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MOVEMENT OF SOLID FILAMENTS IN AXISYMMETRIC FLUID FLOW

by David Meyer

Abstract. — We consider the movement of slender toroidal filaments immersed in a 3D fluid described by the incompressible Euler equations. The filaments are described by Newtonian mechanics and interact with the fluid through the pressure exerted at the boundary. We assume that the filaments are almost rigid in the sense that the only non-rigid movement they can undergo is a change of length and that the fluid is irrotational, but can have a nonzero circulation around the filaments. We show that this kind of system can be described through an ODE in the positions of the bodies and that in the limit, where the bodies shrink to massless filaments, the system converges to an ODE system similar to the dynamics of the corresponding vortex filaments.

Résumé (Mouvement de filaments solides dans un écoulement fluide axisymétrique)

Nous considérons le mouvement de minces filaments toroïdaux immergés dans un fluide à trois dimensions dont la dynamique est décrite par les équations d'Euler incompressibles. Les filaments sont régis par la mécanique newtonienne et interagissent avec le fluide par le biais de la pression exercée à leur bord. Nous supposons que les filaments sont quasiment rigides, dans le sens où le seul mouvement non rigide qu'ils peuvent subir est un changement de longueur, et aussi que le fluide est irrotationnel, avec une circulation non nulle autour de chaque filament. Nous démontrons que ce type d'interaction fluide-structure peut être décrit par un système d'EDO sur les positions des axes des filaments et qu'à la limite où la masse des filaments se réduit à zéro, ce système converge vers un système d'EDO similaire à la dynamique des anneaux tourbillonnaires.

Contents

Mathematical subject classification (2020). — 35Q31, 35Q70, 76B47.
Keywords. — Fluid-Solid system, axisymmetric fluid flow, vortex rings, leapfrogging, vanishing body limit.

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1. Introduction and main results

We would like to understand the movement of toroidal slender filaments in axisymmetric fluid flow with no vorticity and the limiting dynamics where the filaments shrink to massless circles.

To get a nontrivial motion of these bodies, one needs to allow them to change their shape because otherwise, they can only move in one direction without breaking the axisymmetry, which does not allow for any nontrivial dynamics (for instance a single body will never change its speed due to momentum conservation).

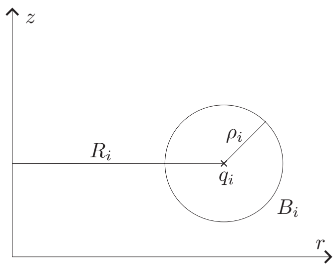
To still keep the dynamics simple, we shall assume that the only non-rigid movement that the bodies may undergo is a change of length, while still keeping their volume fixed. Furthermore, for the sake of simplicity, we shall assume that the bodies have circular cross-sections.

Due to the nonzero circulation around the bodies, one would expect that in the limit these behave similarly to the corresponding vortex filaments. The main result, aside from well-posedness of the system, is that this is indeed the case, which is described in more detail below in Section 1.1.

Mathematically, the setup for this is as follows: We use axisymmetric coordinates (r,z) and denote the right half-plane by $\mathbb{H}$. We fix the number of filaments $k \in \mathbb{N}_{>0}$ and numbers $v_1, \dots, v_k > 0$, which we interpret as the volumes of the bodies and which should remain constant along the evolution. We set

$$\rho_i := \sqrt{\frac{v_i}{\pi R_i}}, \qquad B_i(R_i, Z_i) := B_{\rho_i}((R_i, Z_i)) \subset \mathbb{H} \quad \text{for } i = 1, \dots, k, \tag{1.1}$$

here ρ_i denotes the minor radius, R_i the major radius and Z_i the Z-coordinate. With respect to the measure $r\,\mathrm{d}r\,\mathrm{d}z$, which corresponds to $1/2\pi$ times the three-



dimensional volume, the bodies B_i then have the fixed volume $v_i = \pi\rho_i^2 R_i$. See also the figure above.

Formally, we describe the configuration of the bodies through the manifold $M \subset (\mathbb{R}^2)^k$ of all $(R_1, Z_1), \dots, (R_k, Z_k)$ such that the bodies B_i all have positive distance from each other and from $\partial\mathbb{H}$.

For $q \in M$, we shall write q_i for the i-th component and q_{R_i} and q_{Z_i} for the two components of q_i. We shall also write $B_i(q)$ for clarity instead of B_i sometimes. Let n denote the *outer* normal of $\bigcup_i \partial B_i$.

Each body can undergo two different kinds of motion, the rigid one in the Z-direction and the non-rigid change of length. To make this precise, we use the natural correspondence between tangent vectors and normal velocities on the ∂B_i, as every C^1-curve in M corresponds to a continuous movement of each B_i.

For a tangent vector t^*, let $t^*_{R_1}, t^*_{Z_1}, t^*_{R_2}, \ldots$ denote its components. We say that a tangent vector is associated with B_i if only its R_i- and Z_i-component are non-zero, and write $T_{q_i}M$ for the subspace of those tangent vectors.

Then, for a C^1-curve q, the normal velocity is given by

$$u(\dot q) := \dot q_{Z_i} n\cdot e_Z + \dot q_{R_i} n\cdot e_R - \frac{\dot q_{R_i}\rho_i}{2R_i} \quad \text{on } \partial B_i. \tag{1.2}$$

Here e_R and e_Z are the unit vectors and the purpose of the last summand is to make sure that if the major radius changes, the minor radius also changes so that the volume of B_i is conserved under the motion.

Indeed by using Gauss's theorem, we see

$$\begin{aligned}\int_{\partial B_i} r u(\dot q)\,\mathrm{d}x &= \int_B \operatorname{div}\left(r(\dot q_{Z_i} e_Z + \dot q_{R_i} e_R)\right)\,\mathrm{d}x - \frac{\dot q_{R_i}\rho_i}{2R_i}\int_{\partial B_i} r\,\mathrm{d}x\\ &= \pi\rho_i^2 \dot q_{R_i} - \pi\rho_i^2\dot q_{R_i} = 0,\end{aligned} \tag{1.3}$$

and hence by the Reynolds transport theorem this normal velocity keeps the volume (with respect to $r\,\mathrm{d}r\,\mathrm{d}z$) fixed.

Let

$$\mathcal{F} = \mathcal{F}(t) := \mathbb{H} \smallsetminus \bigcup_i B_i$$

denote the (time-dependent) domain of the fluid. Let L^2_R and H^1_R denote the L^2 resp. H^1 spaces with respect to the measure $r\,\mathrm{d}r\,\mathrm{d}z$. We assume that:

Condition 1.0.1. — *In $\mathcal{F}(t)$ the fluid fulfills the axisymmetric Euler equations with zero vorticity and no swirl for $t\in[0,\infty)$, i.e.,*

$$\partial_t u + (u\cdot\nabla)u + \nabla p = 0, \tag{1.4}$$

$$\operatorname{div}(ru) = 0, \tag{1.5}$$

$$\operatorname{curl}(u) = 0, \tag{1.6}$$

$$u_r = 0 \quad \text{on } \partial\mathbb{H}. \tag{1.7}$$

Here the div *and* curl *are taken with respect to the variables (r,z) and u is $\mathbb{R}^2$-valued.*

These equations are equivalent to the usual Euler equations for u with no vorticity and no swirl (that is, no velocity in the direction perpendicular to the (r,z)-plane) after going back to three-dimensional coordinates. The boundary condition (1.7) encodes the fact that there should be no singularity at $\{r=0\}$.

Condition 1.0.2. — *We assume that the solution of* (1.4)–(1.7) *is strong in the sense that $u, \nabla u, \partial_t u \in L^2_R \cap C^1(\mathcal{F})$ and that q is C^2 in time. In particular this should hold for the initial data.*

We define the circulation around each body as

$$\gamma_i := \int_{\partial B_i} u \cdot \tau \, \mathrm{d}x,$$

where $\tau = n^{\perp}$. This is a conserved quantity by Kelvin's circulation law. For technical reasons, we will assume:

Condition 1.0.3. — *None of the γ_i are* 0.

This condition is not necessary for the well-posedness (nothing in the proof changes), but necessary for the convergence to the limit system, for instance it follows directly from the ODE reformulation (2.16) that a single body with no circulation which is initially moving in the e_z-direction will keep moving in that direction with the same speed as every term in (2.16) is 0 in that case.

We assume that the normal velocity of the boundary of each B_i matches the velocity of u in the corresponding direction:

Condition 1.0.4. — *For all i we have*

$$u \cdot n = u(\dot{q}) \quad \text{on } \partial B_i, \tag{1.8}$$

where we use the identification between tangent vectors and normal velocities mentioned above.

For simplicity we will assume that all bodies and the fluid have constant density 1, though all the arguments still work for different densities.

Condition 1.0.5. — *We assume that in the z-direction, the momentum is (formally) preserved, which yields the condition*

$$v_i \ddot{q}_{Z_i} = -\int_{\partial B_i} r p n \cdot e_Z \, \mathrm{d}x \tag{1.9}$$

for all i.

It remains to derive a condition for the interaction of the fluid and the solids in the r-component. As the solids can change their shape, we make the Ansatz of prescribing an interior velocity field and assuming that the kinetic energy of each B_i only changes through the force exerted by the pressure at the boundary.

We associate to each tangent vector/normal velocity t_i^* associated with B_i such an interior velocity field $u_{i,\mathrm{int}}(t_i^*) \in H^1(B_i)$ with

$$\operatorname{div} r u_{i,\mathrm{int}} = 0 \quad \text{in } B_i, \tag{1.10}$$

$$\operatorname{curl} u_{i,\mathrm{int}} = 0 \quad \text{in } B_i, \tag{1.11}$$

$$u_{i,\mathrm{int}} \cdot n = u(t_i^*) \quad \text{on } \partial B_i. \tag{1.12}$$

Existence and uniqueness can be obtained by standard elliptic theory [11, Chap. 6], as $u_{i,\mathrm{int}}(t_i^*)$ can be written as $\nabla\phi$ with $\operatorname{div}(r\nabla\phi) = 0$ and the boundary condition $\partial_n \phi = u(t_i^*)$ by the assumption of curl-freeness. The compatibility condition for the

Neumann boundary condition $\int_{\partial B_i} r\partial_n\phi\,\mathrm{d}x = 0$ is fulfilled since $\int_{\partial B_i} ru(t_i^*)\,\mathrm{d}x = 0$ by (1.3). Note that this is linear in t_i^*, and that

$$u_{i,\mathrm{int}}(t_{Z_i}^*) = t_{Z_i}^* e_Z. \tag{1.13}$$

We can use this to associate a quadratic form on $T_{q_i}M$ by

$$(t_i^*)^T E_{q_i} t_i^+ := \int_{B_i} r\langle u_{i,\mathrm{int}}(t_i^*), u_{i,\mathrm{int}}(t_i^+)\rangle\,\mathrm{d}x, \tag{1.14}$$

for tangent vectors t_i^*, t_i^+, associated with B_i, which describes the kinetic energy associated with $u_{i,\mathrm{int}}$. Clearly, this is symmetric and positive definite. Because of the explicit form (1.13), we know that

$$(e_Z)_i^T E_{q_i}(e_Z)_i = v_i,$$

where $(e_Z)_i \in (\mathbb{R}^2)^k$ is the vector with e_Z in the i-th component. We set

$$(e_R)_i^T E_{q_i}(e_R)_i =: f_i(q_{R_i}), \tag{1.15}$$

here the function f_i depends on the volume v_i and $(e_R)_i$ is the vector with e_R in the i-th component.

In summary, the quadratic form can be written in components as

$$(t_i^*)^T \begin{pmatrix} f_i(q_{R_i}) & 0 \\ 0 & v_i \end{pmatrix} (t_i^+), \tag{1.16}$$

as one can see from the explicit formula (1.13) and the fact that $u_{i,\mathrm{int}}(t_{R_i}^*)$ must be asymmetric in z in the second component and hence $t_{R_i}^*$ and $t_{Z_i}^*$ are orthogonal to each other with respect to E_{q_i}.

We define the kinetic energy of B_i as

$$\mathcal{E}_{B_i} := \frac{1}{2}\dot{q}_i^T E_{q_i}\dot{q}_i. \tag{1.17}$$

We assume that the kinetic energy of each B_i only changes through the force exerted at the boundary, that is,

$$\mathcal{E}_{B_i}' = -\int_{\partial B_i} rpu(\dot{q}_i)\,\mathrm{d}x. \tag{1.18}$$

After using the decomposition (1.16) and subtracting the condition (1.9), one obtains that

$$f_i(q_{R_i})\ddot{q}_{R_i}\dot{q}_{R_i} + \frac{1}{2}\partial_{q_{R_i}} f_i(q_{R_i})(\dot{q}_{R_i})^3 = -\int_{\partial B_i} rp\dot{q}_{R_i}\Big(n\cdot e_R - \frac{\rho_i}{2R_i}\Big)\,\mathrm{d}x.$$

We make the extra assumption that one can divide out $\dot{q}_{R_i}$, which is equivalent to saying that whenever $\dot{q}_{R_i} = 0$ and the force at the boundary is nonzero, then $\ddot{q}_{R_i}$ is not zero, which rules out bodies with fixed R_i-coordinate. This gives the final equation:

Condition 1.0.6. — *For all i we have*

$$f_i(q_{R_i})\ddot{q}_{R_i} + \frac{1}{2}\partial_{q_{R_i}} f_i(q_{R_i})(\dot{q}_{R_i})^2 = -\int_{\partial B_i} rp\Big(n\cdot e_R - \frac{\rho_i}{2R_i}\Big)\,\mathrm{d}x. \tag{1.19}$$

We can also write (1.9) and (1.19) as a single equation

$$(t_i^*)^T E_{q_i}\ddot{q}_i + \frac{1}{2} t_i^* (\partial_{\dot{q}_i} E_{q_i} \cdot \dot{q}_i)\dot{q}_i = -\int_{\partial B_i} rpu(t_i^*)\,\mathrm{d}x, \tag{1.20}$$

where t_i^* is an arbitrary tangent vector associated with B_i.

1.1. Main results. — Our first main result is well-posedness:

Theorem 1.1.1 (Informal). — *For every initial datum $q(0)$, $\dot{q}(0)$ the system detailed in the previous section has a unique solution up to some time $T > 0$. If $T < \infty$, then q blows up at T in the sense that some of the bodies either collide with each other, the boundary or escape to ∞. The solution is completely determined by the circulations γ_i and the initial data $q(0)$, $\dot{q}(0)$.*

Furthermore, the system preserves energy.

The more precise statements can be found in Corollary 2.2.7, Lemma 2.2.8 and Theorem 2.2.10. In Theorem 2.2.2 the system is recast as a second-order ODE in q. We remark that singularities in the form of collisions can indeed occur for Euler-rigid body systems, see for instance [25].

For the zero-radius limit, we shall first introduce some notation. We will use a rescaling parameter ε and denote the manifold of configurations associated with the bodies with the "volumes" $v_1\varepsilon^2, \dots, v_k\varepsilon^2$ by $\widetilde{M}_\varepsilon$ (recall that the "volumes" were defined in (1.1)). We still denote the minor radii with $\rho_1, \dots \rho_k$. We write $\widetilde{\rho}_i$ for the unrescaled radii ρ_i/ε.

Generally, what we would expect for the limiting dynamics is that the fluid velocity and the velocity of the bodies behaves like the solution to the system

$$\begin{aligned}\operatorname{div}(ru) &= 0,\\ \operatorname{curl}(u) &= \sum_i \gamma_i \delta_{q_i}.\end{aligned}$$

This u can be recovered for instance by going back to three-dimensional coordinates and using the Biot-Savart law, which after some computations (see for instance [7, Lem. 4] for a detailed derivation) yields that

$$u(x) = \sum_i \gamma_i \frac{(x-q_i)^\perp}{2\pi|q_i - x|^2} - \gamma_i \frac{1}{4\pi R_i} \log\Big(\frac{1+|x-q_i|}{|x-q_i|}\Big) e_z + \text{lower order terms} \tag{1.21}$$

(here "$^\perp$" is defined in axisymmetric coordinates as the linear map with $e_r^\perp = e_z$ and $e_z^\perp = -e_r$).

Here we would expect that the term of order -1 only leads to interaction of the rings with each other, but no self-interaction, while the log-term induces a self-interaction, but is of lower order regarding the interaction between different rings. We would like to work in a critical regime where both effects are of the same order. For this, we need the rings to be very close to each other, hence we fix some $\binom{R_0}{Z_0}$ with $R_0 > 0$.

There are now two different regimes that one can consider.

1.1.1. *The first regime*. — For vortex rings, this one is also considered in [9] and [29]. We set all γ_i to be equal to one and set the centers to be

$$q_i = \Big(R_0 + \frac{\widetilde{q}_{R_i}}{\sqrt{|\log\varepsilon|}}, Z_0 + \frac{\widetilde{q}_{Z_i}}{\sqrt{|\log\varepsilon|}}\Big), \tag{1.22}$$

where the rescaled initial positions $\widetilde{q}(0) := (\widetilde{q}_{R_1}(0), \widetilde{q}_{Z_1}(0), \dots)$, should be independent of ε and of order 1. We will further rescale time by a factor $|\log\varepsilon|$ and work with the rescaled positions $\widetilde{q}_i := (\widetilde{q}_{R_i}, \widetilde{q}_{Z_i})$.

In this regime, the main part of the self-induced velocity (in rescaled time and space), that is $(-1/4\pi R_0)|\log\varepsilon|^{1/2}e_Z$, is the same for all rings, hence we may neglect this part. The next order part is of the form $(\widetilde{q}_{R_i}/4\pi R_0^2)e_Z$ in rescaled time and space by Taylor's theorem. The velocity induced by the i-th ring on the j-th ring is of the form $\frac{1}{2\pi}(q_j - q_i)^\perp/|q_i - q_j|^2$.

We hence expect that in the limit $\varepsilon \to 0$, the velocities $\widetilde{q}_i$ should solve the system

$$\widetilde{q}_i' = \frac{1}{2\pi}\sum_{j\neq i}\frac{(q_i - q_j)^\perp}{|q_i - q_j|^2} + \frac{\widetilde{q}_{R_i}}{4\pi R_0^2}e_Z \tag{1.23}$$

in the rescaled time, up to the subtracted term $-(1/4\pi R_0)|\log\varepsilon|^{1/2}e_Z$.

Theorem 1.1.2. — *Assume that the solution of* (1.23) *exist until time T (in the sense that no components of the solution go to ∞ and the distance between the different components stays positively bounded from below). Assume that the shifted initial velocities $\widetilde{q}_i' + (|\log\varepsilon|^{1/2}/4\pi R_0^2)e_Z$ (in the rescaled time and space) are bounded uniformly in ε.*

Then $\widetilde{q} + t(1/4\pi R_0)|\log\varepsilon|^{1/2}e_Z$ converges to the solution of (1.23) *weakly* in $W^{1,\infty}_{\mathrm{loc}}([0,T))$ in rescaled time.*

This ODE system has been studied for instance in [29], where the existence of periodic solutions for two rings has been shown. Such periodic solutions correspond to a "leapfrogging" motion of the rings, which was predicted already by Helmholtz in his famous work [22, 23].

1.1.2. *The second regime*. — We can also consider the regime where the self-induced motion and the motion induced by other rings is of the same order. For this we set

$$q_i = \Big(R_0 + \frac{\widetilde{q}_{R_i}}{|\log\varepsilon|}, Z_0 + \frac{\widetilde{q}_{Z_i}}{|\log\varepsilon|}\Big). \tag{1.24}$$

Here the rescaled initial position $\widetilde{q}(0)$ should again be independent of ε. We will further rescale time by a factor $|\log\varepsilon|^2$ and will again work with the rescaled positions $\widetilde{q}$.

In this regime, all expected velocities are of order 1 in rescaled time and space. We hence expect that in the limit $\varepsilon \to 0$ and in the rescaled time, the rescaled velocities $\widetilde{q}$ should solve the system

$$\widetilde{q}_i' = \frac{1}{2\pi}\sum_{j\neq i}\gamma_j\frac{(\widetilde{q}_i - \widetilde{q}_j)^\perp}{|\widetilde{q}_i - \widetilde{q}_j|^2} - \frac{\gamma_i}{4\pi R_0}e_Z. \tag{1.25}$$

This system was studied in [31] and one can show that if all γ_i have the same sign, then solutions cannot blow up in finite time [31, Th. 2.1].

Theorem 1.1.3. — *Assume that the solution of* (1.25) *exists until some time T in the same sense as in the previous theorem. Further assume that the initial velocities $\widetilde{q}_i'(0)$ (in the rescaled time and space) are bounded uniformly in ε.*

Then $\widetilde{q}$ converges to the solution of (1.25) *weakly* in $W^{1,\infty}_{\mathrm{loc}}([0,T))$ in rescaled time.*

Comparison with the existing literature. — In a two-dimensional setting, the similar convergence of fluid-body systems with shrinking bodies to point vortex systems in a bounded domain has been studied in [15] and [17], while further work on fluid-body systems has been done for instance in [14, 16, 25, 33], see also the references therein. The simpler problem of a stationary shrinking obstacle has been studied for example in [20, 26, 27]. The vanishing body limit for the viscous fluid-solid system was studied for instance in [21, 30].

Somewhat similar problems for filaments immersed in 3D Stokes flow have been considered for instance in [24, 32].

Much effort has gone to determining whether the approximation of the Euler equations by the ODE systems (1.23) and (1.25) is true for initial data whose vorticity is strongly localized around a few points (so-called vortex rings), see for instance [4, 5, 7]. Recently it was shown in [6] that this system indeed describes the correct asymptotic. Special solutions which behave like these systems have been constructed in [9] for the Euler equations and in [29] for the similar Gross-Pitiaevskii equation. In [2] it was shown that traveling wave solutions of (1.25) (i.e., solutions which are stationary in a moving frame) can be lifted to a traveling wave solution of the Euler equations.

Similar models for the Euler equations with a helical symmetry have been justified e.g. in [10].

For general three-dimensional filaments the justification of similar asymptotic models (such as the local induction approximation [3, §2]) is more or less completely open, though some conditional results exist, see for instance [28].

Outline of the paper. — In Section 2 we will describe the fluid velocity through potentials and streamfunctions, which are completely determined by the positions and velocities of the bodies. As a result, we can reduce the system to a second order ODE, with coefficients determined by the potentials and streamfunctions, for the positions of the bodies. This lets us show that our system is well-posed.

The proof of the convergence of the system for shrinking bodies then requires analyzing the limit of this system. For this, we study the asymptotics of the potentials and the streamfunctions in Sections 3 and 4. We will be able to show that these converge to the corresponding two-dimensional functions in the zero radius limit and that the interaction of the different parts of the stream functions produces the same terms as the Biot-Savart law in the limit. A brief outline of the strategies for these convergences is given at the beginning of the respective sections.

In Section 5, we will study the convergence in the zero radius limit of the ODE, which is quite intricate, as the equation degenerates due to the vanishing mass. In order to still get estimates, we use a modulated version of the kinetic energy of the system, whose evolution only depends on the degenerating terms, which allows one to obtain uniform estimates on the velocity and to pass to the limit.

General notation. — We will use the notation $\widetilde{q}$ for the rescaled positions in both regimes, as most estimates work completely similarly for both regimes. We also denote the manifold of $\widetilde{q}$ for which the corresponding q is in $\widetilde{M_\varepsilon}$ by M_ε.

We write $A \lesssim B$ if there is a constant $C > 0$ such that $A \leqslant CB$, where the constant C is allowed to depend on the number k and on q resp. $\widetilde{q}$ and on which of the regimes we are in, but not on any other quantities.

Similarly, we write ℓ for irrelevant (finite) exponents, which are allowed to depend on which regime we are in, but not on any other quantities and are allowed to change their value from line to line.

If $\Omega \subset \mathbb{H}$ we write $L^2_0(\Omega)$ for the space of all functions $f \in L^2(\Omega)$ such that for every connected component Ω_i of Ω we have $\int_{\Omega_i} f \,\mathrm{d}x = 0$.

If $S \subset \mathbb{H}$, we write $S^{\mathbb{R}^3}$ for its figure of revolution in $\mathbb{R}^3$.

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2. Well-posedness of the system

We will follow the approach in [14] to show that our system can be reduced to an ODE in q only (see Theorem 2.2.2 below), which in particular shows well-posedness. The main additional difficulty is that we are in an unbounded domain and hence need decay estimates to justify partial integrations.

2.1. Representation of u.

— In this subsection, we show that u can be recovered from a potential and a streamfunction (defined in 2.1.4), which is shown in Proposition 2.1.8 below.

Definition 2.1.1. — For $t^* \in T_{q_i}M$, let $\phi_{i,t^*} = \phi_{i,t^*}(t)$ be defined as the unique solution of the Neumann problem

$$\begin{aligned}
\operatorname{div}(r\nabla\phi_{i,t^*}) &= 0 \quad \text{in } \mathcal{F}(t),\\
\partial_n\phi_{i,t^*} &= u(t^*) \quad \text{on } \partial B_i,\\
\partial_n\phi_{i,t^*} &= 0 \quad \text{on } \partial B_j \text{ for } j \neq i \text{ and on } \partial\mathbb{H},\\
\phi_{i,t^*} &\in \dot{H}^1_R,\\
\phi_{i,t^*} &\longrightarrow 0 \quad \text{at } \infty.
\end{aligned}$$

We first need to check that this is well-defined.

Lemma 2.1.2. — *Let $b \in L^2(\cup_i \partial B_i)$ be such that $\int_{\cup_i \partial B_i} rb \, \mathrm{d}x = 0$, then the equations*

$$\begin{aligned} \operatorname{div}(r\nabla\phi) &= 0 \quad \text{in } \mathcal{F}(t), \\ \partial_n \phi &= b \quad \text{on } \textstyle\bigcup_i \partial B_i, \\ \partial_n \phi &= 0 \quad \text{on } \partial\mathbb{H}, \\ \phi &\in \dot{H}^1_R, \\ \phi &\longrightarrow 0 \quad \textit{at } \infty \end{aligned}$$

have a unique solution ϕ. Furthermore

$$|\nabla^m \phi(x)| \lesssim \frac{\|b\|_{H^m}}{1+|x|^{2+m}} \quad \forall m \in \mathbb{N}_{\geqslant 0}, \tag{2.1}$$

where the implicit constant is bounded locally uniformly in q.

This implies the well-definedness of ϕ_{i,t^*} and that the estimate (2.1) holds locally uniformly in q for ϕ_{i,t^*}, since $\int_{\cup_i \partial B_i} ru(t^*) \, \mathrm{d}x = 0$, as computed in (1.3).

Proof. — We go back to three-dimensional coordinates and set $\phi^{\mathbb{R}^3}(r,z,0) = \phi(r,z)$, where $\phi^{\mathbb{R}^3}$ is axisymmetric. Then ϕ solves the system above iff $\phi^{\mathbb{R}^3}$ solves the corresponding Neumann problem for Δ in $\mathbb{R}^3 \smallsetminus \bigcup_j B_j^{\mathbb{R}^3}$. By standard techniques (see e.g. [1]), we obtain a unique solution $\phi^{\mathbb{R}^3} \in \dot{H}^1(\mathbb{R}^3 \smallsetminus \bigcup B_j^{\mathbb{R}^3})$. We furthermore obtain from this that $\partial_n \phi = 0$ on $\partial\mathbb{H}$.

The decay rate then directly follows from the lemma below. □

Lemma 2.1.3

(a) *Let $\zeta \in \dot{H}^1(\mathcal{F}^{\mathbb{R}^3})$ be axisymmetric such that $\Delta\zeta = 0$ in $\mathcal{F}^{\mathbb{R}^3}$ and $\zeta(x) \to 0$ as $|x| \to \infty$. Then it holds that*

$$|\nabla^m \zeta(x)| \lesssim \frac{\|\partial_n \zeta\|_{H^m(\partial\mathcal{F}^{\mathbb{R}^3})}}{1+|x|^{1+|m|}} \qquad \forall m \in \mathbb{N}_{\geqslant 0}.$$

The implicit constant is bounded locally uniformly in q.

(b) *Let ζ be as in* (a). *If additionally*

$$\int_{\cup_j \partial B_j^{\mathbb{R}^3}} \partial_n \zeta \, \mathrm{d}x = 0,$$

then

$$|\nabla^m \zeta(x)| \lesssim \frac{\|\partial_n \zeta\|_{H^m(\partial\mathcal{F}^{\mathbb{R}^3})}}{1+|x|^{2+|m|}} \qquad \forall m \in \mathbb{N}_{\geqslant 0},$$

where the implicit constant is controlled as in (a).

Proof. — If we extend ζ to $\mathbb{R}^3$ by solving the Dirichlet problem for $\zeta|_{\partial B_j^{\mathbb{R}^3}}$ on each $B_j^{\mathbb{R}^3}$, then the (distributional) Laplacian of this extension is of the form

$$[\partial_n \zeta] \mathcal{H}^2 \llcorner \bigcup \partial B_j^{\mathbb{R}^3},$$

where $[\cdot]$ denotes the jump across the boundary, as a direct calculation shows. By elliptic regularity (cf. [19, Chap. 2]), this is a finite measure.

We claim that we can recover this extension by convoluting the distributional Laplacian with the Newtonian potential. Indeed for any $f \in C_c^\infty(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} \int_{\cup_i \partial B_i^{\mathbb{R}^3}} \frac{-[\partial_n \zeta]}{4\pi|x-y|} \Delta f(x) \, \mathrm{d}\mathcal{H}^2(y) \, \mathrm{d}x = \int_{\cup_i \partial B_i^{\mathbb{R}^3}} [\partial_n \zeta] f(y) \, \mathrm{d}\mathcal{H}^2(y) = \int_{\mathbb{R}^3} \zeta \Delta f \, \mathrm{d}x,$$

where in the second step, we used that the Newtonian potential is the inverse Laplacian. Hence the difference $[\partial_n \zeta]\mathcal{H}^2 \llcorner (\bigcup_i \partial B_i^{\mathbb{R}^3}) * (-1/4\pi|x|) - \zeta$ is a harmonic tempered distribution, i.e., a polynomial.

Now

$$[\partial_n \zeta]\mathcal{H}^2 \llcorner (\textstyle\bigcup_i \partial B_i^{\mathbb{R}^3}) * \dfrac{-1}{4\pi|x|} - \zeta \longrightarrow 0,$$

by the assumption on ζ and because $[\partial_n \zeta]\mathcal{H}^2 \llcorner (\bigcup_i \partial B_i^{\mathbb{R}^3})$ is a finite measure, so this difference is zero, which shows the claim.

We hence obtain that $|\nabla^m \zeta(x)| \lesssim \|\partial_n \zeta\|_{L^2(\partial \mathcal{F}^{\mathbb{R}^3})}/(1+|x|^{1+m})$ for all $m \geqslant 0$ and for $\operatorname{dist}(x, \bigcup B_i^{\mathbb{R}^3}) \geqslant 1$. For x close to the B_j the estimate follows from elliptic regularity theory. This shows the estimate in part (a), part (b) works exactly the same way, except that the integral of $[\partial_n \zeta]$ vanishes by the assumption and partial integration, which gives one order more of decay.

