of the above set with the set $\{w : -(K_{\mathbb{P}^n} + \Delta) = D\}$. Note first that

$$\big(-(K_{\mathbb{P}^n} + \textstyle\sum_{i=1}^m w_i H_i)\big)^n = \big(n + 1 - \textstyle\sum_{i=1}^m w_i\big)^n.$$

Let $C := n + 1 - D^{1/n}$, so that $\{w : -(K_{\mathbb{P}^n} + \Delta) = D\} = \{w : \sum_{i=1}^m w_i = C\}$. Thus with S defined as in Lemma 16

$$S = \left\{ w : \textstyle\sum_{i=1}^m w_i = C,\ \begin{cases} 0 \leqslant w_i < 1 & \forall i = 1, \dots, m \\ w_i \leqslant C/(n+1) & \forall i = 1, \dots, m \end{cases} \right\}.$$

Observe that since $0 \leqslant C < n + 1$, the inequality $w_i < 1$ is superfluous. After a convenient rescaling we get

$$\frac{n+1}{C} S = \{w : \textstyle\sum_{i=1}^m w_i = n + 1,\ 0 \leqslant w_i \leqslant 1\ \forall i = 1, \dots, m\}.$$

Clearly if $m < n + 1$, $(n + 1/C)S$ is empty. For $m \geqslant n + 1$, any vertex of $(n + 1/C)S$ is given by the intersection of $(n + 1/C)S$ with some collection of the inequalities put to equality. But clearly all such points must be of the form of having $n + 1$ ones and $m - (n + 1)$ zeros. And on the other hand any such point is a vertex. ☐

Fixing the volume $\mathrm{vol}(X, \Delta)$ is, when $X = \mathbb{P}^n$, tantamount to fixing the isomorphism class of the $\mathbb{Q}$-line bundle $-(K_X + \Delta)$ (since the rank of the Picard group of $\mathbb{P}^n$ is one). The following lemma shows that, in this case, the maximal height is convex with respect to the weights of Δ.

Lemma 17. — *Consider an arithmetic Fano variety $\mathcal{X}$ and a curve $t \mapsto (\mathcal{X}, \mathcal{D}_t)$ of arithmetic log Fano varieties where $\mathcal{D}_t = \sum_{i=1}^m w_i(t)\mathcal{D}_i$ for some $m \geqslant 1$, irreducible divisors $\mathcal{D}_i$ over $\mathbb{Z}$ and $w : [0, 1] \to \mathbb{R}^m$ an affine function. Additionally assume that all the $\mathcal{D}_t$ are linearly equivalent, which equivalently means that $-(\mathcal{K}+\mathcal{D}_t)$ isomorphic to $\mathcal{L}$ for a line bundle $\mathcal{L} \to \mathcal{X}$ independent of t. Then the function $h : [0, 1] \to \left]-\infty, \infty\right]$ defined as*

$$t \longmapsto h_{\mathrm{can}}(\mathcal{X}, \mathcal{D}_t) \tag{4.3}$$

is strictly convex. Equivalently the function $t \mapsto \widehat{h}_{\mathrm{can}}(\mathcal{X}, \mathcal{D}_t)$ is strictly convex.

Proof. — By assumption we can identify $-(\mathcal{K} + \mathcal{D}_t)$ with $\mathcal{L}$ for a line bundle $\mathcal{L}$ independent of ϕ. Thus the height $h_\phi(\mathcal{X}, \mathcal{D}_t)$ for a fixed metric on $\mathcal{L}$ is independent of t.

Likewise, $h_{\mathrm{can}}(\mathcal{X},\mathcal{D}_t)$ coincides with $\widehat{h}_{\mathrm{can}}(\mathcal{X},\mathcal{D}_t)$ up to multiplication by a constant independent of t. Next, express

$$\widehat{h}_{\mathrm{can}}(\mathcal{X},\mathcal{D}_t)=\sup_{\phi}\widehat{h}(\mathcal{X},\mathcal{D}_t)+\frac{1}{2}\log\int_X\mu_{(\phi,\mathcal{D}_t)},$$

where the sup ranges over all continuous psh metrics on $\mathcal{L}$. Introducing an arbitrary volume form $\mathrm{d}V$ on X we can rewrite

$$\int_X\mu_{(\phi,\mathcal{D}_t)}=\int_X\exp(-\phi-\sum_{i=1}^m w_i(t)\psi_{D_i}-\log\mathrm{d}V)dV.$$

By Hölder's inequality this expression is convex in t, since $w_i(t)$ is affine. It is even strictly convex since the D_i are distinct. This means that $\widehat{h}_{\mathrm{can}}(\mathcal{X},\mathcal{D}_t)$ is the supremum over a set independent of t, of a collection of strictly convex functions and thus is itself, strictly convex. □

The above lemmas will reduce the proof of Theorem 3 to the case of when the support of Δ consists of $n+1$ distinct hyperplanes. The following lemma shows that we can further reduce to the case when these hyperplanes are the standard toric ones.

Lemma 18. — *Assume that the log pair $(\mathbb{P}^n_{\mathbb{Z}},\mathcal{D})$ is isomorphic to the standard toric one $(\mathbb{P}^n_{\mathbb{Z}},\mathcal{D}_0)$ over $\mathbb{C}$. Then, for any $t\in[0,1]$,*

$$h_{\mathrm{can}}(\mathbb{P}^n_{\mathbb{Z}},(1-t)\mathcal{D})\leqslant h_{\mathrm{can}}(\mathbb{P}^n_{\mathbb{Z}},(1-t)\mathcal{D}_0).$$

Proof. — Denote by $s_0,\dots,s_n$ the integral sections of $\mathcal{O}(1)$ cutting out the irreducible components of $\mathcal{D}$. We can express $s_i:=\sum_j A_{ij}x_j$ with $A_{ij}\in\mathbb{Z}$, where $(x_0,\dots,x_n)$ are the standard affine coordinates on $\mathbb{C}^{n+1}$. It will be enough to show that

$$\widehat{h}_{\mathrm{can}}(\mathbb{P}^n_{\mathbb{Z}},(1-t)\mathcal{D})=\widehat{h}_{\mathrm{can}}(\mathbb{P}^n_{\mathbb{Z}},(1-t)\mathcal{D}_0)+(t-1)\log|\det A| \tag{4.4}$$

(since $t\leqslant 1$ and $|\det A|\geqslant 1$). To this end, denote by F the invertible $\mathbb{C}$-linear map from $\mathbb{C}^{n+1}_x\to\mathbb{C}^{n+1}_y$ satisfying

$$F^*(\textstyle\sum_j A_{ij}y_j)=x_i$$

(the existence of F is equivalent to the invertibility of the matrix A which, in turn, is equivalent to the assumption about an isomorphism over $\mathbb{C}$). We will use the same symbol F for the induced map $\mathbb{P}^n\to\mathbb{P}^n$ and its standard lift to $\mathcal{O}(1)$ (as well as its tensor powers). By basic linear algebra

$$F^*dy_0\wedge\cdots\wedge dy_n=(\det A)^{-1}dx_0\wedge\cdots\wedge dx_n.$$

Now, observe that $-K_{(\mathbb{P}^n_{\mathbb{Z}},(1-t)\mathcal{D})}\simeq t\mathcal{O}(n+1)$, using standard isomorphisms over $\mathbb{Z}$. In particular, any given metric ϕ on $\mathcal{O}(n+1)$ induces a metric $t\phi$ on $-K_{(\mathbb{P}^n_{\mathbb{Z}},(1-t)\mathcal{D})}$ and thus, using the log pair $(\mathbb{P}^n_{\mathbb{Z}},(1-t)\mathcal{D})$, a measure $F^*\mu_{t\phi}$ on $\mathbb{P}^n_{\mathbb{Z}}$. Note that

$$\int_{\mathbb{P}^n}\mu_{t\phi}=|\det A|^{-2}\int_{\mathbb{P}^n}\mu_{tF^*\phi}, \tag{4.5}$$

where $\mu_{tF^*\phi}$ is the measure associated to the metric $tF^*\phi$ on $-K_{(\mathbb{P}^n_{\mathbb{Z}},(1-t)\mathcal{D}_0)}$ (now using the toric log pair $(\mathbb{P}^n_{\mathbb{Z}},(1-t)\mathcal{D}_0)$). Indeed, consider the (singular) metric ψ on $(n+1)\mathcal{O}(1)$ defined by

$$\psi := t\phi + (1-t)\sum_i \log|s_i|^2.$$

The measure $\mu_{t\phi}$ coincides with the measure μ_ψ attached to the metric on $-K_{\mathbb{P}^n}$ induced by ψ. Hence, denoting by G the canonical lift of the map F to $-K_{\mathbb{P}^n}$, we have, by definition, that

$$G^*\mu_\psi = \mu_{G^*\psi},\quad G^*\psi = F^*\psi + \log(|\det A|^2),\quad F^*\psi := tF^*\phi + (1-t)\sum_i \log|F^*s_i|^2.$$

This proves formula (4.5). Next, note that, by Lemma 31 in the appendix,

$$\widehat{h}(\mathcal{O}(n+1), G^*\phi) = \widehat{h}(\mathcal{O}(n+1), \phi).$$

Hence, we get, as above, that

$$\begin{aligned}\widehat{h}(\mathcal{O}(n+1), F^*\phi) &= \widehat{h}(\mathcal{O}(n+1), G^*\phi - \log(|\det A|))\\ &= \widehat{h}(\mathcal{O}(n+1), G^*\phi) - \log(|\det A|^2)\\ &= \widehat{h}(\mathcal{O}(n+1), \phi) - \log(|\det A|)),\end{aligned}$$

giving

$$\widehat{h}(\mathcal{O}(t(n+1)), t\phi) = \widehat{h}(\mathcal{O}(t(n+1)), F^*)(t\phi)) + t\log(|\det A|).$$

