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A MORSE COMPLEX FOR AXIOM A FLOWS

by Antoine Meddane

Abstract. — On a smooth compact Riemannian manifold without boundary, we construct a finite dimensional cohomological complex of currents that are invariant by an Axiom A flow satisfying the strong transversality assumption. The cohomology of that complex is isomorphic to the de Rham cohomology via certain spectral projectors. This construction is achieved by defining anisotropic Sobolev spaces adapted to the global dynamics of Axiom A flows. In the particular case of Morse-Smale gradient flows, this complex coincides with the classical Morse complex.

Résumé (Un complexe de Morse pour les flots Axiome A). — Sur une variété riemannienne compacte, lisse et sans bord, nous construisons un complexe cohomologique de courants, de dimension finie, invariant par un flot Axiome A vérifiant l'hypothèse de forte transversalité. La cohomologie de ce complexe est isomorphe à la cohomologie de de Rham via certains projecteurs spectraux. Cette construction est réalisée en définissant des espaces de Sobolev anisotropes adaptés à la dynamique globale des flots Axiome A. Dans le cas particulier des flots de gradient de Morse-Smale, ce complexe coïncide avec le complexe de Morse classique.

Contents

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Keywords. — Hyperbolic dynamical systems, Axiom A flows, Pollicott-Ruelle resonances, Morse complex, anisotropic Sobolev spaces, microlocal analysis.

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1. Introduction

Axiom A flows are a class of dynamical systems introduced by Smale [68] to describe chaotic dynamical systems. It arises in numerous physical problems and it contains two very interesting examples: Morse gradient flows and Anosov flows. On the one hand, the first one is well-known for its link with topology: notably through the Morse inequalities [55], stated by Morse in 1920, which relate the Betti numbers of the manifold and the number of critical points of a Morse function. Given a Riemannian metric on the manifold, Smale [67] later gave another proof using dynamical arguments and ideas going back to Thom [69]. On the other hand, Anosov flows were defined first by Anosov in [1] to describe the properties satisfied by geodesic flows on negatively curved manifolds and their links with the topology of the manifold are more subtle.

From a purely dynamical point of view, Axiom A flows are interesting because, once they satisfy a dynamical assumption called the *strong transversality assumption*, they form an open subset of the set of vector fields and the flow induced by any small perturbation of the vector field X is topologically conjugated to the flow induced by X. Moreover, we have for any vector field on a compact manifold without boundary the equivalence:

$$\text{Axiom A} + \text{strong transversality assumption} \iff \mathcal{C}^1\text{-structurally stable},$$

which motivated for a long time the study of Axiom A flows. This equivalence is often referred to as the $\mathcal{C}^1$-structural stability conjecture and it was solved by Robinson [60] for the first implication and by Hu [48] (in dimension 3) and Hayashi [45], Wen [72] (in all dimensions) for the converse statement. The proof of the $\mathcal{C}^1$-structural stability conjecture for diffeomorphisms was previously obtained by Robbin [59] for the first implication and by Mañe [54] for the converse statement. A good review on structural stability conjectures can be found in the book of Wen [73]. Up to now, a proof of the $\mathcal{C}^r$-stability conjecture for $r > 1$ is out of reach due to regularity needed in the closing lemma and in the Franks lemma. There exist other transversality assumptions, such as the Kupka-Smale transversality, which was originally introduced (in the case of Axiom A flows) by Smale under the terminology "Axiom B' ". The Kupka-Smale transversality is weaker than the strong transversality since its definition only deals with the periodic orbits of an Axiom A flow. It also has the nice property to be generic, as it was proved by Palis-de Melo [56] on surfaces and by Peixoto [57] in the general case. Using the contributions of many authors for the proof of the stability conjectures, Aoki [2] (for Axiom A diffeomorphisms) and Gan [39] (for Axiom A flows) proved that the set of Axiom A flows satisfying the strong transversality assumption is exactly the interior of the set of Kupka-Smale flows(1). If we denote by $\mathrm{KS}(M)$ the set of Kupka-Smale flows on M, by $\mathrm{ST}(M)$ the set of Axiom A flows satisfying the strong transversality assumption and by $\mathrm{SS}(M)$ the set of $\mathcal{C}^1$-structurally stable

(1) These are the flows with a countable set of periodic orbits and with a finite set of fixed points which are all hyperbolic and which satisfy the Kupka-Smale transversality assumption.

flows, then the previous discussion can be summarized by the equality

$$\mathrm{SS}(M) = \mathrm{ST}(M) = \mathrm{Int}\,\mathrm{KS}(M).$$

Equivalently, an Axiom A flow satisfying the Kupka-Smale transversality assumption is structurally stable if and only if it lies in the interior of $\mathrm{KS}(M)$.

In another direction, the concept of currents(2) turns out to be very useful in the study of gradient flows. More precisely, Laudenbach [51] and Harvey-Lawson [44] gave a new interpretation of Morse homology in terms of currents by proving the following statement. Let us consider a smooth compact Riemannian manifold (M,g) of dimension n and a smooth Morse function f. If x denotes a critical point of index $k\in[\![1,n]\!]$, then the stable manifold $W^{\mathrm{s}}(x)$ for the flow induced by $\nabla_g f$ is an embedded submanifold of dimension k and we have (in the sense of currents)

$$\partial[W^{\mathrm{s}}(x)] = [\partial W^{\mathrm{s}}(x)] = \sum_{\substack{\mathrm{ind}\,y=\mathrm{ind}\,x-1\\ y\ \text{critical point}}} n(x,y)[W^{\mathrm{s}}(y)] \tag{1}$$

for some $n(x,y)\in\mathbb{Z}$ often called the *instanton numbers*. For every $\ell\in[\![0,n]\!]$, the space $\mathcal{D}'^{,n-\ell}(M)$ of currents of degree $n-\ell$ is defined as the topological dual of the space of differential ℓ-forms $\Omega^\ell(M)$, i.e., the space of smooth sections $\Gamma(M;\Lambda^\ell T^*M)$. An equivalent formulation for equation (1) is:

$$\forall\omega\in\Omega^{k-1}(M),\quad \int_{W^{\mathrm{s}}(x)} d\omega = \sum_{\substack{\mathrm{ind}\,y=\mathrm{ind}\,x-1\\ y\ \text{critical point}}} n(x,y)\int_{W^{\mathrm{s}}(y)}\omega.$$

This relation is often presented in the following algebraic form. Consider the differential on the complex of critical points defined by

$$\partial x = \sum_{\substack{\mathrm{ind}\,y=\mathrm{ind}\,x-1\\ y\ \text{critical point}}} n(x,y)\cdot y \quad \text{for every } x\in C^k(f)=\{\text{critical points of index } k\},$$

with the same numbers $n(x,y)$ as before. The cohomological complex $(C^*(f),\partial)$ is referred to as the Morse complex and is in fact quasi-isomorphic to the de Rham complex, in the sense that the cohomology groups are the same. A remarkable feature of (1) is that it gives a representation of this algebraic complex in terms of currents that are invariant by the gradient flow. Morse inequalities have been generalized to more general dynamical systems as one can witness in the book of Franks [36]. Yet, to the best of our knowledge, the previous algebraic procedure does not extend to Axiom A flows and there is no analogue of its analytical version as constructed by Laudenbach. In this direction, we can mention the article of Ruelle and Sullivan [63] in which they constructed similar closed invariant currents for Axiom A diffeomorphisms. Nevertheless, their construction was only local (near a basic set) and was not enough to recover the whole de Rham cohomology. More recently, Dang and Rivière showed how to use the theory of Ruelle resonances [15] to define a natural cohomological complex

(2)In coordinates, differential forms with value in the set of distributions. We refer to the book of Schwartz [65] and the lecture notes of Laudenbach [52] for a comprehensive introduction.

of currents associated with Morse-Smale and Anosov flows which are two particular examples of Axiom A flows. Precisely, for a Morse-Smale gradient vector field $V = \nabla_g f$, they interpreted the Morse complex $(C^*(f), \partial)$ as the complex $(C^*(V), d)$ where

$$C^k(V) = \mathrm{Ran}\bigg(\pi_0^{(k)} : \omega \in \Omega^k(M;\mathbb{C}) \mapsto \frac{1}{2i\pi}\int_{\gamma_0} (\mathcal{L}_V^{(k)} + z)^{-1}(\omega)dz \in \mathcal{D}'^{,k}(M;\mathbb{C})\bigg). \tag{2}$$

Here, the right term is the set of generalized eigencurrents associated with the Ruelle resonance 0 of the Lie derivative of $V = \nabla_g f$ which is usually defined by

$$\mathcal{L}_V^{(k)} : u \in \Omega^k(M;\mathbb{C}) \longmapsto \frac{d}{dt}\Big|_{t=0} \left(\varphi^t\right)^* u \in \Omega^k(M;\mathbb{C}).$$

Moreover, γ_0 denotes a sufficiently small closed curve surrounding 0 and d is the exterior derivatives acting on currents. In order to make sense of this linear map $\pi_0^{(k)}$ for every $k \in [\![0,n]\!]$, they proved in [15, Prop. 4.2] the meromorphic continuation of the resolvent

$$z \longmapsto \int_M \left(\mathcal{L}_V^{(k)} + z\right)^{-1}(\phi)\wedge\psi = \int_M \int_0^{+\infty} e^{-tz}(\varphi^{-t})^*(\phi)dt \wedge \psi$$

for all $(\phi,\psi) \in \Omega^k(M;\mathbb{C}) \times \Omega^{n-k}(M;\mathbb{C})$ and for all $k \in [\![0,n]\!]$. Moreover, they verified that their Morse complex coincides with the complex of currents defined by Laudenbach in [51] by

$$C^k(\nabla_g f) = \mathrm{Vect}_{\mathbb{C}}\big\{[W^{\mathrm{u}}(x)] \in \mathcal{D}'^{,k}(M),\ \nabla_g f(x) = 0,\ \mathrm{ind}(x) = k\big\}. \tag{3}$$

This complex is in fact quasi-isomorphic to the de Rham complex, i.e., the cohomology induced by the Morse complex is isomorphic to the cohomology of the de Rham complex. In this article, we associate a natural cohomological complex to every Axiom A flow satisfying the strong transversality assumption using similar ideas. Namely, we first prove:

Theorem 1. — *Let V be an Axiom A vector field which satisfies the strong transversality assumptions* (11). *There exists $C_0 > 0$ such that, for every $k \in [\![0,n]\!]$, the resolvent operator*

$$z \longmapsto (\mathcal{L}_V^{(k)} + z)^{-1} : \Omega^k(M;\mathbb{C}) \longrightarrow \mathcal{D}'^{,k}(M;\mathbb{C})$$

is holomorphic on $\mathrm{Re}(z) \geqslant C_0$ and continues meromorphically to the whole complex plane $\mathbb{C}$.

We call *resonances* the poles of the resolvent operators $(\mathcal{L}_V^{(k)} + z)^{-1}$ viewed as a meromorphic function on $\mathbb{C}$. In order to prove that $(\mathcal{L}_V^{(k)} + z)^{-1}$ admits a meromorphic extension, one needs to find good Hilbert spaces $\mathcal{H}_k$, called anisotropic Sobolev spaces, on which the Lie derivative operators $-\mathcal{L}_V^{(k)}$ have discrete spectrum of resonances. Moreover, due the anisotropic order of the Sobolev spaces, the generalized eigencurrents in such spaces are very regular along the unstable manifolds and are very irregular along the stable manifolds. Although the anisotropic Sobolev spaces strongly depend on the dynamics of the flow, the spectrum of $-\mathcal{L}_V^{(k)}$ is somehow intrinsic and does not depend on such spaces (as claimed in Theorem 1). This spectral approach has a long history that we shall briefly recall after the statement of the main theorems. Furthermore, using *analytic Fredholm theory* [26, App. C], we will extend the resolvent operator on the half planes $\mathrm{Re}(z) \geqslant -C_k$ (for arbitrarily large C_k) as a meromorphic family of Fredholm operators. In particular, we will find that the residue of $(\mathcal{L}_V^{(k)} + z)^{-1}$ at $z = 0$ is a finite rank projector whose explicit expression is given by

$$\pi_0^{(k)} := \mathrm{Res}_{z=0}(-\mathcal{L}_V^{(k)}) = \frac{1}{2i\pi} \int_{\gamma_0} (\mathcal{L}_V^{(k)} + z)^{-1} dz : \Omega^k(M;\mathbb{C}) \longrightarrow \mathcal{D}'^{,k}(M;\mathbb{C}).$$

Here, γ_0 denotes a positively oriented closed curve which surrounds the resonance 0 and no other resonances. The quasi-isomorphism with the de Rham complex is then given by the maps $\pi_0^{(k)}$. Precisely, we prove:

Theorem 2. — *Let V be an Axiom A vector field which satisfies the strong transversality assumption* (11). *The complex $(C^*(V), d)$ defined in* (2) *is finite dimensional, is quasi-isomorphic to the de Rham complex $(\Omega^*(M;\mathbb{C}), d)$ and the quasi-isomorphism is given by the complex morphism $\pi_0^{(*)}$.*

Moreover, for every $k \in [\![0, n]\!]$, there exist a Hilbert space $\Omega^k(M;\mathbb{C}) \subset \mathcal{H}_k \subset \mathcal{D}'^{,k}(M;\mathbb{C})$ (with continuous injection) and a positive integer $m_k(0)$ such that

$$C^k(V) = \mathrm{Ker}\Big((-\mathcal{L}_V^{(k)})^{m_k(0)}\big|_{\mathcal{H}_k}\Big).$$

In addition, the complex $(C^*(V), d)$ of Theorem 2 generalizes the usual Morse complex, in the sense that (3) is satisfied, when V is reduced to a Morse-Smale gradient flow $\nabla_g f$. Theorem 2 is a consequence of Theorem 1 which relies on the construction of the Hilbert space $\mathcal{H}_k$ presented earlier and which is the main technical issue of this article. This last theorem shows the existence of a finite dimensional complex representing the de Rham cohomology and generated by dynamical currents that are almost invariant by the Axiom A flow, in the sense there may be Jordan blocks (the constant $m_k(0)$ may be greater than 1).

The Hilbert spaces $\mathcal{H}_k$ of the statement are anisotropic Sobolev spaces adapted to the spectral analysis of transfer operators,

$$L^t(u) = u \circ \varphi^{-t} \qquad \forall u \in \mathcal{C}^\infty(M),$$

as it was initiated by Ruelle [62, 61]. Such an operator extends to the space of differential forms by setting

$$L^t_{(k)}(u) := (\varphi^{-t})^*(u), \qquad \forall u \in \Omega^k(M)$$

and is related with the Lie derivative operator by the formula $\frac{d}{dt} L^t_{(k)} = \mathcal{L}_V^{(k)} L^t_{(k)}$ for every $t \in \mathbb{R}$.

1.1. Earlier results on meromorphic continuations for Axiom A flows

These functional constructions were originally made by Ruelle using Markov partitions in view of studying the mixing properties of these dynamical systems and the analytical properties of the zeta functions associated with their periodic orbits. Similar results have been obtained by Bowen, Fried, Rugh, Dolgopyat and others [7], [38], [64], [20] also relying on symbolic dynamics.

Building on some earlier work [5] with Blank and Keller for Anosov diffeomorphisms, Liverani introduced in [53] Banach spaces of distributions with anisotropic Hölder regularity adapted to contact Anosov flows. Among other things, these spaces enabled him to prove the meromorphic continuation of the resolvent in some half plane slightly beyond $\mathrm{Re}(z)=0$. This approach was further developed in subsequent works with Butterley [9] and Giuletti-Pollicott [40] to show the meromorphic continuation of $(\mathcal{L}_V^{(k)}+z)^{-1}$ to the whole complex plane for every $k\in[\![0,n]\!]$ and for any smooth Anosov flow. We also refer to [41, 42] for earlier results with Gouëzel in that spirit for Axiom A diffeomorphisms, where the resolvent acts on compactly supported functions near a compact locally maximal hyperbolic set. We can also mention [4] for a different approach (still for diffeomorphisms and still locally near a hyperbolic set called basic) by Baladi and Tsujii using anisotropic Sobolev spaces and methods from Fourier/microlocal analysis.

From another perspective, the general theory of semiclassical resonances [26], [46] was used to derive alternative approaches to construct Hilbert spaces adapted to the dynamics. First, for smooth Anosov diffeomorphisms, Faure, Roy and Sjöstrand recovered in [29] the existence of a discrete spectrum for the transfer operator. Then for general Anosov flows, Faure and Sjöstrand constructed in [30] Hilbert spaces, referred to as *anisotropic Sobolev spaces*, on which the Lie derivative $\mathcal{L}_V^{(0)}$ has discrete spectrum on a large part of the complex plane. Their analysis used the machinery of microlocal analysis as a toolbox and it reduced in some sense the problem to a dynamical question, i.e., constructing an escape (or Lyapunov) function adapted to the dynamics on the cotangent space. In [30], the meromorphic extension of the resolvent was obtained for $k=0$ and this result was extended to every $k\in[\![0,n]\!]$ in [24] by Dyatlov and Zworski in view of applications to the Ruelle zeta function. More information on the spectrum (e.g. band structure) has also been obtained by Faure and Tsujii [70, 71, 31, 32, 33, 34] in the context of Anosov flows and contact Anosov flows using these kinds of methods.

Subsequently, the meromorphic extension of the resolvent has been extended to Axiom A flows by Dyatlov-Guillarmou [22, 23] under the assumption that the resolvent acts on differential forms supported near a fixed basic set. Although this analysis was enough to prove the meromorphic continuation of the Ruelle zeta function for Axiom A flows, it does not seem to be sufficient to deduce Theorem 1 (and thus Theorem 2) which does not require any support restriction. In that direction, Dang and Rivière [15, 16, 17, 19] proved meromorphic continuation for globally supported test forms in the case of Morse-Smale gradient flows, a generic subset of Morse gradient flows which satisfy the strong transversality assumption. In this series of articles, Dang and Rivière gave a complete description of Pollicott-Ruelle resonances, giving a band structure for the spectrum, computed the dimensions of eigenspaces by making explicit the eigenvectors in terms of de Rham's currents and gave a new proof of Morse-Smale inequalities. In particular, the link with the topology was made possible through their *global* construction of Sobolev spaces adapted to the dynamics of Morse-Smale flows. In this article, we gather the approaches of Dyatlov-Guillarmou and Dang-Rivière so that we obtain the meromorphic continuation of the resolvent acting on globally supported forms for general Axiom A flows. Finally, we emphasize that, besides its applications to topology, Theorem 1 also answers a question raised by Baladi in [3, p. 152] for hyperbolic diffeomorphisms.

1.2. Back to topology. — Recently, the developments of these analytical tools led to much progress on the link between the topology of the manifold and the spectrum of the Lie derivative, at least for the examples where the functional setup was globally defined, namely Morse-Smale and Anosov flows. We recall here some of these advances.

– *Contact Anosov flows in dimension* 3. In that geometric framework, Dyatlov and Zworski [25] computed the dimension of $\mathrm{Ker}(\mathcal{L}_V^{(k)})^{m_k(0)}$ for every $k \in [\![0, 3]\!]$ and expressed it in terms of the Betti numbers of the manifold. They used this to generalize earlier results of Fried [37] on the order of vanishing of the Ruelle zeta function. In particular, their computation holds true for any geodesic flow acting on the unitary cotangent bundle $S^*\Sigma =: M$ of a compact negatively curved surface Σ. Borns-Weil and Shen [6] extended [25] to the non-orientable case and Hadfield [43] showed a similar result for compact negatively curved surfaces with boundaries.

– *Anosov flows in high dimension.* Küster-Weich [50] computed the dimension of $\mathrm{Ker}(\mathcal{L}_V^{(1)})^{m_1(0)}$ in terms of the first Betti number for hyperbolic manifolds of dimension $\neq 3$. Their result also holds for perturbations of hyperbolic metrics.

– *Perturbation of Anosov flows.* Cekić-Paternain [10] gave the first examples of Anosov flows in dimension 3 which preserve a volume form where the vanishing order of the Ruelle zeta function jumps under perturbation of the flow. Again this was achieved by computing explicitly the dimension of the spaces appearing in the cohomological complex of Theorem 2. In dimension 5, Cekić-Dyatlov-Küster-Paternain [11] found a similar result for geodesic flows on nearly hyperbolic 3-manifold (the unitary cotangent bundle is 5-dimensional).

– *Fried's conjecture* [37, 38]. Dang-Guillarmou-Rivière-Shen [14] established, in the case of Anosov flows, a criterion in terms of the spaces appearing in Theorem 2 to ensure that the value at 0 of the twisted Ruelle zeta function is locally constant. It allowed them to prove Fried's conjecture on the Reidemeister torsion for nearly hyperbolic 3-manifolds. This was further pursued by Chaubet and Dang [12] who used the cohomological complex of Theorem 2 to define a dynamical torsion for contact Anosov flows in any dimension.

– *Morse-Smale flows.* Dang-Rivière [19] proved Theorem 2 in the case of Morse-Smale and Anosov flows. In the specific case of Morse-Smale gradient flows [18], they also considered the Lie derivative operator as a limit of the Witten Laplacian and they obtained the Ruelle spectrum as a limit of the Witten spectrum. It enabled them to recover the Witten-Helffer-Sjöstrand instanton formula and to prove the Fukaya conjecture on Witten deformation of the wedge product.

1.3. Outline of the proof. — We use the microlocal approach to Pollicott-Ruelle resonances of the Lie derivative operator $\mathcal{L}_V$ as it was developed by Faure and Sjöstrand. Recall that the proof of Theorem 1 relies on the construction of Hilbert spaces adapted to the dynamics. Following [30], defining such spaces can be reduced through some microlocal procedure to the construction of an escape function. More precisely, one has to exhibit a family of functions that are decreasing along the Hamiltonian flow of $H(x,\xi) = \xi(V(x))$ on the cotangent bundle T^*M of M. The existence of such decreasing functions, called energy or Lyapunov functions, is already known for the flow on M as soon as V is an Axiom A flow satisfying the strong transversality assumption. We can cite for example the articles of Conley [13], Wilson [74] for flows and Pugh-Shub [58] for Axiom A diffeomorphisms satisfying Smale's transversality assumptions. One of the main novelty of this article is to do the same for the induced Hamiltonian flow on T^*M. It was already done by Faure-Sjöstrand [30] for Anosov flows, by Dyatlov-Guillarmou [22] near a basic set and by Dang-Rivière in [15, 16] for Morse-Smale flows. To construct a decreasing function along this Hamiltonian flow, Dang and Rivière highlighted in the case of gradient flows [15] that one needs to prove the compactness of the conormal distribution

$$\bigcup_{x\in M} \{\xi \in S^*_x M \;:\; \xi(T_x W^{\mathrm{u}}(x_-)) = 0, \text{ for } x_- \text{ the critical point s.t } x \in W^{\mathrm{u}}(x_-)\},$$

where $W^{\mathrm{u}}(x_-)$ denotes the unstable manifold of the critical point x_-. Nevertheless, to do so, they made a restriction on the class of Morse-Smale flows, namely, the existence of $\mathcal{C}^1$-*linearization charts* near critical points. Such a restriction is not available for more general Axiom A flows and we need to proceed differently. In particular, we note that our proof allows to remove this linearization assumption in the specific case of Morse-Smale flows. To prove a similar result for Axiom A flows, we proceed in three steps.

– We recall the definition of the strong transversality assumption for Axiom A flows which generalizes the one used for Morse-Smale gradient flows.

– Then, we generalize the compactness result for conormal distributions without using $\mathcal{C}^1$-linearizing charts. This step will require an analysis similar to the local analysis near basic sets performed by Dyatlov-Guillarmou [22].

– We deduce the existence of a *global* escape functions for Axiom A flows which satisfies the strong transversality assumption by adapting the construction of Faure-Sjöstrand [30].

Concerning the proof of Theorem 2, we recall that that there is a strong analogy with the Hodge-de Rham Laplace operator(3) $\Delta = d\circ d^* + d^*\circ d = (d+d^*)^2$ acting on differential forms $\Omega^*(M)$ if we remark that

$$\mathcal{L}_V = d\circ\iota_V + \iota_V\circ d = (d+\iota_V)^2.$$

(3)The derivative d^* denotes the formal adjoint of d in $L^2(M;\Lambda^k T^*M)$.

Note also that both operators Δ and $\mathcal{L}_V$ commute with the exterior derivative d. These analogies are at the heart of the proof of Theorem 2.

1.4. Organization of the article

– In Section 2, we recall the definition of an Axiom A flow and introduce the dynamical tools we will need. Furthermore, we present in this part a few key notions for our analysis which turn out to be related: Smale's order relation on basic sets, strong transversality assumption, filtrations (with open sets) and unrevisited neighborhoods. We also explain how to bypass the $\mathcal{C}^1$-linearizing charts used in Dang-Rivière's articles.

– In Section 3, we present a possible construction of an escape function and we state a generalization of the compactness result for conormal distributions which takes into account the neutral direction given by the flow direction. The results stated in this part were in fact the most challenging ones to prove.

– In Section 4, we define anisotropic Sobolev spaces, in which the Lie derivative operators $\mathcal{L}_V^{(k)}$ have discrete spectrum (see Theorem 5 from which Theorem 1 derives).

– In Section 5, we recall how the methods from [15, 19] can be adapted to deduce Theorem 2 from Theorem 1.

– In Section 6, 7 and 8 we give the proof of the dynamical results such as the construction of energy functions for Axiom A flows, the proof of the compactness of conormal distributions and the construction of the global escape functions.

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2. Dynamical preliminaries.

Throughout this paper, we denote by (M^n, g) a smooth compact Riemannian manifold without boundary of dimension $n \geqslant 1$ together with some smooth Riemannian metric g. We also denote by d_g the geodesic distance associated to the metric g and by $|.|_g = \sqrt{g(\,.\,,.\,)}$ the norm induced on the fibers of the tangent bundle TM or on the cotangent bundle T^*M. To a smooth vector field $V \in \Gamma(TM)$, we can associate a flow $(\varphi^t)_{t\in\mathbb{R}}$ which solves the Cauchy problem:

$$\forall x \in M,\ \forall t \in \mathbb{R}, \quad \begin{cases} \dfrac{d}{dt}\varphi^t(x) = V(\varphi^t(x)), \\ \varphi^0(x) = x. \end{cases} \tag{4}$$

The system (4) is highly non-linear in general, which makes difficult to predict the large-time behavior of trajectories, especially in the case of hyperbolic dynamics.

Definition 2.1 ([68, p. 796]). — A point $x \in M$ is said to be *non-wandering* if for every neighborhood $\mathcal{U}$ of x and every $T > 0$ there exists $t \in \mathbb{R}$ such that $|t| \geqslant T$ and

$\varphi^t(\mathcal{U}) \cap \mathcal{U} \neq \varnothing$. The non-wandering points form a closed invariant subset of M, called the *non-wandering set*, that we will denote by $\Omega := \Omega(\varphi^t)$.

We refer to Appendix A or to the books [56], [49], [35] for a definition of hyperbolic set.

Definition 2.2 (Axiom A flow, [68, p. 803], [35, Def. 5.3.29]). — A flow $\varphi^t : M \to M$ is said to be *Axiom A* if its non-wandering set Ω is hyperbolic and can be written as the union of the fixed points and of the closure of its periodic orbits with positive period, i.e.,

$$\Omega = \mathcal{F} \sqcup \overline{\mathrm{Per}(\varphi^t)}.$$

From the definition, one can remark that an Axiom A flow on a compact manifold must have a finite number of fixed points (since they all are hyperbolic) which are isolated in Ω. It is known from the works of Smale and Bowen that an Axiom A flow has a non-wandering set which splits into a finite number of hyperbolic invariant compact sets called basic sets:

Proposition 2.3 (Spectral decomposition, [68, §II.5], [7], [35, Th. 5.3.37])

If φ^t is an Axiom A flow, then its non-wandering set Ω decomposes into a finite union of basic *sets K_i:*

$$\Omega = K_1 \sqcup K_2 \sqcup \cdots \sqcup K_N, \tag{5}$$

where K basic means:

– *K is compact and hyperbolic;*
– *K is locally maximal: there exists some open set $O \subseteq M$ such that*

$$K = \bigcap_{t\in\mathbb{R}} \varphi^t(O);$$

– *K is topologically transitive, i.e., there exists a point $x \in K$ such that $\overline{(\varphi^t(x))_{t\in\mathbb{R}_+}} = K$.*

From now on, φ^t will denote an Axiom A flow on (M, g). We call *attractor* for φ^t a basic set K which satisfies

$$K = \bigcap_{t\in\mathbb{R}_+} \varphi^t(O),$$

for some open set $O \supset K$. Similarly, we call *repeller* for φ^t a basic set K which satisfies

$$K = \bigcap_{t\in\mathbb{R}_-} \varphi^t(O),$$

for some open set $O \supset K$.

Remark 2.4

– The basic sets K_i of the decomposition (5) are the maximal (for the inclusion) locally maximal, φ^t-invariant, compact, hyperbolic sets which are topologically transitive. A basic set is equal to the closure of its periodic orbits which can possibly be reduced to a fixed point, i.e., have period 0.

– If $N = 1$, then $M = \Omega(\varphi^t) = K_1$. This result is a consequence of the local maximality of basic sets. It can be proved using Proposition B.3 (which uses the definition of a stable/unstable set of a basic set introduced in the next paragraph) together with the definition of the non-wandering set.

2.1. Stable and unstable manifolds. — We begin by recalling some well-known facts concerning uniformly hyperbolic dynamics which can be found in [49] or [21]. Fix a basic set K. For all $\varepsilon > 0$ and all $z \in K$, the *stable manifold*, *weak stable manifold*, *local stable manifold* and *local weak stable manifold* at the point z are defined by

$$\begin{aligned} W^{\mathrm{s}}(z) &:= \{x \in M : \ d_g(\varphi^t(x), \varphi^t(z)) \underset{t\to+\infty}{\longrightarrow} 0\}, \quad W^{\mathrm{so}}(z) := \bigcup_{t\in\mathbb{R}} \varphi^t\left(W^{\mathrm{s}}(z)\right), \\ W^{\mathrm{s}}_{\varepsilon}(z) &:= \{x \in W^{\mathrm{s}}(z) : \ d_g(\varphi^t(x), \varphi^t(z)) < \varepsilon, \ \forall t \in \mathbb{R}_+\}, \\ W^{\mathrm{so}}_{\varepsilon}(z) &:= \{x \in M : \ d_g(\varphi^t(x), \varphi^t(z)) < \varepsilon, \ \forall t \in \mathbb{R}_+\}. \end{aligned}$$

By replacing s by u and φ^t by φ^{-t} in the previous equalities, we could have defined similarly the $\varnothing$/weak/local/local weak unstable manifolds. From this remark, let us only deal with stable manifolds by keeping in mind that everything can be adapted for unstable manifolds. Thanks to the Hadamard-Perron theorem, also called stable manifold theorem, there exists $\varepsilon_0 \ll 1$ such that, for all $z \in K$, the sets $W^{\mathrm{s/so}}_{\varepsilon_0}(z)$ are smooth submanifolds of M of dimension $d_{\mathrm{s/so}}$, which is constant on each basic set. Precise statements and proof of this result can be found in [49, Th. 6.4.9, p. 267] for the case of diffeomorphisms and in [21, Th. 5, p. 34] for the case of flows. In general, stable manifolds are not embedded submanifolds but only immersed submanifolds, except in the case of Morse flows. Moreover, the stable manifold is related to the local stable manifold thanks to the following formula [21, p. 24]:

$$W^{\mathrm{s}}(z) = \bigcup_{n\geqslant 0} \varphi^{-n}\left(W^{\mathrm{s}}_{\varepsilon_0}(\varphi^n(z))\right), \quad (n \in \mathbb{N}),$$

which does not depend on ε_0 given by the stable manifold theorem. If K denotes a basic set, then we define its *stable set* by

$$W^{\mathrm{s}}(K) := \{x \in M : \ d(\varphi^t(x), K) \underset{t\to+\infty}{\longrightarrow} 0\}.$$

Thanks to the shadowing lemma [49, Th. 18.1.6 p. 569] and to the local maximality of basic sets, this last set decomposes into the stable manifolds of elements of K, namely

$$W^{\mathrm{s}}(K) = \bigcup_{z\in K} W^{\mathrm{s}}(z).$$

A proof can be found in [35, Th. 5.3.25] for Axiom A flows and in [8, Prop. 3.10, p. 53], [72, Th. 6.26, p. 131] for Axiom A diffeomorphisms. For every $\varepsilon > 0$, we can define the local stable set of K by setting

$$W^{\mathrm{s}}_{\varepsilon}(K) = \bigcup_{x\in K} W^{\mathrm{s}}_{\varepsilon}(x). \tag{6}$$

Now, let us present a lemma which was originally given by Smale.

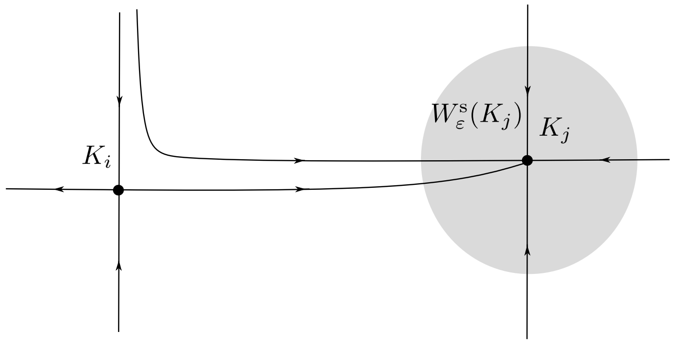


Figure 1. Illustration of Lemma 2.6. Note that the closure of the global stable set of K_j contains the global stable set of K_i.

Lemma 2.5 (Partition by stable manifolds, [68, Cor. II.5.3]). — *We have the following decomposition of M in stable sets:*

$$M = \bigsqcup_{i=1}^{N} W^{\mathrm{s}}(K_i). \tag{7}$$

Lemma 2.5 dictates the behavior of the trajectories outside the non-wandering set. Precisely, if we take an element $x \in M$, then there exists a unique pair $(i,j) \in [\![1,N]\!]^2$ such that $x \in W^{\mathrm{u}}(K_i) \cap W^{\mathrm{s}}(K_j)$ and the decomposition (7) provides elements $x_- \in K_i$ and $x_+ \in K_j$ such that $x \in W^{\mathrm{u}}(x_-) \cap W^{\mathrm{s}}(x_+)$. The point x_+ is unique modulo the equivalence relation on K_j given by:

$$z_1 \sim_j z_2 \iff z_1, z_2 \in K_j \text{ and } z_1 \in W^{\mathrm{s}}(z_2).$$

A similar remark holds for x_-.

Let us mention one more lemma concerning the closure of stable sets. The following lemma is illustrated in Figure 1 and emphasizes the fact that the closure of the local stable manifold of a basic set is still contained in the stable set of this basic set.

Lemma 2.6. — *For every $\varepsilon < \varepsilon_0$, where ε_0 is the positive constant given by the stable manifold theorem for the basic set K, we have $\overline{W^{\mathrm{s}}_{\varepsilon}(K)} \subseteq W^{\mathrm{s}}_{\varepsilon_0}(K)$.*

Proof. — Thanks to the stable manifold theorem, there exist $C, \lambda > 0$ such that for every $z \in K$ and every $\varepsilon < \varepsilon_0$,

$$x \in W^{\mathrm{s}}_{\varepsilon}(z) \iff \forall t \geqslant 0,\ d_g\left(\varphi^t(x), \varphi^t(z)\right) < \min\left(Ce^{-\lambda t} d_g(x,z), \varepsilon\right). \tag{8}$$

Now, if we fix $\varepsilon > 0$ and if we consider a sequence $((x_n, z_n))_n \in M \times K$ which converges to some limit (x_∞, z_∞) such that $x_n \in W^{\mathrm{s}}_{\varepsilon}(z_n)$ for all $n \in \mathbb{N}$, then the relation (8) applied to (x_n, z_n) passes to the limit when $n \to +\infty$ and gives in particular $x_\infty \in W^{\mathrm{s}}_{\varepsilon_0}(z_\infty)$. □

2.2. Lifting the dynamics on the cotangent. — Since we will use the analysis through escape functions developed in [30, p. 329], we start by introducing the lifted flow on the cotangent bundle T^*M

$$\Phi^t(x,\xi) = (\varphi^t(x), \left(D\varphi^t(x)^{-1}\right)^\top(\xi)), \quad \forall (x,\xi) \in T^*M$$

and its restriction to the unitary cotangent bundle $S^*M = \{(x,\xi) \in T^*M,\ |\xi|_g = 1\}$

$$\widetilde{\Phi}^t = \kappa \circ \Phi^t \text{ on } S^*M, \text{ where } \kappa : T^*M \smallsetminus 0_M \longrightarrow S^*M,\ (x,\xi) \longmapsto (x, \xi/|\xi|_g). \tag{9}$$

Note that the flow Φ^t extends the flow φ^t to the cotangent bundle in the sense that $\pi \circ \Phi^t = \varphi^t \circ \pi$, where π denotes the projection from T^*M to M identified with the zero section, and that it is linear on each fiber. Moreover, the flow Φ^t sends $T^*M \smallsetminus 0_M$ into $T^*M \smallsetminus 0_M$ because the linear map $D\varphi^t(x)$ is invertible. Therefore, the flow $\widetilde{\Phi}^t$ is well-defined on S^*M and is generated by a smooth vector field which will be denoted by $\widetilde{X}_H$. To summarize, we have the following commutative diagrams:

$$\begin{array}{ccc} T^*M & \xrightarrow{\ \Phi^t\ } & T^*M \\ \pi\downarrow & & \downarrow\pi \\ M & \xrightarrow{\ \varphi^t\ } & M \end{array} \qquad\qquad \begin{array}{ccc} T^*M \smallsetminus 0_M & \xrightarrow{\ \Phi^t\ } & T^*M \smallsetminus 0_M \\ \kappa\downarrow & & \downarrow\kappa \\ S^*M & \xrightarrow{\ \widetilde{\Phi}^t\ } & S^*M \end{array}$$

Since our analysis will take place in T^*M, we also define the dual distributions associated with the neutral E_{o}, stable E_{s} and unstable E_{u} distributions(4) which appear in the definition of hyperbolicity (see Appendix A) at any point $z \in \Omega$ by

$$(E_{\mathrm{o}}^*(z))(E_{\mathrm{u}}(z) \oplus E_{\mathrm{s}}(z)) = 0,$$
$$(E_{\mathrm{u}}^*(z))(E_{\mathrm{u}}(z) \oplus E_{\mathrm{o}}(z)) = 0, \qquad (E_{\mathrm{s}}^*(z))(E_{\mathrm{s}}(z) \oplus E_{\mathrm{o}}(z)) = 0.$$

This gives us a hyperbolic splitting of the cotangent bundle:

$$T_z^*M = E_{\mathrm{s}}^*(z) \oplus E_{\mathrm{u}}^*(z) \oplus E_{\mathrm{o}}^*(z).$$

2.3. Extension of the invariant distributions outside the non-wandering set

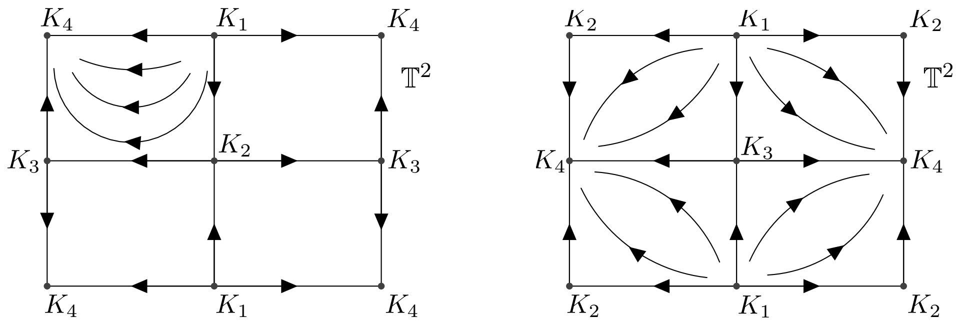
Thanks to the partition's lemma 7 by stable manifolds, we can extend the previous definitions outside the non-wandering set.