To see that these bounds are locally uniform in q, we need uniform estimates on $\|[\partial_n \zeta]\|_{L^1(\cup_i \partial B_i)}$. Note that for this we only need up-to-the-boundary estimates for a neighborhood of each B_i, locally uniform in q and a locally uniform L^2 estimate. By going back to axisymmetric coordinates, one obtains the former, as the geometries of these neighborhoods (in axisymmetric coordinates) only change by rescaling with a bounded factor. On the other hand, one can obtain from the energy equality (which is justified by the decay estimates we have already proved) in axisymmetric coordinates that

$$\int_{\mathcal{F}} r|\nabla \zeta|^2 \, \mathrm{d}x = -\int_{\cup_i \partial B_i} r\zeta \partial_n \zeta \, \mathrm{d}x \gtrsim \|\zeta\|^2_{H^1(\cup_i (B_i + B_1(0) \setminus B_i))}. \tag{2.2}$$

Here we used the (three-dimensional) Sobolev inequality to control the L^2-norm on the right-hand side. The constant of the Sobolev inequality is locally uniform in q, as one can e.g. see by using a diffeomorphism between the different instances of $\mathcal{F}$.

Now we can use the trace inequality from $H^1(B_i + B_1(0) \setminus B_i)$ to $L^2(\partial B_i)$ in (2.2), whose constant is also locally uniform in q, as the geometry only changes through rescaling by a bounded factor. This implies the desired locally uniform estimate

$$\|\partial_n \zeta\|_{L^2(\cup_i \partial B_i)} \gtrsim \|\zeta\|_{L^2(\cup_i (B_i + B_1(0) \setminus B_i))}. \qquad \square$$

The argument for the existence of the stream function is more complicated, as there is no easy algebraic relation between the three-dimensional stream function and the axisymmetric stream function.

Definition 2.1.4. — Let $\psi_i = \psi_i(t)$ be the solution of the elliptic equation

$$\begin{aligned}\operatorname{div}\Big(\frac{1}{r}\nabla\psi_i\Big) &= 0 \quad \text{in} \quad \mathcal{F}(t),\\ \psi_i|_{\partial B_j} &\text{ is constant } \forall j,\\ \int_{\partial B_j} \frac{1}{r}\partial_n\psi_i \,\mathrm{d}x &= \delta_{ij},\\ \psi_i|_{\{r=0\}} &= 0,\\ \lim_{r\to 0}\frac{1}{r}\partial_z\psi_i &= 0.\end{aligned} \tag{2.3}$$

Here δ_{ij} is a Kronecker delta. We refer to the constant boundary values on ∂B_j as C_{ij}.

Lemma 2.1.5. — *Such a ψ_i always exists and is unique under the constraint $\frac{1}{\sqrt{r}}\nabla\psi_i \in L^2$. Furthermore $\frac{1}{r}\nabla\psi_i$ is continuous at $r=0$ and we have the estimate*

$$\Big|\nabla^m\Big(\frac{1}{r}\nabla\psi_i(x)\Big)\Big| \lesssim \frac{1}{1+|x|^{2+m}} \quad \forall m\in\mathbb{N}_{\geqslant 0}.$$

The implicit constant in the estimate is locally uniformly bounded in q.

Proof. — We first show that an auxiliary function $u_{2,i}$ can be constructed by going back to three-dimensional coordinates and later show that one can recover ψ_i from it.

Step 1. — We define $u_{2,i}$ on $\mathcal{F}^{\mathbb{R}^3}$ as the axisymmetric vector field with no azimuthal component which solves (in three-dimensional variables) the system

$$\operatorname{div} u_{2,i} = 0, \tag{2.4}$$
$$\operatorname{curl} u_{2,i} = 0, \tag{2.5}$$
$$u_{2,i}\cdot n = 0 \quad \text{on } \partial B_j^{\mathbb{R}^3} \text{ for all } j, \tag{2.6}$$
$$u_{2,i} \in L^2(\mathcal{F}^{\mathbb{R}^3}). \tag{2.7}$$

We make the ansatz $\operatorname{curl}\Psi_i = u_{2,i}$ for a purely azimuthal field $\Psi_i = \widetilde{\Psi}_i e_\theta$, where e_θ is the unit vector in the θ-direction. This gives the equations

$$(u_{2,i})_r = -\partial_z\widetilde{\Psi}_i, \quad (u_{2,i})_z = \partial_r\widetilde{\Psi}_i + \frac{1}{r}\partial_r\widetilde{\Psi}_i,$$

using that the last term can be rewritten as $\frac{1}{r}\partial_r(r\Psi_i)$, this turns the equations (2.4)–(2.6) into the equations

$$\Delta\Psi_i = 0 \quad \text{in } \mathcal{F}^{\mathbb{R}^3}, \tag{2.8}$$
$$r\widetilde{\Psi}_i|_{\partial B_j^{\mathbb{R}^3}} \text{ is constant for all } j. \tag{2.9}$$

For each fixed set of constant boundary values $(\widetilde{C}_{ij})_{j=1,\dots,k}$ for $r\widetilde{\Psi}_i$, this system has a unique solution $\Psi_i(\widetilde{C}_{ij}) \in H^1$ by standard techniques. Also, because Ψ_i is purely azimuthal, it must vanish at $r=0$ and hence it holds that

$$(u_{2,i})_r = 0 \tag{2.10}$$

at $r=0$.

Step 2. — To show uniqueness of $u_{2,i}$ for given $(\widetilde{C}_{ij})$, we note that we can recover a Ψ_i fulling the system (2.8),(2.9) from $u_{2,i}$. Indeed we may extend $u_{2,i}$ to the full space by zero as $\overline{u}_{2,i}$, which preserves the condition that the divergence vanishes, as the distributional divergence on the boundary equals $[\overline{u}_{2,i} \cdot n] = 0$.

It is well known that on the full space, every divergence-free field can be written as a curl, indeed we have $-\operatorname{curl} \Delta^{-1} \operatorname{curl} g = g$ for every divergence-free g, as a straightforward calculation shows. Since we also have $\operatorname{curl} \overline{u}_{2,i} \in H^{-1}$ and the fundamental solution of the Laplacian maps H^{-1} to H^1, we see that the field obtained this way lies in H^1. Hence two different solutions $u_{2,i}$ for the same boundary values would give rise to two different Ψ_i in H^1 with the same boundary values, which is impossible.

Step 3. — Now we can view $u_{2,i}$ as a function in (r, z) again, it fulfills $\operatorname{div}(r u_{2,i}) = 0$ (with the two-dimensional divergence). Then we can find a $\psi_i(\widetilde{C}_{ij})$ such that

$$u_{2,i} = \frac{1}{r} \nabla^{\perp} \psi_i,$$

because $r u_{2,i}^{\perp}$ is curl-free. This can be done with the usual path integral construction, it is easy to check that the condition $u_{2,i} \cdot n = 0$ ensures that even paths which are not homotopy equivalent yield the same values. One can then check by direct calculation that $\operatorname{div}(\frac{1}{r} \nabla \psi_i(\widetilde{C}_{ij})) = 0$ holds, and by the boundary condition for $u_{2,i}$, we see that $\psi_i(\widetilde{C}_{ij})$ must be constant on each ∂B_j. Furthermore, we have that $u_{2,i}$ is continuous at $r = 0$ by elliptic regularity and hence $\frac{1}{r} \nabla \psi_i(\widetilde{C}_{ij})$ is continuous at $r = 0$ and $\frac{1}{r} \partial_z \psi_i = 0$ at $r = 0$ by (2.10). This $\psi_i(\widetilde{C}_{ij})$ is unique up to an additive constant under the condition $\frac{1}{\sqrt{r}} \nabla \psi_i(\widetilde{C}_{ij}) \in L^2$ (here one gets an additional factor r from the coordinate change), as one can recover the unique $u_{2,i}$ from it.

Next, we argue that we can pick the boundary values $(\widetilde{C}_{ij})$ uniquely such that the condition (2.3) holds. It suffices to show that the linear map that sends the boundary values $(\widetilde{C}_{ij})$ to the integrals $\int_{\partial B_j} \frac{1}{r} \partial_n \psi_i(\widetilde{C}_{ij}) \, \mathrm{d}x$ is invertible for each fixed i.

First note that the $\widetilde{C}_{ij}$ are also the boundary values of ψ_i (up to an additive constant) because we have that $r u_{2,i} = \nabla^{\perp} r \widetilde{\Psi}_i$ (in (r, z)-coordinates) and hence by applying the fundamental theorem of calculus, we see that $(r\widetilde{\Psi})(x) - (r\widetilde{\Psi})(y) = \psi_i(x) - \psi_i(y)$ (in axisymmetric coordinates).

Assume there is a nonzero vector of $\widetilde{C}_{ij}$ such that all integrals vanish. Without loss of generality, we may assume that $\widetilde{C}_{i1}$ is the biggest one of the $\widetilde{C}_{ij}$. Then the normal derivative of ψ_i on ∂B_1 must be non-positive by the maximum principle and hence must be zero everywhere on ∂B_1. Since the tangential derivative also vanishes, the constant extension of $\psi_i(\widetilde{C}_{ij})$ to B_1 still fulfills $\operatorname{div}(\frac{1}{r}(\psi_i(\widetilde{C}_{ij})) = 0$. But this extension is then a locally, but not globally, constant solution of an elliptic equation, which is a contradiction.

We have $\partial_z \psi_i = 0 \cdot u_{2,i} = 0$ at $r = 0$ because $u_{2,i}$ is continuous by elliptic regularity. We may choose the additive factor that we have leftover such that $\psi_i = 0$ at $r = 0$ holds.

The uniqueness of ψ_i follows from the uniqueness of the $\psi_i(\widetilde{C}_{ij})$.

Step 4. — By elliptic regularity, it is easy to see that the C_{ij} are locally uniformly bounded in q, and hence $\|\partial_n \Psi_i(C_{ij})\|_{H^m}$ is locally uniformly bounded in q. By Lemma 2.1.3 (a), we see that

$$|\nabla^m \Psi_i(x)| \lesssim \frac{1}{1+|x|^{1+m}},$$

for all $m \in \mathbb{N}_{\geqslant 0}$ which implies the decay estimate. □

It is known that the equations

$$\operatorname{div}\Big(\frac{1}{r}\nabla f\Big) = g \quad \text{in } \mathbb{H},$$
$$f = \partial_n f = 0 \quad \text{on } \partial\mathbb{H},$$

have a fundamental solution K such that

$$f(y) = \int_{\mathbb{H}} g(x) K(x,y)\,\mathrm{d}x \tag{2.11}$$

is the unique solution under suitable decay assumptions on f for e.g. $g \in C_c^\infty(\mathbb{H})$, see e.g. [13, §2].

By the following lemma, we can recover solutions to $\operatorname{div}(\frac{1}{r}\nabla\cdot)$ from single-layer potentials with this fundamental solution and hence obtain decay estimates.

Lemma 2.1.6

(a) *Let* $\operatorname{div}(\frac{1}{r}\nabla\zeta) = 0$ *in* $\mathcal{F}$ *with* $\frac{1}{\sqrt{r}}\nabla\zeta \in L^2(\mathcal{F})$, *assume that* $\frac{1}{r}\nabla\zeta$ *is continuous for* $r \to 0$ *and that* $\zeta|_{r=0} = 0$. *Furthermore, assume that* $\zeta|_{\partial\mathcal{F}}$ *is sufficiently smooth, then there is a constant* C, *depending on* ζ *such that*

$$|\zeta(x)| \leqslant \frac{C}{1+|x|}.$$

In particular this holds for ψ_i.

(b) ψ_i *can be represented as*

$$\psi_i(y) = \int_{\cup_i \partial B_i} \frac{1}{r} K(x,y)\partial_n \psi_i \,\mathrm{d}x.$$

Proof. — We claim that if we extend ζ to $\mathbb{H}$ by solving the Dirichlet problem for $\operatorname{div}(\frac{1}{r}\nabla\cdot)$ with boundary values ζ in each B_i, then it holds that

$$\zeta = \int_{\cup_i \partial B_i} \frac{1}{r} K(x,y)[\partial_n \zeta]\,\mathrm{d}x.$$

This directly shows (b) because for ψ_i the extension to each B_i is constant. It also shows (a) by the fact that $[\partial_n\zeta]\mathcal{H}^1 \llcorner \partial B_i$ is a finite measure by elliptic regularity and the fact that the fundamental solution $K(x,y)$ decays like $1/(1+|y|)$ at ∞ locally uniformly in x (see [13, Lem. 2.1 ii)]).

The same argument as in the proof of Lemma 2.1.3 shows that

$$g(y) := \int_{\partial\mathcal{F}} \frac{1}{r} K(x,y)[\partial_n\zeta]\,\mathrm{d}x$$

fulfills $\operatorname{div}(\frac{1}{r}\nabla(g-\zeta))=0$. Furthermore, by the aforementioned decay estimates for K we see that $|g(x)|\lesssim 1/(|1+|x|)$, that $g=0$ at $r=0$ and that $\frac{1}{r}\nabla g$ is continuous at 0. By elliptic regularity, it holds that $g-\zeta$ is smooth in the interior of $\mathbb{H}$.

Now let $h(r,z,\theta)=(\frac{1}{r}\nabla(\zeta-g)(r,z))^{\perp}$ (in axisymmetric coordinates). This function is divergence-free (with respect to three-dimensional variables) as a direct calculation shows for $r>0$, at $r=0$ it is also divergence-free by continuity. Hence there is an H with $\operatorname{curl} H=h$, where the gradient is taken with respect to three-dimensional variables. Then it holds that $\Delta H=0$ and H is a tempered distribution and hence is a polynomial.

Hence we know that $\zeta-g$ is a polynomial as well, however, we have that $\zeta-g=0$ at $r=0$, hence it is also a polynomial in r only. However, $g\to 0$ for $r\to\infty$ and hence if $\zeta-g$ does not vanish, then ζ would have to grow at least linearly in r. For all a and large enough R we would then have

$$1\lesssim \frac{1}{R}\big|\zeta(R,a)-\zeta(R/2,a)\big|\lesssim \frac{1}{R}\int_{R/2}^{R}|\partial_r\zeta(s,a)|\,\mathrm{d}s\lesssim\Big(\int_{R/2}^{R}\frac{1}{s}|\partial_r\zeta(s,a)|^2\,\mathrm{d}s\Big)^{1/2}.$$

By taking a square root and using Fubini's theorem, we obtain that $\frac{1}{\sqrt{r}}\nabla\zeta\notin L^2$, which is a contradiction. □

Lemma 2.1.7. — *The Euler equation* (1.4) *holds if* (1.5) *and* (1.6) *hold and the circulations* $\gamma_i=\int_{\partial B_i}\tau\cdot u\,\mathrm{d}x$ *are conserved in time.*

In the two-dimensional setting this statement is well-known, see e.g. [12].

Proof. — Indeed the vorticity equation always holds and therefore we have

$$\operatorname{curl}(\partial_t u+(u\cdot\nabla)u)=0.$$

However not every curl-free field has to be a gradient in $\mathcal{F}$, as the domain has holes. Using the usual path integral construction, it is easy to see that it is a gradient if

$$\int_\Gamma(\partial_t u+(u\cdot\nabla)u)\cdot\tau_\Gamma\,\mathrm{d}x=0$$

along every closed, non-self-intersecting path $\Gamma\subset\mathcal{F}$ with normalized tangent τ_Γ, which has winding number 1 with respect to exactly one B_i and 0 with respect to all others. If Φ_t is the flow induced by u, then by direct calculation one sees that

$$\partial_s\int_{\Phi_s(\Gamma)}u\cdot\tau_{\Phi_s(\Gamma)}\,\mathrm{d}x\big|_{s=0}=\int_\Gamma(\partial_t u+(u\cdot\nabla)u))\cdot\tau_\Gamma\,\mathrm{d}x.$$

On the other hand we see that for such Γ it holds

$$\int_\Gamma u\cdot\tau_\Gamma\,\mathrm{d}x=\int_{\partial B_i}u\cdot\tau\,\mathrm{d}x,$$

which is constant by assumption, and hence we see that there is a p such that $\partial_t u+(u\cdot\nabla)u=-\nabla p$. □

Proposition 2.1.8. — *The function*

$$u(t)=\sum_{i=1}^{k}\Big(\nabla\phi_{i,\dot q_i}+\gamma_i\frac{1}{r}\nabla^{\perp}\psi_i\Big)=:u_1+u_2 \tag{2.12}$$

is a solution to the axisymmetric Euler equations (1.4)–(1.8).

Proof. — A direct calculation reveals that u is curl-free and fulfills $\operatorname{div}(ru)=0$ and hence u fulfills (1.5) and (1.6) in $\mathcal{F}$. We further observe that

$$\begin{aligned}
n\cdot\sum_{i=1}^{k}\nabla\phi_{i,\dot q_i}&=u(\dot q)\quad\text{on }\textstyle\bigcup_j\partial B_j,\\
n\cdot\frac{1}{r}\nabla^{\perp}\psi_i&=0\quad\text{on }\textstyle\bigcup_j\partial B_j,\\
\int_{\partial B_j}\nabla\phi_{i,\dot q_i}\cdot\tau\,\mathrm{d}x&=\int_{\partial B_j}\partial_\tau\phi_{i,\dot q_i}\,\mathrm{d}x=0,\\
\int_{\partial B_j}\frac{1}{r}\nabla^{\perp}\psi_i\cdot\tau\,\mathrm{d}x&=\int_{\partial B_j}\frac{1}{r}\partial_n\psi_i\,\mathrm{d}x=\delta_{ij},\\
\frac{1}{r}\partial_z\psi_i&=\partial_r\phi_{i,\dot q_i}=0\quad\text{on }\partial\mathbb{H},
\end{aligned}$$

where $\tau=n^{\perp}$.

Hence this u has the prescribed circulations and boundary velocities, which shows the statement by Lemma 2.1.7. □

We shall refer to both the ϕ_{i,t^*} and their sum as potentials of u. We shall refer to both the ψ_i and their weighted sum as streamfunctions of u.

Remark 2.1.9. — This u is uniquely determined (in L^2_R) by q, $\dot q$ and the γ_i. Indeed, if there would be two such different u, then their difference would give rise to a nonzero streamfunction with zero circulation (by the same argument as in the proof of Lemma 2.1.5, Step 3), which is impossible by the uniqueness of the functions ψ_i.

2.1.1. *Representation of* $\partial_t u$. — We will need to show that the potential and stream function are differentiable in q to be able to represent $\partial_t u$. The differentiability of solutions to elliptic equations with respect to changes of the underlying domain is a classical topic and we refer the reader to [35] for further reading.

Lemma 2.1.10

(a) *The function* $\phi_{i,\bullet}$ *is smooth as a map from the tangent bundle* TM *to* H^1_R *(here differentiability can be understood in both the* L^2_{loc}*-sense and the pointwise sense).*

(b) *The derivatives* $\partial_q\phi_{i,t^*}$, $\partial_q^2\phi_{i,t^*}$ *lie in* $H^1_R\cap C^\infty(\mathcal{F})$, *furthermore their* H^1_R*-norm is bounded locally uniformly in* q *and* t^*.

(c) $\operatorname{div}(r\nabla\partial_q\phi_{i,t^*})=0$.

(d) $|\nabla^m\partial_q\phi_{i,t^*}(x)|\lesssim 1/1+|x|^{2+m}$ *for all* $m\in\mathbb{N}_{\geqslant 0}$. *Here the implicit constant is bounded locally uniformly in* q *and* t^*.

Proof. — We can identify the tangent space at every point with $\mathbb{R}^2$ and $\phi_{i,\bullet}$ is linear in t^*, hence it suffices to show smoothness in q for a fixed t^*.

Step 1. — We first want to apply the implicit function theorem to obtain that a derivative of ϕ_{i,t^*} with respect to q exists. Fix some q^0. We use the three-dimensional $\phi^{\mathbb{R}^3} = \phi_{i,t^*}(r,z)$ (in axisymmetrical coordinates) again. We set

$$V := (\dot{H}^1 \cap L^6)(\mathbb{R}^3 \smallsetminus \bigcup_j B_j^{\mathbb{R}^3}(q^0)),$$

and equip this with the standard inner product of $\dot{H}^1$. By the Sobolev embedding this is a Hilbert space.

In order to fit different configurations of the bodies into one space, we introduce C^∞ diffeomorphisms $\Xi : \mathbb{R}^3 \smallsetminus \bigcup_j B_j(q^0) \to \mathbb{R}^3 \smallsetminus \bigcup_j B_j(q^1)$ which map each $\partial B_j(q^0)$ to $\partial B_j(q^1)$. We can assume that the family Ξ is smooth in the parameter q^1 since the B_i are. We may also assume that Ξ is the identity outside a large ball depending on q, but bounded locally uniformly in q. Then $\phi^{\mathbb{R}^3}$ is harmonic on $\mathbb{R}^3 \smallsetminus \bigcup_j B_j(q^1)$ with Neumann boundary values $u(t^*, q^1)$ iff the function $\widehat{\phi} := \phi \circ \Xi$ fulfills

$$\text{(2.13)}\quad \int_{\mathcal{F}(q^0)^{\mathbb{R}^3}} \big\langle \nabla\widehat{\phi}(\mathrm{D}\Xi)^{-1}, \nabla\eta(\mathrm{D}\Xi)^{-1}\big\rangle \,|\det \mathrm{D}\Xi|\,\mathrm{d}x = -\int_{\cup\partial B_j(q^0)^{\mathbb{R}^3}} \frac{\rho_j(q^1)}{\rho_j(q^0)}\, u(t^*, q^1)\eta\,\mathrm{d}x,$$

for all $\eta \in V$, where we have written the minor radius ρ_j as a function of q. We may interpret the difference of the left- and right-hand side as a map $\mathcal{G} : M \times V \to V^*$.

Since Ξ is smooth in q and compactly supported, we obtain that this map is Fréchet-smooth. Furthermore we have that

$$D_V\mathcal{G}(q^0, \widehat{\phi}) \cdot \delta\phi = \int_{\mathcal{F}(q^0)^{\mathbb{R}^3}} \langle \nabla\delta\phi, \nabla\cdot\rangle\,\mathrm{d}x.$$

This is an isomorphism by the Riesz representation theorem. Hence we see that $\widehat{\phi}$ is Fréchet-smooth by the implicit function theorem. This implies that a function $\partial_q\phi^{\mathbb{R}^3}$ exists in V by the smoothness of Ξ in q. Similarly, higher derivatives must exist. This shows (a).

Step 2. — Clearly, $\partial_q\phi^{\mathbb{R}^3}$ must be harmonic and hence smooth away from the boundary. To see smoothness up to the boundary we differentiate (2.13) with respect to q at q^0 and obtain that

$$\text{(2.14)}\quad \int_{\mathcal{F}(q^0)^{\mathbb{R}^3}} \big\langle \nabla\partial_q\widehat{\phi}(\mathrm{D}\Xi)^{-1}, \nabla\eta(\mathrm{D}\Xi)^{-1}\big\rangle\,|\det\mathrm{D}\Xi| + \big\langle \nabla\widehat{\phi}\partial_q((\mathrm{D}\Xi)^{-1}), \nabla\eta(\mathrm{D}\Xi)^{-1}\big\rangle\,|\det\mathrm{D}\Xi| + \big\langle \nabla\widehat{\phi}(\mathrm{D}\Xi)^{-1}, \partial_q\left(\nabla\eta(\mathrm{D}\Xi)^{-1}\,|\det\mathrm{D}\Xi|\right)\big\rangle\,\mathrm{d}x = -\int_{\cup\partial B_j(q^0)^{\mathbb{R}^3}} \Big\langle \partial_q\Big(\frac{\rho_j(q)}{\rho_j(q^0)}u(t^*, q^0)\Big), \eta\Big\rangle\,\mathrm{d}x,$$

for all $\eta \in V$, the differentiation of this equation is justified by the differentiability of $\widehat{\phi}$ and by Ξ being compactly supported and smooth in q.

This is an elliptic equation for $\partial_q\widehat{\phi}$ with Neumann boundary conditions and a smooth and compactly supported term given by the second and third summand on the left hand side. Hence $\partial_q\widehat{\phi}$ is smooth up the boundary. The same argument can be used to show regularity of higher derivatives in q. Again this also shows that $\partial_q\phi^{\mathbb{R}^3}$ (and hence also ϕ_{i,t^*}) is smooth up to the boundary and the same is true for higher derivatives in q.

By the pointwise smoothness that follows from this, it is obvious that (c) holds.

Step 3. — To obtain the decay estimate for the derivative, we note that it is enough to show these estimates for $\partial_q\phi^{\mathbb{R}^3}$. Clearly, it holds that $\Delta\partial_q\phi^{\mathbb{R}^3}=0$. We again employ Lemma 2.1.3, by e.g. going back to $\widehat{\phi}$ and using Equation (2.14), it is easy to see that the boundary values are locally uniformly controlled by q and t^*. To see that the integral over the Neumann boundary values of $\partial_q\phi^{\mathbb{R}^3}$ is 0, we introduce some compact B_j' with smooth boundary, in which $B_j^{\mathbb{R}^3}$ is compactly contained and which intersects no other $B_{j'}^{\mathbb{R}^3}$. Then we rewrite them as

$$\int_{\partial B_j^{\mathbb{R}^3}}\partial_n\partial_q\phi^{\mathbb{R}^3}\,\mathrm{d}x=-\int_{\partial B_j'}\partial_n\partial_q\phi^{\mathbb{R}^3}\,\mathrm{d}x=-\partial_q\int_{\partial B_j'}\partial_n\phi^{\mathbb{R}^3}\,\mathrm{d}x=0.$$

Here pulling out the derivative is justified by the regularity of $\partial_q\phi^{\mathbb{R}^3}$. In particular, the decay estimate also implies that the derivative is in H^1. ☐

Lemma 2.1.11. — *The functions f_i are smooth in R_i, in particular, E_{q_i} is smooth with respect to q.*

Proof. — This can be shown as in the previous lemma by using a similar smooth family of diffeomorphisms. ☐

Lemma 2.1.12

(a) *The derivative of ψ_i with respect to q exists and is smooth up to the boundary (here the derivative can e.g. be taken as a classical pointwise derivative or in the L^2_{loc} sense).*

(b) *We have that $\frac{1}{r}\nabla\partial_q\psi_i$, $\frac{1}{r}\nabla\partial_q^2\psi_i\in L^2_R\cap C^\infty$. Furthermore the L^2-norm of these derivatives is bounded locally uniformly in q.*

(c) *It holds that*

$$|\partial_q\psi_i(x)|\lesssim\frac{1}{1+|x|}\qquad\text{and}\qquad\Big|\nabla^m\partial_q\frac{1}{r}\nabla^\perp\psi_i(x)\Big|\lesssim\frac{1}{1+|x|^{2+m}}\quad\forall m\in\mathbb{N}_{\geqslant 0}.$$

In the second estimate the implicit constant is locally uniformly bounded in q.

(d) *The values C_{ij} are differentiable with respect to q.*

Proof

(a) and (d) The argument uses a similar technique as the existence proof for Lemma 2.1.5. First, we again consider the three-dimensional vector potentials Ψ_i as in said proof, for fixed boundary values $(\widetilde{C}_{ij})$, they have arbitrarily many derivatives in q, which are smooth up to the boundary by the same argument as in the previous proof and the derivatives are in $\dot{H}^1 \cap L^6$. This also shows that for fixed $(\widetilde{C}_{ij})$ there is a (smooth) derivative of $u_{2,i}$ and $\psi_i(\widetilde{C}_{ij})$.

It remains to argue that the values C_{ij} are differentiable. To see this note that the linear map from the $(\widetilde{C}_{ij})$ to the integrals $\int_{\partial B_l} \frac{1}{r}\partial_n \psi_i(\widetilde{C}_{ij})\,\mathrm{d}x$, which was used to show existence of the values C_{ij}, is differentiable in q as well. Indeed we may again introduce some compact B_l', which compactly contains B_l and intersects no other B_j. Then we have

$$\int_{\partial B_l'} \frac{1}{r}\partial_n \partial_q \psi_i(\widetilde{C}_{ij})\,\mathrm{d}x = \partial_q \int_{\partial B_l'} \frac{1}{r}\partial_n \psi_i(\widetilde{C}_{ij})\,\mathrm{d}x = \partial_q \int_{\partial B_l} \frac{1}{r}\partial_n \psi_i(\widetilde{C}_{ij})\,\mathrm{d}x,$$

which shows differentiability. As the $\dot{H}^1$-norm of Ψ_i corresponds to the L^2-norm of $\frac{1}{r}\nabla^\perp \psi_i(\widetilde{C}_{ij})$, we see the boundedness statement (b).

The decay of $\partial_q \psi_i$ again follows from the fact that the derivative fulfills

$$\operatorname{div}(\frac{1}{r}\nabla \partial_q \psi_i) = 0$$

and is smooth up to the boundary by using Lemma 2.1.6.

The decay of $\partial_q \frac{1}{r}\nabla^\perp \psi_i$ follows from the fact that the derivative of the three-dimensional stream function is harmonic and smooth up to the boundary as in the previous proof by Lemma 2.1.3, and can be controlled locally uniformly in q. □

Remark 2.1.13. — Note that if q is C^2 in time and u is a solution of the Euler equations (1.4)–(1.8), then we must have

$$\partial_t u = \sum_{i=1}^{k} \partial_q \phi_{i,\dot{q}} \cdot \dot{q} + \phi_{i,\ddot{q}_i} + \gamma_i \frac{1}{r}\nabla^\perp \partial_q \psi_i \cdot \dot{q}. \tag{2.15}$$

Indeed this follows from the fact that u is uniquely determined through $q, \dot{q}$ and the circulations γ_i (Remark 2.1.9). In particular, u has the regularity required in Condition 1.0.2.

Lemma 2.1.14. — *Assume that u solves the Euler equations* (1.4)–(1.8) *with pressure p. Then we have that*

$$|\nabla p(x)| \lesssim \frac{1}{1+|x|^2},$$

and there is a constant C which may be chosen as 0 *such that*

$$|p(x) - C| \lesssim \frac{1}{1+|x|}.$$

The implicit constants in these estimates are bounded locally uniformly in $q, \dot{q}, \ddot{q}$.

Proof. — By the construction of u in (2.12) and the decay estimates in Lemmas 2.1.2 and 2.1.5 we have that

$$|(u\cdot\nabla)u(x)| \lesssim \frac{1}{1+|x|^5}.$$

By Equation (2.15) and Lemmas 2.1.2, 2.1.10 and 2.1.12, we see that

$$|\partial_t u(x)| \lesssim \frac{1}{1+|x|^2}.$$

Hence $|\nabla p(x)| \lesssim 1/(1+|x|^2)$ and the estimate is locally uniform in $q, \dot{q}, \ddot{q}$, because the estimates for u and its derivatives are.

Since $\int_{\partial B_R(0)\cap \mathbb{H}} |\nabla p| \, \mathrm{d}x \to 0$ for $R \to \infty$ we have

$$\max_{x\in\partial B_R(0)} p(x) - \min_{x\in\partial B_R(0)} p(x) \longrightarrow 0.$$

Furthermore $\int_1^\infty |\nabla p(x,a)| \, \mathrm{d}x < \infty$ for all a for which no B_i intersects this line, hence we obtain that p converges to some finite value at infinity, which then gives the decay statement for p by the fundamental theorem of calculus. □

2.2. Derivation of an ODE for the system. — We reduce the motion of the bodies B_i to an ODE whose coefficients depend on the functions ψ_i and ϕ_i. This will also yield existence and uniqueness of solutions to the system. In two-dimensional bounded domains, a similar calculation can be found e.g. in [14].