All in all, this means that

$$\widehat{h}(\mathcal{O}(t(n+1)), t\phi) + \log\int \mu_{t\phi} = \widehat{h}(\mathcal{O}(t(n+1)), F^*t\phi) + \log\int \mu_{F^*t\phi} + (t-1)\log(|\det A|).$$

Taking the sup over all continuous metrics ϕ on $\mathcal{O}(n+1)$ with positive curvature thus concludes the proof of the desired identity (4.4). □

4.1. Conclusion of the proof of Theorem 3. — In the following we will use the notation $\mathcal{D}_w = \sum_{i=1}^m w_i H_i$ for $w \in \mathbb{R}^m$ and for fixed hyperplanes H_i defined over $\mathbb{Z}$. Let $(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{D}_{w'})$ be a K-semistable snc log Fano hyperplane arrangement. Define for brevity $d := (-(K_{\mathbb{P}^n} + \Delta(w')))^n$. Set, as in Lemma 16,

$$S = \{w \in \mathbb{R}^n : (-(K_{\mathbb{P}^n} + \Delta_w))^n = d \text{ and } (\mathbb{P}^n_{\mathbb{Z}}, \mathcal{D}_w) \text{ is K-semistable}\}.$$

Consider the function $h(w)$ defined by

$$h(w) = h_{\mathrm{can}}(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{D}_w).$$

Restricted to the convex set S, $h|_S$ is convex by Lemma 17. Next by Lemma 16, S is the convex hull of weight vectors $(w^k)_{k=1,\dots,\binom{m}{n+1}}$, each corresponding to toric log Fano pairs, equivalent to $(\mathbb{P}^n_{\mathbb{Z}}, (1-t)\mathcal{D}_0)$ over $\mathbb{C}$, where $\mathcal{D}_0$ is the toric standard anti-canonical divisor and t is the unique number such that $(-(K_{\mathbb{P}^n} + (1-t)\mathcal{D}_0)^n = d$. By the convexity of h,

$$h(w') \leqslant \max_k h(w^k) \leqslant h_{\mathrm{can}}(\mathbb{P}^n_{\mathbb{Z}}, (1-t)\mathcal{D}_0),$$

where the second inequality is the content of Lemma 18. We have thus reduced to the standard toric case, which we have already handled. Specifically, the bound (1.4) follows directly from Lemma 10. For Theorem 3, recall that it was observed in the proof of Lemma 11 that the volume dependent bound in (1.4) is strictly increasing with volume, so that a universal bound is uniquely given for maximal volume, i.e., when $\Delta = 0$, yielding the result.

5. Diagonal hypersurfaces

In this section we will deduce Theorem 4 from the results in the previous sections. The starting point of the proof is the following analytic representation of the height:

Lemma 19 (Restriction formula). — *Let $\mathcal{X}$ be the subscheme of $\mathbb{P}^{n+1}_{\mathbb{Z}}$ cut out by a homogeneous polynomial s of degree d with integer coefficients and ϕ a continuous psh metric on $\mathcal{O}(d) \to \mathbb{P}^{n+1}_{\mathbb{C}}$. Then the height $h_\phi(\mathcal{X}_d, \mathcal{O}(d))$ of the restriction of $(\mathcal{O}(d), \phi)$ to $\mathcal{X}$ may be expressed as*

$$\frac{2h_\phi(\mathcal{X}_d, \mathcal{O}(d))}{(n+1)!} = (n+2)\mathcal{E}(\phi, d\phi_0) + \int_{\mathbb{P}^{n+1}} \log\bigl(\|s\|^2_\phi\bigr)\frac{(dd^c\phi)^{n+1}}{(n+1)!},$$

where ϕ_0 is the Weil metric on $\mathcal{O}(1)$ and $\mathcal{E}$ is the functional defined by formula (2.7), corresponding to $\mathcal{O}(d) \to \mathbb{P}^{n+1}$.

Proof. — This is well-known, but for completeness we provide a proof. Consider first the general situation where $\mathcal{X}$ is a subscheme (of relative dimension n) of a regular projective flat scheme $\mathcal{Y}$ cut out by a section s of a relatively ample line bundle $\mathcal{L} \to \mathcal{Y}$. Then, given a metric ϕ on the complexification L of $\mathcal{L} \to \mathcal{Y}$, the restriction formula for arithmetic intersection numbers [8, Prop. 2.3.1] gives

$$(\mathcal{L}, \phi)^{n+2} \cdot \mathcal{Y} = (\mathcal{L}, \phi)^{n+1} \cdot \mathcal{X} - \int_Y \log ||s||_\phi (dd^c\phi)^{n+1}. \tag{5.1}$$

In particular, setting $\mathcal{Y} = \mathbb{P}^{n+1}_{\mathbb{Z}}$ and $\mathcal{L} = \mathcal{O}(d)$ gives

$$\frac{2h_\phi(\mathcal{X}_d, \mathcal{O}(d))}{(n+1)!} = (n+2)\,\frac{h_\phi(\mathbb{P}^{n+1}_{\mathbb{Z}}, \mathcal{O}(d))}{(n+2)!} + \int \log ||s||^2_\phi \frac{(dd^c\phi)^{n+1}}{(n+1)!}.$$

The proof is thus concluded by invoking the well-known fact that

$$h_\phi(\mathbb{P}^{n+1}_{\mathbb{Z}}, \mathcal{O}(d))/(n+2)! = \mathcal{E}_{\mathbb{P}^{n+1}}(\phi, \phi_0).$$

For example, this is a special case of the toric formula [1, Eq. 3.7]. □

In general, if $\mathcal{X}$ is subscheme of $\mathbb{P}^{n+1}_{\mathbb{Z}}$ of codimension one, then $\mathcal{K}_{\mathcal{X}}$ is well-defined as line bundle over $\mathcal{X}$. More precisely, by the adjunction formula, there is an isomorphism of line bundles over $\mathbb{Z}$,

$$\mathcal{K}_{\mathcal{X}} \simeq \bigl(\mathcal{K}_{\mathbb{P}^{n+1}_{\mathbb{Z}}} - \mathcal{O}(\mathcal{I}/\mathcal{I}^2)\bigr)_{\mathcal{X}},$$

where $\mathcal{I}$ is the ideal sheaf cutting out $\mathcal{X}$ [25, Eq. 1.6.2, p. 8]. In particular, if $\mathcal{X}$ is cut out by a homogeneous polynomial s of degree d, then

$$-\mathcal{K}_{\mathcal{X}} \simeq -\mathcal{K}_{\mathbb{P}^{n+1}_{\mathbb{Z}}} - \mathcal{O}(d) \simeq \mathcal{O}(n+2-d). \tag{5.2}$$

Hence, $-\mathcal{K}_{\mathcal{X}}$ is relatively ample if and only if $d \leqslant n+1$. Now assume that the complex variety X defined by the complex points of $\mathcal{X}$ is non-singular. Then, by the adjunction isomorphism (5.2), a metric ϕ_0 on $\mathcal{O}(n+2-d)|_X$ may be identified with a metric on $-K_X$.

5.1. Reduction to Fermat hypersurfaces. — Given integers a_i consider the subscheme $\mathcal{X}_a$ of $\mathbb{P}^{n+1}_{\mathbb{Z}}$ cut out by the homogeneous polynomial

$$s_a := \sum_{i=0}^{n+1} a_i x_i^d.$$

Denote by X_a the corresponding complex variety, which is non-singular and consider the map

$$F_a : \mathbb{C}^{n+2} \longrightarrow \mathbb{C}^{n+2}, \quad x \longmapsto F_a(x) := (a_0^{-1/d} x_0, \dots, a_{n+1}^{-1/d} x_{n+1}). \tag{5.3}$$

It descends to an automorphism of $\mathbb{P}^{n+1}$ with the property that

$$X_a = F_a(X_1).$$

By identifying $\mathbb{C}^{n+2} \smallsetminus \{0\}$ with the complement of the zero-section of $\mathcal{O}(1) \to \mathbb{P}^{n+1}$, the map F_a induces, for any given integer m, an automorphism of $\mathcal{O}(m) \to \mathbb{P}^{n+1}$ (that we also denote by F_a). In the statement of the following proposition we will use that, by the adjunction isomorphism recalled above, a metric ϕ on $\mathcal{O}(n+2-d)|_{X_a}$ may be identified with a metric on $-K_{X_a}$ and thus induces a measure μ_ϕ on X (Section 2.1.1).