Definition 2.7. — For every $x \in W^{\mathrm{s}}(x_+) \cap W^{\mathrm{u}}(x_-)$ with $x_- \in K_i$ and $x_+ \in K_j$, we define the spaces $E_{\mathrm{s}}^*(x)$, $E_{\mathrm{so}}^*(x)$, $E_{\mathrm{u}}^*(x)$, $E_{\mathrm{uo}}^*(x)$ to be the largest spaces satisfying:

$$E_{\mathrm{s}}^*(x)(T_xW^{\mathrm{so}}(x_+)) = 0, \qquad E_{\mathrm{so}}^*(x)(T_xW^{\mathrm{s}}(x_+)) = 0,$$
$$E_{\mathrm{u}}^*(x)(T_xW^{\mathrm{uo}}(x_-)) = 0, \qquad E_{\mathrm{uo}}^*(x)(T_xW^{\mathrm{u}}(x_-)) = 0\,.$$

Moreover, the definition does not depend on the choice of x_+ and x_- in $W^{\mathrm{s}}(x_+) \cap K_j$ and $W^{\mathrm{u}}(x_-) \cap K_i$ respectively.

(4)On a basic set K, we also use the notations E_{so}^* for $E_{\mathrm{s}}^* + E_{\mathrm{o}}^*$ and E_{uo}^* for $E_{\mathrm{u}}^* + E_{\mathrm{o}}^*$.



(a) Axiom A flow which does not satisfy the (strong) transversality assumption: existence of a saddle-connection.

(b) Axiom A flow which satisfies the (strong) transversality assumption.

Figure 2. Some Axiom A flows on the 2-torus.

Remark 2.8

– If x_+ is a fixed point, then $W^{\mathrm{so}}(x_+) = W^{\mathrm{s}}(x_+)$ and consequently $E_{\mathrm{s}}^*(x) = E_{\mathrm{so}}^*(x) \subseteq \{\xi(V(x)) = 0\}$.

– The distributions $E_{\mathrm{s}}^*, E_{\mathrm{u}}^*, E_{\mathrm{so}}^*$ and E_{uo}^* are defined on the whole manifold and are Φ^t-invariant.

– Recall that fixed points are isolated (Remark 2.4). So if x_+ is not a fixed point then we get that $\dim E_{\mathrm{so}}(x_+) = \dim E_{\mathrm{s}}(x_+) + 1$ and $\dim W^{\mathrm{so}}(x_+) = \dim W^{\mathrm{s}}(x_+) + 1$. Therefore, we obtain $E_{\mathrm{s}}^*(x) = E_{\mathrm{so}}^*(x) \cap \{\xi(V(x)) = 0\}$ and $\dim E_{\mathrm{so}}^*(x) = \dim E_{\mathrm{s}}^*(x) + 1$.

– We can extend the *neutral* distribution outside the non-wandering set by fixing, for every $x \in W^{\mathrm{u}}(x_-) \cap W^{\mathrm{s}}(x_+)$,

$$E_{\mathrm{o}}^*(x) := E_{\mathrm{uo}}^*(x_-) \cap E_{\mathrm{so}}^*(x_+) = \{\xi \in T^*M,\ \xi\left(T_xW^{\mathrm{u}}(x_-) + T_xW^{\mathrm{s}}(x_+)\right) = 0\}.$$

2.4. Strong transversality assumption. — In this part, we briefly recall the definition of the strong transversality assumption presented in the introduction, which turned out to be the good transversality assumption to prove the structural $\mathcal{C}^1$-stability conjecture.

We say that φ^t satisfies the strong transversality assumption if for every $(x_-, x_+) \in K_i \times K_j$ and for every $x \in W^{\mathrm{u}}(x_-) \cap W^{\mathrm{s}}(x_+)$ we have

$$T_xW^{\mathrm{uo}}(x_-) + T_xW^{\mathrm{so}}(x_+) = T_xM. \tag{10}$$

Since both spaces $T_xW^{\mathrm{uo}}(x_-)$ and $T_xW^{\mathrm{so}}(x_+)$ contain the flow direction $\mathbb{R}\cdot V(x)$, an equivalent formulation to (10) is:

$$T_xW^{\mathrm{u}}(x_-) + T_xW^{\mathrm{so}}(x_+) = T_xW^{\mathrm{uo}}(x_-) + T_xW^{\mathrm{s}}(x_+) = T_xM. \tag{11}$$

Moreover, one can remark that the strong transversality assumption does not depend on the choice of $x_- \in K_i$ and $x_+ \in K_j$ such that $x \in W^{\mathrm{u}}(x_-) \cap W^{\mathrm{s}}(x_+)$ in (10) or (11).

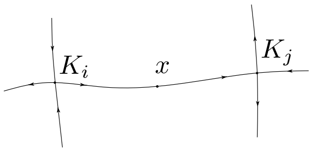


Figure 3. Order relation between two fixed points.

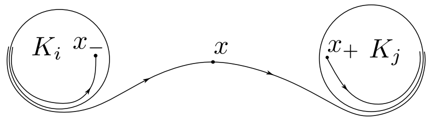


Figure 4. Order relation between two basic sets.

Remark 2.9. — If x_- or x_+ is a fixed point then the strong transversality assumption reads

$$T_x W^{\mathrm{u}}(x_-) + T_x W^{\mathrm{s}}(x_+) = T_x M.$$

This is in particular the case for Morse-Smale gradient flows [15] where both x_- and x_+ are fixed points.

From (11) and directly from the definitions, we can deduce the following *disjointness* properties:

$$E_{\mathrm{s}}^* \cap E_{\mathrm{uo}}^* = 0_M \quad \text{and} \quad E_{\mathrm{so}}^* \cap E_{\mathrm{u}}^* = 0_M,$$

where 0_M denotes the null section of T^*M.

2.5. Order relation. — When Smale [68] defined Axiom A flows, he exhibited a relation between basics sets of an Axiom A flow. Precisely, for two basic sets K_i and K_j, he defined the relation $\leqslant$ by

$$\begin{aligned} K_i \leqslant K_j &\iff W^{\mathrm{u}}(K_i) \cap W^{\mathrm{s}}(K_j) \neq \varnothing \\ &\iff \exists x \in M,\ \exists (x_-, x_+) \in K_i \times K_j,\ \ x \in W^{\mathrm{u}}(x_-) \cap W^{\mathrm{s}}(x_+). \end{aligned} \tag{12}$$

This relation is illustrated in Figures 3 and 4 and gives a graph structure on the family of basic sets, where the edges of the graph are the basic sets and an arrow between two basic sets (from K_i to K_j) is given by the relation $K_i \leqslant K_j$. When φ^t satisfies the strong transversality assumption, then the next theorem (originally due to Smale) states that the relation $\leqslant$ is a partial order relation. Since we were not able to find out an explicit proof in the literature and for the sake of completeness, we present its proof in Appendix B.

Theorem 3 (Smale, [68, Prop. 8.5, p. 784]). — *If φ^t is an Axiom A flow, then we have for every basic set K*

$$W^{\mathrm{u}}(K) \cap W^{\mathrm{s}}(K) = K.$$

Moreover, if φ^t satisfies the strong transversality assumption (11), *then the relation $\leqslant$ defines a partial order relation. An equivalent definition of the relation $\leqslant$ is then given by*

$$K_i \leqslant K_j \iff W^{\mathrm{s}}(K_i) \subseteq \overline{W^{\mathrm{s}}(K_j)} \iff W^{\mathrm{u}}(K_j) \subseteq \overline{W^{\mathrm{u}}(K_i)}, \tag{13}$$

and we have

$$\overline{W^{\mathrm{u}}(K_i)} = \bigcup_{j, K_j \geqslant K_i} W^{\mathrm{u}}(K_j) \ \ \textit{and}\ \ \overline{W^{\mathrm{s}}(K_j)} = \bigcup_{i, K_i \leqslant K_i} W^{\mathrm{s}}(K_i). \tag{14}$$

Figure 5. Both graph correspond to the Axiom A flow on the 2-torus of Figure 2(b). A path in the oriented graph corresponds to a connecting orbit between the basic sets. The left one has its indices compatible with the graph structure contrary to the right one.

As we can see in Figure 2(a), relation (12) may not be an order relation when the strong transversality assumption does not hold. Another consequence of this theorem is that strong transversality implies the *no-cycle property*: the induced graph has no cycle. This fact is necessary to ensure the existence of non-constant Lyapunov functions for the flow φ^t.

2.5.1. *Total order relation*. — In order to use mathematical induction, we need to consider a total order on the family of basic sets. It can be achieved as soon as the Axiom A flow φ^t satisfies the no-cycle property. Therefore, if we assume the no-cycle property, then we define a total order relation from Smale's order relation $\leqslant$ on the basic sets as an order relation on $[\![1, N]\!]$ compatible with the partial order relation $\leqslant$ in the sense that

$$K_i \leqslant K_j \implies i \leqslant j. \tag{15}$$

From now on, we assume the no-cycle property and we fix a total order relation. In Appendix B, we prove that the set $\bigcup_{j\geqslant i} W^{\mathrm{u}}(K_j)$ is compact for every $i \in [\![1, N]\!]$. Be aware that $\overline{W^{\mathrm{u}}(K_i)}$ and $\bigcup_{j\geqslant i} W^{\mathrm{u}}(K_j)$ are not equal in general, see for example Figure 5.

2.6. Filtrations and unrevisited neighborhoods

An important concept throughout our analysis is the concept of filtration. Even though the term of filtration usually refers to an increasing sequence of subcomplexes of a simplicial complex, we give here an open version deeply related to Morse homology where the subcomplexes are open sets, i.e., submanifolds of dimension $n = \dim M$ without boundary. To see the analogy, let us consider a Morse function $f : M \to \mathbb{R}$ which has N critical points $x_1, \ldots, x_N$ that all satisfy $f(x_i) = i$ to simplify. Then, a filtration is given by the family of open sets $(f^{-1}(]-\infty, i + 1/2[))_{0\leqslant i\leqslant N}$. For a general Axiom A flow, we give a definition of a filtration which appeared in the works of Smale [68], Robbin [59, Lem. 7.9, p. 471].

Definition 2.10 (Filtration). — Let φ^t be an Axiom A flow which satisfies the no-cycle property. Let us consider a total order relation on the basic sets in the sense

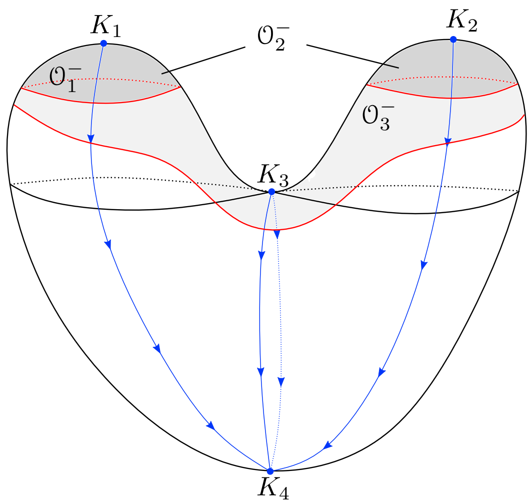


Figure 6. Example of a filtration on the sphere $\mathbb{S}^2$.

of (15). A sequence of open sets $(\mathcal{O}_i^-)_{0\leqslant i\leqslant N}$ is said to be a *filtration* for φ^{-1} if the following conditions hold:

(i) The sequence is increasing:

$$\varnothing=\mathcal{O}_0^-\subseteq\mathcal{O}_1^-\subseteq\cdots\subseteq\mathcal{O}_N^-=M.$$

(ii) For every $i\in[\![1,N]\!]$, the open sets and their closures are φ^{-1}-stable: $\varphi^{-1}\left(\mathcal{O}_i^-\right)\subseteq\mathcal{O}_i^-$, $\varphi^{-1}\left(\overline{\mathcal{O}_i^-}\right)\subseteq\overline{\mathcal{O}_i^-}$.

(iii) For every $i\in[\![1,N]\!]$, we have $K_i\subseteq\mathcal{O}_i^-\smallsetminus\overline{\mathcal{O}_{i-1}^-}$.

Remark 2.11

– Any filtration for φ^{-1} induces a filtration for φ^1 by taking the interior of the complementary of each open set, i.e., by setting $\mathcal{O}_i^+:=\operatorname{Int}\left(\left(\mathcal{O}_{N-i}^-\right)^c\right)$ for every i. Indeed, from $\varphi^{-1}\left(\mathcal{O}_{N-i}^-\right)\subseteq\mathcal{O}_{N-i}^-$, we deduce

$$\varphi^1(\mathcal{O}_i^+)=\varphi^1\left(\operatorname{Int}\left(\left(\mathcal{O}_{N-i}^-\right)^c\right)\right)\subseteq\varphi^1\left(\left(\mathcal{O}_{N-i}^-\right)^c\right)\subseteq\left(\mathcal{O}_{N-i}^-\right)^c.$$

So, $\varphi^1(\mathcal{O}_i^+)$ is an open subset of $\left(\mathcal{O}_{N-i}^-\right)^c$, and by definition of the interior we get $\varphi^1(\mathcal{O}_i^+)\subseteq\operatorname{Int}\left(\left(\mathcal{O}_{N-i}^-\right)^c\right)=\mathcal{O}_i^+$. Moreover, we still have $\varnothing=\mathcal{O}_0^+\subseteq\mathcal{O}_1^+\subseteq\cdots\subseteq\mathcal{O}_N^+=M$ and $K_i\subseteq\mathcal{O}_{N-i+1}^+\smallsetminus\overline{\mathcal{O}_{N-i}^+}$.

– $\mathcal{O}_i^-$ is a neighborhood of $\bigsqcup_{j\leqslant i}W^{\mathrm{s}}(K_i)$. Indeed, $\mathcal{O}_i^-$ contains every basic set K_j for $j\leqslant i$ and if $x\in W^{\mathrm{s}}(K_j)$ then there exists $k\in\mathbb{N}$ such that $\varphi^k(x)\in\mathcal{O}_i^-$. Thus, we deduce $x\in\varphi^{-k}(\mathcal{O}_i^-)\subset\mathcal{O}_i^-$.

– For every $i\in[\![1,N]\!]$, let us define the set

$$\mathcal{V}_i:=\mathcal{O}_i^-\cap\mathcal{O}_{N-i+1}^+.$$

One can check that $\mathcal{V}_i$ is a neighborhood of K_i which satisfies $\overline{\mathcal{V}_i} \cap \Omega = K_i$ and the following property: for all $m \in \mathbb{N}$ and for all $x \in \mathcal{V}_i$, if we have $x \in \mathcal{V}_i$ and $\varphi^m(x) \in \mathcal{V}_i$ then we must have $\varphi^k(x) \in \mathcal{V}_i$ for all $k \in [\![0, m]\!]$. In the example presented in Figure 6, the basic set K_3 belongs to $\mathcal{O}_3^- \cap \mathcal{O}_2^+$.

This last remark brings us to the next definition.

Definition 2.12 (Unrevisited set, [59, p. 463] and [66]). — Let X be a smooth manifold. A set $W \subseteq X$ is called *unrevisited* for a diffeomorphism $f : X \to X$ if for any integer $m \in \mathbb{N}$,

$$x, f^m(x) \in W \implies \forall k \in \{0, \dots, m\},\ f^k(x) \in W.$$

We say that a set is unrevisited for the flow φ^t if it is unrevisited for the time-1 map φ^1. According to the last point of Remark 2.11, the existence of unrevisited neighborhoods is a consequence of the existence of a filtration for φ^t and of a filtration for φ^{-t}. Moreover, the existence of a filtration can be deduced from the existence of a continuous Lyapunov function for the flow which goes back to the work of Conley [13, p. 20] and which can be found in [35, Th. 1.5.44 & 5.3.47]:

Theorem 4 (Conley's fundamental theorem of dynamical systems). — *Let φ^t be an Axiom A flow which satisfies the no-cycle property.*[5] *There exists a continuous function $E : M \to \mathbb{R}$ which is constant on each basic set and take distinct values such that,*

$$E\big(\varphi^t(x)\big) < E(x), \qquad \forall t > 0, \quad \forall x \in M \smallsetminus \Omega,$$

and $E \circ \varphi^t \leqslant E$ everywhere for all $t \geqslant 0$.

Considering the sublevel sets associated with such a function E, it is not hard to deduce

Corollary 2.13 (Existence of a filtration). — *Under the same hypothesis, there exists a filtration for the flow φ^t and there exists a filtration for the flow φ^{-t} in the sense of Definition* 2.10.

Furthermore, intersecting two filtrations (one for φ^t and one for φ^{-t}), we get:

Corollary 2.14 (Definition and existence of unrevisited neighborhoods)
Let K be a basic set. We say that an open set $\mathcal{V}$ of K is an unrevisited neighborhood *of K if $K \subset \mathcal{V}$, $\mathcal{V}$ is an unrevisited set, $\overline{\mathcal{V}}$ is also an unrevisited set and $\Omega \cap \overline{\mathcal{V}} = K$. According to Theorem* 4 *and Corollary* 2.13*, there exist arbitrarily small unrevisited neighborhoods $\mathcal{V}$ of K.*

[5] We use here the fact that the chain recurrent set is equal to the non-wandering set when the Axiom A flow satisfies the no-cycle property, which can be found in [35, Th. 5.3.47]. We refer to [35, Def. 5.3.42] for a definition of the no-cycle property.

Filtrations and unrevisited neighborhoods play an important role in the dynamical proof of this article. The existence of Lyapunov functions, of filtrations and of unrevisited neighborhood are related and almost equivalent. To make it easier to read, we sketch the nonlinear dependencies between these three notions:

(1) We recall that if an Axiom A flow satisfies the strong transversality assumption then it satisfies the no-cycle property.

(2) According to Conley, for an Axiom A flow satisfying the no-cycle property, there exists a continuous Lyapunov function, and this implies the existence of a filtration.

(3) From this filtration, we can obtain arbitrarily small unrevisited neighborhoods.

(4) Once we have one filtration, we can construct filtrations for any total order relation—see Lemma 6.3. This lemma mainly[(6)] relies on Proposition B.3 of Appendix B.

(5) Using filtrations for every total order relation constructed in Lemma 6.3, we can deduce Corollary B.5. Namely, we get the compactness of all the sets $\bigcup_{i\leqslant j} W^{\mathrm{u}}(K_j)$ and $\bigcup_{i\leqslant j} W^{\mathrm{s}}(K_i)$, for every total order relation defined in Section 2.5.1.

(6) Once we have the compactness of these sets, we can construct smooth Lyapunov functions adapted to the total order relation. Precisely, we can construct Lyapunov functions with arbitrary values on the basic sets accordingly to the total order relation—see Proposition 3.1.

Remark 2.15. — Theorem 4 implies actually something stronger. Indeed, the filtration given by the sublevel sets of the continuous Lyapunov function is stable by φ^{-t} for every $t > 0$ and, for every basic set K, there exists an open neighborhood $\mathcal{V} \supset K$ such that: for all $T \geqslant 0$,

$$x, \varphi^T(x) \in \mathcal{V} \implies \forall t \in [0,T],\ \ \varphi^t(x) \in \mathcal{V}.$$

Even if this last definition seems to be more natural for flows, we chose the diffeo-like definition which will be more convenient for the analysis of the lifted Hamiltonian dynamics on the phase space as we will see later on. Unrevisited neighborhoods will be a very important tool of our analysis. They will be a purely dynamical alternative to the $\mathcal{C}^1$ linearizing charts near critical points which were used in [15] for Morse-Smale gradient flows, as we can witness in Figure 7.

Let us mention some properties satisfied by the unrevisited neighborhoods. All the results mentioned below about unrevisited neighborhoods were not precisely stated in the literature. They should be attributed to Conley, Robbin, Hirsch, Palis, Pugh, Shub to the best of our knowledge. Precisely,

(P1) The intersection of two unrevisited sets is also an unrevisited.

(P2) We have a uniform approximation of the stable and unstable manifolds by unrevisited neighborhoods in the sense of the following lemma.

[(6)] In particular, its proof does not use the compactness of the sets $\bigcup_{i\leqslant j} W^{\mathrm{u}}(K_j)$ and $\bigcup_{i\leqslant j} W^{\mathrm{s}}(K_i)$ which is claimed in Theorem 3.

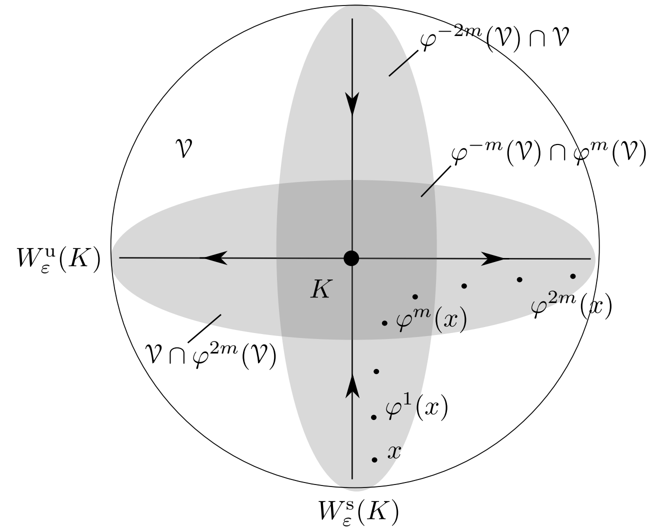


Figure 7. Illustration of some unrevisited neighborhoods near a basic set.

Lemma 2.16 (Uniform convergence of unrevisited neighborhoods). — *Let $\mathcal{V}$ be an unrevisited neighborhood of a basic set K which satisfies $\overline{\mathcal{V}}\cap\Omega=K$. The following equalities are satisfied:*

$$\bigcap_{m\in\mathbb{N}}\overline{\mathcal{V}}\cap\varphi^{m}(\overline{\mathcal{V}})=W^{\mathrm{u}}(K)\cap\overline{\mathcal{V}},\qquad \bigcap_{m\in\mathbb{N}}\mathcal{V}\cap\varphi^{m}(\mathcal{V})=W^{\mathrm{u}}(K)\cap\mathcal{V}. \tag{16}$$

Therefore, the sequence of unrevisited neighborhoods $\mathcal{V}\cap\varphi^{m}(\mathcal{V})$ is decreasing with respect to $m\in\mathbb{N}$ and tends to $W^{\mathrm{u}}(K)\cap\overline{\mathcal{V}}$ as m tends to $+\infty$ for the geodesic distance to a compact set, in the sense that

$$\sup_{y\in\overline{\mathcal{V}}\cap\varphi^{m}(\overline{\mathcal{V}})} d_g\left(y,W^{\mathrm{u}}(K)\cap\overline{\mathcal{V}}\right)\underset{m\to+\infty}{\longrightarrow}0. \tag{17}$$

A similar statement holds for the stable manifolds if we replace φ^{m} by φ^{-m}.

Remark 2.17. — For K and $\mathcal{V}$ as in the previous lemma, we deduce from the first equation in (16) that the sets $W^{\mathrm{s}}(K)\cap\overline{\mathcal{V}}$ and $W^{\mathrm{u}}(K)\cap\overline{\mathcal{V}}$ are compact sets (as claimed implicitly in (17)). This yields the inclusions

$$\overline{W^{\mathrm{s}}(K)\cap\mathcal{V}}\subseteq W^{\mathrm{s}}(K)\cap\overline{\mathcal{V}}\quad\text{and}\quad\overline{W^{\mathrm{u}}(K)\cap\mathcal{V}}\subseteq W^{\mathrm{u}}(K)\cap\overline{\mathcal{V}}.$$

Proof of Lemma 2.16. — Fix a basic set K and an unrevisited neighborhood $\mathcal{V}$ of K such that $\overline{\mathcal{V}}\cap\Omega=K$. Note first that (17) can be deduced from (16) by contradiction. Indeed, let us assume by contradiction that we can find $\varepsilon>0$ and an increasing sequence of integers $(m_i)_{i\in\mathbb{N}}$ satisfying $m_i\to+\infty$ as $i\to+\infty$ such that

$$d_g(y_{m_i},z)>\varepsilon,\quad\forall z\in W^{\mathrm{u}}(K)\cap\overline{\mathcal{V}},\quad\forall i\in\mathbb{N},$$

where

$$y_{m_i} \in \overline{\mathcal{V}} \cap \varphi^{m_i}(\overline{\mathcal{V}}), \quad d_g(y_{m_i}, W^{\mathrm{u}}(K) \cap \overline{\mathcal{V}}) = \sup_{y \in \overline{\mathcal{V}} \cap \varphi^{m_i}(\overline{\mathcal{V}})} d_g\left(y, W^{\mathrm{u}}(K) \cap \overline{\mathcal{V}}\right), \quad \forall i \in \mathbb{N}.$$

By compactness of M, one can extract a subsequence $(y_{\widetilde{m_i}})_i$ of $(y_{m_i})_i$ which converges to a point y_∞ as i tends to $+\infty$. Assuming (16), we have on one hand that

$$y_\infty \in \bigcap_{i \in \mathbb{N}} \overline{\mathcal{V}} \cap \varphi^{m_i}(\overline{\mathcal{V}}) = \bigcap_{m \in \mathbb{N}} \overline{\mathcal{V}} \cap \varphi^{m}(\overline{\mathcal{V}}) = W^{\mathrm{u}}(K) \cap \overline{\mathcal{V}}$$

and on the other hand that

$$d_g(y_\infty, W^{\mathrm{u}}(K) \cap \overline{\mathcal{V}}) \geqslant \varepsilon.$$

It leads to the expected contradiction. Now, it remains to prove (16). We begin by proving the direct inclusion for the first equation in (16). To do so, consider a point y which belongs to $\bigcap_{m \in \mathbb{N}} \overline{\mathcal{V}} \cap \varphi^m(\overline{\mathcal{V}})$ and let us show that y lies in $W^{\mathrm{u}}(K)$. Thanks to Lemma 7, M decomposes into the unstable sets of the basic sets and thus there exists a unique basic set K' such that $y \in W^{\mathrm{u}}(K')$. We claim that $K' = K$. By contradiction, if $K' \neq K$, then we choose $\eta > 0$ sufficiently small so that $\{x,\, d_g(x, K') < \eta\} \cap \overline{\mathcal{V}} = \varnothing$ and we choose $m \in \mathbb{N}$ sufficient large so that $\varphi^{-m}(y) \in W^{\mathrm{u}}_\eta(K') \subset M \smallsetminus \overline{\mathcal{V}}$ (which is possible by definition of the unstable set of K). But, by definition of y, we have for all $m \geqslant 0$ that $\varphi^{-m}(y) \in \varphi^{-m}(\overline{\mathcal{V}} \cap \varphi^m(\overline{\mathcal{V}})) \subset \overline{\mathcal{V}}$, which leads to the expected contradiction. Therefore, we must have $K' = K$ and thus $y \in W^{\mathrm{u}}(K) \cap \overline{\mathcal{V}}$. The reverse inclusion for the first equation of (16) is a consequence of the fact that $\overline{\mathcal{V}}$ is unrevisited together with the definition of $W^{\mathrm{u}}(K)$.

Now, it remains to prove the second equation of (16). To do so, it is enough to remark that the inclusions

$$W^{\mathrm{u}}(K) \cap \mathcal{V} \subseteq \mathcal{V} \cap \bigcap_{m \in \mathbb{N}} \mathcal{V} \cap \varphi^m(\mathcal{V}) \subseteq \mathcal{V} \cap \bigcap_{m \in \mathbb{N}} \overline{\mathcal{V}} \cap \varphi^m(\overline{\mathcal{V}}) = \mathcal{V} \cap W^{\mathrm{u}}(K) \cap \overline{\mathcal{V}} = W^{\mathrm{u}}(K) \cap \mathcal{V}$$

are actually equalities thanks to the first equation of (16). This ends the proof of the lemma. □

Putting together the properties (P1) and (P2) we deduce the next property.

(P3) If $\mathcal{V}$ is an unrevisited neighborhood of a basic set K such that $\overline{\mathcal{V}} \cap \Omega = K$, then the sequence $(\varphi^m(\mathcal{V}) \cap \varphi^{-m}(\mathcal{V}))_{m \in \mathbb{N}}$ tends to

$$\left(W^{\mathrm{s}}(K) \cap \overline{\mathcal{V}}\right) \cap \left(W^{\mathrm{u}}(K) \cap \overline{\mathcal{V}}\right) = (W^{\mathrm{s}}(K) \cap W^{\mathrm{u}}(K)) \cap \overline{\mathcal{V}} = K$$

in the sense that

$$\sup_{y \in \varphi^m(\overline{\mathcal{V}}) \cap \varphi^{-m}(\overline{\mathcal{V}})} d_g\left(y, K\right) \underset{m \to +\infty}{\longrightarrow} 0.$$

Remark 2.18. — The property (P3) rests on the relation $W^{\mathrm{s}}(K) \cap W^{\mathrm{u}}(K) = K$ which is a consequence of Theorem 3, but whose proof does not need the strong transversality assumption. Moreover, they are both related with the local maximality of the basic set K. Indeed, from (P3) we deduce

$$K \subseteq \bigcap_{t \in \mathbb{R}} \varphi^t(\mathcal{V}) \subseteq \bigcap_{m \in \mathbb{Z}} \varphi^m(\mathcal{V}) = \bigcap_{m \in \mathbb{N}} \varphi^m(\mathcal{V}) \cap \varphi^{-m}(\mathcal{V}) = K.$$

3. Escape functions for Axiom A flows

Following the strategy of Faure and Sjöstrand [30], we will construct some function, called an escape function, which will enable us to define anisotropic Sobolev spaces on which the Lie derivative operator $-\mathcal{L}_V$ has nice spectral properties. This function is related to the construction of energy functions (also called Lyapunov functions) whose existence will be stated in paragraph 3.2 and proved in Sections 6 and 7.

According to the strategy of [30], we need to construct an energy function on the unitary cotangent bundle S^*M which is increasing along the (projected) Hamiltonian flow $\widetilde{\Phi}^t$. We choose here to split its construction into that of two energy functions which are slightly easier to build independently: one on the base manifold M and one on the fibers of S^*M.

3.1. Energy functions on M.

— As recalled in Proposition 4, it is known from the work of Conley [13] that any continuous flow on a compact manifold behaves like a gradient flow outside an invariant set called the *chain recurrent set* (see for instance [56] of [35] for a definition). For example, it is the case for gradient flows of Morse functions where the chain recurrent set equals the set of hyperbolic fixed points and an energy function is given by the Morse function itself. For Axiom A flows satisfying the no-cycle property, the chain recurrent set equals the non-wandering set—see [35, Th. 5.3.47]. Later, Wilson [74], Fathi and Pageault [28] explained how to regularized Conley's Lyapunov function. For the sake of completeness and since its proof will be very instructive for the construction of energy functions on S^*M, we will provide a smooth version to Conley's result (recalled in Theorem 4) using filtrations and unrevisited neighborhoods. Furthermore, the next proposition has the advantage to give $\mathcal{C}^\infty$ Lyapunov functions with arbitrary values on the basic sets of any Axiom A flow satisfying the no-cycle property. The proof of next Proposition is given in Section 6 and the analysis on S^*M will be made in Section 7.

Proposition 3.1 (Energy function for Axiom A flows). — *Let φ^t be an Axiom A flow which satisfies the no-cycle property. For every $\varepsilon > 0$ and for every family of pairwise distinct real numbers $(\lambda_i)_{1\leqslant i\leqslant N}$ compatible with the graph structure in the sense that*

$$K_i \leqslant K_j \implies \lambda_i \leqslant \lambda_j,$$

there exist an energy function $E \in \mathcal{C}^\infty(M)$, ε-neighborhoods $\mathcal{N}_i$ of K_i and a constant $\eta > 0$ such that:

$$\mathcal{L}_V E \geqslant 0 \text{ on } M, \quad \text{and} \quad \mathcal{L}_V E > \eta \text{ on } M \smallsetminus \Big(\bigcup_{i=1}^N \mathcal{N}_i\Big).$$

Moreover, for all $i \in [\![1, N]\!]$, the map E is close to λ_i on each $\mathcal{N}_i$ in the sense that

$$E = \lambda_i \text{ on } K_i \quad \text{and} \quad \sup_{x\in\mathcal{N}_i} |E(x) - \lambda_i| < \varepsilon.$$

From now on, the vector field $V\in\Gamma(TM)$ will be considered to be Axiom A and to satisfy the strong transversality assumption (11) and in particular the no-cycle property.

3.2. Energy functions for the Hamiltonian flow. — Let us define the following $\widetilde{\Phi}^t$-invariant subset of S^*M:

$$\Sigma_{\mathrm{uo}}:=\bigcup_{x\in M}\kappa\big(E^*_{\mathrm{so}}(x)\smallsetminus 0_M\big),\qquad \Sigma_{\mathrm{s}}:=\bigcup_{x\in M}\kappa\big(E^*_{\mathrm{u}}(x)\smallsetminus 0_M\big),$$
$$\Sigma_{\mathrm{u}}:=\bigcup_{x\in M}\kappa\big(E^*_{\mathrm{s}}(x)\smallsetminus 0_M\big),\qquad \Sigma_{\mathrm{so}}:=\bigcup_{x\in M}\kappa\big(E^*_{\mathrm{uo}}(x)\smallsetminus 0_M\big),$$

where κ denotes the projection on the unitary cotangent bundle defined in (9). They will be our basic ingredients to construct energy functions on S^*M. Indeed, following the ideas of Faure-Sjöstrand [30] and Dang-Rivière [15, 16], we will see that $(\Sigma_{\mathrm{uo}},\Sigma_{\mathrm{s}})$ and $(\Sigma_{\mathrm{u}},\Sigma_{\mathrm{so}})$ are both a pair of repelling and attracting compact invariant sets for the Hamiltonian flow $\widetilde{\Phi}^t$. It will be enough to construct an energy function on the fiber. First, let us recall that the *strong transversality assumption* implies that

$$\Sigma_{\mathrm{uo}}\cap\Sigma_{\mathrm{s}}=\varnothing=\Sigma_{\mathrm{u}}\cap\Sigma_{\mathrm{so}}.$$

The following lemma proved in Section 7.4 tells us that they are indeed attracting and repelling sets for the Hamiltonian flow:

Lemma 3.2. — *For every $(x,\xi)\in S^*M\smallsetminus(\Sigma_{\mathrm{s}}\cup\Sigma_{\mathrm{uo}})$, we have*

$$d_{S^*M}\big(\widetilde{\Phi}^t(x,\xi),\Sigma_{\mathrm{s}}\big)\underset{t\to+\infty}{\longrightarrow}0\quad\textit{and}\quad d_{S^*M}\big(\widetilde{\Phi}^{-t}(x,\xi),\Sigma_{\mathrm{uo}}\big)\underset{t\to+\infty}{\longrightarrow}0.$$

Similarly, for every $(x,\xi)\in S^*M\smallsetminus(\Sigma_{\mathrm{so}}\cup\Sigma_{\mathrm{u}})$, we have

$$d_{S^*M}\big(\widetilde{\Phi}^t(x,\xi),\Sigma_{\mathrm{so}}\big)\underset{t\to+\infty}{\longrightarrow}0\quad\textit{and}\quad d_{S^*M}\big(\widetilde{\Phi}^{-t}(x,\xi),\Sigma_{\mathrm{u}}\big)\underset{t\to+\infty}{\longrightarrow}0.$$

Moreover, contrary to Anosov flows for which it is rather immediate, we need to make sure that these sets are compact sets. The next proposition is similar to the compactness result of Dang and Rivière [15, Lem. 3.7, p. 15]. Its proof is given in Section 7.3.