We first introduce some additional terminology. We set $\phi(t^*) = \sum \phi_{i,t_i^*}$ if $t^* = t_1^* + \dots + t_k^*$. Furthermore we set $\psi = \sum_i \gamma_i \psi_i$.

Definition 2.2.1. — Let $t^* = t_1^* + \dots + t_k^*$; $s^* = s_1^* + \dots + s_k^*$ and $w^* = w_1^* + \dots + w_k^*$ for t_i^*, s_i^* and w_i^* associated with B_i. We define

$$\begin{aligned}
G_i(q,\gamma) \cdot t_i^* &= \int_{\partial B_i} \frac{1}{2r} \left((\partial_n \psi)^2 \partial_n \phi_{i,t_i^*}\right) \mathrm{d}x,\\
(\mathcal{M}_{ij}(q) t_i^*) \cdot s_j^* &= \int_{\mathcal{F}} r \nabla \phi_{i,t_i^*} \nabla \phi_{j,s_j^*} \, \mathrm{d}x,\\
\langle \Gamma(q), t^*, s^* \rangle \cdot w^* &= \frac{1}{2} \sum_{ij} \Big(\left((\partial_q \mathcal{M}_{ij} \cdot s^*) \, t^* \right) \cdot w^* + \left((\partial_q \mathcal{M}_{ij} \cdot t^*) \, s^* \right) \cdot w^*, \\
&\qquad\qquad - \left((\partial_q \mathcal{M}_{ij} \cdot w^*) \, s^* \right) \cdot t^* \Big),\\
(A(q,\gamma) t^*) \cdot s^* &= \sum_i \int_{\partial B_i} \Big(-\partial_\tau \phi(s^*) \partial_n \phi(t^*) + \partial_\tau \phi(t^*) \partial_n \phi(s^*) \Big) \partial_n \psi \, \mathrm{d}x,
\end{aligned}$$

where in the definition of Γ, the inner dot product refers to the derivative in that direction.

Furthermore, $\mathcal{M}$ shall be the matrix made up of the blocks $\mathcal{M}_{ij}$ and E shall be the diagonal matrix made up of the blocks E_{q_i}. Let $G \in (\mathbb{R}^2)^k \simeq \mathbb{R}^{2k}$ be the vector with the entries $G_1, \dots G_k$.

Theorem 2.2.2. — *The system detailed in the Introduction is equivalent to the system of ODEs given by*

$$E(q)\ddot{q} + \frac{1}{2}\dot{q}(\partial_q E(q) \cdot \dot{q}) + \mathcal{M}(q)\ddot{q} + \langle \Gamma(q), \dot{q}, \dot{q} \rangle = G(q,\gamma) + (A(q,\gamma)\dot{q}). \tag{2.16}$$

Remark 2.2.3. — The equation can be interpreted as the geodesic equation for the metric given by $\mathcal{M} + E$, with extra terms due to the circulation on the right hand side.

The matrix $\mathcal{M}$ describes the "added inertia", which encodes the fact that to accelerate one of the bodies, one also has to accelerate the surrounding fluid.

Proof. — We argued in Remark 2.1.9 that u is uniquely determined by $q, \dot q$, hence it suffices to show that the family of equations in (1.20) is equivalent to this system. Let t_i^* be an arbitrary tangent vector associated with B_i. We set $u_i^* = \nabla(\phi_{i,t_i^*})$.

Then by Equation 1.20 it holds that

$$(t_i^*)^T E_{q_i}\ddot q_i + \frac{1}{2}\dot q_i^T(\partial_{q_i}E_{q_i}\cdot \dot q_i)t_i^* = -\int_{\partial B_i} rpu_i^*\cdot n\,\mathrm{d}x.$$

By the equation for p, partial integration and the identity $u\nabla u = \frac{1}{2}\nabla|u|^2 + u\cdot \operatorname{curl} u$ it follows that

$$\begin{aligned}(t_i^*)^T E_{q_i}\ddot q_i + \frac{1}{2}\dot q_i^T(\partial_{q_i}E_{q_i}\cdot \dot q_i)t_i^* &= \int_{\mathcal{F}} r\nabla p\cdot u_i^*\,\mathrm{d}x\\ &= -\int_{\mathcal{F}} r\Big(\partial_t(u_1+u_2) + \frac{1}{2}\nabla|u_1+u_2|^2\Big)\cdot u_i^*\,\mathrm{d}x.\end{aligned}$$

It follows from the decay estimates in Lemmas 2.1.2, 2.1.5 and 2.1.14 that there are no boundary terms from ∞ in this partial integration.

We now split this into the different contributions and use the proposition below to obtain the equation in the theorem, tested against t_i^*. Since t_i^* was arbitrary this implies the statement. □

Proposition 2.2.4

(a) *We have*

$$-\frac{1}{2}\int_{\mathcal{F}} r\nabla|u_2|^2\cdot u_i^*\,\mathrm{d}x = G_i(q,\gamma)\cdot t_i^*.$$

(b) *It holds that*

$$-\int_{\mathcal{F}} r\big(\partial_t u_2 + \nabla(u_1\cdot u_2)\big)\cdot u_i^*\,\mathrm{d}x = (A(q,\gamma)\dot q)\cdot t_i^*.$$

(c) *We have that*

$$\int_{\mathcal{F}} r\Big(\partial_t u_1 + \frac{1}{2}\nabla|u_1|^2\Big)\cdot u_i^*\,\mathrm{d}x = \ddot q^T\mathcal{M}(q)t_i^* + \langle\Gamma(q),\dot q,\dot q\rangle\cdot t_i^*.$$

Proof

(a) Using that both u_2 and u_i^* decay like $1/|x|^2$ by Lemma 2.1.2 and Lemma 2.1.5 we may partially integrate the left-hand side to obtain equality with

$$\int_{\partial\mathcal{F}} \frac{1}{2}r|u_2|^2\partial_n\phi_{i,t^*}\,\mathrm{d}x. \tag{2.17}$$

To see that this equals the definition of G we note that $\partial_n\phi_{i,t_i^*}$ vanishes on every boundary except ∂B_i and that $|u_2| = \frac{1}{r}|\nabla^\perp\sum_j\gamma_j\psi_j| = \frac{1}{r}|\partial_n\sum_j\gamma_j\psi_j|$ since the tangential derivative of the functions ψ_j vanishes.

(b) We have that

$$-\int_{\mathcal{F}} r\nabla(u_1\cdot u_2)u_i^*\,\mathrm{d}x = \int_{\partial\mathcal{F}} r(u_1\cdot u_2)(u_i^*\cdot n)\,\mathrm{d}x,$$

(this partial integration is justified by the decay estimates from Lemmas 2.1.2 and 2.1.5) which by the construction of u in (2.12) equals

$$\sum_l \int_{\partial B_i} r\Big(\frac{1}{r}\partial_n \sum_j \gamma_j\psi_j\Big)(\partial_\tau\phi_{l,\dot q_l})(\partial_n\phi_{i,t_i^*})\,\mathrm{d}x,$$

because u_2 has no normal component on the boundary.

It holds that $\partial_t u_2 = \frac{1}{r}\nabla^\perp\partial_t\psi$. We have that $\operatorname{div}(\frac{1}{r}\nabla\partial_t\psi) = 0$ and $\partial_t\psi$ has the boundary values

$$\partial_t\psi = \sum_j \gamma_j\partial_q C_{jl}\cdot\dot q - (u_1\cdot n)\partial_n\psi \quad \text{on } \partial B_l$$

as one can see by differentiating the identity $C_{jl}(q) = \psi_j(x_q)(q)$ where x_q is some fixed point on ∂B_l whose derivative in t equals $u_1\cdot n$. Then a partial integration, which can again be justified by the decay estimates in Lemmas 2.1.2 and 2.1.12, reveals that

$$-\int_{\mathcal{F}} r u_i^*\cdot\partial_t u_2\,\mathrm{d}x = \sum_l\sum_j \int_{\partial B_l} \partial_\tau\phi_{i,t_i^*}\,(\gamma_j\partial_q C_{jl}\cdot\dot q - (u_1\cdot n)\partial_n\psi)\,\mathrm{d}x.$$

The first summand vanishes because $\partial_q C_{jl}$ is a constant on each ∂B_l and this proves (b).

(c) We introduce an energy functional for the potential part of the fluid velocity:

$$\mathcal{E}_{u_1} := \frac{1}{2}\int_{\mathcal{F}} r|u_1|^2\,\mathrm{d}x.$$

Following the approach in [33], we will show that

$$(\partial_t\partial_{\dot q} - \partial_q)\mathcal{E}_{u_1}\cdot t_i^* = \int_{\mathcal{F}} u_i^*\cdot(\partial_t u_1 + \frac{1}{2}\nabla|u_1|^2)\,\mathrm{d}x. \tag{2.18}$$

To prove this claim we shall use the following lemma which can be proved exactly as in [33, Lem. 5.1]:

Lemma 2.2.5. — *For $\eta\in\dot H^1_R$ let*

$$\Lambda(\eta) = \int_{\mathcal{F}} r\langle\nabla\phi(\dot q),\nabla\eta\rangle\,\mathrm{d}x.$$

Then it holds that

$$(\partial_t\partial_{\dot q} - \partial_q)(\Lambda) = 0.$$

We now note that $\mathcal{E}_{u_1} = \frac{1}{2}\Lambda(\phi(\dot q))$ and that

$$\partial_{\dot q}\mathcal{E}_{u_1}\cdot t_i^* = \frac{1}{2}\big((\partial_{\dot q}\Lambda)(\phi(\dot q))\cdot t_i^* + \Lambda(\phi(t_i^*))\big).$$

Because $\phi(t_i^*)$ also equals $\partial_{\dot q}\phi(\dot q)\cdot t_i^*$, we see that

$$\partial_{\dot q}\mathcal{E}_{u_1}\cdot t_i^* = (\partial_{\dot q}\Lambda)(\phi(\dot q))\cdot t_i^*. \tag{2.19}$$

Hence we obtain that

$$(\partial_t\partial_{\dot q} - \partial_q)\mathcal{E}_{u_1}\cdot t_i^* \\ = (\partial_t\partial_{\dot q}\cdot t_i^* - \partial_q\cdot t_i^*)\,\Lambda(\phi) + (\partial_{\dot q}\Lambda)(\partial_t\phi(\dot q))\cdot t_i^* + \frac{1}{2}(\partial_q\Lambda)(\phi(\dot q))\cdot t_i^* - \frac{1}{2}\Lambda(\phi^\dagger),$$

where $\phi^\dagger = \partial_q\phi(\dot q)\cdot t_i^*$ and we made use of (2.19).

By Lemma 2.2.5, the first term is 0. By definition, the second term equals

$$\partial_{\dot q}\Lambda(\partial_t\phi(\dot q))\cdot t_i^* = \int_{\mathcal{F}} r\partial_t u_1\nabla\phi(t_i^*)\,\mathrm{d}x.$$

By the Reynolds transport theorem (whose usage is justified by Lemma 2.1.10 (b)), we have that twice the third term equals

$$(\partial_q\Lambda\cdot t_i^*)(\phi) = -\int_{\partial B_i} r|u_1|^2 u_i^*\cdot n\,\mathrm{d}x + \Lambda(\phi^\dagger).$$

This yields the claim (2.18) after another partial integration.

Now we use that $\mathcal{E}_{u_1} = \frac{1}{2}\mathcal{M}(q)\dot q\cdot\dot q$, which follows directly from the definition of $\mathcal{M}$. Then we may compute the Euler-Lagrange equation of this as

$$(\partial_t\partial_{\dot q} - \partial_q)\mathcal{E}_{u_1}\cdot t_i^* = \mathcal{M}(q)\ddot q\cdot t_i^* + ((\partial_q\mathcal{M}(q)\cdot\dot q)\dot q)\cdot t_i^* - \frac{1}{2}((\partial_q\mathcal{M}(q)\cdot t_i^*)\dot q)\cdot\dot q.$$

The last two summands equal the Christoffel symbol Γ as one can directly see by writing them out in components. ☐

2.2.1. *Uniqueness and existence*. — In this subsection, we show that the system is actually well-posed and that energy conservation will imply that solutions exist for all times if q does not blow up.

Lemma 2.2.6. — *The coefficients $\mathcal{M}, G, A, \Gamma$ are all continuously differentiable in q.*

Proof. — One can use the definition of all these terms and Lemmas 2.1.10 and 2.1.12 to obtain that they are smooth in q. We leave the details to the reader. ☐

Corollary 2.2.7. — *For every initial datum $q, \dot q$, there is some $T > 0$ such that the system* (2.16) *and hence also the system introduced in the introduction has a unique solution up to time T, which is C^2 in q.*

Proof. — By the lemma above and the Picard-Lindelöf theorem, we have local existence and uniqueness if the matrix $\mathcal{M} + E$ is invertible, which follows from the fact that both $\mathcal{M}$ and E are positive definite by definition. ☐

The total energy of the system is conserved:

Lemma 2.2.8. — *The kinetic energy*

$$\int_{\mathcal{F}} \frac{1}{2} r|u|^2\,\mathrm{d}x + \sum_i \mathcal{E}_{B_i}$$

($\mathcal{E}$ was defined in (1.17)*) is constant in time.*

Proof. — By Reynolds we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}}\frac{1}{2}r|u|^2\,\mathrm{d}x=\int_{\mathcal{F}}ru\cdot\partial_t u\,\mathrm{d}x-\int_{\partial\mathcal{F}}\frac{1}{2}r|u|^2u\cdot n\,\mathrm{d}x.$$

Here differentiating under the integral sign is justified by the L^2_R-differentiability from Lemmas 2.1.10 and 2.1.12. The first integral also equals

$$\int_{\mathcal{F}}ru\cdot\partial_t u\,\mathrm{d}x=\int_{\mathcal{F}}ru\cdot(-u\cdot\nabla u-\nabla p)\,\mathrm{d}x=\int_{\mathcal{F}}-\frac{1}{2}\operatorname{div}(ru|u|^2)-\operatorname{div}(rpu)\,\mathrm{d}x.$$

Applying Gauss's theorem to this and adding the second integral from the first equation we obtain the statement, as we have by Equation (1.18)

$$\int_{\partial B_i}rpu(\dot{q}_i)\,\mathrm{d}x=-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{B_i}.\qquad\square$$

Lemma 2.2.9. — *The kinetic energy $\int_{\mathcal{F}}\frac{1}{2}r|u|^2\,\mathrm{d}x$ of the fluid decomposes into the energies $\int_{\mathcal{F}}\frac{1}{2}r(|u_1|^2+|u_2|^2)\,\mathrm{d}x$.*

Proof. — We have that

$$\int_{\mathcal{F}}ru_1\cdot u_2\,\mathrm{d}x=\sum_i\gamma_i\int_{\mathcal{F}}r\nabla\phi\frac{1}{r}\nabla^\perp\psi_i\,\mathrm{d}x=\sum_{i,j}\gamma_i\int_{\partial B_j}\partial_\tau\phi C_{ij}\,\mathrm{d}x=0,$$

where we abbreviated the potential of u_1 with ϕ. $\square$

Theorem 2.2.10. — *Solutions of the system exist until q leaves any compact set, i.e., until either some of the bodies collide with each other or the boundary or escape to infinity.*

Proof. — By the energy conservation, we see that $\int r|u|^2\,\mathrm{d}x$ is bounded uniformly in time and hence by Lemma 2.2.9 we also have that $\int r|u_1|^2\,\mathrm{d}x$ is bounded uniformly in time. Now note that there is no q such that for some $t^*\neq 0$ it holds that $\nabla\phi(t^*)=0$. Hence by the continuity of the coefficients we have on compact sets

$$\int r|u_1|^2\,\mathrm{d}x\gtrsim|\dot{q}|^2.$$

This implies that the only way the solution can blow up is if q leaves any compact set. $\square$

3. Convergence of the potential part of the velocity

3.1. Overview and strategy. — In this section, we will consider the limit of the potential velocity and of the interior field in order to compute the limit of the coefficients of the equation.

We will show that all relevant main quantities converge to the corresponding two-dimensional quantities for a single body, which can be explicitly written down, and that the error is an order $\varepsilon|\log\varepsilon|^\ell$ smaller (see Corollary 3.3.4). Furthermore, we will show that quantities that only exist for multiple bodies are even smaller (see Corollary 3.3.9). We will also show that derivatives with respect to q are an order $\varepsilon|\log\varepsilon|^\ell$ smaller as well (see 3.3.10). In Sections 3.4 and 4.6 we will see that $\mathcal{M}$ and A converge to the corresponding two-dimensional quantities for a single body and that Γ and $\partial_q A$ are negligible.

3.1.1. *Proof strategy*. — The basic idea is to compare the coefficients of the elliptic equations defining the potentials with the corresponding equations for the 2D limiting quantities (see e.g. Definition 3.3.1 below). These coefficients converge close to the bodies, which is enough to prove convergence via standard L^2 for the interior field (see Lemma 3.2.1 below). The potentials ϕ_{t^*}, on the other hand, are defined on an exterior domain, and far away from the bodies, the coefficients do not converge. This is dealt with using uniform decay estimates, which are proved using single-layer potentials (see Lemma 3.3.7). The proof is split in the case of a single body and multiple bodies. For a single body, L^2-estimates are sufficient to conclude convergences of the relevant traces via elliptic regularity. For multiple bodies, the method of reflections and decay estimates are sufficient to show that the contribution of a single body is already the leading order contribution (see Section 3.3.2 below). For the derivatives with respect to q, we consider the PDE fulfilled by the derivative of the potential, which can be estimated by the previous a priori estimates.

As we believe it makes the proof more transparent, we rescale space by a factor $1/\varepsilon$ (see the beginning of Section 3.3.1 for details).

We omit the indices of $B, C, q, R, Z, u_{\mathrm{int}}$, etc. when dealing with only a single body. We identify the tangent space of $\mathcal{M}$ with $(\mathbb{R}^2)^k$ via the map $t^* \to (t^*_{R_1}, t^*_{Z_1}, \dots)$.

3.2. The interior field. — For the kinetic energy of each body, we only need to consider a single body as the definition of E_{q_i} (see (1.14)) only depends on B_i. Therefore we drop the indices in this subsection.

We write f_ε for the function f_i, defined with the rescaling parameter ε.

Lemma 3.2.1. — *Consider the energy function f_ε defined in* (1.15).

(a) *We have*

$$\left|f_\varepsilon(R) - \pi R\widetilde{\rho}^2\varepsilon^2\right| \lesssim \varepsilon^3,$$

where the implicit constant depends locally uniformly on R.

(b) *$f_\varepsilon(R)$ is Lipschitz in R with constant $\lesssim \varepsilon^3$, locally uniformly in R.*

In particular this implies that we have $|E_{q_i}| \approx \varepsilon^2$ and $|\nabla_q E_{q_i}| \lesssim \varepsilon^3$.

Proof

(a) We compare the potential of u_{int} with the one of the constant speed movement. Set $\phi_1(x) = r$, which solves the Neumann problem $\Delta\phi_1 = 0$, $\partial_n\phi_1 = e_R \cdot n$.

Similarly, $u_{\mathrm{int}}(e_R)$ can by definition (see (1.10)–(1.12)) be written as $\nabla\phi_2$, where $\operatorname{div}(r\nabla\phi_2) = 0$ and $\partial_n\phi_2 = u(e_R)$. Testing these equations with $\phi_1 - \phi_2$ we obtain that

$$\int_B \langle \nabla\phi_1, \nabla(\phi_1-\phi_2)\rangle \,\mathrm{d}x = \int_{\partial B} e_R\cdot n(\phi_1-\phi_2)\,\mathrm{d}x,$$

$$\int_B r\langle \nabla\phi_2, \nabla(\phi_1-\phi_2)\rangle \,\mathrm{d}x = \int_{\partial B} r\left(e_R\cdot n - \frac{\rho}{2R}\right)(\phi_1-\phi_2)\,\mathrm{d}x.$$

We multiply the first equation with R and subtract the second from it, this yields that

$$\int_B r\langle \nabla(\phi_1-\phi_2), \nabla(\phi_1-\phi_2)\rangle + (R-r)\langle \nabla\phi_1, \nabla(\phi_1-\phi_2)\rangle \,\mathrm{d}x \\ = \int_{\partial B}\left(Re_R\cdot n - r\left(e_R\cdot n-\frac{\rho}{2R}\right)\right)(\phi_1-\phi_2)\,\mathrm{d}x.$$

Note that we may add a constant to $\phi_1-\phi_2$ in the last integral because the other factor is mean-free.

Applying the Cauchy-Schwarz inequality we obtain that

$$\int_B r\,|\nabla(\phi_1-\phi_2)|^2\,\mathrm{d}x \leqslant \rho\,\|\nabla\phi_1\|_{L^2(B)}\,\|\nabla(\phi_1-\phi_2)\|_{L^2(B)} \\ + \left\|Re_R\cdot n - r\left(e_R\cdot n-\frac{\rho}{2R}\right)\right\|_{L^2(\partial B)}\|\phi_1-\phi_2\|_{L^2(\partial B)/\mathrm{constants}}. \tag{3.1}$$

The last factor can be estimated by $c_{\mathrm{trace}}\,\|\nabla(\phi_1-\phi_2)\|_{L^2(B)}$, where c_{trace} is the operator norm of the trace from $\dot{H}^1$ to $L^2(\partial B)/\mathrm{constants}$. By scaling one can see that this constant is $\lesssim \varepsilon^{1/2}$.

This gives us an upper bound on the right-hand side of (3.1) of

$$\rho\,\|\nabla(\phi_1-\phi_2)\|_{L^2}\,\|\nabla\phi_1\|_{L^2} + c_{\mathrm{trace}}\,\|\nabla(\phi_1-\phi_2)\|_{L^2(B)}\left\|\rho+\frac{\rho}{2R}\right\|_{L^2(\partial B)} \\ \lesssim \varepsilon^2\,\|\nabla(\phi_1-\phi_2)\|_{L^2}\,. \tag{3.2}$$

Together with the observation that

$$\int_B r|\nabla(\phi_1-\phi_2)|^2\,\mathrm{d}x \lesssim \|\nabla(\phi_1-\phi_2)\|_{L^2}^2\,,$$

we obtain from (3.1) and (3.2) that

$$\|\nabla(\phi_1-\phi_2)\|_{L^2} \lesssim \varepsilon^2.$$

Now by definition

$$|f_\varepsilon - \pi R\widetilde{\rho}^2\varepsilon^2| = \left|\int_B r\left(|\nabla\phi_2|^2-|\nabla\phi_1|^2\right)\,\mathrm{d}x\right| \\ \lesssim \|\nabla(\phi_1-\phi_2)\|_{L^2}\left(\|\nabla\phi_1\|_{L^2}+\|\nabla\phi_2\|_{L^2}\right) \lesssim \varepsilon^3.$$

(b) We first estimate the derivative of the potential of u_{int} with respect to R and then use this to estimate the Lipschitz constant. We fix some $q^0 = (Z^0, R^0)$ with minor radius ρ^0 and use the family of maps

$$\Xi_q(x) := \frac{\rho^0}{\rho}\left(x - \begin{pmatrix} R \\ Z \end{pmatrix}\right) + \begin{pmatrix} R^0 \\ Z^0 \end{pmatrix},$$

which map $B(q)$ to $B(q^0)$.

Let ϕ^q be defined by $\nabla\phi^q = u^q_{\mathrm{int}}(e_R)$, where the q in the superscript denotes the q-dependence and we use the identification between the tangent space and $\mathbb{R}^2$ mentioned above. Let

$$\widehat{\phi} := \frac{\rho^0}{\rho}\phi^q \circ \Xi_q^{-1}.$$

Then a direct calculation shows that this fulfills the system

$$\begin{aligned} \operatorname{div}\Big(\frac{R^0}{R}\Big(R + \frac{\rho}{\rho^0}(r - R^0)\Big)\nabla\widehat{\phi}\Big) = 0 \quad \text{in } B(q^0),\\ \partial_n\widehat{\phi} = e_R \cdot n - \frac{\rho}{2R} \quad \text{on } \partial B(q^0). \end{aligned}$$

Using for instance the implicit function as in the proof of Lemma 2.1.10, one can easily see that one can differentiate the solution of this equation in R (with respect to the H^1-norm) and that the derivative fulfills the system

$$\begin{aligned} \operatorname{div}\Big(\frac{R^0}{R}\Big(R + \frac{\rho}{\rho^0}(r - R^0)\Big)\nabla\partial_{R_1}\widehat{\phi}\Big) \qquad \qquad (3.3)\\ + \operatorname{div}\Big(\partial_{R_1}\Big(\frac{R^0}{R}\Big(R + \frac{\rho}{\rho^0}(r - R^0)\Big)\Big)\nabla\widehat{\phi}\Big) = 0 \quad \text{in } B(q^0), \end{aligned}$$

$$\partial_n\partial_{R_1}\widehat{\phi} = \partial_{R_1}\Big(e_R \cdot n - \frac{\rho}{2R}\Big) \quad \text{on } \partial B(q^0). \tag{3.4}$$

Here we write $\partial_{R_1}\widehat{\phi}$ for the derivative with respect to the parameter $R = R_1$ in order to prevent confusion with the spatial derivative in the R-direction. Now one can easily check that

$$\Big|\partial_{R_1}\Big(\frac{R^0}{R}\Big(R + \frac{\rho}{\rho^0}(r - R^0)\Big)\Big)\Big| \lesssim \varepsilon, \tag{3.5}$$

$$\|\nabla\widehat{\phi}\|_{L^2} \lesssim \varepsilon, \tag{3.6}$$

$$\Big|\partial_{R_1}\Big(e_R \cdot n - \frac{\rho}{2R}\Big)\Big| \lesssim \varepsilon. \tag{3.7}$$

We can now test the equations (3.3), (3.4) with $\partial_{R_1}\widehat{\phi}$ and obtain after using the Cauchy-Schwarz inequality similarly as in part (a) and the bounds (3.5)–(3.7) that

$$\begin{aligned} \int_{B(q^0)} \frac{R^0}{R}\Big(R + \frac{\rho}{\rho^0}(r - R^0)\Big)\big|\nabla\partial_{R_1}\widehat{\phi}\big|^2 \,\mathrm{d}x \qquad \qquad (3.8)\\ \lesssim \varepsilon^2\big\|\nabla\widehat{\phi}\big\|^2_{L^2} + \varepsilon^{3/2}\Big\|\partial_{R_1}\widehat{\phi}\Big\|_{L^2(\partial B(q^0))/\text{constants}}, \end{aligned}$$

where we again used the mean-freeness of the boundary values to take the L^2-norm modulo constants.

Clearly, the prefactor in the integral on the left-hand side is $\simeq 1$. Again the operator norm of the trace operator from $\dot{H}^1$ to $L^2(\partial B(q^0))/\text{constants}$ is $\simeq \varepsilon^{1/2}$ by scaling, hence we obtain from (3.8) that

$$\big\|\nabla \partial_{R_1} \widehat{\phi}\big\|_{L^2} \lesssim \varepsilon^2,$$

and this bound is locally uniform in R.

By definition it holds that

$$f_\varepsilon(R) = \int_{B(q)} r|\nabla \phi^q|^2 \,\mathrm{d}x = \int_{B(q^0)} \Big(\frac{\rho}{\rho^0}\Big)^2 \Big(R + \frac{\rho}{\rho_0}(r - R^0)\Big) \big|\nabla \widehat{\phi}\big|^2 \,\mathrm{d}x.$$

The prefactor in the second integral is differentiable in R with a derivative $\lesssim \varepsilon$. Now we can differentiate the right-hand side under the integral by the H^1-differentiability of $\widehat{\phi}$ and obtain from the product rule that

$$|\partial_R f_\varepsilon(R)| \lesssim \varepsilon^2 \big\|\nabla \widehat{\phi}\big\|_{L^2} + \varepsilon \big\|\nabla \widehat{\phi}\big\|_{L^2}^2 \lesssim \varepsilon^3.$$

This is locally uniform in R as all the used estimates are. □

3.3. The potential part of the velocity. — We show that the boundary values of the potential converge to the boundary values of the corresponding "two-dimensional" potential.

3.3.1. *The case of a single body*

Definition 3.3.1. — Let $t^*, q \in \mathbb{R}^2$. Let $\rho > 0$. Let n denote the outer normal vector of $\partial B_\rho(q)$. We define a "two-dimensional" potential

$$\check{\phi}_{t^*} = \check{\phi}_{t^*}(q, \rho) := -\rho^2 \frac{t^* \cdot e_1(x - q) + t^* \cdot e_2(y - q)}{(x - q)^2 + (y - q)^2}.$$

One can check that this is the solution of

$$\Delta \check{\phi}_{t^*} = 0 \quad \text{in } \mathbb{R}^2 \smallsetminus B_\rho(q), \tag{3.9}$$

$$\partial_n \check{\phi}_{t^*} = t^* \cdot n \quad \text{on } \partial B_\rho(q), \tag{3.10}$$

$$\check{\phi}_{t^*} \longrightarrow 0 \quad \text{at } \infty. \tag{3.11}$$

Uniqueness of this can be found e.g. in [1, Th. 3.1].

In order to estimate the potentials for $\varepsilon \to 0$, we first use only a single body and again drop the indices. Fix some q and t^*, where we again make use of the identification of the tangent space with $\mathbb{R}^2$ as in the previous subsection. Furthermore, fix some $\widetilde{\rho} > 0$ such that $\varepsilon \widetilde{\rho} = \rho$. We will prove a more general statement for arbitrary normal velocities, which will be useful later to estimate derivatives with respect to q.

It will simplify the argument to rescale everything by a factor of ε. Therefore we let $\grave{B} := B_{\widetilde{\rho}}(\frac{1}{\varepsilon} q)$ and first prove our estimates around this rescaled body.

Proposition 3.3.2. — *Let b_1, b_2 be smooth functions on $\partial \grave{B}$. Further, assume that*

$$\int_{\partial \grave{B}} r b_1 \,\mathrm{d}x = \int_{\partial \grave{B}} b_2 \,\mathrm{d}x = 0.$$

Let $\breve{\phi} \in \dot{H}^1$ *and* $\grave{\phi} \in \dot{H}^1_R$ *be the solutions of*

$$\Delta\breve{\phi} = 0 \quad \text{in } \mathbb{R}^2 \smallsetminus \grave{B}, \tag{3.12}$$

$$\partial_n\breve{\phi} = b_2 \quad \text{on } \partial\grave{B}, \tag{3.13}$$

$$\breve{\phi}(x) \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty, \tag{3.14}$$

and

$$\begin{aligned} \operatorname{div}(r\nabla\grave{\phi}) &= 0, \quad \text{in } \mathbb{H} \smallsetminus \grave{B} \\ \partial_n\grave{\phi} &= b_1, \quad \text{on } \partial\grave{B} \\ \partial_n\grave{\phi} &= 0, \quad \text{on } \partial\mathbb{H} \\ \grave{\phi}(x) &\longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty. \end{aligned}$$

Then for all $m \in \mathbb{N}_{>0}$ *it holds that*

$$\big\|\nabla^m(\grave{\phi} - \breve{\phi})\big\|_{L^2(\partial\grave{B})} \lesssim_m \sqrt{|\log\varepsilon|}\,\Big(\varepsilon\big(\|b_2\|_{H^{m-1}(\partial\grave{B})} + \|b_1\|_{H^{m-1}(\partial\grave{B})}\big) + \|b_1 - b_2\|_{H^{m-1}(\partial\grave{B})}\Big).$$

The implicit constant in these estimates is bounded locally uniform in q.