Proposition 20. — *Let k be a positive integer and ϕ a metric on $\mathcal{O}(n+2-d)|_{X_a}$. Then*

$$\begin{aligned} 2\widehat{h}_\phi(\mathcal{X}_a, \mathcal{O}(n+2-d)) &+ \log \int_{X_a} \mu_\phi \\ &= 2\widehat{h}_{F_a^*\phi}(\mathcal{X}_1, \mathcal{O}(n+2-d)) + \log \int_{X_1} \mu_{F_a^*\phi} + \Big(\frac{n+2-d}{(n+1)} - 1\Big) d^{-1} \sum_i \log(|a_i|^2). \end{aligned}$$

The proof of the proposition follows from combining the following two lemmas:

Lemma 21. — *Let k be a positive integer and ϕ a metric on $\mathcal{O}(k)$. Then*

$$2\widehat{h}_\phi(\mathcal{X}_a, \mathcal{O}(k)) = 2\widehat{h}_{F_a^*\phi}(\mathcal{X}_1, \mathcal{O}(k)) + \frac{1}{d} \sum_i \log(|a_i|^2)\, \frac{k}{(n+1)}.$$

Proof. — We will use that any continuous psh metric ϕ on $\mathcal{O}(k)|_X$ is the restriction of a continuous psh metric on $\mathcal{O}(k) \to \mathbb{P}^{n+1}$, that we shall denote by the same symbol ϕ [14]. First consider the case when $k = d$ and denote by s_a the section of $\mathcal{O}(d)$ cutting out the scheme $\mathcal{X}_a$. By the restriction formula (Lemma 19), setting $\mathrm{MA}(\psi) := (dd^c\psi)^{n+1}/(n+1)!$,

$$\frac{2}{(n+1)!}\, h_\phi(\mathcal{X}_a, \mathcal{O}(k)) = (n+2)\mathcal{E}_{\mathbb{P}^{n+1}}(\phi, d\phi_0) + \int_{\mathbb{P}^{n+1}} \log |s_a|^2_\phi \mathrm{MA}(\phi).$$

Rewriting

$$\int_{\mathbb{P}^{n+1}} \log|s_a|^2_\phi MA(\phi) = \int_{\mathbb{P}^{n+1}} \log|(F_a^{-1})^* s_1|^2_\phi \mathrm{MA}(\phi) = \int_{\mathbb{P}^{n+1}} \log|s_1|^2_{F_a^*\phi} \mathrm{MA}(F_a^*\phi),$$

thus reveals that

$$\frac{2}{(n+1)!}\, h_{F_a^*\phi}(X_1, \mathcal{O}(k)) - \frac{2}{(n+1)!}\, h_\phi(X_a, \mathcal{O}(k)) \\ = (n+2)\big(\mathcal{E}_{\mathbb{P}^{n+1}}(F_a^*\phi, d\phi_0) - \mathcal{E}_{\mathbb{P}^{n+1}}(\phi, d\phi_0)\big).$$

Now, denote by G_a the standard lift of F_a from X_1 to $-K_{X_1}$ and its tensor powers, which has the property that $F_a^*\mu_\phi = \mu_{G_a^*\phi}$ for any metric ϕ on $-K_{X_a}$. We can then express

$$G_a^*\phi = F_a^*\phi + c_a, \quad c_a := \frac{k}{(n+2)}\frac{1}{d}\sum_i \log(|a_i|^2).$$

Indeed,

$$G_a^*(e^{-\frac{n+2}{k}\phi} dz d\bar{z}) = (e^{-\frac{n+2}{k}F_a^*\phi}) F_a^*(dz d\bar{z}) = (e^{-\frac{n+2}{k}F_a^*\phi}) \prod_i |a_i|^{-2/d}(dz d\bar{z}). \tag{5.4}$$

Hence,

$$2h_{F_a^*\phi}(\mathbb{P}^{n+1}, \mathcal{O}(k)) = 2h_{G_a^*\phi - c_a}(\mathbb{P}^{n+1}, \mathcal{O}(k)) = 2h_{G_a^*\phi}(\mathbb{P}^{n+1}, \mathcal{O}(k)) - c_a\, \frac{k^{n+1}}{(n+1)!}.$$

But, by Lemma 31 in the appendix,

$$\mathcal{E}_{\mathbb{P}^{n+1}}(G_a^*\phi, d\phi_0) = \mathcal{E}_{\mathbb{P}^{n+1}}(\phi, d\phi_0).$$

Hence,

$$\frac{2}{(n+1)!}\, h_{F_a^*\phi}(X_1, \mathcal{O}(k)) - \frac{2}{(n+1)!}\, h_\phi(X_a, \mathcal{O}(k)) \\ = (n+2)\left(\mathcal{E}_{\mathbb{P}^{n+1}}(F_a^*\phi, d\phi_0) - \mathcal{E}_{\mathbb{P}^{n+1}}(\phi, d\phi_0)\right) = -(n+2)c_a\, \frac{k^{(n+1)}}{(n+1)!}.$$

As a consequence,

$$\begin{aligned} 2\widehat{h}_{F_a^*\phi}(X_1, \mathcal{O}(k)) - 2\widehat{h}_\phi(X_a, \mathcal{O}(k)) &= -(n+2)c_a\, \frac{k^{n+1}}{(n+1)!} \cdot \frac{1}{dk^n/n!} \\ = -(n+2)c_a\, \frac{k}{d(n+1)} &= -\frac{1}{d}\sum_i \log(|a_i|^2)(n+2)\frac{k}{(n+2)}\, \frac{k}{d(n+1)} \\ &= -\frac{1}{d}\sum_i \log(|a_i|^2)\, \frac{k^2}{d(n+1)}. \end{aligned}$$

Since we have assumed that $k = d$ this means that

$$2\widehat{h}_\phi(X_a, \mathcal{O}(k)) - 2\widehat{h}_{F_a^*\phi}(X_1, \mathcal{O}(k)) = \frac{1}{d}\sum_i \log(|a_i|^2)\, \frac{d}{(n+1)}.$$

Finally, for any given integer k we can express $k = d\lambda$ for $\lambda = k/d$ and use the basic scaling property

$$\widehat{h}_{\lambda\phi}(\mathcal{X}, \lambda\mathcal{L}) = \lambda\widehat{h}_\phi(\mathcal{X}, \mathcal{L}),$$

to get

$$2\widehat{h}_{\phi}(X_a,\mathcal{O}(k))-2\widehat{h}_{F_a^*\phi}(X_1,\mathcal{O}(k))=\frac{1}{d}\sum_i\log(|a_i|^2)\,\frac{k}{(n+1)},$$

which concludes the proof. □

Lemma 22. — *Given a metric ϕ on $\mathcal{O}(n+2-d)$ we have that*

$$\log\int_{X_a}\mu_{\phi}=\log\int_{X_1}\mu_{F_a^*\phi}-d^{-1}\sum_i\log(|a_i|^2).$$

Proof. — Let X be the non-singular hypersurface of $\mathbb{P}^{n+1}$ cut out by a given homogeneous polynomial s in $\mathbb{C}^{n+2}$. Let A_X be the zero-locus of s in $\mathbb{C}^{n+2}\smallsetminus\{0\}$ and assume that $ds\neq 0$ on A_X. Let Ω_s be the holomorphic top form on A_X defined by the relation $\Omega_s\wedge ds=dz$ on A_X, where $dz:=dz_0\wedge\cdots\wedge dz_{n+1}$. We can identify X with $A_X/\mathbb{C}^*$ using the standard $\mathbb{C}^*$-action on $\mathbb{C}^{n+2}$. Denote by δ interior multiplication with the holomorphic vector field generating the $\mathbb{C}^*$-action. Assume now that X is Fano. A given metric ϕ on $-K_X$ then corresponds to a one-homogeneous function r on A_X (using the adjunction isomorphism (5.2) and by identifying $\mathbb{C}^{n+2}\smallsetminus\{0\}$ with the complement of the zero-section in $\mathcal{O}(1)^*\to X$). Moreover, lifting the adjunction isomorphism (5.2) to A_X yields the following well-known formula (which applies in the general setup of Fano varieties over local fields; cf. [35, Lem. 4.2.2]):

$$\int_X\mu_{\phi}=c\int_{\{s=0\}/\mathbb{C}^*}r^{d-(n+2)}(\delta\Omega_s\wedge\overline{\delta\Omega_s})$$

for a non-zero constant c only depending on n and d (since Ω_s has degree $(n+2-d)$ with respect to the $\mathbb{C}^*$-action, the (n,n)-form $r^{d-(n+2)}(\delta\Omega_s\wedge\overline{\delta\Omega_s})$ is $\mathbb{C}^*$-invariant and thus descends to a real top form on X). Hence, setting $F:=F_a$,

$$\int_X\mu_{\phi}=c\int_{\{F^*s=0\}/\mathbb{C}^*}(F^*r)^{d-(n+2)}(\delta F^*\Omega_s\wedge\overline{\delta F^*\Omega_s}).$$

To conclude the proof it will thus be enough to verify that

$$F^*\Omega_s=\Omega_{F^*s}a_0^{-1/d}\cdots a_{n+1}^{-1/d}.$$

To this end, note that applying F^* to the defining relation for Ω_s yields $F^*\Omega_s\wedge d(F^*f)=F^*dz$. Since $F^*dz=a_0^{-1/d}\cdots a_{n+1}^{-1/d}dz$ this concludes the proof. □

It follows directly from Proposition 20 and the definition of $h_{\mathrm{can}}(\mathcal{X})$ that

$$h_{\mathrm{can}}(\mathcal{X}_a)-h_{\mathrm{can}}(\mathcal{X}_1)=(n+1)(n+2-d)^nd\Big(\frac{n+2-d}{(n+1)}-1\Big)d^{-1}\sum_i\log(|a_i|^2).\tag{5.5}$$

In particular, since $n+2-d\leqslant n+1$, this means that

$$h_{\mathrm{can}}(\mathcal{X}_a)\leqslant h_{\mathrm{can}}(\mathcal{X}_1),\tag{5.6}$$

and thus the proof of Theorem 4 is reduced to the case of the Fermat hypersurface $\mathcal{X}_1$.