Proposition 3.3 (Compactness). — *Let φ^t be an Axiom A flow which satisfies the strong transversality assumption* (11)*. Then, the subsets Σ_{u}, Σ_{uo}, Σ_{so}, Σ_{s} of S^*M are $\widetilde{\Phi}^t$-invariant compact sets.*

To construct energy functions, we also need the existence of arbitrarily small stable neighborhoods. The proof of next lemma is given in Section 7.5.

Lemma 3.4 (Invariant neighborhoods). — *For every $\varepsilon>0$, there exist ε-neighborhoods $\mathcal{U}^{\mathrm{s/so}}$ (resp. $\mathcal{U}^{\mathrm{u/uo}}$) of $\Sigma_{\mathrm{s/so}}$ (resp. $\Sigma_{\mathrm{u/uo}}$) which are $\widetilde{\Phi}^1$-stable (resp. $\widetilde{\Phi}^{-1}$-stable).*

As a consequence of these three results, we obtain energy functions on the fiber of S^*M. The proof of the following proposition is given in Section 7.6.

Proposition 3.5 (Energy functions for the Hamiltonian flow)

Let φ^t be an Axiom A flow which satisfies the strong transversality assumption (11). *For every $\varepsilon > 0$, there exist energy functions $E_\pm \in \mathcal{C}^\infty(S^*M;[0,1])$, ε-neighborhoods $\mathcal{W}^{\mathrm{s/so}}$ of $\Sigma_{\mathrm{s/so}}$, $\mathcal{W}^{\mathrm{uo/u}}$ of $\Sigma_{\mathrm{uo/u}}$ and a constant $\eta > 0$ such that:*

$$\mathcal{L}_{X_H} E_+ \geqslant 0 \text{ on } S^*M \text{ and } \mathcal{L}_{X_H} E_+ > \eta \text{ on } S^*M \smallsetminus (\mathcal{W}^{\mathrm{uo}} \cup \mathcal{W}^s),$$
$$\mathcal{L}_{X_H} E_- \geqslant 0 \text{ on } S^*M \text{ and } \mathcal{L}_{X_H} E_- > \eta \text{ on } S^*M \smallsetminus (\mathcal{W}^{\mathrm{u}} \cup \mathcal{W}^{\mathrm{so}}).$$

Moreover, the map $E_\pm$ are constant on each Σ_* and we have the estimate

$$\sup_{(x,\xi)\in\mathcal{W}^{\mathrm{u/uo}}} |E_\pm(x,\xi) - 0| \leqslant \varepsilon \quad \text{and} \quad \sup_{(x,\xi)\in\mathcal{W}^{\mathrm{s/so}}} |E_\pm(x,\xi) - 1| \leqslant \varepsilon.$$

3.3. The escape functions. — Next, we give a global escape function which extends the construction of Dyatlov and Guillarmou [22] to the whole manifold and coincide with the one of Dang and Rivière [15] for Morse-Smale gradient flows and the one of Faure-Sjöstrand [30] for Anosov flows. The proof of next proposition is given in Section 8.

Proposition 3.6 (Escape function). — *Let $u, s, n_0 \in \mathbb{R}$ be such that $u < 0 \leqslant n_0 < s$. There exists a smooth function $m(x,\xi) \in C^\infty(T^*M)$ called an* order function *and an* escape function *$G_m \in \mathcal{C}^\infty(T^*M)$ defined by:*

$$G_m(x,\xi) = m(x,\xi) \log \sqrt{1 + f(x,\xi)^2}$$

where $f \in C^\infty(T^*M)$ is positive everywhere and homogeneous of degree 1 in ξ as soon as $|\xi| \geqslant 1$, and where m is defined by $m(x,\xi) = \chi(|\xi|^2)\, \boldsymbol{E}(x, \xi/|\xi|)$ with χ being a smooth cut-off function such that $\chi = 0$ on $]-\infty, 1/2]$, $\chi = 1$ on $[1, +\infty[$, $\chi \geqslant 0$ everywhere as in Figure 8, and $\boldsymbol{E}$ being a linear combination of previous energy functions:

$$\boldsymbol{E}(x,\xi) := -E(x) + 2s + (2u - n_0)E_+(x,\xi) + (n_0 - 2s)E_-(x,\xi).$$

Moreover, we have the following estimates:

(1) *There exist conical neighborhoods $\widetilde{\mathcal{N}}^{\mathrm{s/o/u}}$ of $\bigcup_{z\in M} E^*_{\mathrm{s/o/u}}(z) \smallsetminus 0_M$ such that $f = |\xi(V)|$ on $\widetilde{\mathcal{N}}^{\mathrm{o}}$ and for $|\xi| \geqslant 1$,*

$$\frac{1}{2} n_0 \leqslant m \leqslant 2n_0 \quad \text{on } \widetilde{\mathcal{N}}^{\mathrm{o}},$$
$$m \geqslant s \ \text{ on } \widetilde{\mathcal{N}}^{\mathrm{s}}, \quad \text{and} \quad m \leqslant u \ \text{ on } \widetilde{\mathcal{N}}^{\mathrm{u}}.$$

Also, the open sets can be chosen arbitrarily close to the invariant distributions E^*_{s}, E^*_{o} and E^*_{u} as in Proposition 3.5.

(2) *The map G_m is strictly decreasing along the flow Φ^t except at points (x,ξ) where $|\xi|$ is small or where x is in a small neighborhood of the non-wandering set and ξ is in a conical neighborhood of E^*_{o}: for all $i \in [\![1, N]\!]$, there exist an open neighborhood $\mathcal{N}_i$ of K_i and a radius $R > 0$ such that for all $(x,\xi) \in \bigcup_{z\in\cup N_i} T^*_z M \smallsetminus \widetilde{\mathcal{N}}^{\mathrm{o}}$ such that $|\xi| \geqslant R$* and *for all $(x,\xi) \in \bigcup_{z\notin\bigcup_i \mathcal{N}_i} T^*_z M$ such that $|\xi| \geqslant R$*

$$X(G_m)(x,\xi) < -c\min(s, |u|) =: -C_m < 0,$$

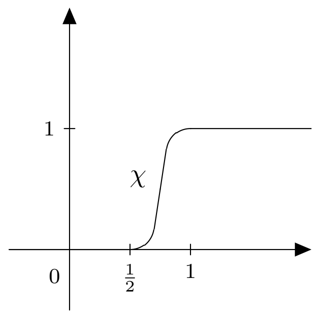


Figure 8. Cut-off function χ.

with $c>0$ being independent of the constants u, n_0, s and of the size of the conical neighborhoods.

(3) *More generally, for every $(x,\xi)\in T^*M$ such that $|\xi|\geqslant R$, we have*

$$X(G_m)(x,\xi)\leqslant 0.$$

4. Anisotropic Sobolev spaces

The purpose of this section is to construct some Hilbert spaces in which the operator $-\mathcal{L}_V^{(k)}$ acting on sections of $\mathcal{E}_k := \Lambda^k T^*M\otimes\mathbb{C}$ has good spectral properties. For $k=0$ and for Anosov flows, Faure and Sjöstrand defined an anisotropic Sobolev space which can be roughly written as $\exp(G_m(x,-iD))^{-1}L^2(M;\mathbb{C})$ using the escape function G_m of Proposition 3.6. This construction can be extended to the fiber bundle $\Lambda^k T^*M\otimes\mathbb{C}$ of differential forms with complex values for every $k\in[\![0,n]\!]$ as explained in [24]. So, we construct a pseudodifferential operator acting on sections of $\mathcal{E}_k$ which has for principal symbol

$$A_m^{(k)}(x,\xi)=\exp(G_m(x,\xi))\,\mathrm{Id}_{\mathcal{E}_k(x)}\in\mathcal{C}^\infty\big(T^*M,\mathrm{Hom}(\pi^*\mathcal{E}_k,\pi^*\mathcal{E}_k)\big),$$

with G_m being the escape function obtained in Proposition 3.6 and were $\pi^*\mathcal{E}_k\to T^*M$ denotes the pull-back bundle by the projection $\pi: T^*M\to M$. We denote this symmetric[(7)] pseudodifferential operators by $A_m^{(k)}:\Omega^k(M;\mathbb{C})\subset L^2(M;\mathcal{E}_k)\to\Omega^k(M;\mathbb{C})$ and we refer for instance to [75, Chap. 4, p. 56] for a definition of the Weyl quantization on $\mathbb{R}^n$ and to [24, App. C.1, p. 29],[16, §9.2 p. 40] for pseudodifferential operators on manifolds and vector bundles. Furthermore, the symbols $A_m^{(k)}(x,\xi)$ belong to a class of symbols with variable order whose properties are discussed in the appendix of [29] and they are *elliptic* in the sense of [29, Def. 8, p. 40]. It implies the existence of a smoothing operator $\widehat{r}:\mathcal{D}'^{,k}(M;\mathbb{C})\to\Omega^k(M;\mathbb{C})$ such that $\widetilde{A}_m^{(k)}:=A_m^{(k)}+\widehat{r}:\Omega^k(M;\mathbb{C})\to\Omega^k(M;\mathbb{C})$ is formally self-adjoint, elliptic and invertible—see [29, Lem. 12, p. 42]. Thus,

(7)This pseudodifferential operator is defined using Weyl quantization in a coordinate system. This quantization is known for mapping a real symbol to a symmetric operator. We refer the reader to the book [75, Chap. 4, Th. 4.1] for Weyl quantization on $\mathbb{R}^n$ and to the book [26, App. E] for Weyl quantization on a variety.

we choose $(\widetilde{A}_m^{(k)})^{-1}$ to be a representative of the inverse of $A_m^{(k)}$ modulo smoothing operators by setting $(\widetilde{A}_k^{(k)})^{-1} := (A_m^{(k)} + \widehat{r})^{-1}$ and we define, following [29] and [30], the anisotropic Sobolev space

$$\mathcal{H}_k^m := (\widetilde{A}_m^{(k)})^{-1} L^2(M; \mathcal{E}_k), \qquad \langle u, v\rangle_{\mathcal{H}_k^m} := \langle \widetilde{A}_m^{(k)} u, \widetilde{A}_m^{(k)} v\rangle_{L^2(M;\mathcal{E}_k)}.$$

The space $\mathcal{H}_k^m$ endowed with $\langle .,.\rangle_{\mathcal{H}_k^m}$ is isometric to the space $L^2(M;\mathcal{E}_k)$. Moreover, it is isomorphic[(8)] to the space $\mathcal{H}_0^m \otimes \Omega^k(M;\mathbb{C})$, and the following inclusions

$$\Omega^k(M;\mathbb{C}) \subset \mathcal{H}_k^m \subset \mathcal{D}'^{,k}(M;\mathbb{C}).$$

are continuous, where $\mathcal{D}'^{,k}(M;\mathbb{C})$ is endowed with the weak topology—see [65].

4.1. Spectral properties. — Adapting the proof of [30, Th. 1.4] to the case of vector bundles and using the properties[(9)] of the escape function stated in Proposition 3.6 (which is the exact analogue of Lemma 1.2 in [30]), one can establish the existence of a discrete spectrum on these anisotropic Sobolev spaces:

Theorem 5 (Discrete spectrum). — *Let φ^t be an Axiom A flow satisfying the strong transversality assumption (*10*). Let G_m be an escape function. For every $k \in [\![0,n]\!]$, the operator $-\mathcal{L}_V^{(k)}$ defines a maximal closed unbounded operator on $\mathcal{H}_k^m$ with domain*

$$\mathcal{D}(-\mathcal{L}_V^{(k)}) = \{u \in \mathcal{H}_k^m : -\mathcal{L}_V^{(k)} u \in \mathcal{H}_k^m\} \ \text{ and } \ \mathcal{L}_V^{(k)} : \mathcal{D}(-\mathcal{L}_V^{(k)}) \subset \mathcal{H}_k^m \longrightarrow \mathcal{H}_k^m.$$

Moreover, there exists a constant $C_0 \in \mathbb{R}$ (which depends on the choice of the escape function G_m) such that

$$-\mathcal{L}_V^{(k)} \text{ has empty spectrum on } \operatorname{Re}(z) \geqslant C_0,$$

and there exists a constant $C_1 > 0$ (which only depends on the vector field V and the metric g) such that

$$-\mathcal{L}_V^{(k)} \text{ has discrete spectrum on } \operatorname{Re}(z) \geqslant -C_m + C_1,$$

where $C_m > 0$ is the constant given by Proposition 3.6*.*

The eigenvalues of $-\mathcal{L}_V^{(k)}$ on the anisotropic Sobolev space are called the *Pollicott-Ruelle resonances* of $-\mathcal{L}_V^{(k)}$. There are many (equivalent) definitions of resonances. In particular, they can be viewed as the poles of the resolvent operator $(-\mathcal{L}_V^{(k)} - z)^{-1} : \Omega^k(M;\mathbb{C}) \to \mathcal{D}'^{,k}(M;\mathbb{C})$. Let us make a few remarks about them:

Remark 4.1

– The discrete spectrum of $-\mathcal{L}_V^{(k)}$ is intrinsic in the sense that it does not depend on the escape function, and the essential spectrum can be chosen as far as we want to the origin by taking m such that $C_m \gg 1$. We refer to [30, Th. 1.5, p. 134] for a proof in the case of Anosov vector fields.

(8) The idea is that any current $u \in \mathcal{D}'^{,k}(M)$ writes in coordinates as a k-form with coefficients in $\mathcal{D}'(M)$, i.e., $u = \sum u_{i_1,\dots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ where $u_{i_1,\dots,i_k} \in \mathcal{D}'(M)$. A partition of unity argument gives the result.

(9) These are the only properties used in the proof of [30].

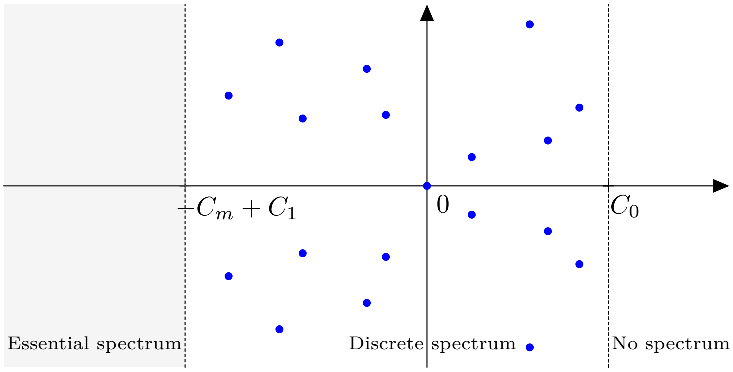


Figure 9. Illustration of Pollicott-Ruelle resonances of $-\mathcal{L}_V^{(k)}$ on the anisotropic Sobolev space $\mathcal{H}_k^m$. The fact that 0 is a resonance or not depends on k.

– The set of resonances is symmetric along the real axis since the vector field V is real.

– When $k = 0$, the resonances are included in the set $\{\mathrm{Re}(z) \leqslant 0\}$ and the point $z = 0$ is a resonance since the constants are solutions of $\mathcal{L}_V u = 0$. This fact is not true in general for $k > 0$ and the optimal constant $h \in \mathbb{R}$ such that there is no spectrum in the set $\mathrm{Re}(z) > h$ is related to the *topological entropy* of the basic sets.

– From the previous remark, we can see that the resonance 0 is somehow related to Morse inequalities:

$$\dim(\mathrm{Ker}(-\mathcal{L}_V)) \geqslant b_0 = \dim \mathrm{H}_0(M),$$

where $b_0(M)$ is the number of connected components of M and where $\mathrm{Ker}(-\mathcal{L}_V)$ denotes the kernel of the Lie derivative viewed as an unbounded operator on $\mathcal{H}_0^m$.

– From the first point, we can deduce that the space $\mathrm{Ker}((-\mathcal{L}_V^{(k)})^\ell)$ does not depend on the space function m for any $\ell \in \mathbb{N}$ (provided m is chosen such that $-C_m + C_1 < 0$).

Before going deeper into topological considerations, let us recall some useful properties of the operators $-\mathcal{L}_V^{(k)}$.

Remark 4.2

– When proving the discrete spectrum theorem for Anosov vector fields, precisely in [30, Lem. 3.3, p. 343], Faure and Sjöstrand obtained a bound on the resolvent operator which remains true in our context. For every z such that $\mathrm{Re}(z) > C_0$, we have

$$\|(\mathcal{L}_V^{(k)} + z)^{-1}\|_{\mathcal{H}_k^m \to \mathcal{H}_k^m} \leqslant \frac{1}{\mathrm{Re}(z) - C_0}.$$

An application of the Hille-Yosida theorem [27, Cor. 3.6, p. 76] yields that

$$(\varphi^{-t})^* : \mathcal{H}_k^m \longrightarrow \mathcal{H}_k^m, \quad \forall t \geqslant 0,$$

generates a *strongly continuous semi-group* whose norm is bounded by $e^{C_0 t}$. Therefore, for every z such that $\mathrm{Re}(z) > C_0$ we can write the resolvent as follows:

$$(\mathcal{L}_V^{(k)} + z)^{-1} = \int_0^{+\infty} e^{-zt}(\varphi^{-t})^* dt : \mathcal{H}_k^m \longrightarrow \mathcal{H}_k^m,$$

where the integral converges absolutely. Note that it is convenient to use the convention $(\varphi^{-t})^* = e^{-t\mathcal{L}_V^{(k)}} : \mathcal{H}_k^m \to \mathcal{H}_k^m$.

– If z_0 is any resonance of $-\mathcal{L}_V^{(k)}$ such that $\mathrm{Re}(z_0) > -C_m + C_1$, then we can define the Riesz projector

$$\pi_{z_0}^{(k)} := \frac{1}{2i\pi} \int_{\gamma_{z_0}} (\mathcal{L}_V^{(k)} + z)^{-1} dz : \mathcal{H}_k^m \longrightarrow \mathcal{H}_k^m,$$

where the integral is over a positively oriented closed curved γ_{z_0} which surrounds the resonance z_0 and no other resonances. Moreover, it commutes with $\mathcal{L}_V^{(k)}$ and it has finite rank. Note that this definition still makes sense when z_0 is not a resonance and in that case, $\pi_{z_0}^{(k)}$ is identically 0. Note also that $\pi_{z_0}^{(k)}$ commutes with the exterior derivatives d because $\mathcal{L}_V^{(k)}$ commutes with d thanks to the Cartan formula.

– The resolvent operator writes as a Laurent series near z_0:

$$(\mathcal{L}_V^{(k)} + z)^{-1} = \sum_{\ell=1}^{m_k(z_0)} (-1)^{\ell-1} \frac{(\mathcal{L}_V^{(k)} + z)^{\ell-1} \pi_{z_0}^{(k)}}{(z - z_0)^\ell} + R_{z_0,k}(z),$$

where $R_{z_0,k}$ is holomorphic near z_0.

5. Construction of the Morse-de Rham complex

Let us define $\mathrm{Res}_k(V)$ as the set of resonances $z \in \mathbb{C}$ of the operator $-\mathcal{L}_V^{(k)}$, i.e., the set of points $z_0 \in \mathbb{C}$ such that we can find an escape function G_m with $\mathrm{Re}(z_0) > -C_m + C_1$ and such that the algebraic multiplicity of z_0, denoted by $m_k(z_0)$, satisfies $m_k(z_0) \neq 0$. We can then define $C_V^k(z_0)$ as the range of the projector $\pi_{z_0}^{(k)}$ defined on the space of k-forms $\Omega^k(M;\mathbb{C})$. Equivalently, we have

$$C_V^k(z_0) = \mathrm{Ker}\big((\mathcal{L}_V^{(k)} + z_0)^{m_k(z_0)}\big),$$

and since $\pi_{z_0}^{(k)}$ has finite rank, the vector space $C_V^k(z_0)$ is *finite dimensional.* Recall that this space is independent of the choice of the escape function used to define Hilbert spaces—see [30, Th. 1.5 p. 334]. Also, the exterior derivative d maps the space $C_V^k(z_0)$ into $C_V^{k+1}(z_0)$ because d commutes with the spectral projector $\pi_{z_0}^{(k)}$, a property which follows from the fact that d commutes with $\mathcal{L}_V^{(k)}$ thanks to Cartan's formula.

We will denote by $H(C_V^*(0), d)$ the cohomology of the spectral complex associated with the eigenvalue 0 and by $H^k(M;\mathbb{C})$ the complex k-th de Rham cohomology:

$$H^k(C_V^*(0), d) = \mathrm{Ker}(d|_{C_V^k(0)}) \big/ \mathrm{Ran}(d|_{C_V^{k-1}(0)}),$$
$$H^k(M;\mathbb{C}) = \mathrm{Ker}(d|_{\Omega^k(M;\mathbb{C})}) \big/ \mathrm{Ran}(d|_{\Omega^{k-1}(M;\mathbb{C})}).$$

Theorem 2 of the introduction is a consequence of the next theorem.

Theorem 6. — *Suppose that the vector field V generates an Axiom A flow which satisfies the strong transversality assumption* (10). *For every integer $k\in[\![0,n]\!]$, the map*

$$\pi_0^{(k)}:\Omega^k(M;\mathbb{C})\longrightarrow C_V^k(0)$$

induces an isomorphism between $\mathrm{H}^k(M;\mathbb{C})$ and $\mathrm{H}^k(C_V^*(0),d)$.

The proof of this theorem is given in [19, Th. 2.1]. It was actually proved in the case of Morse-Smale flows but the proof also works for Axiom A flows since it rests on algebraic arguments. Its starting point is the commutation formula $\pi_0^{(k+1)}\circ d=d\circ\pi_0^{(k)}$ which is a consequence of the fact that the Lie derivative commutes with the exterior derivative, due to the Cartan formula. Then, the (quasi-)isomorphism of the statement can be deduced from the following de Rham theorem.

Theorem 7 (de Rham). — *The following statements are true:*

(1) *Let u be an element in $\mathcal{H}_k^m$ satisfying $du=0$. There exists $\omega\in\Omega^k(M;\mathbb{C})$ such that $u-\omega\in d\left(\mathcal{H}_{k-1}^{m+1}\right)$.*

(2) *If $u=dv$ with $u\in\Omega^k(M;\mathbb{C})$ and $v\in\mathcal{D}'^{,k-1}(M;\mathbb{C})$, then there exists $\omega\in\Omega^{k-1}(M;\mathbb{C})$ such that*

$$u=d\omega.$$

This result can be found in [65, p. 355] or in [19, p. 16] for this version using anisotropic Sobolev spaces.

6. Energy functions and application to Axiom A flows

6.0.1. *A useful lemma*. — In this part, we recall a general analysis introduced in [30] which will be applied to both flows $\varphi^t:M\to M$ and $\widetilde{\Phi}^t:S^*M\to S^*M$.

Let us consider some smooth vector field $v\in\Gamma(TX)$ on a compact Riemannian manifold (X,g_0) and let us denote by $\exp(t\cdot v)$ the flow generated by v. A pair of compact sets (K_+,K_-) is said to be *attractor-repeller* for the flow $\exp(t\cdot v)$ on X if it satisfies the two following conditions:

(i) $\forall x\in X\smallsetminus(K_-\cup K_+)$, $d_{g_0}(\exp(t\cdot v)(x),K_\pm)\underset{t\to\pm\infty}{\longrightarrow}0$.

(ii) There exist open neighborhoods $\mathcal{V}_\pm$ of $K_\pm$ stable by $\exp(\pm1\cdot v)$ such that $\overline{\mathcal{V}_-}\cap\overline{\mathcal{V}_+}=\varnothing$.

Lemma 6.1 (Faure-Sjöstrand, [30]). — *Let (K_-,K_+) be an attractor-repeller pair for the flow $\exp(t\cdot v)$ on the compact Riemannian manifold (X,g_0) and fix $\varepsilon>0$. There exist ε-neighborhoods $\mathcal{W}_\pm$ of $K_\pm$, an energy function $m\in\mathcal{C}^\infty(X;[0,1])$ and a constant $\eta>0$, which depends on ε, such that $v(m)\geqslant0$ everywhere and $v(m)>\eta$ outside $\mathcal{W}_-\cup\mathcal{W}_+$. Moreover, we have $m>1-\varepsilon$ on $\mathcal{W}_+$ and $m=1$ on K_+. Similarly, we have $m<\varepsilon$ on $\mathcal{W}_-$ and $m=0$ on K_-.*

This lemma is proved in a way similar to Lemma 2.1 of Faure-Sjöstrand [30, p. 336].

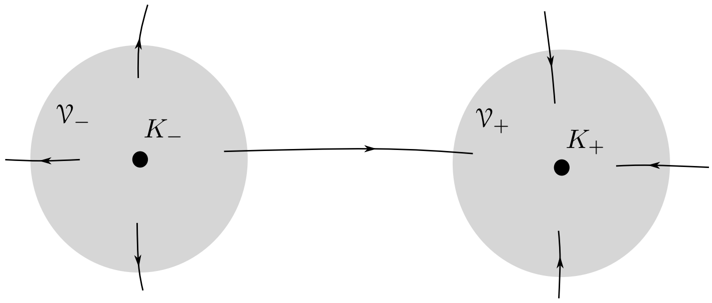


Figure 10. Illustration of an attractor-repeller system.

6.1. Energy functions for Axiom A flows. — In this part, we explain how to construct an energy function for the flow φ^t from the previous lemma. More precisely, we prove Proposition 3.1. The construction of an energy function presented here splits in three parts:

– First, for every total order relation in the sense of Section 2.5.1 and for every $\varepsilon > 0$, we define an ε-filtration from the unrevisited neighborhoods. It is stronger to the existence of a filtration given by Corollary 2.13 since it holds for every total order relation and not only for the one induced by the continuous Lyapunov function of Theorem 4. It will enable us to choose the value of the Lyapunov function on the basic sets, as in the statement of Proposition 3.1.

– Then, we prove that for any total order relation and for every $j \in [\![2, N]\!]$, the pair $(\bigcup_{k\geqslant j} W^{\mathrm{u}}(K_k), \bigcup_{i<j} W^{\mathrm{s}}(K_i))$ is an attractor-repeller pair.

– Finally, as a consequence of Lemma 6.1 we obtain a family of energy functions E_i and a linear combination of them gives the global energy function for φ^t.

Lemma 6.2 (Invariant neighborhoods on the base). — *For every total order relation given in Section* 2.5.1 *and for every* $j \in [\![2, N]\!]$, $\bigcup_{i<j} W^{\mathrm{s}}(K_i)$ *and* $\bigcup_{k\geqslant j} W^{\mathrm{u}}(K_k)$ *are disjoint invariant compact sets such that:*

$$\forall x \notin \bigcup_{i<j} W^{\mathrm{s}}(K_i), \quad d(\varphi^t(x), \bigcup_{k\geqslant j} W^{\mathrm{u}}(K_k)) \underset{t\to+\infty}{\longrightarrow} 0 \tag{18}$$

and

$$\forall x \notin \bigcup_{k\geqslant j} W^{\mathrm{u}}(K_k), \quad d(\varphi^{-t}(x), \bigcup_{i<j} W^{\mathrm{s}}(K_i)) \underset{t\to+\infty}{\longrightarrow} 0. \tag{19}$$

Proof. — The fact that these sets are disjoint and compact is a direct consequence of the order relation's properties. Now, consider $x \notin \bigcup_{i<j} W^{\mathrm{s}}(K_i)$. From the decomposition (7) of M into stable manifolds of the basic sets, there exists $k \in [\![1, N]\!]$ such that $x \in W^{\mathrm{s}}(K_k)$. Thanks to our choice of a total order relation in the sense of Section 2.5.1, we necessarily have $k \geqslant j$. This proves the convergence (18). Up to replacing the flow φ^t by φ^{-t}, we also get the convergence (19). $\square$

Now, the key point is to prove the second property of an attractor-repeller. This is given by the next lemma which is slightly more precise, is a corollary of Theorem 4, and is inspired by the filtration lemma of Robbin [59, 7.9]

Lemma 6.3. — *For every total order relation (Section* 2.5.1*) and for every* $\varepsilon > 0$, *there exists a filtration* $(\mathcal{O}_\ell^-(\varepsilon))_{1\leqslant\ell\leqslant N}$ *on* M *for* φ^{-1} *which is within a distance* ε *of the stable manifolds and there exists a filtration* $(\mathcal{O}_\ell^+(\varepsilon))_{1\leqslant\ell\leqslant N}$ *on* M *for* φ^1 *which is within a distance* ε *of the unstable manifolds, in the sense that* $\forall j \in [\![1, N]\!]$,

$$\sup_{y\in\mathcal{O}_j^-(\varepsilon)} d_g\Big(y, \bigcup_{i\leqslant j} W^{\mathrm{s}}(K_i)\Big) < \varepsilon, \qquad \sup_{y\in\mathcal{O}_{N-j+1}^+(\varepsilon)} d_g\Big(y, \bigcup_{k\geqslant j} W^{\mathrm{u}}(K_k)\Big) < \varepsilon.$$

The proof rests on the next sub-lemma which enables us to construct the filtration by induction from a total order relation on the basic sets. This sub-lemma will also be used in the proof of Proposition 3.5 in order to lift the filtration on S^*M.

Sub-Lemma 6.4 (Uniform traveling time for unstable annulus). — *Let* $\mathcal{V}$ *be an unrevisited neighborhood of a basic set* K *such that* $\mathcal{V}\cap\Omega = K$. *Let* O *be an open set such that* $\mathcal{V} \subset M \smallsetminus O$, $\varphi^1(\overline{O}) \subset O$ *and such that* O *is a neighborhood of* $\{K_j,\ K < K_j\}$. *For every* $m \in \mathbb{N}$, *we define the annulus* $\mathcal{A}(m)$ *by*

$$\mathcal{A}(m) := \varphi^m(\overline{\mathcal{V}}) \cap \overline{\mathcal{V}} \smallsetminus \varphi^{-1}(\mathcal{V}).$$

If K *is not an attractor, i.e.,* $W^{\mathrm{u}}(K) \smallsetminus K \neq \varnothing$, *then* $O \neq \varnothing$, *the compact set* $\mathcal{A}(m)$ *satisfies* $\mathcal{A}(m) \neq \varnothing$, $\mathcal{A}(m) \cap W^{\mathrm{u}}(K) = \mathcal{A}(0) \cap W^{\mathrm{u}}(K)$ *for every integer* $m \geqslant 0$ *and there exist* $k_0, m_0 \in \mathbb{N}$ *such that for every* $m \geqslant m_0$

$$\varphi^{k_0}(\mathcal{A}(m)) \subset O.$$

Proof. — We split the proof in 4 steps.

Step 1. — We show that for every $m \in \mathbb{N}$ we have $\mathcal{A}(m) \neq \varnothing$ and $W^{\mathrm{u}}(K) \cap \mathcal{A}(m) = W^{\mathrm{u}}(K) \cap \mathcal{A}(0)$. Since the relation $W^{\mathrm{u}}(K) \cap \varphi^m(\overline{\mathcal{V}}) \cap \overline{\mathcal{V}} = W^{\mathrm{u}}(K) \cap \overline{\mathcal{V}}$ is satisfied, we must have

$$W^{\mathrm{u}}(K) \cap \mathcal{A}(m) = W^{\mathrm{u}}(K) \cap \mathcal{A}(0) = \{x \in W^{\mathrm{u}}(K) \cap \overline{\mathcal{V}},\ \varphi^1(x) \notin \mathcal{V}\}$$

for all $m \in \mathbb{N}$. Let us prove that the last set is non-empty. If x belongs to $W^{\mathrm{u}}(K) \cap \overline{\mathcal{V}} \smallsetminus K$, which is non-empty as K is not an attractor, then one can find an integer $\ell_0 \geqslant 0$ such that $\varphi^{\ell_0}(x) \in W^{\mathrm{u}}(K) \cap \overline{\mathcal{V}}$ and $\varphi^{\ell_0+1}(x) \notin \overline{\mathcal{V}}$ by fixing $\ell_0 = \sup\{\ell \in \mathbb{N},\ \varphi^\ell(x) \in \overline{\mathcal{V}}\}$ (which is finite by definition of x).

Step 2. — We find for every x in $W^{\mathrm{u}}(K)\cap\mathcal{A}(0)$ an integer $k(x) > 0$ such that we have $\varphi^{k(x)}(x) \in O$. Thanks to Lemma 2.5, we know that there exists a basic set K' such that $x \in W^{\mathrm{s}}(K')$. Thus $x \in W^{\mathrm{u}}(K) \cap W^{\mathrm{s}}(K')$. By definition of the Smale relation, it implies $K \leqslant K'$. Also, we must have $K \neq K'$. Indeed, otherwise we would have $x \in W^{\mathrm{u}}(K)\cap W^{\mathrm{s}}(K) = K$, according to Theorem 3, which is in contradiction with the fact that $x \in \mathcal{A}(0) \subset M \smallsetminus K$. Consequently, we must have $K' \subset O$ and the claimed existence of the integer $k(x)$ follows from the definition of the stable manifold of K' together with the inclusion $K' \subset O$.

Step 3. — We prove by contradiction that there exists $k_0 \in \mathbb{N}$ such that

$$\varphi^{k_0}(W^{\mathrm{u}}(K) \cap \mathcal{A}(0)) \subset O.$$

By contradiction, assume that for every integer $k \in \mathbb{N}^*$ there exists $x_k \in W^{\mathrm{u}}(K) \cap \mathcal{A}(0)$ such that $\varphi^k(x_k) \notin O$. By compactness of $\overline{\mathcal{V}}$, we can extract a subsequence $(x_{k_\ell})_{\ell \in \mathbb{N}}$ which converges to some $x_\infty \in \overline{\mathcal{V}}$. According to Remark 2.17, we must have $x_\infty \in W^{\mathrm{u}}(K) \cap \overline{\mathcal{V}}$. Also, by definition of $\mathcal{A}(0)$, the elements $\varphi^1(x_{k_\ell})$ belong to $M \smallsetminus \mathcal{V}$. So, letting ℓ tend to $+\infty$, we deduce that $\varphi^1(x_\infty) \in M \smallsetminus \mathcal{V}$. Therefore, we have $x_\infty \in W^{\mathrm{u}}(K) \cap \mathcal{A}(0)$ and step 2 implies that $\varphi^{k(x_\infty)}(x_\infty) \in O$. By continuity of $\varphi^{k(x_\infty)}$ and since O is open, there exists $\ell_0 \geqslant 0$ such that $\varphi^{k(x_\infty)}(x_{k_\ell}) \in O$ for every $\ell \geqslant \ell_0$. By stability of O, this leads to

$$\varphi^k(x_{k_\ell}) \in O, \quad \forall k \geqslant k(x_\infty), \quad \forall \ell \geqslant \ell_0,$$

which is in contradiction with the construction of x_{k_ℓ} for ℓ sufficiently large.

Step 4. — We extend step 3 to $\mathcal{A}(m)$ for m large enough. Thanks to Lemma 2.16, we obtain that

$$\sup_{x \in \mathcal{A}(m)} d_g\left(x, \mathcal{A}(0) \cap W^{\mathrm{u}}(K)\right) \underset{m \to +\infty}{\longrightarrow} 0.$$

Therefore, we deduce by continuity from step 3 that there exists an integer $m_0 \in \mathbb{N}$ such that

$$\varphi^{k_0}(\mathcal{A}(m)) \in O, \qquad \forall m \geqslant m_0. \tag{20}$$

This concludes the proof of the sub-lemma. □

Proof of Lemma 6.3. — Consider a total order relation on the family of basic sets. Let us proceed by induction to construct a filtration on M stable by φ^1. The following arguments adapt easily to construct a filtration φ^{-1}-stable if we change φ^1 by φ^{-1}.