Remark 3.3.3

(a) Existence and uniqueness of $\grave{\phi}$ follows by Lemma 2.1.2 and existence and uniqueness of $\breve{\phi}$ are shown in [1, Th. 3.1].

(b) The author strongly believes that the factor $\sqrt{|\log\varepsilon|}$ is an artifact of the proof and can be removed by estimating $\big\|\Delta(\breve{\phi} - \grave{\phi})\big\|_{L^2}$ instead of $\big\|\sqrt{r}\nabla(\breve{\phi} - \grave{\phi})\big\|_{L^2}$ in the proof, which would require more effort.

Corollary 3.3.4. — *For all* $m \in \mathbb{N}_{>0}$ *it holds that*

$$\big\|\nabla^m(\phi_{t^*} - \breve{\phi}_{t^*})\big\|_{L^2(\partial B)} \lesssim_m \varepsilon^{5/2-m}\sqrt{|\log\varepsilon|}\,|t^*|.$$

The implicit constant here is bounded locally uniformly in q.

Proof of the corollary. — Observe that if we set $b_1 = \varepsilon u(t^*)(\varepsilon\cdot)$ and $b_2 = \varepsilon t^* \cdot n$, then it holds that

$$\phi_{t^*}(\varepsilon\cdot) = \grave{\phi} \quad \text{and} \quad \breve{\phi}_{t^*}(\varepsilon\cdot) = \breve{\phi},$$

because these fulfill the same elliptic equation. One easily sees that

$$\|b_1\|_{H^m(\partial B)} \lesssim \varepsilon|t^*|, \quad \|b_2\|_{H^m(\partial B)} \lesssim \varepsilon|t^*|, \quad \|b_1 - b_2\|_{H^m(\partial B)} \lesssim \varepsilon^2|t^*|.$$

The statement then follows from applying the proposition and rescaling. □

Our strategy to prove the proposition is to again apply L^2 estimates as in the previous section, as the coefficients are similar, close to $\grave{B}$, together with decay estimates for the far away behavior.

Lemma 3.3.5. — *Let* b_2 *and* $\breve{\phi}$ *be as in the proposition, then for all* $m \in \mathbb{N}_{\geqslant 0}$ *it holds that*

(a) $\big\|\nabla^m\breve{\phi}\big\|_{L^2(\partial\grave{B})} \lesssim_m \|b_2\|_{H^{\max(0,m-1)}}$,

(b) $|\nabla^m\breve{\phi}|(x) \lesssim_m \|b_2\|_{L^2}/\operatorname{dist}(x, \grave{B})^{1+m}$,

(c) $\|\breve{\phi}\|_{\dot{H}^m(\grave{B}+B_1(0)\smallsetminus\grave{B})} \lesssim_m \|b_2\|_{H^{\max(0,m-1)}}$.

Here all the implicit constants are bounded locally uniformly in q.

Proof. — All three statements are well-known, we sketch the proof here for the convenience of the reader.

One can first repeat the argument in the proof of Lemma 2.1.3 to show that ϕ and $\nabla\phi$ must decay like $|x|^{-1}$ resp. $|x|^{-2}$. Then by testing the PDE (3.12)–(3.14) with $\breve{\phi}$ and partially integrating, we see that

$$\int_{\partial\grave{B}} b_2\breve{\phi}\,\mathrm{d}x = -\|\breve{\phi}\|^2_{\dot{H}^1(\mathbb{R}^2\smallsetminus\grave{B})},$$

where the partial integration is justified by the decay of ϕ. Now b_2 is mean-free, so by using the trace in $\grave{B}+B_1(0)\smallsetminus\grave{B}$, we see that

$$\|\breve{\phi}\|_{\dot{H}^1(\grave{B}+B_1(0)\smallsetminus\grave{B})} \lesssim \|b_2\|_{L^2(\partial\grave{B})}.$$

By using elliptic regularity estimates in $\grave{B}+B_1(0)\smallsetminus\grave{B}$, we see that for $m>0$ we have

$$\big\|\nabla^m\breve{\phi}\big\|_{L^2(\partial\grave{B})} \lesssim \|b_2\|_{H^{m-1}(\partial\grave{B})}.$$

This lets us control $\big\|\nabla\breve{\phi}\big\|_{L^2(\partial\grave{B})}$ and we can repeat the argument in the proof of Lemma 2.1.3 to show (b), where we get one order of decay less from using the two-dimensional Newtonian potential.

(c) follows for $m>0$ similarly by using elliptic regularity. The estimates for $m=0$ in (a) and (c) follow by using the estimate (b) for $\operatorname{dist}(x,\partial\grave{B})\geqslant 1$ and combining the estimate on the gradient in (c) with e.g. the Poincaré inequality. □

Remark 3.3.6. — In particular, by rescaling, we see that we have

$$\big\|\partial_\tau\breve{\phi}_{t^*}\big\|_{L^2(\partial B)} \simeq |t^*|\varepsilon^{1/2}$$

and the same estimate holds for ϕ_{i,t^*} by the proposition.

Lemma 3.3.7. — *We have the following estimates:*

(a) $\big\|\nabla^m\grave{\phi}\big\|_{L^2(\partial\grave{B})} \lesssim_m \|b_1\|_{H^{\max(0,m-1)}}$ *for all* $m\geqslant 0$.

(b) *It holds that*

$$|\nabla^m\grave{\phi}|(x) \lesssim_m \min\Big(\frac{\|b_1\|_{H^m}}{1+\operatorname{dist}(x,\grave{B})^{1+m}}, \frac{\|b_1\|_{H^m}}{\varepsilon(1+\operatorname{dist}(x,\grave{B})^{2+m})}\Big),$$

for all $m\in\mathbb{N}_{\geqslant 0}$ *and all* $x\in\mathbb{H}\smallsetminus\grave{B}$.

(c) *For* $\operatorname{dist}(x,\grave{B})\geqslant 1$ *and* $m\in\mathbb{N}_{\geqslant 0}$ *it holds that*

$$|\nabla^m\grave{\phi}|(x) \lesssim_m \min\Big(\frac{\|b_1\|_{L^2}}{\operatorname{dist}(x,\grave{B})^{1+m}}, \frac{\|b_1\|_{L^2}}{\varepsilon\operatorname{dist}(x,\grave{B})^{2+m}}\Big).$$

The implicit constant in these estimates is bounded locally uniformly in R.

Proof. — (a) follows from the same argument as the previous lemma, where we again use the decay estimate from (b) or (c) for the case $m=0$.

(b) and (c) Our strategy is to quantify the argument of Lemma 2.1.3. If we extend $\grave{\phi}$ to $\grave{B}$ by solving the Dirichlet problem with boundary data $\grave{\phi}$, then by (a) and elliptic regularity

$$\big\|[\partial_n \grave{\phi}]\big\|_{\mathcal{M}(\partial \grave{B})} \lesssim \|b_1\|_{L^2},$$

where $[\cdot]$ denotes the jump across the boundary. We can then proceed as in the proof of Lemma 2.1.3 and set $\phi^{\mathbb{R}^3}(R,\theta,Z) = \grave{\phi}(R,Z)$, where (R,θ,Z) are axisymmetric coordinates in $\mathbb{R}^3$, then as argued there it holds that

$$\phi^{\mathbb{R}^3} = \frac{-1}{4\pi|\cdot|} * [\partial_n \grave{\phi}]\mathcal{H}^2 \llcorner \partial \grave{B}^{\mathbb{R}^3}. \tag{3.15}$$

We first focus on (c), where $\operatorname{dist}(x,\grave{B}) \geqslant 1$. We use three-dimensional axisymmetric coordinates (R,θ,Z) again and fix some x. Then we split $\grave{B}^{\mathbb{R}^3}$ into parts $T_{-n},\dots T_{n-1}$ where we take $T_{-n}\dots T_{n-1}$ as the sets

$$T_i := \grave{B}^{\mathbb{R}^3} \cap \Big\{(R',\theta',Z') \mid \theta'-\theta_x \in [\pi i/n, \pi(i+1)/n)\Big\},$$

where θ_x denotes the azimuthal angle of x and the difference is taken modulo 2π (if $x=0$ then we can take any angle). If we set $n = \lfloor 1/\varepsilon \rfloor$, then each such piece has diameter $\lesssim 1$.

For every i we have

$$\Big|\Big(\frac{1}{4\pi|\cdot|} * [\partial_n \grave{\phi}]\mathcal{H}^2 \llcorner \partial T_i\Big)(x)\Big| \lesssim \frac{\|b_1\|_{L^2}}{1+|i|^2+\operatorname{dist}(x,\widehat{B})^2},$$

where we exploited the facts that due to the rotational symmetry, that the integral of the boundary values over each T_i is 0 and that $\operatorname{dist}(x,T_i) \gtrsim 1+|i|+\operatorname{dist}(x,\grave{B})$. Summing up and using (3.15) gives

$$\begin{aligned}|\grave{\phi}(x)| &\lesssim \sum_{|i|\leqslant 1/\varepsilon} \frac{\|b_1\|_{L^2}}{1+|i|^2+\operatorname{dist}(x,\grave{B})^2}\\ &\lesssim \|b_1\|_{L^2} \min\Big(\frac{1}{1+\operatorname{dist}(x,\grave{B})}, \frac{1}{\varepsilon(1+\operatorname{dist}(x,\grave{B}))^2}\Big),\end{aligned}$$

where we estimated the sum with the integral. This shows (c) if $m=0$.

For $m>0$ the estimates in (c) follow from making the same argument with the derivatives of the fundamental solution. We set

$$D := (\grave{B} + B_1(0)) \smallsetminus \grave{B}. \tag{3.16}$$

In D the estimates in (b) for $m>0$ follow from using elliptic regularity and (a) to obtain that

$$\big\|\nabla^m \grave{\phi}\big\|_{L^\infty(D)} \lesssim \big\|\nabla^m \grave{\phi}\big\|_{H^{4/3}(D)} \lesssim \|b_1\|_{H^m}.$$

The case $\operatorname{dist}(x,\grave{B}) < 1$ and $m=0$ follows from this by using the estimate for $\operatorname{dist}(x,\grave{B}) = 1$ and the fundamental theorem of calculus. □

Proof of Proposition 3.3.2. — By subtracting both PDEs and rearranging we obtain that

$$\int_{\mathbb{H}\smallsetminus\grave{B}} r\langle\nabla\grave{\phi}-\nabla\breve{\phi},\nabla\xi\rangle+\Big(r-\frac{R}{\varepsilon}\Big)\langle\nabla\breve{\phi},\nabla\xi\rangle\,\mathrm{d}x=-\int_{\partial\grave{B}}\xi\Big(rb_1-\frac{R}{\varepsilon}b_2\Big)\,\mathrm{d}x, \tag{3.17}$$

for $\xi\in H^1$ compactly supported in $\mathbb{H}\smallsetminus\grave{B}$.

In order to be able to use both $\breve{\phi}$ and $\grave{\phi}$ as test functions for each equation even though one is defined on the half-space and one on the full space, we introduce smooth cutoff functions η_l, supported in $(\mathbb{H}+\frac{R}{2\varepsilon}e_R)\cap B_{l+1}(0)$, which equal 1 in $(\mathbb{H}+(\frac{R}{2\varepsilon}+1)e_R)\cap B_l(0)$ and whose derivatives have absolute value $\leqslant 2$ everywhere.

By testing with $\eta_l(\grave{\phi}-\breve{\phi})$ we obtain that for large enough l it holds that

$$\begin{aligned}\int_{\mathbb{H}\smallsetminus\grave{B}}\eta_l r\,|\nabla\grave{\phi}-\nabla\breve{\phi}|^2+\eta_l\Big(\frac{R}{\varepsilon}-r\Big)\nabla\breve{\phi}\cdot\nabla(\grave{\phi}-\breve{\phi})+\nabla\eta_l\cdot\Big(r\nabla\grave{\phi}-\frac{R}{\varepsilon}\nabla\breve{\phi}\Big)(\grave{\phi}-\breve{\phi})\,\mathrm{d}x&\\=-\int_{\partial\grave{B}}(\grave{\phi}-\breve{\phi})\Big(rb_1-\frac{R}{\varepsilon}b_2\Big)\,\mathrm{d}x.&\end{aligned}$$

By rearranging and using the Cauchy-Schwarz inequality, we obtain the inequality

$$\begin{aligned}\big\|\sqrt{r\eta_l}\,\nabla(\grave{\phi}-\breve{\phi})\big\|_{L^2}^2\leqslant\Big\|\sqrt{\eta_l}\,\frac{|r-R/\varepsilon|}{\sqrt{r}}\,\nabla\breve{\phi}\Big\|_{L^2}\big\|\sqrt{r\eta_l}\nabla(\grave{\phi}-\breve{\phi})\big\|_{L^2}&\\+2\int_{([R/2\varepsilon,R/2\varepsilon+1]\times\mathbb{R})\cup B_{l+1}(0)\smallsetminus B_l(0)}(|\grave{\phi}|+|\breve{\phi}|)\Big(\big|r\nabla\grave{\phi}\big|+\Big|\frac{R}{\varepsilon}\nabla\breve{\phi}\Big|\Big)\,\mathrm{d}x&\\+2R^{-1/2}\varepsilon^{1/2}c_{\mathrm{trace}}\big\|\sqrt{r\eta_l}\,\nabla(\grave{\phi}-\breve{\phi})\big\|_{L^2}\Big\|rb_1-\frac{R}{\varepsilon}b_2\Big\|_{L^2(\partial\grave{B})}&\\=:(\mathrm{I}+\mathrm{III})\cdot\big\|\sqrt{r\eta_l}\,\nabla(\grave{\phi}-\breve{\phi})\big\|_{L^2}+\mathrm{II},&\end{aligned}$$

where c_{trace} denotes the operator norm of the trace from $\dot{H}^1(\grave{B}+B_1(0)\smallsetminus\grave{B})$ to $L^2(\partial\grave{B})/\text{constants}$, which is $\lesssim 1$ and we have estimated $r^{-1/2}$ with $2R^{-1/2}\varepsilon^{1/2}$ in the last summand. Here I, II and III stand for the factors in the first, second, and third lines of the right-hand side.

Using Lemma 3.3.5, we estimate the first term as

$$\mathrm{I}\lesssim\Big(\int_{\mathbb{H}\smallsetminus\grave{B}}\eta_l\,\frac{1}{1+\mathrm{dist}(x,\grave{B})^4}\frac{(r-R/\varepsilon)^2}{r}\,\mathrm{d}x\Big)^{1/2}\|b_2\|_{L^2}. \tag{3.18}$$

We split into the regions $r\in[\frac{R}{2\varepsilon},\frac{R}{\varepsilon}-1]\cup[\frac{R}{\varepsilon}+1,\frac{3R}{2\varepsilon}]$, $r\in[\frac{R}{\varepsilon}-1,\frac{R}{\varepsilon}+1]$ and $r\geqslant 3R/2\varepsilon$, for other r the integrand is 0. This gives that (3.18) is

$$\begin{aligned}\lesssim\Big(\int_{[1,R/2\varepsilon]\times\mathbb{R}}\frac{1}{|x|^4}\frac{|x_1|^2}{R/\varepsilon}\,\mathrm{d}x+\int_1^\infty\frac{1}{|x|^4}\frac{\varepsilon}{R}\,\mathrm{d}x+\int_{[R/\varepsilon,\infty)\times\mathbb{R}}\frac{1}{|x|^4}\,x_1\,\mathrm{d}x\Big)^{1/2}\|b_2\|_{L^2}&\\\lesssim\varepsilon^{1/2}\sqrt{|\log\varepsilon|}\,\|b_2\|_{L^2}.&\end{aligned}$$

We use Lemma 3.3.7 and Lemma 3.3.5 to obtain that for $l \gg 1/\varepsilon^2$, we have

$$\begin{aligned}\mathrm{II} &\lesssim \frac{1}{\varepsilon}\int_{([R/2\varepsilon,R/2\varepsilon+1]\times\mathbb{R})\cup B_{l+1}(0)\smallsetminus B_l(0)} \frac{1}{\operatorname{dist}(x,\grave{B})}\frac{1}{\operatorname{dist}(x,\grave{B})^2}\,\mathrm{d}x\,\|b_1\|_{L^2}\,\|b_2\|_{L^2}\\ &\lesssim \frac{1}{\varepsilon}\Big(\int_{B_{l+1}(0)\smallsetminus B_l(0)}\frac{1}{|x|^3}\,\mathrm{d}x+\int_{\mathbb{R}}\frac{1}{(|x|+1/\varepsilon)^3}\,\mathrm{d}x\Big)\|b_1\|_{L^2}\,\|b_2\|_{L^2}\\ &\lesssim \Big(\frac{1}{l\varepsilon}+\varepsilon\Big)\|b_1\|_{L^2}\,\|b_2\|_{L^2}\lesssim \varepsilon\,\|b_1\|_{L^2}\,\|b_2\|_{L^2}.\end{aligned}$$

The third term can be estimated as

$$\mathrm{III}\lesssim \varepsilon^{1/2}\Big\|rb_1-\frac{R}{\varepsilon}b_2\Big\|_{L^2(\partial\grave{B})}\lesssim \varepsilon^{1/2}\Big(\|b_1\|_{L^2(\partial\grave{B})}+\frac{1}{\varepsilon}\,\|b_1-b_2\|_{L^2(\partial\grave{B})}\Big).$$

Hence we obtain that

$$\begin{aligned}\big\|\sqrt{r\eta_l}\,\nabla(\grave{\phi}-\breve{\phi})\big\|_{L^2}^2 \lesssim \varepsilon^{1/2}\sqrt{|\log\varepsilon|}\,\big\|\sqrt{r\eta_l}\,\nabla(\grave{\phi}-\breve{\phi})\big\|_{L^2}&\\ \times\Big(\|b_2\|_{L^2(\partial\grave{B})}+\|b_1\|_{L^2(\partial\grave{B})}+\frac{1}{\varepsilon}\,\|b_1-b_2\|_{L^2(\partial\grave{B})}\Big)&\\ +\varepsilon\,\|b_1\|_{L^2(\partial\grave{B})}\,\|b_2\|_{L^2(\partial\grave{B})}.&\end{aligned}$$

This implies that

$$\begin{aligned}&\big\|\sqrt{r\eta_l}\,\nabla(\grave{\phi}-\breve{\phi})\big\|_{L^2}\\ &\qquad\lesssim \varepsilon^{1/2}\sqrt{|\log\varepsilon|}\,\Big(\|b_2\|_{L^2(\partial\grave{B})}+\|b_1\|_{L^2(\partial\grave{B})}+\frac{1}{\varepsilon}\,\|b_1-b_2\|_{L^2(\partial\grave{B})}\Big).\end{aligned}\tag{3.19}$$

Now we can apply elliptic regularity estimates around $\partial\grave{B}$ to the elliptic equation (3.17) to obtain that for $m>0$ it holds that

$$\begin{aligned}\big\|\nabla^m(\grave{\phi}-\breve{\phi})\big\|_{L^2(\partial\grave{B})}&\lesssim \Big\|\frac{(r-R/\varepsilon)}{R/\varepsilon}\nabla\breve{\phi}\Big\|_{H^m(D)}+\varepsilon\,\Big\|rb_1-\frac{R}{\varepsilon}b_2\Big\|_{H^{m-1}(\partial\grave{B})}\\ &\qquad+\big\|\nabla(\grave{\phi}-\breve{\phi})\big\|_{L^2(D)}\\ &\lesssim \varepsilon\sqrt{|\log\varepsilon|}\,\Big(\|b_2\|_{H^{m-1}}+\|b_1\|_{H^{m-1}}+\frac{1}{\varepsilon}\,\|b_1-b_2\|_{H^{m-1}}\Big),\end{aligned}$$

where the neighborhood D was defined in (3.16), for the first term we used that

$$\|\breve{\phi}\|_{H^m(D)}\lesssim\|b_2\|_{H^{m-1}},$$

which follows from Lemma 3.3.5 (c) and in the last step we used the estimate (3.19) and the fact that $r\approx\varepsilon^{-1}$. □

3.3.2. *Multiple bodies*. — We remind the reader of the convention of writing ℓ for irrelevant exponents. We will work in the rescaled setting and keep $\widetilde{q}$ (defined in (1.22) and (1.24)) fixed independently of ε. We set $\grave{B}_i=(1/\varepsilon)B_i$. We will use the method of reflections to construct the potentials for multiple bodies from the potential of a single body.

Fix some sufficiently smooth normal velocity b_i on $\partial\grave{B}_i$ with $\int_{\partial\grave{B}_i} rb_i\,\mathrm{d}x = 0$ and let $\grave{\phi}_i^1 \in H_R^1(\mathbb{H}\smallsetminus\grave{B}_i)$ solve

$$\begin{aligned}\operatorname{div}(r\nabla\grave{\phi}_i^1) &= 0 \quad \text{in } \mathbb{H}\smallsetminus\grave{B}_i,\\ \partial_n\grave{\phi}_i^1 &= b_i \quad \text{on } \partial\grave{B}_i,\\ \partial_n\grave{\phi}_i^1 &= 0 \quad \text{on } \partial\mathbb{H},\end{aligned}$$

and let $\grave{\phi}_i \in H_R^1(\mathbb{H}\smallsetminus\bigcup_j\grave{B}_j)$ solve

$$\begin{aligned}\operatorname{div}(r\nabla\grave{\phi}_i) &= 0 \quad \text{in } \mathbb{H}\smallsetminus\textstyle\bigcup_j\grave{B}_j,\\ \partial_n\grave{\phi}_i &= b_i \quad \text{on } \partial\grave{B}_i,\\ \partial_n\grave{\phi}_i &= 0 \quad \text{on } \partial\grave{B}_j \text{ for } j\neq i,\\ \partial_n\grave{\phi}_i &= 0 \quad \text{on } \partial\mathbb{H}.\end{aligned}$$

We then add corrector functions $\grave{\phi}_1^2, \grave{\phi}_2^2\ldots$ to $\grave{\phi}_i^1$, which for $j\neq i$ fulfill the equations

$$\begin{aligned}\operatorname{div}(r\nabla\grave{\phi}_j^2) &= 0 \quad \text{in } \mathbb{H}\smallsetminus\grave{B}_j,\\ \partial_n\grave{\phi}_j^2 &= -\partial_n\grave{\phi}_i^1 \quad \text{on } \partial\grave{B}_j,\\ \partial_n\grave{\phi}_j^2 &= 0 \quad \text{on } \partial\mathbb{H},\\ \grave{\phi}_j^2 &\in H_R^1.\end{aligned}$$

Existence and uniqueness of these follows from Lemma 2.1.2. We set $\grave{\phi}_i^2 = 0$.

These correction terms then change the normal trace at all other ∂B_l. For this new error we can again construct corrector functions $\grave{\phi}_1^3, \grave{\phi}_2^3\ldots$ with normal boundary values $-\sum_{l\neq j}\partial_n\grave{\phi}_l^2$ and so on. If the sums of the errors and the corrector functions converge, the limit will be $\grave{\phi}_i$, since it is unique.

Proposition 3.3.8. — *This iteration scheme converges for small enough ε to the solution $\grave{\phi}_i$ in both the regimes* (1.22) *and* (1.24)*, the convergence is in $H_R^1(\mathbb{R}^3)$. Furthermore, for all $m\in\mathbb{N}_{>0}$ we have the following estimates, if ε is small enough:*

(a) *For $i\neq j$ it holds that*

$$\big\|\nabla^m\grave{\phi}_i\big\|_{L^2(\partial\grave{B}_j)} \lesssim_m \varepsilon^2|\log\varepsilon|^{\ell(1+m)}\|b_i\|_{L^2}.$$

(b) *For all i it holds that*

$$\big\|\nabla^m(\grave{\phi}_i^1-\grave{\phi}_i)\big\|_{L^2(\partial\grave{B}_i)} \lesssim_m \varepsilon^4|\log\varepsilon|^{\ell(1+m)}\|b_i\|_{L^2}.$$

In particular the estimates for the single body case in Proposition 3.3.2 *and Lemma* 3.3.7 (a) *are still true.*

Proof. — The rescaled bodies have pairwise distances $\gtrsim 1/\varepsilon|\log\varepsilon|$, resp. $\gtrsim 1/\varepsilon\sqrt{|\log\varepsilon|}$, hence by Lemma 3.3.7 (c), for $i\neq j$ we have

$$\big\|\nabla^m\grave{\phi}_i^1\big\|_{L^\infty(\partial\grave{B}_j)} \lesssim_m \varepsilon^{1+m}|\log\varepsilon|^{\ell(1+m)}\|b_i\|_{L^2},$$

for every $m\geqslant 0$.

This lets us estimate the decay of the correctors $\grave{\phi}_j^2$ by the same lemma, which can again be used to estimate the second-order correctors and so on. Iteratively, we obtain from Lemma 3.3.7 that

$$\big\|\nabla^m \grave{\phi}_j^l\big\|_{L^2(\partial \grave{B}_j)} \lesssim_m C^l \varepsilon^{2l-2} |\log\varepsilon|^{\ell(l+m+1)} \|b_i\|_{L^2}, \tag{3.20}$$

here C is a numerical factor, depending on k, but not on l or m, coming from the implicit constant in Lemma 3.3.7 and the fact that we have to sum over k correctors in each step. By integrating over the decay estimate in Lemma 3.3.7 we obtain that

$$\big\|\grave{\phi}_j^l\big\|_{H^1_R(\mathbb{H}\smallsetminus\cup_j \grave{B}_j)} \lesssim \frac{|\log\varepsilon|}{\varepsilon} \big\|\partial_n \grave{\phi}_j^l\big\|_{H^1(\partial\grave{B}_j)}. \tag{3.21}$$

Therefore the scheme converges in H^1_R if ε is small enough.

By Lemma 3.3.7, we obtain from (3.20) that

$$|\nabla^m \grave{\phi}_j^l| \lesssim_m C^l \varepsilon^{2l+m-1} |\log\varepsilon|^{\ell(m+l+1)} \|b_i\|_{L^2} \quad \text{on } \grave{B}_n \text{ with } n \neq j,$$

and for $l \geqslant 2$ we have

$$\big\|\nabla^m \grave{\phi}_j^l\big\|_{L^2(\partial \grave{B}_j)} \lesssim_m \big\|\partial_n \grave{\phi}_j^l\big\|_{H^{m-1}(\partial \grave{B}_j)} \lesssim_m C^l \varepsilon^{2l-2} |\log\varepsilon|^{\ell(m+l+1)} \|b_i\|_{L^2},$$

where we made use of Lemma 3.3.7 (a).

Hence by summing up, we see that for $j \neq i$ and small enough ε we have

$$\begin{aligned}\big\|\nabla^m \grave{\phi}_i\big\|_{L^2(\partial \grave{B}_j)} &\lesssim_m \varepsilon^2 |\log\varepsilon|^{\ell(1+m)} \|b_i\|_{L^2},\\ \big\|\nabla^m (\grave{\phi}_i - \phi_i^1)\big\|_{L^2(\partial \grave{B}_i)} &\lesssim_m \varepsilon^4 |\log\varepsilon|^{\ell(1+m)} \|b_i\|_{L^2}.\end{aligned}$$

After rescaling back to the original balls, we obtain the statement. □

Corollary 3.3.9. — *Fix some $t^* \in T_{q_i}M$ and let ϕ^1_{i,t^*} be the potential for t^* if there is only a single body, let ϕ_{i,t^*} be the potential for k bodies, in one of the two regimes* (1.22) *or* (1.24)*. Then we have the following bounds for all $m \in \mathbb{N}_{>0}$, with implicit constant bounded locally uniformly in $\widetilde{q}$ in both regimes:*

(a) *For $i \neq j$ it holds that*

$$\|\nabla^m \phi_{i,t^*}\|_{L^2(\partial B_j)} \lesssim \varepsilon^{7/2-m} |\log\varepsilon|^{\ell(1+m)} |t^*|.$$

(b) *For all i it holds that*

$$\big\|\nabla^m (\phi^1_{i,t^*} - \phi_{i,t^*})\big\|_{L^2(\partial B_i)} \lesssim \varepsilon^{11/2-m} |\log\varepsilon|^{\ell(1+m)} |t^*|.$$

All these estimates hold locally uniformly in $\widetilde{q}$. In particular, this implies that the estimates from the single body case in Corollary 3.3.9 *still hold for multiple bodies.*

Proof. — This follows directly by rescaling the potentials as in the proof of Corollary 3.3.4 and using Proposition 3.3.8. □

3.3.3. *The derivative with respect to q.* — In the following, we want to obtain bounds on the derivative of the potential with respect to q.

As we have already shown that these are smooth in q in Lemma 2.1.10, it is enough to estimate partial derivatives with respect to the values R_i and Z_i. In order to compare boundary data at different instances of one B_j, we fix some q and use the affine maps

$$c_0(x,s) = \frac{\rho_j(R_j+s)}{\rho_j(R_j)}\,(x-q_j) + q_j + se_R \tag{3.22}$$

and

$$d_0(x,s) = x + se_Z, \tag{3.23}$$

where $\rho_j(\cdot)$ denotes ρ_j as a function R_j. As we do not want to move the other bodies, we define c and d as smooth maps which equal c_0 and d_0 in a neighborhood of B_j and which equal the identity in a neighborhood of the other bodies. For technical reasons, will assume that the neighborhood in which c and d equal the terms in (3.22) resp. (3.23) has a size $\gg |\log\varepsilon|^{-2}$, which is not restrictive.

We also use the convention of writing ϕ_{i,t^*,R_j} resp. ϕ_{i,t^*,R_j+s} for ϕ_{i,t^*}, defined for the position (R_j, Z_j) resp. (R_j+s, Z_j) and also use the same notation with Z_j for the Z_j-derivative.

Proposition 3.3.10. — *Fix some $t^* \in T_{q_i}M$, then for all j, l we have*

$$\left\|\partial_s\left(\left(\partial_\tau\phi_{i,t^*,R_j+s}\right)\circ c(\cdot,s)\right)\right\|_{L^2(\partial B_l(q))} \lesssim \varepsilon^{3/2}|\log\varepsilon|^{\ell}$$

and

$$\left\|\partial_s\left(\left(\partial_\tau\phi_{i,t^*,Z_j+s}\right)\circ d(\cdot,s)\right)\right\|_{L^2(\partial B_l(q))} \lesssim \varepsilon^{3/2}|\log\varepsilon|^{\ell}.$$

Here $\tau = n^\perp$ refers to the tangent both on $B_l(q)$ and on the translated body. The implicit constant is bounded locally uniformly in $\widetilde{q}$ in both the regimes (1.22) *and* (1.24) *in a neighborhood of $s = 0$.*

Proof. — We assume $|t^*| = 1$ and omit the indices i and t^*. We first show the statement for the derivative in the R-direction and explain in the end why the Z-derivative works with the same argument. Without loss of generality, we may also assume that $Z_j = 0$, as the system is invariant under translation.

