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ERGODIC PROPERTIES OF INFINITE EXTENSION OF SYMMETRIC INTERVAL EXCHANGE TRANSFORMATIONS

by Przemysław Berk, Frank Trujillo & Hao Wu

Abstract. — We prove that skew products with the cocycle given by the function $f(x) = a(x - 1/2)$ with $a \neq 0$ are ergodic for every ergodic symmetric IET in the base, thus giving the full characterization of ergodic extensions in this family. Moreover, we prove that under an additional natural assumption of unique ergodicity on the IET, we can replace f with any differentiable function with a non-zero sum of jumps. Finally, by considering weakly mixing IETs instead of just ergodic, we show that the skew products with cocycle given by f have infinite ergodic index.

Résumé (Propriétés ergodiques de l'extension infinie des transformations d'échange d'intervalles symétriques)

Nous prouvons que les produits tordus par le cocycle donné par la fonction $f(x) = a(x-1/2)$ avec $a \neq 0$ sont ergodiques pour chaque transformation d'échange d'intervalles (IET) symétrique ergodique dans la base, donnant ainsi la caractérisation complète des extensions ergodiques dans cette famille. De plus, nous prouvons que sous une hypothèse naturelle supplémentaire d'unique ergodicité sur l'IET, nous pouvons remplacer f par n'importe quelle fonction différentiable avec une somme de sauts non nulle. Enfin, en considérant des IET faiblement mélangeantes au lieu de simplement ergodiques, nous montrons que les produits tordus dont le cocycle est donné par f ont un indice ergodique infini.

Contents

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1. Main results

Let I be a bounded interval, equipped with the Borel σ-algebra and Lebesgue measure λ_I. Let $T = (\pi, \lambda)$ be an interval exchange transformation given by a permutation $\pi \in S_0^{\mathcal{A}}$ and a length vector $\lambda \in \Lambda^{\mathcal{A},I}$ (see Section 2 for precise definitions of these objects). It is not difficult to see that T preserves λ_I. We say that a permutation is *symmetric* if and only if for any $i = 1, \dots, d$ $\pi_1 \circ \pi_0^{-1}(i) = d + 1 - i$.

The main objects of study in this article are (real-valued) skew products over interval exchange transformations (IETs). More precisely, if $(X, \mathcal{B}, \mu)$ is a probability Borel space and $f : X \to \mathbb{R}$ is such that $\int_X f(x)\, d\mu(x) = 0$, then a *skew product* $T_f : X \times \mathbb{R} \to X \times \mathbb{R}$ over a measure-preserving map $(X, \mathcal{B}, \mu, T)$, is a transformation given by

$$T_f(x, r) := (T(x), x + f(x)).$$

We will refer to f as a *cocycle*. It is not difficult to see, that T_f preserves the product measure of μ on I and the Lebesgue measure $\lambda_{\mathbb{R}}$ on $\mathbb{R}$. We will investigate the ergodic properties of T_f with respect to the measure $\lambda_I \otimes \lambda_{\mathbb{R}}$ when T is either an IET or, more generally, a product of $n \geqslant 2$ copies of an IET.

Theorem 1.1. — *Let T be an ergodic symmetric IET on $I = [0, 1)$ and let $f(x) = a(x - 1/2)$ for some $a \in \mathbb{R} \setminus \{0\}$. Then, the skew product $T_f : I \times \mathbb{R} \to I \times \mathbb{R}$ is ergodic with respect to $\lambda_I \otimes \lambda_{\mathbb{R}}$.*

The exceptional nature of the above theorem stems from the fact that we only assume the ergodicity of the IET with respect to λ_I (in contrast to many results in the theory of IETs and of skew-products over IETs where generic conditions related to the Rauzy-Veech renormalization procedure are often imposed). Moreover, this assumption is necessary since we can associate to any non-trivial T-invariant set $A \subseteq I$ the non-trivial T_f-invariant set $A \times \mathbb{R} \subseteq I \times \mathbb{R}$. Theorem 1.1 gives thus a full characterization of the ergodic skew products over symmetric IETs with cocycle of the form $a(x - 1/2)$, for some $a \in \mathbb{R} \setminus \{0\}$.

The skew products over IETs were already researched with various types of cocycles, although not under such weak assumptions. Recently, the first and second author in [1] proved that for almost every symmetric IET on $[0, 1)$ and a cocycle $f(x) = \chi_{[0,1/2)} - \chi_{[1/2,1)}$ the skew product is ergodic. The final arguments in the proof of Theorem 1.1 are partially inspired by this paper. For linear cocycles, the most relevant is the work of Conze and Frączek in [4], where the authors studied piece-wise linear cocycles over IETs of periodic type. However, there are only countably many such IETs.

There are very few results in the literature concerning any ergodic IET. It is worth mentioning here the article [5] by Katok, where he proved that every IET is partially rigid. A variation of his construction of Rokhlin towers serves later in the proof of Theorem 1.1 to construct partially rigid towers needed to establish the ergodicity of skew products.

If we only assume the ergodicity of the underlying IET, one of the main obstacles we face is the impossibility of excluding IETs with *connections* (see Section 2 for a precise definition). Let us point out that the existence of connections does not exclude ergodicity. Indeed, perhaps the most relevant to this article is the example given in [1] which was a symmetric IET of the interval $[0,1)$ with $1/2$ as a discontinuity. There it served as an example of IET taken as a base of a non-ergodic skew product. Here, such examples are also covered in Theorem 1.1. By dealing with symmetric IETs with connections, we obtain interesting side results on their ergodic properties (see Corollary 3.14 and Corollary 3.16).

Since we cannot use the ergodic properties of Rauzy-Veech induction due to the presence of connections, we have to tackle some issues that are usually not a problem if one wants to obtain a result only on a full-measure set of IETs. Notice that, given an IET, we can always induce on a subinterval and obtain another IET but, in general, we do not have control over the combinatorial properties of the induced map. Hence, a major step towards proving our main result is showing that if we choose the induction interval properly, then the induced transformation is a symmetric IET as long as the initial IET T is symmetric. This is the content of Proposition 3.5. The proof of this key property relies on another result that we would like to highlight and which generalizes a well-known property of the Rauzy-Veech induction, namely, the existence of neighborhoods (simplices) around almost every IET so that, for any IET in this neighborhood, the induced map on certain *dynamically defined* induction intervals (given by a fixed number of iterations of the Rauzy-Veech procedure) leads to the same combinatorics and the same *Rokhlin tower decomposition* of the initial intervals (see Section 2.2). We formally state this in Proposition 2.6 (see also Remark 2.7).

By imposing an additional generic condition on the IET, we can largely increase the family of cocycles for which we can deduce the ergodicity of the skew product.

Theorem 1.2. — *Let T be a uniquely ergodic symmetric IET on $I=[0,1)$ and let $f(x)=a(x-1/2)+f_0(x)$, for some $a\in\mathbb{R}\smallsetminus\{0\}$ and some differentiable function f_0 satisfying $f_0(1-x)=-f_0(x)$, and $\int_I Df_0(x)\,dx=0$. Then, the skew product $T_f: I\times\mathbb{R}\to I\times\mathbb{R}$ is ergodic with respect to $\lambda_I\otimes\lambda_{\mathbb{R}}$.*

Finally, one may ask about the ergodic index of the skew product under consideration. Recall that a measure-preserving transformation $(X,\mathcal{B},\mu,T)$ has *infinite ergodic index* if and only if for every $n\in\mathbb{N}$ the transformation $(X^{\times n},\mathcal{B}^{\otimes n},\mu^{\otimes n},T^{\times n})$ is ergodic, where the superscripts $\times n$ and $\otimes n$ denote n-fold products of the objects.

If we consider this property for T_f on $X\times\mathbb{R}$, it is easy to see that $T_f^{\times n}$ is a skew product over $T^{\times n}$ with the cocycle given by the function $f^{\times n}: X^{\times n}\to\mathbb{R}^n$, where $f^{\times n}(x_1,\dots,x_n):=(f(x_1),\dots,f(x_n))$. It is easy to see that a natural obstruction for having an infinite ergodic index is when $T^{\times k}$ is not ergodic, for some $k\in\mathbb{N}$. It turns out that in our case this is the only obstacle.

Theorem 1.3. — *Let T be a weakly mixing symmetric IET on $I = [0,1)$ and let $f(x) = a(x-1/2)$, for some $a \in \mathbb{R} \setminus \{0\}$. Then, the skew product $T_f : I \times \mathbb{R} \to I \times \mathbb{R}$ has infinite ergodic index.*

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2. Interval exchange transformations

2.1. Notations and basic properties. — An *interval exchange transformation (IET)* T on a bounded interval I is a piecewise linear bijection of I, with finite number of intervals of continuity, on which T acts via translation. For convenience and without loss of generality, we assume the interval I to be of the form $[a,b)$, for some $a, b \in \mathbb{R}$, and the IETs to be right-continuous.

More precisely, there exists $\mathcal{A}$ is an alphabet of $d \in \mathbb{N}$ elements, a permutation $\pi = \binom{\pi_0}{\pi_1}$ with $\pi_0, \pi_1 : \mathcal{A} \to \{1, \dots, d\}$ and a collection of left closed and right open subintervals $\{I_\alpha\}_{\alpha \in \mathcal{A}}$ such that and $\bigsqcup_{\alpha \in \mathcal{A}} I_\alpha = I$, $T|_{I_\alpha}$ acts via translation, and π_0 and π_1 describe the order of intervals respectively before and after action of T. It is easy to check that T preserves Lebesgue measure on I and that the parameters (π, λ) fully describe the dynamics of T, where $\lambda = [\lambda_\alpha]_{\alpha \in \mathcal{A}} := [|I_\alpha|]_{\alpha \in \mathcal{A}} \in \Lambda^{\mathcal{A},I}$ is the vector of lengths of intervals I_α, with $\Lambda^{\mathcal{A},I} := \{\lambda \in \mathbb{R}_+^{\mathcal{A}} \mid \sum_{\alpha \in \mathcal{A}} \lambda_\alpha = |I|\}$. Moreover, we always assume that π is *non-reducible*, that is

$$\pi_1 \circ \pi_0^{-1}\{1, \dots, k\} = \{1, \dots, k\} \implies k = d.$$

Otherwise, we can decompose T into two non-trivial IETs and consider their properties separately. We denote by $S_0^{\mathcal{A}}$ the set of all non-reducible permutations of alphabet $\mathcal{A}$.

For every $\alpha \in \mathcal{A}$, we denote by ∂I_α the left endpoint of I_α and by c_α its center point. We say that an IET T has a *connection* if there exist $\alpha, \beta \in \mathcal{A}$ with $\pi_0(\beta) \neq 1$, $\pi_1(\alpha) \neq 1$ and $n \in \mathbb{N}_+$ such that

$$T^{-n}(\partial I_\beta) = \partial I_\alpha.$$

By a connection we often mean the orbit segment $\{T^{-k}(\partial I_\beta)\}_{k=-n,\dots,0}$. If such connection exists, we denote

$$M(\beta) = M(T, \beta) := \min_{\alpha \in \mathcal{A}} \min\{n \in \mathbb{N}_+ \mid T^{-n}(\partial I_\beta) = \partial I_\alpha\}.$$

Otherwise, we write $M(\beta) = \infty$. Similarly, we denote

$$N(\alpha) = N(T, \alpha) := \min_{\beta \in \mathcal{A}} \min\{n \in \mathbb{N}_+ \mid T^{n}(\partial I_\alpha) = \partial I_\beta\}$$

and write $N(\alpha) = \infty$ if such connection does not exist. Note that we always have $T(\partial I_{\pi_1^{-1}(1)}) = \partial I_{\pi_0^{-1}(1)}$, a *trivial* connection. Hence, we define $M(\pi_0^{-1}(1)) := 1$ and $N(\pi_1^{-1}(1)) := 1$. We say that a connection is *irreducible* if it is not a concatenation of two connections, i.e., it contains at most two points of the form ∂I_α, $\alpha \in \mathcal{A}$.

Note that the existence of a non-trivial connection implies that some non-trivial integer combination of lengths of exchanged intervals is equal to 0. Thus, if the length

vector is *rationally independent*, i.e., if $\sum_{\alpha\in\mathcal{A}} r_\alpha\lambda_\alpha = 0$ for some $(r_\alpha)_{\alpha\in\mathcal{A}} \in \mathbb{Q}^{\mathcal{A}}$ implies that $r_\alpha = 0$, for all $\alpha \in \mathcal{A}$, then there cannot be any non-trivial connection. Hence, almost every IET has no non-trivial connections.

However, in this article we consider the class of *all* ergodic IETs, and, let us recall, the existence of non-trivial connections does not exclude ergodicity. Nevertheless, it is well-known that if all ∂I_β are endpoints of connections, then such an IET cannot be ergodic. For the sake of completeness, let us provide a proof of this fact.

Lemma 2.1. — *Let $T : I \to I$ be an interval exchange transformation given by a permutation π and length vector λ. If for every $\beta \in \mathcal{A} \smallsetminus \{\pi_0^{-1}(1)\}$ we have $M(\beta) < \infty$, then T has only periodic orbits. More precisely, the base interval I can be decomposed in a finite number of periodic components, given by semi-closed intervals, such that the period is uniform on each of these components.*

Proof. — Note that by assumption the set of points

$$\{T^n(\partial I_\alpha) \mid n \in \mathbb{Z}, \quad \alpha \in \mathcal{A}\} = \{T^{-n}(\partial I_\alpha) \mid \alpha \in \mathcal{A} \text{ and } 0 \leqslant n \leqslant M(\alpha)\}$$

is finite. Consider the partition given by those points and let $[a, b)$ be an element of this partition.