Base case (construction of $\mathcal{O}_1^+$). — Fix $\varepsilon > 0$. Since the indices of $(K_i)_{1 \leqslant i \leqslant N}$ have been chosen compatible with the relation $\leqslant$, the basic set K_N must be an attractor. So if we consider some unrevisited neighborhood $\mathcal{V}_N$ of K_N small enough so that $\overline{\mathcal{V}_N} \subset W^{\mathrm{s}}(K_N)$, then $\mathcal{V}_N$ and $\overline{\mathcal{V}_N}$ are φ^1-stable and we will simply define

$$\mathcal{O}_1^+ := \mathcal{V}_N \cap \varphi^k(\mathcal{V}_N)$$

for a large enough value of k. Indeed, thanks to Lemma 2.16, we can choose k so that $\sup_{y \in \mathcal{V}_N \cap \varphi^k(\mathcal{V}_N)} d_g(y, W^{\mathrm{u}}(K_N) \cap \mathcal{V}_N) < \varepsilon$. Since K_N is an attractor, the equality $W^{\mathrm{u}}(K_N) = K_N$ holds and it implies

$$\sup_{y \in \mathcal{O}_1^+(\varepsilon)} d_g(y, K_N) < \varepsilon.$$

Induction step. — Fix $\varepsilon > 0$ and assume that the open sets $\mathcal{O}_1^+(\varepsilon) \subseteq \mathcal{O}_2^+(\varepsilon) \subseteq \cdots \subseteq \mathcal{O}_{j-1}^+(\varepsilon) \subseteq \mathcal{O}_j^+(\varepsilon)$ are constructed for some $j \in [\![1, N]\!]$ so that they define a filtration for the family of basic sets $(K_{N-i})_{0\leqslant i\leqslant j-1}$ which is ε-close to the unstable manifolds, in the sense that, for every $i \in [\![1, j]\!]$,

$$\sup_{y\in\mathcal{O}_i^+(\varepsilon)} d_g\Big(y, \bigcup_{k\geqslant N-i+1} W^{\mathrm{u}}(K_k)\Big) < \varepsilon.$$

We want to construct a φ^1-stable ε-neighborhood $\mathcal{O}_{j+1}^+(\varepsilon)$ of $\bigcup_{k\geqslant N-j} W^{\mathrm{u}}(K_k)$. To lighten the proof, let us denote by K the basic set K_{N-j}. If K is an attractor, then we can proceed exactly as in the base case and take the union of this open set with $\mathcal{O}_j^+$. So let us assume that K is not a attractor, i.e., $W^{\mathrm{u}}(K) \neq K$, and consider some unrevisited neighborhood $\mathcal{V}$ of K such that $\overline{\mathcal{V}} \cap \Omega = K$. Applying the sub-lemma 6.4 to K, $\mathcal{V}$, $O = \mathcal{O}_j^+(\varepsilon)$, we obtain integers $k_0, m_0 \geqslant 0$ as in the statement. Therefore, for every $m \geqslant m_0$, we define

$$O(m) := \mathcal{O}_j^+(\varepsilon) \cup \bigcup_{k=0}^{k_0-1} \varphi^k(\varphi^m(\mathcal{V}) \cap \mathcal{V}),$$

which is φ^1-stable by construction of $\mathcal{O}_j^+(\varepsilon)$ if we remark that

$$\begin{aligned}\varphi^{k_0}(\varphi^m(\mathcal{V}) \cap \mathcal{V}) &\subset \varphi^{k_0}\left(\varphi^m(\mathcal{V}) \cap \varphi^{-1}(\mathcal{V})\right) \sqcup \varphi^{k_0}\left(\mathcal{A}(m)\right)\\ &\subset \varphi^{k_0-1}\underbrace{\left(\varphi^{m+1}(\mathcal{V}) \cap \mathcal{V}\right)}_{\subset\varphi^m(\mathcal{V})\cap\mathcal{V}} \sqcup \underbrace{\varphi^{k_0}\left(\mathcal{A}(m)\right)}_{\subset\mathcal{O}_j^+(\varepsilon)} \quad \subset O(m).\end{aligned}$$

Note that $\overline{O(m)}$ is also φ^1-stable. Now, if we choose m sufficiently large according to Lemma 2.16 so that

$$\sup_{y\in\bigcup_{k=0}^{k_0-1}\varphi^k(\varphi^m(\overline{\mathcal{V}})\cap\overline{\mathcal{V}})} d_g\Big(y, W^{\mathrm{u}}(K) \cap \bigcup_{k=0}^{k_0-1} \varphi^k(\overline{\mathcal{V}})\Big) < \varepsilon,$$

then we get the claimed result by setting $\mathcal{O}_{j+1}^+(\varepsilon) := O(m)$. ☐

Remark 6.5. — Thanks to the total order relation (15), the compact sets $\bigcup_{i\leqslant j} W^{\mathrm{s}}(K_i)$ and $\bigcup_{k\geqslant j} W^{\mathrm{u}}(K_k)$ intersect on K_j, i.e.,

$$\bigcup_{i\leqslant j} W^{\mathrm{s}}(K_i) \cap \bigcup_{k\geqslant j} W^{\mathrm{u}}(K_k) = K_j. \tag{21}$$

Therefore, from Remark 2.11, the set

$$\mathcal{V}_j(m) := \varphi^{-m}\left(\mathcal{O}_j^-(\varepsilon)\right) \cap \varphi^m\left(\mathcal{O}_{N-j+1}^+(\varepsilon)\right)$$

defines a decreasing (for the inclusion) family of unrevisited neighborhoods of K_j. Moreover, if we choose m sufficiently large, then $\mathcal{V}_j(m)$ can be made arbitrarily close to K_j in the sense that

$$\sup_{x\in\overline{\mathcal{V}_j(m)}} d_g(x, K_j) \underset{m\to+\infty}{\longrightarrow} 0.$$

This fact can be proved by contradiction (similarly to the proof of Lemma 2.16) by using the relations

$$\bigcap_{m\in\mathbb{N}}\overline{\mathcal{V}_j(m)}=\bigcap_{m\in\mathbb{N}}\varphi^{-m}\big(\overline{\mathcal{O}_j^-(\varepsilon)}\big)\cap\bigcap_{m\in\mathbb{N}}\varphi^{m}\big(\overline{\mathcal{O}_{N-j+1}^+(\varepsilon)}\big)=\bigcup_{i\leqslant j}W^{\mathrm{s}}(K_i)\cap\bigcup_{k\geqslant j}W^{\mathrm{u}}(K_k).$$

A direct application of these lemmas gives what we were looking for. Namely:

Proposition 6.6. — *For every $j\in[\![2,N]\!]$, $(\bigcup_{k\geqslant j}W^{\mathrm{u}}(K_k),\bigcup_{i<j}W^{\mathrm{s}}(K_i))$ defines an attractor-repeller pair.*

Proof. — It is a direct application of Lemmas 6.2 and 6.3 once we have chosen $\varepsilon\ll 1$ small enough to ensure that

$$B_g\Big(\bigcup_{k\geqslant j}W^{\mathrm{u}}(K_k),2\varepsilon\Big)\bigcap B_g\Big(\bigcup_{i<j}W^{\mathrm{s}}(K_i),2\varepsilon\Big)=\varnothing,\quad\forall j,\ 1<j\leqslant N,$$

where $B_g(S,2\varepsilon)$ denotes the geodesic ball at distance 2ε to a compact set S. □

Now, we are ready to construct an energy function for φ^t.

Proof of Proposition 3.1. — Let $\varepsilon>0$ as in the previous proof and fix a sequence of pairwise distinct real numbers $(\lambda_i)_{1\leqslant i\leqslant N}$ compatible with the graph structure in the sense that $\lambda_i\leqslant\lambda_j\Leftrightarrow K_i\leqslant K_j$. Up to a permutation of the indices of the basic sets, we can assume that $\lambda_1<\lambda_2<\cdots<\lambda_N$. In order to find an energy function E such that $E=\lambda_j$ on K_j, we will apply Lemma 6.1 for each attractor-repeller given in Proposition 6.6. Thanks to Lemma 6.3 together with Remark 6.5, there exist filtrations $\mathcal{O}_0^-(\varepsilon)\subset\mathcal{O}_1^-(\varepsilon)\subset\cdots\subset\mathcal{O}_N^-(\varepsilon)$ and $\mathcal{O}_N^+(\varepsilon)\supset\cdots\supset\mathcal{O}_1^+(\varepsilon)\supset\mathcal{O}_0^+(\varepsilon)$ which are respectively φ^{-1}-stable and φ^1-stable such that

$$\forall j,\ 1\leqslant j\leqslant N,\qquad \mathcal{O}_{N-j+1}^+(\varepsilon)\cap\mathcal{O}_j^-(\varepsilon)\text{ is a }\varepsilon\text{-neighborhood of }K_j.$$

Thanks to Lemma 6.1, we obtain for every $j\in[\![2,N]\!]$ a smooth energy function $E_j\in\mathcal{C}^\infty(M;[0,1])$, ε-neighborhoods $\mathcal{W}_j^-\subset\mathcal{O}_{j-1}^-(\varepsilon)$ and $\mathcal{W}_j^+\subset\mathcal{O}_{N-j+1}^+(\varepsilon)$ of $\bigcup_{i<j}W^{\mathrm{s}}(K_i)$ and $\bigcup_{k\geqslant j}W^{\mathrm{u}}(K_k)$ respectively, a constant $\eta_0>0$ (which only depends on ε) such that $\mathcal{L}_V(E_j)\geqslant 0$ on M and

– $\mathcal{L}_V(E_j)>\eta_0$ on $M\smallsetminus(\mathcal{W}_j^-\cup\mathcal{W}_j^+)$;

– $E_j<\varepsilon$ on $\mathcal{W}_j^-$ and $E_j=0$ on $\bigcup_{i<j}W^{\mathrm{s}}(K_i)$; in particular, we have $E_j=0$ on $\bigcup_{i<j}K_i$;

– $E_j>1-\varepsilon$ on $\mathcal{W}_j^+$ and $E_j=1$ on $\bigcup_{k\geqslant j}W^{\mathrm{u}}(K_k)$; in particular, we have $E_j=1$ on $\bigcup_{k\geqslant j}K_k$.

We define a global energy function $E\in\mathcal{C}^\infty(M)$ as a linear combination of previous energy functions:

$$E=\lambda_1+\sum_{j=2}^{N}(\lambda_j-\lambda_{j-1})E_j.$$

Thanks to the properties of the energy functions stated above, we deduce that

$$\mathcal{L}_V(E)=\sum_{j=1}^{N}(\lambda_j-\lambda_{j-1})\mathcal{L}_V(E_j)>\min_{1<j\leqslant N}(\lambda_j-\lambda_{j-1})\eta_0=:\eta\ \text{ on }\ \Big(\bigcap_{j=2}^{N}(\mathcal{W}_j^-\cup\mathcal{W}_j^+)\Big)^c.$$

It remains to prove that $\bigcap_{j=2}^N(\mathcal{W}_j^-\cup\mathcal{W}_j^+)$ is an ε-neighborhood of the non-wandering set and that E is close to λ_i near K_i with equality on K_i. One has:

$$\begin{aligned}\bigcap_{j=2}^N(\mathcal{W}_j^-\cup\mathcal{W}_j^+) &= \bigcup_{\tau:[\![1,N]\!]\to\{\pm\}}\bigcap_{j=2}^N \mathcal{W}_j^{\tau(j)}\\ &= \bigcup_{j=1}^N \mathcal{N}_j := \bigcup_{j=1}^N \left(\mathcal{W}_2^+\cap\cdots\cap\mathcal{W}_j^+\cap\mathcal{W}_{j+1}^-\cap\cdots\cap\mathcal{W}_N^-\right),\end{aligned}$$

using the convention $\mathcal{N}_1=\mathcal{W}_2^-\cap\cdots\cap\mathcal{W}_N^-$. Note that the second equality is a consequence of the fact that the intersection $\mathcal{O}_{j-1}^-(\varepsilon)\cap\mathcal{O}_{N-j+1}^+(\varepsilon)$ is empty for all $j\in[\![2,N]\!]$ for ε small enough, which implies:

$$\text{if } 2\leqslant i<j \text{ and } (\tau(i),\tau(j))=(-,+) \text{ then } \bigcap_{j=2}^N \mathcal{W}_j^{\tau(j)}=\varnothing.$$

Furthermore, thanks to the definition of the sets $W_\ell^\pm$ for $\ell\in[\![2,N]\!]$ and since $(\mathcal{O}_\ell^\pm)_\ell$ is a ε-filtration for $\varphi^{\pm1}$, the sets $\mathcal{N}_j$ satisfy the inclusion

$$\mathcal{N}_j\subset\mathcal{O}_{N-1}^+(\varepsilon)\cap\cdots\cap\mathcal{O}_{N-j+1}^+(\varepsilon)\cap\mathcal{O}_j^-(\varepsilon)\cap\cdots\cap\mathcal{O}_{N-1}^-(\varepsilon)=\mathcal{O}_{N-j+1}^+(\varepsilon)\cap\mathcal{O}_j^-(\varepsilon)$$

and are therefore ε-neighborhoods of the K_j according to remark 6.5. Moreover, we have on each K_i:

$$E=\lambda_1+\sum_{j=2}^N(\lambda_j-\lambda_{j-1})E_j=\lambda_1+\sum_{j=2}^N(\lambda_j-\lambda_{j-1})\delta_{j\leqslant i}=\lambda_i.$$

It remains to prove that E is close to λ_i on the neighborhood $\mathcal{N}_i$ of K_i. To that aim, let us note that for every $x\in M$ and for every $i\in[\![1,N]\!]$, we have

$$|E(x)-\lambda_i|\leqslant\sum_{j=2}^N(\lambda_j-\lambda_{j-1})|E_j(x)-\delta_{j\leqslant i}|.$$

For any $j\in[\![2,N]\!]$ and $x\in\mathcal{N}_i$ we will bound $|E_j(x)-\delta_{j\leqslant i}|$ by a small quantity independent of x in $\mathcal{N}_i$. We have two cases to deal with:

– If $i<j$, then $\delta_{j\leqslant i}=0$ and $|E_j(x)-0|=E_j(x)<\varepsilon$ by definition of E_j.
– If $i\geqslant j$, then $\delta_{j\leqslant i}=1$ and $|E_j(x)-1|=1-E_j(x)<\varepsilon$ again by definition of E_j.

Finally, we obtain the upper bound

$$\sup_{1\leqslant i\leqslant N}\sup_{x\in\mathcal{N}_i}|E(x)-\lambda_i|\leqslant\varepsilon\sum_{j=2}^N(\lambda_j-\lambda_{j-1})\leqslant(\lambda_N-\lambda_1)\varepsilon.$$

Up to dividing ε by $\max(1,\lambda_N-\lambda_1)$ everywhere, we get the claimed result. □

Now, the idea will be to perform the same analysis for the (projected Hamiltonian) flow $\widetilde{\Phi}^t$ acting on S^*M. However, in that case, proving that one has an attractor-repeller structure for $(\Sigma_{\mathrm{s}},\Sigma_{\mathrm{uo}})$ and $(\Sigma_{\mathrm{so}},\Sigma_{\mathrm{u}})$ reveals to be more challenging because we need to understand what happens to the fiber part of $\widetilde{\Phi}^t(x,\xi)$ when the orbit of (x,ξ) comes close to a basic set. The next part is devoted to the local analysis of the

Hamiltonian near basic sets in view of applications to the proofs of Proposition 3.1, Lemmas 6.2 and 6.3 and finally the existence of the energy function on S^*M.

7. Compactness result and energy functions for the Hamiltonian flow

On a basic set K, one can define conical neighborhoods of the unstable distributions E_{u}^* and E_{uo}^* which are stable under the Hamiltonian flow Φ^t as soon as $t > 0$. This characterization of hyperbolicity is sometimes referred to as the Alekseev cone field criterion [35, Prop. 5.1.7]. The goal of this part is to extend cones of the weak/stable/unstable distributions in an invariant fashion on a small neighborhood of the non-wandering set according to the global dynamics given by the Smale ordering of the basic sets.

7.1. Adapted metric on a basic set.

— From the fixed Riemannian metric g on M, we can define on any basic set K not reduced to a fixed point the following new metric called *adapted metric* for the flow φ^t. More precisely, for every $x \in K$ and every $v = v_s + v_u + v_o \in E_{\mathrm{s}}(x) \oplus E_{\mathrm{u}}(x) \oplus E_{\mathrm{o}}(x) = T_xM$, we set

$$\begin{aligned} \widehat{g}(v,v) &= \widehat{g}(v_s,v_s) + \widehat{g}(v_u,v_u) + \widehat{g}(v_o,v_o) \\ &:= \int_0^{+\infty} e^{\lambda t/2}(\varphi^{t*}g)(v_s,v_s)dt + \int_0^{+\infty} e^{\lambda t/2}(\varphi^{-t*}g)(v_u,v_u)dt + \alpha(x)(v_o)^2, \end{aligned} \tag{22}$$

where λ denotes the hyperbolic exponent on the basic set K (see Appendix A) and α denotes the Anosov 1-form defined on K by

$$\operatorname{Ker}\alpha(x) = E_{\mathrm{u}}(x) \oplus E_{\mathrm{s}}(x), \qquad \alpha(x)(V(x)) = 1.$$

In the case where the basic set is reduced to a fixed point, we call adapted metric for the flow a metric which satisfies (22) without the term $\alpha(x)(v_0)^2$.

This new metric is well-defined thanks to hyperbolicity and to the invariance properties of the vector bundles on K. Recall also that the distributions $E_{\mathrm{s}}, E_{\mathrm{u}}$ are only Hölder continuous in general. Precisely, $\widehat{g}$ will be seen as a continuous section of the vector bundle of metrics (i.e., symmetric $(2,0)$-tensors) $\otimes^2_{\mathrm{sym}}TM \to M$ defined on the compact set K. Let us denote by $|.|_{\widehat{g}}$ the norm induced by $\widehat{g}$, i.e., $|v|_{\widehat{g}} := \sqrt{\widehat{g}(v,v)}$. The metric is said to be *adapted* due to the following hyperbolic estimates: for every $x \in K$,

$$\begin{aligned} |D\varphi^t(x)v_s|_{\widehat{g}} &\leqslant e^{-\lambda t/2}|v_s|_{\widehat{g}}, \qquad \forall t \geqslant 0,\ \forall v_s \in E_{\mathrm{s}}(x) \\ |D\varphi^{-t}(x)v_u|_{\widehat{g}} &\leqslant e^{-\lambda t/2}|v_u|_{\widehat{g}}, \qquad \forall t \geqslant 0,\ \forall v_u \in E_{\mathrm{u}}(x) \\ |D\varphi^t(x)v_o|_{\widehat{g}} &= |v_o|_{\widehat{g}}, \qquad \forall t \in \mathbb{R},\ \forall v_o \in E_{\mathrm{o}}(x). \end{aligned} \tag{23}$$

7.2. Extension of the invariant distributions near basic sets.

— In order to analyze the dynamics near basic sets, it will be convenient to extend the previous hyperbolic estimates in some neighborhood of K. To that aim, we need to extend the distributions $E_{\mathrm{s}}, E_{\mathrm{u}}$, E_{o} and $|.|_{\widehat{g}}$ near each basic set. This can be achieved thanks to the following lemma:

Lemma 7.1 (Extension lemma, [47, Lem. 4.4 p. 128]). — *Let X be a smooth manifold and let $\pi : \mathcal{E} \to X$ be some fiber bundle over X. If $\Gamma : K \to \mathcal{E}$ denotes a continuous section defined on a compact set $K \subseteq M$, then Γ extends as a continuous section $\overline{\Gamma} : \mathcal{N} \to E$ on a neighborhood $\mathcal{N}$ of K.*

7.2.1. *Extension of the adapted metric near a basic set.* — Let us apply this lemma with

$$X := M \quad \text{and} \quad \mathcal{E} = T^*M \otimes_{\mathrm{sym}} T^*M.$$

The Riemannian metric $\widehat{g}$ can be seen as a continuous section $\widehat{g} : K \to \mathcal{E}$. Since the basic set K is a compact subset of M, the lemma applies and it enables to extend $\widehat{g}$ continuously on an open neighborhood $\mathcal{N}_0$ of K. Up to considering smaller $\mathcal{N}_0$, we can assume that the extended metric remains Riemannian on $\mathcal{N}_0$ as positivity and semi-definiteness are open conditions. Since we can do this extension near each basic set and since the metric g is Riemannian, a partition of unity argument enables to prove:

Lemma 7.2. — *For any Axiom A flow on a compact manifold, there exists a continuous Riemannian metric (globally defined) which is adapted to the dynamics on each basic set, in the sense that* (23) *holds on each basic set.*

In what follows, we will always assume that g is a continuous Riemannian metric adapted to the dynamics on each basic set and we will denote by $|.|$ its norm on the fibers of TM and T^*M to lighten notations.

7.2.2. *Extension of distributions.* — We now apply Lemma 7.1 in order to extend the distributions E_{s}^*, E_{so}^*, E_{u}^*, E_{uo}^* (defined on K) continuously on a neighborhood of K. Note that we already explained that $E_{\mathrm{s}}^*, E_{\mathrm{u}}^*, \ldots$ are well-defined all over M using the partition into unstable manifolds. The point of this new extension based on the bundles on K (and not on the global dynamics) is that we expect that these new bundles have good hyperbolic properties in the sense of (23). We will also need to make sure that our local analysis is related to the invariant distributions $E_{\mathrm{u/uo}}^*$ and $E_{\mathrm{s/so}}^*$ defined all over M.

Fix a basic set K and recall that the dimension of the distributions $E_{\mathrm{s}}, E_{\mathrm{u}}, E_{\mathrm{o}}$ are constant on K. We denote by $d_{\mathrm{s}} = \dim E_{\mathrm{s}}$ and $d_{\mathrm{u}} = \dim E_{\mathrm{u}}$ their dimension on K. According to Lemma 2.6, the inclusions $\overline{W_\varepsilon^{\mathrm{s}}(K)} \subseteq W_{\varepsilon_0}^{\mathrm{s}}(K)$ and $\overline{W_\varepsilon^{\mathrm{u}}(K)} \subseteq W_{\varepsilon_0}^{\mathrm{u}}(K)$ are satisfied as soon as $\varepsilon < \varepsilon_0$ where $\varepsilon_0 > 0$ is given by the stable manifold theorem.

Lemma 7.3. — *For every $0 < \varepsilon < \varepsilon_0$, with $\varepsilon_0 > 0$ given by the stable manifold theorem, the map*

$$\Gamma : \overline{W_\varepsilon^{\mathrm{s}}(K)} \longrightarrow G_{\mathrm{s}}(M), \quad \Gamma(x) = E_{\mathrm{s}}(x)$$

with value in the Grassmann vector bundle of subspaces of dimension d_{s}, is continuous.

An important remark is that we aim to extend the stable distribution E_{s}, defined on the stable set of K, on a neighborhood of K in a continuous fashion. This extension will be denoted by $\widetilde{E}_{\mathrm{s}}$ to avoid confusion. The subtle point is the following fact: E_{s} is not continuous in general on a neighborhood of K but only on the stable set $W_{\varepsilon_0}^{\mathrm{s}}(K)$

of K. Here, $\widetilde{E}_{\mathrm{s}}$ will be equal to E_{s} on the stable set of K and will be continuous on a neighborhood of K by construction.

Proof. — The result is a direct consequence of the stable manifold theorem when the basic set is reduced to a fixed point. So let us assume that the basic set K is not reduced to a fixed point. In this setting, the proof will be a consequence of the λ-lemma (also called inclination lemma) in a slightly more general version than usual which is recalled in Proposition B.1 from Appendix B. Indeed, we first consider a sequence $(z_n) \in K^{\mathbb{N}}$, a family of elements $x_n \in W^{\mathrm{s}}_{\varepsilon}(z_n)$, $v_n \in E_{\mathrm{s}}(x_n) = T_{x_n} W^{\mathrm{s}}_{\varepsilon}(z_n)$ and we assume that the three sequences $(z_n), (x_n), (v_n)$ are converging respectively to $z \in K$, $x \in M$, $v \in T_x M$. Proceeding as in the proof of Lemma 2.6, we deduce that $x \in W^{\mathrm{s}}_{\varepsilon_0}(z)$. Now, we assume by contradiction that $v \notin E_{\mathrm{s}}(x) = T_x W^{\mathrm{s}}_{\varepsilon_0}(z)$ and we will use the λ-lemma to get the contradiction. Precisely, we are going to prove first that $v \in E_{\mathrm{so}}(x) = T_x W^{\mathrm{so}}_{\varepsilon_0}(z)$ using the λ-lemma and then we will deduce that $v \in E_{\mathrm{s}}(x)$. First, we suppose that $v \notin E_{\mathrm{so}}(x)$ and we consider a $\mathcal{C}^\infty$ disk $\mathcal{D}$ which intersects $W^{\mathrm{so}}_{\varepsilon_0}(z)$ transversally, so that it satisfies the hypothesis of the λ-lemma, such that $\mathcal{D} \cap W^{\mathrm{so}}_{\varepsilon_0}(z) = \{x\}$ and such that $v \in T_x \mathcal{D}$. The λ-lemma implies that $\varphi^k(\mathcal{D})$ is arbitrarily close to the weak-unstable manifold $W^{\mathrm{u}}_{\varepsilon_1}(\varphi^k(z))$ for the $\mathcal{C}^1$ topology when $k \to +\infty$ up to considering a smaller disk $\mathcal{D}$ (of same dimension) containing x, where ε_1 is the constant given in Proposition B.1. In particular, the vector $D\varphi^k(x)v \in D\varphi^k(x)\left(T_x \mathcal{D}\right) = T_{\varphi^k(x)} \varphi^k(\mathcal{D})$ can be made arbitrarily close to $E_{\mathrm{u}}(\varphi^k(z))$ as $k \to +\infty$ in the sense that $D\varphi^k(x)v$ belongs to any conical neighborhood $C_{\mathrm{u}}(\varphi^k(x))$ defined as follows. For all $y \in U$ sufficiently small which is given by Lemma 7.1, we denote by

$$C_{\mathrm{u}}(y) = \Big\{ w = w_{\mathrm{so}} + w_{\mathrm{u}} \in T_y M = \widetilde{E}_{\mathrm{so}}(y) \oplus \widetilde{E}_{\mathrm{u}}(y), |w_{\mathrm{u}}| > |w_{\mathrm{so}}| \Big\}$$

the conical neighborhood of the extended weak-unstable distribution. Here, $T_y M = \widetilde{E}_{\mathrm{so}} \oplus \widetilde{E}_{\mathrm{u}}$ denotes a continuous extension of the hyperbolic splitting of the tangent space in the neighborhood U of K given by Lemma 7.1, which follows from the continuity of the stable/weak-unstable distributions on K [35, Prop. 5.1.4]. Moreover, thanks to the stable manifold theorem, we know that the stable manifold $W^{\mathrm{s}}_{\varepsilon_0}(z')$ at each point $z' \in K$ satisfies by continuity the inclusion $T_y W^{\mathrm{s}}_{\varepsilon_0}(z') \subset C_{\mathrm{s}}(y)$ (which is defined similarly to the weak-unstable cones exchanging u and so) as long as y is close enough to K. Now, we fix k sufficiently large so that $D\varphi^k(x)v \in C_{\mathrm{u}}(\varphi^k(x))$. Since, on one hand, $D\varphi^k(x_\ell)v_\ell$ converges to $D\varphi^k(x)v$, and since we proved that it belongs to $T_{\varphi^k(x_\ell)} W^{\mathrm{s}}_{\varepsilon_0}(\varphi^k(z_\ell)) \subset C_{\mathrm{s}}(\varphi^k(x_\ell))$ on the other hand, we must have by continuity $D\varphi^k(x)v \in \overline{C_{\mathrm{s}}(\varphi^k(x))}$. Finally, putting all together, we obtain $D\varphi^k(x)v \in C_{\mathrm{u}}(\varphi^k(x))$ from the λ-lemma and we obtain $D\varphi^k(x)v \in \overline{C_{\mathrm{s}}(\varphi^k(x))}$ by our continuity argument. Therefore, since $\overline{C_{\mathrm{s}}(\varphi^k(x))} \cap C_{\mathrm{u}}(\varphi^k(x)) = \varnothing$, we get that $v \in E_{\mathrm{so}}(x)$. It remains to prove that $v \in E_{\mathrm{s}}(x)$. To do so, we write $v = v_{\mathrm{s}} + v_{\mathrm{o}} \in E_{\mathrm{s}}(x) \oplus \mathbb{R}V(x)$ $(v_{\mathrm{o}} \neq 0)$ and we see that

$$|D\varphi^t(x)v| = |D\varphi^t(x)v_{\mathrm{s}} + D\varphi^t(x)v_{\mathrm{o}}| \underset{t\to+\infty}{\sim} |D\varphi^t(x)v_{\mathrm{o}}| = |v_{\mathrm{o}}|,$$

where the last equality follows from the choice of an adapted metric on K. Thus, there exists $t_0 \in \mathbb{R}$ such that

$$\forall t \geqslant t_0, \quad D\varphi^t(x)v \in C_{\mathrm{uo}}(\varphi^t(x)).$$

Moreover, since, we also have that $D\varphi^k(x)v \in \overline{C_{\mathrm{s}}(\varphi^k(x))}$ for any sufficiently large integer k, we get that $D\varphi^k(x)v \in \overline{C_{\mathrm{s}}(\varphi^k(x))} \cap C_{\mathrm{uo}}(\varphi^k(x)) = \varnothing$ for such a k. This leads to the expected contradiction and it implies that $v \in E_{\mathrm{s}}(x)$. $\square$

In the next paragraph, we aim to extend the hyperbolic splitting of the tangent space on a neighborhood of the local stable manifold $W^{\mathrm{s}}_\varepsilon(K)$ using the previous lemma together with Lemma 7.1. It will give us a hyperbolic splitting with some invariance properties in the stable manifold and the unstable manifold of K which will be essential in the proof of Proposition 3.3.

If we apply Lemma 7.1 to the continuous section defined in the previous lemma, then we obtain an open set $\mathcal{N}_{\mathrm{s}} \supset \overline{W^{\mathrm{s}}_\varepsilon(K)}$ and an extension $\widetilde{E}_{\mathrm{s}}$ of the distribution $\bigcup_{x\in\overline{W^{\mathrm{s}}_\varepsilon(K)}} E_{\mathrm{s}}(x)$ on $\mathcal{N}_{\mathrm{s}}$. If we replace s by u in the previous construction then we obtain similarly a continuous extension $\widetilde{E}_{\mathrm{u}}$ of $\bigcup_{x\in\overline{W^{\mathrm{u}}_\varepsilon(K)}} E_{\mathrm{u}}(x)$ on a neighborhood $\mathcal{N}_{\mathrm{u}}$ of $\overline{W^{\mathrm{u}}_\varepsilon(K)}$. Next, we define $\widetilde{E}^*_{\mathrm{so}}$ (resp. $\widetilde{E}^*_{\mathrm{uo}}$) by taking the dual orthogonal of $\widetilde{E}_{\mathrm{s}}$ (resp. $\widetilde{E}_{\mathrm{u}}$). The notation $\widetilde{E}^*_{\mathrm{so}}$ can seem a little ambiguous at first, because we extend first and then take the dual orthogonal. Yet, everything is consistent here since $\widetilde{E}^*_{\mathrm{so}}$ also extends continuously the distribution $\bigcup_{x\in\overline{W^{\mathrm{s}}_\varepsilon(K)}} E^*_{\mathrm{so}}(x)$. A similar remark holds for $\widetilde{E}^*_{\mathrm{uo}}$. Moreover, by setting $\widetilde{E}^*_{\mathrm{s}} := \widetilde{E}^*_{\mathrm{so}} \cap \{\xi \in T^*_x M,\ \xi(V(x)) = 0\}$ we obtain a continuous extension of $\bigcup_{x\in\overline{W^{\mathrm{s}}_\varepsilon(K)}} E^*_{\mathrm{s}}(x)$. The different steps can be summarized in the next diagram (which of course also hold if we replace s by u):

$$\bigcup_{x\in\overline{W^{\mathrm{s}}_\varepsilon(K)}} E_{\mathrm{s}}(x) \dashrightarrow \widetilde{E}_{\mathrm{s}} \dashrightarrow \widetilde{E}^*_{\mathrm{so}} \dashrightarrow \widetilde{E}^*_{\mathrm{s}}.$$

Moreover, the distributions $\widetilde{E}^*_{\mathrm{so/s}}$ extend $E^*_{\mathrm{so/s}}$ on a neighborhood of the local stable manifold of K:

$$\forall x \in W^{\mathrm{s}}_\varepsilon(K), \quad \widetilde{E}^*_{\mathrm{so}}(x) = E^*_{\mathrm{so}}(x) \text{ and } \widetilde{E}^*_{\mathrm{s}}(x) = E^*_{\mathrm{s}}(x).$$

Similarly, we have

$$\forall x \in W^{\mathrm{u}}_\varepsilon(K), \quad \widetilde{E}^*_{\mathrm{uo}}(x) = E^*_{\mathrm{uo}}(x) \text{ and } \widetilde{E}^*_{\mathrm{u}}(x) = E^*_{\mathrm{u}}(x).$$

These two last statement will be crucial in the proof of the compactness proposition 3.3.

Finally, in order to extend continuously the neutral direction, we define for all $x \in \mathcal{N} = \mathcal{N}_{\mathrm{s}} \cap \mathcal{N}_{\mathrm{u}}$,

$$\widetilde{E}^*_{\mathrm{o}}(x) := \big\{\xi \in T^*_x M,\ \xi\big(\widetilde{E}_{\mathrm{s}}(x) + \widetilde{E}_{\mathrm{u}}(x)\big) = 0\big\}.$$

Remark 7.4. — It is important to note that for every $x \in W^{\mathrm{s}}_\varepsilon(K) \cap \mathcal{N}$, we have $\widetilde{E}^*_{\mathrm{o}}(x) \subseteq \widetilde{E}^*_{\mathrm{so}}(x)$. Similarly, we have for every $x \in W^{\mathrm{u}}_\varepsilon(K) \cap \mathcal{N}$, the inclusion $\widetilde{E}^*_{\mathrm{o}}(x) \subseteq \widetilde{E}^*_{\mathrm{uo}}(x)$.

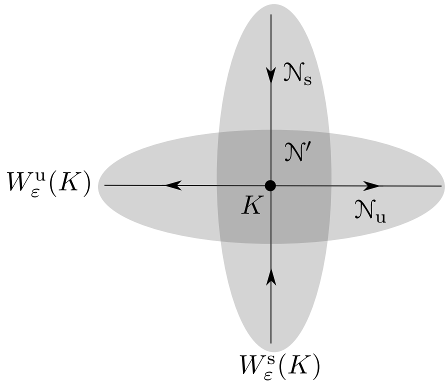


Figure 11. Illustration of some neighborhoods used in the extension of distributions

Up to considering smaller $\mathcal{N}$, we can assume that

$$\widetilde{E}^*_u(x) \oplus \widetilde{E}^*_s(x) \oplus \widetilde{E}^*_o(x) = T^*_x M, \qquad \forall x \in \mathcal{N}. \tag{24}$$

Note that this decomposition of the cotangent space is *not invariant* by the flow Φ^t in general. Despite that, hyperbolic estimates as well as stability of good conical neighborhoods of these bundles should extend by continuity on a neighborhood of K.

7.2.3. *Stability of conical neighborhoods near a basic set*. — We set for all $\delta > 0$ and all $x \in \mathcal{N}$,

$$\begin{aligned} \mathcal{C}^\delta_u(x) &:= \big\{\xi \in T^*_x M,\ \delta|\xi_u| > |\xi_s| + |\xi_o|\big\},\\ \mathcal{C}^\delta_{uo}(x) &:= \big\{x \in T^*_x M,\ \delta(|\xi_u| + |\xi_o|) > |\xi_s|\big\}, \end{aligned} \tag{25}$$

where we used the decomposition (24) on the fibers, i.e., $\xi = \xi_u + \xi_s + \xi_o \in \widetilde{E}^*_u(x) \oplus \widetilde{E}^*_s(x) \oplus \widetilde{E}^*_o(x)$. If we replace s by u in (25), then we can define similarly the stable and weak-stable conical neighborhoods $\mathcal{C}^\delta_s$ and $\mathcal{C}^\delta_{so}$ on $\mathcal{N}$. The main technical statement of this section is:

Lemma 7.5. — *Let K be a basic set and let $\mathcal{N}$ be the open neighborhood appearing in* (24). *There exists $\varepsilon_1 > 0$ such that, for every unrevisited neighborhood $\mathcal{V} \subset \mathcal{N} \cap \varphi^{-1}(\mathcal{N})$ contained in an ε_1-neighborhood of K, the following hold:*

(i) *For every $\delta_0 \in (0,1]$, one can find $m_{\delta_0} \geqslant 0$ so that, for every $\delta \in [0,\delta_0]$, for every $m \geqslant m_{\delta_0}$ and for every $x \in \mathcal{V} \cap \varphi^m(\mathcal{V})$, one has the inclusions*

$$\Phi^1\left(\mathcal{C}^\delta_u(x)\right) \subseteq \mathcal{C}^{\delta'}_u(\varphi^1(x)), \qquad \Phi^1\left(\mathcal{C}^\delta_{uo}(x)\right) \subseteq \mathcal{C}^{\delta'}_{uo}(\varphi^1(x)) \tag{26}$$

where $\delta' = e^{-\lambda/3}\delta$ and with λ being the constant appearing in the definition (22).