We rescale by a factor $1/\varepsilon$ again and use the same notation as in the previous proofs. Clearly, we have $\partial_{R_j}\grave{\phi} = \partial_{R_j}\phi(\varepsilon\cdot)$, in particular, the derivative exists and it holds that

$$\operatorname{div}(r\nabla\partial_{R_j}\grave{\phi}) = 0, \tag{3.24}$$

$$\partial_n\partial_{R_j}\grave{\phi} = 0 \quad \text{on } \partial\grave{B}_l \text{ for } l \neq j \text{ and on } \partial\mathbb{H}. \tag{3.25}$$

We first want to compute the remaining normal derivative. For this we introduce

$$\grave{c}(x,s) = \frac{1}{\varepsilon}\,c(\varepsilon x, s).$$

For $x \in \partial \grave{B}_j$ we have

$$\begin{aligned}(3.26)\quad \partial_s\big(\partial_n \grave{\phi}_{R_j+s}(\grave{c}(x,s))\big)\big|_{s=0} &= \varepsilon\delta_{ij}\partial_s\Big(t^*\cdot n + \frac{\varepsilon}{2(R_j+s)}\sqrt{\frac{\widetilde{\rho}_j^2 R_j}{R_j+s}}\,t^*\cdot e_R\Big)\Big|_{s=0}\\ &= -\varepsilon^2\delta_{ij}\frac{3\widetilde{\rho}_j}{4R_j^2}t^*\cdot e_R.\end{aligned}$$

Here the expressions for the normal boundary values follow from Definition (1.2) and Assumption (1.1) after rearranging. This is $\lesssim \varepsilon^2$ in any H^m-norm.

On the other hand, by the product rule, on $\partial \grave{B}_j$ we also have

$$\begin{aligned}(3.27)\quad &\partial_s\big(\partial_n \grave{\phi}_{R_j+s}(\grave{c}(x,s))\big)\big|_{s=0}\\ &= \partial_s n(\grave{c}(x,s))\big|_{s=0}\cdot\nabla\grave{\phi}_{R_j}(x) + n\cdot\partial_s\nabla\grave{\phi}_{R_j+s}(x)\big|_{s=0} + n\cdot\nabla^2\grave{\phi}_{R_j}(x)\partial_s\grave{c}(x,s)\big|_{s=0}.\end{aligned}$$

Here $n(\grave{c}(x,s))$ stands for the normal at $\grave{c}(x,s)$ of the rescaled body centered at $(1/\varepsilon)(R_j+s)$, which is in fact constant, and hence the first summand drops out.

We have by definition

$$\partial_s\grave{c}(x,s)\big|_{s=0} = \frac{1}{\varepsilon}\,e_R - \frac{1}{2R_j}(x-q_j).$$

Combining (3.26) and (3.27), we see that on $\partial \grave{B}_j$ we have

$$(3.28)\qquad \partial_n\partial_s\grave{\phi}_{R_j+s}\big|_{s=0} = -\frac{1}{\varepsilon}\partial_n\partial_r\grave{\phi}_{R_j} + \frac{\widetilde{\rho}_j}{2R_j}\partial_n^2\grave{\phi}_{R_j} - \varepsilon^2\delta_{ij}\frac{3\widetilde{\rho}_j}{4R_j^2}t^*\cdot e_R.$$

We now estimate the derivatives of the boundary values on $\partial \grave{B}_j$ and on $\partial \grave{B}_l$ for $l \neq j$ differently.

First case: $l \neq j$. — Here $\grave{c}$ is the identity on $\partial \grave{B}_l$ and hence we only need to estimate $\nabla\partial_s\grave{\phi}_{R_j+s}$. By rescaling we know from Corollary 3.3.9 and Lemma 3.3.7 (a) that

$$\big\|\nabla^m\grave{\phi}\big\|_{L^2(\partial \grave{B}_j)} \lesssim_m \varepsilon.$$

By (3.28) it follows that

$$(3.29)\qquad \big\|\partial_n\partial_s\grave{\phi}_{R_j+s}\big|_{s=0}\big\|_{L^2(\partial \grave{B}_j)} \lesssim 1.$$

Hence we conclude from (3.24), (3.25), (3.29) and by Proposition 3.3.8 that for $m \in \mathbb{N}_{>0}$ it holds

$$\big\|\nabla^m\partial_s\grave{\phi}_{R_j+s}\big|_{s=0}\big\|_{L^2(\partial \grave{B}_l)} \lesssim_m \varepsilon^2|\log\varepsilon|^{\ell(m+1)}.$$

By rescaling back to the original bodies B, one obtains the statement.

Second case: $l = j$. — We build an auxiliary function that equals the desired derivative. Consider $\partial_r\grave{\phi}$. It fulfills the PDE

$$(3.30)\qquad \operatorname{div}(r\nabla\partial_r\grave{\phi}) = \operatorname{div}\Big(-\nabla\grave{\phi} + \frac{r\varepsilon}{R_j}\nabla\grave{\phi}\Big)$$

and has the Neumann boundary values $\partial_n\partial_r\grave{\phi}$ on $\partial \grave{B}_j$.

Consider the function $x \cdot \nabla \grave{\phi}$, a direct calculation shows that

$$\begin{aligned} &\operatorname{div}(r\nabla(x\cdot\nabla\grave{\phi}))=0, \\ &\partial_n(x\cdot\nabla\grave{\phi})=\partial_n\grave{\phi}+\frac{R_j}{\varepsilon}\partial_n\partial_r\grave{\phi}+\widetilde{\rho}_j\partial_n^2\grave{\phi}\quad\text{on }\partial\grave{B}_j, \end{aligned}\tag{3.31}$$

Here we used the assumption that $Z_j = 0$.

Now consider the function

$$\widetilde{\phi} := \partial_s\grave{\phi}_{R_j+s}\big|_{s=0} - \frac{1}{2R_j}x\cdot\nabla\grave{\phi} + \frac{1}{2R_j}\grave{\phi} + \frac{3}{2\varepsilon}\partial_r\grave{\phi},\tag{3.32}$$

and set

$$\widehat{\phi}_s(x) = \sqrt{\frac{R_j+s}{R_j}}\ \grave{\phi}_{R_j+s}(\grave{c}(s,x)).$$

Then in a neighborhood of $\grave{B}_j$ it holds that $\partial_s\widehat{\phi}_s|_{s=0} = \widetilde{\phi}$ and hence

$$\partial_s\big((\nabla\grave{\phi}_{R_j+s})(\grave{c}(x,s))\big)\big|_{s=0} = \partial_s\nabla\widehat{\phi}_s(x)\big|_{s=0} = \nabla\widetilde{\phi}(x).\tag{3.33}$$

Hence to prove the proposition it suffices to estimate $\nabla\widetilde{\phi}$.

Let us further reduce this to the case in which $k = 1$. First consider the case where $i \neq j = l$. Then by (3.28) and by rescaling the estimates on ϕ in Corollary 3.3.9 we have

$$\big\|\partial_n\partial_s\grave{\phi}_{R_j+s}\big|_{s=0}\big\|_{H^m(\partial\grave{B}_j)} \lesssim \frac{1}{\varepsilon}\,\big\|\nabla^2\grave{\phi}\big\|_{H^m(\partial\grave{B}_j)} \lesssim_m \varepsilon^2|\log\varepsilon|^\ell.$$

Hence we conclude from Proposition 3.3.8 and (3.24) and (3.25) that

$$\big\|\partial_s\grave{\phi}_{R_j+s}\big|_{s=0}\big\|_{H^m(\partial\grave{B}_j)} \lesssim_m \varepsilon^2|\log\varepsilon|^\ell.$$

Hence by the definition of $\widetilde{\phi}$ (see (3.32)) and Corollary 3.3.9 we have

$$\begin{aligned}\big\|\nabla\widetilde{\phi}\big\|_{H^m(\partial\grave{B}_j)} \lesssim_m \big\|\partial_s\grave{\phi}_{R_j+s}\big|_{s=0}\big\|_{H^m(\partial\grave{B}_j)} + \frac{1}{\varepsilon}\,\big\|\nabla^2\grave{\phi}\big\|_{H^m(\partial\grave{B}_j)} + \big\|\nabla\grave{\phi}\big\|_{H^m(\partial\grave{B}_j)} \\ \lesssim_m \varepsilon^2|\log\varepsilon|^\ell.\end{aligned}$$

After rescaling back to the original bodies B, this shows the statement in the case $i \neq j = l$.

Next, consider the case $i = j = l$ and $k > 1$, and let $\grave{\phi}^1$ be the rescaled potential for a single body and let $\widetilde{\phi}^1$ be the version of $\widetilde{\phi}$ from the single body case. As the formulas (3.25) and (3.28) hold for both $\grave{\phi}$ and $\grave{\phi}^1$, we note that we have

$$\big\|\partial_n\partial_s(\grave{\phi}_{R_j+s} - \grave{\phi}^1_{R_j+s})\big|_{s=0}\big\|_{H^m(\partial\grave{B}_j)} \lesssim \frac{1}{\varepsilon}\,\big\|\nabla^2(\grave{\phi}-\grave{\phi}^1)\big\|_{H^m(\partial\grave{B}_j)} \lesssim_m \varepsilon^4|\log\varepsilon|^\ell,$$

here we used Proposition 3.3.8 (b). Hence we conclude that

$$\big\|\nabla^m\partial_s(\grave{\phi}_{R_j+s} - \grave{\phi}^1_{R_j+s})\big|_{s=0}\big\|_{L^2(\partial\grave{B}_j)} \lesssim_m \varepsilon^4|\log\varepsilon|^\ell.$$

By rescaling Corollary 3.3.9 (b) and the definition of $\widetilde{\phi}$ we conclude that

$$\begin{aligned}\big\|\nabla(\widetilde{\phi}-\widetilde{\phi}^1)\big\|_{H^m(\partial\grave{B}_j)} &\lesssim_m \big\|\nabla^m\partial_s(\grave{\phi}_{R_j+s}-\grave{\phi}^1_{R_j+s})\big|_{s=0}\big\|_{L^2(\partial\grave{B}_j)}\\ &\qquad+\frac{1}{\varepsilon}\big\|\nabla^2(\grave{\phi}-\grave{\phi}^1)\big\|_{H^m(\partial\grave{B}_j)}+\big\|\nabla(\grave{\phi}-\grave{\phi}^1)\big\|_{H^m(\partial\grave{B}_j)}\\ &\lesssim_m \varepsilon^4|\log\varepsilon|^\ell.\end{aligned}$$

This shows the desired estimate once we have proved the case $k=1$ after rescaling back to the original bodies.

It remains to deal with the case $k=1$. We drop the indices on the sets $\grave{B}_j$. By (3.28), (3.30)–(3.32) we have

$$\operatorname{div}(r\widetilde{\phi}) = -\frac{3}{2}\operatorname{div}\Big(\Big(\frac{1}{\varepsilon}-\frac{r}{R_j}\Big)\nabla\grave{\phi}\Big)\quad \text{in a neighborhood of } \grave{B}, \tag{3.34}$$

$$\partial_n\widetilde{\phi} = -\varepsilon^2\frac{3\widetilde{\rho}_j}{4R_j^2}\,t^*\cdot e_R. \tag{3.35}$$

Set $\widetilde{B} := B_{1/\varepsilon|\log\varepsilon|^2}(0)+\grave{B}\smallsetminus\grave{B}$. By assumption the cutoff in the definition of $\grave{c}$ happens outside of $\widetilde{B}$ and hence (3.34) holds in all of $\widetilde{B}$. The crucial observation is that $\partial_{R_j}\grave{\phi}$ will decay an order faster than expected.

Lemma 3.3.11. — *For x with $\operatorname{dist}(x,\grave{B})\geqslant 1$ it holds that*

$$|\partial_s\phi_{R_j+s}(x)| \lesssim \frac{\varepsilon+\|\nabla\widetilde{\phi}\|_{L^2(\widetilde{B})}}{\varepsilon\operatorname{dist}(x,\grave{B})^2}+\frac{1}{\varepsilon\operatorname{dist}(x,\grave{B})^3},$$

and

$$|\nabla\partial_s\widetilde{\phi}_{R_j+s}(x)| \lesssim \frac{\varepsilon+\|\nabla\widetilde{\phi}\|_{L^2(\widetilde{B})}}{\varepsilon\operatorname{dist}(x,\grave{B})^3}+\frac{1}{\varepsilon\operatorname{dist}(x,\grave{B})^4}.$$

The implicit constant is bounded locally uniformly in $\widetilde{q}$.

Proof. — We only show the first estimate, the second works completely similarly, using the derivative of the fundamental solution.

The proof builds on the idea of the proof of Lemma 2.1.3. Recall that in the proof there, we extended the solution to the full space $\mathbb{R}^3$ and used the fundamental solution. We first need to show an estimate on the Neumann boundary values of the solution to the interior problem. As in the proof of Lemma 2.1.3, we extend $\grave{\phi}$ to $\mathbb{H}$ by solving the Dirichlet problem for $\operatorname{div}(r\nabla\cdot)$ in $\grave{B}$. We also set

$$\widehat{\phi}_s(x) := \sqrt{(R_j+s)/R_j}\;\grave{\phi}_{R_j+s}(\grave{c}(x,s))$$

inside of $\grave{B}$. We claim that

$$\big\|\partial_s((\partial_{n_{\mathrm{int}}}\grave{\phi}_{R_j+s})\circ c)\big\|_{L^2(\partial\grave{B})} \lesssim \|\widetilde{\phi}\|_{L^2(\widetilde{B})}+\varepsilon^2, \tag{3.36}$$

here $\partial_{n_{\mathrm{int}}}$ denotes the normal derivative from the inside.

By applying elliptic regularity estimates to the equations (3.34)–(3.35) and using the identity $\partial_s\widehat{\phi}_s = \widetilde{\phi}$, we see that

$$\begin{aligned}\big\|\nabla\partial_s\widehat{\phi}_s\big|_{s=0}\big\|_{L^2(\partial\grave{B})} &= \big\|\nabla\widetilde{\phi}\big\|_{L^2(\partial\grave{B})}\\ &\lesssim \big\|\nabla\widetilde{\phi}\big\|_{L^2(\widetilde{B})} + \big\|\partial_n\widetilde{\phi}\big\|_{L^2(\partial\grave{B})} + \varepsilon\,\Big\|\Big(\frac{1}{\varepsilon} - \frac{r}{R_j}\Big)\nabla\grave{\phi}\Big\|_{H^1(\grave{B}+B_1(0)\smallsetminus\grave{B})}\\ &\lesssim \varepsilon^2 + \big\|\nabla\widetilde{\phi}\big\|_{L^2(\widetilde{B})}.\end{aligned}\tag{3.37}$$

In $\grave{B}$ we have

$$\operatorname{div}\Big(\frac{\grave{c}(x,s)_R}{R_j+s}\nabla\widehat{\phi}_s\Big) = 0,$$

as one can see from a direct computation, where $\grave{c}(x,s)_R$ denotes the R-component of $\grave{c}$. Using for instance the implicit function theorem as in the proof of Lemma 2.1.10, one can justify that this equation can be differentiated in s. Hence we obtain that $\partial_s\widehat{\phi}_s(x)$ fulfills the equation

$$\operatorname{div}\big(r\nabla\partial_s\widehat{\phi}_s\big) = -\operatorname{div}\Big(\partial_s\frac{\grave{c}(x,s)_R}{R_j+s}\nabla\grave{\phi}\Big).$$

Hence by elliptic regularity for this equation and (3.37) we obtain that

$$\begin{aligned}\big\|\partial_{n_{\mathrm{int}}}\partial_s\widehat{\phi}_s\big\|_{L^2(\partial\grave{B})} &\lesssim \big\|\nabla\partial_s\widehat{\phi}_s\big\|_{L^2(\partial\grave{B})} + \varepsilon\,\Big\|\partial_s\frac{\grave{c}(x,s)_R}{R_j+s}\,\nabla\grave{\phi}\Big\|_{H^1(\grave{B})}\\ &\lesssim \varepsilon^2 + \big\|\nabla\widetilde{\phi}\big\|_{L^2(\widetilde{B})} + \varepsilon\,\big\|\partial_n\grave{\phi}\big\|_{H^1(\partial\grave{B})} \lesssim \varepsilon^2 + \big\|\nabla\widetilde{\phi}\big\|_{L^2(\widetilde{B})}.\end{aligned}$$

The claim (3.36) now follows from the fact that $\nabla\widehat{\phi}_s = (\nabla\grave{\phi}_{R_j+s})\circ\grave{c}(\cdot,s)$.

Now fix some $y\in\widetilde{B}$. Then as argued in the proof of Lemma 2.1.3 we have

$$\partial_s\grave{\phi}_{R_j+s}(y) = \partial_s\Big([\partial_n\grave{\phi}_{R_j+s}]\mathcal{H}^2\llcorner\partial\grave{B}(R_j+s)^{\mathbb{R}^3} * \frac{-1}{4\pi|\cdot|}\Big)(y),\tag{3.38}$$

where the convolution is taken with respect to three-dimensional coordinates and $\grave{B}(R_j+s)^{\mathbb{R}^3}$ is the axisymmetric torus corresponding to $\grave{B}$ defined with respect to R_j+s and the factor r in the jump disappears due to the coordinate change. We shall also view the curves $\grave{c}(x,s)$ as curves in $\mathbb{R}^3$, by setting (in axisymmetric coordinates)

$$\grave{c}(x,s) = (\grave{c}(x,s)_R, x_\theta, \grave{c}(x,s)_Z),$$

where x_θ is the azimuthal angle of x.

Using these curves, the convolution in (3.38) can be written as

$$-\partial_s\int_{\partial\grave{B}^{\mathbb{R}^3}}[\partial_n\grave{\phi}_{R_j+s}](\grave{c}(x,s))\,\frac{\grave{c}(x,s)_R}{r}\,\sqrt{R_j/(R_j+s)}\,\frac{1}{4\pi|y-\grave{c}(x,s)|}\,\mathrm{d}x,$$

here the factor in the middle is the determinant due to the coordinate change. Now we can differentiate under the integral, as all derivatives are smooth by Lemma 2.1.10 and use the product rule. We estimate the derivative of the first three factors and of the last factor separately.

First, we deal with

$$-\int_{\partial \grave{B}^{\mathbb{R}^3}} \partial_s\Big([\partial_n \grave{\phi}_{R_j+s}](\grave{c}(x,s))\,\frac{\grave{c}(x,s)_R}{r}\,\sqrt{R_j/(R_j+s)}\Big)\Big|_{s=0}\frac{1}{4\pi|x-y|}\,\mathrm{d}x.$$

Observe that

$$\int_{\partial \grave{B}^{\mathbb{R}^3}} \partial_s\Big([\partial_n \grave{\phi}_{R_j+s}](\grave{c}(x,s))\,\frac{\grave{c}(x,s)_R}{r}\,\sqrt{R_j/(R_j+s)}\Big)\Big|_{s=0}\,\mathrm{d}x \\ = \partial_s\int_{\partial \grave{B}(R_j+s)^{\mathbb{R}^3}} [\partial_n \grave{\phi}_{R_j+s}]\,\mathrm{d}x = 0.$$

Furthermore, we have by (3.36) and (3.26)

$$\big\|\partial_s\big([\partial_n \grave{\phi}_{R_j+s}](\grave{c}(x,s))\big)\big|_{s=0}\big\|_{L^2(\partial\grave{B})} \lesssim \varepsilon^2 + \big\|\nabla\widetilde{\phi}\big\|_{L^2(\widetilde{B})}$$

and

$$\Big|\partial_s\frac{\grave{c}(x,s)_R}{r}\,\sqrt{R_j/(R_j+s)}\Big| \lesssim 1.$$

Since the boundary values of $\grave{\phi}$ are $\lesssim \varepsilon$ in any H^m-norm we have

$$\big\|[\partial_n\grave{\phi}]\big\|_{L^2(\partial\grave{B})} \lesssim \big\|\partial_n\grave{\phi}\big\|_{L^2(\partial\grave{B})} + \big\|\grave{\phi}\big\|_{H^1(\partial\grave{B})} \lesssim \varepsilon. \tag{3.39}$$

Hence we have

$$\Big\|\partial_s\Big([\partial_n \grave{\phi}_{R_j+s}](\grave{c}(x,s))\,\frac{\grave{c}(x,s)_R}{r}\,\sqrt{R_j/(R_j+s)}\Big)\Big|_{s=0}\Big\|_{L^2(\partial\grave{B})} \lesssim \varepsilon + \big\|\nabla\widetilde{\phi}\big\|_{L^2(\partial\grave{B})}.$$

Now we combine this with the mean-freeness to estimate

$$\Big|\int_{\partial \grave{B}^{\mathbb{R}^3}} \partial_s\Big([\partial_n \grave{\phi}_{R_j+s}](\grave{c}(x,s))\,\frac{\grave{c}(x,s)_R}{r}\,\sqrt{R_j/(R_j+s)}\Big)\Big|_{s=0}\frac{1}{4\pi|x-y|}\,\mathrm{d}x\Big| \\ \lesssim \frac{\varepsilon + \|\nabla\widetilde{\phi}\|_{L^2(\partial\grave{B})}}{\varepsilon\,\mathrm{dist}(y,\grave{B})^2}.$$

It remains to estimate the second summand, which is

$$\int_{\partial \grave{B}^{\mathbb{R}^3}} [\partial_n\grave{\phi}]\nabla_x\,\frac{1}{4\pi|y-x|}\cdot\partial_s\grave{c}(x,0)\,\mathrm{d}x.$$

Note that $[\partial_n\grave{\phi}]$ is mean-free and one easily sees that $|\partial_s\grave{c}(x,0)| \lesssim 1/\varepsilon$ and $|\nabla\partial_s\grave{c}(x,0)| \lesssim 1$. Hence by exploiting the mean-freeness of $[\partial_n\grave{\phi}]$ and using (3.39), we see that

$$\Big|\int_{\partial \grave{B}^{\mathbb{R}^3}} [\partial_n\grave{\phi}]\nabla_x\frac{1}{4\pi|y-x|}\cdot\partial_s\grave{c}(x,0)\,\mathrm{d}x\Big| \lesssim \frac{\|\partial_s\grave{c}(0)\|_{\sup}}{\mathrm{dist}(y,\grave{B})^3} + \frac{\|\nabla_x\partial_s\grave{c}(0)\|_{\sup}}{\mathrm{dist}(y,\grave{B})^2} \\ \lesssim \frac{1}{\varepsilon\,\mathrm{dist}(y,\grave{B})^3} + \frac{1}{\mathrm{dist}(y,\grave{B})^2}.$$

The statement follows because $\mathrm{dist}(y,\grave{B}) \ll \varepsilon^{-1}$. □

We continue with the proof of Proposition 3.3.10. By testing the elliptic equation (3.34), (3.35) with $\widetilde{\phi}$ in $\widetilde{B}$ and using partial integration and the Cauchy-Schwarz inequality, we obtain that

$$\|\sqrt{r}\,\nabla\widetilde{\phi}\|^2_{L^2(\widetilde{B})} \leqslant \frac{3}{2}\Big\|\frac{(1/\varepsilon - r/R_j)}{\sqrt{r}}\,\nabla\grave{\phi}\Big\|_{L^2(\widetilde{B})}\|\sqrt{r}\nabla\widetilde{\phi}\|_{L^2(\widetilde{B})} \tag{3.40}$$
$$- \int_{\partial\grave{B}}\Big(\tfrac{3}{2}\,(1/\varepsilon - r/R_j)\,\partial_n\grave{\phi} + r\partial_n\widetilde{\phi}\Big)\widetilde{\phi}\,\mathrm{d}x + \int_{\partial\widetilde{B}\smallsetminus\partial\grave{B}}\Big(\tfrac{3}{2}\,(1/\varepsilon - r/R_j)\,\partial_n\grave{\phi} + r\partial_n\widetilde{\phi}\Big)\widetilde{\phi}\,\mathrm{d}x$$
$$:= \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where the normal n on $\partial\widetilde{B}\smallsetminus\partial\grave{B}$ is taken as the outer normal and I, II and III are defined in the obvious way.

Using the decay estimate from Lemma 3.3.7, we can estimate

$$\mathrm{I} \lesssim \varepsilon\Big(\int_{B_{1/\varepsilon|\log\varepsilon|^2}\smallsetminus B_1(0)} \frac{|x_1|^2}{(|x_1| + 1/\varepsilon)|x|^4}\,\mathrm{d}x\Big)^{1/2}\|\sqrt{r}\,\nabla\widetilde{\phi}\|_{L^2(\widetilde{B})}$$
$$\lesssim \varepsilon\Big(\int_1^{1/\varepsilon|\log\varepsilon|^2} \frac{1}{x(x+1/\varepsilon)}\,\mathrm{d}x\Big)^{1/2}\|\sqrt{r}\,\nabla\widetilde{\phi}\|_{L^2(\widetilde{B})} \lesssim \varepsilon^{3/2}|\log\varepsilon|^{\ell}\|\sqrt{r}\,\nabla\widetilde{\phi}\|_{L^2(\widetilde{B})}.$$

To estimate the second term, we note that by partial integration we have

$$\int_{\partial\grave{B}}\Big(\tfrac{3}{2}\,(1/\varepsilon - r/R_j)\,\partial_n\grave{\phi} + r\partial_n\widetilde{\phi}\Big)\,\mathrm{d}x$$
$$= \lim_{R\to\infty}\int_{\partial(\mathbb{H}\cap B_R(0))}\Big(\tfrac{3}{2}\,(1/\varepsilon - r/R_j)\,\partial_n\grave{\phi} + r\partial_n\widetilde{\phi}\Big)\,\mathrm{d}x = 0.$$

Here the last equality follows from the fact that the integrand is 0 on $\partial\mathbb{H}$ and by using the decay estimates from Lemma 3.3.7 and Lemma 3.3.11 together with the definition of $\widetilde{\phi}$. We can then use that this boundary integral vanishes and the explicit form of the normal derivatives to estimate

$$\mathrm{II} \lesssim \|\tfrac{3}{2}\,(1/\varepsilon - r/R_j)\,\partial_n\grave{\phi} + r\partial_n\widetilde{\phi}\|_{L^2(\partial\grave{B})}\|\nabla\widetilde{\phi}\|_{L^2(\widetilde{B})} \lesssim \varepsilon^{3/2}\|\sqrt{r}\,\nabla\widetilde{\phi}\|_{L^2(\widetilde{B})}.$$

To estimate the third term, we use Lemma 3.3.7 and Lemma 3.3.11 and obtain that

$$\mathrm{III} \lesssim \frac{1}{\varepsilon|\log\varepsilon|^2}\sup_{x\in\partial\widetilde{B}\smallsetminus\partial\grave{B}}|\widetilde{\phi}(x)|\Big(\frac{1}{\varepsilon}\,|\nabla\widetilde{\phi}(x)| + \frac{1}{\varepsilon}\,|\nabla\grave{\phi}(x)|\Big)$$
$$\lesssim \frac{1}{\varepsilon}\big(\varepsilon^2 + \varepsilon\|\nabla\widetilde{\phi}\|_{L^2(\widetilde{B})}\big)\big(\varepsilon^2 + \varepsilon\,\|\nabla\widetilde{\phi}\|_{L^2(\widetilde{B})}\big)\,|\log\varepsilon|^{\ell}$$
$$\lesssim \varepsilon^3|\log\varepsilon|^{\ell} + \varepsilon|\log\varepsilon|^{\ell}\|\nabla\widetilde{\phi}\|_{L^2(\widetilde{B})}.$$

Putting these back into the estimate (3.40), we obtain

$$\|\sqrt{r}\,\nabla\widetilde{\phi}\|_{L^2(\widetilde{B})} \lesssim \varepsilon^{3/2}|\log\varepsilon|^{\ell}.$$

Now we can apply elliptic regularity estimates to (3.34),(3.35) and obtain together with the previous estimates on $\grave{\phi}$ that

$$\|\nabla\widetilde{\phi}\|_{H^m(\partial\grave{B})} \lesssim_m \varepsilon^2|\log\varepsilon|^{\ell},$$
$$\widetilde{\phi} = \partial_s\grave{\phi}_{Z_j+s} + \frac{1}{\varepsilon}\partial_Z\grave{\phi};\quad \widehat{\phi}_s = \grave{\phi}_{Z_j+s}(\grave{d}(s,x)). \qquad \square$$

3.4. The Christoffel Symbol and the added Inertia

Lemma 3.4.1. — *If we identify the tangent space T_qM with $\mathbb{R}^{2k}$, then it holds that*

$$\left|\mathcal{M} - \pi\begin{pmatrix} R_1\varepsilon^2\widetilde{\rho}_1^2 & 0 & \dots & \\ 0 & R_1\varepsilon^2\widetilde{\rho}_1^2 & \dots & \\ 0 & 0 & R_2\varepsilon^2\widetilde{\rho}_2 & 0 \\ \dots & & & \end{pmatrix}\right| \lesssim \varepsilon^3|\log\varepsilon|^\ell,$$

where $\mathcal{M}$ *was defined in* 2.2.1. *Here the implicit constant is bounded locally uniformly in* $\widetilde{q}$.

Proof. — By partial integration we have for $t^* \in T_{q_i}M$ and $s^* \in T_{q_j}M$ that

$$(\mathcal{M}t^*)\cdot s^* = -\int_{\partial B_j} r\partial_n\phi_{i,t^*}\phi_{j,s^*}\,\mathrm{d}x.$$

If $i \neq j$ this is $\lesssim \varepsilon^3|\log\varepsilon|^\ell|s^*||t^*|$ by Corollary 3.3.9 (a). If $i = j$, then it holds that

$$\begin{aligned}\left|\int_{\partial B_i} r\partial_n\phi_{i,t^*}\phi_{j,s^*}\,\mathrm{d}x - R_i\int_{\partial B_i} t^*\cdot n\check{\phi}_{s^*}\,\mathrm{d}x\right| \\ \lesssim \|ru(t^*)\|_{L^2}\left\|\phi_{i,s^*} - \check{\phi}_{s^*}\right\|_{L^2(\partial B_i)/\mathrm{constants}} + \|ru(t^*) - R_it^*\cdot n\|_{L^2}\left\|\check{\phi}_{s^*}\right\|_{L^2(\partial B_i)},\end{aligned}$$

where we used the fact that $ru(t^*)$ is mean-free and $\check{\phi}$ was defined in 3.3.1. Now by the Corollaries 3.3.4 and 3.3.9 and the definition of $u(t^*)$ this is $\lesssim \varepsilon^3|\log\varepsilon|^\ell|s^*||t^*|$.

Now observe that if $t^* = e_1$ and $s^* = e_2$ then $t^*\cdot n$ is orthogonal to $\check{\phi}_{s^*}$ as the former is symmetric with respect to the e_2-direction while the latter is antisymmetric in that direction.

If $t^* = s^*$ then it follows from the explicit form of $\check{\phi}_{s^*}$ that

$$\int_{\partial B_i} t^*\cdot n\check{\phi}_{s^*}\,\mathrm{d}x = -\pi|t^*|^2\rho_i^2. \qquad \square$$

Lemma 3.4.2. — *It holds that*

$$|\Gamma| \lesssim \varepsilon^3|\log\varepsilon|^\ell,$$

where the Christoffel symbol Γ *was defined in* 2.2.1 *and the implicit constant is bounded locally uniformly in* $\widetilde{q}$.

Proof. — By the definition of Γ, it suffices to estimate the derivative of $\mathcal{M}$. If we are differentiating $\mathcal{M}$ with respect to q_l for $l \neq i$, then it holds that

$$\left|-\partial_{q_l}\int_{\partial B_i} r\partial_n\phi_{i,t^*}\phi_{j,t^*}\,\mathrm{d}x\right| \lesssim \varepsilon\,\|ru(t^*)\|_{L^2}\,\|\partial_{q_l}\partial_\tau\phi_{j,s^*}\|_{L^2} \lesssim \varepsilon^3|\log\varepsilon|^\ell|t^*||s^*|,$$

where we used that $u(t^*)$ does not depend on q_l, and used the mean-freeness of $ru(t^*)$ to estimate $\partial_{q_l}\phi_{j,s^*}$ with its derivative, and in the last step we used Proposition 3.3.10. If $i = l$ we can switch the roles of i and j unless $i = j = l$. For simplicity we only consider the derivative with respect to R_i in this case, the other derivative is easier. We again use the diffeomorphism c, defined at the beginning of Section 3.3.3.