Note that T^n acts continuously on $[a, b)$ for all $n \in \mathbb{N}$. Indeed, the only possible points of discontinuity of T are $\{\partial I_\alpha\}_{\alpha\in\mathcal{A}}$, hence, if for some $n \in \mathbb{N}$ the map T^n did not act continuously on $[a, b)$, there would exist $\beta \in \mathcal{A}$ such that $\partial I_\beta \in T^{n-1}\big([a, b)\big)$, which contradicts the choice of $[a, b)$. In particular, it follows that $T^n\big([a, b)\big)$ is an interval, for any $n \in \mathbb{N}$.

By Poincaré's recurrence theorem, there exists $N \in \mathbb{N}$ such that

$$T^N\big([a, b)\big) \cap [a, b) \neq \varnothing.$$

This implies that $T^N\big([a, b)\big) = [a, b)$. Indeed, otherwise either $T^{-N}(a)$ or $T^N(a)$ belongs to $[a, b)$. Since a is in the orbits of one of the points $\{\partial I_\alpha\}_{\alpha\in\mathcal{A}}$, this yields a contradiction.

To sum up, $T^N\big([a, b)\big) = [a, b)$ and T^N acts continuously on $[a, b)$. Since T is a piecewise translation, then so is T^N. Thus $T^N|_{[a,b)}$ is the identity map on $[a, b)$, which finishes the proof. □

Remark 2.2. — By proceeding symmetrically, one can replace in Lemma 2.1 the endpoints of connections with their initial points.

One of the main consequences of the above lemma is the following.

Corollary 2.3

Assume that T is an ergodic IET. Then there exists $\beta \in \mathcal{A} \smallsetminus \{\pi_0^{-1}(1)\}$ such that $M(\beta) = +\infty$.

Proof. — Assume, for the sake of contradiction, that T is ergodic but that the conclusion does not hold. Then, by Lemma 2.1, there exists a non-empty semi-closed

interval $[a,b) \subseteq I$ and $N \geqslant 1$ such that $T^N|_{[a,b)}$ is the identity map in $[a,b)$. Therefore, the set $\bigcup_{i=0}^{N-1} T^i\big([a,(a+b)/2]\big)$ is a non-trivial T-invariant set, which contradicts the ergodicity of T. □

2.2. Induced IETs. — Throughout the proofs of the main results of this paper, we will often use the first return map of T to a subinterval $J \subseteq I$, which we denote by $T_J : J \to J$. More precisely, we define T_J as $x \mapsto T^{r_J(x)}(x)$, where $r_J : J \to \mathbb{N}$ is given by

$$r_J(x) := \min\{n \geqslant 1 \mid T^n(x) \in J\}.$$

We sometimes refer to T_J as the *induced map of T to J*.

A priori, the map T_J is not necessarily well-defined for *all* points in J, although Poincaré's recurrence theorem guarantees that T_J is well-defined in a full Lebesgue measure subset of J. However, it is well known (see, e.g., [6, §3]) that for any subinterval $J = [a_J, b_J) \subseteq I$ the induced map T_J is an IET of at most $d+2$ intervals, where the possible discontinuities are given by preimages of the discontinuities of T (at most $d-1$ points) and of the endpoints of J (at most 2 points, not necessarily disjoint with the previous set).

More precisely, the possible discontinuity points of T_J are given by

$$\{T^{-m_{J,\alpha}}(\partial I_\alpha)\}_{\alpha \in \mathcal{A} \smallsetminus \{\pi_0^{-1}(1)\}}, \quad m_{J,\alpha} := \inf\{n \geqslant 0 \mid T^{-n}(\partial I_\alpha) \in \mathring{J}\} \text{ for } \alpha \in \mathcal{A},$$

together with

$$T^{-m_l}(a_J), \quad m_l := \inf\{n \geqslant 0 \mid T^{-n}(a_J) \in \mathring{J}\},$$

if a_J is different from the left endpoint of I, and

$$T^{-m_r}(b_J), \quad m_r := \inf\{n \geqslant 0 \mid T^{-n}(b_J) \in \mathring{J}\},$$

if b_J is different from the right endpoint of I, where the preimages for which $m_{J,\alpha}$ (resp. m_l or m_r) is $+\infty$ are disregarded. However, note that if T is minimal, which is the case if T is ergodic (see Lemma 2.5), all the above notions are finite.

Moreover, if T is ergodic and $J = [a_J, b_J) \subseteq I$ is chosen so that

$$\begin{aligned} &a_J = T^{m_0}(\partial I_{\alpha_J}) \text{ and } b_J = T^{n_0}(\partial I_{\beta_J}), \quad \text{for some } \alpha_J, \beta_J \in \mathcal{A} \text{ and } m_0, n_0 \in \mathbb{Z}, \\ &T^m(\partial I_{\alpha_J}) \notin J, \qquad \text{for any } m \in \{0, \dots, m_0\} \text{ with } m \neq m_0 \\ &T^n(\partial I_{\beta_J}) \notin J \qquad \text{for any } n \in \{0, \dots, n_0\} \text{ with } n \neq n_0, \end{aligned} \tag{1}$$

then the induced map T_J can be seen as an IET of at most d intervals. Indeed, in this case, the discontinuities of T_J belong to the set $\{T^{-m_{J,\alpha}}(\partial I_\alpha)\}_{\alpha \in \mathcal{A} \smallsetminus \{\pi_0^{-1}(1)\}}$. Analogously, if J is of the form (1) the discontinuities of T_J^{-1} are contained in

$$\{T^{n_{J,\alpha}}(\partial I_\alpha)\}_{\alpha \in \mathcal{A} \smallsetminus \{\pi_1^{-1}(1)\}}, \quad \text{where } n_{J,\alpha} := \min\{n \geqslant 1 \mid T^n(\partial I_\alpha) \in J\} \text{ for any } \alpha \in \mathcal{A}.$$

If, in addition, T has no non-trivial connections, then the previous two sets have exactly $d-1$ elements, and T_J can be naturally seen as an IET on $d = \#\mathcal{A}$ intervals and identified with an element $(\pi_J, \lambda_J) \in S_0^{\mathcal{A}} \times \Lambda^{\mathcal{A},J}$.

The following simple auxiliary fact tells us what happens if T has non-trivial connections.

Lemma 2.4. — *Let T be an ergodic interval exchange transformation of $d = \#\mathcal{A}$ intervals and let J be a subinterval of the form (1). Then T_J can be considered as an interval exchange of d_J intervals, where*

$$d_J := d - \#\{\alpha \in \mathcal{A} \smallsetminus \{\pi_0^{-1}(1)\} \mid m_{J,\alpha} \geqslant M(\alpha)\}.$$

In particular, if J does not contain any point from any connection, then $d - d_J$ is equal to the number of non-trivial irreducible connections of T.

Proof

Since we know that the discontinuities of T_J belong to $\{T^{-m_{J,\alpha}}(\partial I_\alpha)\}_{\alpha\in\mathcal{A}\smallsetminus\{\pi_0^{-1}(1)\}}$ which has at most $d-1$ elements, to prove that T_J can be seen as an interval exchange of d_J intervals it is sufficient to show that this set has exactly $d_J - 1$ elements.

Assume that $\alpha \in \mathcal{A} \smallsetminus \{\pi_0^{-1}(1)\}$ is such that $m_{J,\alpha} \geqslant M(\alpha)$ and let $\beta \in \mathcal{A} \smallsetminus \{\pi_0^{-1}(1)\}$ be such that $T^{-M(\alpha)}(\partial I_\alpha) = \partial I_\beta$. Then, by the assumption on α, we have that

$$T^{-m_{J,\alpha}}(\partial I_\alpha) = T^{-m_{J,\beta}}(\partial I_\beta).$$

This shows that the connection that ends in the point ∂I_α decreases the number of discontinuities of T_J by 1. To conclude the proof of the first statement it remains to repeat the above reasoning for all $\alpha \in \mathcal{A}$ satisfying $m_{J,\alpha} \geqslant M(\alpha)$.

To prove the second assertion, first remark that

$$\#\{\alpha \in \mathcal{A} \smallsetminus \{\pi_0^{-1}(1)\} \mid M(\alpha) < +\infty\}$$

is equal to the number of non-trivial irreducible connections. Then it suffices to notice that

$$\#\{\alpha \in \mathcal{A} \smallsetminus \{\pi_0^{-1}(1)\} \mid m_{J,\alpha} \geqslant M(\alpha)\} = \#\{\alpha \in \mathcal{A} \smallsetminus \{\pi_0^{-1}(1)\} \mid M(\alpha) < +\infty\},$$

if J does not contain any point from any connection. □

In view of the previous lemma, throughout this work, if $T : I \to I$ is an ergodic IET and $J \subseteq I$ is of the form (1), we will consider the induced IET T_J as an IET on d_J intervals, where

$$\begin{aligned} d_J &= 1 + \#\{T^{-m_{J,\alpha}}(\partial I_\alpha) \mid \alpha \in \mathcal{A} \smallsetminus \{\pi_0^{-1}(1)\} \\ &= d - \#\{\alpha \in \mathcal{A} \smallsetminus \{\pi_0^{-1}(1)\} \mid m_{J,\alpha} \geqslant M(\alpha)\} \leqslant d. \end{aligned}$$

We will also identify T_J with an element $(\pi_J, \lambda_J) \in S_0^{\mathcal{A}_J} \times \Lambda^{\mathcal{A}_J, J}$ of a possibly smaller alphabet $\mathcal{A}_J$ and denote by $\{I_\gamma^J\}_{\gamma\in\mathcal{A}_J}$ the intervals exchanged by T_J.

Let us point out that, in the same way that an IET T with d intervals might have less than $d-1$ discontinuities, the induced map T_J might have less than $d_J - 1$ discontinuities, that is, some of the points in $\{T^{-m_{J,\alpha}}(\partial I_\alpha)\}_{\alpha\in\mathcal{A}\smallsetminus\{\pi_0^{-1}(1)\}}$ might not be real discontinuity points of T_J.

Notice that given an ergodic IET $T : I \to I$ and a subinterval $J \subseteq I$ of the form (1), by the minimality of T and T^{-1}, we can express I as a disjoint union of the form

$$I = \bigsqcup_{\gamma\in\mathcal{A}_J} \bigsqcup_{i=0}^{h_\gamma - 1} T^i(I_\gamma^J), \tag{2}$$

where, for any $\gamma \in \mathcal{A}_J$, h_γ denotes the first return time to J by T of any point in I_γ^J. We refer to the above decomposition as the *Rokhlin tower decomposition associated with T_J and to $(h_\gamma)_{\gamma\in\mathcal{A}_J}$ as the corresponding* height *vector*.

We finish this section by recalling a well-known fact, which we prove for completeness.

Lemma 2.5. — *If $T : I \to I$ is ergodic with respect to the Lebesgue measure then it is minimal.*

Proof. — We will show that for every $x, y \in I$ and every $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $|T^{-m}y - x| < 2\varepsilon$. Take an interval $J := [x, x + \varepsilon)$ and consider the first return map T_J. Let I_β^J be intervals exchanged by T_J and h_β the corresponding first return times. Since T is ergodic, the set $\widetilde{I} := \bigcup_{\beta\in\mathcal{B}} \bigcup_{k=0}^{h_\beta-1} T^k(I_\beta^J)$ is of full Lebesgue measure.

Define $h := \max\{h_\beta \mid \beta \in \mathcal{B}\}$. Consider the set

$$C := \{T^j(\partial I_\alpha) \mid \alpha \in \mathcal{A},\ j = 0, \dots, h\}.$$

Pick $0 < \delta < \varepsilon$ such that $(y, y + \delta] \cap C = \varnothing$. Since $\widetilde{I}$ is of full measure, there exists $\widetilde{y} \in [y, y + \delta] \cap \widetilde{I}$. By the choice of δ and by the fact that T is right-continuous, the sets $\{T^{-j}[y, \widetilde{y}] \mid j = 0, \dots, h\}$ are a family of pairwise disjoint intervals and T^{-1} acts on each of them by translation. Since $\widetilde{y} \in \widetilde{I}$, there exists $m \leqslant h$ such that $T^{-m}\widetilde{y} \in J$ and $T^{-m}[y, \widetilde{y}] = [T^m y, T^{-m}\widetilde{y}]$. Thus

$$|T^{-m}y, x| < \varepsilon + \delta < 2\varepsilon,$$

which finishes the proof. □

2.3. Parametrizing IETs with similar induced maps. — For every IET $T : I \to I$ given by $\pi \in S_0^{\mathcal{A}}$ and $\lambda \in \Lambda^{\mathcal{A},I}$, we consider the set $\Lambda_T^{\mathcal{A},I}$ given by

$$(3)\quad \left\{ \widetilde{\lambda} \in \Lambda^{\mathcal{A},I} \;\middle|\; \begin{array}{l} M(T_{\widetilde{\lambda}}, \beta) = M(T, \beta) \text{ and } N(T_{\widetilde{\lambda}}, \beta) = N(T, \beta), \quad \text{for } \beta \in \mathcal{A}, \\ T_{\widetilde{\lambda}}^{-M(\beta)}(\partial I_\beta^{\widetilde{\lambda}}) = \partial I_\gamma^{\widetilde{\lambda}} \iff T^{-M(\beta)}(\partial I_\beta) = \partial I_\gamma, \\ \qquad\qquad \text{for } \beta \in \mathcal{A} \text{ with } M(\beta) < \infty, \\ T_{\widetilde{\lambda}}^{N(\beta)}(\partial I_\beta^{\widetilde{\lambda}}) = \partial I_\gamma^{\widetilde{\lambda}} \iff T^{N(\beta)}(\partial I_\beta) = \partial I_\gamma, \\ \qquad\qquad \text{for } \beta \in \mathcal{A} \text{ with } N(\beta) < \infty. \end{array} \right\}$$

In the above definition, the conditions on $N(\cdot)$ and $M(\cdot)$ are equivalent, nevertheless we write both of them for completeness. The set $\Lambda_T^{\mathcal{A},I}$ denotes all length vectors $\widetilde{\lambda}$ in $\Lambda^{\mathcal{A},I}$ for which the IET $(\pi, \widetilde{\lambda})$ has the same connection pattern as T. Obviously $\lambda \in \Lambda_T^{\mathcal{A},I}$. The following proposition is one of the crucial tools used later in the proofs of the main results. In loose words, it states that by starting with any IET and considering a Rokhlin tower configuration obtained by inducing on a properly chosen interval, we can obtain a new IET by perturbing the parameters of this configuration, which has the same combinatorial and connection data as the initial map.