(ii) *Moreover, there exists $\delta_1 \in (0,1]$ such that for every $\delta \leqslant \delta_1$, one can find $m_\delta \geqslant 0$ so that, for every $m \geqslant m_\delta$, for every $x \in \mathcal{V} \cap \varphi^m(\mathcal{V})$ and for every $\xi \in C^\delta_u(x)$, we have*

$$|\Phi^1(x,\xi)| \geqslant e^{\lambda/4}|\xi|.$$

Proof. — The first step of the proof consists in extending the hyperbolic estimates on the unstable manifold of K. Let us denote by π_α the continuous projector on $\widetilde{E}_\alpha^*$ for each $\alpha\in\{\mathrm{s},\mathrm{o},\mathrm{u}\}$ and let us define

$$\forall \alpha,\beta\in\{\mathrm{s},\mathrm{o},\mathrm{u}\},\quad A_{\alpha\beta}:=\pi_\alpha\circ\Phi^1\circ\pi_\beta \text{ and } B_{\alpha\beta}:=\pi_\alpha\circ\Phi^{-1}\circ\pi_\beta.$$

We will also use the notation ξ_α^1 for $\pi_\alpha\circ\Phi^1(\xi)$. For every $x\in\mathcal{N}\cap\varphi^{-1}(\mathcal{N})$, we have $\xi=\xi_\mathrm{u}+\xi_\mathrm{s}+\xi_\mathrm{o}\in\widetilde{E}_\mathrm{u}^*(x)\oplus\widetilde{E}_\mathrm{s}^*(x)\oplus\widetilde{E}_\mathrm{o}^*(x)=T_x^*M$ and $\Phi_x^1(\xi)=\xi_\mathrm{u}^1+\xi_\mathrm{s}^1+\xi_\mathrm{o}^1\in\widetilde{E}_\mathrm{u}^*(\varphi^1(x))\oplus\widetilde{E}_\mathrm{s}^*(\varphi^1(x))\oplus\widetilde{E}_\mathrm{o}^*(\varphi^1(x))$. The dynamics of Φ^1 and Φ^{-1} can be encode within the two following matrices of linear morphisms:

$$\begin{pmatrix}\xi_\mathrm{u}^1\\ \xi_\mathrm{o}^1\\ \xi_\mathrm{s}^1\end{pmatrix}=\begin{pmatrix}A_\mathrm{uu} & A_\mathrm{ou} & A_\mathrm{su}\\ A_\mathrm{uo} & A_\mathrm{oo} & A_\mathrm{so}\\ A_\mathrm{us} & A_\mathrm{os} & A_\mathrm{ss}\end{pmatrix}\begin{pmatrix}\xi_\mathrm{u}\\ \xi_\mathrm{o}\\ \xi_\mathrm{s}\end{pmatrix}\qquad\text{and}\qquad\begin{pmatrix}\xi_\mathrm{u}\\ \xi_\mathrm{o}\\ \xi_\mathrm{s}\end{pmatrix}=\begin{pmatrix}B_\mathrm{uu} & B_\mathrm{ou} & B_\mathrm{su}\\ B_\mathrm{uo} & B_\mathrm{oo} & B_\mathrm{so}\\ B_\mathrm{us} & B_\mathrm{os} & B_\mathrm{ss}\end{pmatrix}\begin{pmatrix}\xi_\mathrm{u}^1\\ \xi_\mathrm{o}^1\\ \xi_\mathrm{s}^1\end{pmatrix}.$$

To be more precise, we should have written $A_{\alpha\beta}(x)$ and $B_{\alpha\beta}(\varphi^1(x))$ to indicate that $\xi\in T_x^*M$. In the particular case where $x\in W^\mathrm{u}(K)\cap\mathcal{N}\cap\varphi^{-1}(\mathcal{N})$, both matrices are upper triangular block matrices thanks to our definition of $\widetilde{E}_\mathrm{u}^*$, $\widetilde{E}_\mathrm{o}^*$ and $\widetilde{E}_\mathrm{s}^*$, i.e.,

$$A_\mathrm{uo}=A_\mathrm{us}=A_\mathrm{os}=B_\mathrm{uo}=B_\mathrm{us}=B_\mathrm{os}=0\quad\text{on } W^\mathrm{u}(K)\cap\mathcal{N}\cap\varphi^{-1}(\mathcal{N}).$$

However, on the whole open set $\mathcal{N}\cap\varphi^{-1}(\mathcal{N})$ this is not the case anymore.[(10)] Now, we extend the hyperbolic estimates (23) on the unstable manifold of K. For every $\varepsilon\geqslant 0$, we define the set $\mathcal{N}(\varepsilon)$ of points $y\in\mathcal{N}\cap\varphi^{-1}(\mathcal{N})$ on which we have for all $\xi\in T_y^*M$,

$$\begin{gathered}|A_\mathrm{uu}(\xi_\mathrm{u})|\geqslant(e^{\lambda/2}-\varepsilon)|\xi_\mathrm{u}|,\quad |B_\mathrm{ss}(\xi_\mathrm{s}^1)|\geqslant(e^{\lambda/2}-\varepsilon)|\xi_\mathrm{s}^1|,\\ |A_\mathrm{oo}(\xi_\mathrm{o})|\geqslant(1-\varepsilon)|\xi_\mathrm{o}|,\quad |B_\mathrm{oo}(\xi_\mathrm{o}^1)|\geqslant(1-\varepsilon)|\xi_\mathrm{o}^1|\\ \|A_\mathrm{ou}\|\leqslant\varepsilon,\quad \|A_\mathrm{su}\|\leqslant\varepsilon,\quad \|A_\mathrm{so}\|\leqslant\varepsilon,\\ \|B_\mathrm{ou}\|\leqslant\varepsilon,\quad \|B_\mathrm{su}\|\leqslant\varepsilon,\quad \|B_\mathrm{so}\|\leqslant\varepsilon,\\ A_\mathrm{uo}=A_\mathrm{us}=A_\mathrm{os}=B_\mathrm{uo}=B_\mathrm{us}=B_\mathrm{os}=0,\end{gathered}\tag{27}$$

where $\|.\|$ denotes the operator norm defined by:

$$\forall\alpha,\beta\in\{\mathrm{u},\mathrm{o},\mathrm{s}\},\quad \|A_{\alpha\beta}\|:=\sup_{\xi_\alpha\in\widetilde{E}_\alpha^*(y)\smallsetminus\{0\}}\frac{|A_{\alpha\beta}(\xi_\alpha)|}{|\xi_\alpha|}.$$

Note that the map $A_{\alpha\beta}$ and the norm $\|.\|$ depend on the point y, but we will not precise the point y if everything is clear. For every $\varepsilon>0$ and for every unrevisited neighborhood $\mathcal{V}$ sufficiently close to K, we have $\mathcal{V}\cap W^\mathrm{u}(K)\subset\mathcal{N}(\varepsilon)$. Let first check that the inclusions (26) hold on the unstable manifold of K for some $\delta'\leqslant\delta e^{-\lambda/2}$ as soon as ε is sufficiently small. The result on the whole neighborhood will then follow by continuity.

(10) This follows from the fact that in the general case (where $\widetilde{E}_\mathrm{s}^*$ and $\widetilde{E}_\mathrm{u}$ are both non trivial) none of the extended distribution $\widetilde{E}_\mathrm{s}^*$, $\widetilde{E}_\mathrm{u}^*$ or $\widetilde{E}_\mathrm{o}^*$ are invariant by Φ^t on $\mathcal{N}$.

First inclusion, on unstable cones $\mathcal{C}_{\mathrm{u}}^{\delta}$. — Let us fix $\delta \in (0,1]$, $x \in W^{\mathrm{u}}(K) \cap \mathcal{N} \cap \varphi^{-1}(\mathcal{N})$ and $\xi \in \mathcal{C}_{\mathrm{u}}^{\delta}(x)$. Our goal is to find a parameter $\delta' \leqslant \delta e^{-\lambda/2}$ such that $\xi^1 := \Phi_x^1(\xi) \in \widetilde{\mathcal{C}}_{\mathrm{u}}^{\delta'}(\varphi^1(x))$. The first step consists in computing $|\xi_{\mathrm{u}}^1|^2$ in order to find a lower bound which depends on $|\xi_{\mathrm{s}}|$ and $|\xi_{\mathrm{o}}|$. Precisely, we have

$$\begin{aligned}\delta|\xi_{\mathrm{u}}^1| = \delta|A_{\mathrm{uu}}\xi_{\mathrm{u}} + A_{\mathrm{su}}\xi_{\mathrm{s}} + A_{\mathrm{ou}}\xi_{\mathrm{o}}| &\geqslant \delta|A_{\mathrm{uu}}\xi_{\mathrm{u}}| - \delta|A_{\mathrm{su}}\xi_{\mathrm{s}} + A_{\mathrm{ou}}\xi_{\mathrm{o}}|\\ &\geqslant (e^{\lambda/2}-\varepsilon)\delta|\xi_{\mathrm{u}}| - \delta\varepsilon(|\xi_{\mathrm{s}}|+|\xi_{\mathrm{o}}|)\\ &> (e^{\lambda/2}-\varepsilon-\delta\varepsilon)(|\xi_{\mathrm{s}}|+|\xi_{\mathrm{o}}|) =: C_1(\varepsilon)(|\xi_{\mathrm{s}}|+|\xi_{\mathrm{o}}|).\end{aligned} \tag{28}$$

The second inequality is obtained thanks to the estimate on A_{uu} in (27) and the third one follows from our choice of $\xi \in \mathcal{C}_{\mathrm{u}}^{\delta}(x)$. Note that we implicitly show the following estimate which will imply the exponential estimate (as we will see later on) for $\delta \leqslant 1$:

$$|\xi_{\mathrm{u}}^1| > C_1(\varepsilon)|\xi_{\mathrm{u}}| \geqslant \big(e^{\lambda/2}-2\varepsilon\big)|\xi_{\mathrm{u}}|. \tag{29}$$

Now, let us do a similar computation for the matrix B. Again using Cauchy-Schwarz inequality on J and using again estimates (27), we deduce

$$\begin{aligned}|\xi_{\mathrm{s}}|+|\xi_{\mathrm{o}}| = |B_{\mathrm{ss}}\xi_{\mathrm{s}}^1| + |B_{\mathrm{so}}\xi_{\mathrm{s}}^1 + B_{\mathrm{oo}}\xi_{\mathrm{o}}^1| &\geqslant |B_{\mathrm{ss}}\xi_{\mathrm{s}}^1| + |B_{\mathrm{oo}}\xi_{\mathrm{o}}^1| - |B_{\mathrm{so}}\xi_{\mathrm{s}}^1|\\ &\geqslant (e^{\lambda/2}-\varepsilon)|\xi_{\mathrm{s}}^1| + (1-\varepsilon)|\xi_{\mathrm{o}}^1| - \varepsilon|\xi_{\mathrm{s}}^1|\\ &\geqslant (1-\varepsilon)(|\xi_{\mathrm{s}}^1|+|\xi_{\mathrm{o}}^1|) =: C_2(\varepsilon)(|\xi_{\mathrm{s}}^1|+|\xi_{\mathrm{o}}^1|).\end{aligned} \tag{30}$$

where the last inequality holds for ε small enough such that $e^{\lambda/2}-2\varepsilon \geqslant 1-\varepsilon$. Putting together equations (28) and (30), we deduce

$$\delta|\xi_{\mathrm{u}}^1| > C_1(\varepsilon)C_2(\varepsilon)(|\xi_{\mathrm{s}}^1|+|\xi_{\mathrm{o}}^1|).$$

Finally, we obtain $\xi^1 = \Phi_x^1(\xi) \in \mathcal{C}_{\mathrm{u}}^{\delta'}(\varphi^1(x))$ for

$$\delta'(\varepsilon) = \frac{\delta}{C_1(\varepsilon)C_2(\varepsilon)},$$

where the polynomial functions C_1 and C_2 (with respect to the variable ε) satisfy $C_1(\varepsilon) \leqslant e^{\lambda}$ for every $\varepsilon > 0$ and

$$C_1(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} e^{\lambda/2} \quad\text{and}\quad C_2(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 1.$$

If we fix ε sufficiently small so that

$$C_1(\varepsilon)C_2(\varepsilon) \geqslant e^{\lambda/3}, \quad C_1(\varepsilon) \geqslant e^{\lambda/3}, \tag{31}$$

then we get for every $\delta \in (0,1]$ and for every $x \in \mathcal{N}(\varepsilon) \cap W^{\mathrm{u}}(K)$ the inclusion

$$\Phi^1\left(\mathcal{C}_{\mathrm{u}}^{\delta}(x)\right) \subseteq \mathcal{C}_{\mathrm{u}}^{\delta'}(\varphi^1(x))$$

for $\delta'(\varepsilon) \leqslant \delta e^{-\lambda/3}$. Moreover, we deduce from (29) the bound

$$|\xi_{\mathrm{u}}^1| \geqslant e^{\lambda/3}|\xi_{\mathrm{u}}| \tag{32}$$

Second inclusion, on weak unstable cones $\mathcal{C}^{\delta}_{\mathrm{uo}}$. — Let us fix $x \in W^{\mathrm{u}}(K) \cap \mathcal{N} \cap \varphi^{-1}(\mathcal{N})$ and $\xi \in \mathcal{C}^{\delta}_{\mathrm{uo}}(x)$. By definition of the conical neighborhood, we have $\delta(|\xi_{\mathrm{u}}|+|\xi_{\mathrm{o}}|) > |\xi_{\mathrm{s}}|$. The idea of the proof is very similar to the proof of the inclusion for the unstable conical neighborhood $\mathcal{C}^{\delta}_{\mathrm{u}}$. However, we won't have an exponential estimate because of the neutral direction $\widetilde{E}^{*}_{\mathrm{o}}$. Let us check the computations for this case. Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|\xi^1_{\mathrm{u}}| + |\xi^1_{\mathrm{o}}| &= |A_{\mathrm{uu}}\xi_{\mathrm{u}} + A_{\mathrm{su}}\xi_{\mathrm{s}} + A_{\mathrm{ou}}\xi_{\mathrm{o}}| + |A_{\mathrm{so}}\xi_{\mathrm{s}} + A_{\mathrm{oo}}\xi_{\mathrm{o}}| \\
&\geqslant |A_{\mathrm{uu}}\xi_{\mathrm{u}}| - |A_{\mathrm{su}}\xi_{\mathrm{s}}| - |A_{\mathrm{ou}}\xi_{\mathrm{o}}| + |A_{\mathrm{oo}}\xi_{\mathrm{o}}| - |A_{\mathrm{so}}\xi_{\mathrm{s}}| \\
&\geqslant (e^{\lambda/2} - \varepsilon)|\xi_{\mathrm{u}}| + (1-\varepsilon)|\xi_{\mathrm{o}}| - 2\varepsilon|\xi_{\mathrm{s}}| \\
&\geqslant (1-\varepsilon)(|\xi_{\mathrm{u}}| + |\xi_{\mathrm{o}}|) - 2\varepsilon|\xi_{\mathrm{s}}|.
\end{aligned}$$

Now, multiplying by $\delta \leqslant 1$ on both side of the previous inequalities and using the fact that $\xi \in \mathcal{C}^{\delta}_{\mathrm{uo}}(x)$, we get

$$\delta(|\xi^1_{\mathrm{u}}| + |\xi^1_{\mathrm{o}}|) \geqslant (1-3\varepsilon)|\xi_{\mathrm{s}}| =: C_3(\varepsilon)|\xi_{\mathrm{s}}|. \tag{33}$$

It remains to find a lower bound for $|\xi_{\mathrm{s}}|$. This case is much simpler since we have

$$|\xi_{\mathrm{s}}| = |B_{\mathrm{ss}}\xi^1_{\mathrm{s}}| \geqslant (e^{\lambda/2} - \varepsilon)|\xi^1_{\mathrm{s}}| =: C_4(\varepsilon)|\xi^1_{\mathrm{s}}|. \tag{34}$$

Putting together equations (33) and (34), we deduce that

$$\delta(|\xi^1_{\mathrm{u}}| + |\xi^1_{\mathrm{o}}|) \geqslant C_3(\varepsilon)C_4(\varepsilon)|\xi^1_{\mathrm{s}}|.$$

Therefore, we have $\xi^1 = \Phi^1_x(\xi) \in \mathcal{C}^{\delta'}_{\mathrm{u}}(\varphi^1(x))$ for

$$\delta'(\varepsilon) = \frac{\delta}{C_3(\varepsilon)C_4(\varepsilon)},$$

where the polynomial functions C_3 and C_4 (w.r.t the variable ε) satisfy $|C_3(\varepsilon)| \leqslant 1$ for all $\varepsilon \in (0, 1/3]$, $C_3(\varepsilon) \underset{\varepsilon\to 0}{\to} 1$ and $C_4(\varepsilon) \underset{\varepsilon\to 0}{\to} e^{\lambda}$. For ε sufficiently small so that

$$C_3(\varepsilon)C_4(\varepsilon) \geqslant e^{\lambda/3}, \tag{35}$$

we get for every $\delta \in [0,1]$ and for every $x \in \mathcal{N}(\varepsilon) \cap W^{\mathrm{u}}(K)$ the inclusion

$$\Phi^1\left(\mathcal{C}^{\delta}_{\mathrm{uo}}(x)\right) \subseteq \mathcal{C}^{\delta'}_{\mathrm{uo}}(\varphi^1(x)),$$

with $\delta'(\varepsilon) \leqslant \delta e^{-\lambda/3}$. It ends the proof of the inclusions along the unstable manifolds.

Now, fix a value of $\varepsilon > 0$ sufficiently small so that (31) and (35) are satisfied. There exists $\varepsilon_1 > 0$ such that $\mathcal{V} \cap W^{\mathrm{u}}(K) \subset \mathcal{N}(\varepsilon)$ hold for every unrevisited neighborhood $\mathcal{V}$ contained in a ε_1-neighborhood of K. Let $\mathcal{V}$ be an unrevisited neighborhood contained in a ε_1-neighborhood of K. Thanks to our choice of ε, the inclusions (26) are satisfied on $\mathcal{V} \cap W^{\mathrm{u}}(K)$ and we would like to extend them to $\mathcal{V} \cap \varphi^m(\mathcal{V})$ as stated in (26).

Recall that each point of $\mathcal{V} \cap \varphi^m(\mathcal{V})$ converges to the compact set $\overline{\mathcal{V}} \cap W^{\mathrm{u}}(K)$ as $m \to +\infty$ according to Lemma 2.16. The remaining of the proof consists in extending the inclusions in (26) by continuity on $\mathcal{V}\cap\varphi^m(\mathcal{V})$ for m large enough. Fix $\delta_0\in(0,1]$. By continuity of the maps

$$F(x,\xi)=\Big(\frac{|\xi_{\rm s}|+|\xi_{\rm o}|}{|\xi_{\rm u}|}\Big)^{-1}\Big(\frac{|\xi^1_{\rm s}|+|\xi^1_{\rm o}|}{|\xi^1_{\rm u}|}\Big),\quad G(x,\xi)=\frac{|\xi^1_{\rm s}|+|\xi^1_{\rm o}|}{|\xi^1_{\rm u}|},$$
$$\widetilde F(x,\xi)=\Big(\frac{|\xi_{\rm s}|}{|\xi_{\rm u}|+|\xi_{\rm o}|}\Big)^{-1}\Big(\frac{|\xi^1_{\rm s}|}{|\xi^1_{\rm u}|+|\xi^1_{\rm o}|}\Big),\quad \widetilde G(x,\xi)=\frac{|\xi^1_{\rm s}|}{|\xi^1_{\rm u}|+|\xi^1_{\rm o}|},$$

there exists an integer $m_{\delta_0}\in\mathbb{N}$ such that the claimed inclusions are satisfied on $\mathcal{V}\cap\varphi^{m_{\delta_0}}(\mathcal{V})$ for the parameter $\delta'=\delta e^{-\lambda/4}$. This ends the proof of (i).

Exponential estimate (proof of (ii)*).* — Applying (i) to $\delta=\delta_0\in(0,1]$, we get the existence of $m_\delta\in\mathbb{N}$ such that the inclusions (26) are satisfied on $\mathcal{V}\cap\varphi^{m_\delta}(\mathcal{V})$. Extending (32) by continuity, we can find $m_0\in\mathbb{N}$ such that for every $x\in\mathcal{V}\cap\varphi^{m_0}(\mathcal{V})$ and for every $\xi\in\mathcal{C}^\delta_{\rm u}(x)$, we have

$$|\xi^1_u|\geqslant e^{2\lambda/7}|\xi_u|.$$

Up to considering a larger m_δ, we can assume that $m_\delta\geqslant m_0$. The exponential estimate follows by equivalence of the norms $|\xi|_u:=|\xi_u|$ and $|.|$ on the conical neighborhoods $\mathcal{C}^\delta_{\rm u}(x)$ and $\mathcal{C}^\delta_{\rm u}(\varphi^1(x))$. Indeed, we have $\xi^1\in\mathcal{C}^\delta_{\rm u}(\varphi^1(x))$ thanks to our previous analysis and therefore

$$|\Phi^1(x,\xi)|=|\xi^1|\geqslant\frac{e^{2/7\lambda}}{1+\delta}|\xi|\geqslant e^{\lambda/4}|\xi|.$$

for every $\delta\leqslant\delta_1:=\min(1,e^{2/7\lambda}-1)$ and every $x\in\mathcal{V}\cap\varphi^{m_\delta}(\mathcal{V})$. □

The following corollary states in a quantitative manner that, if the trajectory of a point (x,ξ) stays for a long time near a basic set, then the fiber part of $\Phi^t(x,\xi)$ gets attracted to the $\widetilde E^*_{\rm u}$ or $\widetilde E^*_{\rm uo}$ distribution:

Corollary 7.6. — *Let K be a basic set and let $\mathcal{N}$ be the open neighborhood appearing in* (24)*. There exists $\varepsilon_1>0$ such that, for every unrevisited neighborhood $\mathcal{V}\subset\varphi^1(\mathcal{N})\cap\varphi^{-1}(\mathcal{N})$ contained in an ε_1-neighborhood of K, the following hold:*

– *For every $\delta,\delta'\in(0,1]$ satisfying $\delta'\leqslant\delta$, one can find $m_0\geqslant 0$ such that, for every $m\geqslant m_0$ and for every $x\in\varphi^{-m}(\mathcal{V})\cap\mathcal{V}$, one has*

$$\xi\notin\mathcal{C}^\delta_{\rm so/s}(x)\implies\Phi^m(x,\xi)\in\mathcal{C}^{\delta'}_{\rm u/uo}(\varphi^m(x)).$$

– *For every $\delta\in(0,1]$, there exist $C>0$, $m_1\in\mathbb{N}$ such that, for every $m\geqslant m_1$, for every $x\in\varphi^{-m}(\mathcal{V})\cap\mathcal{V}$ and for every $\xi\in T^*_xM$ such that $\xi\notin\mathcal{C}^\delta_{\rm so}(x)$, we have the inequality*

$$|\Phi^m(\xi)|\geqslant Ce^{m\lambda/4}|\xi|,\tag{36}$$

with λ being the constant appearing in the definition (22)*.*

The idea of the proof is the following. Thanks to Lemma 7.5, we know that the conical neighborhoods are Φ^1-stable near the unstable manifold of the basic set, so our goal is to use the different unrevisited neighborhoods represented in Figure 7 to show that we can iterate the lemma.

Proof of the corollary. — Fix $\delta, \delta' \in (0,1]$ satisfying $\delta' \leqslant \delta$ and let $\mathcal{V}$ be some unrevisited ε_1-neighborhood of K contained in $\varphi^1(\mathcal{N}) \cap \varphi^{-1}(\mathcal{N})$ so that we can apply Lemma 7.5 for the flows φ^t and φ^{-t} (with ε_1 given by the lemma). Doing so for the flows φ^t and φ^{-t}, we obtain the existence of an integer $m_{\delta'}$ as in the statement of the lemma (we take $\delta_0 := \delta'$ and we assume that $m_{\delta'}$ is the same integer obtained for φ^t and φ^{-t}). Now, fix an even integer $m_0 \geqslant 2m_{\delta'}$ which will be chosen sufficiently large later on. For all $m \geqslant m_0$ and all $x \in \mathcal{V} \cap \varphi^{-m}(\mathcal{V})$, we have the inclusions

$$\forall k \in [\![0, m_0/2]\!], \quad \varphi^k(x) \in \varphi^k(\mathcal{V}) \cap \varphi^{-m+k}(\mathcal{V}) \subset \mathcal{V} \cap \varphi^{-m_0/2}(\mathcal{V})$$

and

$$\forall k \in [\![m_0/2, m]\!], \quad \varphi^k(x) \in \varphi^k(\mathcal{V}) \cap \varphi^{-m+k}(\mathcal{V}) \subset \varphi^{m_0/2}(\mathcal{V}) \cap \mathcal{V},$$

see Figure 7. The remaining of the proof consists in interacting m times the lemma for $(x,\xi) \in T^*M$ such that $x \in \mathcal{V} \cap \varphi^{-m}(\mathcal{V})$ and $\xi \notin \mathcal{C}^{\delta}_{\mathrm{so/s}}(x)$: $m_0/2$ times for the backward flow φ^{-t} when the orbit of x belongs to $\mathcal{V} \cap \varphi^{-m_0/2}(\mathcal{V})$ and $m - m_0/2$ times for φ^t when the orbit of x goes in $\varphi^{m_0/2}(\mathcal{V}) \cap \mathcal{V}$. For m sufficiently large, we will get the result. Let us consider (x,ξ) as above. Two analyses are needed: first for $0 \leqslant k \leqslant m_0/2$ and then for $m_0/2 \leqslant k \leqslant m$.

– First, since $\varphi^k(x) \in \mathcal{V} \cap \varphi^{-m_0/2}(\mathcal{V}) \subset \mathcal{V} \cap \varphi^{-m_{\delta'}}(\mathcal{V})$ for all $k \in [\![0, m_0/2-1]\!]$, a straight application of the lemma for the flow φ^{-t} instead of φ^t yields

$$\xi \notin \overline{\mathcal{C}^{\delta}_{\mathrm{so/s}}(x)} \implies \Phi^{m_0/2}(x,\xi) \notin \overline{\mathcal{C}^{\delta_{m_0/2}}_{\mathrm{so/s}}(\varphi^{m_0/2}(x))}$$

where $\delta_{m_0/2} = \min(e^{\lambda m_0/8}\delta, 1)$. In particular, if we choose $m_0 \in \mathbb{N}$ large enough so that

$$e^{-m_0\lambda/8} \leqslant \delta' \leqslant \delta,$$

then we get $\delta_{m_0/2} = 1$,

$$\Phi^{m_0/2}(x,\xi) \notin \overline{\mathcal{C}^{1}_{\mathrm{so/s}}(\varphi^{m_0/2}(x))} \quad \text{and thus } \Phi^{m_0/2}(x,\xi) \in \mathcal{C}^{1}_{\mathrm{u/uo}}(\varphi^{m_0/2}(x)).$$

– For every $k \in [\![m_0/2, m-1]\!]$, we can apply the lemma[11] to the flow φ^t and it yields

$$\varphi^k(x) \in \mathcal{V} \cap \varphi^{m_{\delta'}}(\mathcal{V}) \quad \text{and} \quad \Phi^k(x,\xi) \in \mathcal{C}^{\mu^{k-m_0/2}}_{\mathrm{u/uo}}(\varphi^k(x)) \implies \Phi^{k+1}(x,\xi) \in \mathcal{C}^{\mu^{k+1-m_0/2}}_{\mathrm{u/uo}}(\varphi^{k+1}(x)),$$

with $\mu = e^{-\lambda/4}$. Finally, we obtain $\Phi^m(x,\xi) \in \mathcal{C}^{\mu^{m-m_0/2}}_{\mathrm{u/uo}}(\varphi^m(x))$. Thanks to our choice of m and m_0, we have $\mu^{m-m_0/2} \leqslant \mu^{m_0/2} = e^{-m_0\lambda/8} \leqslant \delta'$. This gives the first point.

To prove the exponential estimate (36), we first need to introduce the constant $\delta_1 \in (0,1]$ given by Lemma 7.5(ii). From the above analysis, there exists an integer $m_1 \geqslant m_0$ such that for every $m \geqslant m_1$ and for every $k \in [\![m_1, m]\!]$, we have $\Phi^k(x,\xi) \in \mathcal{C}^{\delta_1}_{\mathrm{u/uo}}(\varphi^k(x))$. The result is a direct application of Lemma 7.5(ii) and the constant C is obtained by continuity of Φ^{m_1}. □

[11] Note that we use here the uniformity in $\delta \in [\delta', 1]$ stated in Lemma 7.5.

7.3. Proof of proposition 3.3: compactness. — We are now in position to prove the compactness of $\Sigma_{u/uo}$. Compactness of $\Sigma_{s/so}$ can be proved similarly if we exchange φ^1 with φ^{-1}. Let $((x_m,\xi_m))_{m\in\mathbb{N}}$ be a sequence of elements in $\Sigma_{u/uo}$. Up to extraction of a subsequence, we can assume that there exists an integer $j\in[\![1,N]\!]$ such that every point x_m belongs to $W^s(K_j)$. Up to another extraction, we can suppose that $((x_m,\xi_m))_m$ has a limit in S^*M that we denote by $(x_\infty,\xi_\infty)\in S^*M$. Our goal will be to prove that (x_∞,ξ_∞) belongs to $\Sigma_{u/uo}$. First of all, we know from our assumption on x_m that the limit x_∞ must lie in the closure of the set $W^s(K_j)$, i.e., $x_\infty\in\overline{W^s(K_j)}$. Since $\overline{W^s(K_j)}=\bigcup_{j_0,K_{j_0}\leqslant K_j}W^s(K_{j_0})$, we can find some integer $j_0\leqslant j$ such that $x_\infty\in W^s(K_{j_0})$. Now, let us assume by contradiction that $(x_\infty,\xi_\infty)\notin\Sigma_{u/uo}$.

To obtain a contradiction, we split the analysis in four steps. First, we deal with the case where $j=j_0$. In steps 2–4, we treat the case $j\neq j_0$. Precisely, in the second step, we construct by induction a family of integers $j_0<j_1<\cdots<j_\ell=j$ such that for each $k\in[\![1,\ell]\!]$ one can find an element $(x_\infty^{(k)},\xi_\infty^{(k)})$ with $x_\infty^{(k)}\in W^s(K_{j_k})\cap W^u(K_{j_{k-1}})$ and a sequence $(x_m^{(k)},\xi_m^{(k)})=\widetilde{\Phi}^{\tau_{m,k}}(x_m,\xi_m)$, for some parameter $\tau_{m,k}\geqslant 0$ satisfying $\tau_{m,k+1}-\tau_{m,k}\underset{m\to+\infty}{\to}+\infty$, which accumulates to $(x_\infty^{(k)},\xi_\infty^{(k)})$ as m tends to infinity. In a third step, we apply our previous analysis near the basic sets of Corollary 7.6 to prove that if $(x_\infty^{(k)},\xi_\infty^{(k)})\notin\Sigma_{u/uo}$, then $(x_\infty^{(k+1)},\xi_\infty^{(k+1)})$ must belong to $\Sigma_{so/s}$. In the last step, we deduce the expected contradiction.

Step 1: we assume that $j=j_0$. — The contradiction follows from an application of Lemma 7.3 and more precisely from the continuity of $\Sigma_{u/uo}$ on $W^s_\varepsilon(K_j)$ for $\varepsilon<\varepsilon_0$ as in the statement of Lemma 7.3 (for example $\varepsilon=\varepsilon_0/2$), where ε_0 is given by the stable manifold theorem. Let $\mathcal{V}$ be an unrevisited neighborhood of K_j such that $\mathcal{V}\subset B_g(K_j,\varepsilon')=\{z\in M,\ d_g(z,K_j)<\varepsilon'\}$ for $\varepsilon'>0$ which will be chosen sufficiently small. If $x_\infty\in W^s(K_j)$, then there exists $T(\varepsilon')\geqslant 0$ such that $\varphi^{T(\varepsilon')}(x_\infty)\in W^s(K_j)\cap\mathcal{V}$ (by definition of the stable set). Since $x_m\in W^s(K_j)$ and since the sequence $(x_m)_{m\in\mathbb{N}}$ converges to x_∞ as m tends to $+\infty$, we can find an integer $m_0(\varepsilon')\geqslant 0$ such that $\varphi^{T(\varepsilon')}(x_m)\in W^s(K_j)\cap\mathcal{V}$ for every $m\geqslant m_0(\varepsilon')$. Since $\mathcal{V}$ is unrevisited, we get that

$$\forall m\in[\![m_0(\varepsilon'),+\infty]\!],\ \forall k\geqslant 0,\quad \varphi^{T(\varepsilon')+k}(x_m)\in W^s(K_j)\cap\overline{\mathcal{V}}.$$

In particular, we get that

$$\forall m\in[\![m_0(\varepsilon'),+\infty]\!],\ \forall k\geqslant 0,\quad d_g\big(\varphi^{T(\varepsilon')+k}(x_m),K_j\big)\leqslant\varepsilon'.$$

As an application of the shadowing lemma [35, Th. 5.3.3] similar to the one used in the proof of the In-Phase theorem [35, Th. 5.3.25], we get that $\varphi^{T(\varepsilon')}(x_m)\in W^s_\varepsilon(K_j)$ for all $m\in[\![m_0(\varepsilon'),+\infty]\!]$ as soon as ε' is sufficiently small (depending on ε).

From the continuity of $\Sigma_{u/uo}$ on $W^s_\varepsilon(K_j)$, we deduce that (x_∞,ξ_∞) must belong to $\Sigma_{u/uo}$. Therefore, we get

$$(x_\infty,\xi_\infty)\in\Sigma_{u/uo},$$

which is in contradiction with the assumption on (x_∞,ξ_∞). *From now on*, we assume that $j_0<j$.

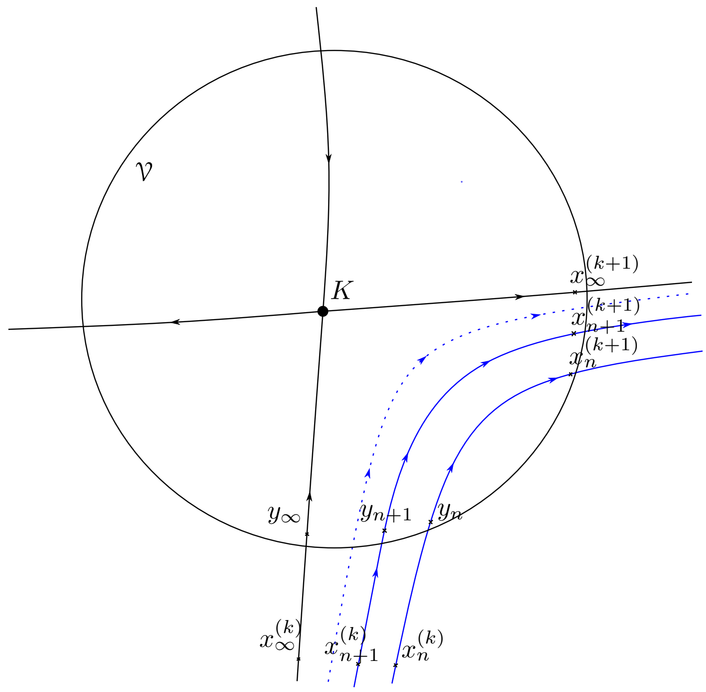


Figure 12. Illustration of some element used in the proof

Step 2: construction of the sequence by induction. — Since j_0 has already been defined, we just have to set $(x_\infty^{(0)}, \xi_\infty^{(0)}) := (x_\infty, \xi_\infty)$ to end the base case. Now, let us assume the integers $j_1 < j_2 < \cdots < j_k (< j)$, the elements of the unitary cotangent bundle $(x_\infty^{(1)}, \xi_\infty^{(1)}), \ldots, (x_\infty^{(k)}, \xi_\infty^{(k)})$ and the parameters $\tau_{m,1}, \ldots, \tau_{m,k}$ are constructed as in the above discussion. Our goal is to define a sequence $(x_m^{(k+1)}, \xi_m^{(k+1)}) := \widetilde{\Phi}^{\tau_{m,k+1}}(x_m, \xi_m)$, the family $\tau_{m,k+1} \geqslant 0$ such that $\tau_{m,k+1} - \tau_{m,k}$ tends to $+\infty$ as m goes to $+\infty$ and to exhibit a new accumulation point $(x_\infty^{(k+1)}, \xi_\infty^{(k+1)})$ with $x_\infty^{(k+1)} \in W^{\mathrm{u}}(K_{j_k}) \cap W^{\mathrm{s}}(K_{j_{k+1}})$.

Definition of $\tau_{m,k+1}$. — Let us consider an unrevisited neighborhood $\mathcal{V}$ of K_{j_k} so that $\overline{\mathcal{V}} \cap \Omega = K_{j_k}$ and sufficiently small to be in the range of application of Corollary 7.6. Recall that $x_\infty^{(k)}$ belongs to $W^{\mathrm{s}}(K_{j_k})$, so there exists $T > 0$ such that $\varphi^T(x_\infty^{(k)}) \in W^{\mathrm{s}}(K_{j_k}) \cap \mathcal{V}$. Since every $x_1^{(k)}, x_2^{(k)}, \ldots, x_m^{(k)}, \ldots$ are in $W^{\mathrm{s}}(K_j)$ and since the sequence converges to $x_\infty^{(k)}$ as $m \to +\infty$, there exists an integer $m_0 \geqslant 0$ such that for every $m \geqslant m_0$ we have $\varphi^T(x_m^{(k)}) \in \mathcal{V}$. To lighten notations, let us denote by y_m the point $\varphi^T(x_m^{(k)})$ and by η_m the cotangent vector $\kappa((D\varphi^T(x_m^{(k)})^{-1})^\top(\xi_m^{(k)})) \in S^*_{y_m}M$ for every $m \in \mathbb{N} \cup \{\infty\}$. For every $m \geqslant m_0$, we can define the exit time of the unrevisited set $\mathcal{V}$ for the point y_m as follows:

$$\tau_m := \inf\{p \in \mathbb{N},\ \varphi^p(y_m) \notin \mathcal{V}\} - 1.$$

It is well-defined by definition of y_m and it depends on k. Note that τ_m is finite for all m since $y_m \notin W^{\mathrm{s}}(K_{j_k})$ (because $y_m \in W^{\mathrm{s}}(K_j)$ with $j > j_k$) and since $\bigcap_{p\in\mathbb{N}} \varphi^{-p}(\mathcal{V}) = W^{\mathrm{s}}(K_{j_k}) \cap \mathcal{V}$. Also, the convergence $y_m \underset{m\to+\infty}{\to} y_\infty \in W^{\mathrm{s}}(K_{j_k})$ implies that τ_m goes to $+\infty$ when m tends to $+\infty$. Now, we define $y_m^{(1)} = \varphi^{\tau_m}(y_m)$ and $\eta_m^{(1)} = \kappa((D\varphi^{\tau_m}(y_m)^{-1})^\top(\eta_m))$. Up to extraction, we can assume that $(y_m^{(1)}, \eta_m^{(1)})$ converges to an element $(y_\infty^{(1)}, \eta_\infty^{(1)}) \in S^*M$ by compactness of S^*M. Let us define

$$(x_m^{(k+1)}, \xi_m^{(k+1)}) := (y_m^{(1)}, \eta_m^{(1)}) = \widetilde{\Phi}^{\tau_{m,k}+\tau_m+T}(x_m, \xi_m)$$

for every $m \in \mathbb{N} \cup \{\infty\}$ and also

$$\tau_{m,k+1} := \tau_{m,k} + \tau_m + T \geqslant 0.$$

Thanks to Lemma 2.16 and to Remark 2.17, the point $y_\infty^{(1)}$ belongs to $\overline{W^{\mathrm{u}}(K_{j_k}) \cap \mathcal{V}} = W^{\mathrm{u}}(K_{j_k}) \cap \overline{\mathcal{V}}$. Indeed, it follows from the convergence $y_m^{(1)} \underset{m\to+\infty}{\to} y_\infty^{(1)}$ together with the fact that $y_m^{(1)} \in \varphi^{\tau_m}(\mathcal{V}) \cap \mathcal{V}$ and $\tau_m \underset{m\to+\infty}{\to} +\infty$. Moreover, we have $\varphi^1(y_\infty^{(1)}) \notin \mathcal{V}$ and thus $y_\infty^{(1)} \notin K_{j_k}$ because $\varphi^1(y_m^{(1)}) \notin \mathcal{V}$ by definition. Furthermore, there exists an integer $j_{k+1} \in [\![1, N]\!]$ such that $y_\infty^{(1)} \in W^{\mathrm{s}}(K_{j_{k+1}})$. Now, using the properties of the Smale order relation given in Theorem 3 together with the total order on the indices of Section 2.5.1, we deduce from $y_\infty^{(1)} \in W^{\mathrm{u}}(K_{j_k}) \cap W^{\mathrm{s}}(K_{j_{k+1}})$ and $y_\infty^{(1)} \in \overline{W^{\mathrm{s}}(K_j)}$ (given by construction) that $j_k < j_{k+1} \leqslant j$. This ends the induction step. Note that the algorithm stops once we have reached K_j, in which case we have defined $j_0 < j_1 < \cdots < j_\ell = j$.