Setting $\mathcal{M}_{R_i+s}$ for $\mathcal{M}$ defined with respect to R_i+s we have

$$(\mathcal{M}_{R_i+s}t^*)\cdot s^* = -\int_{\partial B_i} c(x,s)_R\,\frac{\rho_i(R_i+s)}{\rho_i(R_i)}\,u(t^*)\circ c\phi_{i,s^*,R_i+s}\circ c\,\mathrm{d}x, \tag{3.41}$$

where $\rho_i(R_i)$ refers to ρ_i as a function of R_i and $c(x,s)_R$ is the R-component of c. Using the fundamental theorem of calculus and Proposition 3.3.10, we see that

$$\Big\|\partial_s\Big(\frac{\rho_i(R_i)}{\rho_i(R_i+s)}\phi_{i,s^*,R_i+s}\circ c\Big)\Big\|_{L^2/\text{constants}} \lesssim \varepsilon^{5/2}|\log\varepsilon|^\ell|s^*|. \tag{3.42}$$

Furthermore, we have

$$\|\partial_s u(t^*)\circ c\|_{L^2}\lesssim\varepsilon^{3/2}, \tag{3.43}$$

as one sees by rescaling (3.26). It is easy to see that

$$\Big|\partial_s\Big(c(x,s)_R\frac{\rho_i(R_i+s)^2}{\rho_i(R_i)^2}\Big)\Big|\lesssim\varepsilon, \tag{3.44}$$

uniformly in x. We can now use this to estimate the derivative of (3.41) by the product rule:

$$\begin{aligned}
\Big|\partial_s\int_{\partial B_i} c(x,s)_R\,\frac{\rho_i(R_i+s)}{\rho_i(R_i)}\,u(t^*)\circ c\phi_{i,s^*,R_i+s}\circ c\,\mathrm{d}x\Big|_{s=0}\Big| \\
\leqslant \|ru(t^*)\|_{L^2}\,\Big\|\partial_s\Big(\frac{\rho_i(R_i)}{\rho_i(R_i+s)}\phi_{i,s^*,R_i+s}\circ c\Big)\Big|_{s=0}\Big\|_{L^2/\text{constants}} \\
+\Big\|\partial_s\Big(c(x,s)_R\frac{\rho_i(R_i+s)^2}{\rho_i(R_i)^2}\Big)\Big\|_{L^\infty}\|u(t^*)\|_{L^2}\,\|\phi_{i,s^*}\|_{L^2(\partial B_i)} \\
+\|r\phi_{i,s^*}\|_{L^2(\partial B_i)}\,\big\|\partial_s(u(t^*)\circ c)\big|_{s=0}\big\|_{L^2}\lesssim\varepsilon^3|\log\varepsilon|^\ell.
\end{aligned}$$

Here we used (3.43),(3.42),(3.44) and used Lemma 3.3.7 (a) (after rescaling) to estimate $\|\phi_{i,s^*}\|_{L^2(\partial B_i)}\leqslant\varepsilon^{3/2}$. □

4. The stream function

4.1. Overview and strategy. — In this section, we want to compute the asymptotics of G and A and their derivatives with respect to q, which requires us to compute the asymptotics of the streamfunction ψ. The streamfunction will, up to lower order terms, resemble the asymptotics of the Biot-Savart law (1.21). Plugged in the definition of G, the leading order term of the stream function gives 0, so for a direct computation, one would need a higher order expansion of the stream function.

As we need to compute a derivative of the streamfunction anyway we take the alternative approach of expressing G as the derivative of the energy of the streamfunction, which gives the asymptotics of G just from the highest order term of the stream function at the expense of requiring an estimate on the second derivative with respect to q, which is not much more complicated than just computing the first derivative.

Unlike for the potential function, the interaction between the different bodies will matter and we will obtain an interaction term in G, which can be computed from the highest order parts alone. This will yield the asymptotic of G in Propositions 4.5.2 and 4.5.3

The computation of A on the other hand will be more straightforward and the highest order term suffices (see Proposition 4.6.1 below).

The main difficult here is that the limiting object $\ln|x|$ does not lie in H^1, so we can not expect L^2-based estimates to work for the streamfunction. As we are only interested in the boundary data anyway, we will directly characterize it in terms of the single-layer potential on the boundary.

In this section, we will make massive use of the fundamental solution K of the operator $\operatorname{div}(\frac{1}{r}\nabla\cdot)$, as introduced in (2.11). By an abuse of notation, we will also denote the linear operator

$$f \longmapsto \int_{\cup_i \partial B_i} K(x,y) f(x)\,\mathrm{d}x$$

by K. Similarly, we will write

$$\overline{K}_R(x,y) = \frac{R}{2\pi}\big(\log(|x-y|) - \log(8) + 2 - \log(R)\big),$$

and also write $\overline{K}_R$ for the associated integral operator. Recall that in Lemma 2.1.6, we showed that the function $\frac{1}{r}\partial_n\psi_i$ is a solution of the system

$$\int_{\cup_j \partial B_j} K(x,y)\mu(x)\,\mathrm{d}x \quad \text{is constant on each } B_j \tag{4.1}$$

$$\int_{\partial B_j} \mu(x)\,\mathrm{d}x = \delta_{ij}. \tag{4.2}$$

Our goal is to show that for a single body $\frac{1}{r}\partial_n\psi$ converges to a constant by showing that the kernel K converges to $\overline{K}_R$ (for which the solution of the analogous system is constant).

For multiple bodies we will show that the "cross-terms" in K are an order lower and that the corresponding lower order terms in $\frac{1}{r}\partial_n\psi_i$ are essentially given through the derivatives of K itself.

The derivatives with respect to q are estimated by differentiating the equation (4.1)–(4.2) with respect to q and estimating the derivative of K in Propositions 4.4.1 and 4.4.2.

Recall that the kernel K can be written as

$$K(x,y) = \frac{-1}{2\pi}\sqrt{x_R y_R}\; F\Big(\frac{|x-y|^2}{x_R y_R}\Big), \tag{4.3}$$

where x_R and y_R stand for the R-component and

$$F(s) = \int_0^\pi \frac{\cos(t)}{\sqrt{2(1-\cos(t)) + s}}\,\mathrm{d}t$$

(see [13, §2]). This integral cannot be elementarily evaluated, however it has a series expansion at 0, which we will make use of:

Lemma 4.1.1. — *For small enough $s>0$ there is an expansion*

$$F(s)=-\frac{1}{2}\log(s)+\log(8)-2+\sum_{j\geqslant 1}a_js^j+b_js^j\log(s). \tag{4.4}$$

This series has a positive radius of convergence, in particular, we also have the corresponding asymptotics for the derivatives of F.

Proof. — The statement is known, see e.g. [36, Footnote 101] and the computation of the explicit terms can be found there. We provide the proof of the convergence of the expansion here as we were not able to find it in the literature.

By elementary manipulations, one sees that

$$F(s)=(1+s/2)\int_0^{\pi/2}\frac{1}{\sqrt{1+s/4-\sin^2(t)}}\,\mathrm{d}t-2\int_0^{\pi/2}\sqrt{1+s/4-\sin^2(t)}\,\mathrm{d}t.$$

These integrals can be rewritten by using complete elliptic integrals of the first and second kind [8]. These are defined as

$$K_{\text{elliptic}}(m)=\int_0^{\pi/2}(1-m^2\sin^2(t))^{-1/2}\,\mathrm{d}t,$$
$$E_{\text{elliptic}}(m)=\int_0^{\pi/2}(1-m^2\sin^2(t))^{1/2}\,\mathrm{d}t.$$

With this it holds that

$$F(s)=\Big(1+\frac{s}{2}\Big)\frac{1}{\sqrt{1+s/4}}K_{\text{elliptic}}\Big(\sqrt{1-s/(4+s)}\Big)-2\sqrt{1+s/4}\;E_{\text{elliptic}}\Big(\sqrt{1-s/(4+s)}\Big).$$

It suffices to show that the functions $K_{\text{elliptic}}(\sqrt{1-t^2})$ and $E_{\text{elliptic}}(\sqrt{1-t^2})$ have an expansion of the type

$$\sum_{j\geqslant 0}c_jt^{2j}+d_jt^{2j}\log(t) \tag{4.5}$$

for small enough t, because close to $s=0$ the functions $1/\sqrt{1+s/4}$; $\sqrt{1+s/4}$ and $\sqrt{s/(4+s)}$ are analytic and one easily sees that the class of functions with an expansion of the type (4.5) with a positive radius of convergence are stable under composition and multiplication with analytic functions.

It is known [34, p. 53] that there is an expansion

$$K_{\text{elliptic}}(\sqrt{1-t^2})=\log(4/t)-2\Big(\sum_{j\geqslant 1}\frac{1}{2j(2j-1)}\sum_{l=j}^{\infty}\frac{(2l)!}{2^{2l}(l!)^2}\,t^{2l}\Big). \tag{4.6}$$

By the facts that $\sum_j 1/2j(2j-1)<\infty$ and $(2l)!/2^{2l}(l!)^2<1$, we see that this converges for $|t|<1$. By using the definition and elementary calculations, one can see that

$$E_{\text{elliptic}}(m)=m(1-m^2)K'_{\text{elliptic}}(m)+mK_{\text{elliptic}}(m).$$

This implies

$$(4.7)\quad E_{\mathrm{elliptic}}(\sqrt{1-t^2}) = t^2\sqrt{1-t^2}\,\big(K_{\mathrm{elliptic}}(\sqrt{1-t^2})\big)'\frac{\sqrt{1-t^2}}{t} + \sqrt{1-t^2}\,K_{\mathrm{elliptic}}\big(\sqrt{1-t^2}\big).$$

One then obtains the desired expansion by combining (4.6), (4.7) and using a binomial series for the prefactor $\sqrt{1-t^2}$. □

We set

$$h(s) := F(s) + \frac{1}{2}\log(s) - \log 8 + 2,$$
$$g(x,y) := -h\Big(\frac{|x-y|^2}{x_R y_R}\Big)$$

for the remainder. For all n and small enough $|x-y|$ it holds that

$$(4.8)\qquad |\nabla^n g(x,y)| \lesssim_n |x-y|^{2-n}\big|\log|x-y|\big|,$$

locally uniformly in x_R and y_R by Lemma 4.1.1 above.

4.2. The case of a single body. — In this subsection we drop the index i.

Lemma 4.2.1

(a) *The linear map K is invertible from $L^2_0(\partial B)$ to $\dot H^1(\partial B)$ with operator norm $\lesssim 1$ for small enough ε.*

(b) *We have*

$$\|K\|_{L^2(\partial B)\to L^2(\partial B)} \lesssim \varepsilon|\log\varepsilon|.$$

(c) *We have that*

$$\big\|K - \overline{K}_R\big\|_{L^2(\partial B)\to H^1(\partial B)} \lesssim \varepsilon|\log\varepsilon|$$

and

$$\big\|K - \overline{K}_R\big\|_{L^2(\partial B)\to L^2(\partial B)} \lesssim \varepsilon^2|\log\varepsilon|.$$

All these estimates are locally uniform in q.

Proof

(a) and (b) Observe that for (a) it is enough to show (c) and to show that $\overline{K}_R$ is invertible with operator norm $\lesssim 1$, as one sees e.g. by using a geometric series. Similarly, for (b) it is enough to show that

$$(4.9)\qquad \big\|\overline{K}_R\big\|_{L^2(\partial B)\to L^2(\partial B)} \lesssim \varepsilon|\log\varepsilon|.$$

Let $\theta \in \mathbb{T} := [0,1)$ parametrize ∂B, then we claim that the kernel $\overline{K}_R$ acts as

$$(4.10)\qquad \begin{aligned} e^{2\pi\theta in} &\longmapsto -\frac{R}{2|n|}\,\varepsilon\widetilde{\rho}e^{2\pi\theta in} \quad \text{for } n\neq 0,\\ 1 &\longmapsto -R\varepsilon\widetilde{\rho}(-\log(\varepsilon\widetilde{\rho}) + \log(8) - 2 + \log(R)), \end{aligned}$$

which clearly has the desired operator norm and is invertible with norm $\lesssim 1$ from $L^2(\partial B)$ to $H^1(\partial B)$, as one loses the factor ε again due to the derivative.

To show the claim (4.10), we first observe that the constant $\frac{1}{2\pi}(\log(8)-2+\log(R))$ acts as multiplication with $\varepsilon\widetilde{\rho}(\log(8)-2+\log(R))$ on constants and maps other frequencies to 0. The claim (4.10) then follows from the lemma below.

Lemma 4.2.2. — *The kernel* $\log|x-y|$ *acts as the Fourier multiplier*

$$1 \longmapsto 2\pi\varepsilon\widetilde{\rho}\log(\varepsilon\widetilde{\rho}),$$
$$e^{2\pi in\theta} \longmapsto -\frac{\varepsilon\widetilde{\rho}\pi}{|n|}\, e^{2\pi in\theta}.$$

Here $\theta\in\mathbb{T}=[0,1)$ *is a constant speed parametrization of the boundary* ∂B.

Proof. — Note that when parametrizing the boundary with $\theta\in\mathbb{T}$, the action of the kernel corresponds to convolution with

$$2\pi\varepsilon\widetilde{\rho}\big(\log(\varepsilon\widetilde{\rho})+\log|1-e^{2\pi i\theta}|\big).$$

We have that

$$\log|1-e^{2\pi i\theta}| = \operatorname{Re}\log 1-e^{2\pi i\theta}$$

and this can be approximated in L^2 by $\operatorname{Re}\log(1-(1-\delta)e^{2\pi i\theta})$ for $\delta\searrow 0$ by e.g. dominated convergence. Now it holds that

$$\operatorname{Re}\log(1-(1-\delta)e^{2\pi i\theta}) = -\operatorname{Re}\sum_{j=1}^{\infty}\frac{(1-\delta)^j}{j}\, e^{2\pi ij\theta} = -\sum_{j=1}^{\infty}\frac{(1-\delta)^j}{j}\cos(2\pi j\theta),$$

where we used the Taylor series of the logarithm around 1.

By the Plancherel theorem, we can take the limit $\delta\searrow 0$ in this Fourier series and obtain the statement by the well-known formula $\widehat{f_1 * f_2}=\widehat{f}_2\widehat{f}_2$. □

Proof of part (c) *of Lemma* 4.2.1. — It suffices to show that the kernels $K-\overline{K}_R$ and $\partial_y(K-\overline{K}_R)$ are bounded on $L^2(\partial B)$. We can write

$$\begin{aligned}2\pi(K-\overline{K}_R)(x,y) &= -(\sqrt{x_Ry_R}-R)(-\log(|x-y|)+\log(8)-2+\log(R))\\ &\quad -\frac{1}{2}\sqrt{x_Ry_R}\log\left(x_Ry_R/R^2\right)+\sqrt{x_Ry_R}\; g(x,y),\end{aligned}\tag{4.11}$$

where we used the expansion (4.4) and the definition of $\overline{K}_R$. It is easy to see that

$$|\sqrt{x_Ry_R}-R|\lesssim\varepsilon \quad\text{and}\quad \big|\log(x_Ry_R/R^2)\big|\lesssim\varepsilon,$$

hence one obtains L^2-boundedness from Schur's lemma (cf. [18, App. A.1]). Taking a y-derivative in (4.11), we get

$$\begin{aligned}2\pi\partial_y(K-\overline{K}_R)(x,y) &= -\partial_y(\sqrt{x_Ry_R}-R)\big(-\log(|x-y|)+\log(8)-2+\log(R)\big)\\ &+(\sqrt{x_Ry_R}-R)\partial_y\log(|x-y|)-\frac{1}{2}\partial_y\left(\sqrt{x_Ry_R}\log(x_Ry_R/R^2)\right)+O(|x-y||\log|x-y|).\end{aligned}$$

Clearly, the O-term is bounded from L^2 to L^2 and of the desired order. It is easy to check that

$$|\partial_y(\sqrt{x_Ry_R}-R)|\lesssim 1 \quad\text{and}\quad \big|\partial_y\sqrt{x_Ry_R}\log x_Ry_R/R^2)\big|\lesssim 1.$$

This shows boundedness of all terms by Schur's lemma except for

$$(\sqrt{x_R y_R} - R)\partial_y \log|x-y|$$

by direct estimates. For this we use the lemma below to conclude. □

Lemma 4.2.3. — *Let $j \in C^1(\partial B \times \partial B)$, then for all $f \in L^2(\partial B)$ it holds that*

$$\Big\| \int_{\partial B} j(x,y) f(x) \partial_y \log|x-y| \, \mathrm{d}x \Big\|_{L^2_y(\partial B)} \lesssim \Big(\|j\|_{\sup} + \varepsilon \|j\|_{C^1} \Big) \|f\|_{L^2}.$$

This estimate holds locally uniformly in q.

Proof. — We can write $j(x,y) = p_1(x) + p_2(x,y)|x-y|$ with $\|p_1\|_{\sup} \lesssim \|j\|_{\sup}$ and $\|p_2\|_{\sup} \lesssim \|j\|_{C^1}$. Then the kernel $p_1(x)\partial_y \log|x-y|$ has operator norm $\lesssim \|p_1\|_{\sup}$ by applying Lemma 4.2.2 to the function $p_1 f$. The other part has operator norm $\lesssim \varepsilon \|p_2\|_{\sup}$, as one can easily check that

$$|(x-y)\partial_y \log|x-y|| \lesssim 1.$$ □

Proposition 4.2.4. — *It holds that*

$$\Big\| \frac{1}{r}\partial_n \psi - \frac{1}{2\pi\varepsilon\widetilde{\rho}} \Big\|_{L^2(\partial B)} \lesssim \varepsilon^{1/2} |\log \varepsilon|,$$

where the implicit constant is bounded locally uniformly in q.

Proof. — We have that

$$K \frac{1}{r} \partial_n \psi \quad \text{and} \quad \overline{K}_R \frac{1}{2\pi\varepsilon\widetilde{\rho}}$$

are constant on ∂B and $\frac{1}{r}\partial_n\psi - 1/2\pi\varepsilon\widetilde{\rho}$ is mean-free by the definition of ψ, hence we may subtract these two identities from each other and obtain that

$$\Big\| \frac{1}{r}\partial_n \psi - \frac{1}{2\pi\varepsilon\widetilde{\rho}} \Big\|_{L^2_0(\partial B)} \lesssim \Big\| (K - \overline{K}_R) \frac{1}{2\pi\varepsilon\widetilde{\rho}} \Big\|_{\dot{H}^1(\partial B)} \lesssim \varepsilon^{1/2} |\log \varepsilon|,$$

here we made use of Lemma 4.2.1 (a) in the first estimate and of (c) in the second. □

4.3. Multiple bodies. — Next we consider multiple bodies again. Recall that we defined

$$L^2_0(\textstyle\bigcup_i \partial B_i) := \Big\{ f \in L^2(\bigcup_i \partial B_i) \mid \displaystyle\int_{\partial B_i} f \, \mathrm{d}x = 0 \quad \forall i \Big\}.$$

We denote the space $H^1(\bigcup_i \partial B_i)$ modulo *locally* constant functions with $\dot{H}^1(\bigcup_i \partial B_i)$ with the norm $\|\partial_\tau \cdot\|_{L^2(\cup_i \partial B_i)}$ where $\tau = n^\perp$. We set

$$\widetilde{K}(x,y) = K(x,y) I_{\{\exists i \text{ with } x,y \in \partial B_i\}}.$$

and also denote the associated linear operator with $\widetilde{K}$.

Lemma 4.3.1. — *We have*

$$\|K - \widetilde{K}\|_{L^2(\partial B_i) \to \dot{H}^1(\partial B_j)} \lesssim \varepsilon |q_i - q_j|^{-1}$$

locally uniformly in $\widetilde{q}$ (in both regimes (1.22) *and* (1.24)*).*

Proof. — The statement is nontrivial only for $i \neq j$. By the expansions (4.3) and (4.4), we have

$$|\partial_y K(x,y)| = \Big|\partial_y \sqrt{x_R y_R}\, F\Big(\frac{|x-y|^2}{x_R y_R}\Big) + \sqrt{x_R y_R}\, F'\Big(\frac{|x-y|^2}{x_R y_R}\Big)\partial_y \frac{|x-y|^2}{x_R y_R}\Big| \lesssim \frac{1}{|x-y|}.$$

The statement immediately follows since the bodies have pairwise distance $\approx |q_i - q_j|$. □

Corollary 4.3.2. — *The operator K is invertible from $L^2_0(\bigcup_i \partial B_i)$ to $\dot H^1(\bigcup_i \partial B_i)$ for small enough ε with operator norm $\lesssim 1$, where the implicit constant and the smallness requirement for ε are locally uniform in $\widetilde q$. Furthermore, for $i \neq j$, it holds that*

$$\|K^{-1}\|_{\dot H^1(\partial B_j)\to L^2_0(\partial B_i)} \lesssim \varepsilon |q_i - q_j|^{-1}. \tag{4.12}$$

Proof. — By Lemma 4.2.1 (a), invertibility holds for the operator $\widetilde K$. By using e.g. a geometric series, this implies invertibility and by Lemma 4.3.1, we have that

$$\|K^{-1} - \widetilde K^{-1}\|_{\dot H^1(\cup_i \partial B_i)\to L^2_0(\cup_i \partial B_i)} \lesssim \varepsilon |q_i - q_j|^{-1}.$$

This shows the statement, since $\|\widetilde K^{-1}\|_{\dot H^1(\partial B_j)\to L^2_0(\partial B_i)} = 0$ by definition. □

Let ψ_i^1 denote ψ_i in case B_i is the only body present.

Proposition 4.3.3. — *For all i we have*

$$\Big\|\frac{1}{r}\partial_n(\psi_i^1 - \psi_i)\Big\|_{L^2(\partial B_i)} \lesssim \varepsilon^{3/2} |\log \varepsilon|^\ell,$$

where the implicit constant is bounded locally uniformly in $\widetilde q$. In particular the estimate from the single body case in Proposition 4.2.4 still holds.

Proof. — We have that

$$K\frac{1}{r}\partial_n \psi_i = \text{const} \quad \text{on all } \partial B_j$$

and

$$\widetilde K\Big(\frac{1}{r}\partial_n \psi_i^1\big|_{\partial B_i}\Big) = \text{const} \quad \text{on all } \partial B_j.$$

By subtracting the two equations, we see that

$$K\Big(\frac{1}{r}\big(\partial_n \psi_i - \partial_n \psi_i^1\big|_{\partial B_i}\big)\Big) + (K - \widetilde K)\Big(\frac{1}{r}\partial_n \psi_i^1\big|_{\partial B_i}\Big) = \text{const}.$$

Now we can use Corollary 4.3.2 and that $\frac{1}{r}(\partial_n \psi - \partial_n \psi_i^1|_{\partial B_i})$ is mean-free and (4.12) and that

$$(K - \widetilde K)\Big(\frac{1}{r}\partial_n \psi_i^1\big|_{\partial B_i}\Big) = 0$$

on ∂B_i by definition to obtain that

$$\Big\|\frac{1}{r}\big(\partial_n \psi_i - \partial_n \psi_i^1\big|_{\partial B_i}\big)\Big\|_{L^2_0(\partial B_i)} \lesssim \varepsilon |\log \varepsilon|^\ell \Big\|(K - \widetilde K)\Big(\frac{1}{r}\partial_n \psi_i^1\big|_{\partial B_i}\Big)\Big\|_{\dot H^1(\cup_l \partial B_l)} \lesssim \varepsilon^{3/2} |\log \varepsilon|^\ell. \quad □$$

Proposition 4.3.4. — *For $i \neq j$ we have that*

$$\Big\| \frac{1}{r}\partial_n \psi_i - \frac{2}{r} n \cdot \nabla_y K(q_i, q_j) \Big\|_{L^2(\partial B_j)} \lesssim \varepsilon^{3/2} |\log \varepsilon|^\ell,$$

where "∇_y" refers to the gradient in the second variable and the implicit constant is bounded locally uniformly in $\widetilde{q}$.

In particular, it follows from the asymptotics (4.3) *and* (4.4) *that* $\|\partial_n \psi_i\|_{L^2(\partial B_j)} \lesssim \varepsilon^{1/2} |q_i - q_j|^{-1}$ *for $i \neq j$, locally uniformly in $\widetilde{q}$.*

Proof. — Let ψ_i^1 be the potential in case there is only a single body B_i. Let $\mu \in L^2_0(\partial B_j)$ be such that

$$K\mu + K\Big(\frac{1}{r}\partial_n \psi_i^1|_{\partial B_i}\Big) = \text{const} \quad \text{on } \partial B_j.$$

This is well-defined by Lemmas 4.2.1 and 4.3.1 and it holds that $\|\mu\|_{L^2(\partial B_j)} \lesssim \varepsilon^{1/2} |q_i - q_j|^{-1}$. Then for $m \neq j$ by Lemma 4.3.1, it holds that

$$\|K\mu\|_{\dot H^1(\partial B_m)} \lesssim \varepsilon^{3/2} |\log \varepsilon|^\ell$$

and hence

$$\Big\| K\mu + K\Big(\frac{1}{r}\partial_n \psi_i^1|_{\partial B_i}\Big)\Big\|_{\dot H^1(\partial B_m)} \lesssim \varepsilon^{1/2} |q_i - q_j|^{-1}.$$

Subsequently we conclude by (4.12) that

$$\Big\| \mu - \frac{1}{r}\partial_n \psi_i \Big\|_{L^2_0(\partial B_j)} \lesssim \varepsilon^{3/2} |\log \varepsilon|^\ell, \tag{4.13}$$

and therefore it suffices to compute μ.

We first estimate $\nabla \psi_i^1$. We know from the definition of ψ_i^1 and the maximum principle that $\frac{1}{r}\partial_n \psi_i^1 \geqslant 0$ on ∂B_i. Hence we can use the mean value theorem to estimate

$$\Big\| \nabla K\Big(\frac{1}{r}\partial_n \psi_i^1|_{\partial B_i}\Big) - \nabla_y K(q_i, \cdot) \Big\|_{C^0(\partial B_j)} \lesssim \varepsilon \sup_{\substack{z \in B_i \\ y \in \partial B_j}} \nabla^2_{z,y} K(z, y) \lesssim \varepsilon |\log \varepsilon|^\ell, \tag{4.14}$$

where we used the asymptotics (4.3) and (4.4). Similarly we have

$$\|\nabla K(q_i, q_j) - \nabla_y K(q_i, \cdot)\|_{C^0(\partial B_j)} \lesssim \varepsilon \sup_{y \in B_j} |\nabla^2_y K(q_i, y)| \lesssim \varepsilon |\log \varepsilon|^\ell.$$

Hence we have that

$$\big\|\psi_i^1 - x \cdot \nabla_y K(q_i, q_j)\big\|_{\dot H^1(\partial B_j)} \lesssim \varepsilon^{3/2} |\log \varepsilon|^\ell. \tag{4.15}$$

We have

$$\overline{K}_{R_j}^{(-1)} x = -\frac{2}{R_j}\, n + \text{const} \tag{4.16}$$

by (4.10) (where n denotes the normal as usual). By Lemma 4.2.1 and the definition of μ we have

$$\big\|\mu + \overline{K}_{R_j}^{(-1)}\,(\psi_i^1)\big\|_{L^2_0(\partial B_j)} = \big\| \big(\overline{K}_{R_j}^{(-1)} - K^{-1}\big)\,(\psi_i^1)\big\| \lesssim \varepsilon^{3/2} |\log \varepsilon|^\ell. \tag{4.17}$$

Together (4.15), (4.16) and (4.17) imply that

$$\Big\| \mu - \frac{2}{R_j}\, n \cdot \nabla_y K(q_i, q_j) \Big\|_{L^2_0(\partial B_j)} \lesssim \varepsilon^{3/2} |\log \varepsilon|^\ell.$$

Finally we can replace the R_j in the denominator by r as $\|n\cdot\nabla_y K(q_i,q_j)\|_{L^2(\partial B_j)}\lesssim \varepsilon^{1/2}|q_i-q_j|^{-1}$ by (4.3) and (4.4). □

4.4. The derivative with respect to q. — To compute derivatives with respect to q, we only need to consider partial derivatives with respect to a single R_i or Z_i, as everything is smooth by Lemma 2.1.12. In the following we write ψ_{i,R_j+s} instead of ψ_i to emphasize with respect to which R_j the function ψ_i is defined and analogously write ψ_{i,Z_j+s}. For mixed derivatives we write ψ_{i,R_j+s_1,R_m+s_2} if we want to indicate multiple positions. We set

$$\delta_j^s(x) := \begin{cases}\rho_j(R_j+s)/\rho_j(R_j) & \text{if } x\in\partial B_j,\\ 1 & \text{else,}\end{cases}$$

where we again write $\rho_j(\cdot)$ for ρ_j as a function of R_j.

Proposition 4.4.1. — *For all i,j,l it holds that*

$$\Big\|\partial_s\Big(\delta_j^s\Big(\frac{1}{r}\partial_n\psi_{i,R_j+s}\Big)\circ c\Big)\Big\|_{L^2(\partial B_l)}\lesssim \varepsilon^{1/2}\Big(\min_{a\neq b}|q_a-q_b|\Big)^{-2},$$

where the diffeomorphism c was introduced in the beginning of Section 3.3.3 and corresponds to the change of R_j.

Similarly it holds that

$$\Big\|\partial_s\Big(\Big(\frac{1}{r}\partial_n\psi_{i,Z_j+s}\Big)\circ d\Big)\Big\|_{L^2(\partial B_l)}\lesssim \varepsilon^{1/2}\Big(\min_{a\neq b}|q_a-q_b|\Big)^{-2},$$

where the diffeomorphism d was introduced in the beginning of Section 3.3.3 and corresponds to the change of Z_j.

The implicit constant in both estimates is locally uniform in $\widetilde{q}$.

Proposition 4.4.2

(a) *For all i,j,l we have*

$$\Big\|\partial_s^2\Big(\delta_j^s\Big(\frac{1}{r}\partial_n\psi_{i,R_j+s}\Big)\circ c\Big)\Big\|_{L^2(\partial B_l)}\lesssim \varepsilon^{1/2}|\log\varepsilon|^{\ell},$$

and the same holds for the second derivative with respect to Z_j and the mixed second derivative. The implicit constant is bounded locally uniformly in $\widetilde{q}$.

(b) *For all i,j,l,m with $j\neq m$ we have*

$$\Big\|\partial_{s_1}\partial_{s_2}\Big(\delta_j^{s_1}\delta_m^{s_2}\Big(\frac{1}{r}\partial_n\psi_{i,R_j+s_1,R_m+s_2}\Big)\circ c_{jm}(s_1,s_2)\Big)\Big\|_{L^2(\partial B_l)}\lesssim \varepsilon^{1/2}|\log\varepsilon|^{\ell},$$

where c_{jm} is the composition of the map c defined for j and m with arguments s_1 and s_2 respectively. The same estimate holds for the derivatives with respect to the values Z_j or mixed derivatives. The implicit constant is bounded locally uniformly in $\widetilde{q}$.