Proposition 2.6. — *Let $T = (\pi, \lambda) \in S_0^{\mathcal{A}} \times \Lambda^{\mathcal{A},I}$ be an ergodic IET and let $J \subseteq I$ be a subinterval of the form* (1) *with endpoints $T^m(\partial I_{\alpha_J})$, $T^n(\partial I_{\beta_J})$, for some $\alpha_J, \beta_J \in \mathcal{A}$ and $m, n \in \mathbb{Z}$. Assume that J does not contain any point from any connection of T and let*

$$I = \bigsqcup_{\gamma \in \mathcal{A}_J} \bigsqcup_{i=0}^{h_\gamma - 1} T^i(I_\gamma^J), \tag{4}$$

be the associated Rokhlin tower decomposition of I.

Then, for any $v \in \mathbb{R}_+^{\mathcal{A}_J}$ satisfying $\sum_{\gamma \in \mathcal{A}_J} v_\gamma h_\gamma = |I|$, there exists $\widetilde{\lambda} \in \Lambda_T^{\mathcal{A},I}$ such that the IET $\widetilde{T} = (\pi, \widetilde{\lambda})$ and the interval $\widetilde{J}$ with endpoints $\widetilde{T}^m(\partial \widetilde{I}_{\alpha_J})$, $\widetilde{T}^n(\partial \widetilde{I}_{\beta_J})$, where $\{\widetilde{I}_\alpha\}_{\alpha \in \mathcal{A}}$ denote the intervals exchanged by $\widetilde{T}$, satisfy the following.

– *The induced IET $\widetilde{T}_{\widetilde{J}} = (\widetilde{\pi}^{\widetilde{J}}, \widetilde{\lambda}^{\widetilde{J}})$ is defined on the alphabet $\mathcal{A}_J$,*
– *$\widetilde{\lambda}^{\widetilde{J}} = v$ and $\widetilde{\pi}^{\widetilde{J}} = \pi^J$,*
– *the vectors of heights in the tower decomposition associated with $\widetilde{T}_{\widetilde{J}}$ and $T_J = (\pi^J, \lambda^J)$, as in* (2)*, are identical.*

In the following, given two intervals $J_1, J_2 \subseteq \mathbb{R}$, we denote

$$J_1 < J_2 \iff x < y \text{ for any } x \in J_1 \text{ and any } y \in J_2. \tag{5}$$

Notice that given a collection of disjoint intervals, we can order it according to the relation above.

Proof of Proposition 2.6. — Fix $v \in \mathbb{R}_+^{\mathcal{A}_J}$ satisfying $\sum_{\gamma \in \mathcal{A}_J} v_\gamma h_\gamma = |I|$. We will define the desired IET $\widetilde{T}$ as follows. First, we will change the lengths of the intervals in the Rokhlin tower decomposition of I associated with T_J, while keeping their order in I, to express I as a disjoint union of intervals whose lengths are given by v. Then we will define a transformation $\widetilde{T}$ on this union so that it defines a Rokhlin tower decomposition for the new transformation. Finally, we will check that $\widetilde{T}$ Is an IET with the desired properties.

Since $\sum_{\gamma \in \mathcal{A}_J} v_\gamma h_\gamma = |I|$, we can represent the interval I as the following union:

$$I = \bigsqcup_{\gamma \in \mathcal{A}_J} \bigsqcup_{i=0}^{h_\gamma - 1} \widetilde{I}_{\gamma,i}^J, \tag{6}$$

where $\widetilde{I}_{\gamma,i}$ is a left-closed right-open interval of length v_γ and

$$\widetilde{I}_{i,\gamma}^J < \widetilde{I}_{j,\beta}^J \iff T^i(I_\gamma^J) < T^j(I_\beta^J), \tag{7}$$

for every $\gamma, \beta \in \mathcal{A}_J$ and every $i = 0, \dots, h_\gamma - 1$, $j = 0, \dots, h_\beta - 1$.

Let

$$\widetilde{J} = \bigsqcup_{\gamma \in \mathcal{A}_J} \widetilde{I}_{\gamma,0}^J.$$

Notice that since $J = \bigsqcup_{\gamma \in \mathcal{A}_J} I_\gamma^J$ is an interval, by (7), $\widetilde{J}$ is also an interval. Clearly $|J| = \sum_{\gamma \in \mathcal{A}_J} |\widetilde{I}_{\gamma,0}^J| = \sum_{\gamma \in \mathcal{A}_J} v_\gamma = |v|_1$. Moreover, since $J = \bigsqcup_{\gamma \in \mathcal{A}_J} T^{h_\gamma}(I_\gamma^J)$ we can also express $\widetilde{J}$ as a disjoint union of intervals $\{L_\gamma^J\}_{\gamma \in \mathcal{A}_J}$ of lengths given by v and such that $\{L_\gamma^J\}_{\gamma \in \mathcal{A}_J}$ and $\{T^{h_\gamma}(I_\gamma^J)\}_{\gamma \in \mathcal{A}_J}$ have the same order inside I.

We define a transformation $\widetilde{T}$ on I by setting, for any $\gamma \in \mathcal{A}$,
– $\widetilde{T}(\widetilde{I}^J_{\gamma,i}) = \widetilde{I}^J_{\gamma,i+1}$ for $0 \leqslant i < h_\gamma - 1$,
– $\widetilde{T}(I^J_{\gamma,h_\gamma-1}) = L^J_\gamma$,
and requiring $\widetilde{T}$ to act via a translation when restricted to these subintervals.

Notice that with these definitions the images, by T and $\widetilde{T}$ respectively, of the intervals in the decompositions (4) and (6) are ordered in the same way, that is,

$$\widetilde{T}(\widetilde{I}^J_{i,\gamma}) < \widetilde{T}(\widetilde{I}^J_{j,\beta}) \iff T\big(T^i(I^J_\gamma)\big) < T\big(T^j(I^J_\beta)\big),$$

We will show that $\widetilde{T}$ can be seen as an IET with the same combinatorial data as π and that its length vector belongs to $\Lambda_T^{\mathcal{A},I}$.

Denote $H := \sum_{\gamma\in\mathcal{A}_J} h_\gamma$. Let $\{I_k\}_{k=0}^{H-1}$ be the intervals in the tower decomposition (4) ordered according to their order in I, i.e.,

$$I_{k_1} = T^{j_1}\big(I^J_{\gamma_1}\big) \text{ and } I_{k_2} = T^{j_2}\big(I^J_{\gamma_2}\big) \text{ with } k_1 < k_2 \iff T^{j_1}\big(I^J_{\gamma_1}\big) < T^{j_2}\big(I^J_{\gamma_2}\big),$$

and let $\{I^+_k\}_{k=0}^{H-1}$ be the intervals in the same tower decomposition ordered according to the order of their images, i.e.,

$$I^+_{k_1} = T^{j_1}\big(I^J_{\gamma_1}\big) \text{ and } I^+_{k_2} = T^{j_2}\big(I^J_{\gamma_2}\big) \text{ with } k_1 < k_2 \iff T\left(T^{j_1}\big(I^J_{\gamma_1}\big)\right) < T\left(T^{j_2}\big(I^J_{\gamma_2}\big)\right).$$

Similarly, let $\{\widetilde{I}_k\}_{k=0}^{H-1}$ and $\{\widetilde{I}^+_k\}_{k=0}^{H-1}$ denote the intervals in the tower decomposition (6) ordered according to their order in I and the order of their images by $\widetilde{T}$ in I, respectively.

For any $\alpha \in \mathcal{A}$ let $0 \leqslant k_\alpha < k'_\alpha < H$ and $0 \leqslant l_\alpha < l'_\alpha < H$ be such that

$$I_k \subseteq I_\alpha \iff k_\alpha \leqslant k \leqslant k'_\alpha, \qquad I^+_\ell \subseteq T(I_\alpha) \iff \ell_\alpha \leqslant \ell \leqslant \ell'_\alpha.$$

Then for every $k_\alpha \leqslant k \leqslant k'_\alpha$, the IET T acts as a translation on I_k by $\sum_{j<\ell_\alpha} |I^+_j| - \sum_{j<k_\alpha} |I_j|$.

We claim that for any $\alpha \in \mathcal{A}$, $\widetilde{I}_\alpha := \bigsqcup_{j=k_\alpha}^{k'_\alpha} \widetilde{I}_j$ is an interval exchanged by $\widetilde{T}$. Indeed, since the intervals $\widetilde{I}_k$ and $\widetilde{I}^+_k$ are ordered in the same way as the intervals I_k and I^+_k, respectively, the transformation $\widetilde{T}$ acts on $\widetilde{I}_k$ via translation by $\sum_{j<\ell_\alpha} |\widetilde{I}^+_j| - \sum_{j<k_\alpha} |\widetilde{I}_j|$. Since the translation value does not depend on k, $\widetilde{T}$ acts as a translation on the whole interval $\widetilde{I}_\alpha$.

Therefore, we can see $\widetilde{T}$ as an IET on I with $\#\mathcal{A}$ intervals. Moreover, $\widetilde{T}$ has the same combinatorics as T since the intervals (and their images) in both tower decompositions are ordered in the same way. Thus we can identify $\widetilde{T}$ with $(\pi, \widetilde{\lambda})$ for some $\widetilde{\lambda} \in \Lambda^{\mathcal{A},I}$, and we denote by $\{\widetilde{I}_\alpha\}_{\alpha\in\mathcal{A}}$ the intervals exchanged by $\widetilde{T}$.

Notice that the tower structure associated with $\widetilde{T}_{\widetilde{J}} = (\widetilde{\pi}^{\widetilde{J}}, \widetilde{\lambda}^{\widetilde{J}})$ is given by (6), that is, if we denote by $\{\widetilde{I}^{\widetilde{J}}_\gamma\}_{\gamma\in\mathcal{A}_J}$ the intervals exchanged by $\widetilde{T}_{\widetilde{J}}$ then $\widetilde{I}^{\widetilde{J}}_\gamma = \widetilde{I}_{\gamma,0}$, for every $\gamma \in \mathcal{A}_J$, and we have

$$I = \bigsqcup_{\gamma\in\mathcal{A}_J} \bigsqcup_{i=0}^{h_\gamma-1} \widetilde{I}^J_{\gamma,i} = \bigsqcup_{\gamma\in\mathcal{A}_J} \bigsqcup_{i=0}^{h_\gamma-1} \widetilde{T}(\widetilde{I}^{\widetilde{J}}_\gamma).$$

Since the intervals $\{\widetilde{T}_{\widetilde{J}}(\widetilde{I}^{\widetilde{J}}_\gamma)\}_{\gamma\in\mathcal{A}_J} = \{\widetilde{T}^{h_\gamma}(\widetilde{I}^{\widetilde{J}}_\gamma)\}_{\gamma\in\mathcal{A}_J} = \{L^J_\gamma\}_{\gamma\in\mathcal{A}}$ have the same order as the intervals $\{T_J(I^J_\gamma)\}_{\gamma\in\mathcal{A}_J} = \{T^{h_\gamma}(I^J_\gamma)\}_{\gamma\in\mathcal{A}_J}$ it follows that $\widetilde{\pi}_{\widetilde{J}} = \pi^J$. Finally, since the lengths of the intervals $\{\widetilde{I}^{\widetilde{J}}_\gamma\}_{\gamma\in\mathcal{A}_J}$ are given by v, it follows that $\widetilde{\lambda}^{\widetilde{J}} = v$.

Moreover, notice that the endpoints of the intervals exchanged by $\widetilde{T}$ and those exchanged by T belong to the same tower floors in their respective decompositions, that is,

$$\partial\widetilde{I}_\alpha \in \widetilde{T}^i(\widetilde{I}^{\widetilde{J}}_\gamma) \iff \widetilde{I}_\alpha \in T^i(I^J_\gamma),$$

for any $\gamma \in \mathcal{A}_J$ and any $0 \leqslant i < h_\gamma$. From this, and since J verifies (1), it follows that $\widetilde{J}$ is an interval with endpoints $\widetilde{T}^m(\partial\widetilde{I}_{\alpha_J})$, $\widetilde{T}^n(\partial\widetilde{I}_{\beta_J})$.

