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**On self-similar blow up for the energy supercritical semilinear wave equation**

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# ON SELF-SIMILAR BLOW UP FOR THE ENERGY SUPERCRITICAL SEMILINEAR WAVE EQUATION

BY JIHOI KIM

ABSTRACT. — We analyse the energy supercritical semilinear wave equation

$$\Phi_{tt} - \Delta\Phi - |\Phi|^{p-1}\Phi = 0$$

in  $\mathbb{R}^d$  space. We first prove in a suitable regime of parameters the existence of a countable family of self-similar profiles which bifurcate from the soliton solution. We then prove the non-radial finite codimensional stability of these profiles by adapting the functional setting of [22].

RÉSUMÉ (Sur l'explosion auto-similaire pour l'équation d'onde semi-linéaire supercritique en énergie)

Nous analysons l'équation d'onde semi-linéaire supercritique en énergie

$$\Phi_{tt} - \Delta\Phi - |\Phi|^{p-1}\Phi = 0$$

dans l'espace  $\mathbb{R}^d$ . Nous prouvons d'abord, dans un régime approprié de paramètres, l'existence d'une famille dénombrable de profils auto-similaires qui bifurquent à partir de la solution du soliton. Nous prouvons ensuite la stabilité non radiale en codimension finie de ces profils en adaptant le cadre fonctionnel de [22].

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## 1. INTRODUCTION

1.1. SETTING OF THE PROBLEM. — We consider the semi-linear focusing wave equation

$$(1.1) \quad \begin{cases} \Phi_{tt} - \Delta\Phi - |\Phi|^{p-1}\Phi = 0, \\ \Phi|_{t=0} = \Phi_0, \quad \partial_t\Phi|_{t=0} = \Phi_1, \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d.$$

This model admits a scaling invariance: if  $\Phi(t, x)$  is a solution, then so is

$$\Phi_\lambda(t, x) = \lambda^\alpha \Phi(\lambda t, \lambda x), \quad \lambda > 0, \quad \alpha := \frac{2}{p-1}.$$

This transformation is an isometry on the homogeneous Sobolev space with critical exponent:

$$(1.2) \quad \|\Phi_\lambda(t, \cdot)\|_{\dot{H}^{s_c}} = \|\Phi(t, \cdot)\|_{\dot{H}^{s_c}}, \quad s_c := \frac{d}{2} - \frac{2}{p-1}.$$

In this paper, we focus on the energy super-critical case where space dimension  $d \geq 3$  and  $s_c > 1$ . The question we address is the existence and stability of self-similar blow up regimes.

The problem of singularity formation in semi-linear dispersive equations has attracted a considerable attention in the last fifty years both in the physics and mathematics communities, with a substantial acceleration in the last twenty years. The series of works by Merle and Zaag [26, 25, 24, 23] give a detailed description of singularity formation mechanism in energy sub-critical ranges  $s_c < 1$  where the leading order expected behaviour is the self-similar ODE blow up. In the energy critical range, the situation is very different and new so called type II blow up scenario was discovered in the setting of the energy-critical wave and Schrödinger map [16, 29, 28, 20] and semi-linear problems [15]. The soliton solution

$$\begin{cases} \Delta Q + Q^p = 0 \\ \lim_{|x| \rightarrow +\infty} Q(x) = 0 \end{cases}$$

plays a distinguished role in the analysis as it serves as blow up profile for the main part of the singular bubble. The stability analysis of the obtained type II blow up bubbles then relies on delicate energy estimates built on repulsivity properties of the linearized self-similar flow near the soliton.

In the energy super-critical range, and in analogy with the pioneering results for the non-linear heat equation [14, 19, 18, 6], the situation is quite different. Solitonic type II bubbles still exist but only for  $p > p_{JL}$  large enough, [21, 5] where Joseph-Lundgren exponent  $p_{JL}$  is defined in (3.2), and a new type of self-similar blow up arises, different from the ODE blow up, as governed by explicit stationary self-similar solutions. More explicitly, the ansatz

$$(1.3) \quad \Phi(t, r) = (T-t)^{-\alpha} u(\rho), \quad \rho := |y|, \quad y := \frac{x}{T-t}$$

maps (1.1) onto the radially symmetric non-linear ODE

$$(1.4) \quad (1 - \rho^2)u'' + \left[ \frac{d-1}{\rho} - 2(1+\alpha)\rho \right] u' - \alpha(1+\alpha)u + |u|^{p-1}u = 0.$$

The program of existence of self-similar dynamics then becomes a two-step analysis. First construct solutions to the non-linear ODE (1.4) with regularity at the origin and good boundary condition at  $+\infty$

$$u(\rho) \sim \frac{c}{\rho^{2/(p-1)}} \quad \text{as } \rho \rightarrow +\infty.$$

These solutions however never belong to the energy space in which (1.1) is naturally well posed, hence a global in space stability analysis is required to ensure that a suitable truncation of these profiles can be stabilized, at least for a finite dimensional manifold of initial data. This second step relies on both a linear and non-linear analysis of the linearized flow around self-similar profiles.

Let us stress that the program of constructing self-similar solutions and showing their finite codimensional stability goes way beyond the scope of non-linear wave equations, and is in particular a very active field of research in fluid related problems, [22], hence the need for robust analytic methods.

1.2. EXISTENCE OF SELF-SIMILAR PROFILES. — The existence of self-similar profiles with suitable boundary conditions is in general a delicate problem, and here we take advantage of symmetry reductions to transform the problem into the non-linear ODE problem (1.4) which is of shooting type. However the understanding of solutions is non trivial, and relies on the derivation of explicit monotonicity formulas to follow the non-linear flow. The existence of a countable family of solutions to (1.4) is obtained in [3, 8] in the expected range

$$(1.5) \quad 1 < s_c < \frac{3}{2} \iff 1 + \frac{4}{d-2} < p < 1 + \frac{4}{d-3}.$$

Our first result in this paper describes the asymptotic behaviour of the branch of solutions to (1.4) leading to an explicit sequence of solutions that concentrate at the origin to a soliton profile. Our approach generalizes the analogous result for the semi-linear heat equation implemented in [2, 7]. The advantage of this method is its robustness as it can be applied to more complicated problems, see e.g. [1], and also allows for a full description of the profile in space.

**THEOREM 1** (Existence and asymptotes of excited self-similar solutions)

*Assume (1.5). There exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ , there exists a smooth radially symmetric self-similar solution to equation (1.1) such that for*

$$\Lambda = \alpha + y \cdot \nabla,$$

$\Lambda u_n$  vanishes exactly  $n$  times on  $(0, \infty)$ . Moreover:

(i) Behaviour at infinity: as  $n \rightarrow \infty$  the solutions  $u_n$  converge to the explicit singular solution

$$u_\infty(\rho) := b_\infty \rho^{-\alpha}, \quad b_\infty := (\alpha(d-2-\alpha))^{\alpha/2}$$

to (1.4) in the following sense: for all  $\rho_0 > 0$ ,

$$\lim_{n \rightarrow \infty} \sup_{\rho \geq \rho_0} (1 + \rho^\alpha) |u_n(\rho) - u_\infty(\rho)| = 0.$$

(ii) Behaviour at the origin: *There exist  $0 < \rho_0 \ll 1$  and  $\mu_n \rightarrow 0$  such that*

$$\lim_{n \rightarrow \infty} \sup_{\rho \leq \rho_0} |u_n(\rho) - \mu_n^{-\alpha} Q(\rho/\mu_n)| = 0,$$

where the soliton  $Q$  is the unique non trivial radially symmetric solution to

$$\Delta Q + Q^p = 0, \quad Q(\rho) = b_\infty \rho^{-\alpha} + \mathcal{O}_{\rho \rightarrow \infty}(\rho^{1-d/2}).$$

1.3. NON-LINEAR STABILITY. — The non-linear stability of self-similar blow up is a classical problem. It has been addressed for the energy super-critical non-linear heat equation in [7] and the stability proof relies on a two-step argument: linear exponential decay in time for local in space norms around the singularity which in the parabolic case rely on self-adjoint spectral methods, and then propagation of space time decay using energy estimates which provide strong enough control to close the non-linear terms.

In the setting of energy super-critical non-linear wave equations, a non-self adjoint spectral method is developed in the pioneering works by Donninger and Schörkhuber for wave maps [9], see also [13] and references therein, but decay is restricted to the light cone only  $|x| < T - t$  and hence does not allow the full control of the solution. In [22], a full linear and non-linear analysis is performed for the stability study of quasilinear self-similar blow up. Our claim in this paper is that this robust framework can be adapted to (1.1) to show the stability of any self-similar profile, modulo a finite number of unstable modes. We moreover claim that full non-radial perturbations can be considered as opposed to previous works which restrict to data with radial symmetries.

THEOREM 2 (Non-linear stability). — *Let  $d = 3$  and  $u_n$  be the self-similar profiles constructed in Theorem 1 with corresponding initial data  $(\Phi(0), \Phi_t(0)) = P_n$  for*

$$(1.6) \quad P_n := \left( \frac{1}{T^\alpha} u_n(r/T), \frac{1}{T^{\alpha+1}} \Lambda u_n(r/T) \right).$$

For  $T \ll 1$ , there exists a finite codimensional Lipschitz manifold of smooth initial data<sup>(1)</sup>  $(\Phi(0), \Phi_t(0)) \in \bigcap_{m \geq 0} H^m(\mathbb{R}^3, \mathbb{R}^2)$  such that in the neighbourhood of  $P_n$ , the corresponding solution  $(\Phi, \Phi_t)$  to (1.1) develops a Type I blow up at time  $T$  at the origin i.e., as  $t \rightarrow T$ ,

$$\|\Phi(t)\|_{L^\infty} \sim (T - t)^{-\alpha}.$$

More precisely, there holds the decomposition:

$$(\Phi, \Phi_t) = \left( \frac{1}{(T - t)^\alpha} (u_n + \Psi)(t, r/(T - t)), \frac{1}{(T - t)^{\alpha+1}} (\Lambda u_n + \Omega)(t, r/(T - t)) \right)$$

with the asymptotic behaviour in the limit  $t \rightarrow T$ : there exist  $c_n, d_n > 0$  with

(1) Subcritical norms:

$$(1.7) \quad \limsup_{t \rightarrow T} \|\Phi\|_{H^s}^2 + \mathbb{1}_{s \geq 1} \|\Phi_t\|_{H^{s-1}}^2 < \infty \quad \text{for } 0 \leq s < s_c,$$

<sup>(1)</sup>See comments on the results below for the precise definition of the Lipschitz manifold.

(2) Critical norm:

$$(1.8) \quad (\|\Phi\|_{\dot{H}^{s_c}}^2, \|\Phi_t\|_{\dot{H}^{s_c-1}}^2) = (c_n, d_n)(1 + o_{t \rightarrow T}(1))|\log(T - t)|,$$

(3) Supercritical norms:

$$(1.9) \quad \lim_{t \rightarrow T} \|\Psi\|_{\dot{H}^s}^2 + \|\Omega\|_{\dot{H}^{s-1}}^2 = 0 \quad \text{for } s_c < s \leq 2.$$

*Comments on the results*

(1) *Stability of the self-similar blow up.* As in [22], a key step in the analysis is to realize the linearized operator close to a self-similar profile as a compact perturbation of a maximally dissipative operator in a *global in space weighted Sobolev space with supercritical regularity*. Using sufficient regularity and propagating additional weighted energy estimates then allows to close bound for the nonlinear terms. Hence counting the exact number of instability is reduced to an explicit spectral problem.

(2) *Restriction on the parameters.* The second restriction  $s_c < 3/2$  in (1.5) is essential in the proof of Theorem 1. In particular it is used to ensure that the second integrand in the definition of the resolvent map (3.12) is integrable (see (3.9)). Note also that in Theorem 2, there is a further restriction on the parameters:

$$d = 3 \iff p > 5.$$

This is due to the poor regularity of the nonlinearity. In particular, the nonlinearity  $\Phi \mapsto |\Phi|^{p-1}\Phi$  has  $C^{\lfloor p \rfloor}$  regularity for  $p \notin 2\mathbb{N} + 1$ . The role of this constraint is to allow us to take  $k \leq \lfloor p \rfloor - 1$  derivatives when closing the nonlinear estimates. We are only able to take one less derivative than the regularity of  $|\Phi|^{p-1}\Phi$  since the Lipschitz dependence of the nonlinear term on  $\Phi$  in the weighted  $H^k$  space means we lose one more power in the nonlinear term (see Lemma D.1). Furthermore, we require  $k \geq d/2$  by Sobolev embedding which is what we use to bound the nonlinear term. Since (1.5) implies that  $p - 1 \ll 1$  for large values of  $d$ , the codimensional stability result cannot be generalized into higher dimensions. Also, note that the constraint  $p + 1 > s_c$  which is implied by (1.5) is essential in the development of the local theory (see [12] for the related well-posedness result).

(3) *Manifold structure of the initial data.* Let

$$B_\varepsilon^{\mathbb{H}} = \{X \mid \|X\|_{\mathbb{H}} < \varepsilon\}, \quad B_\delta^H = \{X \mid \|X\|_H < \delta\}$$

with  $\varepsilon, \delta \ll 1$ , where

$$\mathbb{H} = H_4 \times H_3,$$

where the spaces  $H_k$  are defined in Section 2 and  $H$  is the weighted  $W^{k,\infty}$ -space defined in the proof of Proposition 8.1. Consider the self-similar profile and the dampened profile in self-similar variables:

$$(1.10) \quad P_n = (u_n(\rho), \Lambda u_n(\rho)), \quad P_n^D = (\eta(e^{-s_0}\rho)u_n(\rho), \eta(e^{-s_0}\rho)\Lambda u_n(\rho)),$$

where  $\eta$  is a smooth, rapidly decaying function defined in (8.1). Profiles are dampened to achieve finite energy. We then, construct the finite codimensional manifold of initial

data in Theorem 2 as follows: consider a direct sum decomposition

$$\mathbb{H} = U \oplus V$$

into subspaces  $U$  and  $V$  stable and unstable under the semigroup action of the linearized operator with  $\dim V < \infty$ . Then consider the Lipschitz map

$$\Phi : B_\varepsilon^{\mathbb{H}} \cap (B_\delta^H + P_n^D - P_n) \cap U \longrightarrow V$$

obtained by solving a Brouwer type fixed point problem and a linear map  $\Xi : V \rightarrow U$  on the finite dimensional space  $V$  such that

$$\text{Id} + \Xi : V \longrightarrow (B_\delta^H + P_n^D - P_n).$$

Then, the finite codimensional manifold can be realized as

$$\mathcal{M} = P_n + (\text{Id} + (\text{Id} + \Xi) \circ \Phi) (B_\varepsilon^{\mathbb{H}} \cap (B_\delta^H + P_n^D - P_n) \cap U) \subset H + P_n^D.$$

Note that the modifier  $\Xi$  is there to ensure that our initial data does not leave the neighbourhood  $H + P_n^D$  which is essential in obtaining finite energy initial data. Also, in Lemma D.1, it is proved that  $\Phi$  is a Lipschitz map with respect to the topology of  $\mathbb{H}$ . Similar properties of the stable manifold are proved in [13], [7], [5].

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## 2. NOTATIONS

Let us introduce some notations before we start. We write for the generator of scaling operator  $\Lambda$ :

$$\Lambda = \alpha + y \cdot \nabla, \quad \alpha := \frac{2}{p-1}.$$

We will denote by  $(t, x)$  the original variables and  $(s, y)$  for the self-similar variables:

$$s = -\log(T-t), \quad y = \frac{x}{T-t}$$

and denote their modulus:

$$r = |x|, \quad \rho = |y|.$$

For real-valued functions  $f$  and  $g$ , we write

$$f \propto g$$

if there exists  $\lambda \in \mathbb{R}$  such that  $f = \lambda g$ .

Define the Japanese bracket

$$\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}.$$

We also write

$$\nabla^j = \begin{cases} \Delta^i & j = 2i, \\ \nabla \Delta^i & j = 2i + 1, \end{cases}$$

and for scalar (or vector) valued functions  $f, g$  on  $\mathbb{R}^d$ ,

$$(f, g) = \int_{\mathbb{R}^d} f \cdot g \, dy.$$

Now fix  $d = 3$ . Let  $\chi \in C_c^\infty(\mathbb{R}^3, [0, \infty))$  be a radial smooth cut-off function with

$$\chi(y) = \begin{cases} 1 & |y| \leq 1, \\ 0 & |y| \geq 2. \end{cases}$$

For  $k \in \mathbb{N}$ , denote by  $H_k$  the completion of  $C_c^\infty(\mathbb{R}^3)$  with respect to the norm induced by the inner product

$$(\Psi, \tilde{\Psi})_{H_k} = (\nabla^k \Psi, \nabla^k \tilde{\Psi}) + \int_{\mathbb{R}^3} \chi \Psi \tilde{\Psi} dy.$$

### 3. CONSTRUCTION OF EXTERIOR SOLUTIONS

Our aim in this section is to construct a family of outer solutions to the self-similar equation (1.4). The key is that the outer spectral problem, including the singularity through the renormalized light cone  $\rho = 1$ , is explicit.

We introduce relevant notations for this section.

*Linearized operator.* — Recall the generator of scaling operator  $\Lambda$ :

$$\Lambda = \alpha + y \cdot \nabla.$$

Introduce the linearized operator

$$(3.1) \quad \mathcal{L}_\infty = (1 - \rho^2) \frac{d^2}{d\rho^2} + \left[ \frac{d-1}{\rho} - 2(1 + \alpha)\rho \right] \frac{d}{d\rho} - \alpha(1 + \alpha) + p\alpha(d - 2 - \alpha)\rho^{-2}$$

for (1.4) near the singular solution  $u = u_\infty$  where we recall

$$u_\infty(\rho) = b_\infty \rho^{-\alpha}, \quad b_\infty = (\alpha(d - 2 - \alpha))^{\alpha/2}.$$

Also, let

$$\omega = \sqrt{pb_\infty^{p-1} - \frac{(d-2)^2}{4}}.$$

Note that  $\omega \in \mathbb{R}$  if

$$(3.2) \quad 1 + \frac{4}{d-2} < p < p_{JL} := \begin{cases} \infty & \text{for } d \leq 10, \\ 1 + \frac{4}{d-4-2\sqrt{d-1}} & \text{for } d \geq 11, \end{cases}$$

with sufficient condition being  $1 < s_c < 3/2$ .  $p_{JL}$  is known as the Joseph-Lundgren exponent.

*Hypergeometric functions.* — We denote by  ${}_2F_1$  the Gauss hypergeometric functions:

$$(3.3) \quad {}_2F_1(a, b, c; z) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where  $(a)_n = a(a + 1) \cdots (a + n - 1)$ .



3.1. FUNDAMENTAL SOLUTIONS AND EXTERIOR RESOLVENT. — Recall the definition of linearized operator  $\mathcal{L}_\infty$  above. In this section, we compute the fundamental solutions of the linearized operator  $\mathcal{L}_\infty$  and use calculus of variation to invert  $\mathcal{L}_\infty$  in a suitable space of functions.

LEMMA 3.1 (Fundamental solutions of  $\mathcal{L}_\infty$ ). — Recall the definition (1.2) of  $s_c$ .