We shall only prove the statements for the R_j-derivative and focus on the first derivative and occasionally comment on the slight changes needed for the Z_j-derivative, which is generally easier. The second derivatives can be handled with the same technique, but the involved calculations become a lot more tedious, so we omit most of them. We set

$$K_s^{R_j}(x,y) = \Big(1 + I_{\partial B_j}(x)\Big(\frac{R_j}{R_j+s} - 1\Big)\Big) K(c(x,s), c(y,s)),$$

and similarly

$$K_s^{Z_j} := K(d(x,s), d(y,s)).$$

We also write $K_s^{R_j}$ for the associated linear map. Note that for f supported on ∂B_j it holds that

$$K_s^{R_j} f(y) = \frac{R_j \rho_j}{(R_j+s)\rho_j(R_j+s)} K(f \circ c^{-1})(c(y,s)),$$

where we again write $\rho_j(\cdot)$ to denote ρ_j as a function of R_j and where the additional prefactor comes from the change of the minor radius. For f supported on any other ∂B_i it holds

$$K_s^{R_j} f(y) = K(f)(c(y,s)).$$

Similarly, for mixed second derivatives with respect to different indices, one would use the kernel

$$K_{s_1,s_2}^{R_j,R_m}(x,y) = \Big(1 + I_{\partial B_j}(x)\Big(\frac{R_j}{R_j+s} - 1\Big) + I_{\partial B_m}(x)\Big(\frac{R_m}{R_m+s} - 1\Big)\Big) \\ \times K(c_{jm}(x,s_1,s_2), c_{jm}(y,s_1,s_2)).$$

Lemma 4.4.3. — *The linear operator $K_s^{R_j}$ is Fréchet differentiable in s as a map from $L^2(\partial B_j)$ to $\dot H^1(\partial B_j)$ for all i, and we have*

$$\big\|\partial_s K_s^{R_j}\big\|_{L^2(\partial B_j)\to \dot H^1(\partial B_j)}, \big\|\partial_s^2 K_s^{R_j}\big\|_{L^2(\partial B_j)\to \dot H^1(\partial B_j)} \lesssim \varepsilon|\log\varepsilon|,$$

furthermore, the Fréchet derivative is given by integration against the pointwise derivative in s.

Note that the corresponding derivatives of $K_s^{Z_j}$ are trivially 0 by the explicit form of K in (4.3). The statement for mixed second derivatives also trivially reduces to the derivative with respect to a single index, as only the change of R_j matters.

Proof. — It is easy to see that the kernel is pointwise smooth in s for $x \neq y$ by using the Expansions (4.3) and (4.4) and the differentiability of c. We shall estimate the operator norm of the first pointwise derivative. A similar, but tedious calculation, which we omit here can be made to show that the second and third pointwise derivatives are bounded, which by the mean value theorem justifies that the first two pointwise derivatives agree with the Fréchet derivatives.

Let us estimate the first derivative with respect to s of the different parts of the kernel:

$$\partial_s \log|c(x,s)-c(y,s)| = \partial_s \log\left(\rho_j(R_j+s)/\rho_j\right) = -\frac{1}{2(R_j+s)}, \tag{4.18}$$

$$\Big|\partial_s \frac{R_j}{R_j+s}\sqrt{c(x,s)_R c(y,s)_R}\Big| \lesssim \varepsilon, \tag{4.19}$$

$$\Big\|\partial_s \frac{R_j}{R_j+s}\sqrt{c(x,s)_R c(y,s)_R}\Big\|_{C^1_{x,y}(\partial B_j\times\partial B_j)} \lesssim 1, \tag{4.20}$$

$$\partial_s \log(R_j+s) = \frac{1}{R_j+s}, \tag{4.21}$$

$$\Big|\partial_s \log\frac{(R_j+s)^2}{c(x,s)_R c(y,s)_R}\Big| \lesssim \varepsilon, \tag{4.22}$$

$$\Big\|\partial_s \log\frac{(R_j+s)^2}{c(x,s)_R c(y,s)_R}\Big\|_{C^1_{x,y}(\partial B_j\times\partial B_j)} \lesssim 1. \tag{4.23}$$

Furthermore we have

$$\Big|\partial_s \frac{(c(x,s)-c(y,s))^2}{c(x,s)_R c(y,s)_R}\big|_{s=0}\Big| = \Big|\partial_s |x-y|^2 \frac{\rho_j(R_j+s)^2}{\rho_j^2 c(x,s)_R c(y,s)_R}\big|_{s=0}\Big| \lesssim |x-y|^2, \tag{4.24}$$

and similarly it holds

$$\Big|\partial_y\partial_s \frac{(c(x,s)-c(y,s))^2}{c(x,s)_R c(y,s)_R}\big|_{s=0}\Big| \lesssim |x-y|. \tag{4.25}$$

Now we may use the expansions (4.3) and (4.4) to write

$$\begin{aligned}
2\pi\partial_y\partial_s K_s^{R_j}(x,y)\big|_{s=0} &= \partial_s\partial_y\Big(\frac{R_j}{R_j+s}\sqrt{c(x,s)_R c(y,s)_R}\Big)\\
&\qquad\times\Big(\log(|x-y|)-\log(8)+2-\frac12\log(x_R y_R)+g(x,y)\Big)\\
&\quad+\partial_s\Big(\frac{R_j}{R_j+s}\sqrt{c(x,s)_R c(y,s)_R}\Big)\partial_y\Big(\log(|x-y|)-\frac12\log\left(x_R y_R/R_j^2\right)+g(x,y)\Big)\\
&\quad+\partial_y\left(\sqrt{x_R y_R}\right)\partial_s\Big(\log(|c(x,s)-c(y,s)|)-\frac12\log\Big(\frac{c(x,s)_R c(y,s)_R}{(R_j+s)^2}\Big)+\log(R_j+s)\Big)\\
&\quad+\sqrt{x_R y_R}\;\partial_y\partial_s\Big(\log(|c(x,s)-c(y,s)|)-\frac12\log\frac{c(x,s)_R c(y,s)_R}{(R_j+s)^2}\Big)\\
&\quad-\partial_y\left(\sqrt{x_R y_R}\right)\partial_s h\Big(\frac{(c(x,s)-c(y,s))^2}{c(x,s)_R c(y,s)_R}\Big)\\
&\quad-\sqrt{x_R y_R}\;\partial_s\partial_y h\Big(\frac{(c(x,s)-c(y,s))^2}{c(x,s)_R c(y,s)_R}\Big)\\
&= \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI}.
\end{aligned}$$

Here I–VI stand for the obvious terms and we dropped some constants whose derivative vanishes.

– The boundedness of the terms I and II follows from the estimates (4.19) and (4.20) above and Schur's lemma and further from using Lemma 4.2.3 for the derivative of the logarithm.

– The boundedness of III follows directly from (4.18) and (4.22) by using Schur's lemma.

– The boundedness of IV follows from (4.18) and (4.23).

– The boundedness of V and VI follows from (4.24) resp. (4.25) and the estimate (4.8) on g. □

Lemma 4.4.4. — *The kernel $K_s^{R_j}$ is Fréchet-differentiable in s as a map from $L^2(\partial B_i)$ to $\dot H^1(\partial B_l)$ for $i \neq l$ and, locally uniformly in $\widetilde q$, we have that*

$$\big\|\partial_s K_s^{R_j}\big\|_{L^2_0(\partial B_i)\to \dot H^1(\partial B_l)} \lesssim \varepsilon |q_i - q_l|^{-2}$$

and

$$\big\|\partial_s^2 K_s^{R_j}\big\|_{L^2_0(\partial B_i)\to \dot H^1(\partial B_l)} \lesssim \varepsilon |q_i - q_l|^{-3}.$$

Furthermore, the Fréchet derivative is given by integration against the pointwise derivative in s. The same estimates also hold for the Kernel $K_s^{Z_j}$ and the second derivatives of $K_{s_1,s_2}^{R_j,R_m}$.

Proof. — We only consider the case $l = j$ and the first derivative, the other cases and the second derivative are very similar. In this case, the kernel is smooth in (x, y, s) and hence the Fréchet- and pointwise derivative agree by e.g. the mean value theorem.

By (4.3) and (4.4), we have that

$$\begin{aligned}
-2\pi\partial_s\partial_y K_s^{R_j}(x,y) = {}& \partial_s\partial_y \sqrt{x_R c(y,s)_R}\; F\Big(\frac{|x-y|^2}{x_R y_R}\Big) \\
&+ \partial_s \sqrt{x_R c(y,s)_R}\; F'\Big(\frac{|x-y|^2}{x_R y_R}\Big)\partial_y \frac{|x-y|^2}{x_R y_R} \\
&+ \partial_y \sqrt{x_R y_R}\; F'\Big(\frac{|x-c(y,s)|^2}{x_R c(y,s)_R}\Big)\partial_s \frac{|x-c(y,s)|^2}{x_R c(y,s)_R} \\
&+ \sqrt{x_R y_R}\Big[F'\Big(\frac{|x-c(y,s)|^2}{x_R c(y,s)_R}\Big)\partial_y\partial_s \frac{|x-c(y,s)|^2}{x_R c(y,s)_R} \\
&+ F''\Big(\frac{|x-c(y,s)|^2}{x_R c(y,s)_R}\Big)\partial_s \frac{|x-c(y,s)|^2}{x_R c(y,s)_R}\partial_y \frac{|x-y|^2}{x_R y_R}\Big].
\end{aligned}$$

It is easy to see that all relevant derivatives of the prefactor are $\lesssim 1$ and that $|\partial_s c| \lesssim 1$. Hence the absolute value of this is

$$\lesssim \Big|F\Big(\frac{|x-y|^2}{x_R y_R}\Big) + F'\Big(\frac{|x-y|^2}{x_R y_R}\Big)(1+|x-y|) + F''\Big(\frac{|x-y|^2}{x_R y_R}\Big)|x-y|^2\Big|.$$

From the asymptotics of F (see (4.4)), we see that for $|x-y|$ small this is $\lesssim 1/|x-y|^2$. Hence we see that

$$\big\|\partial_s K_s^{R_j}\big\|_{L^2(\partial B_i)\to \dot H^1(\partial B_j)} \lesssim \varepsilon |q_i - q_l|^{-2}. \qquad \square$$

Proof of Propsition 4.4.1. — We know from Lemma 2.1.12 that $\frac{1}{r}\psi_{j,R_j+s}$ is differentiable, hence we may differentiate the equation

$$K_s^{R_j}\delta_j^s\frac{1}{r}\partial_n\psi_{i,R_j+s}\circ c = \text{const} \quad \text{on each } \partial B_m$$

with respect to s, as $K_s^{R_j}$ is differentiable by Lemma 4.4.3. This yields that

$$\partial_s K_s^{R_j}\Big(\frac{1}{r}\partial_n\psi_i\Big) + K\Big(\partial_s\Big(\delta_j^s\Big(\frac{1}{r}\partial_n\psi_{i,R_j+s}\Big)\circ c\Big)\Big) = \text{const} \quad \text{on each } \partial B_m.$$

Note that $\partial_s\delta_j^s\frac{1}{r}\partial_n\psi_{i,R_j+s}\circ c$ is mean-free on each ∂B_m, because the integral of $\delta_j^s\frac{1}{r}\partial_n\psi_{i,R_j+s}\circ c$ over ∂B_m is either 1 or 0 for all s by definition. Furthermore by Lemma 4.4.3 and Lemma 4.4.4, we have

$$\Big\|\partial_s K_s^{R_j}\Big(\frac{1}{r}\partial_n\psi_i\Big)\Big\|_{\dot H^1(\cup_m\partial B_m)} \lesssim \varepsilon^{1/2}\Big(|\log\varepsilon| + \min_{a\neq b}|q_a-q_b|^{-2}\Big).$$

We can absorb the logarithm into the second summand by definition of the regimes.

By Corollary 4.3.2, we conclude. □

Proof of Proposition 4.4.2. — We only consider the second derivative with respect to R_j, all others work the same because one has the same or better estimates. We have

$$\partial_s^2 K_s^{R_j}\Big(\frac{1}{r}\partial_n\psi_i\Big) + 2\partial_s K_s^{R_j}\Big(\partial_s\delta_j^s\Big(\frac{1}{r}\partial_n\psi_{i,R_j+s}\Big)\circ c\Big) + K\Big(\partial_s^2\delta_j^s\Big(\frac{1}{r}\partial_n\psi_{i,R_j+s}\Big)\circ c\Big) = \text{const}$$

on all ∂B_m. Note that by Lemmas 4.4.3, 4.4.4 and Proposition 4.4.1 it holds that

$$\Big\|\partial_s^2 K_s^{R_j}\frac{1}{r}\partial_n\psi_i\Big\|_{\dot H^1(\cup_m\partial B_m)} + \Big\|\partial_s K_s^{R_j}\Big(\partial_s\delta_j^s\Big(\frac{1}{r}\partial_n\psi_{i,R_j+s}\Big)\circ c\Big)\Big\|_{\dot H^1(\cup_m\partial B_m)} \lesssim \varepsilon^{1/2}|\log\varepsilon|^{\ell}.$$

Hence from Corollary 4.3.2 we conclude the statement. □

4.4.1. *Further estimates on the derivative of* K. — To compute the force G, we will also need estimates in the $L^2\to L^2$-topology.

Lemma 4.4.5. — *For all j it holds that*

$$\big\|\partial_s K_s^{R_j}\big\|_{L^2(\partial B_j)\to L^2(\partial B_j)} \lesssim \varepsilon,$$

and

$$\big\|\partial_s^2 K_s^{R_j}\big\|_{L^2(\partial B_j)\to L^2(\partial B_j)} \lesssim \varepsilon,$$

locally uniformly in $\widetilde q$ and furthermore, the pointwise and Fréchet derivatives agree.

Here we only need the estimate in the R_j-direction, because the derivative in the Z_j-direction is 0. Also note that changes of the other values R_m trivially give a derivative of zero here.

Proof. — We only show this for the first derivative with respect to R_j, the second derivative is very similar. We may expand the kernel as

$$\begin{aligned}2\pi \left(\partial_s K_s^{R_j}(x,y)\big|_{s=0}\right) \\ = \partial_s\Big(\frac{R_j}{R_j+s}\sqrt{c(x,s)_R c(y,s)_R}\Big)\Big(\log(|x-y|)-\log(8)+2-\frac{1}{2}\log(x_R y_R)+g(x,y)\Big) \\ + \sqrt{x_R y_R}\,\partial_s\Big[\log(|c(x,s)-c(y,s)|)-\frac{1}{2}\log(c(x,s)_R c(y,s)_R) \\ - h\Big(\frac{|c(x,s)-c(y,s)|^2}{c(x,s)_R c(y,s)_R}\Big)\Big].\end{aligned}$$

It is easy to see that

$$\Big|\partial_s\Big(\frac{R_j}{R_j+s}\sqrt{c(x,s)_R c(y,s)_R}\Big)\Big| \lesssim \varepsilon$$

and by reusing (4.18) and (4.22) and (4.21) one sees that

$$\begin{aligned}\Big|\partial_s\Big(\log(|c(x,s)-c(y,s)|)-\frac{1}{2}\log(c(x,s)_R c(y,s)_R) \\ - h\Big(\frac{|c(x,s)-c(y,s)|^2}{c(x,s)_R c(y,s)_R}\Big)\Big)\Big| \lesssim 1.\end{aligned} \tag{4.26}$$

The boundedness follows by Schur's Lemma. The Fréchet differentiability follows from the boundedness of the second derivative and the mean value theorem. □

4.5. The force G. — We compute the asymptotics of G, defined in 2.2.1. The crucial lemma for the "self-interaction" terms is the following:

Lemma 4.5.1. — *We have*

$$\int_{\mathcal{F}}\frac{1}{2}r\Big\langle\frac{1}{r}\nabla^\perp\psi_i,\frac{1}{r}\nabla^\perp\psi_j\Big\rangle\,\mathrm{d}x = -\frac{1}{2}C_{ij},$$

and for any t_l^* associated with B_l it holds that

$$\partial_q\frac{1}{2}C_{ij}\cdot t_l^* = \frac{1}{2}\int_{\partial B_l}\frac{1}{r}\partial_n\phi_{l,t_l^*}\big\langle\nabla^\perp\psi_i,\nabla^\perp\psi_j\big\rangle\,\mathrm{d}x,$$

where the values C_{ij} were defined in 2.1.4.

Proof. — We have that

$$\frac{1}{2}\int_{\mathcal{F}} r\frac{1}{r^2}\langle\nabla^\perp\psi_i,\nabla^\perp\psi_j\rangle\,\mathrm{d}x = -\sum_l\frac{1}{2}\int_{\partial B_l}\frac{1}{r}C_{il}\partial_n\psi_j\,\mathrm{d}x = -\frac{1}{2}C_{ij}.$$

Here the partial integration is justified by Lemmas 2.1.5 and 2.1.6. Note that it holds

$$\partial_q\psi_j\cdot t_l^* = \partial_q C_{jm}\cdot t_l^* - \partial_n\psi_j u(t_l^*) \tag{4.27}$$

on ∂B_m as one can see by differentiating $C_{jm} = \psi_j(x_l)$ by q for some point x_l on ∂B_m moving with normal velocity $u(t_l^*)$.

Now by the Reynolds transport theorem (which can be used by the integrability statement in Lemma 2.1.12) the derivative of this with respect to q in direction $t_l^* \in T_{q_l}M$ equals

$$\begin{aligned}\frac{1}{2}\partial_q C_{ij}\cdot t_l^* &= -\int_{\mathcal{F}}\frac{1}{r}\langle\nabla^\perp\psi_i,\nabla^\perp\partial_q\psi_j\cdot t_l^*\rangle\,\mathrm{d}x+\frac{1}{2}\int_{\partial B_l}\frac{1}{r}\langle\nabla^\perp\psi_i,\nabla^\perp\psi_j\rangle u(t_l^*)\,\mathrm{d}x\\ &=\sum_m\int_{\partial B_m}\frac{1}{r}\partial_q\psi_j\cdot t_l^*\partial_n\psi_i\,\mathrm{d}x+\frac{1}{2}\int_{\partial B_l}\frac{1}{r}\langle\nabla^\perp\psi_i,\nabla^\perp\psi_j\rangle u(t_l^*)\,\mathrm{d}x\\ &=\sum_m\int_{\partial B_m}\frac{1}{r}\big(\partial_q C_{jm}\cdot t_l^*-\partial_n\psi_j u(t_l^*)\big)\partial_n\psi_i\,\mathrm{d}x+\frac{1}{2}\int_{\partial B_l}\frac{1}{r}\langle\nabla^\perp\psi_i,\nabla^\perp\psi_j\rangle u(t_l^*)\,\mathrm{d}x\\ &=\partial_q C_{ji}\cdot t_l^*-\int_{\partial B_l}\frac{1}{r}\langle\nabla^\perp\psi_i,\nabla^\perp\psi_j\rangle u(t_l^*)\,\mathrm{d}x+\frac{1}{2}\int_{\partial B_l}\frac{1}{r}\langle\nabla^\perp\psi_i,\nabla^\perp\psi_j\rangle u(t_l^*)\,\mathrm{d}x,\end{aligned}$$

where we have made use of equation (4.27) in the third line and of the facts that $\partial_q C_{jm}$ is a constant function and that the matrix C is symmetric by the first statement. □

Proposition 4.5.2. — *For every tangent vector t^*, we have that*

$$\Big|G(q,e_i)\cdot t^*-\frac{1}{4\pi}\log(\varepsilon\widetilde{\rho})(t_i^*)\cdot e_R\Big|\lesssim 1,\quad \textit{and}\quad |\partial_q G(q,e_i)|\lesssim 1.$$

These estimate are locally uniform in $\widetilde{q}$.

Proof. — By Lemma 4.5.1 and the definition of $G(q,e_i)$ in 2.2.1, it equals $\frac{1}{2}$ times the derivative of the energy

$$\int_{(\cup_l\partial B_l)^2}K(x,y)\frac{1}{r^2}\partial_n\psi_i(x)\partial_n\psi_i(y)\,\mathrm{d}x\,\mathrm{d}y$$

with respect to q. We first consider the partial derivative in the direction R_i.

Note that $K\frac{1}{r}\partial_n\psi_i$ is constant and that $\frac{1}{r}\partial_n\psi_i$ is mean-free on all boundaries except ∂B_i, hence the integral over all boundaries except $(\partial B_i)^2$ is zero.

Using the diffeomorphism c, we can rewrite the energy with respect to R_i+s as

$$\int_{(\partial B_i)^2}\frac{R_i+s}{R_i}K_s^{R_i}(\delta_i^s)^2\Big(\frac{1}{r}\partial_n\psi_{i,R_i+s}\Big)\circ c\Big(\frac{1}{r}\partial_n\psi_{i,R_i+s}\Big)\circ c\,\mathrm{d}x\,\mathrm{d}y.$$

Here the factor δ_i^s is the determinant due to the change of coordinates.

We can differentiate under the integral as everything is smooth by Lemma 2.1.12. We first show that the parts where a derivative falls on $\frac{1}{r}\partial_n\psi_i$ are small. Indeed by Lemma 4.2.1 and Proposition 4.4.1 we have

$$\begin{aligned}\Big\|K\partial_s\Big(\delta_i^s\Big(\frac{1}{r}\partial_n\psi_{i,R_i+s}\Big)\circ c\Big)\Big\|_{L^2(\partial B_i)}&\lesssim\varepsilon|\log\varepsilon|^\ell\Big\|\partial_s\Big(\delta_i^s\Big(\frac{1}{r}\partial_n\psi_{i,R_i+s}\Big)\circ c\Big)\Big\|_{L^2(\partial B_i)}\\ &\lesssim\varepsilon^{3/2}|\log\varepsilon|^\ell.\end{aligned}$$

Hence

$$\begin{aligned}&\Big|\int_{(\partial B_i)^2}K(x,y)\partial_s\Big(\delta_i^s\Big(\frac{1}{r}\partial_n\psi_{i,R_i+s}\Big)\circ c\Big)\frac{1}{r}\partial_n\psi_i(y)\,\mathrm{d}x\,\mathrm{d}y\Big|\\ &\qquad\lesssim\varepsilon^{3/2}|\log\varepsilon|^\ell\Big\|\frac{1}{r}\partial_n\psi_i\Big\|_{L^2(\partial B_i)}\lesssim\varepsilon|\log\varepsilon|^\ell.\end{aligned}$$

The same argument can also be made for all terms involving one or two derivatives of $\frac{1}{r}\partial_n\psi_i$ or $K_s^{R_j}$ by the estimates in Lemmas 4.2.1, 4.4.5 and Propositions 4.4.1 and 4.4.2 and also for derivatives with respect to R_j or Z_j for $j \neq i$. As the second derivative of $(R_i+s)/R_i$ vanishes, this shows the estimate for the derivative, in the direction R_j.

Hence we are left with the main contribution where the derivative falls on $(R_i+s)/R_i$, which is

$$\int_{(\partial B_i)^2} \frac{1}{R_i} K(x,y)\frac{1}{r^2}\partial_n\psi_i(x)\partial_n\psi_i(y)\,\mathrm{d}x\,\mathrm{d}y.$$

It can be rewritten as

$$\fint_{(\partial B_i)} \frac{1}{R_i}\overline{K}_{R_i}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ + O\Big(\varepsilon^{-1}\left\|K-\overline{K}_{R_i}\right\|_{L^2\to L^2} + \|K\|_{L^2\to L^2}\left\|\frac{1}{r}\partial_n\psi_i - \frac{1}{2\pi\varepsilon\widetilde{\rho}_i}\right\|_{L^2}\left\|\frac{1}{r}\partial_n\psi_i\right\|_{L^2}\Big).$$

The O-term is $\lesssim \varepsilon|\log\varepsilon|^\ell$ by Lemma 4.2.1 and Proposition 4.3.3. We computed in the claim (4.10) that the main integral equals

$$\frac{1}{2\pi}\big(\log(\varepsilon\widetilde{\rho}_i) - \log(8) - 2 - \log R_i\big). \qquad \square$$

Next, we consider the "cross-terms" in G, given by the interaction between ψ_i and ψ_j.

Proposition 4.5.3. — *Let $t^* = (t_1^*,\dots t_k^*)$ be a tangent vector, identified with a vector in $\mathbb{R}^{2k}$ as usual, then for $i\neq j$ we have that*

$$\left|\int_{\cup_m\partial B_m} \frac{1}{r}u(t^*)\partial_n\psi_i\partial_n\psi_j\,\mathrm{d}x - t_j^*\cdot\nabla_y K(q_i,q_j) - t_i^*\cdot\nabla_y K(q_j,q_i)\right| \lesssim \varepsilon|\log\varepsilon|^\ell,$$

where the normal velocity $u(t^*)$ was defined in (1.2). Furthermore it holds that

$$|\partial_q G(q,\gamma)| \lesssim \max_{i,j}|q_i-q_j|^{-2}|\gamma|^2.$$

Both of these estimates are locally uniform in $\widetilde{q}$.

Proof. — We first prove the first statement for the contribution of ∂B_j, which also covers the contribution of ∂B_i by symmetry. We have

$$\int_{\partial B_j}\frac{1}{r}\Big(t_j^*\cdot n - \frac{\rho_j}{2R_j}t_j^*\cdot e_R\Big)\partial_n\psi_j\partial_n\psi_i\,\mathrm{d}x \\ = \int_{\partial B_j}\frac{1}{r}t_j^*\cdot n\partial_n\psi_i\partial_n\psi_j\,\mathrm{d}x + O\big(\varepsilon\left\|\partial_n\psi_i\right\|_{L^2(\partial B_j)}\left\|\partial_n\psi_j\right\|_{L^2(\partial B_j)}\big). \tag{4.28}$$

By Propositions 4.3.3 and 4.3.4, the error term here is $\lesssim \varepsilon|\log\varepsilon|^\ell$.

We can now use Proposition 4.3.4 and (4.28) to obtain that

$$\int_{\partial B_j}\frac{1}{r}u(t^*)\partial_n\psi_i\partial_n\psi_j\,\mathrm{d}x = 2\int_{\partial B_j}(t^*\cdot n)n\cdot\nabla_y K(q_i,q_j)\frac{1}{r}\partial_n\psi_j\,\mathrm{d}x + O\left(\varepsilon|\log\varepsilon|^\ell\right).$$

We further have

$$2\int_{\partial B_j}(t^*\cdot n)n\cdot\nabla_y K(q_i,q_j)\frac{1}{r}\partial_n\psi_j\,\mathrm{d}x = 2\fint_{\partial B_j}(t^*\cdot n)n\cdot\nabla_y K(q_i,q_j)\,\mathrm{d}x + O\Big(\varepsilon^{1/2}\big|\nabla_y K(q_i,q_j)\big|\,\Big\|\frac{1}{r}\partial_n\psi_j-\frac{1}{2\pi\varepsilon\widetilde{\rho}_j}\Big\|_{L^2(\partial B_j)}\Big).$$

By Proposition 4.3.3, the error term is $\lesssim\varepsilon|\log\varepsilon|^{\ell}$. The main integral equals

$$t^*\cdot\nabla_y K(q_i,q_j).$$

For m with $m\neq i,j$, one can directly see by Proposition 4.3.4 that the integral is $\lesssim\varepsilon|\log\varepsilon|^{\ell}$. It remains to estimate the derivative. Note that G is a quadratic form in γ and that we have already shown the statement for $\gamma=e_l$ in Proposition 4.5.2, and that the "off-diagonal" coefficients in G are exactly the integrals we estimated in the first step, so we need to estimate their derivatives.

For notational simplicity we only consider the derivative of the integral on ∂B_j with respect to R_l, as the derivative with respect to Z_l enjoys the same estimates, this is not restrictive. We begin with the derivative with respect to R_l for $l\neq j$.

By Propositions 4.4.1 it holds that

$$\Big\|\partial_s\frac{1}{r}\partial_n\psi_{i,R_l+s}\Big\|_{L^2(\partial B_j)}\lesssim\varepsilon^{1/2}|q_i-q_j|^{-2}\quad\text{and}\quad\Big\|\partial_s\frac{1}{r}\partial_n\psi_{j,R_i+s}\Big\|_{L^2(\partial B_j)}\lesssim\varepsilon^{1/2}|\log\varepsilon|^{\ell},$$

which by the Cauchy-Schwarz inequality and Propositions 4.3.3 and 4.3.4 implies the statement.

Finally, consider the derivative with respect to R_j. We can rewrite the integral as

$$\int_{\partial B_j(R_j)}\Big(t^*\cdot n-\frac{\rho_j(R_j+s)}{2(R_j+s)}t^*\cdot e_R\Big)\Big(\delta_j^s\frac{1}{r}\partial_n\psi_{j,R_j+s}\Big)\circ c\,\big(\partial_n\psi_{i,R_j+s}\big)\circ c\,\mathrm{d}x.$$

Here the factor δ_j^s is the Jacobian due to the coordinate change. Using Proposition 4.4.1, we see that

$$\Big\|\partial_s\Big(\delta_j^s\Big(\frac{1}{r}\partial_n\psi_{j,R_j+s}\Big)\circ c\Big)\Big\|_{L^2(\partial B_j)}\lesssim\varepsilon^{1/2}|\log\varepsilon|^{\ell}.$$

Furthermore by Proposition 4.4.1 we have

$$\Big\|\partial_s\Big(\Big(\frac{1}{r}\partial_n\psi_{i,R_j+s}\Big)\circ c\Big)\Big\|_{L^2(\partial B_j)}\lesssim\varepsilon^{1/2}|q_i-q_j|^{-2}.$$

Hence, by the Cauchy-Schwarz inequality we conclude. □

4.6. The mixed term A. — We estimate the force A (defined in 2.2.1), which contains both the stream function and the potentials.

Proposition 4.6.1. — *For all $s^*,t^*\in T_qM$ we have that*

$$(A(q,\gamma)t^*)\cdot s^*\longrightarrow(t^*)^T\begin{pmatrix}0 & R_1\gamma_1 & 0 & \dots & \\ -R_1\gamma_1 & 0 & \dots & & \\ \dots & & & & \\ & & \dots & 0 & R_k\gamma_k\\ & & \dots & -R_k\gamma_k & 0\end{pmatrix}s^*,$$

with a rate of $O(\varepsilon|\log\varepsilon|^{\ell})$ locally uniformly in $\widetilde{q}$. Furthermore, it holds that

$$|\partial_q A| \lesssim \varepsilon|\log\varepsilon|^{\ell},$$

locally uniformly in $\widetilde{q}$.

In particular, A is invertible for small enough ε with an inverse of order $\lesssim 1$ by the assumption that all γ_i are $\neq 0$.