Since J does not contain any point from any connection, then every connection is contained in one of the towers of the form $\bigsqcup_{i=0}^{h_\gamma-1} T^i(I^J_\gamma)$, for some $\gamma \in \mathcal{A}_J$. Then, it follows from the definition of $\widetilde{T}$ (and the previous remarks concerning the endpoints of $\widetilde{T}$) that $\widetilde{T}$ possesses the same connection pattern as T, that is, $\widetilde{\lambda} \in \Lambda_T^{\mathcal{A},I}$. □

Remark 2.7. — The previous proposition defines a $(d-d'-1)$-dimensional simplex $\Delta_J \subseteq \Lambda^{\mathcal{A},I}$ around λ such that for any $\widetilde{\lambda} \in \Delta_J$ the IET $(\pi,\widetilde{\lambda})$ verifies the conclusions of the proposition, where d' denotes the number of non-trivial irreducible connections of (π,λ).

Indeed, It follows from the proof that the map $v \mapsto \widetilde{\lambda}(v)$ given by the previous proposition is linear on the simplex $\widetilde{\Delta}_J := \left\{v \in \mathbb{R}_+^{\mathcal{A}_J} \;\middle|\; \sum_{\gamma\in\mathcal{A}_J} v_\gamma h_\gamma = |I|\right\}$ and that $\widetilde{\lambda}(\lambda^J) = \lambda$. Moreover, the map is also injective since we can recover v by inducing the IET associated to $(\pi,\widetilde{\lambda}(v))$ to the interval $\widetilde{J}$. Notice that, by Lemma 2.4, the simplex $\widetilde{\Delta}_J$ has dimension $d-d'-1$. Thus $\Delta_J = \widetilde{\lambda}(\widetilde{\Delta}_J)$ satisfies the statement above.

3. Symmetric interval exchange transformations

3.1. Notations and basic properties. — Let $I = [a,b)$ be a bounded interval and $T = (\pi,\lambda) \in S_0^{\mathcal{A}} \times \Lambda^{\mathcal{A},I}$ be an IET on I with $d := \#\mathcal{A}$ intervals. The permutation π (and any IET having π as permutation) is said to be *symmetric* if $\pi_1\circ\pi_0^{-1}(i) = d+1-i$, for any $1 \leqslant i \leqslant d$. We say that T is *non-degenerate* if ∂I_α is a discontinuity of T, for every $\alpha \in \mathcal{A}\smallsetminus\{\pi_0^{-1}(1)\}$. If T is *degenerate*, i.e., if there exists ∂I_α which is not a real discontinuity, then whenever we refer to *intervals of continuity* of T, we mean maximal intervals of continuity. Notice that the inverse of a symmetric IET is also a symmetric IET.

We denote the *symmetric reflection* or *involution* on the open interval $\mathring{I} = (a,b)$ by $\mathcal{I}_I$, where $\mathcal{I}_I : \mathring{I} \to \mathring{I}$ is given by $\mathcal{I}_I(x) = a+b-x$. We omit the endpoints of the interval in this definition so that the domain and codomain of the involution are well-defined subsets of I. It is well-known and easy to verify that if T is a symmetric IET then

$$\mathcal{I}_I \circ T(x) = T^{-1} \circ \mathcal{I}_I(x) \quad \text{if } x \neq \partial I_\alpha \text{ for any } \alpha \in \mathcal{A}. \tag{8}$$

More generally, the equation above implies

$$\mathcal{I}_I \circ T^n(x) = T^{-n} \circ \mathcal{I}_I(x) \quad \text{if } x \neq T^{-i}(\partial I_\alpha) \text{ for any } \alpha \in \mathcal{A} \text{ and any } i \text{ such that } 0 \leqslant i < n. \tag{9}$$

Notice that $\mathcal{I}_I \circ T$ and $T^{-1} \circ \mathcal{I}_I$ are not defined everywhere on I since the $\mathcal{I}_I \circ T$ is not defined at $\partial I_{\pi_0^{-1}(d)}$ while $T^{-1} \circ \mathcal{I}_I$ is not defined at $\partial I_{\pi_0^{-1}(1)}$. Moreover, a direct calculation shows that

$$\mathcal{I}_I \circ T(\partial I_\alpha) = \partial I_{\widehat{\alpha}}, \text{ where } \pi_0(\widehat{\alpha}) = \pi_0(\alpha) + 1, \tag{10}$$

for $\alpha \in \mathcal{A}$ with $\pi_0(\alpha) \neq d$, and

$$T^{-1} \circ \mathcal{I}_I(\partial I_\alpha) = \partial I_{\widehat{\alpha}}, \text{ where } \pi_0(\widehat{\alpha}) = \pi_0(\alpha) - 1, \tag{11}$$

for $\alpha \in \mathcal{A}$ with $\pi_0(\alpha) \neq 1$.

Let us point out that π being symmetric is not a necessary condition for (8) to hold. Indeed, if π is not symmetric but its intervals of continuity are exchanged symmetrically (e.g., by adding a 'fake discontinuity' in one of the exchanged intervals of a symmetric IET with d intervals and considering it as an IET on $d+1$ intervals) then (8) still holds. Moreover, there are examples of IETs that do not exchange their intervals of continuity symmetrically but still satisfy (8). These examples arise from two-covers of quadratic differentials (see, e.g., [3]).

The following lemma provides a simple sufficient condition for an IET satisfying (8) to be symmetric.

Lemma 3.1. — *Let $T = (\pi, \lambda) : I \to I$ be a non-degenerate IET. If T satisfies* (8) *and $\lambda_\alpha \neq \lambda_\beta$ for any distinct $\alpha, \beta \in \mathcal{A}$, then T is symmetric.*

Proof. — Arrange the intervals $\{I_\alpha\}_{\alpha \in \mathcal{A}}$ according to their lengths, such that $\lambda_{\alpha_1} > \lambda_{\alpha_2} > \cdots > \lambda_{\alpha_d}$. Note that $\mathcal{I}_I \circ T$ is continuous on $\mathring{I}_{\alpha_1}$, so $T^{-1} \circ \mathcal{I}$ is also continuous on $\mathring{I}_{\alpha_1}$. Since the discontinuity points of T^{-1} are given by $\{T(\partial I_\beta) \mid \beta \in \mathcal{A} \smallsetminus \{\pi_1^{-1}(1)\}\}$, so by the maximal length of I_{α_1}, non-degenericity of T and the continuity of $T^{-1} \circ \mathcal{I}_I$, we must have

$$\mathcal{I}_I(I_{\alpha_1}) = T(I_{\alpha_1}). \tag{12}$$

By induction on the index $\{\alpha_i, 1 \leqslant i \leqslant d\}$, the above identity holds for every $\alpha \in \mathcal{A}$. Because the involution $\mathcal{I}$ reverses the order of $\{I_\alpha, \alpha \in \mathcal{A}\}$, that is:

$$I_\alpha < I_\beta \implies \mathcal{I}_I(I_\alpha) > \mathcal{I}_I(I_\beta),$$

where the order of intervals is given by (5), the identity (12) implies that T also reverses the order of $\{I_\alpha, \alpha \in \mathcal{A}\}$, hence T is symmetric. □

We have an immediate consequence for IETs with general combinatorial data.

Corollary 3.2. — *Let $T : I \to I$ be an IET satisfying* (8) *such that all its continuity intervals are of different lengths. Then T exchanges its continuity intervals symmetrically.*

Proof. — The result follows from Lemma 3.1 by replacing the intervals exchanged by T with the continuity intervals of T (thus possibly reducing the number of exchanged intervals). □

Given an IET $T : I \to I$ with exchanged intervals $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ we denote by c_α the middle point of the interval I_α, for any $\alpha \in \mathcal{A}$. Note that if T is symmetric, then

$$\mathcal{I}_I \circ T(c_\alpha) = c_\alpha, \tag{13}$$

for every $\alpha \in \mathcal{A}$. These points, as well as the middle point of the interval I, which we denote by

$$c_{1/2} := \frac{a+b}{2},$$

will play an important role in our proofs. Notice that $c_{1/2}$ is the only fixed point of $\mathcal{I}_I$ while the points $\{c_\alpha\}_{\alpha\in\mathcal{A}}$ are the only ones satisfying (13). Moreover, the backward and forward iterates of these points are closely related.

Lemma 3.3. — *Let $T : I \to I$ be a symmetric IET and $\alpha \in \mathcal{A} \cup \{1/2\}$. If $m \geqslant 1$ is such that $\{c_\alpha, T(c_\alpha), \dots, T^{m-1}(c_\alpha)\} \cap \{\partial I_\beta\}_{\beta\in\mathcal{A}} = \varnothing$, then*

$$T^{-m}(c_\alpha) = T^{-1} \circ \mathcal{I}_I\big(T^{m-\delta_{1/2}(\alpha)}(c_\alpha)\big), \tag{14}$$

where

$$\delta_{1/2}(\beta) = \begin{cases} 1 & \text{if} \quad \beta = 1/2, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Proof. — This result follows by directly applying (9) and noticing that $\mathcal{I}_I(c_{1/2}) = c_{1/2}$ and that $\mathcal{I}_I(c_\alpha) = T(c_\alpha)$, for any $\alpha \in \mathcal{A}$. □

The following result concerning Birkhoff sums over symmetric IETs follows directly from [2, Lem. 3.11]. Although not immediately useful for us, this fact will be crucial in one of the final arguments of the proof of the main result of this paper.

Lemma 3.4. — *Let $T : I \to I$ be a symmetric IET, $\alpha \in \mathcal{A} \cup \{1/2\}$ and $f : I \to \mathbb{R}$ satisfying $f \circ \mathcal{I}_I = -f$ on $\mathring{I}$. If c_α is not part of any connection, then*

$$S_{2n+\delta_{1/2}(\alpha)} f(T^{-n}(c_\alpha)) = 0$$

for any $n \in \mathbb{N}$, where $\delta_{1/2}$ is given by (15).

3.2. Induced symmetric IETs. — Given an IET $T : I \to I$ and a subinterval $J \subseteq I$ with associated induced map $T_J = (\pi^J, \lambda^J) \in S_0^{\mathcal{A}_J} \times \Lambda^{\mathcal{A}_J, J}$ (see Section 2.2 and Lemma 2.4), we denote by $\{I_\gamma^J\}_{\gamma\in\mathcal{A}_J}$ the intervals exchanged intervals by T_J and by $\{c_\gamma^J\}_{\gamma\in\mathcal{A}_J}$ the middle points of these intervals. We denote by $p_J : I \to J$ the first return map to J by T^{-1}, that is, p_J is given by $x \mapsto T^{-b_J(x)}(x)$, where

$$b_J(x) := \min\{m \geqslant 1 \mid T^{-m}(x) \in J\}. \tag{16}$$

We say that a subinterval $J \subseteq I$ is *symmetric* if there exists $\alpha \in \mathcal{A}$ and $\Delta > 0$ such that $J = [c_\alpha - \Delta, c_\alpha + \Delta) \subseteq I_\alpha$ or $J = \big[c_{1/2} - \Delta, c_{1/2} + \Delta\big) \subseteq I$. To differentiate between the two cases we will refer to the former as *α-symmetric* and to the latter as *$1/2$-symmetric*.

As we shall see, inducing a symmetric IET on symmetric intervals defines IETs satisfying (8), which, as seen in the previous section, is closely related with the symmetry of IETs (see Lemma 3.9). Moreover, it is possible to construct α-symmetric subintervals of the form (1), for any $\alpha \in \mathcal{A} \cup \{1/2\}$ (see Lemma 3.12). Under additional conditions on the symmetric subinterval, we can guarantee that associated induced maps are actually symmetric.

Proposition 3.5. — *Let $T : I \to I$ be an ergodic symmetric IET and fix $\alpha \in \mathcal{A} \smallsetminus \{\pi_0^{-1}(1)\}$ such that $M(\alpha) = +\infty$ (see Corollary 2.3). Fix $m \geqslant 1$ and let $J \subseteq I$ be the left-closed right-open subinterval with endpoints $T^{-m}(\partial I_\alpha)$, $T^m(\partial I_{\widehat{\alpha}})$ (resp. $T^{-m+1}(\partial I_\alpha)$, $T^m(\partial I_{\widehat{\alpha}})$), where $\pi_0(\widehat{\alpha}) = \pi_0(\alpha) - 1$. Then J is β-symmetric for some $\beta \in \mathcal{A}$ (resp. $1/2$-symmetric) and T_J satisfies (8).*

If, in addition, J does not contain points from any connection. Then T_J is an ergodic symmetric IET on $d - d'$ intervals, where d' is the number of non-trivial irreducible connections of T.

In the setting of the previous proposition, the exchanged intervals' middle points of the induced map and of the original IET will be closely related.

Proposition 3.6. — *Let T, $\alpha, \widehat{\alpha}, \beta$ and J as in Proposition 3.5. Assume that J does not contain points from any connection. Then the following holds:*

(1) *For any $\gamma \in \mathcal{A}_J$ there exists unique $\sigma \in \mathcal{A} \cup \{1/2\}$, with $\sigma \neq \beta$, and $\ell \geqslant 1$ such that $c_\gamma^J = p_J(c_\sigma) = T^{-\ell}(c_\sigma)$ and $T_J(p_J(c_\sigma)) = T^{\ell - \delta_{1/2}(\sigma)}(c_\sigma)$, where $\delta_{1/2}$ is given by (15). Moreover, c_σ does not belong to any non-trivial connection.*

(2) *For any $\delta \in \mathcal{A} \cup \{1/2\}$ with $\delta \neq \beta$, either $p_J(c_\delta) = c_\gamma^J$ or $p_J(c_\delta) = \partial I_\gamma^J$ for some $\gamma \in \mathcal{A}_J$. In the latter case, c_δ lies inside a connection of T.*

For the sake of clarity, we postpone the proofs of Propositions 3.5 and 3.6 to the end of this section and rather start by proving several preliminary lemmas.