(i) Interior solution: In the region  $\rho \in (0, 1)$ , the homogeneous equation  $\mathcal{L}_\infty(\psi) = 0$  has a basis of solutions

$$(3.4) \quad \begin{aligned} \psi_1^L &= \operatorname{Re} \left[ \rho^{1-(d)/2+i\omega} {}_2F_1 \left( \frac{1-s_c+i\omega}{2}, \frac{2-s_c+i\omega}{2}, 1+i\omega, \rho^2 \right) \right], \\ \psi_2^L &= \operatorname{Im} \left[ \rho^{1-(d)/2+i\omega} {}_2F_1 \left( \frac{1-s_c+i\omega}{2}, \frac{2-s_c+i\omega}{2}, 1+i\omega, \rho^2 \right) \right]. \end{aligned}$$

(ii) Exterior solution: In the region  $\rho \in (1, \infty)$ , the homogeneous equation  $\mathcal{L}_\infty(\psi) = 0$  has a basis of solutions

$$(3.5) \quad \begin{aligned} \psi_1^R &= \rho^{-\alpha-1} {}_2F_1 \left( \frac{2-s_c-i\omega}{2}, \frac{2-s_c+i\omega}{2}, \frac{3}{2}, \rho^{-2} \right), \\ \psi_2^R &= \rho^{-\alpha} {}_2F_1 \left( \frac{1-s_c-i\omega}{2}, \frac{1-s_c+i\omega}{2}, \frac{1}{2}, \rho^{-2} \right). \end{aligned}$$

*Proof.* — For  $\rho \in (0, 1)$ , consider solutions of the form  $\psi = \rho^\gamma \sum_{n=0}^\infty a_n \rho^n$  for  $(a_n)_{n=0}^\infty$  bounded sequence in  $\mathbb{R}$  with  $a_0 \neq 0$  so the sum is absolutely convergent in  $(0, 1)$ . Then

$$\begin{aligned} \mathcal{L}_\infty(\psi) &= [\gamma(\gamma + d - 2) + pb_\infty^{p-1}]a_0\rho^{\gamma-2} + [(\gamma + 1)(\gamma + d - 1) + pb_\infty^{p-1}]a_1\rho^{\gamma-1} \\ &\quad + \sum_{n=0}^\infty \rho^{\gamma+n} \left\{ [(\gamma + n + 2)(\gamma + n + d) + pb_\infty^{p-1}]a_{n+2} \right. \\ &\quad \left. - [(\gamma + n)(\gamma + n + 1 + 2\alpha) + \alpha(1 + \alpha)]a_n \right\}. \end{aligned}$$

Equating first two terms to 0, we infer  $\gamma = 1 - (d/2) \pm i\omega$  and  $a_1 = 0$ . Equating higher order terms to 0,

$$a_{n+2} = \frac{(\gamma + n + \alpha)(\gamma + n + 1 + \alpha)}{(\gamma + n + (d/2) + 1 + i\omega)(\gamma + n + (d/2) + 1 - i\omega)} a_n.$$

The cases  $\gamma = 1 - (d/2) + i\omega$  and  $1 - (d/2) - i\omega$  give rise to complex conjugate solutions. Thus, real and imaginary parts of the complex solution satisfying the recursion relation above:

$$\rho^{1-(d)/2+i\omega} {}_2F_1 \left( \frac{1-s_c+i\omega}{2}, \frac{2-s_c+i\omega}{2}, 1+i\omega, \rho^2 \right)$$

yield two linearly independent real solutions. In the region  $(1, \infty)$ , consider solutions of the form  $\psi = \rho^{-\gamma} \sum_{n=0}^\infty a_n \rho^{-n}$  and proceed as in the region  $(0, 1)$ . □

We now investigate the regularity of the fundamental solutions at the singular point  $\rho = 1$ . First, we recall some results on the singular ODEs.

**PROPOSITION 3.2** (Solutions to singular ODEs, [31]). — *Let  $f \in C^m([0, T], \mathbb{R}^n)$ ,  $A \in C^m([0, T], \mathbb{R}^{n \times n})$  for an  $m \geq 1$ ,  $m > \max_{\lambda_k \in \sigma(A(0))} \operatorname{Re}(\lambda_k)$ , where  $\sigma(M)$  denotes the spectrum of a matrix  $M$ . Suppose we have  $1 \leq \ell \leq m$  such that*

$$\sigma(A(0)) \cap \{\ell, \ell + 1, \dots\} = \emptyset$$

and vectors  $u_0^{(0)}, \dots, u_0^{(\ell-1)} \in \mathbb{R}^n$  such that

$$(3.6) \quad (kI - A(0))u_0^{(k)} = f^{(k)}(0) + \sum_{j=0}^{k-1} \binom{k}{j} A^{(k-j)}(0)u_0^{(j)}, \quad k = 0, \dots, \ell - 1$$

holds, then there exists a unique solution  $u \in C^m([0, T], \mathbb{R}^n)$  of the problem

$$tu'(t) = A(t)u(t) + f(t), \quad 0 < t \leq T, \quad u^{(j)}(0) = u_0^{(j)}, \quad j = 0, \dots, \ell - 1.$$

**COROLLARY 3.3.** — *There exists a unique  $\psi_1 \in C^1((0, \infty))$  solution to  $\mathcal{L}_\infty(\psi) = 0$  with  $\psi(1) = 1$ . Moreover,  $\psi_1$  is smooth.*

*Proof.* — We write  $\mathcal{L}_\infty(\psi) = 0$  in the form required by Proposition 3.2, so for  $(\Psi_1, \Psi_2) = (\psi, \partial_\rho \psi)$ ,

$$\begin{cases} (\rho - 1)\partial_\rho \Psi_1 = (\rho - 1)\Psi_2, \\ (\rho - 1)\partial_\rho \Psi_2 = \frac{1}{1 + \rho} \left[ \frac{p\alpha(d - 2 - \alpha)}{\rho^2} - \alpha(1 + \alpha) \right] \Psi_1 + \frac{1}{1 + \rho} \left[ \frac{d - 1}{\rho} - 2(1 + \alpha)\rho \right] \Psi_2. \end{cases}$$

Hence, we can write for some constant  $c(\alpha)$ ,

$$(\rho - 1)\partial_\rho \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = A(\rho) \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \quad A(0) = \begin{pmatrix} 0 & 0 \\ c(\alpha) & s_c - 3/2 \end{pmatrix}$$

for  $A$  smooth in  $(0, \infty)$ . Then since  $\sigma(A(1)) = \{s_c - 3/2, 0\}$ , by Proposition 3.2 with the values  $\ell = 1$ ,  $f \equiv 0$  and  $n = 2$ , we infer for  $a \in \mathbb{R}$ , there exists a unique  $\psi_a \in C^1((0, \infty))$  solving  $\mathcal{L}_\infty(\psi_a) = 0$  with

$$(\psi_a(1), \psi'_a(1)) = a(1, c(\alpha)/(s_c - 3/2))$$

and in fact,  $\psi_a \in C^\infty((0, \infty))$  so done by setting  $a = 1$ . □

For  $0 < \rho_0 < 1$ , define the spaces of functions on which we invert our linearized operator  $\mathcal{L}_\infty$ :

$$(3.7) \quad \begin{aligned} X_{\rho_0} &= \left\{ w : (\rho_0, \infty) \rightarrow \mathbb{R} \mid \|w\|_{X_{\rho_0}} := \sup_{\rho_0 \leq \rho \leq 1} \rho^{(d/2)-1} |w| + \sup_{\rho \geq 1} \rho^{\alpha+1} |w| < \infty \right\}, \\ Y_{\rho_0} &= \left\{ w : (\rho_0, \infty) \rightarrow \mathbb{R} \mid \|w\|_{Y_{\rho_0}} := \int_{\rho_0}^1 \frac{\rho^{d/2}}{|1 - \rho|^{s_c-1/2}} |w| + \int_1^\infty \frac{\rho^{(d-1)/2}}{|1 - \rho|^{s_c-1/2}} |w| < \infty \right\}. \end{aligned}$$

**PROPOSITION 3.4** (Exterior resolvent)

(i) **Basis of fundamental solutions:** *There exists  $\psi_2$  given by*

$$(3.8) \quad \psi_2 := \begin{cases} c_1 \psi_1^L & \text{if } \rho \in (0, 1), \\ c_2 \psi_1^R & \text{if } \rho \in (1, \infty), \end{cases}$$

for some  $c_i \in \mathbb{R}$  which is linearly independent of the smooth homogeneous solution  $\psi_1$  found in Corollary 3.3 and with the Wronskian given by

$$(3.9) \quad W := \psi_1' \psi_2 - \psi_2' \psi_1 = \rho^{1-d} |1 - \rho^2|^{s_c-3/2}.$$

The fundamental solutions have asymptotic behaviors:

$$(3.10) \quad \psi_i \propto \rho^{1-d/2} \sin(\omega \log \rho + \delta_i) [1 + \mathcal{O}_{\rho \rightarrow 0}(\rho^2)]$$

and

$$(3.11) \quad \rho^{-1} \psi_1, \psi_2, \Lambda \psi_1 \propto \rho^{-\alpha-1} [1 + \mathcal{O}_{\rho \rightarrow \infty}(\rho^{-1})]$$

for some  $\delta_i \in \mathbb{R}$ .

(ii) **Continuity of the resolvent:** *There exists a bounded linear operator  $\mathcal{T}: Y_{\rho_0} \rightarrow X_{\rho_0}$  such that  $\mathcal{L}_\infty \circ \mathcal{T} = \text{id}_{Y_{\rho_0}}$  given by*

$$(3.12) \quad \mathcal{T}(f) = -\psi_1 \int_\rho^\infty \frac{f \psi_2}{(1-r^2)W} dr - \psi_2 \int_1^\rho \frac{f \psi_1}{(1-r^2)W} dr$$

with  $\|\mathcal{T}\|_{\mathcal{L}(Y_{\rho_0}, X_{\rho_0})} \lesssim 1$  for all  $\rho_0 \in (0, 1)$ .

*Proof*

(i) Since  $\mathcal{L}_\infty(\psi_i^L) = 0$  and  $\mathcal{L}_\infty(\psi_i^R) = 0$ , we have from the definition of the Wronskian that

$$(1 - \rho^2)W' + \left[ \frac{d-1}{\rho} - 2(1 + \alpha)\rho \right] W = 0, \quad \rho \in (0, \infty) \setminus \{1\}.$$

Then  $W \propto \rho^{1-d} |1 - \rho^2|^{s_c-3/2}$  in  $(0, 1)$ . Also, in view of the asymptotic behavior of the hypergeometric functions at  $\rho = 1$  (see [27, §15.4(ii)]),  $\partial_\rho \psi_1^L$  is singular. Then,  $\psi_1^L$  and  $\psi_1$  are linearly independent, so there exists  $c_1 \in \mathbb{R}$  such that (3.9) holds. Similarly,  $W \propto \rho^{1-d} |1 - \rho^2|^{s_c-3/2}$  in  $(1, \infty)$  and  $\psi_1^R$  and  $\psi_1$  are linearly independent, so we can choose  $c_2$  with (3.9). The asymptotic behaviors then follow from the definitions (3.4).

(ii) Integrals in (3.12) are well-defined since

$$\psi_1 = \begin{cases} \mathcal{O}_{\rho \rightarrow 1}(1), \\ \mathcal{O}_{\rho \rightarrow \infty}(\rho^{-\alpha}), \end{cases} \quad \psi_2 = \begin{cases} \mathcal{O}_{\rho \rightarrow 1}(1), \\ \mathcal{O}_{\rho \rightarrow \infty}(\rho^{-\alpha-1}), \end{cases} \quad \frac{1}{(1 - \rho^2)W} = \begin{cases} \mathcal{O}_{\rho \rightarrow 1}(|\rho - 1|^{(1/2)-s_c}), \\ \mathcal{O}_{\rho \rightarrow \infty}(\rho^{2\alpha}), \end{cases}$$

(see [27, §15.4(ii)]). Using variation of constants,

$$w = \psi_1 \left( a_1 - \int_\rho^\infty \frac{f \psi_2}{(1-r^2)W} dr \right) - \psi_2 \left( a_2 + \int_1^\rho \frac{f \psi_1}{(1-r^2)W} dr \right)$$

solves

$$\mathcal{L}_\infty(w) = f.$$

Since we require  $\mathcal{T} : Y_{\rho_0} \rightarrow X_{\rho_0}$ , we choose  $a_1 = 0$ . Since  $\psi'_2 = \mathcal{O}(|\rho - 1|^{s_c - 3/2})$  as  $\rho \rightarrow 1$  (see [27, §15.4(ii)]), by requiring  $\mathcal{T}(f)$  to be differentiable at  $\rho = 1$  we take  $a_2 = 0$ . It suffices to prove that  $\mathcal{T}$  is bounded. For all  $\rho \geq 1$ ,

$$\begin{aligned} \rho^{1+\alpha} |\mathcal{T}(f)(\rho)| &\lesssim \rho^{1+\alpha} \left( |\psi_1| \int_{\rho}^{\infty} \left| \frac{f\psi_2}{(1-r^2)W} \right| dr + |\psi_2| \int_1^{\rho} \left| \frac{f\psi_1}{(1-r^2)W} \right| dr \right) \\ &\lesssim \sup_{\rho \geq 1} \left( \rho \int_{\rho}^{\infty} r^{(d-3)/2} (r-1)^{(1/2)-s_c} |f| dr + \int_1^{\rho} r^{(d-1)/2} (r-1)^{(1/2)-s_c} |f| dr \right) \lesssim \|f\|_{Y_{\rho_0}}. \end{aligned}$$

For all  $\rho$  such that  $\rho_0 \leq \rho \leq 1$ ,

$$\begin{aligned} \rho^{(d/2)-1} |\mathcal{T}(f)(\rho)| &\lesssim \rho^{(d/2)-1} \left( |\mathcal{T}(f)(1)| + |\psi_1| \int_{\rho}^1 \left| \frac{f\psi_2}{(1-r^2)W} \right| dr + |\psi_2| \int_{\rho}^1 \left| \frac{f\psi_1}{(1-r^2)W} \right| dr \right) \\ &\lesssim \|f\|_{Y_{\rho_0}} + \sup_{\rho_0 \leq r \leq 1} \int_r^1 s^{d/2} (s-1)^{(1/2)-s_c} |f| ds \lesssim \|f\|_{Y_{\rho_0}}, \end{aligned}$$

where in the final inequality, we used  $\psi_i = \mathcal{O}(\rho^{1-d/2})$  and  $1/(1-\rho^2)W = \mathcal{O}(\rho^{d-1})$  as  $\rho \rightarrow 0$ . Thus,  $\|\mathcal{T}(f)\|_{X_{\rho_0}} \lesssim \|f\|_{Y_{\rho_0}}$ .  $\square$

3.2. EXTERIOR SOLUTIONS. — We now solve (1.4) in the exterior region  $\rho > \rho_0$  as a fixed point problem involving  $\mathcal{L}_{\infty}$ . We first prove a Lipschitz type bound on the nonlinear term.

LEMMA 3.5 (Non-linear bounds). — For  $w \in X_{\rho_0}$  and  $\varepsilon > 0$ , define

$$(3.13) \quad G[\psi_1, \varepsilon]w = \underbrace{(\psi_1 + w)^2}_{:=A[\psi_1]w} \underbrace{\int_0^1 (1-s)(u_{\infty} + s\varepsilon(\psi_1 + w))^{p-2} ds}_{:=B[\psi_1, \varepsilon]w}.$$

Then for all  $\varepsilon \ll \rho_0^{s_c-1}$  and  $w_1, w_2 \in B_{X_{\rho_0}} = \{w \in X_{\rho_0} \mid \|w\|_{X_{\rho_0}} < 1\}$ ,

$$(3.14) \quad \begin{aligned} \|G[\psi_1, \varepsilon]w_1\|_{Y_{\rho_0}} &\lesssim \rho_0^{1-s_c}, \\ \|G[\psi_1, \varepsilon]w_1 - G[\psi_1, \varepsilon]w_2\|_{Y_{\rho_0}} &\lesssim \rho_0^{1-s_c} \|w_1 - w_2\|_{X_{\rho_0}}. \end{aligned}$$

Proof. — Note that for all  $\rho \geq 1$ ,

$$|\psi_1(\rho)| + |w_1(\rho)| \lesssim |u_{\infty}(\rho)|.$$

Since  $\psi_1 = \mathcal{O}(\rho^{-\alpha})$  as  $\rho \rightarrow \infty$  and  $\varepsilon \lesssim 1$ ,

$$|G[\psi_1, \varepsilon]w_1(\rho)| \lesssim (|\psi_1| + |w_1|)^2 \left[ |u_{\infty}| + \varepsilon(|\psi_1| + |w_1|) \right]^{p-2} \lesssim |u_{\infty}|^p \lesssim \rho^{-\alpha-2},$$

so

$$\int_1^{\infty} \rho^{(d-1)/2} |1 - \rho|^{(1/2)-s_c} |G[\psi_1, \varepsilon]w_1| d\rho \lesssim \int_1^{\infty} \rho^{s_c-5/2} |1 - \rho|^{(1/2)-s_c} d\rho \lesssim 1.$$

Note that since  $\psi_1 = \mathcal{O}(\rho^{1-d/2})$  as  $\rho \rightarrow 0$ , for all  $\rho_0 \leq \rho \leq 1$ ,

$$|\psi_1(\rho)| + |w_1(\rho)| \lesssim \rho^{1-d/2} \lesssim \rho^{1-s_c} |u_{\infty}(\rho)|.$$

Then since  $\varepsilon \ll \rho_0^{s_c-1}$ ,

$$|G[\psi_1, \varepsilon]w_1| \lesssim \rho^{2-d} \{(1 + \varepsilon \rho^{1-s_c})|u_\infty(\rho)|\}^{p-2} \lesssim \rho^{\alpha-d}.$$

Then

$$\int_{\rho_0}^1 \rho^{d/2} (1 - \rho)^{(1/2)-s_c} |G[\psi_1, \varepsilon]w_1| d\rho \lesssim \int_{\rho_0}^1 \rho^{-s_c} (1 - \rho)^{(1/2)-s_c} d\rho \lesssim \rho_0^{1-s_c}.$$

Hence, the first bound in (3.14) holds. For the contraction estimate, recall the definition of nonlinear functionals  $A$  and  $B$  in (3.13) and observe that

$$\begin{aligned} |G[\psi_1, \varepsilon]w_1 - G[\psi_1, \varepsilon]w_2| &\leq |Aw_1 - Aw_2| |Bw_1| + |Aw_2| |Bw_1 - Bw_2| \\ &\lesssim |2\psi_1 + w_1 + w_2| |w_1 - w_2| \left[ |u_\infty| + \varepsilon(|\psi_1| + |w_1|) \right]^{p-2} \\ &\quad + \varepsilon |w_1 - w_2| (\psi_1 + w_2)^2 I_{w_1, w_2}, \end{aligned}$$

where

$$\begin{aligned} I_{w_1, w_2} &:= \left| \int_0^1 \varepsilon^{-1} \partial_w B[\psi_1, \varepsilon]w \Big|_{w_2 + \sigma(w_1 - w_2)} d\sigma \right| \\ &\lesssim \left| \int_0^1 s(1-s) \int_0^1 (u_\infty + s\varepsilon(\psi_1 + w_2) + \sigma\varepsilon(w_1 - w_2))^{p-3} d\sigma ds \right| \\ &\lesssim \left[ |u_\infty| + \varepsilon(|\psi_1| + |w_1| + |w_2|) \right]^{p-3} \lesssim u_\infty^{p-3}, \end{aligned}$$

where the final inequality follows since  $\varepsilon \ll \rho_0^{s_c-1}$ . Then

$$\begin{aligned} |G[\psi_1, \varepsilon]w_1 - G[\psi_1, \varepsilon]w_2| &\lesssim \left[ (|\psi_1| + |w_1| + |w_2|)|u_\infty|^{p-2} + \varepsilon(|\psi_1| + |w_2|)^2 |u_\infty|^{p-3} \right] |w_1 - w_2|. \end{aligned}$$

Since  $\psi_1 = \mathcal{O}(\rho^{-\alpha})$  as  $\rho \rightarrow \infty$ ,

$$\begin{aligned} \int_1^\infty \rho^{(d-1)/2} |1 - \rho|^{(1/2)-s_c} |G[\psi_1, \varepsilon]w_1 - G[\psi_1, \varepsilon]w_2| d\rho &\lesssim \int_1^\infty \rho^{s_c-7/2} |1 - \rho|^{(1/2)-s_c} d\rho \|w_1 - w_2\|_{X_{\rho_0}}. \end{aligned}$$

Since  $\psi_1 = \mathcal{O}(\rho^{1-d/2})$  as  $\rho \rightarrow 0$ , for all  $\rho$  such that  $\rho_0 \leq \rho \leq 1$ ,

$$\begin{aligned} |G[\psi_1, \varepsilon]w_1 - G[\psi_1, \varepsilon]w_2| &\lesssim \left( \rho^{2(1-d/2)-\alpha(p-2)} + \varepsilon \rho^{3(1-d/2)-\alpha(p-3)} \right) \sup_{\rho_0 \leq r \leq 1} r^{(d/2)-1} |w_1 - w_2| \\ &\lesssim \rho^{\alpha-d} \|w_1 - w_2\|_{X_{\rho_0}}, \end{aligned}$$

where the final inequality holds because  $\varepsilon \ll \rho_0^{s_c-1}$ . Thus,

$$\int_{\rho_0}^1 \rho^{d/2} |1 - \rho|^{(1/2)-s_c} |G[\psi_1, \varepsilon]w_1 - G[\psi_1, \varepsilon]w_2| d\rho \lesssim \rho_0^{1-s_c} \|w_1 - w_2\|_{X_{\rho_0}}.$$

Hence, the second bound in (3.14) holds. □

We are now in position to solve (1.4). We in particular, prove the existence of a one-parameter family of smooth solutions in the region  $\rho > \rho_0$ .