Step 3: local analysis near a basic set. — For every $k \in [\![1, \ell-1]\!]$, we will prove the implication

(37) $$(x_\infty^{(k)}, \xi_\infty^{(k)}) \notin \Sigma_{\mathrm{u/uo}} \text{ and } \tau_{m,k+1} - \tau_{m,k} \underset{m\to+\infty}{\longrightarrow} +\infty \implies (x_\infty^{(k+1)}, \xi_\infty^{(k+1)}) \in \Sigma_{\mathrm{so/s}}.$$

Let us fix $k \in [\![1, \ell-1]\!]$. To simplify, we will use the same notations as the one used in the previous induction. By definition, we have $(x_\infty^{(k+1)}, \xi_\infty^{(k+1)}) \in \Sigma_{\mathrm{so/s}}$ if and only if $\eta_\infty^{(1)} \in E_{\mathrm{u/uo}}^*(y_\infty^{(1)})$. Since $y_\infty \in W^{\mathrm{s}}(K_{j_k}) \cap \overline{\mathcal{V}}$, since $x_\infty^{(k+1)} = y_\infty^{(1)} \in W^{\mathrm{u}}(K_{j_k}) \cap \overline{\mathcal{V}} \smallsetminus \varphi^{-1}(\mathcal{V})$ and since the conical neighborhoods $\mathcal{C}_*^\delta$ are well-defined on $\overline{\mathcal{V}}$, we have by construction

$$E_{\mathrm{so/s}}^*(y_\infty) = \widetilde{E}_{\mathrm{so/s}}^*(y_\infty) \quad \text{and} \quad E_{\mathrm{u/uo}}^*(y_\infty^{(1)}) = \widetilde{E}_{\mathrm{u/uo}}^*(y_\infty^{(1)}).$$

Assume that $\eta_\infty \notin E_{\mathrm{so/s}}^*(y_\infty)$ and let us prove that $\eta_\infty^{(1)}$ belongs to $E_{\mathrm{u/uo}}^*(y_\infty^{(1)})$ or, equivalently, that for any $\delta' > 0$ we have

(38) $$\eta_\infty^{(1)} \in \mathcal{C}_{\mathrm{u/uo}}^{\delta'}(y_\infty^{(1)}).$$

Fix $\delta' > 0$. By hypothesis, we can find a small constant $\delta > 0$ such that

$$\eta_\infty \notin \mathcal{C}_{\mathrm{so/s}}^{\delta}(y_\infty).$$

Since $\tau_{m,k+1} - \tau_{m,k} \underset{m\to+\infty}{\to} +\infty$, we can apply Corollary 7.6 which gives (38) and thus $(x_\infty^{(k+1)}, \xi_\infty^{(k+1)}) \in \Sigma_{\mathrm{so/s}}$.

Step 4: at the end of the algorithm. — According to step 2, we have defined $(x_\infty^{(\ell)}, \xi_\infty^{(\ell)}) \in S^*M$ with $x_\infty^{(\ell)} \in W^{\mathrm{s}}(K_j)$ as well as $x_m^{(\ell)} \in W^{\mathrm{s}}(K_j)$ for every $m \in \mathbb{N}$. Furthermore, since we have assumed at the beginning $(x_\infty, \xi_\infty) \notin \Sigma_{\mathrm{u/uo}}$, we deduce from the (37) of step 3 that

$$(x_\infty, \xi_\infty) \notin \Sigma_{\mathrm{u/uo}} \implies (x_\infty^{(1)}, \xi_\infty^{(1)}) \in \Sigma_{\mathrm{so/s}} \implies (x_\infty^{(1)}, \xi_\infty^{(1)}) \notin \Sigma_{\mathrm{u/uo}} \implies \cdots \implies (x_\infty^{(\ell)}, \xi_\infty^{(\ell)}) \in \Sigma_{\mathrm{so/s}},$$

where the strong transversality assumption is used extensively through the relation $\Sigma_{\mathrm{u/uo}} \cap \Sigma_{\mathrm{so/s}} = \varnothing$. However, proceeding similarly to the first step, we also have the convergence

$$\Sigma_{\mathrm{u/uo}} \ni (x_m^{(\ell)}, \xi_m^{(\ell)}) = \widetilde{\Phi}^{\tau_{m,\ell}} \underbrace{(x_m, \xi_m)}_{\in \Sigma_{\mathrm{u/uo}}} \underset{m\to+\infty}{\longrightarrow} (x_\infty^{(\ell)}, \xi_\infty^{(\ell)}) \in \Sigma_{\mathrm{so/s}},$$

which leads by continuity of $\Sigma_{\mathrm{u/uo}}$ on $W^{\mathrm{s}}_\varepsilon(K_j)$ for $\varepsilon \ll 1$ (according to Lemma 7.3) to $(x_\infty^{(\ell)}, \xi_\infty^{(\ell)}) \in \Sigma_{\mathrm{u/uo}}$ and thus gives the expected contradiction.

Conclusion. — In all cases we obtained a contradiction and therefore we must have $(x_\infty, \xi_\infty) \in \Sigma_{\mathrm{u/uo}}$. □

Let us state a corollary which will be used to construct the map f which appears in the definition of the escape function—see Proposition 3.6.

Corollary 7.7. — *For every $\varepsilon > 0$, there exists $\varepsilon' > 0$ such that, for every point x which is ε'-close to K, every element of $\kappa\big(E^*_{\mathrm{u/uo}}(x) \smallsetminus 0_M\big)$ is ε-close to the compact set $\bigcup_{z\in K} \kappa\big(E^*_{\mathrm{u/uo}}(z) \smallsetminus 0_M\big)$ for the distance d_{S^*M} associated with the Sasaki metric on S^*M.*

Proof. — By contradiction, assume that there exists $\varepsilon > 0$ such that for every $m \in \mathbb{N}^*$, there exist x_m and $\xi_m \in \kappa\big(E^*_{\mathrm{u/uo}}(x_m) \smallsetminus 0_M\big)$ which satisfy

$$\text{(39)} \qquad d_g(x_m, K) \leqslant 1/m \quad \text{and} \quad d_{S^*M}\Big((x_m, \xi_m), \bigcup_{z\in K} \kappa\big(E^*_{\mathrm{u/uo}}(z) \smallsetminus 0_M\big)\Big) \geqslant \varepsilon.$$

By compactness of S^*M, we can extract a converging subsequence $((x_{m_k}, \xi_{m_k}))_k$ which converges to an element $(x_\infty, \xi_\infty) \in S^*M$ as $k \to +\infty$. Taking the limit in the inequalities (39) implies that $x_\infty \in K$ and $\xi_\infty \notin \kappa\big(E^*_{\mathrm{u/uo}}(x_\infty) \smallsetminus 0_M\big)$. However, the set $\Sigma_{\mathrm{s/so}} = \kappa\big(E^*_{\mathrm{u/uo}} \smallsetminus 0_M\big)$ is a compact set thanks to the compactness proposition 3.3. Therefore, we must have $(x_\infty, \xi_\infty) \in \Sigma_{\mathrm{s/so}}$ or, equivalently, $\xi_\infty \in \kappa\big(E^*_{\mathrm{u/uo}}(x_\infty) \smallsetminus 0_M\big)$, which gives the contradiction. □

7.4. The compact sets are attracting and repelling sets for the Hamiltonian flow

Now that we have proved the compactness of $\Sigma_{\mathrm{uo/u}}$ and $\Sigma_{\mathrm{s/so}}$, it remains to show that $(\Sigma_{\mathrm{s/so}}, \Sigma_{\mathrm{uo/u}})$ defines an attractor-repeller pair. Equivalently, we have to prove Lemmas 3.2 and 3.4. In the upcoming argument, we generalize the convergence presented in Figure 13 to Axiom A flows satisfying the strong transversality assumption (10). The proof is very similar to that given in [22, Lem. 2.10 (4), p. 13], except we allow our phase point ξ to have a neutral component, i.e., $\xi(V)$ can be non zero.

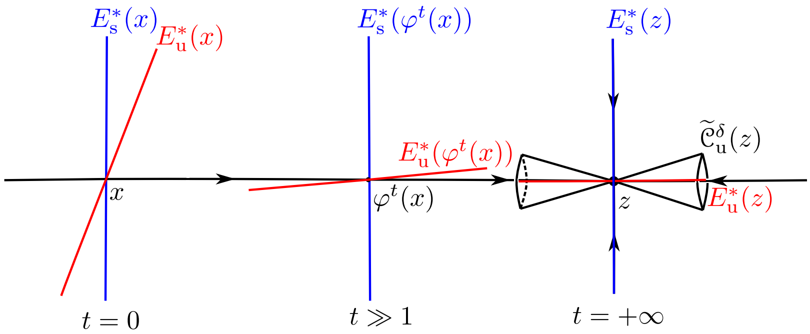


Figure 13. Convergence of the dual unstable distribution. The point z denotes an hyperbolic fixed point and the point x belongs to the stable manifold of z.

Proof of Lemma 3.2. — Let us take $(x,\xi) \in S^*M$ such that $(x,\xi) \notin \Sigma_{\mathrm{uo/u}}$ and let us prove that $\widetilde{\Phi}^t(x,\xi)$ tends to $\Sigma_{\mathrm{s/so}}$ as $t \to +\infty$. The case $t \to -\infty$ is obtained by applying the result to the vector field $-V$. From the spectral decomposition of M (Lemma 2.5), we have $M = \bigsqcup_{i=1}^N W^{\mathrm{s}}(K_i)$ and consequently there exists a unique integer $i \in [\![1,N]\!]$ such that $x \in W^{\mathrm{s}}(K_i)$. Let $\mathcal{V}$ be an unrevisited neighborhood of K_i sufficiently small to be in the range of application of Corollary 7.6. Also, there exists $T>0$ such that $\varphi^t(x) \in W^{\mathrm{s}}(K_i) \cap \mathcal{V}$ for every $t \geqslant T$. Now, let us recall that for every $y \in W^{\mathrm{s}}(K_i) \cap \mathcal{V}$ we have

$$\widetilde{E}_{\mathrm{s}}^*(y) = E_{\mathrm{s}}^*(y) \quad \text{and} \quad \widetilde{E}_{\mathrm{so}}^*(y) = E_{\mathrm{so}}^*(y)$$

by construction. Since $\Sigma_{\mathrm{uo/u}}$ is $\widetilde{\Phi}^t$-invariant, the condition $(x,\xi) \notin \Sigma_{\mathrm{uo/u}}$ implies that $\widetilde{\Phi}^t(x,\xi) \notin \Sigma_{\mathrm{uo/u}} = \kappa(E_{\mathrm{so/s}}^*)$ for every $t \geqslant 0$. Therefore, there exists $\delta>0$ such that $\widetilde{\Phi}^T(x,\xi) \notin \mathcal{C}_{\mathrm{so/s}}^\delta$ and Corollary 7.6 implies that

$$d_{S^*M}\Big(\widetilde{\Phi}^t(x,\xi), \bigcup_{z\in K_i} \kappa\big(\mathcal{C}_{\mathrm{u/uo}}^{\delta'}(z)\big)\Big) \underset{t\to+\infty}{\longrightarrow} 0, \qquad \forall \delta'>0,$$

or, equivalently, that

$$d_{S^*M}\Big(\widetilde{\Phi}^t(x,\xi), \bigcup_{z\in K_i} \kappa\big(E_{\mathrm{u/uo}}^*(z)\big)\Big) \underset{t\to+\infty}{\longrightarrow} 0.$$

This ends the proof. □

7.5. Proof of Lemma 3.4: invariant neighborhoods for the Hamiltonian flow

Now, we need to find invariant neighborhoods of $\Sigma_{\mathrm{s/so}}$ and $\Sigma_{\mathrm{uo/u}}$, i.e., we have to prove Lemma 3.4. The idea of the proof follows from two remarks:

– In the case where K is an attractor, we can construct a $\widetilde{\Phi}^1$-stable neighborhood of $\{(x,\xi) \in \Sigma_{\mathrm{s/so}} : x \in K\}$ as follows: if $\mathcal{V}$ is an unrevisited neighborhood of K small in the sense that the conical neighborhood $\mathcal{C}_*^\delta$ are well-defined on $\mathcal{V}$, then for all $\delta \in (0,1]$ the sets

$$\bigcup_{z\in\mathcal{V}} \kappa\left(\mathcal{C}^\delta_{\mathrm{u}}(z)\right) \quad \text{and} \quad \bigcup_{z\in\mathcal{V}} \kappa\left(\mathcal{C}^\delta_{\mathrm{uo}}(z)\right)$$

are $\widetilde{\Phi}^1$-stable neighborhoods of $\{(x,\xi)\in\Sigma_{\mathrm{s}} : x\in K\}$ and $\{(x,\xi)\in\Sigma_{\mathrm{so}} : x\in K\}$.

– In the general case, thanks to Lemma 7.5 we can note that if $\mathcal{V}$ is an unrevisited neighborhood of K (to be in the setting of this lemma), then the set

$$\bigcup_{z\in\varphi^m(\mathcal{V})\cap\mathcal{V}} \kappa\big(\mathcal{C}^\delta_{\mathrm{u/uo}}(z)\big)$$

defines an unrevisited neighborhood containing $\bigcup_{x\in W^{\mathrm{u}}(K)\cap\mathcal{V}} \kappa\big(E^*_{\mathrm{u/uo}}(x)\big)$ for every $\delta\in(0,1]$ and for every $m\gg 1$.

Our strategy will be to construct invariant neighborhoods $\Sigma_{\mathrm{s/so}}$ and $\Sigma_{\mathrm{uo/uo}}$ by induction from the unrevisited neighborhoods exhibited above. In the upcoming proof, we construct a filtration[12] by open sets for the Hamiltonian flow $\widetilde{\Phi}^t$ starting from the filtration on the base.

Proof of Lemma 3.4. — Let us only treat the case of $\Sigma_{\mathrm{s/so}}$, as the other cases can again be treating similarly by considering Φ^{-1} instead of Φ^1. Consider a filtration $(\mathcal{O}^+_j)_j$ of the manifold M for φ^t which is arbitrarily close to the unstable manifolds. The distance of the filtration to the unstable manifolds will be chosen sufficiently small in order to have the claimed quantitative result for the lifted filtration. First, we want to construct by induction a filtration for the diffeomorphism $\widetilde{\Phi}^1$

$$\mathcal{U}^{\mathrm{s/so}}_j \ \varepsilon\text{-close to} \bigcup_{z\in\bigcup_{\ell>N-j} W^{\mathrm{u}}(K_\ell)} \Sigma_{\mathrm{s/so}}(z) \tag{40}$$

For $j=1$, we define $\mathcal{U}^{\mathrm{s/so}}_1$ as a neighborhood of the attractor $\bigcup_{z\in K_N}\Sigma_{\mathrm{s/so}}(z)$ by fixing

$$\mathcal{U}^{\mathrm{s/so}}_1 := \bigcup_{z\in\varphi^m(\mathcal{V}_N)\cap\mathcal{V}_N} \kappa\big(\mathcal{C}^\delta_{\mathrm{u/uo}}(x)\big),$$

where $\mathcal{V}_N$ denotes an unrevisited neighborhood of the attractor K_N sufficiently small to be in the range of application of Corollary 7.6 and where m is an integer. By choosing δ small enough and then m sufficiently large, i.e., $m\geqslant m_\delta$, the conical neighborhoods and $\mathcal{U}^{\mathrm{s/so}}_1$ are $\widetilde{\Phi}^1$-stable and we can assume that

$$\max_{(x,\xi)\in\mathcal{U}^{\mathrm{s/so}}_1} d_{S^*M}\Big((x,\xi), \bigcup_{z\in K_N}\Sigma_{\mathrm{s/so}}(z)\Big) < \varepsilon.$$

Note that this construction only uses the fact that K_N is an attractor.

Now, let us deal with the induction step and assume that we can find for every $\varepsilon>0$ some open sets $\mathcal{U}^{\mathrm{s/so}}_j$ satisfying (40) for any $j<i$. Fix $\varepsilon>0$. We want to construct $\mathcal{U}^{\mathrm{s/so}}_i$. But before doing that, we consider an unrevisited neighborhood $\mathcal{V}$ of $K:=K_{N-i+1}$ small enough in the range of application of Corollary 7.6. Also, we consider the family of unstable annuli $\mathcal{A}(m)$ of Sub-lemma 6.4 and we choose $\mathcal{V}$

(12) Even if the definition of filtration was given for the flow φ^t, it can be adapted for the Hamiltonian without too much effort, see [66] or [68].

sufficiently small as in the sub-lemma. According to Lemmas 6.2 and 3.2 and thanks to the fact that $\Sigma_{\mathrm{s/so}} \cap \Sigma_{\mathrm{uo/u}} = 0_M$ (given by the strong transversality assumption), we have for all $m \geqslant 0$

$$\forall (x,\xi) \in \bigcup_{z\in\mathcal{A}(m)} \Sigma_{\mathrm{s/so}}(z), \quad d_{S^*M}\Big(\widetilde{\Phi}^k(x,\xi), \bigcup_{z\in\bigcup_{\ell>N-i+1} W^{\mathrm{u}}(K_\ell)} \Sigma_{\mathrm{s/so}}(z)\Big) \underset{k\to+\infty}{\longrightarrow} 0.$$

Furthermore, from the compactness proposition 3.3, thanks to Lemma 3.2 and by construction of $\mathcal{U}_{i-1}^{\mathrm{s/so}}$, there exist integers $k_0, m_0 \geqslant 0$ (potentially larger than the one of the sub-lemma) such that for all $m \geqslant m_0$,

$$\forall (x,\xi) \in \bigcup_{z\in\mathcal{A}(m)} \Sigma_{\mathrm{s/so}}(z), \qquad \widetilde{\Phi}^{k_0}(x,\xi) \in \mathcal{U}_{i-1}^{\mathrm{s/so}}. \tag{41}$$

By continuity and since $\mathcal{U}_{i-1}^{\mathrm{s/so}}$ is an open set, we can extend (41) on small conical neighborhoods: there exists $\delta_0 > 0$ such that

$$\forall (x,\xi) \in \bigcup_{z\in\mathcal{A}(m)} \kappa(\mathcal{C}^{\delta_0}_{\mathrm{u/uo}}(z)), \qquad \widetilde{\Phi}^{k_0}(x,\xi) \in \mathcal{U}_{i-1}^{\mathrm{s/so}}.$$

Now, for all $m \geqslant 0$ and for all $\delta \in (0,1]$, we define the set

$$\mathcal{W}(m,\delta) := \bigcup_{z\in\mathcal{V}\cap\varphi^m(\mathcal{V})} \kappa(\mathcal{C}^{\delta}_{\mathrm{u/uo}}(z)).$$

According to Lemma 7.5, for every $\delta > 0$ there exists an integer $m(\delta) > 0$ such that for all $m \geqslant m(\delta)$ the open set $\mathcal{W}(m,\delta)$ is an arbitrarily small (as $m \to +\infty$ and $\delta \to 0$) unrevisited neighborhood containing $\bigcup_{z\in\mathcal{V}\cap W^{\mathrm{u}}(K)} \kappa\big(E^*_{\mathrm{u/uo}}(z)\big)$. Actually, we have something better. For every $m \geqslant m(\delta)$, the conical neighborhood $\mathcal{C}^{\delta}_{\mathrm{u/uo}}$ is Φ^1-stable on $\mathcal{V} \cap \varphi^m(\mathcal{V})$ in the sense that, for all $\ell \in \mathbb{N}$,

$$(x,\xi) \in \mathcal{W}(m,\delta),\ \varphi^\ell(x) \in \mathcal{V}\cap\varphi^m(\mathcal{V}) \implies \forall k \in [\![0,\ell]\!],\ \widetilde{\Phi}^k(x,\xi) \in \mathcal{W}(m,\delta). \tag{42}$$

Therefore, we define for all $\delta \leqslant \delta_0$ and all $m \geqslant m(\delta)$ the set

$$\mathcal{U}_i^{\mathrm{s/so}}(m,\delta) := \mathcal{U}_{i-1}^{\mathrm{s/so}} \cup \bigcup_{k=0}^{k_0-1} \widetilde{\Phi}^k(\mathcal{W}(m,\delta)),$$

which is $\widetilde{\Phi}^1$-stable because $\mathcal{U}_{i-1}^{\mathrm{s/so}}$ is. Also, $\widetilde{\Phi}^{k_0}(\mathcal{W}(m,\delta))$ is equal to

$$\begin{aligned}\widetilde{\Phi}^{k_0}\Big(\bigcup_{z\in\varphi^m(\mathcal{V})\cap\varphi^{-1}(\mathcal{V})} \kappa(\mathcal{C}^{\delta}_{\mathrm{u/uo}}(z))\Big) &\cup \widetilde{\Phi}^{k_0}\Big(\bigcup_{z\in\varphi^m(\mathcal{V})\cap\mathcal{V}\cap\mathcal{A}(m)} \kappa(\mathcal{C}^{\delta}_{\mathrm{u/uo}}(z))\Big)\\ &\subset \widetilde{\Phi}^{k_0-1}(\mathcal{W}(m,\delta)) \cup \widetilde{\Phi}^{k_0}\Big(\bigcup_{z\in\mathcal{A}(m)} \kappa(\mathcal{C}^{\delta}_{\mathrm{u/uo}}(z))\Big) \subset \mathcal{U}_i^{\mathrm{s/so}}(m,\delta),\end{aligned}$$

where we used in the first inclusion the fact that $\varphi^1(\varphi^m(\mathcal{V})\cap\varphi^{-1}(\mathcal{V})) \subset \varphi^m(\mathcal{V})\cap\mathcal{V}$ (since $\mathcal{V}$ is unrevisited) together with (42). For the second inclusion, we used (41).

Finally, choosing δ small enough and then m large enough, we can assume the set $\mathcal{U}_i^{\mathrm{s/so}}(m,\delta)$ to be a ε-neighborhood of $\bigcup_{z\in\bigcup_{j>N-i} W^{\mathrm{u}}(K_j)} \Sigma_{\mathrm{s/so}}(z)$. This ends the induction and the proof. □

Remark 7.8. — We can note that the set $\mathcal{U}_N^{\mathrm{s/so}}$ constructed in the previous proof defines an arbitrarily small neighborhood of $\Sigma_{\mathrm{s/so}}$ which is stable by $\widetilde{\Phi}^1$.

7.6. Proof of Proposition 3.5: energy function for the Hamiltonian flow

Thanks to Proposition 3.3, Lemmas 3.2 and 3.4, we deduce that $(\Sigma_{\mathrm{s}}, \Sigma_{\mathrm{uo}})$ and $(\Sigma_{\mathrm{so}}, \Sigma_{\mathrm{u}})$ define attractor-repeller pairs. Therefore, a straight application of Lemma 6.1 gives the result.

8. Construction of the escape function: proof of Proposition 3.6

Let us begin with some candidate for the escape function that depends on two auxiliary functions. We will see how the properties of the maps $\boldsymbol{E}$ and f will be related to those of the escape function. First of all, let us assume that the maps $\boldsymbol{E}$ and f have been defined properly and let us recall the expression of the escape function:

$$G_m(x,\xi) = m(x,\xi)\log\sqrt{1+f(x,\xi)^2},$$

with $m(x,\xi) = \boldsymbol{E}\,(x,\xi/|\xi|)\,\chi(|\xi|^2)$. To lighten notations, we will use the Japanese bracket $\langle r\rangle := \sqrt{1+r^2}$ and the shortcut $\widetilde{\xi}$ for $\xi/|\xi|$. Now, let us compute $\mathcal{L}_{X_H}(G_m)$, for $\|\xi\| \geqslant 1$

$$\mathcal{L}_{X_H}(G_m)(x,\xi) = \mathcal{L}_{\widetilde{X}_H}(\boldsymbol{E})(x,\widetilde{\xi})\log\langle f(x,\xi)\rangle + \boldsymbol{E}(x,\widetilde{\xi})\mathcal{L}_{X_H}(\log\langle f\rangle).$$

Our goal is to make sure that this quantity is non-positive and is negative outside a conical neighborhood of E_{o}^*.

Definition of $\boldsymbol{E}$. — Fix $\varepsilon > 0$ such that

$$\varepsilon \leqslant \min\Big(\frac{|u|}{2s-2u};\ \frac{7}{4}\frac{s}{2s-2u};\ \frac{n_0}{2s-n_0};\ \frac{1}{4}\frac{n_0}{n_0-2u}\Big).$$

We define $E \in \mathcal{C}^\infty(S^*M)$ as

$$\boldsymbol{E}(x,\xi) := -E(x) + 2s + (2u-n_0)E_+(x,\xi) + (n_0-2s)E_-(x,\xi),$$

where E_+, E_- are given by Proposition 3.5 for the constant ε, namely there exist smooth energy functions $E_\pm \in \mathcal{C}^\infty(S^*M,[0,1])$, ε-neighborhoods $\mathcal{W}^{\mathrm{s/so}}$ of $\Sigma_{\mathrm{s/so}}$, $\mathcal{W}^{\mathrm{uo/u}}$ of $\Sigma_{\mathrm{uo/u}}$ in S^*M and a constant $\eta > 0$ such that:

$$\begin{aligned}
&\mathcal{L}_{\widetilde{X}_H}E_+ \geqslant 0 \quad \text{on } S^*M \quad \text{and} \quad \mathcal{L}_{\widetilde{X}_H}E_+ > \eta \quad \text{on } S^*M \smallsetminus (\mathcal{W}^{\mathrm{uo}} \cup \mathcal{W}^{s}),\\
&\mathcal{L}_{\widetilde{X}_H}E_- \geqslant 0 \quad \text{on } S^*M \quad \text{and} \quad \mathcal{L}_{\widetilde{X}_H}E_- > \eta \quad \text{on } S^*M \smallsetminus (\mathcal{W}^{\mathrm{u}} \cup \mathcal{W}^{\mathrm{so}}).
\end{aligned}$$

We also have the estimates $E_+ \geqslant 1-\varepsilon$ on $\mathcal{W}^s$, $E_+ \leqslant \varepsilon$ on $\mathcal{W}^{\mathrm{uo}}$, $E_- \geqslant 1-\varepsilon$ on $\mathcal{W}^{\mathrm{so}}$ and $E_- \leqslant \varepsilon$ on $\mathcal{W}^{\mathrm{u}}$. In order to use these estimates together, let us introduce new open sets in S^*M:

$$\mathcal{N}^{\mathrm{s}} := \mathcal{W}^{\mathrm{s}} \cap \mathcal{W}^{\mathrm{so}}, \quad \mathcal{N}^{\mathrm{o}} := \mathcal{W}^{\mathrm{so}} \cap \mathcal{W}^{\mathrm{uo}}, \quad \mathcal{N}^{\mathrm{u}} := \mathcal{W}^{\mathrm{uo}} \cap \mathcal{W}^{\mathrm{u}}.$$

Moreover, $E \in \mathcal{C}^\infty(M)$ denotes the energy function on the basis given by Proposition 3.1 for some parameter $\varepsilon' > 0$ and

$$\lambda_1 = 0, \dots, \lambda_j = \frac{n_0(j-1)}{4(N-1)}, \dots, \lambda_N = \frac{n_0}{4}.$$

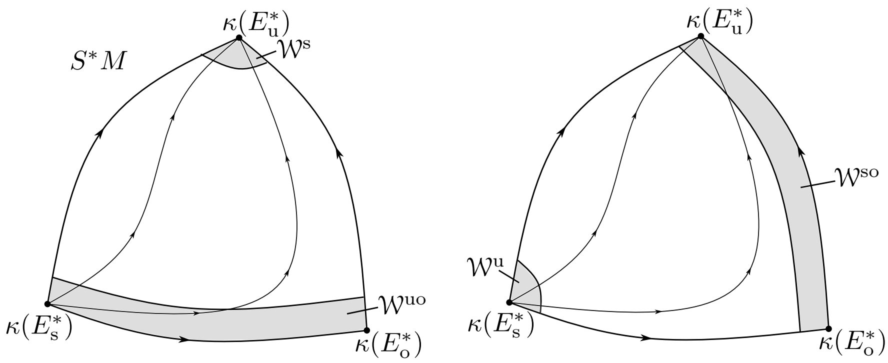


Figure 14. Illustration of some elements used in the proof. Inspired from the picture of [30, Fig. 6].

Thus, the map $E(x)$ has value in the interval $[0, n_0/4]$ and there exists a family $\mathcal{N}_i$ of ε'-neighborhoods of K_i and a constant $\eta_0 > 0$ (which only depends on ε') such that[(13)]

$$-\mathcal{L}_V(E) < -\eta_0, \quad \text{on } M \smallsetminus \bigcup_{i=1}^{N} \mathcal{N}_i.$$

In addition, we choose ε, ε' sufficiently small so that, for every $i \in [\![1, N]\!]$,

$$\{z,\ d_g(z, K_i) < \varepsilon'\} \subset \varphi^m(\mathcal{V}_i) \cap \varphi^{-m}(\mathcal{V}_i),$$
$$\bigcup_{d_g(z,K_i)<\varepsilon'} \big\{(z,\xi) \in \mathcal{N}^{\mathrm{s/u}}\big\} \subset \bigcup_{d_g(z,K_i)<\varepsilon'} \kappa\big(\mathcal{C}^1_{\mathrm{u/s}}(z)\big),$$

for some given unrevisited neighborhood $\mathcal{V}_i$ of K_i as in the statement of Corollary 7.6 and for a fixed integer m large enough ($\geqslant m_1$ of the corollary) such that $Ce^{m\lambda/4} \geqslant 3$ (where C, λ are the constant given in the corollary). These two inclusions will be used in the construction of f. But first, let us briefly explain why these inclusions are satisfied for $\varepsilon, \varepsilon' \ll 1$. Since m is fixed and since $\varphi^m(\mathcal{V}_i) \cap \varphi^{-m}(\mathcal{V}_i)$ is a neighborhood of K_i, the first inclusion is true for ε' sufficiently small. The second inclusion is given implicitly in the proof of Lemma 3.4. To be more precise, we apply Corollary 7.7 which states that, up to a choice of a smaller ε',

$$\kappa\big(E^*_{\mathrm{u/s}}(x) \smallsetminus 0_M\big) \text{ is arbitrarily close to } \bigcup_{z\in K_i} \kappa\big(E^*_{\mathrm{u/s}}(z) \smallsetminus 0_M\big)$$

for all x satisfying $d_g(x, K_i) < \varepsilon'$. If we note that

$$\bigcup_{d_g(z,K_i)<\varepsilon'} \kappa\big(\mathcal{C}^1_{\mathrm{s/u}}(z)\big) \text{ is an open neighborhood of } \bigcup_{z\in K_i} \kappa\big(E^*_{\mathrm{u/s}}(z) \smallsetminus 0_M\big),$$

and if we recall that $\mathcal{N}^{\mathrm{s/u}}$ is an ε-neighborhood of $\Sigma_{\mathrm{s/u}} = \bigcup_{z\in M} \kappa\big(E^*_{\mathrm{u/s}}(z) \smallsetminus 0_M\big)$, then we get that

$$\bigcup_{d_g(z,K_i)<\varepsilon'} \big\{(z,\xi) \in \mathcal{N}^{\mathrm{s/u}}\big\} \text{ is contained in } \bigcup_{d_g(z,K_i)<\varepsilon'} \kappa\big(\mathcal{C}^1_{\mathrm{s/u}}(z)\big),$$

(13)We make the assumption that $N \geqslant 2$. The case $N = 1$ corresponds to the Anosov case.

as soon as ε is small enough. Another consequence of these two inclusions is that

$$\bigcup_{z\in\mathcal{N}_i}\{(z,\xi)\in\mathcal{N}^{\mathrm{s/u}}\}\subset\bigcup_{\varphi^m(\mathcal{V}_i)\cap\varphi^{-m}(\mathcal{V}_i)}\kappa\big(\mathcal{C}^1_{\mathrm{s/u}}(z)\big). \tag{43}$$

This last inclusion together with Corollary 7.6 (second point) will be essential for the definition of f.

Let us introduce another notation which will be useful: for any open set U in S^*M, we denote by $\mathrm{Cone}(U)$ the conical neighborhood[(14)] of U in $T^*M\smallsetminus 0_M$. In particular, the sets $\mathrm{Cone}(\mathcal{N}^{\mathrm{s}})$, $\mathrm{Cone}(\mathcal{N}^{\mathrm{o}})$ and $\mathrm{Cone}(\mathcal{N}^{\mathrm{u}})$ define conical neighborhoods of $\bigcup_{x\in M}E^*_{\mathrm{u}}(x)$, $\bigcup_{x\in M}E^*_{\mathrm{o}}(x)$ and $\bigcup_{x\in M}E^*_{\mathrm{s}}(x)$ respectively (outside the null section). Now that the setting is done, we need to check that the escape function satisfies the desired properties.

– On $\mathcal{N}^{\mathrm{s}}$, we have

$$\begin{aligned}\boldsymbol{E}(x,\xi)&\leqslant\underbrace{-E(x)}_{\leqslant 0}+2s+(2u-n_0)(1-\varepsilon_1)+(n_0-2s)(1-\varepsilon_1)\\&\leqslant 2s\varepsilon+2u(1-\varepsilon)\leqslant u.\end{aligned}$$

– On $\mathcal{N}^{\mathrm{u}}$, we obtain

$$\begin{aligned}\boldsymbol{E}(x,\xi)&\geqslant\underbrace{-E(x)}_{\geqslant -s/4}+2s+(2u-n_0)\varepsilon+(n_0-2s)\varepsilon\\&\geqslant 2s(1-\varepsilon-\frac{1}{8})+2u\varepsilon\geqslant s.\end{aligned}$$

– On $\mathcal{N}^{\mathrm{o}}$, we obtain

$$\frac{n_0}{2}\leqslant-\frac{n_0}{4}+n_0(1-\varepsilon)+2u\varepsilon\leqslant\boldsymbol{E}(x,\xi)\leqslant n_0(1-\varepsilon)+2s\varepsilon\leqslant 2n_0.$$

We now explain how to construct the function f following the strategy of [30, 24]. We want to construct a function $f\in\mathcal{C}^\infty(T^*M)$ which is a homogeneous polynomial of degree 1 for $|\xi|\geqslant 1$, so that $\mathcal{L}_{X_H}(\log\langle f\rangle)$ is a bounded function. Since we want $\mathcal{L}_{X_H}(G_m)$ to be negative everywhere, we need to make sure that our definition of f gives the right sign in the term $\boldsymbol{E}\cdot\mathcal{L}_{X_H}(\log\langle f\rangle)$. We will also seek f such that $\mathcal{L}_{X_H}(f)$ vanishes near E^*_{o} and such that $f(x,\xi)=\chi(|\xi|^2)\widetilde{f}(x,\xi)$, with χ as before and with $\widetilde{f}\in\mathcal{C}^\infty(T^*M\smallsetminus 0_M)$ which is 1-homogeneous with respect to the variable ξ. We can already check that $\mathcal{L}_{X_H}(\log\langle f\rangle)$ is a bounded function on T^*M for $|\xi|\geqslant 1$. Take an arbitrary basic set K_i.