Given a symmetric IET $T : I \to I$, it is easy to verify that the interior of $1/2$-symmetric intervals are invariant by $\mathcal{I}_I$. On the other hand, as we shall see below, the interior of α-symmetric intervals are invariant by $\mathcal{I}_I \circ T$.

Lemma 3.7. — *Let $T : I \to I$ be a symmetric IET and $J \subseteq I$ be an α-symmetric interval, for some $\alpha \in \mathcal{A}$. Then $\mathcal{I}_I \circ T(\mathring{J}) = \mathring{J}$. Moreover,*

$$\mathcal{I}_J = \mathcal{I}_I \circ T|_{\mathring{J}},$$

that is, $\mathcal{I}_I \circ T|_{\mathring{J}}$ is the symmetric reflection on J.

Proof. — Let $\alpha \in \mathcal{A}$ and $\Delta > 0$ such that $J = [c_\alpha - \Delta, c_\alpha + \Delta) \subseteq I_\alpha$. Fix $x \in J$ and let $-\Delta < \delta < \Delta$ be such that $x = c_\alpha + \delta$. Notice that $\mathcal{I}_J(x) = c_\alpha - \delta$.

Since $T|_{I_\alpha}$ is a translation, by (13), we have

$$\mathcal{I}_I \circ T(x) = \mathcal{I}_I \circ T(c_\alpha + \delta) = \mathcal{I}_I \left(T(c_\alpha) + \delta\right) = \mathcal{I}_I \circ T(c_\alpha) - \delta = c_\alpha - \delta \in J,$$

which finishes the proof. □

Notice that any of the exchanged intervals of a symmetric IET $T : I \to I$ defines a symmetric interval. Hence, we obtain the following as a direct consequence of Lemmas 3.3 and 3.7.

Corollary 3.8. — *Let $T : I \to I$ be a symmetric IET, $\alpha \in \mathcal{A}$ and $m \geqslant 1$. Then*
$$\{c_\alpha, \dots, T^{m-1}(c_\alpha)\} \cap \{\partial I_\beta\}_{\beta\in\mathcal{A}} = \varnothing \iff \{c_\alpha, \dots, T^{-m+1}(c_\alpha)\} \cap \{\partial I_\beta\}_{\beta\in\mathcal{A}} = \varnothing.$$

We now relate the induced IET on a symmetric interval J with the associated involution $\mathcal{I}_J$.

Lemma 3.9. — *Let $T : I \to I$ be a symmetric IET and $J \subseteq I$ be a symmetric interval. Then T_J satisfies* (8).

Proof. — Assume first that J is α-symmetric for some $\alpha \in \mathcal{A}$. Let $x \in J$ be not an endpoint of an interval exchanged by T_J and let $h \in \mathbb{N}$ be such that $T_J(x) = T^h(x)$. Notice that since x is not an endpoint of the exchanged intervals, by (9) applied to T,
$$(\mathcal{I}_I \circ T) \circ T_J(x) = (\mathcal{I}_I \circ T) \circ T^h(x) = T^{-h} \circ (\mathcal{I}_I \circ T)(x).$$
Thus, to prove (8), it suffices to notice that
$$T_J^{-1} \circ (\mathcal{I}_I \circ T)(x) = T^{-h} \circ (\mathcal{I}_I \circ T)(x).$$
Indeed, by Lemma 3.7, for any $n \geqslant 1$, $T^n(x) \in J$ if and only if $(\mathcal{I}_I \circ T) \circ T^n(x) \in J$. Thus the equation above follows since $T_J(x) = T^h(x)$ and $T^{-h} \circ (\mathcal{I}_I \circ T)(x) = (\mathcal{I}_I \circ T) \circ T^h(x)$.

Assume now that J is 1/2-symmetric. Let $x \in J$ be not an endpoint of an interval exchanged by T_J and let $h \in \mathbb{N}$ be such that $T_J(x) = T^h(x)$. Again by (9) we get
$$\mathcal{I}_I \circ T_J(x) = \mathcal{I}_I \circ T^h(x) = T^{-h} \circ \mathcal{I}_I(x)$$
and, similarly to the previous case, to show (8) it is sufficient to notice that $T_J^{-1} \circ \mathcal{I}_I(x) = T^{-h} \circ \mathcal{I}_I(x)$. □

By the result above and given Corollary 3.2, if J is symmetric and the lengths of the continuity intervals of T_J are pairwise distinct, then T_J exchanges its continuity intervals symmetrically.

The following lemma shows that if the orbit of a middle point intersects the discontinuities of the IET, then the middle point must be part of a connection.

Lemma 3.10. — *Let $T : I \to I$ be a symmetric IET and $\beta \in \mathcal{A} \cup \{1/2\}$. Assume there exists $m \in \mathbb{Z}$ and $\alpha \in \mathcal{A}$ such that $T^m(c_\beta) = \partial I_\alpha$ and $T^k(c_\beta) \notin \{\partial I_\gamma\}_{\gamma\in\mathcal{A}}$, for any $-|m| < k < |m|$.*

Then if $m \geqslant 0$, we have $\alpha \neq \pi_0^{-1}(1)$ and
$$T^{-m-\delta_{1/2}(\beta)}(c_\beta) = \partial I_{\widehat{\alpha}},$$
where $\delta_{1/2}$ is given by (15) *and $\pi_0(\widehat{\alpha}) = \pi_0(\alpha) - 1$.*

If on the other hand $m < 0$, then $\alpha \neq \pi_1^{-1}(1)$ and
$$T^{-m-\delta_{1/2}(\beta)}(c_\beta) = \partial I_{\widehat{\alpha}},$$
with $\pi_0(\widehat{\alpha}) = \pi_0(\alpha) + 1$.

In particular, c_β lies inside a non-trivial connection.

Proof. — If $m = 0$, then necessarily $\beta = 1/2$ and we have $c_{1/2} = \partial I_\alpha$ with $\pi_1(\alpha) \neq 1$. By (11), $T^{-1}(c_{1/2}) = \partial I_{\widehat{\alpha}}$, where $\pi_0(\widehat{\alpha}) = \pi_0(\alpha) - 1$.

Without loss of generality, let us assume $m < 0$, the case $m > 0$ being analogous. By (14),

$$T^m(c_\beta) = T^{-1} \circ \mathcal{I}_I(T^{-m-\delta_{1/2}(\beta)}) \iff T^{-m-\delta_{1/2}(\beta)}(c_\beta) = \mathcal{I}_I \circ T\big(T^m(c_\beta)\big).$$

Notice that since $T^{m+1}(c_\beta) \notin \{\partial I_\gamma\}_{\gamma\in\mathcal{A}}$ then $T^m(c_\beta) = \partial I_\alpha \neq \partial I_{\pi_0^{-1}(1)}$. Hence, by (10), the equation above implies $T^{-m-\delta_{1/2}(\beta)}(c_\beta) = \partial I_{\widehat{\alpha}}$, where $\pi_0(\widehat{\alpha}) = \pi_0(\alpha) + 1$. □

The previous lemma immediately implies the following.

Corollary 3.11. — *Let $T : I \to I$ be a symmetric IET. Then any non-trivial irreducible connection of T contains at most one point from the set $\{c_\alpha \mid \alpha \in \mathcal{A} \cup \{1/2\}\}$. In particular, there exists $\alpha \in \mathcal{A}$ such that c_α does not belong to any connection.*

The following result provides a 'recipe' to construct symmetric intervals dynamically by using iterates of the endpoints of the exchanged intervals.

Lemma 3.12. — *Let $T : I \to I$ be a symmetric IET. Let $\alpha \in \mathcal{A} \smallsetminus \{\pi_0^{-1}(1)\}$ and $m < M(\alpha)$. Then*

$$\mathcal{I}_I \circ T(T^{-m}(\partial I_\alpha)) = \mathcal{I}_I \circ (T^{-m+1}(\partial I_\alpha)) = T^m(\partial I_{\widehat{\alpha}}), \tag{17}$$

where $\pi_0(\widehat{\alpha}) = \pi_0(\alpha) - 1$.

In particular, the left-closed right-open interval with endpoints $T^{-m}(\partial I_\alpha)$ and $T^m(\partial I_{\widehat{\alpha}})$ is β-symmetric for some $\beta \in \mathcal{A}$, while the left-closed right-open interval with endpoints $T^{-m+1}(\partial I_\alpha)$ and $T^m(\partial I_{\widehat{\alpha}})$ is $1/2$-symmetric.

Moreover, $M(\alpha) = N(\widehat{\alpha})$.

Proof. — First, notice that (17) is equivalent to (10) if $m = 1$. This implies that $\mathcal{I}_I(\partial I_\alpha) = T(\partial I_{\widehat{\alpha}})$. In the following, we assume $m > 1$. If $m < M(\alpha)$, by (9),

$$\mathcal{I} \circ T(T^{-m}(\partial I_\alpha)) = \mathcal{I} \circ T^{-m+1}(\partial I_\alpha) = T^{m-1} \circ \mathcal{I}(\partial I_\alpha) = T^m(\partial I_{\widehat{\alpha}}).$$

This proves (17), which easily implies that the symmetry properties of the intervals in the statement.

Assume now that $m = M(\alpha)$ and $T^{-m}(\partial I_\alpha) = \partial I_\beta$ for some $\beta \in \mathcal{A}$. Since $M(\alpha)$ is the minimal number that makes a connection, by (17) we have

$$\mathcal{I} \circ T(T(\partial I_\beta)) = \mathcal{I} \circ T(T^{-m+1}\partial I_\alpha) = T^{m-1}(\partial I_{\widehat{\alpha}}).$$

Moreover, noticing that $T(\partial I_\beta) \notin \{\partial I_\alpha\}_{\alpha\in\mathcal{A}}$ since $m > 1$, by (8)

$$\mathcal{I} \circ T(T(\partial I_\beta)) = T^{-1} \circ \mathcal{I} \circ T(\partial I_\beta) = T^{-1}(\partial I_{\widehat{\beta}}),$$

where $\pi_0(\widehat{\beta}) = \pi_0(\beta)+1$. Thus we get $T^m(\partial I_{\widehat{\alpha}}) = \partial I_{\widehat{\beta}}$. This proves $N(\widehat{\alpha}) \leqslant M(\alpha)$. The opposite inequality is proved analogously, by exchanging the roles of T and T^{-1}. □

Using the previous lemma, it is not difficult to construct symmetric intervals as in the statement of Proposition 3.5.

Lemma 3.13. — *Let $T : I \to I$ be an ergodic symmetric IET. Then there exists $\beta \in \mathcal{A}$ such that c_β is not part of any non-trivial connection from T and, for any $\varepsilon > 0$, there exists a β-symmetric subinterval $J \subseteq I$ disjoint from the connections of T satisfying $|J| < \varepsilon$.*

Proof. — By Corollary 3.11, there exists $\beta \in \mathcal{A}$ such that c_β is not part of any connection of T and, by Corollary 2.3, there exists $\alpha \in \mathcal{A} \smallsetminus \{\pi_0^{-1}(1)\}$ such that $M(\alpha) = +\infty$.

Since T is ergodic and hence minimal, there exists $m \geqslant 1$ such that

$$\left|T^{-m}(\partial I_\alpha) - c_\beta\right| < \varepsilon/2.$$

By taking ε smaller if necessary, we may assume that $\varepsilon < \min_{\delta\in\mathcal{A}} |I_\delta|$ and that $(c_\beta - \varepsilon, c_\beta + \varepsilon)$ does not contain any point from any connection.

Then, by Lemma 3.12, the left-closed right-open subinterval J with endpoints $T^{-m}(\partial I_\alpha)$, $T^m(\partial I_{\widehat{\alpha}})$ is β-symmetric, where $\pi_0(\widehat{\alpha}) = \pi_0(\alpha) - 1$. In particular $J \subseteq (c_\beta - \varepsilon/2, c_\beta + \varepsilon/2)$. □

We are now in a position to prove Propositions 3.5 and 3.6.

Proof of Proposition 3.5. — Let $T = (\pi, \lambda), J, m, \alpha, \widehat{\alpha}$ as in the statement of the proposition, with J not necessarily disjoint from the connections of T.

By Lemma 3.12, there exists $\beta \in \mathcal{A} \cup \{1/2\}$ such that J is β-symmetric, where $\beta = 1/2$ only if the endpoints of J are of the form $T^{-m+1}(\partial I_\alpha)$, $T^m(\partial I_{\widehat{\alpha}})$. Then, by Lemma 3.9, the induced IET T_J satisfies (8).

From now on, let us assume that J does not contain any point from any connection. By Lemma 2.4, it follows that $T_J = (\pi^J, \lambda^J)$ is an IET on $d - d'$ intervals, where d' is the number of non-trivial irreducible connections of T. In view of Proposition 2.6, there exists $\widetilde{\lambda} \in \Lambda_T^{\mathcal{A},I}$ (see (3)) such that $\widetilde{T} := (\pi, \widetilde{\lambda})$ is symmetric, the IET $\widetilde{T}_{\widetilde{J}} = (\widetilde{\pi}^J, \widetilde{\lambda}^J)$ obtained by inducing $\widetilde{T}$ to the interval $\widetilde{J}$ with endpoints $T^{-m}(\partial \widetilde{I}_\alpha)$ and $T^m(\partial \widetilde{I}_{\widetilde{\alpha}})$ has the same permutation as T_J, and $\widetilde{\lambda}^J$ has intervals of rationally independent lengths.