**PROPOSITION 3.6** (Exterior solutions). — *For all  $0 < \varepsilon \ll \rho_0^{s_c-1}$ , there exists a smooth solution to (1.4) of the form*

$$u = u_\infty + \varepsilon(\psi_1 + w)$$

with

$$(3.15) \quad \|w\|_{X_{\rho_0}} \lesssim \varepsilon \rho_0^{1-s_c}, \quad \|\Lambda w\|_{X_{\rho_0}} \lesssim \varepsilon \rho_0^{1-s_c}.$$

Furthermore,

$$w|_{\varepsilon=0} = 0, \quad \|\partial_\varepsilon w|_{\varepsilon=0}\|_{X_{\rho_0}} \lesssim \rho_0^{1-s_c}.$$

*Proof.* — The function  $u = u_\infty + \varepsilon v > 0$  solves (1.4) if and only if

$$\begin{aligned} \mathcal{L}_\infty(v) &= \varepsilon^{-1}[u_\infty^p + p u_\infty^{p-1} \varepsilon v - (u_\infty + \varepsilon v)^p] \\ &= -p(p-1)\varepsilon v^2 \int_0^1 (1-s)(u_\infty + s\varepsilon v)^{p-2} ds. \end{aligned}$$

We further decompose  $v = \psi_1 + w$ . Since  $\mathcal{L}_\infty(\psi_1) = 0$ , it suffices to construct  $w$  satisfying

$$(3.16) \quad w = -p(p-1)\varepsilon \mathcal{T} \circ G[\psi_1, \varepsilon]w.$$

Lemma 3.5 together with Proposition 3.4 states precisely that for  $\varepsilon \ll \rho_0^{s_c-1}$ ,

$$-p(p-1)\varepsilon \mathcal{T} \circ G[\psi_1, \varepsilon] : B_{X_{\rho_0}} \longrightarrow B_{X_{\rho_0}}$$

is a contraction map. From the Banach fixed point theorem, there exists a unique solution  $w$  to (3.16) with  $\|w\|_{X_{\rho_0}} \lesssim \varepsilon \rho_0^{1-s_c}$ . Clearly,  $w$  is smooth in  $(0, \infty) \setminus \{1\}$ . In view of (3.16),  $w \in C^1((0, \infty))$  so  $u \in C^1((0, \infty))$ . Writing (1.4) in the form required by Proposition 3.2, for  $(\Psi_1, \Psi_2) = (u, u')$ ,

$$\begin{cases} (\rho-1)\partial_\rho \Psi_1 = (\rho-1)\Psi_2, \\ (\rho-1)\partial_\rho \Psi_2 = -\frac{\alpha(\alpha+1)}{1+\rho}\Psi_1 + \frac{1}{1+\rho}\left[\frac{d-1}{\rho} - 2(\alpha+1)\rho\right]\Psi_2 + \frac{u^p}{1+\rho}. \end{cases}$$

Hence,

$$(\rho-1)\partial_\rho \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = A(\rho) \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} + \frac{1}{\rho+1} \begin{pmatrix} 1 \\ u^p \end{pmatrix},$$

where  $A$  is smooth in  $(0, \infty)$  and

$$A(1) = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ -\alpha(\alpha+1) & 2s_c-3 \end{pmatrix},$$

with  $\sigma(A(1)) = \{s_c - 3/2, 0\}$ . By Proposition 3.2, since  $u \in C^1((0, \infty))$ ,  $(u, u') \in C^1((0, \infty))$  so  $u \in C^2((0, \infty))$ . Iterating this, we conclude that  $u$  is smooth.

Applying  $\Lambda$  to (3.16), we infer

$$\Lambda w = -p(p-1)\varepsilon \left[ (\Lambda\psi_1) \int_\rho^\infty \frac{G[\psi_1, \varepsilon](w)\psi_2}{(1-r^2)W} dr - (\Lambda\psi_2) \int_1^\rho \frac{G[\psi_1, \varepsilon](w)\psi_1}{(1-r^2)W} dr \right].$$

Hence, by considering the asymptotes of  $\Lambda\psi_i$  and proceeding as in the proof of Proposition 3.4, we infer

$$\|\Lambda w\|_{X_{\rho_0}} \lesssim \varepsilon \|G[\psi_1, \varepsilon]w\|_{Y_{\rho_0}} \lesssim \varepsilon \rho_0^{1-s_c}.$$

In view of (3.16),  $w|_{\varepsilon=0} = 0$ . Differentiating (3.16) in  $\varepsilon$ ,

$$\begin{aligned} \partial_\varepsilon w|_{\varepsilon=0} &= -p(p-1) \left( \mathcal{T} \circ G[\psi_1, 0]w|_{\varepsilon=0} + \varepsilon \mathcal{T}(\partial_\varepsilon G[\psi_1, \varepsilon]w)|_{\varepsilon=0} \right) \\ &= -p(p-1) \mathcal{T} \circ G[\psi_1, 0]w|_{\varepsilon=0} = -\frac{p(p-1)}{2} \mathcal{T}(u_\infty^{p-2}\psi_1^2), \end{aligned}$$

so by continuity of the resolvent and the asymptotic behaviour of  $\psi_1$  as  $\rho \rightarrow 0$  and  $\rho \rightarrow \infty$ ,

$$\begin{aligned} \|\partial_\varepsilon w|_{\varepsilon=0}\|_{X_{\rho_0}} &\lesssim \|u_\infty^{p-2}\psi_1^2\|_{Y_{\rho_0}} \\ &\lesssim \int_{\rho_0}^1 \rho^{-s_c} |1 - \rho|^{(1/2)-s_c} d\rho + \int_1^\infty \rho^{s_c-5/2} |1 - \rho|^{(1/2)-s_c} d\rho \lesssim \rho_0^{1-s_c}. \quad \square \end{aligned}$$

#### 4. CONSTRUCTION OF INTERIOR SOLUTIONS

In this section, we construct inner solutions to the self-similar equation (1.4) which are perturbations of a rescaled soliton. The steps are similar to that of the previous section.

Let us first introduce some notations for this section.

*Linearized operator.* — Recall the definition of soliton solution

$$(4.1) \quad \Delta Q + Q^p = 0, \quad Q(\rho) = b_\infty \rho^{-\alpha} + \mathcal{O}_{\rho \rightarrow \infty}(\rho^{1-d/2}).$$

We let the linearized operator  $\mathcal{H}_\infty$  near

$$Q_\lambda(\rho) := \lambda^{-\alpha} Q(\rho/\lambda), \quad \lambda > 0.$$

for the profile equation (1.4) be

$$(4.2) \quad \mathcal{H}_\infty = -\Delta - pQ^{p-1} = -\frac{d^2}{d\rho^2} - \frac{d-1}{\rho} \frac{d}{d\rho} - pQ^{p-1}.$$

LEMMA 4.1 (Fundamental solutions of  $\mathcal{H}_\infty$ ). — *Recall from above the definition of the soliton  $Q$ . We then have a basis of fundamental solutions*

$$\mathcal{H}_\infty(\Lambda Q) = 0, \quad \mathcal{H}_\infty \varphi = 0$$

*with the following asymptotic behavior as  $\rho \rightarrow \infty$ :*

$$(4.3) \quad \Lambda Q, \varphi \propto \rho^{1-d/2} \sin(\omega \log \rho + \delta_\bullet) + \mathcal{O}(\rho^{2-d+\alpha})$$

*for some  $\delta_{\Lambda Q}, \delta_\varphi \in \mathbb{R}$ . By scaling  $\varphi$  if necessary, we assume that the Wronskian is given by*

$$W := (\Lambda Q)' \varphi - \varphi' \Lambda Q = -\rho^{1-d}.$$

*Proof.* — Recall the definition of  $Q_\lambda$  above. Then, for all  $\lambda > 0$ ,

$$\Delta Q_\lambda + Q_\lambda^p = 0$$

and differentiating with respect to  $\lambda$  and evaluating at  $\lambda = 1$  yields  $\mathcal{H}_\infty(\Lambda Q) = 0$ . Let  $\varphi$  be another solution to  $\mathcal{H}_\infty(\varphi) = 0$  which does not depend linearly on  $\Lambda Q$ , we aim at deriving the asymptotic of both  $\Lambda Q$  and  $\varphi$  as  $\rho \rightarrow \infty$ . We first solve

$$(4.4) \quad -\tilde{\varphi}'' - \frac{d-1}{\rho} \tilde{\varphi}' - \frac{pb_\infty^{p-1}}{\rho^2} \tilde{\varphi} = f.$$

The homogeneous problem admits the explicit basis of solutions

$$(4.5) \quad \varphi_1 = \rho^{1-d/2} \sin(\omega \log \rho), \quad \varphi_2 = \rho^{1-d/2} \cos(\omega \log \rho),$$

and the corresponding Wronskian is given by

$$W = \varphi_1' \varphi_2 - \varphi_2' \varphi_1 = \omega \rho^{1-d}.$$

Using the variation of constants, the solutions to (4.4) are given by

$$\tilde{\varphi}(\rho) = \varphi_1 \left( a_1 + \int_\rho^\infty f \varphi_2 \frac{r^{d-1}}{\omega} dr \right) + \varphi_2 \left( a_2 - \int_\rho^\infty f \varphi_1 \frac{r^{d-1}}{\omega} dr \right).$$

Then, we rewrite the equation  $\mathcal{H}_\infty(\varphi) = 0$ :

$$-\varphi'' - \frac{2}{\rho} \varphi' - \frac{pb_\infty^{p-1}}{\rho^2} \varphi = p \left( Q^{p-1} - \frac{b_\infty^{p-1}}{\rho^2} \right) \varphi,$$

and hence

$$(4.6) \quad \varphi = a_1 \varphi_1 + a_2 \varphi_2 + \tilde{\phi}, \quad \tilde{\phi} = \mathcal{G}(\tilde{\phi})$$

where

$$\begin{aligned} \mathcal{G}(\tilde{\phi})(\rho) &= \varphi_1 \int_\rho^\infty p \left( Q^{p-1} - \frac{b_\infty^{p-1}}{r^2} \right) (a_1 \varphi_1 + a_2 \varphi_2 + \tilde{\phi}) \varphi_2 \frac{r^{d-1}}{\omega} dr \\ &\quad - \varphi_2 \int_\rho^\infty p \left( Q^{p-1} - \frac{b_\infty^{p-1}}{r^2} \right) (a_1 \varphi_1 + a_2 \varphi_2 + \tilde{\phi}) \varphi_1 \frac{r^{d-1}}{\omega} dr. \end{aligned}$$

In view of the asymptotic behaviour (4.1) for  $Q$ , we infer for all  $\rho \geq 1$ ,

$$\left| p \left( Q^{p-1} - \frac{b_\infty^{p-1}}{\rho^2} \right) \right| \lesssim \rho^{-1-s_c}.$$

We infer for  $\rho \geq 1$

$$|\mathcal{G}(\tilde{\phi})(\rho)| \lesssim \rho^{1-d/2} \int_\rho^\infty (r^{-s_c} + r^{\alpha-1} |\tilde{\phi}|) dr \lesssim \rho^{2-d+\alpha} + \rho^{1-d/2} \int_\rho^\infty r^{\alpha-1} |\tilde{\phi}| dr$$

and similarly,

$$|\mathcal{G}(\tilde{\phi}_1)(\rho) - \mathcal{G}(\tilde{\phi}_2)(\rho)| \lesssim \rho^{1-d/2} \int_\rho^\infty r^{\alpha-1} |\tilde{\phi}_1 - \tilde{\phi}_2| dr.$$

Thus, for  $R \geq 1$  large enough, the Banach fixed point theorem applies and yields a unique solution  $\tilde{\phi}$  to (4.6) in the space corresponding to the norm

$$\sup_{\rho \geq R} \rho^{d-\alpha-2} |\tilde{\phi}|.$$



In particular, in view of the explicit formula (4.5) for  $\varphi_1$  and  $\varphi_2$ , and in view of the fact that  $\mathcal{H}_\infty(\Lambda Q) = 0$  and  $\mathcal{H}_\infty(\varphi) = 0$ , we infer (4.3)  $\square$

For  $\rho_1 \geq 1$ , we define the space of functions on which we invert our linearized operator  $\mathcal{H}_\infty$ :

$$(4.7) \quad \begin{aligned} \tilde{X}_{\rho_1} &= \left\{ w : (0, \rho_1) \rightarrow \mathbb{R} \mid \right. \\ &\quad \left. \|w\|_{\tilde{X}_{\rho_1}} := \sup_{0 \leq \rho \leq \rho_1} (1 + \rho)^{(d/2)-3} (|w| + \rho|w'| + \rho^2|w''|) < \infty \right\}, \\ \tilde{Y}_{\rho_1} &= \left\{ w : (0, \rho_1) \rightarrow \mathbb{R} \mid \|w\|_{\tilde{Y}_{\rho_1}} := \sup_{0 \leq \rho \leq \rho_1} (1 + \rho)^{(d/2)-1} |w| < \infty \right\}. \end{aligned}$$

**PROPOSITION 4.2** (Interior resolvent). — *There exists a bounded linear operator  $\mathcal{S} : \tilde{Y}_{\rho_1} \rightarrow \tilde{X}_{\rho_1}$  such that  $\mathcal{H}_\infty \circ \mathcal{S} = \text{id}_{\tilde{Y}_{\rho_1}}$  given by*

$$(4.8) \quad \mathcal{S}(f) = \Lambda Q \int_0^\rho f \varphi r^{d-1} dr - \varphi \int_0^\rho f \Lambda Q r^{d-1} dr$$

with  $\|\mathcal{S}\|_{\mathcal{L}(\tilde{Y}_{\rho_1}, \tilde{X}_{\rho_1})} \lesssim 1$  for all  $\rho_1 \geq 1$ .

*Proof.* — We recall from the previous lemma that  $W = -\rho^{1-d}$ . Let  $R_0 > 0$  be sufficiently small so that  $\Lambda Q > 0$  in  $[0, R_0]$ . Then solving the Wronskian equation, we assume without loss of generality that for  $\varphi$ ,

$$\varphi = -\Lambda Q \int_\rho^{R_0} \frac{dr}{(\Lambda Q)^2 r^{d-1}}$$

on  $(0, R_0]$  which ensures that as  $\rho \rightarrow 0$ ,

$$(4.9) \quad |\varphi| \lesssim \rho^{2-d}, \quad |\varphi'| \lesssim \rho^{1-d}, \quad |\varphi''| \lesssim \rho^{-d},$$

where we have used that  $Q$  and hence,  $\Lambda Q$  is a smooth radial function. Using the variation of constants

$$w = \Lambda Q \left( a_1 + \int_0^\rho f \varphi r^{d-1} dr \right) + \varphi \left( a_2 - \int_0^\rho f \Lambda Q r^{d-1} dr \right)$$

solves

$$\mathcal{H}_\infty(w) = f.$$

In particular,  $\mathcal{S}(f)$  corresponds to the choice  $a_1 = a_2 = 0$ . Finally, using the estimates (4.3), (4.9), we estimate for  $0 \leq \rho \leq 1$ :

$$\begin{aligned} |\mathcal{S}(f)| &= \left| \Lambda Q \int_0^\rho f \varphi r^{d-1} dr - \varphi \int_0^\rho f \Lambda Q r^{d-1} dr \right| \\ &\lesssim \left( \int_0^\rho r dr + \rho^{2-d} \int_0^\rho r^{d-1} dr \right) \sup_{0 \leq \rho \leq 1} |f| \lesssim \|f\|_{\tilde{Y}_{\rho_1}}. \end{aligned}$$

Similarly, taking derivatives,

$$\begin{aligned} |\rho \mathcal{S}(f)'| &= \rho \left| (\Lambda Q)' \int_0^\rho f \varphi r^{d-1} dr - \varphi' \int_0^\rho f \Lambda Q r^{d-1} dr \right| \\ &\lesssim \left( \rho^2 \int_0^\rho r dr + \rho^{2-d} \int_0^\rho r^{d-1} dr \right) \sup_{0 \leq r \leq 1} |f| \lesssim \|f\|_{\tilde{Y}_{\rho_1}}, \end{aligned}$$

and

$$\begin{aligned} |\rho^2 \mathcal{S}(f)''| &= \rho^2 \left| (\Lambda Q)'' \int_0^\rho f \varphi r^{d-1} dr - \varphi'' \int_0^\rho f \Lambda Q r^{d-1} dr - f \right| \\ &\lesssim \left( \rho^2 \int_0^\rho r dr + \rho^{2-d} \int_0^\rho r^{d-1} dr + \rho^2 \right) \sup_{0 \leq \rho \leq 1} |f| \lesssim \|f\|_{\tilde{Y}_{\rho_1}}. \end{aligned}$$

For  $1 \leq \rho \leq \rho_1$ ,

$$\begin{aligned} (1 + \rho)^{(d/2)-3} |\mathcal{S}(f)| &= (1 + \rho)^{(d/2)-3} \left| \Lambda Q \int_0^\rho f \varphi r^{d-1} dr - \varphi \int_0^\rho f \Lambda Q r^{d-1} dr \right| \\ &\lesssim (1 + \rho)^{-2} \int_0^\rho (1 + r)^{d/2} |f| dr \\ &\lesssim (1 + \rho)^{-2} \int_0^\rho (1 + r) dr \sup_{0 \leq \rho \leq \rho_1} (1 + \rho)^{(d/2)-1} |f| \lesssim \|f\|_{\tilde{Y}_{\rho_1}}. \end{aligned}$$