– On $\widetilde{\mathcal{N}}^{\mathrm{s}}_i:=\mathrm{Cone}\big(\bigcup_{z\in\mathcal{N}_i}\{(z,\xi)\in\mathcal{N}^{\mathrm{s}}\}\big)$, we define the map $\widetilde{f}$ by

$$\widetilde{f}(x,\xi):=\int_0^m|\Phi^t(x,\xi)|dt.$$

Thanks to Corollary 7.6, thanks to our choice of m (such that $Ce^{m\lambda/4}\geqslant 3$) and thanks to the inclusion (43), we get that

$$\mathcal{L}_{X_H}(\widetilde{f})(x,\xi)=|\Phi^m(x,\xi)|-|\xi|\geqslant Ce^{m\lambda/4}|\xi|-|\xi|\geqslant 2|\xi|,\quad\forall(x,\xi)\in\widetilde{\mathcal{N}}^{\mathrm{s}}_i.$$

(14) defined by $\mathrm{Cone}(U)=\{(x,\lambda\xi)\in T^*M,\ \lambda\in\mathbb{R}^*,\ (x,\xi)\in U\}$.

Also, f is a homogeneous polynomial of degree 1 in the variable ξ, so we can find a constant $c>0$, which does not depend on $x\in K_i$, such that $0<c^{-1}|\xi|<\widetilde{f}(x,\xi)<c|\xi|$ on $\widetilde{\mathcal{N}}^{\mathrm{s}}\cap\{|\xi|\geqslant 1\}$. For $|\xi|\geqslant 1$, we get

$$\mathcal{L}_{X_H}(f)=\chi(|\xi|^2)\mathcal{L}_{X_H}(\widetilde{f})\geqslant\frac{2}{c}\cdot c|\xi|\geqslant\frac{2}{c}\,\widetilde{f}=\frac{2}{c}\,f.$$

Therefore, there exists a universal constant $\gamma>0$ such that

$$\mathcal{L}_{X_H}(\log\langle f\rangle)=\frac{f}{\langle f\rangle^2}\,\mathcal{L}_{X_H}(f)\geqslant\frac{2}{c}\,\frac{f^2}{\langle f\rangle^2}\geqslant\gamma>0. \tag{44}$$

– On $\widetilde{\mathcal{N}}_i^{\mathrm{u}}:=\operatorname{Cone}\big(\bigcup_{z\in\mathcal{N}_i}\{(z,\xi)\in\mathcal{N}^{\mathrm{u}}\}\big)$, we define similarly

$$\widetilde{f}(x,\xi):=\int_0^m|\Phi^{-t}(x,\xi)|dt.$$

With the same remark, up to reducing the constant γ, we obtain

$$\mathcal{L}_{X_H}(\log\langle f\rangle)\leqslant-\gamma<0, \tag{45}$$

as soon as $|\xi|\geqslant 1$.

– On $\widetilde{\mathcal{N}}_i^{\mathrm{o}}:=\operatorname{Cone}\big(\bigcup_{z\in\mathcal{N}_i}\{(z,\xi)\in\mathcal{N}^{\mathrm{o}}\}\big)$, we fix $\widetilde{f}(x,\xi)=|\xi(V(x))|$. This ensures that

$$\mathcal{L}_{X_H}(\log\langle f\rangle)=0.$$

– On $\bigcup_{z\in\bigcup_i\mathcal{N}_i}\{(z,\xi)\in S^*M\smallsetminus(\mathcal{N}^{\mathrm{s}}\sqcup\mathcal{N}^{\mathrm{o}}\sqcup\mathcal{N}^{\mathrm{u}})\}$ and on $\bigcup_{z\notin\bigcup_i\mathcal{N}_i}S^*_zM$, we let $\widetilde{f}$ take arbitrary positive values on S^*M.

It now remains to show that G_m has the expected decaying properties (namely points (2) and (3)) of Proposition 3.6.

Decaying estimates. — To compute the derivative of the map E along the flow $\widetilde{\Phi}^t$, we will need at some point the next relation:

$$\mathcal{L}_{\widetilde{X}_H}\boldsymbol{E}=-\mathcal{L}_V(E)+(2u-n_0)\mathcal{L}_{\widetilde{X}_H}E_++(n_0-2s)\mathcal{L}_{\widetilde{X}_H}E_-$$

and we can already see that it is non-positive everywhere. For $|\xi|\geqslant 1$, we can estimate the quantity $\mathcal{L}_{X_H}G_m$ in different directions.

– On $\widetilde{\mathcal{N}}^{\mathrm{s}}:=\operatorname{Cone}\big(\bigcup_{z\in\bigcup_i\mathcal{N}_i}\{(z,\xi)\in\mathcal{N}^{\mathrm{s}}\}\big)$, we get

$$\begin{aligned}\mathcal{L}_{X_H}G_m&=\underbrace{\mathcal{L}_{\widetilde{X}_H}(\boldsymbol{E})}_{\leqslant 0}\underbrace{\log\langle f\rangle}_{>0}+\underbrace{\boldsymbol{E}(x,\widetilde{\xi})}_{\leqslant u}\underbrace{\mathcal{L}_{X_H}(\log\langle f\rangle)}_{\geqslant\gamma}\\&\leqslant-\gamma|u|.\end{aligned}$$

– On $\widetilde{\mathcal{N}}^{\mathrm{u}}:=\operatorname{Cone}\big(\bigcup_{z\in\bigcup_i\mathcal{N}_i}\{(z,\xi)\in\mathcal{N}^{\mathrm{u}}\}\big)$, we get

$$\begin{aligned}\mathcal{L}_{X_H}G_m&=\underbrace{\mathcal{L}_{\widetilde{X}_H}(\boldsymbol{E})}_{\leqslant 0}\underbrace{\log\langle f\rangle}_{>0}+\underbrace{\boldsymbol{E}(x,\widetilde{\xi})}_{\geqslant s}\underbrace{\mathcal{L}_{X_H}(\log\langle f\rangle)}_{\leqslant-\gamma}\\&\leqslant-\gamma s.\end{aligned}$$

– On $\widetilde{\mathcal{N}}^{\mathrm{o}}:=\operatorname{Cone}\big(\bigcup_{z\in\mathcal{N}_i}\{(z,\xi)\in\mathcal{N}^{\mathrm{o}}\}\big)$, we obtain

$$\mathcal{L}_{X_H}G_m=\mathcal{L}_{\widetilde{X}_H}(\boldsymbol{E})\log\langle f\rangle\leqslant 0.$$

– On $\mathrm{Cone}\big(\bigcup_{z\in\bigcup_i \mathcal{N}_i}\{(z,\xi)\in S^*M\smallsetminus(\mathcal{N}^{\mathrm{s}}\sqcup\mathcal{N}^{\mathrm{o}}\sqcup\mathcal{N}^{\mathrm{u}})\}\big)$. Or equivalently, for $x\in\bigcup_{1\leqslant i\leqslant N}\mathcal{N}_i$ with $(x,\xi)\notin\mathrm{Cone}(\mathcal{W}^{\mathrm{so}}\cup\mathcal{W}^{\mathrm{u}})$ or $(x,\xi)\notin\mathrm{Cone}(\mathcal{W}^{s}\cup\mathcal{W}^{\mathrm{uo}})$. Since $\boldsymbol{E}\,|_{S^*M}$ and $\mathcal{L}_{X_H}(\log\langle f\rangle)$ are bounded and since f is 1-homogeneous with respect to ξ for $|\xi|\geqslant 1$, there exist constants $C_1,C_2>0$ such that for all $|\xi|\geqslant 1$,

$$\mathcal{L}_{X_H}(G_m)=\mathcal{L}_{\widetilde{X}_H}(\boldsymbol{E})\log\langle|\xi|C_1\rangle+C_2.$$

Moreover, according to the construction of $\boldsymbol{E}$, one can find a positive constant $\eta>0$ such that $\mathcal{L}_{\widetilde{X}_H}(\boldsymbol{E})(x,\widetilde{\xi})<-\eta<0$. Therefore, there exists a positive radius $R>0$ such that, for every (x,ξ) in the set $T^*M\smallsetminus(\mathrm{Cone}(\mathcal{N}^{\mathrm{s}})\cup\mathrm{Cone}(\mathcal{N}^{\mathrm{o}})\cup\mathrm{Cone}(\mathcal{N}^{\mathrm{u}}))$ with $x\in\bigcup_{1\leqslant i\leqslant N}\mathcal{N}_i$ and $|\xi|\geqslant R$, we have

$$\mathcal{L}_{X_H}(G_m)\leqslant-\gamma\min(s,|u|). \tag{46}$$

Therefore, we define $C_m:=\gamma\min(s,|u|)$.

– Outside a small neighborhood of the non-wandering set, i.e., for $x\in M\smallsetminus\big(\bigcup_{i=1}^N\mathcal{N}_i\big)$. In this case, we have $\mathcal{L}_{\widetilde{X}_H}(\boldsymbol{E})\leqslant-(n_0/4(N-1))\eta_0<0$. So, using a similar argument as in the previous point, we deduce that equation (46) still holds far away from the null section. ☐

Appendix A. Hyperbolic sets

In this appendix, we recall the definition of a hyperbolic set.

Definition A.1. — A φ^t-invariant compact set K is said to be *hyperbolic* for the flow φ^t on M if K is the union of isolated fixed points and compact sets on which the induced vector field V never vanishes, where

– for each $x\in K$, we have the following decomposition

$$T_xM=E_{\mathrm{u}}(x)\oplus E_{\mathrm{s}}(x)\oplus E_{\mathrm{o}}(x), \tag{47}$$

where $E_{\mathrm{o}}(x)=\mathbb{R}V(x)$ and E_{s} (resp. E_{u}) is called the stable (resp. unstable) distribution. Moreover, this decomposition is continuous with respect to $x\in K$;

– the decomposition (47) is invariant by the flow φ^t:

$$\forall x\in K,\quad (D_x\varphi^t)(E_{\mathrm{u}}(x))=E_{\mathrm{u}}(\varphi^t(x))\quad\text{and}\quad(D_x\varphi^t)(E_{\mathrm{s}}(x))=E_{\mathrm{s}}(\varphi^t(x))\,;$$

– there are constants $C>0$ and $\lambda>0$ such that for every $x\in K$ and for every $t\geqslant 0$, the following inequalities are satisfied:

$$\begin{aligned}|D_x\varphi^t(v_{\mathrm{s}})|_g&\leqslant Ce^{-\lambda t}|v_{\mathrm{s}}|_g, &\quad \forall v_{\mathrm{s}}\in E_{\mathrm{s}}(x),\\ |D_x\varphi^{-t}(v_{\mathrm{u}})|_g&\leqslant Ce^{-\lambda t}|v_{\mathrm{u}}|_g, &\quad \forall v_{\mathrm{u}}\in E_{\mathrm{u}}(x).\end{aligned} \tag{48}$$

When K is reduced to a singleton $\{z\}$, we must have $V(z)=0$ and thus $E_{\mathrm{o}}(x)=\{0\}$. In that case, we say that z is a hyperbolic fixed point. When the whole manifold is a hyperbolic compact set on which the vector field V never vanishes, the flow is said to be Anosov. It was first introduced by Anosov in [1] and this formal definition of hyperbolicity was motivated by the properties of the geodesic flow on negatively curved manifolds. Another famous example of Anosov flow is given by the suspension of an Anosov diffeomorphism. The notion of hyperbolicity was later extended by

Smale who defined the notion of an Axiom A flow which is at the heart of this article. There are many other examples of hyperbolic sets and we refer the reader to [49] and [56] for a comprehensive study of hyperbolic dynamics. We also refer to [21], [35] for the case of flows.

Remark A.2

– The definition of hyperbolicity does not depend on the continuous metric g on M. Indeed, if g' denotes another smooth metric then, by compactness of M, g' is equivalent to g and (48) still holds with some constant C' instead of C.

– The distributions E_{u} and E_{s} are only Hölder-continuous in general. Let $d_{\mathrm{u}}(x)$ and $d_{\mathrm{s}}(x)$ be the dimensions of $E_{\mathrm{u}}(x)$ and $E_{\mathrm{s}}(x)$ at any point $x \in K$. The maps $d_{\mathrm{s}}(x)$ and $d_{\mathrm{u}}(x)$ are locally constants on K and E_{u} (resp. E_{s}) define a Hölder-continuous section of the Grassmann bundle $G_{d_{\mathrm{u}},n}$ (resp. $G_{d_{\mathrm{u}},n}$) of vector subspaces of dimension d_{u} (resp. d_{s}).

Appendix B. About Smale ordering: proof of Theorem 3

In this section, we give a proof of Theorem 3. Although it seems to be well known to specialists—see [60, p. 158] or the proof of Lemma 9.1.11 in [35, p. 536], we were not able to locate a precise reference for the proof in the Axiom A case. In the particular case of Morse-Smale flows, a proof was given by Smale in [67] and a detailed proof can for instance be found in [16, App. B]. The most technical part of the proof consists in proving the transitivity of Smale's relation. This step is an application of a classical lemma of dynamical systems called the λ-lemma, which can be found in [35, Prop. 6.1.10, p. 335]. However, in the usual λ-lemma for hyperbolic flows (as referenced), the statement only involves hyperbolic periodic orbit and its proof relies on the analysis of diffeomorphisms near hyperbolic fixed points. Since we aim at dealing with hyperbolic orbit which are not only periodic, we begin by presenting a version of the λ-lemma for non periodic hyperbolic orbits which will allow us to proceed similarly to [16, App. B].

B.1. Proof of Smale's theorem.

— In this part, we prove Theorem 3 by using the following dynamical lemma whose proof will be given in paragraph B.2.

Proposition B.1 (Generalized λ-lemma). — *Let z be a point of a topologically transitive hyperbolic set K. Let $\mathcal{D}$ be an embedded disk intersecting the local stable manifold $W^{\mathrm{so}}_{\varepsilon_0}(z)$ (for some small $\varepsilon_0 > 0$ given by the stable manifold theorem) transversally at some point $q \in W^{\mathrm{s}}_{\varepsilon_0}(z)$ such that* $\dim(\mathcal{D}) = \dim(E_{\mathrm{u}})$*, where* $\dim(E_{\mathrm{u}})$ *denotes the dimension of the unstable distribution on the topologically transitive hyperbolic set K. Then for any $\varepsilon > 0$, there exists $k_1 \in \mathbb{N}$ such that for each $k \geqslant k_1$ there is a non-empty embedded disk $\mathcal{D}_k \subset \mathcal{D}$ (of the same dimension) containing q such that $\varphi^k(\mathcal{D}_k)$ is $\mathcal{C}^1$ ε-close*[15] *to $W^{\mathrm{u}}_{\varepsilon_1}(\varphi^k(z))$, for some $\varepsilon_1 \in (0, \varepsilon_0]$ which only depends on φ^t, K.*

[15] It means that $\varphi^k(\mathcal{D}_k)$ is the graph of a $\mathcal{C}^1$ map over $W^{\mathrm{u}}_{\varepsilon_1}(\varphi^k(z))$ whose distance for the $\mathcal{C}^1$ norm is lower than ε. Particularly, it implies that $\varphi^k(\mathcal{D}_k)$ cannot be too small.

Lemma B.2 (Technical lemma). — *Let K be a basic set and fix $\varepsilon_0 > 0$ given by the stable manifold theorem. Let x be in $W^{\mathrm{s}}_{\varepsilon_0}(x_+)$ with $x_+ \in K$ and let y be in $W^{\mathrm{u}}_{\varepsilon_0}(y_-)$ with $y_- \in K$, where ε_0 is given by the stable manifold theorem. If $\mathcal{D}_-$, $\mathcal{D}_+$ denote disks which respectively contain y, x and intersect $W^{\mathrm{uo}}_{\varepsilon_0}(y_-)$, $W^{\mathrm{so}}_{\varepsilon_0}(x_+)$ transversally at a point, then, for all $T > 0$, there exists $t > T$ and there exist disks $\widetilde{\mathcal{D}}_-$, $\widetilde{\mathcal{D}}_+$ contained respectively in $\mathcal{D}_-$, $\mathcal{D}_+$, of same dimension, such that $y \in \widetilde{\mathcal{D}}_-$, $x \in \widetilde{\mathcal{D}}_+$ and*

$$\varphi^t(\widetilde{\mathcal{D}}_+) \text{ and } \widetilde{\mathcal{D}}_- \text{ intersect transversally.}$$

Before giving the proof of this technical lemma, we present some applications.

Proposition B.3. — *Let K be a basic set for the Axiom A flow φ^t (we do not assume the strong transversality nor the no-cycle property here). Then, we have $W^{\mathrm{s}}(K) \cap W^{\mathrm{u}}(K) = K$.*

Proof. — Assume by contradiction that the inclusion $K \subset W^{\mathrm{s}}(K) \cap W^{\mathrm{u}}(K)$ is strict, i.e., there exists an element $x \in M \smallsetminus K$ such that $x \in W^{\mathrm{s}}(K) \cap W^{\mathrm{u}}(K)$. By definition of $W^{\mathrm{s}}(K)$ and $W^{\mathrm{u}}(K)$, there exist elements $x_-, x_+ \in K$ such that $x \in W^{\mathrm{s}}(x_+) \cap W^{\mathrm{u}}(x_-)$. Since $x \in W^{\mathrm{s}}(x_+)$, we also know that for all $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that $\varphi^{t_\varepsilon}(x)$ is ε-close of $\varphi^{t_\varepsilon}(x_+)$. For ε small enough, the point x cannot belong to the non-wandering set, because otherwise $\varphi^{t_\varepsilon}(x)$ would belong to Ω (by invariance) and would also be ε-close to K and thus belong to K (since the basic sets are isolated). Then, our goal is to deduce from the fact that K is basic that x belongs to Ω. It will lead to the contradiction. To prove that $x \in \Omega$, we come back to the definition and consider an open set O which contains x. Fix $T > 0$. It is sufficient to exhibit a $t > T$ such that $\varphi^t(O)$ intersects O. To do so, we consider small disks $\mathcal{D}_-$, $\mathcal{D}_+$ contained in O which contain x and intersect respectively the unstable manifold of x_-, the stable manifold of x_+ in the sense that $\varphi^{t_{\varepsilon_0}}(\mathcal{D}_+)$ intersects transversally $W^{\mathrm{so}}_{\varepsilon_0}(\varphi^{t_{\varepsilon_0}}(x_+))$ to a point[16] and $\varphi^{-t_{\varepsilon_0}}(\mathcal{D}_-)$ intersects transversally $W^{\mathrm{uo}}_{\varepsilon_0}(\varphi^{-t_{\varepsilon_0}}(x_-))$ to a point[17], where ε_0 is given by the stable manifold theorem. Now, the result follows from an application of the technical Lemma B.2 which gives the existence of a $t > T$ and of disks $\widetilde{\mathcal{D}}_+$, $\widetilde{\mathcal{D}}_+$ contained respectively in $\mathcal{D}_+$, $\mathcal{D}_+$, such that $x \in \widetilde{\mathcal{D}}_+ \cap \widetilde{\mathcal{D}}_-$ and

$$\varphi^t(\widetilde{\mathcal{D}}_+) \cap \widetilde{\mathcal{D}}_- \neq \varnothing.$$

Therefore $x \in \Omega$, which leads to the expected contradiction. □

Corollary B.4 (Transitivity of Smale relation). — *Let us assume that the flow φ^t satisfies the strong transversality assumption* (11). *Let K_i, K_j, K_k be three basic sets such that $K_i \leqslant K_j$ and $K_j \leqslant K_k$, then we have transitivity of the Smale relation in the sense that $K_i \leqslant K_k$.*

Proof of Corollary B.4. — Let x be in $W^{\mathrm{u}}(x_-) \cap W^{\mathrm{s}}(x_+)$ with $(x_-, x_+) \in K_i \times K_j$ and let y be in $W^{\mathrm{u}}(y_-) \cap W^{\mathrm{s}}(y_+)$ with $(y_-, y_+) \in K_j \times K_k$. Up to replacing (x, x_-, x_+)

[16]This means that the disk $\mathcal{D}_+$ has the same dimension that $W^{\mathrm{u}}_{\varepsilon_0}(\varphi^{t_{\varepsilon_0}}(x_+))$.

[17]Similarly, the disk $\mathcal{D}_-$ has the same dimension that $W^{\mathrm{s}}_{\varepsilon_0}(\varphi^{-t_{\varepsilon_0}}(x_-))$.

by $(\varphi^{t_{\varepsilon_0}}(x), \varphi^{t_{\varepsilon_0}}(x_-), \varphi^{t_{\varepsilon_0}}(x_+))$ and (y, y_-, y_+) by $(\varphi^{-t_{\varepsilon_0}}(y), \varphi^{-t_{\varepsilon_0}}(y_-), \varphi^{-t_{\varepsilon_0}}(y_+))$, where ε_0 is given by the stable manifold theorem and where $t_{\varepsilon_0} > 0$ is sufficiently large so that $\varphi^{t_{\varepsilon_0}}(x) \in W^{\mathrm{s}}_{\varepsilon_0}(\varphi^{t_{\varepsilon_0}}(x_+))$ and $\varphi^{-t_{\varepsilon_0}}(y) \in W^{\mathrm{s}}_{\varepsilon_0}(\varphi^{-t_{\varepsilon_0}}(y_-))$, we can assume that $x \in W^{\mathrm{u}}(x_-) \cap W^{\mathrm{s}}_{\varepsilon_0}(x_+)$ and $y \in W^{\mathrm{u}}_{\varepsilon_0}(y_-) \cap W^{\mathrm{s}}(y_+)$. Since the intersections $W^{\mathrm{u}}(x_-) \cap W^{\mathrm{so}}(x_+)$, $W^{\mathrm{uo}}(y_-) \cap W^{\mathrm{s}}(y_+)$ are transverse, then, for any disk $\mathcal{D}_+$ of dimension $n - \dim W^{\mathrm{so}}(x_+)$ transverse to $W^{\mathrm{so}}(x_+)$ which contains x and which is contained in $W^{\mathrm{u}}(x_-)$ and for any disk $\mathcal{D}_-$ of dimension $n - \dim W^{\mathrm{uo}}(y_-)$ transverse to $W^{\mathrm{uo}}(y_-)$ which contains y and which is contained in $W^{\mathrm{s}}(y_+)$, Lemma B.2 ensures that for every $T \geqslant 0$ there exists a $t > T$ and there exist smaller disks $\widetilde{\mathcal{D}}_-$, $\widetilde{\mathcal{D}}_+$ (as in the statement of the lemma) such that

$$\varphi^t(\widetilde{\mathcal{D}}_+) \cap \widetilde{\mathcal{D}}_- \neq \varnothing.$$

In particular, there exists z such that $z \in W^{\mathrm{u}}(K_i) \cap W^{\mathrm{s}}(K_k)$. □

In the next corollary of Lemma B.2, we give a proof of the claimed relations (13) and (14).

Corollary B.5. — *The following statements are satisfied.*

(1) *If φ^t satisfies the no-cycle property, then for every total order relation on the basic sets (in the sense of Section* 2.5.1*) and for every $i \in [\![1, N]\!]$ the set $\bigcup_{i\leqslant j} W^{\mathrm{u}}(K_j)$ is compact.*

(2) *If φ^t satisfies the strong transversality assumption, then for every basic set K, the set $\bigcup_{K',\, K\leqslant K'} W^{\mathrm{u}}(K')$ is compact and is equal to $\overline{W^{\mathrm{u}}(K)}$. Moreover, we have $\overline{W^{\mathrm{u}}(K)} \supseteq W^{\mathrm{u}}(K')$ if and only if $K \leqslant K'$.*

Proof of Corollary B.5(1). — This step only rests on the existence of filtrations given in Lemma 6.3 for any total order relation (Section 2.5.1). Indeed, according to Lemma 6.3, for every $i \in [\![1, N]\!]$, there exists an open set $\mathcal{O}^+_{N-i+1}$ which contains $\bigcup_{i\leqslant j} W^{\mathrm{u}}(K_j)$, whose closure is φ^1 stable and does not intersect any K_ℓ with $\ell < i$. Now, we fix some $i \in [\![1, N]\!]$. From the previous fact, the closure of $\bigcup_{i\leqslant j} W^{\mathrm{u}}(K_j)$ must be contained in $\overline{\mathcal{O}^+_{N-i+1}}$. Compactness follows from the equality

$$\bigcup_{i\leqslant j} W^{\mathrm{u}}(K_j) = \bigcap_{n\geqslant 0} \varphi^n\big(\overline{\mathcal{O}^+_{N-i+1}}\big).$$

Proof of Corollary B.5(2). — Point (2) is a consequence of (1) and of Lemma B.2, as we shall explain. First, we fix some basic set K. As a consequence of Corollary B.4, we can relabel the basic sets so that the set $\{K' \text{ basic}, K \leqslant K'\}$, of cardinal $N - i$ (for some $i \in [\![0, N-1]\!]$), is sent bijectively to $\{K_j,\ i \leqslant j \leqslant N\}$ (with $K = K_i$) and so that the new label on $(K_\ell)_{1\leqslant \ell\leqslant N}$ defines a total order relation in the sense of Section 2.5.1. Now, using (1) for this new total order relation, we deduce that $\bigcup_{i\leqslant j} W^{\mathrm{u}}(K_j) = \bigcup_{K',\, K\leqslant K'} W^{\mathrm{u}}(K')$ is a compact set.

In order to prove the claimed statement, we apply Lemma B.2 to show that this last set is included in $\overline{W^{\mathrm{u}}(K)}$, or equivalently that for every basic set K' such that $K \leqslant K'$ we have $W^{\mathrm{u}}(K') \subset \overline{W^{\mathrm{u}}(K)}$. Let K' be a basic set such that $K \leqslant K'$, fix $y \in W^{\mathrm{u}}(y_-)$

with $y_- \in K'$ and choose $x \in W^{\mathrm{u}}(x_-) \cap W^{\mathrm{s}}(x_+)$ with $(x_-, x_+) \in K \times K'$. As in the proof of Corollary B.4, we can assume that $x \in W^{\mathrm{u}}(x_-) \cap W^{\mathrm{s}}_{\varepsilon_0}(x_+)$ and $y \in W^{\mathrm{u}}_{\varepsilon_0}(y_-)$. Moreover, for every $\varepsilon > 0$, if we choose some disks $\mathcal{D}_+$ and $\mathcal{D}_-$ such that

$$x \in \mathcal{D}_+ \subset B_g(x, \varepsilon), \qquad y \in \mathcal{D}_- \subset B_g(y, \varepsilon),$$

as in the statement of Lemma B.2, there exists an element $z \in \varphi^t(\mathcal{D}_+) \cap \mathcal{D}_-$ which satisfies by construction

$$d_g(z, y) < \varepsilon \quad \text{ and } \quad z \in \varphi^t(\mathcal{D}_+) \subset W^{\mathrm{u}}(K).$$

Since this last result holds for every $\varepsilon > 0$, we deduce that every $y \in W^{\mathrm{u}}(K')$ belongs to $\overline{W^{\mathrm{u}}(K)}$. Finally, for every basic set K' such that $K \leqslant K'$ we have $\overline{W^{\mathrm{u}}(K)} \supseteq W^{\mathrm{u}}(K')$ and thus

$$\overline{W^{\mathrm{u}}(K)} \supseteq \bigcup_{K',\, K \leqslant K'} W^{\mathrm{u}}(K'). \qquad \square$$

Proof of the technical lemma B.2. — The result is an application of the λ-lemma. Consider a disk $\mathcal{D}_+$ which is transverse to $W^{\mathrm{so}}_{\varepsilon_0}(x_+)$ and consider a disk $\mathcal{D}_-$ which is transverse to $W^{\mathrm{uo}}_{\varepsilon_0}(y_-)$ such that

$$\mathcal{D}_+ \cap W^{\mathrm{so}}_{\varepsilon_0}(x_+) = \{q_+\}, \qquad \mathcal{D}_- \cap W^{\mathrm{uo}}_{\varepsilon_0}(y_-) = \{q_-\}.$$

Up to replacing x_+ and y_- by respectively $\varphi^t(x_+)$ and $\varphi^{t'}(y_-)$ for some $t, t' \in \mathbb{R}$, we can assume that $q_+ \in W^{\mathrm{s}}_{\varepsilon_0}(x_+)$ and $q_- \in W^{\mathrm{u}}_{\varepsilon_0}(y_-)$. Note that the disks $\mathcal{D}_+$, $\mathcal{D}_-$ are respectively of dimension $d_{\mathrm{u}} = \dim E_{\mathrm{u}}|_K$, $d_{\mathrm{s}} = \dim E_{\mathrm{s}}|_K = n - 1 - d_{\mathrm{u}}$ (which are constant since K is topologically transitive). In this proof, we are going to apply the λ-lemma (Lemma B.1) three times and thus consider smaller disks (as in the statement of the lemma) at each application. To lighten notations, we still denote them by $\mathcal{D}_+$, $\mathcal{D}_-$. Now, applying the λ-lemma of Proposition B.1, we deduce that, for every $\varepsilon > 0$, there exists $t(\varepsilon) > 0$ such that

$$\varphi^{t(\varepsilon)}(\mathcal{D}_+) \text{ is } \mathcal{C}^1 \ \varepsilon\text{-close to } W^{\mathrm{u}}_{\varepsilon_1}(\varphi^{t(\varepsilon)}(x_+))$$

up to considering a smaller disk $\mathcal{D}_+$ whose size depends on ε, where $\varepsilon_1 > 0$ is the constant appearing in the λ-lemma. If we apply the λ-lemma for the flow φ^{-t} instead of φ^t then we obtain similarly

$$\varphi^{-t(\varepsilon)}(\mathcal{D}_-) \text{ is } \mathcal{C}^1 \ \varepsilon\text{-close to } W^{s}_{\varepsilon_1}(\varphi^{-t(\varepsilon)}(y_-)), \tag{49}$$

up to taking a larger $t(\varepsilon)$ and up to considering a smaller disk $\mathcal{D}_-$ (whose size also depends on ε). Now, we choose ε sufficiently small so that it ensures the uniform transversality of $\mathcal{C}^1$ ε-perturbations of the (weak) stable and unstable manifolds on K. Such a choice is always possible because transversality of submanifolds is an open condition for the $\mathcal{C}^1$ topology—we refer to [52, §5.3.3, p. 88] for this result[18]—and

[18] Although the referenced result deals with the transverse intersection of a graph of a $\mathcal{C}^1$ map with a fixed submanifold, it can be applied to the case of transverse intersection of two graphs of $\mathcal{C}^1$ maps. Indeed, since both graphs are submanifolds of M (of dimension d_{uo} and d_{s} with $n = d_{\mathrm{uo}} + d_{\mathrm{s}}$ for example), we can see them as the images of two injective maps $i_1 : \mathbb{R}^{d_{\mathrm{uo}}} \to M$ and $i_2 : \mathbb{R}^{d_{\mathrm{s}}} \to M$.

the uniformity comes from the compactness of the basic set K and Hölder regularity of the stable and unstable foliations on K.

The main idea consists in connecting the points $\varphi^{t(\varepsilon)}(x_+)$ and $\varphi^{-t(\varepsilon)}(y_-)$ using periodic orbits whose distance to both points will be chosen sufficiently small. From the assumption that K is topologically transitive, for every $\epsilon > 0$, there exists a point $z_\epsilon \in K$ whose positive orbit is dense in K and a time $T(\epsilon) > 0$ such that

$$d_g(\varphi^{t(\varepsilon)}(x_+), z_\epsilon) < \frac{\epsilon}{2}, \qquad d_g(\varphi^{-t(\varepsilon)}(y_-), \varphi^{T(\epsilon)}(z_\epsilon)) < \frac{\epsilon}{2}.$$

By continuity of $\varphi^{T(\epsilon)}$ and since K is the closure of its periodic orbits, we deduce that there exists a periodic point p_ϵ (which depends on ϵ and ε) of period $T_0(\epsilon) > 0$ such that

$$d_g(\varphi^{t(\varepsilon)}(x_+), p_\epsilon) < \epsilon, \qquad d_g(\varphi^{-t(\varepsilon)}(y_-), \varphi^{T(\epsilon)}(p_\epsilon)) < \epsilon.$$

Now, we are going to use the Bowen brackets and their continuity, namely the continuity of the well-defined maps $[.,.]_{\flat/\sharp} : \{(z_1, z_2) \in K \times K,\ d_g(z_1, z_2) \leqslant \epsilon\} \to K$ given by the transverse intersections

$$W^{\mathrm{u}}_\epsilon(z_1) \cap W^{\mathrm{so}}_\epsilon(z_2) = \{[z_1, z_2]_\flat\}, \qquad W^{\mathrm{uo}}_\epsilon(z_1) \cap W^{s}_\epsilon(z_2) = \{[z_1, z_2]_\sharp\}$$

for all $z_1, z_2 \in K$ and for all sufficiently small $\epsilon > 0$ (which only depends on K and φ^t) as recalled[(19)] in [35, Prop. 6.2.2]. Now, we fix such a $\epsilon > 0$ and we choose ϵ sufficiently small so that $\epsilon < \varepsilon_1/2$. If we note that $W^{\mathrm{u}}([\varphi^{t(\varepsilon)}(x_+), p_\epsilon]_\flat) = W^{\mathrm{u}}(\varphi^{t(\varepsilon)}(x_+))$ and that $W^{\mathrm{so}}([\varphi^{t(\varepsilon)}(x_+), p_\epsilon]_\flat) = W^{\mathrm{so}}(p_\epsilon)$ and if we consider the adapted chart at the point $[\varphi^{t(\varepsilon)}(x_+), p_\epsilon]_\flat$ which sends $W^{\mathrm{u}}_\epsilon([\varphi^{t(\varepsilon)}(x_+), p_\epsilon]_\flat)$ to $\mathbb{R}^{d_{\mathrm{u}}} \times \{0\}^{d_{\mathrm{s}}+1}$ and $W^{\mathrm{so}}_\epsilon([\varphi^{t(\varepsilon)}(x_+), p_\epsilon])$ to $\{0\}^{d_{\mathrm{u}}} \times \mathbb{R}^{d_{\mathrm{s}}+1}$, then we get that

$$\text{the disk } \varphi^{t(\varepsilon)}(\mathcal{D}_+) \text{ intersects } W^{\mathrm{so}}_\epsilon([\varphi^{t(\varepsilon)}(x_+), p_\epsilon]_\flat) \text{ transversally at a point.}$$

Furthermore, since $W^{\mathrm{so}}_{\varepsilon_1}(p_\epsilon)$ contains $W^{\mathrm{so}}_\epsilon([\varphi^{t(\varepsilon)}(x_+), p_\epsilon]_\flat)$ thanks to the choice of $\epsilon < \varepsilon_1/2$, we deduce that

$$\varphi^{t(\varepsilon)}(\mathcal{D}_+) \text{ intersects } W^{\mathrm{so}}_{\varepsilon_1}(p_\epsilon) \text{ transversally at the same point.}$$

Transporting this intersection along the diffeomorphism $\varphi^{T(\epsilon)}$ implies that

$$\varphi^{t(\varepsilon)+T(\epsilon)}(\mathcal{D}_+) \text{ intersects } \varphi^{T(\epsilon)}\left(W^{\mathrm{so}}_{\varepsilon_1}(p_\epsilon)\right) \subset W^{\mathrm{so}}_{\varepsilon_1}(\varphi^{T(\epsilon)}(p_\epsilon)) \text{ transversally.}$$

Now, if we apply again the λ-lemma for the disk $\varphi^{t(\varepsilon)+T(\epsilon)}(\mathcal{D}_+)$ which intersects transversally the stable manifold of the periodic point $\varphi^{T(\epsilon)}(p_\epsilon)$, then there exists a

---

Moreover, it is not hard to see that the two manifolds are transverse if and only if the map (i_1, i_2) is transverse to the diagonal $\Delta_M := \{(x, x), x \in M\}$

(19) Note that the Bowen bracket has values in K means that K has a local product structure. This local product structure follows from the fact that the basic set K is locally maximal (by definition). Indeed, it is not hard to see that $[z_1, z_2]_\flat, [z_1, z_2]_\sharp \in K$ as soon as ϵ is chosen small enough so that $\{x,\ d_g(x, K) < \epsilon\} \subset O$, for O open set such that $\bigcap_{t\in\mathbb{R}} \varphi^t(O) = K$. Moreover, the continuity of the Bowen bracket follows from the continuity of the stable and unstable leaves.

disk $D \subset \varphi^{t(\varepsilon)+T(\epsilon)}(\mathcal{D}_+)$ of same dimension than $\mathcal{D}_+$ which contains $\varphi^{T(\epsilon)}(p_\epsilon)$ and there exists an integer k sufficiently large so that

$$\varphi^{kT_0(\epsilon)}(D) \text{ is } \mathcal{C}^1\ \varepsilon\text{-close to } W^{\mathrm{u}}_{\varepsilon_1}(\varphi^{T(\epsilon)}(p_\epsilon)). \tag{50}$$

On the other hand, let us go back to the disk $\varphi^{-t(\varepsilon)}(\mathcal{D}_-)$ that we moved backward along the flow. We consider an adapted chart near the point $[\varphi^{T(\epsilon)}(p_\epsilon), \varphi^{-t(\varepsilon)}(y_-)]_\sharp$ such that $W^{\mathrm{uo}}_\epsilon([\varphi^{T(\epsilon)}(p_\epsilon), \varphi^{-t(\varepsilon)}(y_-)]_\sharp)$ is sent to $\mathbb{R}^{d_{\mathrm{u}}+1} \times \{0\}^{d_{\mathrm{s}}}$ and such that $W^s_\epsilon([\varphi^{T(\epsilon)}(p_\varepsilon), \varphi^{-t(\varepsilon)}(y_-)]_\sharp)$ is sent to $\{0\}^{d_{\mathrm{u}}+1} \times \mathbb{R}^{d_{\mathrm{s}}}$. If we note that

$$W^{\mathrm{uo}}([\varphi^{T(\epsilon)}(p_\epsilon), \varphi^{-t(\varepsilon)}(y_-)]_\sharp) = W^{\mathrm{uo}}(\varphi^{T(\epsilon)}(p_\epsilon))$$

and

$$W^{\mathrm{s}}([\varphi^{T(\epsilon)}(p_\varepsilon), \varphi^{-t(\varepsilon)}(y_-)]_\sharp) = W^{\mathrm{s}}(\varphi^{-t(\varepsilon)}(y_-)),$$

then we deduce from (49) that

$$\varphi^{-t(\varepsilon)}(\mathcal{D}_-) \text{ is } \mathcal{C}^1\ \varepsilon\text{-close to } W^s_\epsilon([\varphi^{T(\epsilon)}(p_\epsilon), \varphi^{-t(\varepsilon)}(y_-)]),$$

up to considering a smaller disk $\mathcal{D}_-$ (of same dimension and whose size depends on ε, ϵ). Finally, thanks to (50) and thanks to the choice of ε, we deduce that

$$\bigcup_{|t|<\varepsilon_1} \varphi^t\big(\varphi^{kT_0(\epsilon)}(D)\big) \text{ and } \varphi^{-t(\varepsilon)}(\mathcal{D}_-) \text{ intersect each other transversally.}$$ □

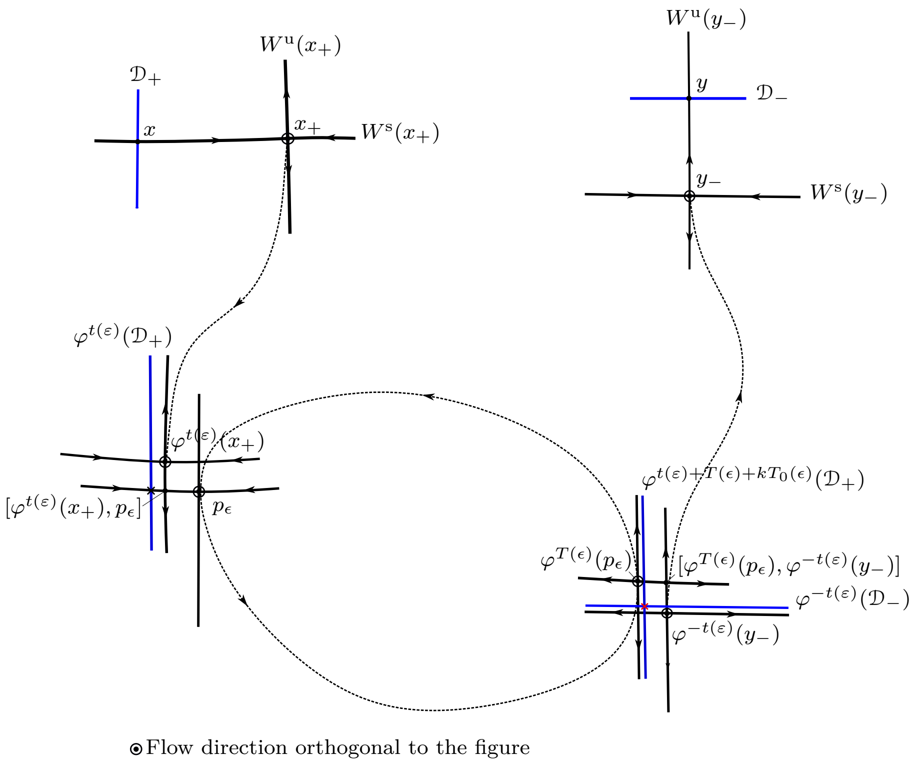


Figure 15. Illustration of some elements used in the proof of Lemma B.2.