Since $\widetilde{J}$ is a symmetric interval for $\widetilde{T}$, by Lemma 3.9 the induced IET $\widetilde{T}_J$ satisfies (8). Thus, by Corollary 3.2, $\widetilde{T}_{\widetilde{J}}$ exchanges its maximal continuity intervals symmetrically, and therefore so does T_J.

Hence, to prove that T_J is a symmetric IET, it suffices to show that it possesses exactly $d - d'$ maximal continuity intervals. By replacing T_J with $\widetilde{T}_{\widetilde{J}}$ if necessary, we may assume that the intervals exchanged by T_J are of rationally independent lengths. Note that since $\widetilde{\lambda} \in \Lambda_T^{\mathcal{A},I}$, this change would not affect the connection pattern. Then also the intervals of continuity of T_J are of rationally independent lengths. Let $\{I_\alpha^J\}_{\alpha\in\mathcal{A}_J}$ be the intervals exchanged by T_J and let $\{\widehat{I}_\alpha^J\}_{\alpha\in\mathcal{C}}$ be the maximal continuity intervals of T_J. Then, to finish the proof, it suffices to show that $\#\mathcal{C} = \#\mathcal{A}_J$.

For every $\gamma \in \mathcal{C}$ consider the point $\widehat{c}_\gamma^J$, the center-point of the interval $\widehat{I}_\gamma^J$. Since T_J interchanges the intervals of continuity symmetrically, we have $T_J(\widehat{c}_\gamma^J) = \mathcal{I}_J(\widehat{c}_\gamma^J)$ and no

other point satisfies this equation. Moreover, since the lengths of intervals exchanged by T_J are rationally independent, we also have $\widehat{c}_\gamma^J \notin \{\partial I_\alpha^J\}_{\alpha\in\mathcal{A}_J}$.

Claim 1. — *Let $\sigma \in \mathcal{A}\cup\{1/2\}$ with $\sigma \neq \beta$ and $\ell := b_J(c_\sigma) \geqslant 1$ be the first backwards return time of c_σ to J, that is, such that $p_J(c_\sigma) = T^{-\ell}(c_\sigma)$. Then, the following dichotomy holds.*

– *either $\{c_\sigma, T(c_\sigma), \dots, T^{\ell-\delta_{1/2}(\sigma)}(c_\sigma)\} \cap \{\partial I_\delta\}_{\delta\in\mathcal{A}} = \varnothing$ and*

$$p_J(c_\sigma) = \widehat{c}_\gamma^J, \qquad T_J(\widehat{c}_\gamma^J) = T^{\ell-\delta_{1/2}(\sigma)}(c_\sigma) = T^{2\ell-\delta_{1/2}(\sigma)}(\widehat{c}_\gamma^J), \quad \text{for some } \gamma\in\mathcal{C},$$

– *or $\{c_\sigma, T(c_\sigma), \dots, T^{\ell-\delta_{1/2}(\sigma))}(c_\sigma)\} \cap \{\partial I_\delta\}_{\delta\in\mathcal{A}} \neq \varnothing$ and*

$$p_J(c_\sigma) = \partial I_\gamma^J, \quad \text{for some } \gamma\in\mathcal{A}_J,$$

where $\delta_{1/2}$ is given by (15)*. Moreover, in the latter case, c_σ lies in a non-trivial connection of T.*

Proof. — First, let us assume that $\{c_\sigma, T(c_\sigma), \dots, T^{\ell-\delta_{1/2}(\sigma)}(c_\sigma)\} \cap \{\partial I_\delta\}_{\delta\in\mathcal{A}} = \varnothing$.

By (14) and Lemma 3.7, $T^{-k}(c_\sigma) \in J$ if and only if $T^{k-\delta_{1/2}}(c_\sigma) \in J$, for any k such that $0 \leqslant k \leqslant \ell$. Hence, the first visit time of c_σ to J via T is $\ell - \delta_{1/2}(\sigma)$, which implies

$$T_J(p_J(c_\sigma)) = T^{\ell-\delta_{1/2}(\sigma)}(c_\sigma).$$

Moreover, $p_J(c_\sigma) = T^{-\ell}(c_\sigma)$ is a fixed point of $\mathcal{I}_J \circ T_J$. Indeed, by (9) and Lemma 3.7,

$$\begin{aligned}\mathcal{I}_J \circ T_J(p_J(c_\sigma)) &= \mathcal{I}\circ T\big(T^{\ell-\delta_{1/2}(\sigma)}(c_\sigma)\big)\\ &= T^{-\ell+\delta_{1/2}(\sigma)}\circ T^{-1}\circ \mathcal{I}(c_\sigma) = T^{-\ell+\delta_{1/2}(\sigma)}(c_\sigma) = p_J(c_\sigma).\end{aligned}$$

Therefore, since $p_J(c_\sigma)$ is a fixed point of $\mathcal{I}_J \circ T_J$ and T_J exchanges its continuity intervals symmetrically, $p_J(c_\sigma) = \widehat{c}_\gamma^J$, for some $\gamma \in \mathcal{C}$.

Now, let us assume that $\{c_\sigma, T(c_\sigma), \dots, T^{\ell-\delta_{1/2}(\sigma)}(c_\sigma)\} \cap \{\partial I_\delta\}_{\delta\in\mathcal{A}} \neq \varnothing$.

Let k with $0 \leqslant k \leqslant \ell - \delta_{1/2}(\sigma)$ be the minimum such that $T^k(c_\sigma) = \partial I_\delta$ for some $\delta \in \mathcal{A}$. By Lemma 3.10, $T^{-k-\delta_{1/2}(\sigma)}(c_\sigma) = \partial I_{\widehat{\delta}}$, where $\pi_0(\widehat{\delta}) = \pi_0(\delta) - 1$. Since $-k - \delta_{1/2}(\sigma) \geqslant -\ell$, it follows that $p_J(\partial I_{\widehat{\delta}}) = p_J(c_\sigma)$ which, by definition of T_J (see Section 2.2 and Lemma 2.4), coincides with ∂I_γ^J, for some $\gamma \in \mathcal{A}_J$. Notice that, in this case, c_σ lies in a non-trivial connection of T. □

The claim above shows that p_J maps the set $\{c_\sigma \mid \sigma \in \mathcal{B}\}$, where

$$\mathcal{B} := \{\sigma \in \mathcal{A}\cup\{1/2\} \mid \sigma \neq \beta \text{ and } c_\sigma \text{ does not belong to any non-trivial irreducible connection}\},$$

injectively to the set $\{\widehat{c}_\gamma^J \mid \gamma \in \mathcal{C}\}$.

Indeed, if for $\sigma, \sigma' \in \mathcal{B}$ we have $p_J(c_\sigma) = \widehat{c}_\gamma^J = p_J(c'_\sigma)$ then it follows from the previous claim that $r_J(\widehat{c}_\gamma^J) = 2b_J(c_\sigma) - \delta_{1/2}(\sigma) = 2b_J(c'_\sigma) - \delta_{1/2}(\sigma')$. Hence, since all the terms in the previous equality must have the same parity, it follows that either $\sigma = 1/2 = \sigma'$ or $\sigma, \sigma' \in \mathcal{A}$. In the latter case, it follows from the claim that $c_\sigma = T^{r_J(\widehat{c}_\gamma^J)/2}(\widehat{c}_\gamma^J) = c'_\sigma$.

Therefore, Claim 1 implies that $\#\mathcal{C} \geqslant \#\mathcal{B}$. Since, by Corollary 3.11, any non-trivial irreducible connection contains at most one point of the form $\{c_\sigma \mid \sigma \in \mathcal{A} \cup \{1/2\}\}$ it follows that $\#\mathcal{B} \geqslant d - d'$. Therefore $\#\mathcal{C} \geqslant d - d'$, which together with $\#\mathcal{C} \leqslant \#\mathcal{A}_J = d - d' \leqslant \#\mathcal{A}$, implies

$$\#\mathcal{C} = \#\mathcal{B} = d - d' = \#\mathcal{A}_J. \qquad \square$$

Proof of Proposition 3.6. — By Proposition 3.5, it follows that the maximal continuity intervals of T_J, which in the proof of Proposition 3.5 we denoted by $\{\widehat{I}_\alpha^J\}_{\alpha\in\mathcal{C}}$, coincide with the intervals exchanged by T_J, which we denoted by $\{I_\alpha^J\}_{\alpha\in\mathcal{A}_J}$.

Therefore, Proposition 3.6 follows from Claim 1 and the fact that p_J maps the set

$$\{c_\sigma \mid \sigma \in \mathcal{A} \cup \{1/2\},\ \sigma \neq \beta \text{ and } c_\sigma \text{ does not belong to any non-trivial irreducible connection}\},$$

injectively to the set $\{\widehat{c}_\gamma^J \mid \gamma \in \mathcal{C}\} = \{c_\gamma^J \mid \gamma \in \mathcal{A}_J\}$, which we showed at the end of the proof of Proposition 3.5. $\square$

The following is a direct consequence of Proposition 3.6.

Corollary 3.14. — *If a symmetric IET T has a non-trivial connection that does not contain a point from the set $\{c_\alpha \mid \alpha \in \mathcal{A} \cup 1/2\}$, then T is not ergodic.*

Proof. — By Proposition 3.6, we have that the sets

$$\{c_\sigma \mid \sigma \in \mathcal{A} \cup \{1/2\},\ \sigma \neq \beta \text{ and } c_\sigma \text{ does not belong to any non-trivial irreducible connection}\},$$

and $\mathcal{A}_J$ have the same cardinality (see the end of the proof of Proposition 3.6). Recalling that $d - \#\mathcal{A}_J$ is equal to the number of non-trivial irreducible connections and that each non-trivial irreducible connection contains at most one point from the set $\{c_\sigma \mid \sigma \in \mathcal{A} \cup \{1/2\}\}$ (see Corollary 3.11), the Corollary follows by a simple counting argument. $\square$

The following example illustrates the situation described in the previous corollary.

Example 3.15. — Let $\mathcal{A} = \{1, 2, 3, 4\}$, and let T be a symmetric 4-IET with permutation $\pi_0(i) = i,\ 1 \leqslant i \leqslant 4$ and lengths $|I_i| = \lambda_i > 0$, for $i = 1, 2, 3$, and $|I_4| = 2(\lambda_1 + \lambda_2) + \lambda_3$. Choose $\lambda = (\lambda_i)_{i\in\mathcal{A}}$ such that $|\lambda|_1 = 1$. Note that $T(\partial I_2) = \lambda_3 + \lambda_4 = 1 - (\lambda_1 + \lambda_2)$ and is not the middle point of I_4. Also note that $T^2(\partial I_2) = \lambda_1 + \lambda_2 + \lambda_3 = \partial I_4 < \frac{1}{2}$. Hence T has a connection (of length 2), which does not contain any center point or $\frac{1}{2}$. In this case, we have an invariant set $I_3 \cup T(I_3)$, where $T^2(I_3) = I_3$. Thus T is not ergodic. This is represented in the figure below.

The following corollary is of independent interest.

Corollary 3.16. — *Assume that T is an ergodic symmetric IET such that $c_{1/2}$ lies inside a connection. Then -1 is an eigenvalue for the Koopman operator associated with T. In particular, T is not weak mixing.*

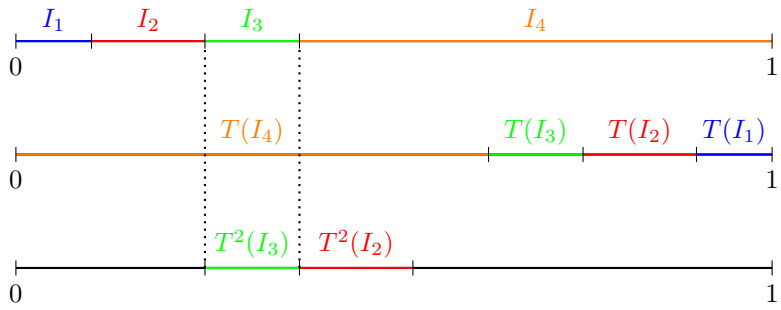

Figure 1. Plot of the exchanged intervals (and some of their iterates) for the symmetric IET T described in Example 3.15. The set $\{\partial I_2, T(\partial I_2), T^2(\partial I_2) = \partial I_4\}$ defines a connection disjoint from the set $\{c_\alpha \mid \alpha \in \mathcal{A} \cup 1/2\}$. By Corollary 3.14, T is not ergodic.

Proof. — Let $J \subseteq I$ as in Proposition 3.5 such that J does not contain any point from any connection. Such an interval exists by Lemma 3.13. Then, by Propositions 3.5 and 3.6, T_J is symmetric and the middle points $\{c_\gamma^J\}_{\gamma\in\mathcal{A}_J}$ of the exchanged intervals $\{I_\gamma^J\}_{\gamma\in\mathcal{A}_J}$ are preimages of the middle points of the intervals exchanged by T_J. Moreover, again by Proposition 3.6, the Rokhlin towers associated to T_J are all of even height since for any $\gamma \in \mathcal{A}_J$ there exists $\sigma \in \mathcal{A}$ such that $c_\gamma^J = p_J(c_\sigma)$ and $T_J(c_\gamma^J) = T^{2b_J(c_\sigma)}(c_\gamma^J)$, where b_J is given by (16).