Similarly, taking derivatives,

$$\begin{aligned} (1 + \rho)^{(d/2)-3} |\rho \mathcal{S}(f)'| &= (1 + \rho)^{(d/2)-3} \rho \left| (\Lambda Q)' \int_0^\rho f \varphi r^{d-1} dr - \varphi' \int_0^\rho f \Lambda Q r^{d-1} dr \right| \\ &\lesssim (1 + \rho)^{-2} \int_0^\rho (1 + r) dr \sup_{0 \leq \rho \leq \rho_1} (1 + \rho)^{(d/2)-1} |f| \lesssim \|f\|_{\tilde{Y}_{\rho_1}} \end{aligned}$$

and

$$\begin{aligned} (1 + \rho)^{(d/2)-3} |\rho^2 \mathcal{S}(f)''| &= (1 + \rho)^{(d/2)-3} \rho^2 \left| (\Lambda Q)'' \int_0^\rho f \varphi r^{d-1} dr - \varphi'' \int_0^\rho f \Lambda Q r^{d-1} dr - f \right| \\ &\lesssim (1 + \rho)^{-2} \int_0^\rho (1 + r)^{d/2} |f| dr + (1 + \rho)^{(d/2)-1} |f| \lesssim \|f\|_{\tilde{Y}_{\rho_1}}. \end{aligned}$$

Thus,  $\|\mathcal{S}(f)\|_{\tilde{X}_{\rho_1}} \lesssim \|f\|_{\tilde{Y}_{\rho_1}}$ . □

LEMMA 4.3 (Non-linear bounds). — For  $w \in \tilde{X}_{\rho_1}$  and  $\lambda > 0$ , define

$$(4.10) \quad F[Q, \lambda]w = \underbrace{p(p-1)\lambda^2 w^2}_{:= \tilde{A}[\lambda]w} \underbrace{\int_0^1 (1-s)(Q + \lambda^2 s w)^{p-2} ds}_{:= \tilde{B}[Q, \lambda]w} - \mathcal{F}(Q + \lambda^2 w),$$

where

$$\mathcal{F} = \rho^2 \frac{d^2}{d\rho^2} + 2(1 + \alpha)\rho \frac{d}{d\rho} + \alpha(1 + \alpha).$$

Then there exists  $C > 0$  such that for all  $\rho_1 \lambda \ll 1$  and  $\|w_1\|_{\tilde{X}_{\rho_1}}, \|w_2\|_{\tilde{X}_{\rho_1}} \leq C$ ,

$$(4.11) \quad \begin{aligned} \|F[Q, \lambda]w_1\|_{\tilde{Y}_{\rho_1}} &\leq C \|\mathcal{S}\|_{\mathcal{L}(\tilde{Y}_{\rho_1}, \tilde{X}_{\rho_1})}^{-1}, \\ \|F[Q, \lambda]w_1 - F[Q, \lambda]w_2\|_{\tilde{Y}_{\rho_1}} &\lesssim \rho_1^2 \lambda^2 \|w_1 - w_2\|_{\tilde{X}_{\rho_1}}. \end{aligned}$$

*Proof.* — We first bound  $\mathcal{F}(Q)$ . In view of (4.1),

$$\rho^2 Q^{p-1} = b_\infty^{p-1} + \mathcal{O}_{\rho \rightarrow \infty}(\rho^{1-s_c}).$$

Then in view of (4.3), since  $Q'' + \frac{d-1}{\rho}Q' + Q^p = 0$ , we infer

$$\begin{aligned} \mathcal{F}(Q) &= -\rho^2 Q^p + (3 - 2s_c)\rho Q' + \alpha(1 + \alpha)Q \\ &= (b_\infty^{p-1} - \rho^2 Q^{p-1})Q + (3 - 2s_c)\Lambda Q = \mathcal{O}_{\rho \rightarrow \infty}(\rho^{1-d/2}). \end{aligned}$$

Note also that since  $s_c > 1$ , we have that for all  $0 \leq \rho \leq \rho_1$ ,

$$|w_1(\rho)| \lesssim (1 + \rho_1)^{3-d/2} \|w_1\|_{\tilde{\mathcal{X}}_{\rho_1}} \lesssim (1 + \rho_1)^2 |Q(\rho)| \|w_1\|_{\tilde{\mathcal{X}}_{\rho_1}},$$

so by our choice of  $\lambda$ ,

$$\lambda^2 |w_1(\rho)| \lesssim |Q(\rho)| \|w_1\|_{\tilde{\mathcal{X}}_{\rho_1}}.$$

With these estimates, for all  $\rho$  such that  $0 \leq \rho \leq \rho_1$ ,

$$\begin{aligned} |F[Q, \lambda]w_1| &\lesssim \lambda^2 |w_1|^2 \left( |Q| + \lambda^2 |w_1| \right)^{p-2} + |\mathcal{F}(Q)| + \lambda^2 |\mathcal{F}(w_1)| \\ &\lesssim \lambda^2 (1 + \rho)^{6-d-\alpha(p-2)} \left( \|w_1\|_{\tilde{\mathcal{X}}_{\rho_1}}^2 + \|w_1\|_{\tilde{\mathcal{X}}_{\rho_1}}^p \right) + (1 + \rho)^{1-d/2} + \frac{\lambda^2 \|w_1\|_{\tilde{\mathcal{X}}_{\rho_1}}}{(1 + \rho)^{(d/2)-3}} \\ &\lesssim \left[ \rho_1^{3-s_c} \lambda^2 \left( \|w_1\|_{\tilde{\mathcal{X}}_{\rho_1}}^2 + \|w_1\|_{\tilde{\mathcal{X}}_{\rho_1}}^p \right) + 1 + \rho_1^2 \lambda^2 \right] (1 + \rho)^{1-d/2} \\ &\lesssim \left[ 1 + \rho_1^2 \lambda^2 \left( 1 + \|w_1\|_{\tilde{\mathcal{X}}_{\rho_1}}^p \right) \right] (1 + \rho)^{1-d/2}, \end{aligned}$$

where we have used that  $s_c > 1$  in the last inequality. Choose  $C > 0$  such that

$$|F[Q, \lambda]w_1| \leq \frac{C}{2} \|\mathcal{S}\|_{\mathcal{L}(\tilde{\mathcal{Y}}_{\rho_1}, \tilde{\mathcal{X}}_{\rho_1})}^{-1} \left[ 1 + \rho_1^2 \lambda^2 \left( \|w_1\|_{\tilde{\mathcal{X}}_{\rho_1}} + \|w_1\|_{\tilde{\mathcal{X}}_{\rho_1}}^p \right) \right] (1 + \rho)^{1-d/2}.$$

Then for  $\rho_1 \lambda \ll 1$  and  $\|w_1\|_{\tilde{\mathcal{X}}_{\rho_1}} \leq C$ ,

$$\|F[Q, \lambda]w_1\|_{\tilde{\mathcal{Y}}_{\rho_1}} \leq C \|\mathcal{S}\|_{\mathcal{L}(\tilde{\mathcal{Y}}_{\rho_1}, \tilde{\mathcal{X}}_{\rho_1})}^{-1}.$$

Hence, the first bound in (4.11) holds. For the second bound, we have

$$\begin{aligned} |F[Q, \lambda]w_1 - F[Q, \lambda]w_2| &\leq |\tilde{A}w_1 - \tilde{A}w_2| |\tilde{B}w_1| + |\tilde{A}w_2| |\tilde{B}w_1 - \tilde{B}w_2| + \lambda^2 |\mathcal{F}(w_1 - w_2)| \\ &\lesssim \lambda^2 |w_1 + w_2| |w_1 - w_2| (|Q| + \lambda^2 |w|)^{p-2} + \lambda^4 |w_1 - w_2| |w_2|^2 \tilde{I}_{w_1, w_2} \\ &\quad + \lambda^2 (1 + \rho)^{3-d/2} \|w_1 - w_2\|_{\tilde{\mathcal{X}}_{\rho_1}}, \end{aligned}$$

where

$$\begin{aligned} \tilde{I}_{w_1, w_2} &:= \left| \int_0^1 \lambda^{-2} \partial_w \tilde{B}[Q, \lambda]w|_{w_2 + \sigma(w_1 - w_2)} d\sigma \right| \\ &\lesssim \left| \int_0^1 s(1-s) \int_0^1 (Q + s\lambda^2 w_2 + \sigma s \lambda^2 (w_1 - w_2))^{p-3} d\sigma ds \right| \\ &\lesssim \left[ |Q| + \lambda^2 (|w_1| + |w_2|) \right]^{p-3} \lesssim (1 + \rho)^{-\alpha(p-3)}. \end{aligned}$$

Thus,

$$\begin{aligned} & |F[Q, \lambda]w_1 - F[Q, \lambda]w_2| \\ & \lesssim \left[ \lambda^2(1 + \rho)^{6-d-(p-2)\alpha} + \lambda^4(1+\rho)^{9-(3d/2)-(p-3)\alpha} + \lambda^2(1+\rho)^{3-d/2} \right] \|w_1 - w_2\|_{\tilde{X}_{\rho_1}} \\ & \lesssim \left( \rho_1^{3-s_c} \lambda^2 + \rho_1^{6-2s_c} \lambda^4 + \rho_1^2 \lambda^2 \right) (1 + \rho)^{1-d/2} \|w_1 - w_2\|_{\tilde{X}_{\rho_1}} \\ & \lesssim \rho_1^2 \lambda^2 (1 + \rho)^{1-d/2} \|w_1 - w_2\|_{\tilde{X}_{\rho_1}}, \end{aligned}$$

where again, we have used that  $s_c > 1$ . Hence the second bound in (4.11) holds.  $\square$

We prove the existence of a one-parameter family of smooth solutions to (1.4) in the region  $\rho < \rho_0$ .

**PROPOSITION 4.4 (Interior solutions).** — *For all  $\rho_0, \lambda$  such that  $0 \leq \rho_0 \ll 1, 0 < \lambda \leq \rho_0$ , there exists a solution to (1.4) on  $0 \leq \rho \leq \rho_0$  of the form*

$$u = \lambda^{-\alpha}(Q + \lambda^2 w)(\rho/\lambda)$$

with  $\|w\|_{\tilde{X}_{\rho_1}} \lesssim 1$  where  $\rho_1 = \rho_0/\lambda \geq 1$ .

*Proof.* —  $u = \lambda^{-\alpha}(Q + \lambda^2 w)(\rho/\lambda)$  solves (1.4) if and only if

$$(4.12) \quad \mathcal{H}_\infty(w) = \lambda^{-2} \left[ (Q + \lambda^2 w)^p - Q^p - pQ^{p-1} \lambda^2 w \right] - \mathcal{F}(Q + \lambda^2 w) = F[Q, \lambda]w.$$

Lemma 4.3 together with Proposition 4.2 states precisely that for  $\rho_1 \lambda = \rho_0 \ll 1$ ,

$$S \circ F[Q, \lambda] : B_{\tilde{X}_{\rho_1}}(C) := \{w \in \tilde{X}_{\rho_1} \mid \|w\|_{\tilde{X}_{\rho_1}} \leq C\} \longrightarrow B_{\tilde{X}_{\rho_1}}(C)$$

is a contraction map. Thus, Banach fixed point theorem applies and yields a unique solution  $w$  to (4.12) with  $\|w\|_{\tilde{X}_{\rho_1}} \leq C$ .  $\square$

### 5. THE MATCHING

We are now in position to “glue” inner and outer solutions to produce exact solutions to (1.1).

**PROPOSITION 5.1 (Existence of countably many smooth self-similar profiles)**

*There exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ , there exists a smooth solution  $u_n$  to (1.1) such that  $\Lambda u_n$  vanishes exactly  $n$  times.*

*Proof*

*Step 1 (Matching).* — Recall that

$$\begin{aligned} (5.1) \quad \psi_1 &= c_1 \rho^{1-d/2} \sin(\omega \log \rho + \delta_1) + \mathcal{O}_{\rho \rightarrow 0}(\rho^{3-d/2}), \\ \Lambda \psi_1 &= c_1 \rho^{1-d/2} \left[ (1 - s_c) \sin(\omega \log \rho + \delta_1) + \omega \cos(\omega \log \rho + \delta_1) \right] \\ &\quad + \mathcal{O}_{\rho \rightarrow 0}(\rho^{3-d/2}), \end{aligned}$$

for some  $c_1 \in \mathbb{R}$  which is non-zero by definition (3.4). Then, we can choose  $\rho_0$  with  $0 < \rho_0 \ll 1$  such that

$$\begin{aligned} (5.2) \quad \psi_1(\rho_0) &= c_1 \rho_0^{1-d/2} + \mathcal{O}_{\rho \rightarrow 0}(\rho^{3-d/2}), \\ \Lambda \psi_1(\rho_0) &= c_1 (1 - s_c) \rho_0^{1-d/2} + \mathcal{O}_{\rho \rightarrow 0}(\rho^{3-d/2}), \end{aligned}$$

and Propositions 3.6 and 4.4 apply. In particular, let

$$\begin{aligned} u_{\text{ext}}[\varepsilon] &= u_\infty + \varepsilon\psi_1 + \varepsilon w_{\text{ext}}, \\ u_{\text{int}}[\lambda] &= \lambda^{-\alpha}(Q + \lambda^2 w_{\text{int}})(\rho/\lambda), \end{aligned}$$

be solutions to (1.4) in the regions  $[\rho_0, \infty)$  and  $[0, \rho_0]$  respectively. Define

$$\mathcal{J}[\rho_0](\varepsilon, \lambda) = u_{\text{ext}}[\varepsilon](\rho_0) - u_{\text{int}}[\lambda](\rho_0).$$

Then

$$\partial_\varepsilon \mathcal{J}[\rho_0](\varepsilon, \lambda) = \partial_\varepsilon u_{\text{ext}}[\varepsilon](\rho_0) = \psi_1(\rho_0) + w_{\text{ext}}(\rho_0) + \varepsilon \partial_\varepsilon w(\rho_0).$$

In view of Proposition 3.6, since  $\psi_1(\rho_0) \neq 0$ ,

$$\partial_\varepsilon \mathcal{J}[\rho_0](0, 0) = \psi_1(\rho_0) \neq 0.$$

From the asymptotic behaviour of  $Q$  as  $\rho \rightarrow \infty$ , as  $\lambda \rightarrow 0$ ,

$$\begin{aligned} (5.3) \quad |\lambda^{-\alpha}(Q - u_\infty + \lambda^2 w_{\text{int}})(\rho_0/\lambda)| &\lesssim \lambda^{-\alpha}[(\rho_0/\lambda)^{1-d/2} + \lambda^2(\rho_0/\lambda)^{3-d/2}] \\ &\lesssim \lambda^{s_c-1} \rho_0^{1-d/2} (1 + \rho_0^2) \rightarrow 0. \end{aligned}$$

Since  $u_{\text{ext}}[0] = u_\infty$  is self-similar, this implies

$$\mathcal{J}[\rho_0](0, 0) = u_\infty(\rho_0) - \lim_{\lambda \rightarrow 0} \lambda^{-\alpha} u_\infty(\rho_0/\lambda) = 0.$$

Applying the implicit function theorem to

$$\tilde{\mathcal{J}}(\varepsilon, \mu) := \mathcal{J}[\rho_0](\varepsilon, \mu^{2/(s_c-1)}),$$

we have that if  $\tilde{\mathcal{J}}$  is  $C^1$  in the neighbourhood of  $(0, 0)$ , then there exist  $\lambda_0 > 0$  and  $\tilde{\varepsilon} \in C^1([0, \lambda_0^{(s_c-1)/2}])$  such that  $\tilde{\mathcal{J}}(\tilde{\varepsilon}(\mu), \mu) = 0$ . Then, it suffices to check existence of  $\partial_\mu \tilde{\mathcal{J}}(\varepsilon, \mu)$  at  $\mu = 0$ . Indeed, from (5.3), we have

$$\tilde{\mathcal{J}}(\varepsilon, \mu) = u_{\text{ext}}[\varepsilon](\rho_0) - u_{\text{int}}[\mu^{2/(s_c-1)}](\rho_0) = (\varepsilon\psi_1 + \varepsilon w_{\text{ext}})(\rho_0) + \mathcal{O}_{\mu \rightarrow 0}(\mu^2),$$

so  $\partial_\mu \tilde{\mathcal{J}}(\varepsilon, 0) = 0$  and we have the existence of  $\tilde{\varepsilon}$ . Then, for  $\varepsilon(\lambda) := \tilde{\varepsilon}(\lambda^{(s_c-1)/2})$ , we have  $\mathcal{J}[\rho_0](\varepsilon(\lambda), \lambda) = 0$  and  $\varepsilon \in C^{s_c-1}([0, \lambda_0])$ . Hence,

$$u_{\text{ext}}[\varepsilon(\lambda)](\rho_0) = u_{\text{int}}[\lambda](\rho_0)$$

on  $[0, \lambda_0)$  i.e.,

$$(5.4) \quad \varepsilon(\lambda)(\psi_1(\rho_0) + w_{\text{ext}}(\rho_0)) = \lambda^{-\alpha}(Q - u_\infty + \lambda^2 w_{\text{int}})(\rho_0/\lambda).$$

By the definition of  $\rho_0$  and from the bounds on  $w_{\text{ext}}$  and  $w_{\text{int}}$  in Propositions 3.6 and 4.4, we infer for some  $c \in \mathbb{R}$ ,

$$\begin{aligned} \varepsilon(\lambda) \rho_0^{1-d/2} [c + \mathcal{O}(\rho_0^2 + \varepsilon(\lambda) \rho_0^{s_c-1})] &= \varepsilon(\lambda)(\psi_1(\rho_0) + w_{\text{ext}}(\rho_0)) \\ &= \lambda^{-\alpha}(Q - u_\infty + \lambda^2 w_{\text{int}})(\rho_0/\lambda) \\ &\lesssim \lambda^{s_c-1} \rho_0^{1-d/2} [1 + \mathcal{O}(\rho_0^2)] \end{aligned}$$

as  $\rho_0 \rightarrow 0$ , so that, as  $\lambda \rightarrow 0$ ,

$$|\varepsilon(\lambda)| \lesssim \lambda^{s_c-1}.$$

It then follows from (5.4) and (3.15) that

$$(5.5) \quad \varepsilon(\lambda) = \psi_1^{-1}(\rho_0)\lambda^{-\alpha}(Q - u_\infty)(\rho_0/\lambda) + \mathcal{O}(\lambda^{s_c-1}(\rho_0^2 + \lambda^{s_c-1}\rho_0^{1-s_c})).$$

Consider now the spatial derivative

$$\mathcal{J}'[\rho_0](\varepsilon(\lambda), \lambda) = \varepsilon(\lambda)(\psi_1'(\rho_0) + w'_{\text{ext}}(\rho_0)) - \lambda^{-1-\alpha}(Q' - u'_\infty + \lambda^2 w'_{\text{int}})(\rho_0/\lambda).$$