Remark 23. — For any sufficiently large finite field extension $\mathbb{F}$ of $\mathbb{Q}$, the schemes $\mathcal{X}_a \otimes_{\mathbb{Z}} \mathbb{F}$ and $\mathcal{X}_1 \otimes_{\mathbb{Z}} \mathbb{F}$ are isomorphic over $\mathbb{F}$ and, in particular, over $\mathbb{C}$ – the isomorphism is provided by the map F_a. Moreover, since X_1 admits a Kähler-Einstein metric [45, §6.3], $h_{\mathrm{can}}(\mathcal{X}_1) = h_{\phi_1}(\mathcal{X}_1)$, for any volume-normalized Kähler-Einstein metric ϕ_1 on $-K_{X_1}$. Denoting by ϕ_a the corresponding metric on $-K_{X_a}$, induced by the natural lift G_a of F_a from $-K_{X_1}$ to $-K_{X_a}$, the identity (5.5) thus holds for $h_{\phi_a}(\mathcal{X}_a) - h_{\phi_1}(\mathcal{X}_1)$ (this also follows directly from Proposition 20). Such a difference is, by general principles [2, Lem. 2.4], a purely arithmetic quantity (independent of the choice of metric ϕ_1), since G_a induces an isomorphism $\mathcal{K}_{\mathcal{X}_a} \otimes_{\mathbb{Z}} \mathbb{F} \simeq \mathcal{K}_{\mathcal{X}_a} \otimes_{\mathbb{Z}} \mathbb{F}$. It seems thus natural to ask if there is a direct scheme-theoretic proof of the identity (5.5)? Interestingly, the corresponding inequality (5.6) is reminiscent of Odaka's minimization conjecture [33, 22]. But an important difference is that $h_{\mathrm{can}}(\mathcal{X}_1)$ does not, in general, maximize $h_{\mathrm{can}}(\mathcal{X})$ over *all* integral models $\mathcal{X}$ of $\mathcal{X}_1 \otimes_{\mathbb{Z}} \mathbb{Q}$ (and likewise when $\mathbb{Q}$ is replaced by a number field $\mathbb{F}$). For example, when $d = 2$, $\mathcal{X}_1 \otimes_{\mathcal{O}_{\mathbb{F}}} \mathbb{F}$ is isomorphic to $\mathbb{P}^1_{\mathbb{F}}$ for $\mathbb{F} = \mathbb{Q}(\sqrt{-1})$, but $h_{\mathrm{can}}(\mathcal{X}_1 \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{F}}) < h_{\mathrm{can}}(\mathbb{P}^1_{\mathcal{O}_{\mathbb{F}}})$, as follows from the conclusion of Theorem 3 (after a base change). Alternatively, the strict inequality follows from Corollary 6, applied to $\Delta = 0$, since $\mathcal{X}_1 \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{F}}$ is not isomorphic to $\mathbb{P}^1_{\mathcal{O}_{\mathbb{F}}}$ over $\mathcal{O}_{\mathbb{F}}$.

Before continuing we also make a final remark.

Remark 24. — Let $\mathcal{X}$ be a hypersurface in $\mathbb{P}^{n+1}_{\mathbb{Z}}$ cut-out by a homogeneous polynomial s of degree d of the form T^*s_1 where $T \in \mathrm{GL}(n+2, \mathbb{C})$. Then formula (5.5) can be generalized as follows (shown in essentially the same manner as before):

$$h_{\mathrm{can}}(\mathcal{X}) = h_{\mathrm{can}}(\mathcal{X}_1) + (n+1)(n+2-d)^n d\Big(\frac{n+2-d}{(n+1)} - 1\Big)\log(|\det T|^2).$$

5.2. Reduction to log hyperplane arrangements.

— Fix a degree $d(\leqslant n+1)$ and denote by $\mathcal{X}$ the corresponding Fermat hypersurface. The Fermat hypersurface of degree one will be denoted by $\mathcal{Y}$. We will next express the canonical height $h_{\mathrm{can}}(\mathcal{X})$ in terms of the canonical height $h_{\mathrm{can}}(\mathcal{Y}, \mathcal{D})$ where $\mathcal{D}$ is the divisor on $\mathcal{Y}$ defined by

$$\mathcal{D} = (1-1/d)\,[x_0 = 0] + \cdots + (1-1/d)\,[x_n = 0] + (1-1/d)\big[\textstyle\sum_{i=0}^n x_i = 0\big],$$

where x_i denotes the homogeneous coordinates on $\mathbb{P}^{n+1}_{\mathbb{Z}}$ restricted to $\mathcal{Y}$.

Proposition 25. — *Denote by $\mathcal{X}$ the Fermat hypersurface of a given degree $m(\leqslant n+1)$ and by $\mathcal{Y}$ the Fermat hypersurface of degree one, endowed with the divisor $\mathcal{D}$. Then*

$$\widehat{h}_{\mathrm{can}}(\mathcal{X}) = \widehat{h}(\mathcal{Y}, \mathcal{D}) - \frac{1}{2}\log\frac{V(X)}{V(Y,\Delta)}. \tag{5.7}$$

Proof. — By the adjunction formula we have isomorphisms $-K_{\mathcal{X}} \simeq (n+2-m)\mathcal{O}(1)|_{\mathcal{X}}$ and $-\mathcal{K}_{(\mathcal{Y},\Delta)} \simeq (n+2-m)(1/m)\mathcal{O}(1)|_{\mathcal{Y}}$. Consider the following morphism:

$$F : \mathbb{C}^{n+2} \longrightarrow \mathbb{C}^{n+2}, \quad (x_0, \ldots, x_{n+1}) \longmapsto (x_0^m, \ldots, x_{n+1}^m),$$

which induces a map $\mathbb{P}^{n+1} \to \mathbb{P}^{n+1}$ and a lift to $\mathcal{O}(1)$, which is naturally defined over $\mathbb{Z}$ and satisfies $F^*\mathcal{O}(1) \simeq m\mathcal{O}(1)$. In particular, it induces a morphism

$$F : \mathcal{X} \longrightarrow \mathcal{Y}, \quad F^*(-\mathcal{K}_{(\mathcal{Y},\Delta)}) \simeq -K_{\mathcal{X}},$$

under the adjunction isomorphisms. We recall that, by basic functorial properties of heights,

$$\widehat{h}(\mathcal{X}, F^*\mathcal{L}) = \widehat{h}(\mathcal{Y}, \mathcal{L}). \tag{5.8}$$

In fact, in this case this formula follows directly from the analytic representation of the height in Lemma 19, using that F preserves the Weil metric ϕ_0. In particular, setting $\mathcal{L} := (n+2-m)(1/m)\mathcal{O}(1)_{|\mathcal{Y}}$ and using the adjunction isomorphisms yields $\widehat{h}(\mathcal{X}, -K_{\mathcal{X}}, F^*\phi) = \widehat{h}(\mathcal{Y}, -\mathcal{K}_{(\mathcal{Y},\Delta)}, \phi)$. Thus, all that remains is to show that

$$\int_X \mu_{F^*\phi} = m^{-(n+1)} \int_Y \mu_\phi,$$

where we have used that $F^*\phi$ induces a metric on $-K_X$ and ϕ induces a metric on $-K_{(Y,\Delta)}$. Since F has topological degree $m^{(n+1)}$ we have $F_*[X_m] = m^{(n+1)}[Y]$ as homology classes and thus it will be enough to show that

$$F^*\mu_\phi = m^{2(n+1)}\mu_{F^*\phi}.$$

To this end consider the affine piece $\mathbb{C}^{n+1}$ of $\mathbb{P}^{n+1}$ where $x_0 \neq 0$. Setting $z_i = x_i/x_0$ for $i = 1, \ldots, n+1)$ we can, locally, parametrize X by the coordinates $z_1, \ldots, z_n$. In these coordinates a metric ψ on the restriction of $\mathcal{O}(1)$ to X induces, by the adjunction isomorphism, a metric on $-K_X$ and thus a measure on X locally expressed as

$$\mu_\psi = \frac{e^{-(n+2-m)\psi}}{(m|z_{n+1}|^{m-1})^2} \frac{i}{2}\, \mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 \cdots \frac{i}{2}\, \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n. \tag{5.9}$$

To see this, recall that, by definition,

$$\mu_\psi := \|\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n\|_\psi^{-2} \frac{i}{2}\, \mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 \cdots \frac{i}{2}\, \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n.$$

In the affine piece $\mathbb{C}^{n+1}$ of $\mathbb{P}^{n+1}$ we can express

$$s = f x_0^{\otimes 2}, \quad f = 1 + \sum_{i=1}^n z_i^m + z_{n+1}^m.$$

By the adjunction isomorphism (5.2) we have

$$\|\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n\| := \|\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}s\| = \|\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}f\|\, \|x_0^{\otimes m}\|.$$

Since $\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}f = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}z_{n+1}\partial f/\partial z_{n+1}$ this means that

$$\|\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n\|^2 := \Big|\frac{\partial f}{\partial z_{n+1}}\Big|^2 \|\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}z_{n+1}\|\, \|x_0^{\otimes m}\| = e^{(n+2)\psi} e^{-m\psi},$$

giving

$$\mu_\phi = \frac{e^{-(n+2-m)\psi}}{|\partial f/\partial z_{n+1}|^2} \frac{i}{2}\, \mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 \cdots \frac{i}{2}\, \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n, \tag{5.10}$$

which proves (5.9). Likewise, we can parametrize the affine piece of Y by the coordinates $z_1, \ldots, z_n$. A given metric ϕ on the restriction of $\mathcal{O}(1)$ to Y induces, by the adjunction isomorphism a measure (defined with respect to the divisor $\mathcal{D}$)

$$\mu_\phi = e^{-(n+2-m)m^{-1}\phi} |z_{n+1}|^{-2(1-1/m)} |z_1|^{-2(1-1/m)} \cdots |z_n|^{-2(1-1/m)} \cdot \frac{i}{2} \mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 \cdots \frac{i}{2} \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n.$$