By defining a function f that equals 1 (resp. -1) on the odd (resp. even) levels of each Rokhlin tower, we get an eigenfunction of T with eigenvalue -1, that is, such that $f \circ T = -f$. □

Example 3.17. — Let $\mathcal{A} = \{1, 2, 3, 4\}$, and let T be a symmetric 4-IET with permutation $\pi_0(i) = i$, $1 \leqslant i \leqslant 4$ and lengths $|I_i| = \lambda_i > 0$, for $i = 1, 2, 3$, and $|I_4| = 1/2 + \lambda_1 + \lambda_2$, where $2(\lambda_1 + \lambda_2) + \lambda_3 = 1/2$ and $\lambda_2 > \lambda_1 + \lambda_3$. With this configuration, we have that $T(\partial I_2) = 1 - \lambda_1 - \lambda_2$, $T^2(\partial I_2) = 1/2$, and $T^3(\partial I_2) = \partial I_3$. Thus we have a connection containing $1/2$.

By inducing on the interval $J = I_2 = [\partial I_2, \partial I_3)$, we obtain a symmetric 3-IET, with initial order of J_1, J_2, J_3 and the parameters as follows: $|J_1| = \lambda_3, |J_2| = \lambda_1, |J_3| = \lambda_2 - \lambda_1 - \lambda_3 > 0$.

As shown in the figure below, we can check that $T^4(J_3) \subseteq I_2$, $T^3(J_1) = I_3$, $T^3(I_3) \subseteq I_2$, $T^4(J_2) = I_1$, $T^4(I_1) \subseteq I_2$. Thus the first return times are $r_J(J_1) = 6$, $r_J(J_2) = 8$, $r_J(J_3) = 4$. So all the towers are of even heights, and we can give value 1 to the odd levels and -1 to the even levels of each tower, which gives us an eigenfunction of eigenvalue -1. To guarantee the ergodicity of the IETs, it suffices to ask that λ_2 and λ_3 are rationally independent. Indeed, then the induced map is a 3-IET, whose image after a single step of the classical Rauzy-Veech induction yields an irrational rotation.

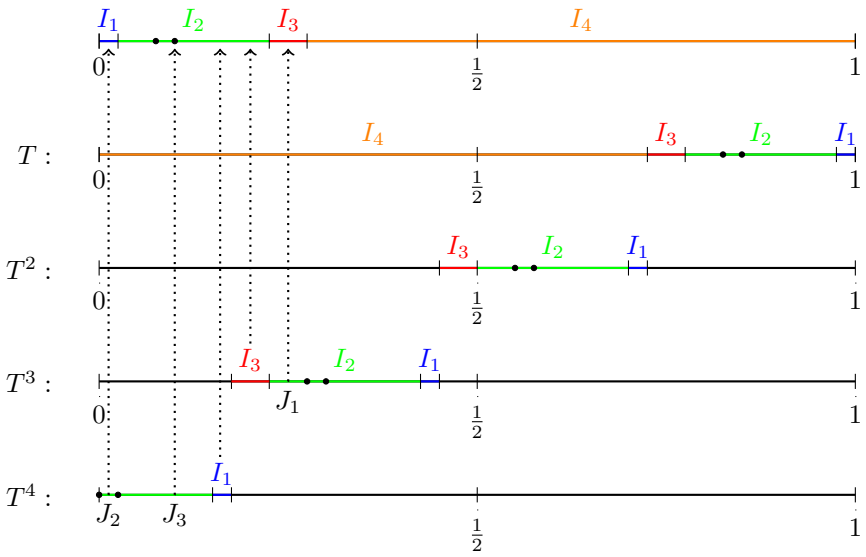

Figure 2. Plot of the exchanged intervals (and some of their iterates) for the symmetric IET T described in Example 3.17. Every Rokhlin tower in the decomposition associated with the induced map T_J (see (2)) is of even height.

4. The essential values criterion

In this section, we recall and state the standard notion of essential value, which is a classical tool to study skew products' ergodicity. Let $(X, \mathcal{B}, \mu)$ be a standard probability space. Let $T : X \to X$ be μ-measure preserving automorphism and let $f : X \to \mathbb{R}^m$, where $m \geqslant 1$. Consider the skew product T_f on $X \times \mathbb{R}^n$ given by

$$T_f(x, r) = (Tx, r + f(x)).$$

We say that $a \in \mathbb{R}^m$ is an *essential value* of T_f if for every $\varepsilon > 0$ and every measurable subset $E \subseteq X$ with $\mu(E) > 0$, there exists $n \in \mathbb{N}$ such that

$$\mu\{x \in E \mid T^n(x) \in E \text{ and } |S_n f(x) - a| < \varepsilon\} > 0,$$

where $S_n f(x)$ denotes the n-th Birkhoff sum of f evaluated at x, which is given by

$$S_n f(x) := \begin{cases} \sum_{i=0}^{n-1} f(T^i(x)) & \text{if } n \geqslant 1, \\ 0 & \text{if } n = 0, \\ -\sum_{i=-n}^{-1} f(T^i(x)) & \text{if } n \leqslant 1. \end{cases} \tag{18}$$

We denote the set of essential values of T_f by $\mathrm{Ess}(T_f)$.

The following classical fact links this notion to the ergodicity of the skew product.

Theorem 4.1. — *With the notation as above, the set* $\mathrm{Ess}(T_f)$ *is a closed subgroup of* $\mathbb{R}^m$*. Moreover,* T_f *is ergodic with respect to* $\mu \otimes \mathrm{Leb}_{\mathbb{R}^m}$ *if and only if* T *is ergodic and* $\mathrm{Ess}(T_f) = \mathbb{R}^m$*.*

We will use a simplified version of this criterion, as introduced by Conze and Frączek in [4].

Theorem 4.2 ([4, Lem. 2.7]). — *Let* $a \in \mathbb{R}^m$*. Assume that for every* $\varepsilon > 0$ *there exists a sequence of subsets* $\{\Xi_n\}_{n\in\mathbb{N}}$ *and an increasing sequence* $\{q_n\}_{n\in\mathbb{N}}$ *of natural numbers such that*

(1) $\liminf_{n\to\infty} \mu(\Xi_n) > 0$,
(2) $\lim_{n\to\infty} \sup_{x\in\Xi_n} |T^{q_n}(x) - x| = 0$,
(3) $\lim_{n\to\infty} \mu(\Xi_n \triangle T(\Xi_n)) = 0$,
(4) $|S_{q_n} f(x) - a| < \varepsilon$.

Then $a \in \mathrm{Ess}(T_f)$*.*

Lemma 2.7 in [4] actually states that the topological support of the limit distribution $P := \lim_{n\to\infty} \frac{1}{\mu(\Xi_n)} (S_{q_n} f(x)|_{\Xi_n})_* \mu|_{\Xi_n}$, which exists up to taking a subsequence due to tightness guaranteed by Condition (4), is contained in $\mathrm{Ess}(T_f)$. However, again by (4), the topological support of P is contained in $\overline{B(a,\varepsilon)} \subset \mathbb{R}^m$. By passing with ε to 0 and by the fact that $\mathrm{Ess}(T_f)$ is a closed subset of $\mathbb{R}^m$, we get that $a \in \mathrm{Ess}(T_f)$.

We now provide a version of the above criterion, that is going to be effective for our purposes.

Proposition 4.3. — *Let* $T : I \to I$ *be an ergodic IET and let* $m \in \mathbb{N}_+$*. Assume that the function* $f : I^{\times m} \to \mathbb{R}^m$ *satisfies the following:*

(i) f *is of the form* $(x_1, \dots, x_m) \mapsto (f_1(x_1), \dots, f_m(x_m))$ *with each* f_j *being differentiable over the exchanged intervals,*

(ii) *there exist sequences* $\{\Xi_n\}_{n\in\mathbb{N}}$ *and* $\{q_n\}_{n\in\mathbb{N}}$ *satisfying* (1)-(3) *in Theorem* 4.2*, with the additional assumption that, for any* $n \in \mathbb{N}$*,* $\Xi_n = \bigsqcup_{i=0}^{h_n-1} T^i(I_n)$ *is a Rokhlin tower of* $h_n \leqslant q_n$ *intervals with* $|I_n| \leqslant 1/q_n$ *and* $\sup_{x\in\Xi_n} |T^{q_n}(x) - x| \leqslant D/q_n$ *for some* $D > 1$*,*

(iii) *for every* $x \in \Xi_n$*, the interval* $[x, T^{q_n}(x)]$ *(or* $[T^{q_n}(x), x]$*) is a continuity interval of* f_j*, for every* $j \in \{1, \dots, m\}$*,*

(iv) *there exists* $C > 1$ *such that*

$$|S_{q_n} f_j(x)| \leqslant C, \qquad C^{-1} q_n \leqslant S_{q_n} f_j'(x) \leqslant C q_n,$$

for any $j = 1, \dots, m$ *and any* $x \in \Xi_n$*, and*

$$|f_j'(x)| \leqslant C,$$

for any $x \in I$*.*

If additionally $T^{\otimes m}$ *is ergodic, then so is* $T_f^{\otimes m}$*.*

Proof. — In view of Theorem 4.1, it is enough to show that we can obtain an m-dimensional cube of essential values. We will prove the proposition for $m = 1$ by realizing the essential values through Rokhlin towers, which are subsets of Ξ_n. For $m \geqslant 2$ we first obtain the same result for every $j = 1, \dots, m$, i.e., we find an interval (a_j, b_j) of essential values, realized through Rokhlin towers inside Ξ_n. Hence, the set $(a_1, b_1) \times \cdots \times (a_m, b_m)$ is the set of essential values realized via sequences of Rokhlin towers inside $\bigsqcup_{i=0}^{h_n-1}(T^{\otimes m})^i(I_n^{\times m})$.

That being said, we assume from now on that $m = 1$. Note that due to our assumption on Ξ_n and the derivative of f, each of the sets $S_{q_n}f(\Xi_n) \subseteq [-2C, 2C]$ is a uniformly bounded union of intervals. Note that, since

$$\liminf_{n\to\infty} h_n|I_n| = \liminf_{n\to\infty} \mu(\Xi_n) > 0$$

and $h_n \leqslant q_n$, we have that, by passing to a subsequence if necessary, there exists $E > 1$ with

$$E^{-1} \leqslant q_n|I_n| \leqslant 1.$$

Hence, given the assumption on the derivative, we have

$$\mathrm{Leb}_{\mathbb{R}}\left(S_{q_n}f(\Xi_n)\right) \geqslant \frac{1}{CE} > 0$$

for every $n \in \mathbb{N}$. By passing to a subsequence if necessary, we may assume that the sets $S_{q_n}f(\Xi_n)$ have a nonempty intersection. Let $y \in \bigcap_{i=1}^{\infty} S_{q_n}f(\Xi_n)$. We claim that y is an essential value. For this purpose, we will construct a subtower $\Xi_n^y \subseteq \Xi_n$, i.e., a subset of Ξ_n which itself is a Rokhlin tower of intervals, depending on $\varepsilon > 0$, such that (1) and (4) in Theorem 4.2 is satisfied. This is enough, since (2) and (3) are automatically satisfied by any sequence of subtowers of Ξ_n.

Fix $\varepsilon > 0$ and consider the point $x_n \in \Xi_n$ such that $S_{q_n}f(x_n) = y$. Let $\ell_n \in \{0, \dots, h_n - 1\}$ be such that $x_n \in T^{\ell_n}(I_n)$ and assume without loss of generality that $\ell_n < h_n/2$, the other case being treated symmetrically. Since each level of the tower is an interval then

$$\text{either} \qquad \left(x_n, x_n + \frac{1}{\max\{C, D, E\}q_n}\right) \subseteq T^{\ell_n}(I_n),$$

$$\text{or} \qquad \left(x_n - \frac{1}{\max\{C, D, E\}q_n}, x_n\right) \subseteq T^{\ell_n}(I_n).$$

Again, without loss of generality, let us assume that it is the former. Consider the tower

$$\Xi_n^y := \bigsqcup_{i=0}^{\varepsilon h_n/4CD} T^i\left(x_n, x_n + \frac{\varepsilon}{2\max\{C, D, E\}q_n}\right) \subseteq \Xi_n.$$

We now show that for every $x \in \Xi_n^y$ we have $S_{q_n}f(x) \in (y - \varepsilon, y + \varepsilon)$. If $x \in T^{\ell_n}(I_n)$, then by the mean value theorem, we have

$$|S_{q_n}f(x) - y| = |S_{q_n}f(x) - S_{q_n}f(x_n)| \leqslant Cq_n|x - x_n| \leqslant \varepsilon/2.$$

If $x \in T^j(T^{\ell_n}(I_n))$ with $j = 1, \dots, \varepsilon h_n/4CD$, then we split the Birkhoff sum into two pieces $S_{q_n}f(x) = S_{q_n-j}f(x) + S_jf(T^{q_n-j}(x))$, which we estimate separately. For the

first term, we have

$$\left|S_{q_n-j}f(x) - S_{q_n-j}f(T^j(x_n))\right| \leqslant Cq_n|x - T^j(x_n)| \leqslant \varepsilon/2.$$

For every $k \in \{0, \dots, h_n/2\}$ we have that $T^k(x_n)$ and $T^k(T^{-j}(x))$ belong to the same level of Ξ_n and, since Ξ_n is a tower of intervals as such, belong to the same interval continuity of f. Moreover, by (iii), $T^k(T^{q_n-j}(x)) = T^k(T^{q_n}(T^{-j}(x)))$ and $T^k(T^{-j}(x))$ belong to the same continuity interval of f for every $k \in \{0, \dots, h_n/2\}$. Hence $T^k(T^{q_n-j}(x))$ and $T^k(x_n)$ belong to the same continuity intervals of f. Hence, by (4) and mean value theorem we get

$$\left|S_j f(T^{q_n-j}(x)) - S_j f(x_n)\right| \leqslant \frac{C\varepsilon h_n}{4D}\,|T^{q_n-j}(x) - x_n| \leqslant \frac{C\varepsilon h_n}{4CD}\,\frac{2D}{q_n} \leqslant \varepsilon/2$$

and thus $|S_{q_n}f(x) - y| \leqslant \varepsilon$. It remains to notice that

$$\begin{aligned}\liminf_{n\to\infty} \mathrm{Leb}(\Xi_n^y) &\geqslant \liminf_{n\to\infty} \frac{\varepsilon h_n}{4CD}\,\frac{1}{\max\{C,D,E\}}\,|I_n| \\ &\geqslant \liminf_{n\to\infty} \frac{1}{4(\max\{C,D,E\})^2}\,\mathrm{Leb}(\Xi_n) > 0.\end{aligned}$$

Thus we have proved that $y \in \mathrm{Ess}(T_f)$. Note that for every $n \in \mathbb{N}$ we have

$$x \in \Big(x_n, x_n + \frac{1}{2\max\{C,D,E\}q_n}\Big).$$

Hence,

$$S_{q_n}f(x) - S_{q_n}f(x_n) \geqslant C^{-1}q_n|x - x_n|.$$

Therefore, by applying similar reasoning as to y, we obtain that every

$$z \in \Big[y, y + \frac{1}{2C^{-1}\max\{C,D,E\}}\Big]$$

is an essential value of T_f. This finishes the proof of the proposition. □

5. Proofs of main results.

This section contains the proof of Theorems 1.1, 1.2 and 1.3. In all three proofs, we will apply the ergodicity criterion described in Proposition 4.3. For this reason, we start this section by outlining a construction that will be common to all of the proofs since it concerns only the underlying IET T and not the cocycle being considered. At the end of this construction, we will describe in detail why the assumptions of Proposition 4.3 are fulfilled in each setting.