From the bound on  $\varepsilon(\lambda)$  above and the bound on  $w'_{\text{ext}}$  and  $w'_{\text{int}}$  in Propositions 3.6 and 4.4, we infer

$$\begin{aligned} \mathcal{J}'[\rho_0](\varepsilon(\lambda), \lambda) &= \varepsilon(\lambda)\psi_1'(\rho_0) - \lambda^{-1-\alpha}(Q' - u'_\infty)(\rho_0/\lambda) + \mathcal{O}(\lambda^{s_c-1}(\rho_0^{2-d/2} + \lambda^{s_c-1}\rho_0^{1-d+\alpha})) \\ &= \frac{\lambda^{s_c-1}}{\rho_0^{(d/2)-1}\psi_1(\rho_0)} \left[ (\rho_0/\lambda)^{(d/2)-1}(Q - u_\infty)(\rho_0/\lambda)\psi_1'(\rho_0) \right. \\ &\quad \left. - (\rho_0/\lambda)^{d/2}(Q' - u'_\infty)(\rho_0/\lambda)\frac{\psi_1(\rho_0)}{\rho_0} \right] + \mathcal{O}(\lambda^{s_c-1}(\rho_0^{2-d/2} + \lambda^{s_c-1}\rho_0^{1-d+\alpha})), \end{aligned}$$

where in the final inequality we inject (5.5) for  $\varepsilon(\lambda)$ . From the asymptotic behaviors (5.1) for  $\psi_1$  and knowing that as  $\rho \rightarrow \infty$ :

$$(5.6) \quad \begin{aligned} (Q - u_\infty)(\rho) &= c_2\rho^{1-d/2} \sin(\omega \log \rho + \delta_2) + \mathcal{O}(\rho^{2-d+\alpha}), \\ (Q' - u'_\infty)(\rho) &= c_2\rho^{-d/2} [(1 - d/2) \sin(\omega \log \rho + \delta_2) + \omega \cos(\omega \log \rho + \delta_2)] \\ &\quad + \mathcal{O}(\rho^{1-d+\alpha}), \end{aligned}$$

for some  $c_2 \in \mathbb{R}$  which is non-zero (see [17]), it follows that

$$\begin{aligned} &\frac{\rho_0^{(d/2)-1}\psi_1(\rho_0)}{\lambda^{s_c-1}} \mathcal{J}'[\rho_0](\varepsilon(\lambda), \lambda) \\ &= c_1c_2\omega\rho_0^{-d/2} \left[ \sin(\omega \log \rho_0 - \omega \log \lambda + \delta_2) \cos(\omega \log \rho_0 + \delta_1) \right. \\ &\quad \left. - \cos(\omega \log \rho_0 - \omega \log \lambda + \delta_2) \sin(\omega \log \rho_0 + \delta_1) \right] + \mathcal{O}(\rho_0^{2-d/2} + \lambda^{s_c-1}\rho_0^{1-d+\alpha}) \\ &= c_1c_2\omega\rho_0^{-d/2} \sin(-\omega \log \lambda + \delta_2 - \delta_1) + \mathcal{O}(\rho_0^{2-d/2} + \lambda^{s_c-1}\rho_0^{1-d+\alpha}). \end{aligned}$$

Thus,

$$(5.7) \quad \begin{aligned} \mathcal{J}'[\rho_0](\varepsilon(\lambda), \lambda) &= c_1c_2\omega\lambda^{s_c-1} \left[ \frac{\sin(-\omega \log \lambda + \delta_2 - \delta_1)}{\rho_0^{d-1}\psi_1(\rho_0)} + \mathcal{O}(\rho_0^{2-d/2} + \lambda^{s_c-1}\rho_0^{1-d+\alpha}) \right]. \end{aligned}$$

Let

$$(5.8) \quad \lambda_{n,+} = \exp \left[ \frac{-n\pi + \delta_2 - \delta_1 + \delta_0}{\omega} \right], \quad \lambda_{n,-} = \exp \left[ \frac{-n\pi + \delta_2 - \delta_1 - \delta_0}{\omega} \right].$$

Then,  $\lambda_{n,\pm} \rightarrow 0$  as  $n \rightarrow \infty$  and

$$0 < \dots < \lambda_{n,+} < \lambda_{n,-} < \lambda_{n-1,+} < \lambda_{n-1,-} < \dots$$

Then,

$$\mathcal{J}'[\rho_0](\varepsilon(\lambda_{n,\pm}), \lambda_{n,\pm}) = \pm(-1)^n \lambda_{n,\pm}^{s_c-1} \left[ \frac{c_1 c_2 \omega}{\rho_0^{d-1} \psi_1(\rho_0)} \sin \delta_0 + \mathcal{O}\left(\rho_0^{2-d/2} + \lambda_{n,\pm}^{s_c-1} \rho_0^{1-d+\alpha}\right) \right].$$

For  $\rho_0 \ll 1$ , and  $n \gg 1$ ,

$$\mathcal{J}'[\rho_0](\varepsilon(\lambda_{n,\pm}), \lambda_{n,-}) \mathcal{J}'[\rho_0](\varepsilon(\lambda_{n,\pm}), \lambda_{n,+}) < 0.$$

Since  $\lambda \mapsto \mathcal{J}'[\rho_0](\varepsilon(\lambda), \lambda)$  is continuous, it follows from intermediate value theorem that for all  $n \geq N \gg 1$ , there exists  $\lambda_{n,+} < \mu_n < \lambda_{n,-}$  such that  $\mathcal{J}'[\rho_0](\varepsilon(\mu_n), \mu_n) = 0$  i.e.,

$$u_{\text{ext}}[\varepsilon(\mu_n)](\rho_0) = u_{\text{int}}[\mu_n](\rho_0), \quad u'_{\text{ext}}[\varepsilon(\mu_n)](\rho_0) = u'_{\text{int}}[\mu_n](\rho_0).$$

Hence, the function

$$u_n(\rho) := \begin{cases} u_{\text{int}}[\mu_n](\rho) & 0 \leq \rho \leq \rho_0, \\ u_{\text{ext}}[\varepsilon(\mu_n)](\rho) & \rho_0 \leq \rho, \end{cases}$$

is a smooth solution to (1.4) in  $[0, \infty)$  for all  $n \geq N$ .

*Step 2 (Counting the zeros).* — The remaining part of the proof is devoted to counting the number of zeros of  $\Lambda u_n$ . We first claim that for  $\rho_0 \ll 1$ ,

$$(5.9) \quad \Lambda u_{\text{ext}}[\varepsilon] \text{ has as many zeros as } \Lambda \psi_1 \text{ on } \rho \geq \rho_0.$$

Indeed,  $\Lambda \psi_1 + \Lambda w_{\text{ext}}$  does not vanish on  $[R_0, \infty)$  for  $R_0$  large enough from (3.11) and the uniform bound (3.15). Moreover,  $\Lambda \psi_1(\rho_0) \neq 0$  from the normalization (5.2), and the absolute value of the derivative of  $\Lambda \psi_1$  at any of its zeros is uniformly lower bounded using (3.10) and hence the uniform smallness (3.15) yields the claim.

We now claim that for  $\rho_0 \ll 1$ ,

$$(5.10) \quad \Lambda u_{\text{int}}[\mu_n] \text{ has as many zeros as } \Lambda Q \text{ on } 0 \leq r \leq \frac{\rho_0}{\mu_n}.$$

Indeed, recall that

$$\Lambda u_{\text{int}}[\mu_n](\rho) = \mu_n^{-\alpha} (\Lambda Q + \mu_n^2 \Lambda w_{\text{int}})(\rho/\mu_n).$$

We now claim

$$(5.11) \quad (\rho_0/\mu_n)^{(d/2)-1} |\Lambda Q(\rho_0/\mu_n)| \gtrsim 1.$$

Assume (5.11), then since the zeros of  $\Lambda Q$  are simple and since

$$\|\Lambda w_{\text{int}}\|_{\tilde{X}_{\frac{\rho_0}{\mu_n}}} = \sup_{0 \leq \rho \leq \rho_0/\mu_n} (1 + \rho)^{(d/2)-3} |\Lambda w_{\text{int}}| \lesssim 1$$

so that

$$\sup_{0 \leq \rho \leq \rho_0/\mu_n} (1 + \rho)^{(d/2)-1} |\mu_n^2 \Lambda w_{\text{int}}| \lesssim \rho_0^2,$$

and similarly for  $\Lambda^2 w_{\text{int}}$ , and since

$$\Lambda Q(0) = \frac{2}{p-1} \neq 0,$$

we conclude for  $\rho_0 \ll 1$  that  $\Lambda Q + \mu_n^2 \Lambda w_{\text{int}}$  has as many zeros as  $\Lambda Q$  on  $0 \leq \rho \leq \rho_0/\mu_n$ . We deduce that on  $0 \leq \rho \leq \rho_0$ ,  $\Lambda u_{\text{int}}[\mu_n]$  has as many zeros as  $\Lambda Q$  on  $0 \leq \rho \leq \rho_0/\mu_n$ .

*Proof of (5.11).* — Recall that

$$u_{\text{ext}}[\varepsilon(\mu_n)](\rho_0) = u_{\text{int}}[\mu_n](\rho_0), \quad u_{\text{ext}}[\varepsilon(\mu_n)]'(\rho_0) = u_{\text{int}}'[\mu_n](\rho_0),$$

which implies

$$\Lambda u_{\text{ext}}[\varepsilon(\mu_n)](\rho_0) = \Lambda u_{\text{int}}[\mu_n](\rho_0).$$

This yields, using (5.5):

$$\frac{\varepsilon(\mu_n)}{\mu_n^{s_c-1}} = \frac{1}{\psi_1(\rho_0)\mu_n^{(d/2)-1}}(Q - u_\infty)(\rho_0/\mu_n) + O(\mu_n^{s_c-1}\rho_0^{s_c-1} + \rho_0^2)$$

and taking  $\Lambda$  of (5.4):

$$\frac{\varepsilon(\mu_n)}{\mu_n^{s_c-1}} = \frac{1}{\Lambda\psi_1(\rho_0)\mu_n^{(d/2)-1}}\Lambda Q(\rho_0/\mu_n) + O(\mu_n^{s_c-1}\rho_0^{s_c-1} + \rho_0^2).$$

We infer

$$\frac{1}{\psi_1(\rho_0)\mu_n^{(d/2)-1}}(Q - u_\infty)(\rho_0/\mu_n) = \frac{1}{\Lambda\psi_1(\rho_0)\mu_n^{(d/2)-1}}\Lambda Q(\rho_0/\mu_n) + O(\mu_n^{s_c-1}\rho_0^{s_c-1} + \rho_0^2).$$

In view of the asymptotics (5.2) of  $\psi_1$ , we infer

$$(5.12) \quad \left| (\rho_0/\mu_n)^{(d/2)-1}(Q - u_\infty)(\rho_0/\mu_n) \right| \leq \frac{2}{s_c - 1} \left| (\rho_0/\mu_n)^{(d/2)-1}\Lambda Q(\rho_0/\mu_n) \right| + O(\mu_n^{s_c-1} + \rho_0^2).$$

On the other hand, from (5.6),

$$(5.13) \quad \begin{aligned} \Lambda Q(\rho) &= \frac{c_2}{\rho^{(d/2)-1}} \left[ (1 - s_c) \sin(\omega \log \rho + \delta_2) + \omega \cos(\omega \log \rho + \delta_2) \right] \\ &\quad + \mathcal{O}_{\rho \rightarrow \infty}(\rho^{2-d+\alpha}) \\ &= \frac{c_2 \sqrt{(s_c - 1)^2 + \omega^2}}{\rho^{(d/2)-1}} \sin(\omega \log \rho + \delta_2 + \alpha_0) + \mathcal{O}_{\rho \rightarrow \infty}(\rho^{2-d+\alpha}), \end{aligned}$$

where

$$\cos(\alpha_0) = \frac{1 - s_c}{\sqrt{(s_c - 1)^2 + \omega^2}}, \quad \sin(\alpha_0) = \frac{\omega}{\sqrt{(s_c - 1)^2 + \omega^2}}, \quad \alpha_0 \in (\pi/2, \pi).$$

Thus, in view of (5.6) and (5.13), there exist  $\rho_2 > 0$  sufficiently small and a constant  $\delta > 0$  sufficiently small only depending on  $\omega$  and  $s_c - 1$  such that for  $0 < \rho < \rho_2$ , we have

$$\begin{aligned} \text{dist}(\omega \log \rho + \delta_2 + \alpha_0, \pi\mathbb{Z}) &< \delta \\ \implies \rho^{(d/2)-1}|Q(\rho) - u_\infty(\rho)| &\geq \frac{4}{s_c - 1} \rho^{(d/2)-1} |\Lambda Q(\rho)| + \frac{c_1 \sin(\alpha_0)}{2}. \end{aligned}$$

In view of (5.12), we infer for  $n \geq n_0$  large enough

$$(5.14) \quad \text{dist}(\omega \log(\rho_0/\mu_n) + \delta_2 + \alpha_0, \pi\mathbb{Z}) \geq \delta,$$

and (5.11) is proved.



Combining the two claims proved above, we infer

$$\#\{\rho \geq 0 \mid \Lambda u_n(\rho) = 0\} = \#\left\{0 \leq \rho \leq \frac{\rho_0}{\mu_n} \mid \Lambda Q(\rho) = 0\right\} + \#\{\rho > \rho_0 \mid \Lambda \psi_1(\rho) = 0\},$$

which implies

$$\#\{\rho \geq 0 \mid \Lambda u_{n+1}(\rho) = 0\} = \#\{\rho \geq 0 \mid \Lambda u_n(\rho) = 0\} + \#A_n,$$

with

$$A_n := \left\{ \frac{\rho_0}{\mu_n} < \rho \leq \frac{\rho_0}{\mu_{n+1}} \mid \Lambda Q(r) = 0 \right\}.$$

We claim for  $n \geq n_1$  large enough:

$$(5.15) \quad \#A_n = 1$$

which by possibly shifting the numeration by a fixed amount ensures that  $\Lambda u_n$  vanishes exactly  $k$  times.

*Upper bound.* — We first claim

$$(5.16) \quad \#A_n \leq 1$$

Recall (5.13) so that there exists  $R \geq 1$  large enough such that

$$\left\{ \rho \geq R \mid \Lambda Q(\rho) = 0 \right\} = \{r_q \mid q \geq q_1\}, \quad \omega \log(r_q) + \delta_2 + \alpha_0 = q\pi + \mathcal{O}_{r_q \rightarrow \infty}(r_q^{1-s_c})$$

and hence, together with (5.14), we infer

$$(5.17) \quad \inf_{q \geq q_1, n \geq n_1} |\log(\rho_0/\mu_n) - \log(r_q)| \geq \frac{\delta}{2\omega}.$$

This implies for  $n \geq n_1$

$$(5.18) \quad A_n \subset \left\{ q \geq q_1 \mid \log(\rho_0/\mu_n) + \delta/2\omega \leq \log(r_q) \leq \log(\rho_0/\mu_{n+1}) - \delta/2\omega \right\}.$$

Since  $\lambda_{n,+} < \mu_n < \lambda_{n,-}$  with  $\lambda_{n,\pm}$  given by (5.8), we have for  $k \geq k_1$

$$\begin{aligned} \log(\rho_0/\mu_{n+1}) - \frac{\delta}{2\omega} - \left( \log(\rho_0/\mu_n) + \frac{\delta}{2\omega} \right) &= \log(\mu_n) - \log(\mu_{n+1}) - \frac{\delta}{\omega} \\ &\leq \log(\lambda_{n,-}) - \log(\lambda_{n+1,+}) - \frac{\delta}{\omega} \leq \frac{\pi + 2\delta_0 - \delta}{\omega}. \end{aligned}$$

Also, we have for  $q \geq q_1$

$$\log(r_{q+1}) - \log(r_q) = \frac{\pi}{\omega} + \mathcal{O}_{r_q \rightarrow \infty}(r_q^{1-s_c}).$$

We now choose  $\delta_0$  such that

$$(5.19) \quad 0 < \delta_0 < \frac{\delta}{4}.$$

Then, we infer for  $n \geq n_1$  and  $q \geq q_1$ ,

$$\log(\rho_0/\mu_{n+1}) - \frac{\delta}{2\omega} - \left( \log(\rho_0/\mu_n) + \frac{\delta}{2\omega} \right) \leq \frac{\pi}{\omega} - \frac{\delta}{2\omega} < \log(r_{q+1}) - \log(r_q),$$

which, in view of (5.18), implies (5.16).

*Lower bound.* — We now prove (5.15) and assume for a contradiction:  $\#A_{n_2} = 0$ . Then, let  $q_2 \geq q_1$  be such that

$$r_{q_2} < \frac{\rho_0}{\mu_{n_2}} < \frac{\rho_0}{\mu_{n_2+1}} < r_{q_2+1}.$$

We infer from (5.17):

$$(5.20) \quad \log(r_{q_2}) \leq \log(\rho_0/\mu_{n_2}) - \frac{\delta}{2\omega} < \log(\rho_0/\mu_{n_2+1}) + \frac{\delta}{2\omega} \leq \log(r_{q_2+1}).$$

However, we have for  $n_2 \geq n_1$  and  $q_2 \geq q_1$ ,

$$\begin{aligned} \log(\rho_0/\mu_{n_2+1}) + \frac{\delta}{2\omega} - \left( \log(\rho_0/\mu_{n_2}) - \frac{\delta}{2\omega} \right) &= \log(\mu_{n_2}) - \log(\mu_{n_2+1}) + \frac{\delta}{\omega} \\ &\geq \log(\lambda_{n_2,-}) - \log(\lambda_{n_2+1,+}) + \frac{\delta}{\omega} \geq \frac{\pi - 2\delta_0 + \delta}{\omega} \geq \frac{\pi}{\omega} + \frac{\delta}{2\omega} > \log(r_{q_2+1}) - \log(r_{q_2}), \end{aligned}$$

which contradicts (5.20). This concludes the proof of Proposition 5.1.  $\square$

**COROLLARY 5.2.** — *Let  $u_n$  be the solution to (1.4) constructed in Proposition 5.1. For  $\rho_0 \ll 1$ , the following holds.*

(i) Convergence to  $u_\infty$  as  $n \rightarrow \infty$ :

$$(5.21) \quad \lim_{n \rightarrow \infty} \sup_{\rho \geq \rho_0} (1 + \rho^\alpha) |u_n(\rho) - u_\infty(\rho)| = 0.$$

(ii) Convergence to  $Q$  at the origin: *there exists  $\mu_n \rightarrow 0$  such that*

$$(5.22) \quad \lim_{n \rightarrow \infty} \sup_{\rho \leq \rho_0} |u_n(\rho) - \mu_n^{-\alpha} Q(\rho/\mu_n)| = 0.$$

(iii) Last zeros: *let*

$$\rho_{0,n} := \max \{ \rho \mid \Lambda u_n(\rho) = 0, \rho < \rho_0 \}, \quad \rho_{\Lambda Q,n} := \max \{ \rho \mid \Lambda Q(\rho) = 0, \rho < \rho_0/\mu_n \}.$$

*Then*

$$\rho_{0,n} = \mu_n \rho_{\Lambda Q,n} \left[ 1 + \mathcal{O}_{\rho_0 \rightarrow 0}(\rho_0^2) \right].$$

*Furthermore, for  $n \geq N$ ,*

$$e^{-2\pi/\omega} \rho_0 < \rho_{0,n} < \rho_0.$$

*Proof.* — Choose  $\rho_0 \ll 1$  as in the proof of Proposition 5.1.