Since $F^* z_i = z_i^m$ this means that

$$F^*\mu_\phi = e^{-(n+2-m)f^*(m^{-1}\phi)} |z_{n+1}|^{-2(m-1)} |z_1|^{-2(m-1)} \cdots |z_n|^{-2(m-1)} \cdot \frac{i}{2} \mathrm{d}(z_1^m) \wedge \mathrm{d}(\overline{z}_1^m) \cdots \frac{i}{2} \mathrm{d}(z_n^m) \wedge \mathrm{d}(\overline{z}_n^m).$$

Finally, since $\mathrm{d}(z^m) = mz^{m-1}$ this proves the desired identity (5.7), using the representation (5.10) with $\psi = F^*(m^{-1}\phi)$. $\square$

5.3. Conclusion of the proof of Theorem 4. — The affine projection

$$(x_0, \ldots, x_{n+1}) \longmapsto (x_0, \ldots, x_n)$$

induces an isomorphism from $\mathcal{Y}$ to $\mathbb{P}^n_{\mathbb{Z}}$, identifying $(\mathcal{Y}, \mathcal{D})$ with a hyperplane arrangement $(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{D})$ with simple normal crossings. It follows readily from the definition of $\mathcal{D}$ and the criterion (4.1) that $(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{D})$ is K-semistable. Hence, combining Proposition 25 with refined bound following the statement of Theorem 3 yields

$$\widehat{h}_{\mathrm{can}}(\mathcal{X}) \leqslant \widehat{h}_{\mathrm{can}}(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{D}_t) - \frac{1}{2} \log \frac{V(X)}{V(\mathbb{P}^n, \Delta_t)},$$

where $\mathcal{D}_t$ is the toric divisor on $\mathbb{P}^n_{\mathbb{Z}}$ such that $(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{D}_t)$ is K-semistable and $V(\mathbb{P}^n, \Delta_t) = V(\mathbb{P}^n, \Delta)$. The explicit formula for $\widehat{h}_{\mathrm{can}}(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{D}_t)$ thus yields

$$\widehat{h}_{\mathrm{can}}(\mathcal{X}) \leqslant \widehat{h}_{\mathrm{can}}(\mathbb{P}^n_{\mathbb{Z}}) - \frac{1}{2} \log \frac{V(X)}{V(\mathbb{P}^n)}.$$

Multiplying both sides with $V(X)$ reveals that

$$h_{\mathrm{can}}(\mathcal{X}) \leqslant \lambda h_{\mathrm{can}}(\mathbb{P}^n_{\mathbb{Z}}) - \frac{1}{2} V(X) \log \lambda, \qquad \lambda := V(X)/V(\mathbb{P}^n).$$

Since $\lambda \in \,]0, 1[$ it thus follows from Lemma 10 that the right hand side above is increasing with respect to λ and thus maximal when $\lambda = 1$, giving $h_{\mathrm{can}}(\mathcal{X}, \mathcal{D}) \leqslant h_{\mathrm{can}}(\mathbb{P}^n_{\mathbb{Z}})$. Moreover, the equality is strict if $d \geqslant 2$ since then $\lambda < 1$.

6. Arithmetic log surfaces

In this section we will prove Theorem 5. Throughout the section $\mathcal{X}$ will be assumed normal. Given an effective divisor $\Delta_{\mathbb{Q}}$ on $\mathbb{P}^1_{\mathbb{Q}}$ supported at three points, such that $-K_{(\mathbb{P}^1, \Delta)}$ is ample, we define the *canonical model* of $(\mathbb{P}^1_{\mathbb{Q}}, \Delta_{\mathbb{Q}}; -K_{(\mathbb{P}^1_{\mathbb{Q}}, \Delta_{\mathbb{Q}})})$ over $\mathbb{Z}$ as $(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{D}_c; -K_{(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{D}_c)})$ where $\mathcal{D}_c$ is the Zariski closure of the divisor on $\mathbb{P}^1_{\mathbb{Q}}$ supported on $\{0, 1, \infty\}$, having the same coefficients as $\Delta_{\mathbb{Q}}$. That this is a model over $\mathbb{Z}$ follows from the basic fact that any three points on $\mathbb{P}^1_{\mathbb{Q}}$ can be mapped to $\{0,1,\infty\}$, by an automorphism of $\mathbb{P}^1_{\mathbb{Q}}$,

As explained in Section 1.1.4, by Theorem 3, the proof is reduced to showing that for any fixed metric on $-K_{(\mathbb{P}^1,\Delta)}$:

– The canonical integral model $(\mathbb{P}^1_{\mathbb{Z}},\mathcal{D}_c;-\mathcal{K}_{(\mathbb{P}^1_{\mathbb{Z}},\mathcal{D}_c)})$ of $(\mathbb{P}^1_{\mathbb{Q}},\Delta_{\mathbb{Q}};-K_{(\mathbb{P}^1_{\mathbb{Q}},\Delta_{\mathbb{Q}})})$ minimizes $\mathcal{M}_{(\mathcal{X},\mathcal{D})}(\overline{\mathcal{L}})$ over all integral models $(\mathcal{X},\mathcal{D};\mathcal{L})$ of $(\mathbb{P}^1_{\mathbb{Q}},\Delta_{\mathbb{Q}};-K_{(\mathbb{P}^1_{\mathbb{Q}},\Delta_{\mathbb{Q}})})$.

– When $\Delta_{\mathbb{Q}}=0$, the minimum is uniquely attained for $\mathcal{X}=\mathbb{P}^1_{\mathbb{Z}}$, up to isomorphisms over $\mathbb{Z}$.

6.1. Preliminaries on log canonical thresholds. — Following [25, 42] a log pair $(\mathcal{X},\mathcal{D})$ is said to be *log canonical (lc)* if for any normal blow-up morphism $p:\mathcal{Y}\to\mathcal{X}$

$$\mathcal{K}_{\mathcal{Y}/\mathcal{X}}-p^*\mathcal{D}=\sum_i a_iE_i,\qquad a_i\geqslant -1,\quad \mathcal{K}_{\mathcal{Y}/\mathcal{X}}:=\mathcal{K}_{\mathcal{Y}}-p^*\mathcal{K}_{\mathcal{X}},$$

where the prime divisor E_i is either an exceptional divisor of p or the proper transform of a component of $\mathcal{D}$. The *log canonical threshold* of a $\mathbb{Q}-$divisor F on $\mathcal{X}$ with respect to the log pair $(\mathcal{X},\mathcal{D})$ is defined by

$$\operatorname{lct}(\mathcal{X},\mathcal{D};F):=\sup_{t>0}\{t:(\mathcal{X},tF+\mathcal{D})\text{ is lc}\}.$$

The following lemma follows readily from the definition:

Lemma 26. — *For any normal blow-up morphism $p:\mathcal{Y}\to\mathcal{X}$*

$$\operatorname{lct}(\mathcal{X},\mathcal{D};F)\leqslant\inf_i\frac{a_i-b_i+1}{c_i},$$

where a_i,b_i and c_i denote the order of vanishing along the p-exceptional prime divisor E_i of $\mathcal{K}_{\mathcal{Y}/\mathcal{X}}$, $p^*\mathcal{D}$ and p^*F, respectively and i ranges over all p-exceptional prime divisors.

6.2. Preparations for the proof of Theorem 5. — The following result is a logarithmic generalization of [33, Th. 2.14 (3)] (in the case of arithmetic surfaces).

Lemma 27. — *Let (X,D) be a log Fano curve over $\mathbb{C}$ and $(\mathcal{X},\mathcal{D})$ an arithmetic log Fano model for (X,D) such the fibers $\mathcal{X}_b$ of $\mathcal{X}$ are reduced and irreducible and the divisor $\mathcal{D}$ is horizontal (i.e., $\mathcal{D}$ is the Zariski closure of D). Assume that*

$$\alpha(\mathcal{X},\mathcal{D}):=\inf_{b,F}\operatorname{lct}(\mathcal{X},\mathcal{D}+\mathcal{X}_b;\,F)\geqslant 1/2,$$

where the inf runs over all effective $\mathbb{Q}-$divisors F on $\mathcal{X}$ linearly equivalent to $-\mathcal{K}_{(\mathcal{X},\mathcal{D})}$ and closed points b in the base $\mathcal{B}:=\operatorname{Spec}\mathbb{Z}$ such that F does not contain the support of $\mathcal{X}_b$. Then

$$\frac{1}{2}\overline{\mathcal{L}}^2+\overline{\mathcal{K}}_{(\mathcal{X}',\mathcal{D})}\cdot\overline{\mathcal{L}}\geqslant\frac{1}{2}\overline{\mathcal{L}}^2+\overline{\mathcal{K}}_{(\mathcal{X},\mathcal{D})}\cdot\overline{\mathcal{L}}$$

for any relatively ample model $(\mathcal{X}',\mathcal{D}';\mathcal{L}')$ of $(X,D;-K_{(X,D)})$ and given metrics on $-K_{(X,D)}$ and L.

In the proof we fix once and for all metrics on $-K_{(X,D)}$ and L and set

$$\mathcal{M}_{(\mathcal{X},\mathcal{D})}(\mathcal{L}) := \frac{1}{2}\overline{\mathcal{L}}^2 + \overline{\mathcal{K}}_{(\mathcal{X},\mathcal{D})} \cdot \overline{\mathcal{L}} \tag{6.1}$$

for the corresponding metrized lines bundles $\overline{\mathcal{L}}$ and $\overline{\mathcal{K}}_{(\mathcal{X},\mathcal{D})}$. Thus $\mathcal{M}_{(\mathcal{X},\mathcal{D})}(\mathcal{L})$ specializes to the arithmetic log Mabuchi functional $\mathcal{M}_{(\mathcal{X},\mathcal{D})}(\overline{\mathcal{L}})$ (formula (1.5)) precisely when the metric on $\overline{\mathcal{K}}_{(\mathcal{X},\mathcal{D})}$ is the one induced from the curvature form ω of $\overline{\mathcal{L}}$. But here it will be convenient to consider the present more general setup.