B.2. Proof of the generalized λ-lemma. — When the orbit $z \in K$ along the flow is periodic (of period $T > 0$), the analysis takes place on a chart near z on which the diffeomorphism φ^T acts naturally. In the case of non periodic orbit, we need to consider a moving chart defined at each point of the orbit $(\varphi^k(z))_{k\geqslant 0}$. We in fact use $\mathcal{C}^1$-adapted charts for this trajectory used and constructed in [21, Def. 4.8, p. 36].

Definition B.6 (Adapted charts). — Let $x \in K$. A $\mathcal{C}^1$ diffeomorphism

$$\zeta_x : \mathcal{U}_x \longrightarrow \mathcal{V}_x, \quad x \in \mathcal{U}_x \subset M, \quad 0 \in \mathcal{V}_x \subset \mathbb{R}^n, \quad \zeta_x(x) = 0,$$

is called an adapted chart for φ^t centered at x if:

(1) For each $y \in \mathcal{U}_x$, we have $D\zeta_x(y)V(y) = \partial_{x_1} = (1, 0, \dots, 0)^\top \in T_{\zeta_x(y)}\mathbb{R}^n = \mathbb{R}^n$.

(2) $D\zeta_x(x)E_\mathrm{u}(x) = E_\mathrm{u}(0_{\mathbb{R}^n}) := \{0\}^{d_\mathrm{s}+1} \times \mathbb{R}^{d_\mathrm{u}}$ and the restriction of $D\zeta_x(x)$ to $E_\mathrm{u}(x)$ is an isometry from the adapted metric defined in Section 7.1 to the Euclidean metric.

(3) $D\zeta_x(x)E_\mathrm{s}(x) = E_\mathrm{s}(0_{\mathbb{R}^n}) := \{0\} \times \mathbb{R}^{d_\mathrm{s}} \times \{0\}^{d_\mathrm{u}}$ and the restriction of $D\zeta_x(x)$ to $E_\mathrm{u}(x)$ is an isometry from the adapted metric to the euclidean metric.

For the sake of our analysis, we consider such a family of adapted charts and we explain how to modify it to straighten the weak-stable manifold and the unstable manifold. Precisely, we construct a family of charts satisfying the following assumptions.

Definition B.7 (Straightening charts). — A family of $\mathcal{C}^1$ diffeomorphisms

$$\chi_x : U_x \longrightarrow V_x, \quad x \in K, \quad x \in U_x \subset M, \quad 0 \in V_x \subset \mathbb{R}^n, \quad \chi_x(x) = 0,$$

is called a family of straightening charts for φ^t at K if: for all $x \in K$, we have

$$\chi_x(U_x \cap W^{\mathrm{so}}_{\varepsilon_0}(x)) \subset \mathbb{R}^{d_\mathrm{s}+1} \times \{0\}^{d_\mathrm{u}}$$
$$\chi_x(U_x \cap W^{\mathrm{u}}_{\varepsilon_0}(x)) \subset \{0\}^{d_\mathrm{s}} \times \mathbb{R}^{d_\mathrm{u}},$$

where ε_0 is given by the stable manifold theorem, where $W^{\mathrm{so}}_{\varepsilon_0}(x)$ denotes the local weak-stable manifold at x and where $W^{\mathrm{u}}_{\varepsilon_0}(x)$ denotes the local unstable manifold at x.

First of all, we begin with explaining how to deduce a family of straightening charts from a family of adapted charts. Arguing as in [21, p. 36], we first consider a family of adapted charts $(\zeta_x : \mathcal{U}_x \subset M \to \mathcal{V}_x \subset \mathbb{R}^n)_{x\in K}$. Then, we consider the rescaled chart

$$\widetilde{\zeta}_x := T \circ \zeta_x : \mathcal{U}_x \longrightarrow \widetilde{\mathcal{V}}_x := T(\mathcal{V}_x), \quad x \in K, \quad T = \delta_1^{-1}\,\mathrm{Id}_{\mathbb{R}_n},$$

for some $\delta_1 > 0$ and we define the family of diffeomorphisms

$$\widetilde{\psi}_x := \widetilde{\zeta}_{\varphi^1(x)} \circ \varphi^1 \circ \widetilde{\zeta}_x^{-1} : \widetilde{\mathcal{V}}_x \subset \mathbb{R}^n \longrightarrow \widetilde{\mathcal{V}}_{\varphi^1(x)} \subset \mathbb{R}^n, \qquad x \in K.$$

As explained in [21, §3.4], the stable manifold theorem applied to the family of maps $(\widetilde{\psi}_{\varphi^k(x)})_{k\in\mathbb{Z}}$ gives the existence of a stable manifold $W^{\mathrm{s}}_{\mathrm{loc},x}(0_{\mathbb{R}^n})$ and of an unstable manifold $W^{\mathrm{u}}_{\mathrm{loc},x}(0_{\mathbb{R}^n})$ which are invariant under the family of diffeomorphisms $(\widetilde{\psi}_{\varphi^k(x)})_{k\in\mathbb{Z}}$ as soon as $\delta_1 > 0$ is sufficiently small (δ_1 only depends on φ^t, K). We also assume that δ_1 is small enough so that $[-2,2]^n \subset \widetilde{\mathcal{V}}_x$ for every $x \in K$. Recall that the local stable and unstable manifolds on M are defined by pullback from these local stable and unstable manifolds on $\mathbb{R}^n$. Moreover, the unstable and stable manifolds restricted to $[-2,2]^n$ can be seen as graph of functions $F_{\mathrm{u}} : [-2,2]^{d_{\mathrm{u}}} \to [-2,2]^{d_{\mathrm{s}}+1}$, $F_s : [-2,2]^{d_{\mathrm{s}}} \to [-2,2]^{d_{\mathrm{u}}+1}$ which are of class $\mathcal{C}^{1,1}$, defined as the set of $\mathcal{C}^1$ functions whose differential is Lipschitz[20], and which satisfy the bounds

$$\|F_{\mathrm{u}}\|_{\mathcal{C}^1} \leqslant C\delta_1, \quad \|F_s\|_{\mathcal{C}^1} \leqslant C\delta_1 \tag{51}$$

for some constant $C > 0$ depending only on φ^t, K. Here, $\|.\|_{\mathcal{C}^1}$ denotes the maximum of the $\mathcal{C}^1$ norm and the Lipschitz norm of the differential. Following the construction of [21], the weak-unstable is then defined by

$$W^{\mathrm{uo}}_{\mathrm{loc},x}(0) := \bigl\{x + (s, 0, \dots, 0), \text{ for } x \in W^{\mathrm{u}}_{\mathrm{loc},x}(0) \text{ and } -2 \leqslant s \leqslant 2\bigr\} \cap [-2,2]^n$$

and the weak-stable manifold $W^{\mathrm{so}}_{\mathrm{loc},x}(0)$ has a very similar definition. Note that the weak-unstable manifold $W^{\mathrm{so}}_{\mathrm{loc},x}(0)$ is the graph of the $\mathcal{C}^1$ map

$$F_{\mathrm{so}} : (x_1, x_{\mathrm{s}}) \in [-2,2]_{x_1} \times [-2,2]^{d_{\mathrm{s}}}_{x_{\mathrm{u}}} \longmapsto \pi_{\mathrm{u}}(F_{\mathrm{s}}(x_{\mathrm{s}})) \in [-2,2]^{d_{\mathrm{u}}},$$

where $\pi_{\mathrm{u}}(x_1, x_{\mathrm{u}}) = x_{\mathrm{u}}$ for all $(x_1, x_{\mathrm{u}}) \in [-2,2]_{x_1} \times [-2,2]^{d_{\mathrm{u}}}_{x_{\mathrm{s}}}$. Now, we consider the straightening operator

$$\begin{gathered} S : (x_{\mathrm{o}}, x_{\mathrm{s}}, x_{\mathrm{u}}) \in [-2,2]^n \longmapsto (\widetilde{x}_{\mathrm{o}}, \widetilde{x}_{\mathrm{s}}, \widetilde{x}_{\mathrm{u}}), \\ \text{with } (\widetilde{x}_{\mathrm{o}}, \widetilde{x}_{\mathrm{s}}) = (x_{\mathrm{o}}, x_{\mathrm{s}}) - F_{\mathrm{u}}(x_{\mathrm{u}}), \quad \widetilde{x}_{\mathrm{u}} = x_{\mathrm{u}} - F_{\mathrm{so}}(x_{\mathrm{o}}, x_{\mathrm{s}}), \end{gathered}$$

where $(x_{\mathrm{o}}, x_{\mathrm{s}}, x_{\mathrm{u}})$ denotes the decomposition $\mathbb{R} \times \mathbb{R}^{d_{\mathrm{s}}} \times \mathbb{R}^{d_{\mathrm{u}}}$ of $\mathbb{R}^n$. Moreover, the bounds (51) ensure that S is a diffeomorphism on its image (which contains the ball $[-2 + C\delta_1, 2 - C\delta_1]$ and is contained in $[-2 - C\delta_1, 2 + C\delta_1]$). Furthermore, it is not hard to see that S is indeed straightening the weak-stable and the unstable manifolds in the sense that $S(W^{\mathrm{so}}_{\mathrm{loc},x}(0)) \subset \mathbb{R}^{d_{\mathrm{s}}+1} \times \{0\}^{d_{\mathrm{u}}}$ and $S(W^{\mathrm{u}}_{\mathrm{loc},x}(0)) \subset \{0\}^{d_{\mathrm{s}}+1} \times \mathbb{R}^{d_{\mathrm{u}}}$. Finally, we introduce the adapted chart

$$\chi_x = S \circ \widetilde{\zeta}_x \; : \; U_x := \mathcal{U}_x \cap \widetilde{\zeta}_x^{-1}(]-2,2[^n) \longrightarrow V_x := \mathrm{Ran}(\chi_x)$$

which satisfies all the claimed properties.

Another zoom for local estimates. — For $\delta > 0$, we consider the family diffeomorphisms

$$\psi_x = \widetilde{\chi}_{\varphi^1(x)} \circ \varphi^1 \circ \widetilde{\chi}_x^{-1} : \widetilde{V}_x \longrightarrow \widetilde{V}_{\varphi^1(x)}, \quad \psi_x(0_{\mathbb{R}^n}) = 0_{\mathbb{R}^n}, \quad x \in K,$$

where $\widetilde{V}_x = \delta^{-1} V_x$, $\widetilde{\chi}_x = \delta^{-1}\chi_x$. denoted for all $y \in [-1,1]^n$ by

$$D\psi_x(y) = \begin{pmatrix} A^x_{\mathrm{so},\mathrm{so}}(y) & A^x_{\mathrm{u},\mathrm{so}}(y) \\ A^x_{\mathrm{so},\mathrm{u}}(y) & A^x_{\mathrm{u},\mathrm{u}}(y) \end{pmatrix}$$

[20] The regularity is not optimal here and could be improved using adapted charts with higher regularity, as it was done in [21]. But it will be enough for the proof of the λ-lemma.

has for coefficients the linear maps

$$A^x_{\mathrm{so,so}}(y) : \mathbb{R}^{d_\mathrm{s}+1} \longrightarrow \mathbb{R}^{d_\mathrm{s}+1}, \qquad A^x_{\mathrm{so,u}}(y) : \mathbb{R}^{d_\mathrm{s}+1} \longrightarrow \mathbb{R}^{d_\mathrm{u}},$$
$$A^x_{\mathrm{u,so}}(y) : \mathbb{R}^{d_\mathrm{u}} \longrightarrow \mathbb{R}^{d_\mathrm{s}+1}, \qquad A^x_{\mathrm{u,u}}(y) : \mathbb{R}^{d_\mathrm{u}} \longrightarrow \mathbb{R}^{d_\mathrm{u}},$$

which satisfy for all $y \in [-1,1]^n$:

$$\|A^x_{\mathrm{so,so}}(y)\| \leqslant 1 + C'\delta, \quad \|(A^x_{\mathrm{u,u}}(y))^{-1}\| \leqslant e^{-\lambda/2} + C'\delta, \\ \|A^x_{\mathrm{so,u}}(y)\| \leqslant C'\delta|y_u|, \quad \|A^x_{\mathrm{u,so}}(y)\| \leqslant C'\delta|y|, \tag{52}$$

for some constant $C' > 0$. These estimates are a consequence of (23). Moreover, it is important to remark that $A^{(k)}_{\mathrm{so,u}}$ vanishes on the stable manifold $W^{\mathrm{so}}_{\mathrm{loc},x}(0) \cap [-1,1]^n = [-1,1]^{d_\mathrm{s}+1} \times \{0\}^{d_\mathrm{u}}$.

Proof of Proposition B.1. — We work in the family of charts $(\widetilde{\chi}_x)_{x\in K}$ defined just before the proof from a family of straightening charts $(\chi_x)_{x\in K}$ (see Definition B.7) for some $\delta > 0$ sufficiently small so that

$$(1+3C'\delta)e^{-\lambda/4} \leqslant e^{-\lambda/8}, \quad (1+C'\delta)(e^{-\lambda/2}+C'\delta) < e^{-\lambda/4}, \\ (e^{-\lambda/2}+C'\delta)\Big(1 - \frac{C'\delta}{2}(e^{-\lambda/2}+C'\delta)\Big)^{-1} < e^{-\lambda/4}. \tag{53}$$

Consider a point $z \in K$ whose orbit $(\varphi^k(z))_{k\in\mathbb{Z}}$ is not reduced to a fixed point and consider a disk $\mathcal{D} \subset [-1,1]^n$ which intersects the stable manifold $W^{\mathrm{so}}_{\mathrm{loc},z}(0) \cap [-1,1]^n = [-1,1]^{d_\mathrm{s}+1} \times \{0\}^{d_\mathrm{u}}$ at some point $q \in W^s_{\mathrm{loc},z}(0)$ in the sense that

$$\mathcal{D} \cap W^{\mathrm{so}}_{\mathrm{loc},z}(0) = \{q\} \quad \text{and} \quad \mathbb{R}^n = T_q\mathcal{D} \oplus T_q W^{\mathrm{so}}_{\mathrm{loc},z}(0).$$

Since q belongs to the stable manifold $W^{\mathrm{s}}_{\mathrm{loc},z}(0)$, we deduce from the stable manifold theorem that there exists $c_0 > 0$ which depends on φ^t, K and δ such that

$$\forall k \in \mathbb{Z}, \qquad |\widetilde{\chi}_{\varphi^k(z)} \circ \varphi^k \circ \widetilde{\chi}_z^{-1}(q)| \leqslant c_0 e^{-k\lambda/2}|q|,$$

and therefore

$$\big|\psi^k_z(q)\big| \leqslant c_0 e^{-k\lambda/2} \quad \text{for } \psi^k_z := \psi_{\varphi^{k-1}(z)} \circ \cdots \circ \psi_z,$$

since $\psi_x = \widetilde{\chi}_{\varphi^1(x)} \circ \varphi^1 \circ \widetilde{\chi}_x^{-1}(q)$ $(x \in K)$. Contrary to the usual version of the λ-lemma, the diffeomorphism we apply to the disk $\mathcal{D}$ changes at each iteration. But since all the diffeomorphisms $\psi_{\varphi^k(z)}$ satisfy hyperbolic estimates which are uniform with respect to k, we claim that the usual proof can be adapted in this setting. For the sake of completeness, we briefly check that claim.

Step 1: Lipschitz bound for the tangent space of the iterated disks $\varphi^k(\mathcal{D})$ at $\psi^k_z(q)$

First, we can describe the tangent space $T_q\mathcal{D}$ at q as the graph of a linear map $E_0 : \mathbb{R}^{d_\mathrm{u}} \to \mathbb{R}^{d_\mathrm{s}+1}$ whose norm is bounded by some constant $\kappa > 0$. Similarly, for every $k \in \mathbb{N}$, we describe $T_{\varphi^k(q)}\varphi^k(\mathcal{D})$ as the graph of a linear map $E_k : \mathbb{R}^{d_\mathrm{u}} \to \mathbb{R}^{d_\mathrm{s}+1}$. The usual strategy consists in using the recursive relation

$$D\psi_{\varphi^k(z)}(\psi^k_z(q))T_{\psi^k_z(q)}\psi^k_z(\mathcal{D}) = T_{\psi^{k+1}_z(q)}\psi^{k+1}_z(\mathcal{D}), \quad \forall k \in \mathbb{N}, \tag{54}$$

to deduce a recursive relation on the norm of E_k. To simplify the notations, let us denote by q_k the point $\psi_z^k(q)$ and z_k the point $\varphi^k(z)$. As a consequence of (54) and

$$D\psi_{z_k}(q_k)\begin{pmatrix}E_k\\ I_{d_\mathrm{u}}\end{pmatrix}\begin{pmatrix}A^{z_k}_{\mathrm{so,so}}(q_k)E_k + A^{z_k}_{\mathrm{u,so}}(q_k)\\ A^{z_k}_{\mathrm{so,u}}(q_k)E_k + A^{z_k}_{\mathrm{u,u}}(q_k)\end{pmatrix} = \begin{pmatrix}A^{z_k}_{\mathrm{so,so}}(q_k)E_k + A^{z_k}_{\mathrm{u,so}}(q_k)\\ A^{z_k}_{\mathrm{u,u}}(q_k)\end{pmatrix}, \tag{55}$$

since $A^{z_k}_{\mathrm{so,u}}$ vanishes on $[-1,1]^{d_\mathrm{s}+1}\times\{0\}^{d_\mathrm{u}}$ by stability of the weak stable manifold. But since the terms in (55) belong to the graph of E_{k+1}, we must have

$$E_{k+1} = \big(A^{z_k}_{\mathrm{so,so}}(q_k)E_k + A^{z_k}_{\mathrm{u,so}}(q_k)\big)\,A^{z_k}_{\mathrm{u,u}}(q_k)^{-1}. \tag{56}$$

Now, it remains to see that the norms of E_k satisfy an exponential bound. Thanks to (56) and thanks to (52) again, we deduce that

$$\begin{aligned}\|E_{k+1}\| &\leqslant \big(\|A^{z_k}_{\mathrm{so,so}}(q_k)\|\|E_k\| + \|A^{z_k}_{\mathrm{u,so}}(q_k)\|\big)\times\|A^{z_k}_{\mathrm{u,u}}(q_k)^{-1}\|\\ &\leqslant \big((1+C'\delta)\|E_k\| + C'\delta|q_k|\big)(e^{-\lambda/2}+C'\delta)\\ &\leqslant e^{-\lambda/4}\|E_k\| + e^{-\lambda/4}c_0e^{-k\lambda/2},\end{aligned}$$

according to the estimates on δ given in (53). Now, if we consider the auxiliary sequence $u_k = \|E_k\|e^{k\lambda/4}$, then it is not hard to see that $u_{k+1}\leqslant u_k + c_0e^{-k\lambda/4}$ and, thus, that

$$\forall k\in\mathbb{N},\quad \|E_k\|\leqslant e^{-k\lambda/4}\Big(\kappa + \frac{c_0}{1-e^{-\lambda/4}}\Big).$$

In particular, we get that $\|E_k\|$ converges to 0 as $k\to+\infty$ and there exists $k_0\in\mathbb{N}$, such that

$$\forall k\geqslant k_0,\quad |q_k|\leqslant\frac{1}{2},\qquad \|E_k\|\leqslant\frac{1}{4}.$$

By continuity, there exists a disk $\widetilde{\mathcal{D}}\subset\mathcal{D}$ (of same dimension) which contains q such that $\psi_z^{k_0}(\widetilde{\mathcal{D}})$ is contained in the adapted chart near $\varphi^{k_0}(z)$ and which is the graph of a $\mathcal{C}^1$ map F_{k_0}. Precisely, there exists $F_{k_0}:[-\eta,\eta]^{d_\mathrm{u}}\to[-1,1]^{d_\mathrm{s}+1}$ for some $\eta\in(0,1)$ such that

$$\mathrm{Graph}(F_{k_0}) = \psi_z^{k_0}(\widetilde{\mathcal{D}}),\quad (F_{k_0}(0_{\mathbb{R}^{d_\mathrm{u}}}),0_{\mathbb{R}^{d_\mathrm{u}}}) = q_k,\quad \sup_{x_\mathrm{u}\in[-\eta,\eta]^{d_\mathrm{u}}}\|DF_{k_0}(x_\mathrm{u})\|\leqslant\frac{1}{2}.$$

Now, we consider a $\mathcal{C}^1$ extension(21) of F_{k_0}, that we denote by $\widetilde{F}_{k_0}$, such that

$$\widetilde{F}_{k_0}\in\mathcal{C}^1\left([-1,1]^{d_\mathrm{u}};[-1,1]^{d_\mathrm{s}+1}\right),\quad \widetilde{F}_{k_0}\big|_{[-\eta,\eta]^{d_\mathrm{u}}} = F_{k_0},\quad \sup_{x_\mathrm{u}\in[-1,1]^{d_\mathrm{u}}}\|D\widetilde{F}_{k_0}(x_\mathrm{u})\|\leqslant\frac{1}{2}.$$

Step 2: a graph transform argument for the map $\widetilde{F}_{k_0}$. — We first introduce the complete metric space

$$\begin{aligned}\mathcal{E} := \Big\{F\in\mathcal{C}^1\big([-1,1]^{d_\mathrm{u}};[-1,1]^{d_\mathrm{s}+1}\big),\ (F(0_{\mathbb{R}^{d_\mathrm{u}}}),0_{\mathbb{R}^{d_\mathrm{u}}})\in W^{\mathrm{s}}_{\mathrm{loc},z}(0),&\\ |F(0_{\mathbb{R}^{d_\mathrm{u}}})|\leqslant\frac{1}{2},\quad \sup_{x\in[-1,1]^n}\|DF(x)\|\leqslant\frac{1}{2}\Big\}.&\end{aligned}$$

(21)Thanks to the expansivity of the unstable manifold, the $\mathcal{C}^1$ convergence of the disk will not depend on the extension $\widetilde{F}_{k_0}$ of F_{k_0}. This fact will be detailed in step 3.

endowed with the distance $d_{\mathcal{E}}(.,.)$ defined, for all $F, \widetilde{F} \in \mathcal{E}$, by

$$d_{\mathcal{E}}(F, \widetilde{F}) = \max\Big(|F(0_{\mathbb{R}^{d_{\mathrm{u}}}}) - \widetilde{F}(0_{\mathbb{R}^{d_{\mathrm{u}}}})|, \sup_{x_{\mathrm{u}} \in [-1,1]^n} \|DF(x_{\mathrm{u}}) - D\widetilde{F}(x_{\mathrm{u}})\|\Big).$$

Now, we consider the family of graph transforms defined by

$$\text{(57)} \qquad \operatorname{Graph}(\Psi_k(F)) = \psi_{z_k}(\operatorname{Graph} F) \cap [-1,1]^n,$$

for all $k \in \mathbb{Z}$ and for all $F \in \mathcal{E}$. The fact that $[-1,1]^{d_{\mathrm{u}}} \subset \pi_{\mathrm{u}}(\psi_{z_k}(\operatorname{Graph} F))$ (which is implicitly used in (57)), where $\pi_{\mathrm{u}} : [-1,1]^n \to [-1,1]^{d_{\mathrm{u}}}$ denotes the orthogonal projection to $[-1,1]^{d_{\mathrm{u}}}$, follows from the stronger fact: there exists $\Lambda > 1$ (which depends on φ^t, K, δ) such that

$$\text{(58)} \qquad \forall F \in \mathcal{E},\ \forall k \in \mathbb{Z},\ \forall x_{\mathrm{u}} \in [-1,1]^{d_{\mathrm{u}}}, \quad \frac{d}{dx_{\mathrm{u}}} \pi_{\mathrm{u}}(\psi_{z_k}(F(x_{\mathrm{u}}), x_{\mathrm{u}})) \geqslant \Lambda.$$

This last inequality is a consequence of the expansivity of the unstable manifold. Now that the definition of the graph transforms given in (57) makes sense, our goal is to prove that for every $F \in \mathcal{E}$, for all $k \in \mathbb{Z}$ and for all $\ell \in \mathbb{N}$,

$$\text{(59)} \qquad d_{\mathcal{E}}\left(\Psi_k^{\ell}(F), 0\right) \leqslant \max(c_0, 1) e^{-\ell\lambda/8} d_{\mathcal{E}}(F, 0),$$

where Ψ_k^{ℓ} denotes the composed map $\Psi_{k+\ell} \circ \cdots \circ \Psi_k$. We can remark that (59) implies the $\mathcal{C}^1$ convergence of $\Psi_k^{\ell}(F)$ to the null map, which is geometrically the unstable manifold $W^{\mathrm{u}}_{\mathrm{loc},z}(0)$.

In order to prove (59), let $k \in \mathbb{Z}$, $F \in \mathcal{C}^1([-1,1]^{d_{\mathrm{u}}}, [-1,1]^{d_{\mathrm{s}}+1})$ and let $x = (F(x_{\mathrm{u}}), x_{\mathrm{u}}) \in [-1,1]^n$ for some $x_{\mathrm{u}} \in [-1,1]^{d_{\mathrm{u}}}$. By definition of the graph transform given in (57), we have

$$\text{(60)} \qquad D\psi_{z_k}(x) \begin{pmatrix} DF(x_{\mathrm{u}}) \\ I_{d_{\mathrm{u}}} \end{pmatrix} = \begin{pmatrix} A^{z_k}_{\mathrm{so,so}}(x) DF(x_{\mathrm{u}}) + A^{z_k}_{\mathrm{u,so}}(x) \\ A^{z_k}_{\mathrm{so,u}}(x) DF(x_{\mathrm{u}}) + A^{z_k}_{\mathrm{u,u}}(x) \end{pmatrix}.$$

From the definition of the graph transform Ψ_k, we get that

$$D\Psi_k(F)(\psi_{z_k}(x)_u) = \left(A^{z_k}_{\mathrm{so,so}}(x) DF(x_{\mathrm{u}}) + A^{z_k}_{\mathrm{u,so}}(x)\right)\left(A^{z_k}_{\mathrm{so,u}}(x) DF(x_{\mathrm{u}}) + A^{z_k}_{\mathrm{u,u}}(x)\right)^{-1},$$

where $\psi_{z_k}(x) = (\psi_{z_k}(x)_{\mathrm{so}}, \psi_{z_k}(x)_u)$. Taking the operator norm of $D\Psi_k(F)(\psi_{z_k}(x)_u)$, we get

$$\|D\Psi_k(F)(\psi_{z_k}(x)_u)\| \leqslant \left\|A^{z_k}_{\mathrm{so,so}}(x) DF(x_{\mathrm{u}}) + A^{z_k}_{\mathrm{u,so}}(x)\right\| \times I,$$

with $I = \left\|\left(A^{z_k}_{\mathrm{so,u}}(x) DF(x_{\mathrm{u}}) + A^{z_k}_{\mathrm{u,u}}(x)\right)^{-1}\right\|$. Moreover, we have

$$\text{(61)} \qquad I \leqslant \left\|\left(A^{z_k}_{\mathrm{u,u}}(x)\right)^{-1}\right\| \underbrace{\left\|\left(\mathrm{Id} + A^{z_k}_{\mathrm{so,u}}(x) DF(x_{\mathrm{u}}) \left(A^{z_k}_{\mathrm{u,u}}(x)\right)^{-1}\right)^{-1}\right\|}_{J}.$$

Since $F \in \mathcal{E}$, we have that

$$\begin{aligned} \left\|A^{z_k}_{\mathrm{so,u}}(x) DF(x_{\mathrm{u}}) \left(A^{z_k}_{\mathrm{u,u}}(x)\right)^{-1}\right\| &\leqslant \left\|A^{z_k}_{\mathrm{so,u}}(x)\right\| \|DF(x_{\mathrm{u}})\| \left\|\left(A^{z_k}_{\mathrm{u,u}}(x)\right)^{-1}\right\| \\ &\leqslant \frac{C'\delta}{2}(e^{-\lambda/2} + C'\delta) < 1. \end{aligned}$$

Therefore, we deduce that

$$J \leqslant \sum_{k=0}^{+\infty} \left\| A_{\mathrm{so,u}}^{z_k}(x) DF(x_{\mathrm{u}}) \left(A_{\mathrm{u,u}}^{z_k}(x)\right)^{-1} \right\|^k \leqslant \sum_{k=0}^{+\infty} \Big(\frac{C'\delta}{2}(e^{-\lambda/2}+C'\delta)\Big)^k$$
$$\leqslant \Big(1-\frac{C'\delta}{2}(e^{-\lambda/2}+C'\delta)\Big)^{-1}.$$

Putting all together the last inequalities with (61) and (53), we get that

$$I \leqslant (e^{-\lambda/2}+C'\delta)\Big(1-\frac{C'\delta}{2}(e^{-\lambda/2}+C'\delta)\Big)^{-1} < e^{-\lambda/4}.$$

Finally, if we denote by $\|F'\|_\infty$ the quantity $\sup_{y_u\in[-1,1]^n} \|DF(y_u)\|$ to simplify, then we get that

$$\begin{aligned} \|D\Psi_k(F)\,(\psi_{z_k}(x)_u)\| &\leqslant \left(\left\|A_{\mathrm{so,so}}^{z_k}(x)\right\| \|DF(x_{\mathrm{u}})\| + \left\|A_{\mathrm{u,so}}^{z_k}(x)\right\|\right) e^{-\lambda/4} \\ &\leqslant ((1+C'\delta)\|F'\|_\infty + C'\delta|F(x_{\mathrm{u}})|)\, e^{-\lambda/4} \\ &\leqslant ((1+C'\delta)\|F'\|_\infty + C'\delta\,(|F(0_{\mathbb{R}^{d_{\mathrm{u}}}})| + \|F'\|_\infty))\, e^{-\lambda/4} \\ &\leqslant e^{-\lambda/8} d_{\mathcal{E}}(F,0), \end{aligned} \tag{62}$$

where we used the estimates on δ given in (53).

Furthermore, since $(F(0_{\mathbb{R}^{d_{\mathrm{u}}}}), 0_{\mathbb{R}^{d_{\mathrm{u}}}}) \in W^{\mathrm{s}}_{\mathrm{loc},z}(0)$, we get that

$$\forall k\in\mathbb{Z},\quad \psi_{z_k}(F(0_{\mathbb{R}^{d_{\mathrm{u}}}}), 0_{\mathbb{R}^{d_{\mathrm{u}}}}) = (\Psi_k F(0_{\mathbb{R}^{d_{\mathrm{u}}}}), 0_{\mathbb{R}^{d_{\mathrm{u}}}}) \in W^{\mathrm{s}}_{\mathrm{loc},z}(0).$$

Since

$$|\psi_{z_k}^{\ell}(F(0_{\mathbb{R}^{d_{\mathrm{u}}}}), 0_{\mathbb{R}^{d_{\mathrm{u}}}})| \leqslant c_0 e^{-\ell\lambda/2} |(F(0_{\mathbb{R}^{d_{\mathrm{u}}}}), 0_{\mathbb{R}^{d_{\mathrm{u}}}})| = c_0 e^{-\ell\lambda/2}|F(0_{\mathbb{R}^{d_{\mathrm{u}}}})|,$$

we deduce that

$$\forall k\in\mathbb{Z},\quad \forall \ell\in\mathbb{N}\qquad |\Psi_k^{\ell}(F)(0_{\mathbb{R}^{d_{\mathrm{u}}}})| \leqslant c_0 e^{-\ell\lambda/2}|F(0_{\mathbb{R}^{d_{\mathrm{u}}}})|. \tag{63}$$

Fix $\ell_0\in\mathbb{N}$ such that $c_0 e^{-\ell_0\lambda/8} \leqslant 1$. A mean value inequality leads to the stability of the graph transforms:

$$\forall F\in\mathcal{E},\quad \forall k\in\mathbb{Z},\quad \forall \ell\geqslant \ell_0 \qquad \Psi_k^{\ell} F\in\mathcal{E}, \tag{64}$$

and the inequalities (62) and (63) leads to the claimed inequality (59).

Step 3: coming back to the disk. — Applying step 2 to the map $\widetilde{F}_{k_0}$ defined at the end of step 1, we get in particular that

$$d_{\mathcal{E}}\big(\Psi_{k_0}^{\ell}(\widetilde{F}_{k_0}),0\big) \leqslant \max(c_0,1) e^{-\ell\lambda/8} d_{\mathcal{E}}(\widetilde{F}_{k_0},0) \leqslant \max(c_0,1)\,\frac{e^{-\ell\lambda/8}}{2}.$$

In particular, thanks to the mean value inequality, we deduce that

$$\big\|\Psi_{k_0}^{\ell}(\widetilde{F}_{k_0})\big\|_{\mathcal{C}^1} \underset{\ell\to+\infty}{\longrightarrow} 0.$$

Finally, thanks to the expansivity of the unstable manifold, to the stability given in (64) and to the above $\mathcal{C}^1$-convergence, we obtain

$$\mathrm{Graph}\big(\Psi_{k_0}^{\ell}(\widetilde{F}_{k_0})\big)\cap[-1,1]^n = \mathrm{Graph}\left(\Psi_{k_0}^{\ell}(F_{k_0})\right)\cap[-1,1]^n = \psi_z^{k_0+\ell}(\widetilde{\mathcal{D}})\cap[-1,1]^n,$$

for all $\ell\geqslant\ell_0$.

Now, if we fix $\varepsilon_1 > 0$ such that $\varepsilon_1 < \varepsilon_0$ and

$$\forall y \in K, \qquad W^{\mathrm{u}}_{\varepsilon_1}(y) \subset \widetilde{\chi}_y\bigl(\{0\}^{d_{\mathrm{s}}+1} \times [-1,1]^{d_{\mathrm{u}}}\bigr),$$

then we can deduce the following statement: for all $\varepsilon > 0$, there exists $k_1 \in \mathbb{N}$ such that for all $k \geqslant k_1$, there exists a disk $\widetilde{\mathcal{D}}_k \subset \widetilde{\mathcal{D}}$ (of same dimension) which contains q such that $\varphi^k(\widetilde{\mathcal{D}}_k)$ is $\mathcal{C}^1$ ε-close to $W^{\mathrm{u}}_{\varepsilon_1}(\varphi^k(z))$. $\square$

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