(i) In view of (5.1) and (3.15), we infer

$$\begin{aligned} \sup_{\rho \geq \rho_0} (1 + \rho^\alpha) |u_n(\rho) - u_\infty(\rho)| &= \sup_{\rho \geq \rho_0} (1 + \rho^\alpha) |\varepsilon(\mu_n)(\psi_1(\rho) + w_{\text{ext}}(\rho))| \\ &\lesssim \varepsilon(\mu_n) \left[ \sup_{\rho_0 \leq \rho \leq 1} (|\psi_1(\rho)| + |w_{\text{ext}}(\rho)|) + \sup_{\rho \geq 1} \rho^\alpha (|\psi_1(\rho)| + |w_{\text{ext}}(\rho)|) \right] \\ &\lesssim \varepsilon(\mu_n) \rho_0^{1-d/2}. \end{aligned}$$

Since  $\varepsilon(\mu_n) \rightarrow 0$  as  $n \rightarrow \infty$ , result follows.

(ii) In view of Proposition 4.4, we infer

$$\sup_{\rho \leq \rho_0} |u_n(\rho) - \mu_n^{-\alpha} Q(\rho/\mu_n)| \leq \mu_n^{2-\alpha} \sup_{\rho \leq \rho_0} |w_{\text{int}}(\rho/\mu_n)| \lesssim \mu_n^{s_c-1}.$$

Since  $\mu_n \rightarrow 0$  as  $n \rightarrow \infty$ , result follows.

(iii) In view of (4.3),

$$\Lambda Q(e^{-3\pi/2\omega} \rho_0/\mu_n) \Lambda Q(\rho_0/\mu_n) < 0,$$

so by intermediate value theorem, there exists a zero of  $\Lambda Q$  in the interval  $[e^{-3\pi/2\omega} \rho_0/\mu_n, \rho_0/\mu_n]$ . In particular,

$$(5.23) \quad e^{-3\pi/2\omega} \frac{\rho_0}{\mu_n} \leq \rho_{\Lambda Q, n} \leq \frac{\rho_0}{\mu_n}.$$

Also, if

$$e^{-2\pi/\omega} \rho_0 \leq \rho \leq \rho_0,$$

then  $\rho/\mu_n \gg 1$  for  $n \geq N \gg 1$ . Thus, from (4.3) and Proposition 4.4 since

$$\sup_{0 \leq \rho \leq \rho_0/\mu_n} (1 + \rho)^{(d/2)-3} |\Lambda w_{\text{int}}| \lesssim 1,$$

it follows that

$$\begin{aligned} \Lambda u_n(\rho) &= \mu_n^{-\alpha} (\Lambda Q + \mu_n^2 \Lambda w_{\text{int}})(\rho/\mu_n) \\ &\propto \mu_n^{s_c-1} \rho^{1-d/2} \left[ \sin(\omega \log \rho - \omega \log \mu_n + \delta_2) + \mathcal{O}_{\rho \rightarrow 0}(\rho_0^2) \right]. \end{aligned}$$

Thus,

$$\left| \omega \log \rho_{0,n} - \omega \log \mu_n - \omega \log \rho_{\Lambda Q, n} \right| \lesssim \rho_0^2.$$

Hence,

$$\rho_{0,n} = \mu_n \rho_{\Lambda Q, n} e^{\mathcal{O}(\rho_0^2)} = \mu_n \rho_{\Lambda Q, n} \left[ 1 + \mathcal{O}_{\rho_0 \rightarrow 0}(\rho_0^2) \right].$$

Furthermore, since (5.23) holds, we deduce

$$e^{-2\pi/\omega} \rho_0 < \rho_{0,n} < \rho_0. \quad \square$$

REMARK 1. — The statements of Proposition 5.1 and Corollary 5.2 yield Theorem 1.

### 6. DISSIPATIVITY OF LINEARIZED OPERATOR

We now start the study of the dynamical stability of self-similar profiles. Our aim in this section is to realize the linearized operator as a compact perturbation of a maximal accretive operator in a *global in space* Sobolev norm. From now on, we assume  $d = 3$ .

*Linearized wave equation.* — Recall from Section 2 the definition of similarity transformation variables:

$$(6.1) \quad \tilde{\Psi}(s, y) = (T - t)^\alpha \Phi(t, x), \quad s = -\log(T - t), \quad y = \frac{x}{T - t}.$$

which maps the wave equation (1.1) onto

$$(6.2) \quad \begin{aligned} \partial_s^2 \tilde{\Psi} &= -2y \cdot \nabla \partial_s \tilde{\Psi} - (1 + 2\alpha) \partial_s \tilde{\Psi} \\ &\quad + \sum_{i,j} (\delta_{ij} - y_i y_j) \partial_{y_i} \partial_{y_j} \tilde{\Psi} - 2(1 + \alpha) y \cdot \nabla \tilde{\Psi} - \alpha(1 + \alpha) \tilde{\Psi} + |\tilde{\Psi}|^{p-1} \tilde{\Psi}. \end{aligned}$$

We write the above as a system of linearized equations near  $u_n$ . For the perturbations:

$$\Psi = \tilde{\Psi} - u_n, \quad \Omega = -\partial_s \Psi - \Lambda \Psi,$$

we have

$$(6.3) \quad \partial_s X = \mathcal{M}X + G, \quad X = \begin{pmatrix} \Psi \\ \Omega \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ -|\tilde{\Psi}|^{p-1}\tilde{\Psi} + u_n^p + pu_n^{p-1}\Psi \end{pmatrix},$$

where

$$(6.4) \quad \mathcal{M} = - \begin{pmatrix} \Lambda & 1 \\ \Delta + pu_n^{p-1} & \Lambda + 1 \end{pmatrix}.$$

From now on, we write

$$(6.5) \quad \Psi_j = \nabla^j \Psi, \quad \Omega_j = \nabla^j \Omega,$$

where

$$\nabla^j = \begin{cases} \Delta^i & j = 2i, \\ \nabla \Delta^i & j = 2i + 1. \end{cases}$$

LEMMA 6.1 (Commuting with derivatives). — For  $k \in \mathbb{N}$ , there holds

$$\nabla^k \mathcal{M}X = \mathcal{M}_k \nabla^k X + \tilde{\mathcal{M}}_k X,$$

where

$$(6.6) \quad \mathcal{M}_k = - \begin{pmatrix} \Lambda + k & 1 \\ \Delta & \Lambda + k + 1 \end{pmatrix},$$

and  $\tilde{\mathcal{M}}_k$  satisfies the pointwise bound

$$(6.7) \quad |\tilde{\mathcal{M}}_k X| \lesssim_k \begin{pmatrix} 0 \\ \sum_{j=0}^k \langle \rho \rangle^{-2+j-k} |\nabla^j \Psi| \end{pmatrix}.$$

*Proof.* — Direct computation yields the following formulas

$$[\nabla^k, V] = \sum_{j \leq k-1} c_j \nabla^{k-j} V \nabla^j, \quad [\nabla^k, \Lambda] = k \nabla^k.$$

Hence, by Lemma A.1, since  $\partial_\rho^k(u_n^{p-1}) = \mathcal{O}(\rho^{-2-k})$  as  $\rho \rightarrow \infty$  for all  $k$ ,

$$\nabla^k(\Delta + pu_n^{p-1})\Psi = \Delta \Psi_k + \mathcal{O}\left(\sum_{j=0}^k \langle \rho \rangle^{-2+j-k} |\nabla^j \Psi|\right)$$

and

$$\nabla^k \Lambda \Omega = (\Lambda + k)\Omega_k, \quad \nabla^k(\Lambda + 1)\Omega = (\Lambda + k + 1)\Omega_k. \quad \square$$

6.1. SUBCOERCIVITY. — Let us introduce some notations. First, recall the definition of  $H_k$  from Section 2.

*Weighted  $L^2$ -space.* — We also define for  $\gamma > 0$ , the weighted  $L^2$ -space  $L_\gamma^2$  as the completion of  $C_c^\infty(\mathbb{R}^3)$  with respect to the norm induced by the inner product

$$(\Psi, \tilde{\Psi})_{L_\gamma^2} = \int_{\mathbb{R}^3} \Psi \tilde{\Psi} \langle \rho \rangle^{-2\gamma} dy$$

where  $\langle \cdot \rangle$  denotes the Japanese bracket. We write  $\|\Psi\|_{L_\gamma^2}^2 = (\Psi, \Psi)_{L_\gamma^2}$ .

LEMMA 6.2. — Recall the notations for the spaces  $H_k$  and  $L_{k+2}^2$  above. Then for all  $k \in \mathbb{N}$ , the embedding  $\iota : H_{k+1} \hookrightarrow L_{k+2}^2$  is compact.

*Proof.* — An improved Hardy's inequality (see [4]) states that for all  $\alpha \in 2\mathbb{Z}$  and  $f \in C_c^\infty(\mathbb{R}^3 \setminus B_1(0))$ ,

$$\int_{\mathbb{R}^3} \frac{|f|^2}{|y|^{2+\alpha}} dy \lesssim \int_{\mathbb{R}^3} \frac{|\nabla f|^2}{|y|^\alpha} dy.$$

Also an improved Hardy-Rellich inequality (see [4]) states that for all  $\beta \in 2\mathbb{Z}$  and  $f \in C_c^\infty(\mathbb{R}^3 \setminus B_1(0))$

$$\int_{\mathbb{R}^3} \frac{|f|^2}{|y|^{4+\beta}} dy \lesssim \int_{\mathbb{R}^3} \frac{|\Delta f|^2}{|y|^\beta} dy.$$

By repeatedly applying these inequalities, starting with  $f = (1 - \chi)\Psi$  for the cut-off function  $\chi$  defined in Section 2, we infer for all  $\Psi \in C_c^\infty(\mathbb{R}^3)$ ,

$$\begin{aligned} \|\Psi\|_{L_{k+1}^2(\mathbb{R}^3 \setminus B_1(0))} &\lesssim \int_{\mathbb{R}^3} \frac{|(1 - \chi)\Psi|^2}{|y|^{2(k+1)}} dy \lesssim \int_{\mathbb{R}^3} \frac{|\Delta((1 - \chi)\Psi)|^2}{|y|^{2(k-1)}} dy \\ &\lesssim \dots \lesssim \int_{\mathbb{R}^3} \frac{|\nabla^k((1 - \chi)\Psi)|^2}{|y|^2} dy \lesssim \int_{\mathbb{R}^3} |\nabla^{k+1}((1 - \chi)\Psi)|^2 dy \lesssim \|\Psi\|_{H_{k+1}}^2. \end{aligned}$$

By density, above inequality holds also for all  $\Psi \in H_{k+1}$ . On the other hand, by Rellich-Kondrachov theorem, the embedding

$$\iota : H_{k+1} \hookrightarrow L_{\text{loc}}^2(\mathbb{R}^3) := \{\Psi | \chi\Psi \in L^2(\mathbb{R}^3) \text{ for all } \chi \in C_c^\infty(\mathbb{R}^3)\}$$

is compact. Combining the two and using smallness of  $\langle \rho \rangle^{-2}$  for large  $\rho$ , the result follows.  $\square$

LEMMA 6.3 (Subcoercivity estimate). — *For all  $k \geq 0$ , there exists  $\mu_n > 0$  with  $\lim_{n \rightarrow \infty} \mu_n = \infty$  and  $(\Pi_i)_{i=1}^n \in H_{k+1}$ ,  $c_n > 0$  such that for all  $\Psi \in H_{k+1}$ ,*

$$(6.8) \quad \|\Psi\|_{H_{k+1}}^2 \geq \mu_n \sum_{j=0}^k \int_{\mathbb{R}^3} |\nabla^j \Psi|^2 \langle \rho \rangle^{-2(k+2-j)} dy - c_n \sum_{i=1}^n (\Psi, \Pi_i)_{L_{k+2}^2}^2.$$

*Proof.* — Given  $T \in L_{k+2}^2$ , the antilinear map  $h \mapsto (T, h)_{k+2}$  is continuous on  $H_{k+1}$  since

$$(h, h)_{L_{k+2}^2} \lesssim (h, h)_{H_{k+1}}$$

by Lemma 6.2. By Riesz, there exists a unique  $L(T) \in H_{k+1}$  such that

$$(6.9) \quad \forall h \in H_{k+1}, \quad (L(T), h)_{H_{k+1}} = (T, h)_{L_{k+2}^2}$$

and by setting  $h = L(T)$ , we infer that  $L : L_{k+2}^2 \rightarrow H_{k+1}$  is a bounded linear map. By Lemma 6.2, the map  $\iota \circ L : L_{k+2}^2 \rightarrow L_{k+2}^2$  is compact. If  $\Psi_i = L(T_i)$ ,  $i = 1, 2$ , then

$$(L(T_1), T_2)_{L_{k+2}^2} = (\Psi_1, T_2)_{L_{k+2}^2} = (\Psi_1, L(T_2))_{H_{k+1}} = (\Psi_1, \Psi_2)_{H_{k+1}}.$$

Similarly,

$$(T_1, L(T_2))_{L_{k+2}^2} = (\Psi_1, \Psi_2)_{H_{k+1}} = (L(T_1), T_2)_{L_{k+2}^2},$$

i.e.,  $L$  is self-adjoint on  $L_{k+2}^2$ . Since (6.9) implies that  $L$  is a positive operator, there exists an  $L_{k+2}^2$ -orthonormal eigenbasis  $(\Pi_{n,i})_{1 \leq i \leq I(n)}$  of  $L$  with positive eigenvalues  $\lambda_n \rightarrow 0$ . The eigenvalue equation implies  $\Pi_{n,i} \in H_{k+1}$ . Let

$$\mathcal{A}_n = \left\{ \Psi \in H_{k+1} \mid (\Psi, \Psi)_{L_{k+2}^2} = 1, (\Psi, \Pi_{j,i})_{L_{k+2}^2} = 0, 1 \leq i \leq I(j), 1 \leq j \leq n \right\}$$

and consider the minimization problem

$$I_n = \inf_{\Psi \in \mathcal{A}_n} (\Psi, \Psi)_{H_{k+1}},$$

whose infimum is attained at some  $\Psi \in \mathcal{A}_n$  since the embedding  $\iota : H_{k+1} \hookrightarrow L^2_{k+2}$  is compact. Also, by a standard Lagrange multiplier argument,

$$\forall h \in H_{k+1}, \quad (\Psi, h)_{H_{k+1}} = \sum_{j=1}^n \sum_{i=1}^{I(j)} \beta_{i,j} (\Pi_{j,i}, h)_{L^2_{k+2}} + \beta (\Psi, h)_{L^2_{k+2}}.$$

Set  $h = \Pi_{j,i}$  and since  $\Pi_{j,i}$  is an eigenvector of  $L$ , we infer  $\beta_{i,j} = 0$  and in view of (6.9),  $L(\Psi) = \beta^{-1}\Psi$ . Together with the orthogonality conditions,  $\beta^{-1} \leq \lambda_{n+1}$ . Hence

$$(6.10) \quad I_n = (\Psi, \Psi)_{H_{k+1}} = \beta (\Psi, \Psi)_{L^2_{k+2}} \geq \frac{1}{\lambda_{n+1}}.$$

For all  $\varepsilon > 0$ ,  $k \geq 1$ , from Gagliardo-Nirenberg interpolation inequality with weight (see [10]) together with Young's inequality, we infer

$$\begin{aligned} \sum_{j=0}^k \int_{\mathbb{R}^3} |\nabla^j \Psi|^2 \langle \rho \rangle^{-2(k+2)(k+1-j)/(k+1)} dy \\ \leq \varepsilon \int_{\mathbb{R}^3} |\nabla^{k+1} \Psi|^2 dy + c_{\varepsilon,k} \int_{\mathbb{R}^3} |\Psi|^2 \langle \rho \rangle^{-2(k+2)} dy. \end{aligned}$$

Together with (6.10), we have that for all  $\Psi$  satisfying orthogonality condition of  $\mathcal{A}_n$ ,

$$\sum_{j=0}^k \int_{\mathbb{R}^3} |\nabla^j \Psi|^2 \langle \rho \rangle^{-2(k+2-j)} dy \leq (\varepsilon + c_{\varepsilon,k} \lambda_{n+1}) \|\Psi\|_{H_{k+1}}^2.$$

Since  $\lambda_n \rightarrow 0$ , there exists a choice of  $\varepsilon_n \rightarrow 0$  such that  $\varepsilon_n + c_{\varepsilon_n,k} \lambda_{n+1} \rightarrow 0$  and this yields (6.8).  $\square$

6.2. DISSIPATIVITY. — We now turn to the fundamental dissipativity property. Let us introduce some notations.

*Sobolev space.* — Recall (6.5) and the definition of  $H_k$  from Section 2. Let

$$(6.11) \quad \mathbb{H}_k := H_{k+1} \times H_k$$

with the inner product:

$$(6.12) \quad \langle X, \tilde{X} \rangle = \underbrace{(\Psi_{k+1}, \tilde{\Psi}_{k+1}) + (\Omega_k, \tilde{\Omega}_k)}_{:= \langle X, \tilde{X} \rangle_1} + \underbrace{\int_{\mathbb{R}^3} \chi(\Psi \tilde{\Psi} + \Omega \tilde{\Omega}) dy}_{:= \langle X, \tilde{X} \rangle_2}$$

for

$$X = \begin{pmatrix} \Psi \\ \Omega \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} \tilde{\Psi} \\ \tilde{\Omega} \end{pmatrix}.$$

Further, we define the domain of  $\mathcal{M}$

$$D(\mathcal{M}) = \{X \in \mathbb{H}_k \mid \mathcal{M}X \in \mathbb{H}_k\},$$

which is a Banach space equipped with the graph norm

$$\|X\|_{D(\mathcal{M})} = \|X\|_{\mathbb{H}_k} + \|\mathcal{M}X\|_{\mathbb{H}_k}.$$

*Spherical harmonics.* — Denote by  $\Delta_{\mathbb{S}^{d-1}}$  the Laplace-Beltrami operator defined on the unit sphere  $\mathbb{S}^{d-1}$ . Then we can write

$$(6.13) \quad \Delta = \frac{\partial^2}{\partial \rho^2} + \frac{d-1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \Delta_{\mathbb{S}^{d-1}} =: \mathcal{L} + \rho^{-2} \Delta_{\mathbb{S}^{d-1}}.$$

Denote by  $Y^{(\ell, m)}$  the orthonormal  $\Delta_{\mathbb{S}^{d-1}}$ -eigenbasis (e.g. spherical harmonics if  $d = 3$ ) of  $L^2(\mathbb{S}^{d-1})$  with discrete eigenvalues

$$(6.14) \quad -\lambda_m = -m(m+d-2)$$

for  $m \geq 0$ . We fix  $d = 3$  and define the space of test functions

$$\mathcal{D} = \left\{ X = \sum_{\ell, m} X_{\ell, m}(\rho) Y^{(\ell, m)} \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^2) \text{ is a finite sum} \right\}.$$

Note then, that  $\mathcal{D}$  is dense in  $\mathbb{H}_k$ .