Proof when D is trivial. — To fix ideas we first consider the case when D is trivial. Set $\mathcal{B} := \operatorname{Spec}\mathbb{Z}$ and $\mathcal{L} := -\mathcal{K}_{\mathcal{X}}$. To simplify the notation we will remove the bar indicating the metric in the notation for the arithmetic intersection numbers. Anyhow, all the arithmetic intersections will be computed over the closed points b in the base $\mathcal{B}$ and are thus independent of the choice of metric (since they are proportional to the algebraic intersections on the scheme $\pi^{-1}(b)$ over the residue field of b). If F_1 and F_2 are $\mathbb{Q}$-divisors we will write $F_1 \geqslant F_2$ if $F_1 - F_2$ is effective.

Step 1. — It is enough to consider the case of a relatively semi-ample model of the form $(\mathcal{X}', \mathcal{L}') = (\mathcal{Y}, p^*\mathcal{L} - E)$ where $p : \mathcal{Y} \to \mathcal{X}$ is the blow-up along a closed subscheme $\mathcal{Z}$ of $\mathcal{X}$ and E is an effective p-exceptional divisor on $\mathcal{Y}$ whose support contains all the p-exceptional prime divisors and such that for any $b \in \pi(\mathcal{Z})$ $p^*\mathcal{L} - E$ admits a global section s_b not vanishing identically along $\mathcal{Y}_b$.

This is shown precisely as in the proof of [32, Prop. 3.10] – for completeness, a proof is given in Step 1 in Section 6.3 below.

Step 2: The inequality holds in the case of Step 1. — First observe that

$$\mathcal{M}_{\mathcal{Y}}(\mathcal{L}') - \mathcal{M}_{\mathcal{X}}(\mathcal{L}) = \mathcal{L}' \cdot \Big(\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - \frac{1}{2}E\Big). \tag{6.2}$$

Indeed, rewriting

$$\mathcal{M}_{\mathcal{X}}(\mathcal{L}) = -\frac{\mathcal{L}^2}{2} + \cdot\mathcal{L} \cdot (\mathcal{L} + \mathcal{K}_{\mathcal{X}})$$

(and likewise for $(\mathcal{Y}, \mathcal{L}')$) the left hand side in formula (6.2) may be expressed as

$$\frac{p^*\mathcal{L}^2 - \mathcal{L}'^2}{2} + \mathcal{L}' \cdot (\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - E) = \frac{E \cdot (p^*\mathcal{L} + \mathcal{L}')}{2} + \mathcal{L}' \cdot (\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - E).$$

Since $E \cdot p^*\mathcal{L} = 0$ this proves formula (6.2).

Since $\mathcal{L}'$ is relatively semi-ample it will thus be enough to show that the vertical exceptional divisor $\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - \frac{1}{2}E$ is effective. This means, by the assumption on $\alpha(\mathcal{X})$ $(:= \alpha(\mathcal{X}, 0))$, that it is enough to show that

$$\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - \alpha(\mathcal{X})E \geqslant 0. \tag{6.3}$$

To fix ideas first assume that $\pi(\mathcal{Z})$ is supported on a single point that we denote by b. By Step 1, we can express $s_b = p^*s$ for a global section s of $\mathcal{L} \to \mathcal{X}$ whose zero-divisor F does not vanish identically on $\mathcal{X}_b$ and such that $p^*F - E$ is effective. Since F is a contender for the inf defining $\alpha(\mathcal{X})$ we have

$$\alpha(\mathcal{X}) \leqslant \operatorname{lct}(\mathcal{X}, \mathcal{X}_b;\, F),$$

Next, since $p^*F \geqslant E$ it follows from Lemma 26 that

$$\operatorname{lct}(\mathcal{X}, \mathcal{X}_b;\, F) \leqslant \inf_i \frac{a_i + 1 - b_i}{c_i},$$

where i runs over the p-exceptional irreducible prime divisors E_i of $\mathcal{Y}$ and a_i, b_i and c_i denote the order of vanishing along E_i of $\mathcal{K}_{\mathcal{Y}/\mathcal{X}}$, $\mathcal{Y}_b$ and E respectively. Note that $c_i > 0$ (since the support of E contains the support of all p-exceptional divisors) and $b_i \geqslant 1$ (since $\mathcal{Z}$ is assumed to be supported in $\mathcal{X}_b$). Thus

$$\alpha(\mathcal{X}) \leqslant \inf_i \frac{a_i + 1 - b_i}{c_i} \leqslant \inf_i \frac{a_i}{c_i},$$

giving

$$\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - \alpha(\mathcal{X})E \geqslant \sum_i a_i E_i - (\min\nolimits_j a_j/c_j)\, E_i = \sum_i (a_i/c_i - (\min\nolimits_j a_j/c_j))\, c_i E_i \geqslant 0,$$

which proves (6.3). Finally, consider the general case when the support of $\pi(\mathcal{Z})$ consists of a finite number of points b_m in $\mathcal{B}$. We then split the vertical divisors $\mathcal{K}_{\mathcal{Y}/\mathcal{X}}$ and E into the components $\mathcal{K}^{(m)}_{\mathcal{Y}/\mathcal{X}}$ and $E^{(m)}$ over b_m:

$$\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - \alpha(\mathcal{X})E = \sum_m \mathcal{K}^{(m)}_{\mathcal{Y}/\mathcal{X}} - \alpha(\mathcal{X})E^{(m)},$$

and apply the previous bound for each fixed m (with b replaced by b_m) to get that $\mathcal{K}^{(m)}_{\mathcal{Y}/\mathcal{X}} - \alpha(\mathcal{X})E^{(m)} \geqslant 0$ and thus $\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - \alpha(\mathcal{X})E \geqslant 0$, as desired.

Proof for log pairs. — Just as in the previous case it is enough to consider the special case of Step 2. In this case formula (6.2) readily generalizes to

$$\mathcal{M}_{(\mathcal{Y}, q^*\mathcal{D}')}(\mathcal{L}') - \mathcal{M}_{(\mathcal{X},\mathcal{D})}(\mathcal{L}) = \mathcal{L}' \cdot \Big(\mathcal{D}' - p^*\mathcal{D} + \mathcal{K}_{\mathcal{Y}/\mathcal{X}} - \frac{1}{2}E\Big). \tag{6.4}$$

As before it will thus be enough to show that

$$\mathcal{K}_{\mathcal{Y}/\mathcal{X}} + q^*\mathcal{D}' - p^*\mathcal{D} - \alpha(\mathcal{X}, \mathcal{D})E \geqslant 0. \tag{6.5}$$

To simplify the exposition we will assume that $\pi(\mathcal{Z})$ is a single closed point in $\mathcal{B}$, denoted by b (the general case is shown in a similar way by decomposing with respect to the components of $\pi(\mathcal{Z})$ as above). By the definition of $\alpha(\mathcal{X}, \mathcal{D})$,

$$\alpha(\mathcal{X}, \mathcal{D}) \leqslant \operatorname{lct}(\mathcal{X}, \mathcal{D} + \mathcal{X}_b;\, F), \tag{6.6}$$

Next, since $p^*F - E$ is effective, i.e., $p^*F \geqslant E$, Lemma 26 yields

$$\operatorname{lct}(\mathcal{X}, \mathcal{D} + \mathcal{X}_b;\, F) \leqslant \inf_i \frac{a_i + 1 - d_i - b_i}{c_i},$$

where a_i, b_i, c_i and d_i are the order of vanishing along E_i of $\mathcal{K}_{\mathcal{Y}/\mathcal{X}}$, $\mathcal{Y}_b$, E and $p^*\mathcal{D}$ respectively. In particular, $b_i \geqslant 1$ since $\mathcal{Z}$ is supported in $\mathcal{X}_b$, Hence,

$$\alpha(\mathcal{X},\mathcal{D}) \leqslant \inf_i \frac{a_i+1-d_i-b_i}{c_i}.$$

Next, we may decompose

$$p^*\mathcal{D} = (p^*\mathcal{D})_{\mathrm{hor}} + (p^*\mathcal{D})_{\mathrm{ex}},$$

where $(p^*\mathcal{D})_{\mathrm{hor}}$ is the horizontal divisor obtained as the proper transform of the horizontal divisor $\mathcal{D}$ and $(p^*\mathcal{D})_{\mathrm{ex}}$ is p-exceptional. By (6.6)

$$\begin{aligned}\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - (p^*\mathcal{D})_{\mathrm{ex}} - \alpha E_{\mathrm{ex}} &\geqslant \sum_i (a_i-d_i)E_i - \Big(\min_j \frac{a_j-b_j+1-d_j}{c_j}\Big)c_iE_i\\ &\geqslant \sum_i \Big(\frac{(a_i-d_i)}{c_i}E_i - \Big(\min_j \frac{a_j-d_j}{c_j}\Big)\Big)c_iE_i \geqslant 0,\end{aligned}$$

using that $b_j \geqslant 1$. Hence,

$$\mathcal{K}_{\mathcal{Y}/\mathcal{X}} + \mathcal{D}' - p^*\mathcal{D} - \alpha(\mathcal{X},\mathcal{D})E_{\mathrm{ex}} \geqslant \mathcal{D}' - (p^*\mathcal{D})_{\mathrm{hor}}.$$

But, since both $(\mathcal{X}',\mathcal{D}')$ and $(\mathcal{X},\mathcal{D})$ are models for (X,Δ) and $\mathcal{D}$ is assumed horizontal it follows that $q^*\mathcal{D}' - (p^*\mathcal{D})_{\mathrm{hor}}$ is an effective vertical divisor and hence

$$q^*\mathcal{D}' - (p^*\mathcal{D})_{\mathrm{hor}} \geqslant 0,$$

which concludes the proof of the inequality (6.5).