Throughout this section, let $T = (\pi, \lambda) : I \to I$ be an ergodic symmetric IET on $d = \#\mathcal{A}$ intervals $\{I_\alpha\}_{\alpha\in\mathcal{A}}$.

By Corollary 3.11, there exists $\beta \in \mathcal{A}$ such that c_β is not a part of any connection of T. By Lemma 3.13, there exists $\alpha \in \mathcal{A}$ and a nested sequence of β-symmetric intervals $\{J_n\}_{n\in\mathbb{N}}$ disjoint from the connections of T, with endpoints $T^{-m_n}(\partial I_\alpha)$ and $T^{m_n}(\partial I_{\widehat{\alpha}})$ for some $m_n \nearrow \infty$, where $\pi_0(\widehat{\alpha}) = \pi_0(\alpha)-1$, and such that $\{c_\beta\} = \bigcap_{n\in\mathbb{N}} J_n$.

By Proposition 3.5, for every $n \in \mathbb{N}$, the induced IET T_{J_n} is a symmetric IET with $d - d'$ intervals, where d' is the number of non-trivial irreducible connections of T.

Without loss of generality we may index their exchanged intervals using the same alphabet $\mathcal{B} \subseteq \mathcal{A}$. Let us denote by $\{I^n_\gamma\}_{\gamma\in\mathcal{B}}$ the intervals exchanged by T_{J_n} and by $\{c^n_\gamma\}_{\gamma\in\mathcal{B}}$ their middle points, where, to avoid the use of double subscripts, we changed our usual notation $I^{J_n}_\gamma$ (resp. $c^{J_n}_\gamma$) to I^n_γ (resp. c^n_γ).

By Proposition 3.6, each of the towers in the tower decomposition of I associated with T_{J_n}

$$I = \bigsqcup_{\gamma\in\mathcal{B}} \bigsqcup_{i=0}^{h^n_\gamma-1} T^i(I^n_\gamma), \tag{19}$$

where $h^n = (h^n_\gamma)_{\gamma\in\mathcal{B}}$ is some vector in $\mathbb{N}_+^{\mathcal{A}}$ (see Section 2.2), contains exactly one point from the set

$$\{c_\sigma \mid \sigma \in \mathcal{A}\cup\{1/2\},\ \sigma\neq\beta \text{ and } c_\sigma \text{ does not belong to any non-trivial irreducible connection}\},$$

in the middle of its *central level* $T^{\lfloor h^n_\gamma/2\rfloor}(I^n_\gamma)$. More precisely, for every $\gamma\in\mathcal{B}$ there exists σ in the set above such that the middle point c^n_γ of the interval I^n_γ verifies $c_\sigma = T^{\lfloor h^n_\gamma/2\rfloor}(c^n_\gamma)$.

Using these facts, we will show how to build towers $\{\Xi_n\}_{n\in\mathbb{N}}$ and a sequence $\{q_n\}_{n\in\mathbb{N}}$ that satisfies the assumptions of Proposition 4.3.

A natural approach would be to consider subtowers of the already constructed towers. However in their current form, the towers may be very unbalanced: the wide towers may be very short and thus of very small measure, while thin towers may be very tall and contain the majority of the interval I. Since we need to construct towers of measure bounded away from 0, we would have to choose them to be inside of the thin towers. This however makes it very difficult to control the rigidity of T inside those towers as well as to estimate the values of Birkhoff sums. We tackle this problem by jumping between the points around which we induce.

Consider a Rokhlin tower $X_n := \bigsqcup_{i=0}^{h^n_{\gamma_n}-1} T^i(I^n_{\gamma_n})$, where $\gamma_n\in\mathcal{B}$ is chosen so that its Lebesgue measure is the largest compared to the other towers in the decomposition (19). In particular

$$\mathrm{Leb}(X_n) \geqslant \frac{1}{\#\mathcal{B}} \geqslant \frac{1}{\#\mathcal{A}}.$$

Let us denote by $\mathfrak{J}_n$ the central level of this tower and recall that it contains a point c_σ for some $\sigma\in\mathcal{A}\cup\{1/2\}$. By Proposition 3.5, the induced transformation $T_{\mathfrak{J}_n}$ is a symmetric IET with $d-d'$ intervals, and we denote its exchanged intervals by $\{\mathfrak{I}^n_\gamma\}_{\gamma\in\mathcal{B}}$. As before, we have a decomposition in Rokhlin towers of the form

$$I = \bigsqcup_{\gamma\in\mathcal{B}} \bigsqcup_{i=0}^{\mathfrak{h}^n_\gamma-1} T^i(\mathfrak{I}^n_\gamma).$$

Let $\Gamma_n\in\mathcal{B}$ be such that $\mathfrak{I}^n_{\Gamma_n}$ is the largest of all intervals exchanged by $T_{\mathfrak{J}_n}$. Up to taking a subsequence, let us assume without loss of generality that there exists $\Gamma\in\mathcal{B}$ such that $\Gamma_n=\Gamma$, for every $n\in\mathbb{N}$.

As before, by Proposition 3.6, the tower $\bigsqcup_{i=0}^{\mathfrak{h}_\Gamma^n-1} T^i(\mathfrak{I}_\Gamma^n)$ contains exactly one point of the form c_σ for some $\sigma \in \mathcal{A} \cup \{1/2\}$ in the middle of its central level. Let us denote this point by $\mathfrak{c}_n$.

Define

$$\Xi_n := \bigsqcup_{i=0}^{h_{\gamma_n}^n/2-1} T^i(\mathfrak{I}_\Gamma^n) \qquad \text{and} \qquad q_n := \mathfrak{h}_\Gamma^n.$$

Before passing to the proofs of each of the theorems, let us show that $\{\Xi_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ satisfy the assumptions (ii) and (iii) in Proposition 4.3.

We start by showing that (ii) in Proposition 4.3 is satisfied, that is, that the sequences above verify (1)–(3) in Theorem 4.2.

First, we can easily check that $\{\Xi_n\}_{n\in\mathbb{N}}$ satisfies (1) in Theorem 4.2. Indeed,

$$\begin{aligned}\operatorname{Leb}(\Xi_n) &= (h_{\gamma_n}^n/2)|\mathfrak{I}_\Gamma^n| \\ &> \frac{1}{\#\mathcal{A}}(h_{\gamma_n}^n/2)|\mathfrak{J}_n| = \frac{1}{\#\mathcal{A}}(h_{\gamma_n}^n/2)|I_\gamma^n| \\ &> \frac{1}{3\#\mathcal{A}} h_{\gamma_n}^n |I_\gamma^n| = \frac{1}{3\#\mathcal{A}} \operatorname{Leb}(X_n) \\ &> \frac{1}{3\#\mathcal{A}^2}.\end{aligned}$$

To see that $\{\Xi_n\}_{n\in\mathbb{N}}$ satisfies (3) in Theorem 4.2 it is enough to notice that

$$\mu(\Xi_n \triangle T(\Xi_n)) \leqslant 2|\mathfrak{I}_\Gamma^n| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

We will now show that

$$\sup_{x\in\Xi_n} |T^{q_n}(x) - x| < D/q_n \quad \text{for some } D > 1 \tag{20}$$

thus showing (2) in Theorem 4.2 as well. The argument uses the same observation as the one used to prove (iii) in Proposition 4.3, which is the following.

Let $j \in \{0, \dots, h_{\gamma_n}^n/2\}$ and let $x \in T^j(\mathfrak{I}_\Gamma^n)$. Then $T^{q_n-j}(x) \in \mathfrak{J}_n$. However, $\mathfrak{J}_n$ is the middle level of the tower X_n. Hence we have that x and $T^{q_n}(x) = T^j(T^{q_n-j}(x))$ belong to the same level of the tower X_n. In particular, they belong to a continuity interval of T (and as such to a continuity interval of any function continuous over exchanged intervals). Moreover, we have

$$|T^{q_n}(x) - x| \leqslant |\mathfrak{J}_n| \leqslant \#\mathcal{A}|\mathfrak{I}_\Gamma^n| = \#\mathcal{A}\frac{1}{q_n} q_n |\mathfrak{I}_\Gamma^n| \leqslant \frac{\#\mathcal{A}}{q_n}.$$

Thus Conditions (ii) and (iii) in Proposition 4.3 are satisfied.

In the following proofs, we will check that the remaining assumptions in Proposition 4.3, namely, Conditions (i) and (iv), are satisfied for the different cocycles considered in each setting. In view of the construction above this is enough to apply Proposition 4.3 and conclude the ergodicity of the skew product under consideration.

Proof of Theorem 1.1. — We assume that $a > 0$ since the other case follows symmetrically. We will apply Proposition 4.3 for $m = 1$. Since $f(x) = a(x - 1/2)$ is continuous, (i) in 4.3 is satisfied. We now show that (iv) is satisfied.

First, trivially, we have

$$f'(x) = a \quad \text{for } x \in I.$$

In particular

$$S_{q_n} f'(x) = aq_n \quad \text{for } x \in I.$$

Recall that the tower $\bigsqcup_{i=0}^{h_\Gamma^n - 1} T^i(\mathfrak{I}_\Gamma^n)$ has $\mathfrak{c}_n$ as its central point. By construction, the first visit time of $\mathfrak{c}_n$ via T^{-1} to $\mathfrak{J}_n$ is $q_n/2$ or $(q_n+1)/2$. In both cases, by Lemma 3.4,

$$S_{q_n} f(T^{-\lfloor (q_n+1)/2 \rfloor}(\mathfrak{c}_n)) = 0.$$

Since $S_{q_n} f$ is continuous in $\mathfrak{I}_\Gamma^n$ and $q_n |\mathfrak{I}_\Gamma^n| < 1$, by the mean value theorem we have that

$$|S_{q_n} f(x)| < a \quad \text{for every } x \in \mathfrak{I}_\Gamma^n.$$

If $x \in T^j(\mathfrak{I}_\Gamma^n)$ for $j \in \{1, \dots, h_{\gamma_n}^n/2\}$, then by (20) and, again, by the mean value theorem,

$$\left| S_{q_n} f(x) - S_{q_n} f(T^{-j}(x)) \right| = \left| S_j f(T^{q_n - j}(x)) - S_j f(T^{-j}(x)) \right| \leqslant aD.$$

Thus we get

$$|S_{q_n} f(x)| < a(D+1) \quad \text{for every } x \in \Xi_n.$$

Since the bound does not depend on n, (iv) is satisfied and, by Proposition 4.3, the skew product T_f is ergodic. □

Proof of Theorem 1.2. — The only real difference between the proof of this result and the proof of Theorem 1.1 is the condition on the derivative. However, since T is now uniquely ergodic, we have

$$\frac{1}{q_n} \sum_{i=0}^{q_n - 1} f_0' \circ T^i \longrightarrow 0 \text{ uniformly.}$$

Thus for any $\varepsilon > 0$ and n large enough we have

$$(a - \varepsilon) q_n \leqslant S_{q_n} f'(x) \leqslant (a + \varepsilon) q_n \quad \text{for } x \in I.$$

By taking $\varepsilon < |a/2|$, we get the desired condition on the derivative. The rest of the proof follows analogously to the proof of Theorem 1.1. □

Proof of Theorem 1.3. — We use again Proposition 4.3, this time for arbitrary $m \in \mathbb{N}$. We apply it to $T^{\times m}$ (which is ergodic by weak mixing) and to $f^{\times m}$. Condition (i) in Proposition 4.3 is easily verified and Condition (iv) is satisfied in the same way as in the proof of Theorem 1.1. As in the previous proofs, the ergodicity of T_f follows from Proposition 4.3. □

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