**PROPOSITION 6.4** (Maximal dissipativity). — *For all  $k \geq 3$ , there exist  $c_k > 0$  and  $(X_i)_{1 \leq i \leq N} \in \mathbb{H}_k$  such that for the finite rank projection operator*

$$\mathcal{P} = \sum_{i=1}^N \langle \cdot, X_i \rangle X_i,$$

*the modified operator*

$$\tilde{\mathcal{M}} = \mathcal{M} - \mathcal{P}$$

*is dissipative:*

$$(6.15) \quad \forall X \in D(\mathcal{M}), \quad \langle -\tilde{\mathcal{M}}X, X \rangle \geq c_k \langle X, X \rangle$$

*and is maximal:*

$$(6.16) \quad \forall R > 0, F \in \mathbb{H}_k, \quad \exists X \in D(\mathcal{M}) \quad \text{such that} \quad (-\tilde{\mathcal{M}} + R)X = F.$$

*Proof*

*Step 1 (Dissipativity on dense subset).* — We claim that the bound (6.15) for functions in

$$(6.17) \quad \mathcal{D}_R = \left\{ X \in C^\infty(\mathbb{R}^3, \mathbb{R}^2) \mid \sum_{m=0}^{k+1} \sup_{\mathbb{R}^3} \rho^{\alpha+R+m} (|\nabla^m \Psi| + \mathbb{1}_{m \geq 1} |\nabla^{m-1} \Omega|) < \infty \right\},$$

where  $R > 0$  is chosen sufficiently large so integrating by parts is justified. Integrate by parts the principal part of the inner product defined in (6.12):

$$\begin{aligned} \langle -\mathcal{M}X, X \rangle_1 &= (\nabla^{k+2}(\mathcal{M}X)_\Psi, \Psi_k) - (\nabla^k(\mathcal{M}X)_\Omega, \Omega_k) \\ &= \int_{\mathbb{R}^3} \left[ \nabla((\Lambda + k)\Psi_k + \Omega_k) \cdot \nabla \Psi_k + (\Delta \Psi_k + (1 + k + \Lambda)\Omega_k - (\tilde{\mathcal{M}}_k X)_\Omega) \cdot \Omega_k \right] dy \\ &= \int_{\mathbb{R}^3} \left[ \nabla((\Lambda + k)\Psi_k) \cdot \nabla \Psi_k + (1 + k + \Lambda)\Omega_k \cdot \Omega_k - (\tilde{\mathcal{M}}_k X)_\Omega \cdot \Omega_k \right] dy \\ &= (-s_c + k + 1) \left[ (\nabla \Psi_k, \nabla \Psi_k) + (\Omega_k, \Omega_k) \right] - \int_{\mathbb{R}^3} (\tilde{\mathcal{M}}_k X)_\Omega \cdot \Omega_k dy, \end{aligned}$$

where in the last equality, we have used the Pohozaev identity. In view of (6.7) and by Young’s inequality, we infer

$$\left| \int_{\mathbb{R}^3} (\tilde{\mathcal{M}}_k X)_\Omega \cdot \Omega_k \, dy \right| \leq \varepsilon \int_{\mathbb{R}^3} |\Omega_k|^2 \, dy + C_{\varepsilon,k} \sum_{j=0}^k \int_{\mathbb{R}^3} |\nabla^j \Psi|^2 \langle \rho \rangle^{-4+2j-2k} \, dy.$$

Taking  $\varepsilon > 0$  small, it follows that

$$\langle -\mathcal{M}X, X \rangle_1 \geq 2c_k \left[ (\Psi_{k+1}, \Psi_{k+1}) + (\Omega_k, \Omega_k) \right] - C_k \sum_{j=0}^k \int_{\mathbb{R}^3} |\nabla^j \Psi|^2 \langle \rho \rangle^{-4+2j-2k} \, dy.$$

We also lower bound the non-principal part:

$$\begin{aligned} \langle -\mathcal{M}X, X \rangle_2 &= - \int_{\mathbb{R}^3} \chi \left[ (\mathcal{M}X)_\Psi \Psi + (\mathcal{M}X)_\Omega \Omega \right] \, dy \\ &= \int_{\mathbb{R}^3} \chi \left[ (\Lambda \Psi + \Omega) \Psi + ((\Delta + pu_n^{p-1}) \Psi + (1 + \Lambda) \Omega) \Omega \right] \, dy \\ &\geq -C \int_{|y| \leq 2} \left[ |\Psi|^2 + |\Delta \Psi|^2 + |\Omega|^2 + |\nabla \Omega|^2 \right] \, dy \end{aligned}$$

where the last inequality follows since  $\chi = 0$  for  $|y| \geq 2$ . Thus, by adding the principal and non-principal parts, we infer

$$\langle -\mathcal{M}X, X \rangle \geq 2c_k \langle X, X \rangle - C_k \sum_{j=0}^k \int_{\mathbb{R}^3} |\nabla^j \Psi|^2 \langle \rho \rangle^{-4+2j-2k} \, dy - C \|X\|_{H^2(|y| \leq 2)}^2.$$

We conclude using (6.8) and an analogous result for  $\Omega$  that

$$\langle -\mathcal{M}X, X \rangle \geq c_k \langle X, X \rangle - C \left[ \sum_{i=1}^N (\Psi, \Pi_i)_{L^2_{k+2}}^2 + \sum_{i=1}^N (\Omega, \Xi_i)_{L^2_{k+1}}^2 \right]$$

for  $(\Pi_i)$  as in Lemma 6.3 and for some  $\Xi_i \in L^2_{k+1}$ . Since the linear form

$$X = (\Psi, \Omega) \mapsto \sqrt{C} (\Psi, \Pi_i)_{L^2_{k+2}}$$

is continuous on  $\mathbb{H}_k$ , by Riesz theorem, there exists  $X_i \in \mathbb{H}_k$  such that

$$\forall X \in \mathbb{H}_k, \quad \langle X, X_i \rangle = (\Psi, \Pi_i)_{L^2_{k+2}}$$

and similarly for  $(\Xi_i)$ . Hence, the claim (6.15) follows for all  $X \in \mathcal{D}_R$ .

*Step 2 (ODE formulation of maximality).* — Next, we claim that for all  $R$  sufficiently large,

$$(6.18) \quad \forall F \in \mathcal{D}, \quad \exists! X \in \mathbb{H}_k \quad \text{such that} \quad (-\mathcal{M} + R)X = F.$$

Furthermore, we claim that  $X \in \mathcal{D}_R$ . Note that this is equivalent to

$$(6.19) \quad \begin{cases} (\Lambda + R)\Psi + \Omega = F_\Psi, \\ (\Delta + pu_n^{p-1})\Psi + (\Lambda + R + 1)\Omega = F_\Omega. \end{cases}$$



Let  $F \in \mathcal{D}$ . Then, solving for  $\Psi$ , we have

$$(6.20) \quad [\Delta - (\Lambda + R + 1)(\Lambda + R) + pu_n^{p-1}]\Psi = \underbrace{F_\Omega - (\Lambda + R + 1)F_\Psi}_{:=H}.$$

Since  $\Lambda$  commutes with  $\Delta_{\mathbb{S}^{d-1}}$ , we can write

$$F = \sum_{\ell, m} F_{\ell, m} Y^{(\ell, m)}, \quad H = \sum_{\ell, m} H_{\ell, m} Y^{(\ell, m)}$$

as a finite sum where  $H_{\ell, m}(\rho) Y^{(\ell, m)} \in C_c^\infty(\mathbb{R}^3)$ . Then, recalling the definitions (6.13) and (6.14) of  $\mathcal{L}$  and  $\lambda_m$ , the solution is of the form

$$(6.21) \quad \begin{aligned} \Psi &= \sum_{\ell, m} Y^{(\ell, m)} \Psi_{\ell, m}, \\ \left[ \mathcal{L} - \rho^{-2} \lambda_m - (\Lambda + R + 1)(\Lambda + R) + pu_n^{p-1} \right] \Psi_{\ell, m}(\rho) &= H_{\ell, m}(\rho). \end{aligned}$$

By Lemma B.2, it follows that for all  $R$  sufficiently large and  $F_{\ell, m} Y^{(\ell, m)} \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^2)$ , there exists unique solution  $\Psi_{\ell, m}(\rho) Y^{(\ell, m)} \in H^{k+1}(\mathbb{R}^3)$  to (6.21). Hence, there exists a unique  $\Omega_{\ell, m}(\rho) Y^{(\ell, m)} \in H^k(\mathbb{R}^3)$  given by first equation of (6.19) so that  $X_{\ell, m}(\rho) Y^{(\ell, m)} = (\Psi_{\ell, m}, \Omega_{\ell, m}) Y^{(\ell, m)} \in \mathbb{H}_k$  smooth. Thus, we have (6.18). Also, from the decay properties of each  $X_{\ell, m}$  proved in Lemma B.2, we infer  $X \in \mathcal{D}_R$ .

*Step 3 (Density of  $\mathcal{D}_R$ ).* — Now, we extend these results from  $\mathcal{D}_R$  to  $D(\mathcal{M})$ . We claim that for  $R$  large,  $\mathcal{D}_R \subset D(\mathcal{M})$  is dense. For  $X \in D(\mathcal{M})$ , we have  $X, \mathcal{M}X \in \mathbb{H}_k$  so there exists a sequence  $(Y_n) \in \mathcal{D}$  such that

$$Y_n \longrightarrow (-\mathcal{M} + R)X \quad \text{in } \mathbb{H}_k.$$

By (6.18) and Lemma B.2, there exists unique  $X_n \in \mathbb{H}_k$  smooth solution to

$$(-\mathcal{M} + R)X_n = Y_n \longrightarrow (-\mathcal{M} + R)X, \quad X_n \in \mathbb{H}_k.$$

It suffices to prove the  $X_n \rightarrow X$  in  $\mathbb{H}_k$ . Recall that for  $R$  sufficiently large all integration by parts used to prove (6.15) is justified. Then since  $X_n \in \mathcal{D}_R$ , (6.15) holds for  $X_n - X_m$  i.e.,

$$\begin{aligned} \langle Y_n - Y_m, X_n - X_m \rangle &= \langle (-\mathcal{M} + R)(X_n - X_m), X_n - X_m \rangle \\ &= \langle (-\mathcal{M} + \mathcal{P})(X_n - X_m), X_n - X_m \rangle - \langle \mathcal{P}(X_n - X_m), X_n - X_m \rangle + R \|X_n - X_m\|_{\mathbb{H}_k}^2 \\ &\geq R \|X_n - X_m\|_{\mathbb{H}_k}^2 - \langle \mathcal{P}(X_n - X_m), X_n - X_m \rangle. \end{aligned}$$

Since  $\mathcal{P}$  is a bounded operator, we infer for  $R$  large,

$$\frac{R}{2} \|X_n - X_m\|_{\mathbb{H}_k} \leq \|Y_n - Y_m\|_{\mathbb{H}_k}.$$

In view of the convergence of  $(Y_n)$  in  $\mathbb{H}_k$ , we deduce that  $(X_n)$  is a Cauchy sequence hence, convergent in  $\mathbb{H}_k$  to say,  $\tilde{X}$ . Then  $\tilde{X} - X \in \mathbb{H}_k$  and

$$(-\mathcal{M} + R)(\tilde{X} - X) = 0$$

as distributions. By the uniqueness statement in (6.18), it follows that  $\tilde{X} = X$  i.e.,

$$X_n \longrightarrow X, \quad \mathcal{M}X_n \longrightarrow \mathcal{M}X \quad \text{in } \mathbb{H}_k \quad \iff \quad X_n \longrightarrow X \quad \text{in } D(\mathcal{M}).$$

Hence,  $\mathcal{D}_R$  is dense in  $D(\mathcal{M})$  as claimed.

*Step 4 (Conclusion).* — Since (6.15) holds for all  $X \in \mathcal{D}_R$ , by density of  $\mathcal{D}_R$ , we have dissipativity i.e., (6.15) holds for all  $X \in D(\mathcal{M})$ . It remains to prove (6.16). Let  $F \in \mathbb{H}_k$ . There exists  $(F_n) \in \mathcal{D}$  such that

$$F_n \longrightarrow F \quad \text{in } \mathbb{H}_k.$$

By (6.18), there exists  $X_n \in \mathbb{H}_k$  solution to

$$(-\mathcal{M} + R)X_n = F_n.$$

Using (6.15) and arguing as in the proof of density, we infer for  $R$  large,

$$\frac{R}{2} \|X_n - X_m\|_{\mathbb{H}_k} \leq \|F_n - F_m\|_{\mathbb{H}_k},$$

so  $X_n$  has a limit say,  $X \in \mathbb{H}_k$ . Since  $F_n$  converges to  $F$  in  $\mathbb{H}_k$  and  $D(\mathcal{M})$  is a Banach space, we infer

$$(-\mathcal{M} + R)X = F, \quad X \in D(\mathcal{M}).$$

Thus we have shown that for  $R$  large,

$$(6.22) \quad \forall F \in \mathbb{H}_k, \quad \exists X \in D(\mathcal{M}) \quad \text{such that} \quad (-\mathcal{M} + R)X = F.$$

Now we prove this for  $\tilde{\mathcal{M}}$ . Let  $F \in \mathbb{H}_k$ . Since  $\mathcal{P}$  is bounded, for  $R$  large, by (6.15), for  $X$  as in (6.22),

$$\langle F, X \rangle = \langle (-\mathcal{M} + R)X, X \rangle = \langle (-\tilde{\mathcal{M}} - \mathcal{P} + R)X, X \rangle \geq \frac{R}{2} \|X\|_{\mathbb{H}_k}^2.$$

Thus, for all  $F \in \mathbb{H}_k$ , solution  $X$  to (6.22) is unique i.e.,  $(-\mathcal{M} + R)^{-1}$  is well-defined on  $\mathbb{H}_k$  with

$$\|(-\mathcal{M} + R)^{-1}\| \lesssim \frac{1}{R}.$$

Hence,

$$-\tilde{\mathcal{M}} + R = -\mathcal{M} + \mathcal{P} + R = (-\mathcal{M} + R)[\text{id} + (-\mathcal{M} + R)^{-1}\mathcal{P}]$$

is invertible on  $\mathbb{H}_k$  for  $R$  large which yields (6.16). An elementary induction argument ensures that (6.16) holds for all  $R > 0$  (see Proposition 3.14 from [11]).  $\square$

### 7. GROWTH BOUNDS FOR DISSIPATIVE OPERATORS

In this section, we recall some classical facts on growth bounds for compact perturbations of maximal accretive operators. We realize the linearized operator defined on the real vector space from previous sections as a real operator on the corresponding complex space. This is essential in the spectral theory of the linearized operator.

In this section,  $(H, \langle \cdot, \cdot \rangle)$  is a Hilbert space and  $A$  is a closed operator defined on a dense domain  $D(A)$ . Define the adjoint operator  $A^*$  on the domain

$$D(A^*) = \{X \in H \mid Y \in D(A) \mapsto \langle X, AY \rangle \text{ extends to an element of } H^*\}$$

to be  $X \mapsto A^*X$  the unique element of  $H$  given by Riesz theorem such that

$$\forall Y \in D(A), \quad \langle A^*X, Y \rangle = \langle X, AY \rangle.$$

For  $\nu \in \mathbb{R}$ , we denote by

$$\Lambda_\nu(A) = \{\lambda \in \sigma(A) \mid \operatorname{Re}(\lambda) \geq \nu\}, \quad V_\nu(A) = \bigoplus_{\lambda \in \Lambda_\nu(A)} \ker(A - \lambda).$$

LEMMA 7.1 (Perturbative exponential decay). — *Let  $T_0$  and  $T$  be the strongly continuous semigroups generated by maximally dissipative operators  $A_0$  and  $A = A_0 + K$  where  $K$  is a compact operator on  $H$ . Then for all  $\nu > 0$ , the following holds:*

(i) *The set  $\Lambda_\nu(A)$  is finite and each eigenvalue  $\lambda \in \Lambda_\nu(A)$  has finite algebraic multiplicity  $k_\lambda$ .*

*We have  $\Lambda_\nu(A) = \overline{\Lambda_\nu(A^*)}$  and  $\dim V_\nu(A^*) = \dim V_\nu(A)$ . The direct sum decomposition*

$$H = V_\nu(A) \oplus V_\nu^\perp(A^*)$$

*is preserved by  $T(s)$  and there holds*

$$\forall X \in V_\nu^\perp(A^*), \quad \|T(s)X\| \leq M_\nu e^{\nu s} \|X\|.$$

(ii) *The restriction of  $A$  to  $V_\nu(A)$  is given by a direct sum of Jordan blocks. Each block corresponds to an invariant subspace  $J_\lambda$  and the semigroup  $T$  restricted to  $J_\lambda$  is given by*

$$T(s)|_{J_\lambda} = \begin{pmatrix} e^{\lambda s} & s e^{\lambda s} & \dots & \frac{s^{m_\lambda - 1} e^{\lambda s}}{(m_\lambda - 1)!} \\ 0 & e^{\lambda s} & \dots & \frac{s^{m_\lambda - 2} e^{\lambda s}}{(m_\lambda - 2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda s} \end{pmatrix}.$$

*Proof.* — See Lemma 3.3 of [22]. □

COROLLARY 7.2 (Exponential decay modulo finitely many instabilities)

*Let  $\nu > 0$ ,  $T_0, T$  be the strongly continuous semigroups generated by maximally dissipative operators  $A_0$  and  $A = A_0 - \nu + K$  respectively where  $K$  is a compact operator on Hilbert space  $H$ . Then  $\Lambda_0(A)$  is finite and let*

$$H = U \oplus V,$$

*where  $V = V_\nu(A)$  and  $U = V_\nu^\perp(A^*)$  so that  $U$  and  $V$  are invariant subspaces for  $A$ . Then there exists  $C, \delta > 0$  such that*

$$\forall X \in U, \quad \|T(s)X\| \leq C e^{-(\delta/2)s} \|X\|.$$

*Proof.* — We apply Lemma 7.1 to  $\tilde{A} = A_0 + K$  which generates the semigroup  $\tilde{T}$ . Note that  $\Lambda_{\nu/4}(\tilde{A})$  is finite and  $\Lambda_0(A) \subset \Lambda_{\nu/4}(\tilde{A})$ . Let

$$H = U_\nu \oplus V_\nu$$

be the invariant decomposition of  $\tilde{A}$  associated to the set  $\Lambda_{\nu/4}$  with  $V_\nu$  being the image of the spectral projection of the set  $\Lambda_{\nu/4}$ . Then  $U_\nu \subset U$  and

$$U = U_\nu \oplus O_\nu,$$

where  $O_\nu$  is the image of the spectral projection of  $A$  associated with the set  $\Lambda_{\nu/4}(\tilde{A}) \setminus \Lambda_0(A)$ . Then by Lemma 7.1,

$$\forall X \in U_\nu, \quad \|T(s)X\| = e^{-\nu s} \|\tilde{T}(s)X\| \leq M_\nu e^{-(3\nu/4)s} \|X\|.$$

Now for  $X \in U$ , since  $U_\nu$  is invariant under  $T$  and we have exponential decay on  $U_\nu$ , so without loss of generality, assume  $X \in O_\nu$ .  $O_\nu$  is an invariant subspace of  $A$  generated by the eigenvalues  $\lambda$  such that  $-3\nu/4 \leq \operatorname{Re}(\lambda) < 0$ . Then for

$$\delta = \inf \{ \operatorname{Re}(\lambda) \mid 0 < -\operatorname{Re}(\lambda) \leq 3\nu/4, \lambda \text{ is an eigenvalue of } A \},$$

Lemma 7.1 implies that

$$\|T(s)X\|_{O_\nu} \lesssim \sup_{\operatorname{Re}(\lambda) < 0} e^{\lambda s} s^{m_\lambda - 1} \|X\| \lesssim e^{-(\delta/2)s} \|X\|. \quad \square$$