Lemma 28. — *If (X,Δ) is a K-semistable log Fano curve over $\mathbb{C}$, then the canonical model $(\mathbb{P}^1_{\mathbb{Z}},\mathcal{D}_c)$ of (X,Δ) satisfies*

$$\alpha(\mathcal{X},\mathcal{D}) \geqslant 1/2,$$

and the inequality is strict if (X,Δ) is K-stable.

Proof. — By inversion of adjunction on surfaces over excellent schemes [42]

$$\mathrm{lct}(\mathcal{X},\mathcal{D}+\mathcal{X}_b;\, F) = \mathrm{lct}(\mathcal{X},\mathcal{D}_{|\mathcal{X}_b}; F_{|\mathcal{X}_b}),$$

if F does not contain the support of the divisor $\mathcal{X}_b$. In the present case $\mathcal{X}_b = \mathbb{P}^1_{\mathbb{F}_b}$, where b has been identified with a prime number and $\mathbb{F}_b$ denotes the field with b elements. Decomposing

$$\mathcal{D}_c = \sum w_i\mathcal{D}_i,$$

the K-semistability assumption is, by (4.2), equivalent to the condition

$$w_j \leqslant \frac{1}{2}\sum_i w_i, \quad \forall j. \tag{6.7}$$

We recall that for any curve C over a perfect field (here taken to be $\mathbb{P}^1_{\mathbb{F}_b}$) an effective $\mathbb{Q}$-divisor F on C is lc if and only if all its coefficients are less then are equal to one [25, 42]. Since $-K_{\mathbb{P}^1_{\mathbb{F}_b}}$ is linearly equivalent to $\mathcal{O}(2)$ and $D_{i|\mathcal{X}_b}$ is linearly equivalent to $\mathcal{O}(1)$ it thus follows from the weight condition (6.7) that $\mathrm{lct}(\mathcal{X},\mathcal{D}_{|\mathcal{X}_b}; F_{|\mathcal{X}_b}) \geqslant 1/2$.

Indeed, it is enough to consider the case when $F = (2 - \sum_i w_i)[x]$, where $[x]$ denotes the prime divisor on $\mathbb{P}^1_{\mathbb{F}_b}$ corresponding to a closed point x in $\mathbb{P}^1_{\mathbb{F}_b}$. Then

$$\frac{1}{2}F + \mathcal{D}_{|\mathcal{X}_b} = \Big(1 - \frac{1}{2}\sum_i w_i\Big)[x] + \sum w_i[x_i].$$

The definition of $\mathcal{D}_c$ ensures that $[x_i] = [x_j]$ if and only if $i = j$. In the case that $x \neq x_i$ for any i the coefficients of $F/2 + \mathcal{D}_{|\mathcal{X}_b}$ are indeed less than or equal to 1, since $w_i \in [0,1]$. Moreover, if $x = x_j$ then the coefficient of index j equals $(1 - \frac{1}{2}\sum_i w_i) + w_j$ which is less than are equal to 1, by the weight condition (6.7).

We will also need the following lemma, shown precisely as in the case when $\mathcal{D} = 0$ considered in [1, Prop. 5.3]. □

Lemma 29. — *Let $(\mathcal{X}, \mathcal{D}; \mathcal{L})$ be a polarized arithmetic log surfaces $(\mathcal{X}, \mathcal{D}; \mathcal{L})$ such that the complexification (X, Δ) of $(\mathcal{X}, \mathcal{D})$ is a log Fano variety and $\mathcal{L} \otimes \mathbb{C} = -K_{(X,\Delta)}$. A metric realizes the infimum*

$$\inf_{\|\cdot\|} \mathcal{M}_{(\mathcal{X},\mathcal{D})}(\mathcal{L}, \|\cdot\|)$$

over all locally bounded metrics on $-(K_{(X,D)})$ with positive curvature current if and only if the metric is a log Kähler-Einstein metric. In particular, in the case when $D = 0$ any minimizer coincides with the Fubini-Study metric up to the application of an automorphism of X and a scaling of the metric. Moreover,

$$\inf_{\|\cdot\|} \mathcal{M}_{(\mathbb{P}^1_{\mathbb{Z}},\mathcal{D})}\big(-\mathcal{K}_{(\mathbb{P}^1_{\mathbb{Z}},\mathcal{D})}, \|\cdot\|\big) = -\sup_{\|\cdot\|} \frac{1}{2}\big(-\mathcal{K}_{(\mathbb{P}^1_{\mathbb{Z}},\mathcal{D})}, \|\cdot\|\big)^2,$$

where the sup in the right-hand side is restricted to volume-normalized metrics.

6.3. Conclusion of the proof of Theorem 2 and Corollary 6. — Combining the previous first two lemmas immediately yields

$$\mathcal{M}_{(\mathcal{X}',\mathcal{D}')}(\overline{\mathcal{L}'}) \geqslant \mathcal{M}_{(\mathbb{P}^1_{\mathbb{Z}},\mathcal{D})}(\overline{-\mathcal{K}_{(\mathbb{P}^1_{\mathbb{Z}},\mathcal{D})}}). \tag{6.8}$$

Applying the third lemma above thus gives

$$\mathcal{M}_{(\mathcal{X}',\mathcal{D}')}(\overline{\mathcal{L}'}) \geqslant -\sup_{\|\cdot\|}\Big(\frac{1}{2}\big(-\mathcal{K}_{(\mathbb{P}^1_{\mathbb{Z}},\mathcal{D})}, \|\cdot\|\big)^2\Big),$$

where the infimum in the left hand side is restricted to volume-normalized metrics. Invoking Theorem 3 and using that the Fubini-Study metric is a minimizer when $\mathcal{D} = 0$ (by Lemma 29) thus proves the inequality in Theorem 5 and it corollary. Moreover, Theorem 3 implies that the inequality is strict, as soon as D is non-trivial.

6.3.1. *The equality case.* — Consider now the case of equality in Theorem 5 (and, as a consequence, $D = 0$):

$$\mathcal{M}_{(\mathcal{X}',\mathcal{D}')}(\overline{\mathcal{L}'}) = \mathcal{M}_{(\mathbb{P}^1_{\mathbb{Z}},0)}(\overline{-\mathcal{K}_{\mathbb{P}^1_{\mathbb{Z}}}}), \tag{6.9}$$

where, in the right-hand side, $-\mathcal{K}_{\mathbb{P}^1_{\mathbb{Z}}}$ is endowed with the Fubini-Study metric. By the minimizing property in Lemma 26, when $D = 0$, the metric on $\mathcal{L}$ coincides with the Fubini-Study metric up to the application of an automorphism of X and scaling of the metric. All that remains is to show is thus that $(\mathcal{X}', \mathcal{D}')$ is isomorphic to $(\mathbb{P}^1_{\mathbb{Z}}, 0)$. To this end first note that since $\mathcal{X}$ $(= \mathbb{P}^1_{\mathbb{Z}})$ and $\mathcal{X}'$ have the same generic fiber they are birationally equivalent. Thus, there exists a normal variety $\mathcal{Y}$, which is flat and projective over $\mathcal{B}$, dominating both $\mathcal{X}$ and $\mathcal{X}'$, with birational morphisms

$$p:\mathcal{Y}\longrightarrow\mathcal{X},\quad q:\mathcal{Y}\longrightarrow\mathcal{X}',\tag{6.10}$$

which are the identity over the generic point in $\mathcal{B}$ (a concrete construction is given in Step 1 below). It will thus be enough to show that the equality (6.9) implies that p can be taken to be an isomorphism. Indeed, if p is an isomorphism we get a birational morphism q from $\mathbb{P}^1_{\mathbb{Z}}$ to $\mathcal{X}'$ and any such morphism is an isomorphism (since the fibers of $\mathbb{P}^1_{\mathbb{Z}}$ over $\mathcal{B}$ are all reduced and irreducible). Moreover, when $\mathcal{X}'$ is equal to $\mathbb{P}^1_{\mathbb{Z}}$ any $\mathcal{L}$ whose complexification equals $-K_{\mathbb{P}^1}$ is isomorphic to $-\mathcal{K}_{\mathbb{P}^1_{\mathbb{Z}}}$ and the components of any divisor $\mathcal{D}'$ on $\mathcal{X}'$ whose complexification is trivial are fibers $\mathcal{X}_{b_i}$ of $\mathbb{P}^1_{\mathbb{Z}}$ (using again that the fibers of $\mathbb{P}^1_{\mathbb{Z}}$ over $\mathcal{B}$ are all reduced and irreducible). Hence, the assumed equality (6.9) implies, since $\mathcal{X}_{b_i}^2=0$ and $-\mathcal{K}_{\mathbb{P}^1_{\mathbb{Z}}}$ is relatively ample that $\mathcal{D}'$ is trivial, i.e., $\mathcal{D}'=0$.

Thus all that remains is to show that the assumed equality in formula (6.9) implies that the morphism $p:\mathcal{Y}\to\mathcal{X}$ (in formula (6.10)) can be taken to be an isomorphism.