Proof. — Recall that A was defined as

$$(A(q,\gamma)t^*)\cdot s^* = \sum_l \int_{\partial B_l} \Big(-\partial_\tau\phi(s^*)\partial_n\phi(t^*) + \partial_\tau\phi(t^*)\partial_n\phi(s^*)\Big)\partial_n \sum_j \gamma_j\psi_j \,\mathrm{d}x,$$

where $\phi(s^*)$ and $\phi(t^*)$ are the summed up potentials. Without loss of generality, we may assume $|t^*| = |s^*| = 1$, $t^* \in T_{q_i}M$, $s^* \in T_{q_j}M$, and that only γ_l is nonzero. We first show the convergence for $i \neq j$. By definition we have

$$\|\partial_n\phi_{i,t^*}\|_{L^\infty(\partial B_i)} \lesssim 1 \quad \text{and} \quad \|\partial_n\phi_{j,s^*}\|_{L^\infty(\partial B_j)} \lesssim 1$$

and on all other boundaries the normal derivatives vanish. By Corollary 3.3.9, we have

$$\|\partial_\tau\phi_{i,t^*}\|_{L^2(\partial B_j)} \lesssim \varepsilon^{5/2}|\log\varepsilon|^{\ell}$$

and vice versa. Furthermore we have

$$\|\partial_n\psi_l\|_{L^2(\partial B_i)}, \|\partial_n\psi_l\|_{L^2(\partial B_j)} \lesssim \varepsilon^{-1/2}$$

for all l by Propositions 4.3.3 and 4.3.4. By the Cauchy-Schwarz inequality we conclude convergence to 0. Similarly, we can directly estimate the derivative. By Propositions 3.3.10 and 4.4.1, we know that all derivatives of the boundary values enjoy estimates which are at worst an order $|\log\varepsilon|^{\ell}$ worse, hence these derivatives are small by the Cauchy-Schwarz inequality and the product rule.

Next, consider the case $i = j \neq l$. Here we again have

$$\|\partial_\tau\phi_{i,t^*}\|_{L^2(\partial B_i)} \lesssim \varepsilon^{1/2}$$

by Corollary 3.3.9 and the same holds for the potential with respect to s^*. On the other hand we also have

$$\|\partial_n\psi_l\|_{L^2(\partial B_i)} \lesssim \varepsilon^{1/2}|\log\varepsilon|^{\ell}$$

by Proposition 4.3.4. By the Cauchy-Schwarz inequality, this implies that

$$\left|\int_{\partial B_i} \partial_n\psi_l\,(-\partial_\tau\phi_{i,s^*}\partial_n\phi_{i,t^*} + \partial_\tau\phi_{i,t^*}\partial_n\phi_{i,s^*})\,\mathrm{d}x\right| \lesssim \varepsilon|\log\varepsilon|^{\ell}.$$

The smallness of the derivative of this term again follows from the fact that all derivatives have estimates which are at worst an order $|\log\varepsilon|^{\ell}$ worse by Propositions 3.3.10 and 4.4.1.

It remains to consider the case $i = j = l$. In this case we have the same estimates as above for the tangential derivatives and furthermore by Corollary 3.3.9 we have

$$\big\|\partial_\tau(\phi_{i,t^*} - \breve{\phi}_{t^*})\big\|_{L^2(\partial B_i)} \lesssim \varepsilon^{3/2}|\log\varepsilon|^{\ell} \quad \text{and} \quad \big\|\partial_n(\phi_{i,t^*} - \breve{\phi}_{t^*})\big\|_{L^2(\partial B_i)} \lesssim \varepsilon^{3/2},$$

where the "two-dimensional" potential $\breve{\phi}$ was defined in 3.3.1 and the same holds for the potentials with respect to s^*.

Also by Proposition 4.3.3, we have

$$\Big\|\frac{1}{r}\partial_n\psi_i - \frac{1}{2\pi\widetilde{\rho}_i\varepsilon}\Big\|_{L^2(\partial B_i)} \lesssim \varepsilon^{1/2}|\log\varepsilon|^{\ell}.$$

Finally, we clearly have $\|r-R_i\|_{L^2(\partial B_i)} \lesssim \varepsilon^{3/2}$. Hence by the Cauchy-Schwarz inequality we see that

$$\int_{\partial B_i} \partial_n\psi_i\left(-\partial_\tau\phi_{i,s^*}\partial_n\phi_{i,t^*} + \partial_\tau\phi_{i,t^*}\partial_n\phi_{i,s^*}\right)\mathrm{d}x \\ = R_i \fint_{\partial B_i}\left(-\partial_\tau\breve{\phi}_{s^*}\partial_n\breve{\phi}_{t^*} + \partial_\tau\breve{\phi}_{t^*}\partial_n\breve{\phi}_{s^*}\right)\mathrm{d}x + O\big(\varepsilon|\log\varepsilon|^{\ell}\big).$$

By the antisymmetry of these integrals with respect to t^* and s^* it suffices to consider the case $t^*=e_1$ and $s^*=e_2$. In this case, we can use the explicit form of $\breve{\phi}_{t^*}$ in 3.3.1 to see that

$$\fint_{\partial B_i}\left(-\partial_\tau\breve{\phi}_{s^*}\partial_n\breve{\phi}_{t^*} + \partial_\tau\breve{\phi}_{t^*}\partial_n\breve{\phi}_{s^*}\right)\mathrm{d}x = \fint \tau\cdot e_2 n\cdot e_1 - \tau\cdot e_1 n\cdot e_2\,\mathrm{d}x = 1.$$

The smallness of the derivative follows again from the fact that all the derivatives of the boundary values have estimates which are an order $\varepsilon|\log\varepsilon|^{\ell}$ better by Propositions 3.3.10 and 4.4.1.

Finally, all these estimates are locally uniform in $\widetilde{q}$ because all the used estimates for the boundary values are. □

Definition 4.6.2. — We let J^1_γ and J^2_γ be the velocities in (1.25) and (1.23), i.e.,

$$(J^1_\gamma(\widetilde{q}))_i := \frac{1}{2\pi}\sum_{j\neq i}\gamma_j\frac{(\widetilde{q}_i-\widetilde{q}_j)^{\perp}}{|\widetilde{q}_i-\widetilde{q}_j|^2} - \frac{\gamma_i}{4\pi R_0}e_Z,$$
$$(J^2_\gamma(\widetilde{q}))_i := \frac{1}{2\pi}\sum_{j\neq i}\gamma_j\frac{(\widetilde{q}_i-\widetilde{q}_j)^{\perp}}{|\widetilde{q}_i-\widetilde{q}_j|^2} + \frac{\widetilde{q}_{R_i}\gamma_i}{4\pi R_0^2}e_Z.$$

Corollary 4.6.3. — *In the regime* (1.24) *(= distances $\approx|\log\varepsilon|$), we have that*

$$\frac{A^{-1}G}{|\log\varepsilon|}\longrightarrow -J^1_\gamma(\widetilde{q})$$

locally uniformly in $\widetilde{q}$.

Proof. — By Propositions 4.5.2 and 4.6.1 have that

$$\frac{A(q,\gamma)^{-1}G(q,e_i)}{|\log\varepsilon|}\longrightarrow\frac{1}{4\pi R_0\gamma_i}e_{Z,i},$$

where $e_{Z,i}\in(\mathbb{R}^2)^k\simeq\mathbb{R}^{2k}$ denotes the vector which has an e_Z in the i-th component and no other entries.

As G is a quadratic form in γ by definition, it remains to show the statement for the "off-diagonal" terms in G. By Propositions 4.5.3 and 4.6.1 we have that

$$\begin{aligned}\lim_{\varepsilon\to 0}&\frac{A(q,\gamma)^{-1}\left(G(q,e_i+e_j)-G(q,e_i)-G(q,e_j)\right)}{|\log\varepsilon|}\\&=-\lim_{\varepsilon\to 0}\frac{1}{R_0|\log\varepsilon|}\Big(\frac{P_j}{\gamma_j}\nabla_y^{\perp}K(q_i,q_j)+\frac{P_i}{\gamma_i}\nabla_y^{\perp}K(q_j,q_i)\Big)\\&=\frac{1}{2\pi}\Big(\frac{P_i(\widetilde{q}_j-\widetilde{q}_i)^{\perp}}{\gamma_i|\widetilde{q}_i-\widetilde{q}_j|^2}+\frac{P_j(\widetilde{q}_i-\widetilde{q}_j)^{\perp}}{\gamma_j|\widetilde{q}_i-\widetilde{q}_j|^2}\Big).\end{aligned}$$

Here $P_l:\mathbb{R}^2\to(\mathbb{R}^2)^k$ is the map to the l-th coordinate and in the last step we used the asymptotics (4.3) and (4.4). □

Corollary 4.6.4. — *In the regime* (1.22) *(= distances $\approx|\log\varepsilon|^{1/2}$), we have that*

$$\frac{A^{-1}G}{|\log\varepsilon|^{1/2}}-\frac{|\log\varepsilon|^{1/2}}{4\pi R_0}v_Z\longrightarrow -J_\gamma^2(\widetilde{q})$$

locally uniformly in $\widetilde{q}$, where $v_Z\in(\mathbb{R}^2)^k$ is the vector $(e_Z,e_Z,\dots)$.

Proof. — The calculation of the "off-diagonal" terms is the same as in the previous proof. For the diagonal terms, we have by Propositions 4.5.2 and 4.6.1 that

$$\frac{A(q,\gamma)^{-1}G(q,e_i)}{|\log\varepsilon|^{1/2}}-\frac{|\log\varepsilon|^{1/2}}{4\gamma_i\pi R_i}\longrightarrow 0.$$

The statement then follows from the definition of $\widetilde{q}$ and the fact that e.g. by the mean value theorem we have

$$\frac{|\log\varepsilon|^{1/2}}{4\pi R_i}-\frac{|\log\varepsilon|^{1/2}}{4\pi R_0}+\frac{(R_i-R_0)|\log\varepsilon|^{1/2}}{4\pi R_0^2}\longrightarrow 0.$$ □

5. Passage to the limit

In this section, we write $\widetilde{q}_\varepsilon$ instead of $\widetilde{q}$ to emphasize the ε-dependence.

Proof of Theorem 1.1.2. — We write the system (2.16) in the rescaled time $s=t|\log\varepsilon|^2$ and the rescaled position $\widetilde{q}_\varepsilon$, defined as in (1.24), it then reads as

$$(5.1)\quad \begin{aligned}|\log\varepsilon|^3\Big(E(q)\widetilde{q}_\varepsilon''+\frac{1}{2}\widetilde{q}_\varepsilon'(\nabla_{\widetilde{q}_\varepsilon}E(q)\cdot\widetilde{q}_\varepsilon')+\mathcal{M}(q)\widetilde{q}_\varepsilon''+|\log\varepsilon|^{-1}\big\langle\Gamma(q),\widetilde{q}_\varepsilon',\widetilde{q}_\varepsilon'\big\rangle\Big)\\=G(q,\gamma)+|\log\varepsilon|(A(q,\gamma)\widetilde{q}'),\end{aligned}$$

where the time derivatives are denoted with a $'$ and all derivatives taken with respect to the rescaled time and space.

We first show that the velocity in rescaled time and space is bounded, until either we approach the boundary or some component of $\widetilde{q}$ goes to infinity.

We take C as some large compact subset of M_ε containing $\widetilde{q}_\varepsilon(0)$. If $\widetilde{q}_\varepsilon\in C$, then this implies that each q_i lies in a compact subset of $\mathbb{H}$. It follows from the definition of M_ε that if we view the manifolds M_ε as subsets of $\mathbb{R}^{2k}$ that for small enough ε

such a set C is also a subset of $M_{\varepsilon'}$ for $\varepsilon' < \varepsilon$. Hence C can be chosen as the same set for all small enough ε.

Recall further that the matrix A is invertible by Proposition 4.6.1 and that its inverse has operator norm $\lesssim 1$ as long as $\widetilde{q}_\varepsilon \in C$. We may hence rewrite the equation as

$$\begin{aligned}|\log\varepsilon|^3\Big[\big(E(q)+\mathcal{M}(q)\big)\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big(\widetilde{q}_\varepsilon' + \frac{A^{-1}G}{|\log\varepsilon|}\Big)\Big) + \frac{1}{2}\big(\nabla_{\widetilde{q}_\varepsilon} E\cdot \widetilde{q}_\varepsilon'\big)\Big(\widetilde{q}_\varepsilon' + \frac{A^{-1}G}{|\log\varepsilon|}\Big) \\ + \frac{1}{2}\big(\nabla_{\widetilde{q}_\varepsilon}\mathcal{M}\cdot \widetilde{q}_\varepsilon'\big)\Big(\widetilde{q}_\varepsilon' + \frac{A^{-1}G}{|\log\varepsilon|}\Big)\Big] \\ - |\log\varepsilon|^3\Big[\big(E(q)+\mathcal{M}(q)\big)\frac{\mathrm{d}}{\mathrm{d}s}\frac{A^{-1}G}{|\log\varepsilon|} + \frac{1}{2}\big(\nabla_{\widetilde{q}_\varepsilon} E\cdot \widetilde{q}_\varepsilon'\big)\frac{A^{-1}G}{|\log\varepsilon|} + |\log\varepsilon|^{-1}\Big\langle \Gamma(q), \widetilde{q}_\varepsilon', \frac{A^{-1}G}{|\log\varepsilon|}\Big\rangle \\ - \Big\langle |\log\varepsilon|^{-1}\Gamma(q) - \frac{1}{2}\nabla_{\widetilde{q}_\varepsilon}\mathcal{M}(q), \widetilde{q}_\varepsilon', \widetilde{q}_\varepsilon' + \frac{A^{-1}G}{|\log\varepsilon|}\Big\rangle\Big] \\ = |\log\varepsilon| A\Big(\widetilde{q}_\varepsilon' + \frac{A^{-1}G}{|\log\varepsilon|}\Big),\end{aligned}$$

where we used the notation $\langle N, a, b\rangle = Na\cdot b$ in the penultimate line.

By testing against $\widetilde{q}_\varepsilon' + A^{-1}G/|\log\varepsilon|$ and dividing out the $|\log\varepsilon|^3$ we obtain that from the antisymmetry of A that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{2}\Big(\Big(\widetilde{q}_\varepsilon' + \frac{A^{-1}G}{|\log\varepsilon|}\Big)^T (E(q)+\mathcal{M}(q))\Big(\widetilde{q}_\varepsilon' + \frac{A^{-1}G}{|\log\varepsilon|}\Big)\Big) \\ = \Big(\widetilde{q}_\varepsilon' + \frac{A^{-1}G}{|\log\varepsilon|}\Big)^T (E(q)+\mathcal{M}(q))\,\frac{\mathrm{d}}{\mathrm{d}s}\frac{A^{-1}G}{|\log\varepsilon|} \\ + \frac{1}{2}\Big(\widetilde{q}_\varepsilon' + \frac{A^{-1}G}{|\log\varepsilon|}\Big)^T \big(\nabla_{\widetilde{q}_\varepsilon} E\cdot\widetilde{q}_\varepsilon'\big)\,\frac{A^{-1}G}{|\log\varepsilon|} \\ + |\log\varepsilon|^{-1}\Big\langle\Gamma(q), \widetilde{q}_\varepsilon', \frac{A^{-1}G}{|\log\varepsilon|}\Big\rangle\Big(\widetilde{q}_\varepsilon' + \frac{A^{-1}G}{|\log\varepsilon|}\Big) \\ - \Big\langle |\log\varepsilon|^{-1}\Gamma(q) - \frac{1}{2}\nabla_{\widetilde{q}_\varepsilon}\mathcal{M}(q), \widetilde{q}_\varepsilon', \widetilde{q}_\varepsilon' + \frac{A^{-1}G}{|\log\varepsilon|}\Big\rangle\Big(\widetilde{q}_\varepsilon' + \frac{A^{-1}G}{|\log\varepsilon|}\Big) \\ =: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},\end{aligned} \tag{5.2}$$

where I–IV stand for the terms in each line. Our goal is to show that each of these terms is $\lesssim \varepsilon^2(1 + |\widetilde{q}_\varepsilon'|^2 + \varepsilon|\log\varepsilon|^\ell|\widetilde{q}_\varepsilon'|^3)$ as long as $\widetilde{q}_\varepsilon \in C$.

For $\widetilde{q}_\varepsilon \in C$ we have the following estimates. By Lemmas 3.2.1 and 3.4.1, we have

$$\varepsilon^2 \lesssim E + \mathcal{M} \lesssim \varepsilon^2 \tag{5.3}$$

(in the sense that the smallest and highest eigenvalues have these bounds). Furthermore by Proposition 4.6.1 we have

$$|A|,\ |\nabla_{\widetilde{q}} A|,\ |A^{-1}|,\ |\nabla_{\widetilde{q}} A^{-1}| \lesssim 1. \tag{5.4}$$

By Propositions 4.5.2 and 4.5.3, we have

$$|G|,\ |\nabla_{\widetilde{q}} G| \lesssim |\log\varepsilon|, \tag{5.5}$$

finally by Propositions 3.2.1 (b) and 3.4.2, we have

$$|\nabla_{\widetilde{q}_\varepsilon} E|,\ |\Gamma| \lesssim \varepsilon^3|\log\varepsilon|^\ell. \tag{5.6}$$

Hence we conclude for $\widetilde{q}_\varepsilon \in C$ that

$$|\mathrm{I}| \lesssim \varepsilon^2(1+|\widetilde{q}_\varepsilon'|^2) \quad \text{and} \quad |\mathrm{II}|,\ |\mathrm{III}| \lesssim \varepsilon^3|\log\varepsilon|^\ell(1+|\widetilde{q}_\varepsilon'|^3).$$

We have that $\mathrm{IV}=0$. Indeed if we set $p := \widetilde{q}_\varepsilon' + A^{-1}G/|\log\varepsilon|$ then, by the definition of Γ (2.2.1), it holds

$$\begin{aligned}\langle\Gamma(q),\dot q,p\rangle p &= \sum_{i,j,k}\frac{1}{2}(\partial_j\mathcal{M}(q)_{ik}+\partial_i\mathcal{M}(q)_{jk}-\partial_k\mathcal{M}(q)_{ij})\dot q_i p_j p_k\\ &= \sum_{i,j,k}\frac{1}{2}\partial_i\mathcal{M}(q)_{jk}\dot q_i p_j p_k = \frac{1}{2|\log\varepsilon|}(\nabla_{\widetilde{q}_\varepsilon}\mathcal{M}\cdot\dot q)p\cdot p.\end{aligned}$$

Hence, plugging these estimates into (5.2), we obtain by (5.3) that as long as $\widetilde{q}_\varepsilon \in C$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\Big(\Big(\widetilde{q}_\varepsilon'+\frac{A^{-1}G}{|\log\varepsilon|}\Big)^T(E(q)+\mathcal{M}(q))\Big(\widetilde{q}_\varepsilon'+\frac{A^{-1}G}{|\log\varepsilon|}\Big)\Big)\\ &\quad\lesssim \varepsilon^2+\Big(\Big(\widetilde{q}_\varepsilon'+\frac{A^{-1}G}{|\log\varepsilon|}\Big)^T(E(q)+\mathcal{M}(q))\Big(\widetilde{q}_\varepsilon'+\frac{A^{-1}G}{|\log\varepsilon|}\Big)\Big)\\ &\qquad+|\log\varepsilon|^\ell\Big(\Big(\widetilde{q}_\varepsilon'+\frac{A^{-1}G}{|\log\varepsilon|}\Big)^T(E(q)+\mathcal{M}(q))\Big(\widetilde{q}_\varepsilon'+\frac{A^{-1}G}{|\log\varepsilon|}\Big)\Big)^{3/2}.\end{aligned}\tag{5.7}$$

As long as we have

$$\Big(\Big(\widetilde{q}_\varepsilon'+\frac{A^{-1}G}{|\log\varepsilon|}\Big)^T(E(q)+\mathcal{M}(q))\Big(\widetilde{q}_\varepsilon'+\frac{A^{-1}G}{|\log\varepsilon|}\Big)\Big)\leqslant\varepsilon^{1/2},$$

the last term in (5.7) can be absorbed in the first two and by Gronwall's lemma we obtain that

$$\begin{aligned}&\Big(\Big(\widetilde{q}_\varepsilon'+\frac{A^{-1}G}{|\log\varepsilon|}\Big)^T(E(q)+\mathcal{M}(q))\Big(\widetilde{q}_\varepsilon'+\frac{A^{-1}G}{|\log\varepsilon|}\Big)\Big)(s)\\ &\qquad\lesssim e^s\Big(\varepsilon^2+\Big(\widetilde{q}_\varepsilon'+\frac{A^{-1}G}{|\log\varepsilon|}\Big)^T(E(q)+\mathcal{M}(q))\Big(\widetilde{q}_\varepsilon'+\frac{A^{-1}G}{|\log\varepsilon|}\Big)(0)\Big).\end{aligned}$$

By (5.3) and the assumption about the initial velocities, this implies that

$$\Big|\widetilde{q}_\varepsilon'+\frac{A^{-1}G}{|\log\varepsilon|}\Big|\lesssim e^s,$$

and hence

$$|\widetilde{q}_\varepsilon'|\lesssim e^s$$

until either $\widetilde{q}_\varepsilon$ leaves the set C or up to a time of order $|\log\varepsilon|$. By (5.4), (5.5) and (5.1), this implies that

$$|\log\varepsilon|^2\Big(E(q)\widetilde{q}_\varepsilon''+\frac{1}{2}\widetilde{q}_\varepsilon'(\nabla_{\widetilde{q}_\varepsilon}E(q)\cdot\widetilde{q}_\varepsilon')+\mathcal{M}(q)\widetilde{q}_\varepsilon''+|\log\varepsilon|^{-1}\big\langle\Gamma(q),\widetilde{q}_\varepsilon',\widetilde{q}_\varepsilon'\big\rangle\Big)\longrightarrow 0$$

in $W^{-1,\infty}$ up to a time of order $|\log\varepsilon|$ or until $\widetilde{q}_\varepsilon$ leaves C. Hence we obtain that

$$A\widetilde{q}_\varepsilon'+\frac{G}{|\log\varepsilon|}\overset{*}{\rightharpoonup}0\quad\text{in } L^\infty.$$

Because A^{-1} and $G/|\log\varepsilon|$ converge strongly, we see by Corollary 4.6.3 that

$$\widetilde{q}_\varepsilon' - J^1_\gamma(\widetilde{q}_\varepsilon)0 \quad \text{in } L^\infty$$

until $\widetilde{q}_\varepsilon$ leaves C, which takes at least $\gtrsim 1$ time, as $\widetilde{q}_\varepsilon'$ is bounded ($J^1_\gamma(\widetilde{q}_\varepsilon)$ was defined in 4.6.2). Hence we have that

$$\widetilde{q}_\varepsilon(v) - \widetilde{q}_\varepsilon(0) - \int_0^v J^1_\gamma(\widetilde{q}_\varepsilon(s))\,\mathrm{d}s \longrightarrow 0$$

for all times $v \lesssim 1$ and $\widetilde{q}_\varepsilon$ converges locally uniformly to some $\widetilde{q}_0$ by compactness, as long as long as $\widetilde{q}_\varepsilon$ lies in C. Therefore we see that $\widetilde{q}_0$ must be a solution of $\widetilde{q}_0' = J^1_\gamma(\widetilde{q}_0)$ because J^1_γ is locally uniformly continuous.

Finally, we may remove the condition that $\widetilde{q}_\varepsilon$ lies in a compact set C, by taking C so large that the solution of $q' = J^1_\gamma(q)$ lies in the interior of C until some time T, which is possible for small enough ε whenever q does not blow until time T. Then we have uniform convergence of $\widetilde{q}_\varepsilon$ as long as it lies in C. As the limit lies in the interior of C, the solution $\widetilde{q}_\varepsilon$ also lies in C for small enough ε up to time T.

Hence we have convergence, as long as the limiting solution does not blow up. □

Proof of Theorem 1.1.3. — The proof is quite similar to the previous one. In the rescaled time $s = |\log\varepsilon|t$, and the rescaled spatial variable $\widetilde{q}_\varepsilon$, defined as in (1.22) the system (2.16) reads as

$$\begin{multline}|\log\varepsilon|^{3/2}\Big(E(q)\widetilde{q}_\varepsilon'' + \frac{1}{2}\widetilde{q}_\varepsilon'(\nabla_{\widetilde{q}_\varepsilon}E(q)\cdot\widetilde{q}_\varepsilon') + \mathcal{M}(q)\widetilde{q}_\varepsilon'' + |\log\varepsilon|^{-1/2}\big\langle\Gamma(q),\widetilde{q}_\varepsilon',\widetilde{q}_\varepsilon'\big\rangle\Big)\\ = G(q,\gamma) + |\log\varepsilon|^{1/2}(A(q,\gamma)\widetilde{q}_\varepsilon').\end{multline} \tag{5.8}$$

Similarly as in the previous proof we can rewrite the equation, tested against $\widetilde{q}_\varepsilon' + A^{-1}G/|\log\varepsilon|^{1/2}$ as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{2}\Big(\Big(\widetilde{q}_\varepsilon' + \frac{A^{-1}G}{|\log\varepsilon|^{1/2}}\Big)^T(E(q)+\mathcal{M}(q))\Big(\widetilde{q}_\varepsilon' + \frac{A^{-1}G}{|\log\varepsilon|^{1/2}}\Big)\Big)&\\ = \Big(\widetilde{q}_\varepsilon' + \frac{A^{-1}G}{|\log\varepsilon|^{1/2}}\Big)^T(E(q)+\mathcal{M}(q))\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{A^{-1}G}{|\log\varepsilon|^{1/2}}\Big)&\\ + \frac{1}{2}\Big(\widetilde{q}_\varepsilon' + \frac{A^{-1}G}{|\log\varepsilon|^{1/2}}\Big)^T(\nabla_{\widetilde{q}_\varepsilon}E\cdot\widetilde{q}_\varepsilon')\Big(\frac{A^{-1}G}{|\log\varepsilon|^{1/2}}\Big)&\\ + |\log\varepsilon|^{-1/2}\Big\langle\Gamma(q),\widetilde{q}_\varepsilon',\frac{A^{-1}G}{|\log\varepsilon|^{1/2}}\Big\rangle\Big(\widetilde{q}_\varepsilon' + \frac{A^{-1}G}{|\log\varepsilon|^{1/2}}\Big)&\\ - \Big\langle|\log\varepsilon|^{-1/2}\Gamma(q) - \frac{1}{2}\nabla_{\widetilde{q}_\varepsilon}\mathcal{M}(q),\widetilde{q}_\varepsilon',\widetilde{q}_\varepsilon' + \frac{A^{-1}G}{|\log\varepsilon|^{1/2}}\Big\rangle\Big(\widetilde{q}_\varepsilon' + \frac{A^{-1}G}{|\log\varepsilon|^{1/2}}\Big)&\\ =: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.&\end{aligned}$$

Let v_Z denote the vector $(e_Z, e_Z, \dots) \in \mathbb{R}^{2k}$. We would like to estimate the shifted velocity $\widetilde{q}_\varepsilon' + (|\log\varepsilon|^{1/2}/4\pi R_0)v_Z$. We again use a compact set $C \subset M_\varepsilon$ containing $\widetilde{q}_\varepsilon(0)$. Then we let $\widetilde{C} := C + v_Z\mathbb{R}$. On this set we still have uniform estimates because the system is invariant in the v_Z direction. If $\widetilde{q}_\varepsilon \in \widetilde{C}$, then we clearly we still have the estimates (5.3) and (5.6). Furthermore by Propositions 4.5.2 and 4.5.3 we then also have

$$|G| \lesssim |\log\varepsilon| \quad\text{and}\quad |\nabla_{\widetilde{q}_\varepsilon} G| \lesssim |\log\varepsilon|^{1/2}.$$

Furthermore by Proposition 4.6.1 we have

$$|A| \lesssim 1, \quad |\nabla_{\widetilde{q}_\varepsilon} A| \lesssim |\log\varepsilon|^{-1/2}$$

and from Corollary 4.6.4 one sees that

$$\Big|\frac{A^{-1}G}{|\log\varepsilon|^{1/2}} - \frac{|\log\varepsilon|^{1/2}}{4\pi R_0} v_Z\Big| \lesssim 1.$$

Hence, we directly see that

$$|\mathrm{II}|,\ |\mathrm{III}| \lesssim \varepsilon^3 |\log\varepsilon|^{\ell}\Big(1 + \Big|\widetilde{q}_\varepsilon' + \frac{|\log\varepsilon|^{1/2}}{4\pi R_0} v_Z\Big|^3\Big).$$

The term IV again drops out by the same calculation as in the previous proof. Note that we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{A^{-1}G}{|\log\varepsilon|^{1/2}}\Big) = \frac{1}{|\log\varepsilon|^{1/2}}\left(\left(\nabla_{\widetilde{q}_\varepsilon} A^{-1}\right) G + A^{-1}\nabla_{\widetilde{q}_\varepsilon} G\right)\Big(\widetilde{q}_\varepsilon' + \frac{|\log\varepsilon|^{1/2}}{4\pi R_0} v_Z\Big),$$

because the derivative of $A^{-1}G$ in the v_Z direction is 0, as the system is invariant in that direction. Hence we see that

$$\begin{aligned}|\mathrm{I}| &\lesssim \frac{\varepsilon^2}{|\log\varepsilon|^{1/2}}\Big(\Big|\widetilde{q}_\varepsilon' + \frac{|\log\varepsilon|^{1/2}}{4\pi R_0} v_Z\Big| + \Big|\frac{A^{-1}G}{|\log\varepsilon|^{1/2}} - \frac{|\log\varepsilon|^{1/2}}{4\pi R_0} v_Z\Big|\Big)\\ &\qquad\times\left(|A^{-1}|^2|\nabla_{\widetilde{q}_\varepsilon} A||G| + |A^{-1}||\nabla_{\widetilde{q}_\varepsilon} G|\right)\Big|\widetilde{q}_\varepsilon' + \frac{|\log\varepsilon|^{1/2}}{4\pi R_0} v_Z\Big|\\ &\lesssim \varepsilon^2\Big(1 + \Big|\widetilde{q}_\varepsilon' + \frac{|\log\varepsilon|^{1/2}}{4\pi R_0} v_Z\Big|^2\Big).\end{aligned}$$

From the assumption that

$$\Big|\widetilde{q}_\varepsilon'(0) + \frac{|\log\varepsilon|^{1/2}}{4\pi R_0} v_Z\Big| \lesssim 1,$$

we see by the same Gronwall argument as in the previous proof that

$$\Big|\widetilde{q}_\varepsilon'(s) + \frac{|\log\varepsilon|^{1/2}}{4\pi R_0} v_Z\Big| \lesssim e^s$$

until $\widetilde{q}_\varepsilon$ leaves $\widetilde{C}$ or until a time of order $|\log\varepsilon|$. From this, we conclude that $\widetilde{q}_\varepsilon''$ is bounded in $W^{-1,\infty}$ and by (5.3) and (5.6) we see that

$$|\log\varepsilon|^{3/2}\Big(E(q)\widetilde{q}_\varepsilon'' + \frac{1}{2}\widetilde{q}_\varepsilon'(\nabla_{\widetilde{q}_\varepsilon} E(q)\cdot\widetilde{q}_\varepsilon') + \mathcal{M}(q)\widetilde{q}_\varepsilon'' + |\log\varepsilon|^{-1/2}\langle\Gamma(q), \widetilde{q}_\varepsilon', \widetilde{q}_\varepsilon'\rangle\Big) \longrightarrow 0$$

in $W^{-1,\infty}$ until $\widetilde{q}_\varepsilon$ leaves $\widetilde{C}$. Hence we see again that

$$\widetilde{q}_\varepsilon' + \frac{|\log\varepsilon|^{1/2}}{4\pi R_0} v_Z + \Big(\frac{A^{-1}G}{|\log\varepsilon|^{1/2}} - \frac{|\log\varepsilon|^{1/2}}{4\pi R_0} v_Z\Big) \overset{*}{\rightharpoonup} 0 \quad\text{in } L^\infty.$$

This implies the statement by the same argument as in the previous proof and Corollary 4.6.4. □

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