Step 1. — In the case of arithmetic surfaces $p:\mathcal{Y}\to\mathcal{X}$ can be taken as the successive blow-ups of $\mathcal{X}$ along a finite number of closed points x_i in regular surfaces $\mathcal{X}_i$ and there exists a p-exceptional and p-ample effective divisor E on $\mathcal{Y}$ and a morphism q from $\mathcal{Y}$ to $\mathcal{X}$ *such that $q^*\mathcal{L}'=p^*\mathcal{L}-E$. In particular, $\mathcal{M}_{\mathcal{X}'}(\mathcal{L}')=\mathcal{M}_{\mathcal{Y}}(p^*\mathcal{L}-E)$.*

This is shown as in the beginning of the proof of [32, Prop. 3.10], as next explained. First note that, since $\mathcal{X}$ and $\mathcal{X}'$ have the same generic fiber, they are birationally equivalent. Since $\mathcal{X}$ is normal this means that there exists a morphism $h:U\to\mathcal{X}'$ from a Zariski open subset U in $\mathcal{X}$ of codimension two. As a consequence, $h^*\mathcal{L}'$ extends to a $\mathbb{Q}$-line bundle $\mathcal{L}''$ on $\mathcal{X}$ coinciding with $-\mathcal{K}_{\mathcal{X}}$ on the generic fiber. Since $\mathcal{X}=\mathbb{P}^1_{\mathbb{Z}}$ this implies that $\mathcal{L}''$ is isomorphic to $-\mathcal{K}_{\mathcal{X}}$ (using that $\pi:\mathcal{X}\to\operatorname{Spec}\mathbb{Z}$ has reduced irreducible fibers). Now fix a positive integer k such that $k\mathcal{L}'$ is a relatively very ample line bundle and take a basis s_i' in the free $\mathbb{Z}-$module $H^0(\mathcal{X}',k\mathcal{L}')$. Then $s_i:=h^*s_i$ extends, since $\mathcal{X}$ is normal, to a unique element in $H^0(\mathcal{X},k\mathcal{L})$. Denote by $\mathcal{J}$ the ideal sheaf on $\mathcal{X}$ generated by the sections s_i. Since $\mathcal{X}$ is a regular surface we get after successive blow-ups (as stated in Step 1) a morphism $p:\mathcal{Y}\to\mathcal{X}$ from a regular surface $\mathcal{Y}$ to $\mathcal{X}$ with the property that $p^*\mathcal{J}$ defines an effective p-exceptional divisor E_k on $\mathcal{Y}$ (using that $\mathbb{Z}$ is an excellent ring) [44]. Set

$$E:=k^{-1}E_k\quad(E_k:=p^*\mathcal{J}).$$

By construction, E_k is p-ample,

$$H^0(\mathcal{Y},kp^*\mathcal{L}-E_k)\cong H^0(\mathcal{X},k\mathcal{L}\otimes\mathcal{J})\cong H^0(\mathcal{X}',k\mathcal{L}'),\tag{6.11}$$

and the global sections of $kp^*\mathcal{L}-E_k$ induce a morphism q to $\mathcal{X}'$ such that $q^*\mathcal{L}'=p^*\mathcal{L}-E$. Finally, note that

$$\mathcal{M}_{\mathcal{X}'}(\mathcal{L}')=\mathcal{M}_{\mathcal{Y}}(q^*\mathcal{L}'),$$

as follows directly from the fact that p is an isomorphism between Zariski open subsets of $\mathcal{X}'$ and $\mathcal{Y}$ and, as a consequence, the $\mathbb{Q}-$line bundle $q^*\mathcal{L}'$ is trivial on the support of the divisor $q^*\mathcal{K}_{\mathcal{X}} - \mathcal{K}_{\mathcal{Y}}$.

Step 2. — $\mathcal{M}_{\mathcal{Y}}(p^*\mathcal{L} - E) = \mathcal{M}_{\mathcal{X}}(\mathcal{L}) \implies$ *p is an isomorphism, when* $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}}$.

Replacing $q^*\mathcal{L}'$ with $p^*\mathcal{L} - E$ in formula (6.2) yields, since $\mathcal{M}_{\mathcal{Y}}(p^*\mathcal{L} - E) = \mathcal{M}_{\mathcal{X}}(\mathcal{L})$,

$$(p^*\mathcal{L} - E) \cdot \Big(\mathcal{K}_{\mathcal{Y}/\mathcal{X}} - \frac{1}{2}E\Big) = 0.$$

It follows, since, by construction, $p^*\mathcal{L} - E$ is p-ample, that

$$\mathcal{K}_{\mathcal{Y}/\mathcal{X}} = \frac{1}{2}E. \tag{6.12}$$

Now, since $p : \mathcal{Y} \to \mathcal{X}$ is the blow-up along a finite number of closed points x_i in regular surfaces $\mathcal{X}_i$,

$$\mathcal{K}_{\mathcal{Y}/\mathcal{X}} = \sum c_i E_i, \quad c_i \geqslant 1, \tag{6.13}$$

where the sum runs over all prime p-exceptional divisors E_i. Hence,

$$E = \sum_i 2c_i E_i \geqslant \sum_i 2E_i.$$

But this contradicts the isomorphisms (6.11), if the number of points x_i is non-zero. Indeed, denote by E_1 the strict transform of the p-exceptional divisor on $\mathcal{Y}$ induced from the exceptional divisor on the first point x_1 blown-up on $\mathcal{X}(= \mathbb{P}^1_{\mathbb{Z}})$. Then it follows from the previous inequality and the construction of E that the restriction of the ideal sheaf $\mathcal{J}$ on $\mathcal{X}$ to a neighbourhood of x_1 in the fiber $\mathcal{X}_{\pi(x_1)}$ is contained in the $2k$th power $\mathfrak{m}_{x_1}^{2k}$ of the maximal ideal $\mathfrak{m}_{x_1}$ on $\mathcal{X}_{\pi(x_1)}$ defined by the point x_1. But, in general, for $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}}$, the line bundle $k\mathcal{L}_{|\mathcal{X}_{\pi(x)}} \otimes \mathfrak{m}_x^{2k}$ on $\mathcal{X}_{\pi(x)}$ is trivial for any closed point x on $\mathcal{X}$ (since $\mathcal{L}_{|\mathcal{X}_b} := -K_{\mathcal{X}_b} = \mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_b}}(2)$). But this contradicts the isomorphism (6.11), since $\mathcal{L}'$ is relatively ample. Hence, the number of points x_i must be zero, as desired.

Combing these two steps thus concludes, as discussed above, the proof of Theorem 5. Finally, Corollary 6 can be deduced from Theorem 5 using a generalization of Lemma 29. But here we instead proceeds as follows. Given an arithmetic log Fano surface $(\mathcal{X}, \mathcal{D})$ set $\mathcal{L} := -\mathcal{K}_{\mathcal{X}}$ and endow $\mathcal{L}$ and $-\mathcal{K}_{\mathcal{X}}$ with the same metric induced from a volume-normalized metric on $-K_X$ with positive curvature current. Then, by definition (6.1),

$$-\frac{1}{2}\overline{\mathcal{K}}^2_{(\mathcal{X},\mathcal{D})} = \mathcal{M}_{(\mathcal{X},\mathcal{D})}(\mathcal{L}).$$

Hence, combining Step one and Step two above yields

$$-\frac{1}{2}\overline{\mathcal{K}}^2_{(\mathcal{X},\mathcal{D})} \geqslant \mathcal{M}_{(\mathbb{P}^1_{\mathbb{Z}},0)}(\overline{-\mathcal{K}_{\mathbb{P}^1_{\mathbb{Z}}}}) = -\frac{1}{2}(\overline{-\mathcal{K}_{\mathbb{P}^1_{\mathbb{Z}}}})^2,$$

and the equality case is deduced precisely as before.

Remark 30. — When $(\mathbb{P}^1, \Delta)$ is K-stable equality holds in the inequality (6.8) if and only if $(\mathcal{X}', \mathcal{D}') = (\mathbb{P}^1, \mathcal{D}^0)$. Indeed, by Lemma 28, the K-stability of $(\mathbb{P}^1, \Delta)$ implies that $\alpha(\mathbb{P}^1, \mathcal{D}^0) > 1/2$. Hence, if equality holds in (6.8), then formula (6.4) forces $E = 0$, showing that p is an isomorphism. We can then conclude precisely as in the beginning of Section 6.3.1.

7. Appendix

In the proof of Lemma 21 we used the following result (applied to $X=\mathbb{P}^n_{\mathbb{C}}$).

Lemma 31. — *Let X be a Fano manifold and V a holomorphic vector field on X. Denote by G_t the flow of the real part of V on X at time t and by $(G_t^V)^*\phi$ its action on a given continuous metric ϕ on $-K_X$ with positive curvature current. If X admits a Kähler-Einstein metric, then*

$$\frac{d}{dt}\mathcal{E}(G_t^*\phi,\psi_0)=0$$

for any fixed metric ψ_0 on $-K_X$.

Proof. — This is well-known and essentially goes back to [19], but for the convenience of the reader we provide a proof in the spirit of the present paper and its precursor [3]. Consider the *Ding functional* $\mathcal{D}_{\phi_0}$ on the space of all continuous metrics on $-K_X$ with positive curvature current, defined by formula (2.9). Since, $\mu_{G^*\phi}=G^*\mu_\phi$ for any biholomorphism G of X it follows that $\mathcal{E}(G_t^*\phi)$ and $\mathcal{D}_{\phi_0}(G_t{}^*\phi)$ have the same derivative. Moreover, in general, the function $t\mapsto\mathcal{D}_{\phi_0}(G_t^*\phi)$ is linear. Indeed, its derivative is the Futaki invariant of V (see the claim in [45, p. 73], where $\mathcal{D}_{\phi_0}$ is denoted by F_ω). Hence, all that remains is to verify that $\mathcal{D}_{\phi_0}$ is bounded from below (since then $t\mapsto\mathcal{D}_{\phi_0}(G_t^*\phi)$ must be constant). But this follows from the existence of a Kähler-Einstein metric, since such a metric minimizes $\mathcal{D}_{\phi_0}$, as recalled in [3, §2.3]. □

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