**COROLLARY 7.3.** — *Let  $A, \delta, U$  and  $V$  as in Corollary 7.2. For  $c > 0$ , let  $G(s) \in V$  be such that*

$$\|G(s)\| \leq e^{-(\delta/2)(1+c)s}, \quad \forall s \geq 0.$$

*Then, for  $s_0$  sufficiently large, there exists  $x \in V$  with  $\|x\| \leq e^{-(\delta/2)(1+c/2)s_0}$  such that the solution  $X(s)$  to*

$$\frac{dX(s)}{ds} = AX(s) + G(s), \quad X(s_0) = x \in V,$$

*satisfies the bound*

$$\|X(s)\| \leq e^{-(\delta/2)(1+c/2)s}, \quad \forall s \geq s_0.$$

*Proof.* — By Lemma 7.1, the subspace  $V$  can be further decomposed into invariant subspaces on which  $A$  is represented by Jordan blocks. Therefore, without loss of generality, assume that  $V$  is irreducible and for  $\operatorname{Re}(\lambda) \geq 0$ ,

$$(7.1) \quad A = \lambda + N, \quad e^{sN} = \begin{pmatrix} 1 & s & \cdots & \frac{s^{m_\lambda - 1}}{(m_\lambda - 1)!} \\ 0 & 1 & \cdots & \frac{s^{m_\lambda - 2}}{(m_\lambda - 2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Now consider

$$Y(s) = e^{-sN} e^{(\delta/2)(1+3c/4)s} X(s), \quad \tilde{G}(s) = e^{-sN} e^{(\delta/2)(1+3c/4)s} G(s).$$

Then since  $N$  and  $A$  commute,

$$\frac{dY(s)}{ds} = \left[ \lambda + \frac{\delta}{2} \left( 1 + \frac{3c}{4} \right) \right] Y(s) + \tilde{G}(s), \quad Y(s_0) = y.$$

For  $s_0$  sufficiently large, for all  $s \geq s_0$ ,

$$\|\tilde{G}(s)\| \leq e^{-(c\delta/16)s}.$$

We now run a standard Brouwer type argument for  $Y$ . For  $\|y\| \leq 1$ , define the exit time

$$s^* = \inf \{ s \geq s_0 \mid \|Y(s)\| \geq 1 \}.$$

If  $s^* = \infty$  for some  $\|y\| \leq 1$ , then we are done. Otherwise, the map

$$\Phi : B = \{\|y\| \leq 1\} \longrightarrow S = \{\|y\| = 1\}$$

given by  $\Phi(y) = Y(s^*)$  is well-defined. Note that  $\Phi|_S = \text{id}_S$  and  $\Phi$  is continuous since

$$\frac{d\|Y\|^2}{ds}(s^*) = 2\text{Re}(\lambda) + \delta\left(1 + \frac{3c}{4}\right) + 2\text{Re}\langle \tilde{G}(s^*), Y(s^*) \rangle \geq \frac{\delta}{2}\left(1 + \frac{3c}{4}\right) > 0$$

i.e., the outgoing condition is met. This is a contradiction by Brouwer fixed point theorem. Thus, there exists  $x$  such that for all  $s \geq s_0$ ,

$$\|e^{-sN} X(s)\| \leq e^{-(\delta/2)(1+3c/4)s}.$$

Since  $e^{-sN}$  is invertible with inverse  $e^{sN}$  bounded by  $s^{m\lambda-1}$ , result follows immediately.  $\square$

## 8. FINITE CODIMENSIONAL STABILITY

We are now in position to prove non linear finite codimensional stability of the self-similar profiles for the full problem.

*Choice of parameters.* — In this section, we set  $d = 3$  and  $k = 3$  so that  $H^{k+1}(\mathbb{R}^3)$  is an algebra which we shall later use in the proof of Theorem 2. For convenience, we write

$$\mathbb{H} = \mathbb{H}_3 := H_4 \times H_3.$$

where we recall from Section 2 the definition of  $H_k$ .

*Stable and unstable subspaces.* — Recall from Proposition 6.4 that  $\mathcal{M} - \mathcal{P} + c_k/2$  is maximally dissipative so Corollary 7.2 applies:

$$\Lambda_0(\mathcal{M}) = \{\lambda \in \sigma(\mathcal{M}) \mid \text{Re}(\lambda) \geq 0\}$$

is a finite set with an associated finite dimensional invariant subspace  $V$ . Consider the invariant decomposition

$$\mathbb{H} = U \oplus V$$

and let  $P$  be the associated projection on  $V$ . We denote by  $\mathcal{N}$  the nilpotent part of the matrix representing  $\mathcal{M}$  on  $V$ . Let  $\delta > 0$  such that the conclusions of Corollary 7.2 and 7.3 hold.

*Dampened profile.* — We produce a finite energy initial value by dampening the tail of the self-similar profiles on  $|x| \geq 1$ : for some constant  $n_p > s_c$ , let  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$  be a smooth function

$$(8.1) \quad \eta(r) = \begin{cases} 1 & r \leq 1, \\ r^{-n_p} & r \geq 2, \end{cases}$$

and for smooth self-similar profiles  $u_n$  solving (1.4), we define the dampened profile

$$u_n^D(s, \rho) = \eta(e^{-s}\rho)u_n(\rho).$$

Note that the requirement  $n_p > s_c$  is precisely the condition for  $u_n^D$  to achieve finite energy. Recall the self-similar transformation (6.1). We introduce the perturbation variables  $(\Psi^D, \Omega^D)$ :

$$(8.2) \quad \tilde{\Psi} = \Psi + u_n = \Psi^D + \underbrace{\eta(e^{-s}\rho)u_n}_{=u_n^D}, \quad \Omega - \Lambda u_n = \Omega^D - \eta(e^{-s}\rho)\Lambda u_n.$$

Then the wave equation (6.2) yields

$$(8.3) \quad \begin{cases} \partial_s \Psi^D = -\Lambda \Psi^D - \Omega^D, \\ \partial_s \Omega^D = -\Delta \Psi^D - (\Lambda + 1)\Omega^D - |\tilde{\Psi}|^{p-1}\tilde{\Psi} + \mathcal{E}(s, \rho), \end{cases}$$

where

$$(8.4) \quad \mathcal{E}(s, \rho) = \eta(e^{-s}\rho)u_n^p - (\Delta\eta(e^{-s}\rho))u_n - 2\nabla\eta(e^{-s}\rho) \cdot \nabla u_n.$$

8.1. BOOTSTRAP BOUND AND PROOF OF THEOREM 2. — The heart of the proof of Theorem 2 is the following bootstrap proposition.

PROPOSITION 8.1 (Bootstrap). — *Recall the definition of  $H_k$  from Section 2. Assume  $d = 3$ ,  $k = 3$  and write*

$$\mathbb{H} = \mathbb{H}_3 = H_4 \times H_3.$$

Given  $c \ll 1$  and  $s_0 \gg 1$  to be chosen in the proof, consider  $X(s_0) \in \mathbb{H}$  such that

$$(8.5) \quad \|(I - P)X(s_0)\|_{\mathbb{H}} \leq e^{-(\delta/2)s_0}, \quad \|PX(s_0)\|_{\mathbb{H}} \leq e^{-(\delta/2)(1+c/2)s_0}$$

and for  $(\Psi^D, \Omega^D)$  as in (8.2) and for  $0 \leq j \leq 4$ ,

$$(8.6) \quad \left\| \frac{\langle \rho \rangle^{j+1} \nabla^j \Psi^D(s_0)}{u_n^D} \right\|_{L^\infty(\mathbb{R}^3)} + \left\| \frac{\langle \rho \rangle^{j+1} \mathbb{1}_{j \geq 1} \nabla^{j-1} \Omega^D(s_0)}{u_n^D} \right\|_{L^\infty(\mathbb{R}^3)} \leq e^{-(\delta/2)s_0}.$$

Define the exit time  $s^*$  to be the maximal time such that the following bootstrap bounds hold on  $s \in [s_0, s^*]$ :

$$(8.7) \quad \|e^{s\mathcal{N}}PX(s)\|_{\mathbb{H}} \leq e^{-(\delta/2)(1+3c/4)s},$$

for  $j = 0, 1$  and  $0 < \kappa < 1/4(p + 1)$ ,

$$(8.8) \quad \left\| \frac{\rho^{j-\kappa} \nabla^j \Psi^D(s)}{u_n^D} \right\|_{L^\infty(|y| \geq 1)} \leq 1,$$

for all  $j$  such that  $0 \leq j \leq 4$ ,

$$(8.9) \quad I_j(s) := \int_{\mathcal{D}_j} \rho^{2j-2s_c} \xi(e^{-s}\rho)^{2n_p+1} \left( |\nabla^j \Psi^D(s)|^2 + \mathbb{1}_{j \geq 1} |\nabla^{j-1} \Omega^D(s)|^2 \right) dy \leq 1,$$

where

$$\xi(r) = \eta(r)^{-1/n_p} = \begin{cases} 1 & r \leq 1, \\ r & r \geq 2, \end{cases} \quad \mathcal{D}_j = \begin{cases} \{y \in \mathbb{R}^3 \mid |y| \geq 1\} & 0 \leq j \leq 2, \\ \mathbb{R}^3 & 3 \leq j \leq 4, \end{cases}$$

and for  $\delta/(1 + c) < \delta_0 < \delta$ ,

$$(8.10) \quad \|X(s)\|_{\mathbb{H}} \leq e^{-(\delta_0/2)s}.$$

Then the bootstrap bounds (8.8), (8.9) and (8.10) can be strictly improved in  $s \in [s_0, s^*]$ . Equivalently, if  $s^* < \infty$ , then equality holds for (8.7) at  $s = s^*$ . Furthermore, the following non-linear bound holds:

$$(8.11) \quad \forall s \in [s_0, s^*], \quad \|G(s)\|_{\mathbb{H}} \leq e^{-(\delta/2)(1+c)s}.$$

Let us assume Proposition 8.1 and conclude the proof of Theorem 2.

*Proof of (Proposition 8.1  $\Rightarrow$  Theorem 2).* — Assume Proposition 8.1 holds. Let  $s_0$  be as in Proposition 8.1. Note that the bootstrap bounds (8.9) and (8.10) imply

$$\int_{\mathbb{R}^3} |\Psi^D|^2 dy \leq \int_{|y| \leq 1} |\Psi|^2 dy + \int_{|y| \geq 1} \rho^{-2s_c+2n_p+1} |\Psi^D|^2 dy < \infty$$

and

$$\int_{\mathbb{R}^3} |\Delta^2 \Psi|^2 dy < \infty.$$

Then

$$\|\tilde{\Psi}\|_{H^4(\mathbb{R}^3)}^2 \leq \|u_n^D\|_{L^2(\mathbb{R}^3)}^2 + \|\Psi^D\|_{L^2(\mathbb{R}^3)}^2 + \|u_n\|_{H^4(\mathbb{R}^3)}^2 + \|\Psi\|_{H^4(\mathbb{R}^3)}^2 < \infty.$$

Similarly for  $\Omega^D$ . Thus, we infer

$$\|u_n + \Psi\|_{H^4(\mathbb{R}^3)} + \|\Lambda u_n - \Omega\|_{H^3(\mathbb{R}^3)} \leq C(s)$$

for  $s \in [s_0, s^*]$  so it follows that

$$\|\Phi\|_{\dot{H}^{s_c}(\mathbb{R}^3)} + \|\partial_t \Phi\|_{\dot{H}^{s_c-1}(\mathbb{R}^3)} \leq C(t),$$

so the exit time  $s^*$  defined in the statement of Proposition 8.1 is strictly smaller than the life time provided by the standard Cauchy theory (see [12]).

We now conclude from the Brouwer fixed point argument. Note that for all initial data satisfying (8.5) and (8.6) in the space

$$H = \left\{ (\Psi^D, \Omega^D) \in (H^4 \times H^3)(\mathbb{R}^3) \mid \sum_{j=0}^4 \|\langle \rho \rangle^{j+\alpha+n_p+1} (|\nabla^j \Psi^D| + |\mathbb{1}_{j \geq 1} \nabla^{j-1} \Omega^D|)\|_{L^\infty} < \infty \right\},$$

the non-linear bound (8.11) and (8.7) have been shown to hold on  $[s_0, s^*]$ . Moreover, as explained in the proof of Corollary 7.3, given  $(I - P)X(s_0)$ , after a choice of projection of initial data on the subspace of unstable nodes  $PX(s_0)$ , the solution can be immediately propagated to any time  $t < T$ . This choice is dictated by Corollary 7.3. Furthermore, this choice of  $PX(s_0)$  is unique and is Lipschitz dependent on  $(I - P)X(s_0)$  from Lemma D.1.  $\square$

The rest of this section is devoted to the proof of the bootstrap Proposition 8.1.

8.2. WEIGHTED SOBOLEV BOUNDS. — Recall that we have set  $d = 3$ ,  $k = 3$ . Then, we write  $\mathbb{H} = \mathbb{H}_3$ .

LEMMA 8.2 (Sobolev embedding). — Let  $(\Psi^D, \Omega^D)$  be such that the right hand side of the bound (8.12) is finite. Then, for  $j = 0, 1$ ,

$$(8.12) \quad \left\| \frac{\rho^{j-\kappa} \nabla^j \Psi^D(s)}{u_n^D} \right\|_{L^\infty(|y| \geq 1)} \lesssim \|\nabla^j \Psi^D\|_{L^\infty(|y|=2)} + \left( \sum_{\ell=0}^4 I_\ell(s) \right)^{1/2}.$$

*Proof*

Step 1 (General bound). — We recall the notations for the spherical harmonics from Section 6.2. In particular, we write the spherical harmonic functions as  $Y^{(\ell, m)}$  with eigenvalues  $-\lambda_m = -m(m + 1)$ . We claim that given  $i \in \mathbb{N}$  and  $\beta \in \mathbb{R}$  and for all  $f \in C_{c, \text{rad}}^\infty(\mathbb{R}^3 \setminus \{0\})$ ,

$$(8.13) \quad \int_{\mathbb{R}^3} r^\beta |\nabla^i f(r) Y^{(\ell, m)}(\theta, \varphi)|^2 dx = (1 + o_{m \rightarrow \infty}(1)) \underbrace{\sum_{j=0}^i \binom{i}{j} \lambda_m^{i-j} \int_0^\infty r^{2+\beta+2(j-i)} |f^{(j)}|^2 dr}_{:=S_{i,m}[f]}.$$

We proceed by induction on  $i$ . The claim for  $i = 1, 2$  is proved in Lemma 2.1 from [4]. If the claim holds for  $i = 2k - 1, 2k$ , then by replacing  $f$  in (8.13) by  $(\mathcal{L} - r^{-2}\lambda_m)f$ , where we recall that  $\mathcal{L}$  is the radial part of the Laplacian, we infer

$$\begin{aligned} & \int_{\mathbb{R}^3} r^\beta |\nabla^{i+2} f(r) Y^{(\ell, m)}(\theta, \varphi)|^2 dx \\ &= (1 + o_{m \rightarrow \infty}(1)) \sum_{j=0}^i \binom{i}{j} \lambda_m^{i-j} \int_0^\infty r^{2+\beta+2(j-i)} |\partial_r^j (\partial_r^2 + 2r^{-1}\partial_r - r^{-2}\lambda_m) f|^2 dr \\ &= \sum_{j=0}^i \binom{i}{j} \lambda_m^{i-j} \int_0^\infty r^{2+\beta+2(j-i)} |(\partial_r^{j+2} - \lambda_m r^{-2} \partial_r^j) f|^2 dr + o_{m \rightarrow \infty}(S_{i+2,m}[f]) \\ &= \sum_{j=0}^i \binom{i}{j} \lambda_m^{i-j} \int_0^\infty r^{2+\beta+2(j-i)} (|f^{(j+2)}|^2 + 2\lambda_m r^{-2} |f^{(j+1)}|^2 + \lambda_m^2 r^{-4} |f^{(j)}|^2) dr \\ & \qquad \qquad \qquad + o_{m \rightarrow \infty}(S_{i+2,m}[f]), \end{aligned}$$

where in the last equality we have used integration by parts:

$$\begin{aligned} & -\lambda_m^{i-j+1} \int_0^\infty r^{\beta+2(j-i)} f^{(j+2)} f^{(j)} dr \\ &= \lambda_m^{i-j+1} \int_0^\infty r^{\beta+2(j-i)} |f^{(j+1)}|^2 dr + C_{i,j,\beta} \lambda_m^{i-j+1} \int_0^\infty r^{-2+\beta+2(j-i)} |f^{(j)}|^2 dr \\ &= \lambda_m^{i-j+1} \int_0^\infty r^{\beta+2(j-i)} |f^{(j+1)}|^2 dr + o_{m \rightarrow \infty}(S_{i+2,m}[f]). \end{aligned}$$



Then, we infer

$$\int_{\mathbb{R}^3} r^\beta |\nabla^{i+2} f(\rho) Y^{(\ell,m)}(\theta, \varphi)|^2 dx = (1 + o_{m \rightarrow \infty}(1)) \times \sum_{j=0}^{i+2} \left[ \binom{i}{j} + 2 \binom{i}{j-1} + \binom{i}{j-2} \right] \lambda_m^{i+2-j} \int_0^\infty r^{2+\beta+2(j-i-2)} |f^{(j)}|^2 dr.$$

Hence, the result follows for  $i + 2$ . This concludes the proof of our claim (8.13).

*Step 2 (Interior Bound).* — From the claim, we have that for  $M$  large, for all  $f \in C_{c,\text{rad}}^\infty(\mathbb{R}^3 \setminus \{0\})$  and  $m \geq M$ ,

$$\sum_{j=0}^i \lambda_m^{i-j} \int_0^\infty \rho^{2j+2\alpha-1} |f^{(j)}|^2 d\rho \lesssim_i \int_{\mathbb{R}^3} \rho^{2i+2\alpha-3} |\nabla^i f(\rho) Y^{(\ell,m)}|^2 dx.$$

Also, by induction on  $i$ , we have that for all  $m < M$ ,

$$(8.14) \quad \sum_{j=0}^i \int_0^\infty \rho^{2j+2\alpha-1} |f^{(j)}|^2 d\rho \lesssim_i C_m \sum_{j=0}^i \int_{\mathbb{R}^3} \rho^{2j+2\alpha-3} |\nabla^j f(\rho) Y^{(\ell,m)}|^2 dx.$$

Thus, (8.14) holds for all  $m \in \mathbb{N}$  with some universal constant independent of  $m$ . We now apply this to a function vanishing at 0 and  $\infty$ . Let  $\chi_s \in C_{\text{rad}}^\infty(\mathbb{R}^3)$  and  $\varphi \in C^\infty(\mathbb{R})$  be such that

$$\varphi(\rho) = \begin{cases} 0 & \rho \leq 1, \\ 1 & \rho \geq 2, \end{cases} \quad \chi_s(y) = \begin{cases} \varphi(|y|) & |y| \leq e^s, \\ 1 - \varphi(e^{-s}|y|) & |y| \geq e^s. \end{cases}$$

Write

$$\Psi^D(y) = \sum_{\ell,m} \Psi_{\ell,m}^D(\rho) Y^{(\ell,m)}(\theta, \varphi),$$

and apply (8.14) to  $f(\rho) = \chi_s \Psi_{\ell,m}^D(\rho)$ , we infer,

$$\begin{aligned} \sum_{j=0}^4 \lambda_m^{4-j} \int_2^{e^s} r^{2j+2\alpha-1} |\partial_\rho^j \Psi_{\ell,m}^D|^2 d\rho &\leq \sum_{j=0}^4 \lambda_m^{4-j} \int_0^\infty r^{2j+2\alpha-1} |\partial_\rho^j (\chi_s \Psi_{\ell,m}^D)|^2 d\rho \\ &\lesssim \sum_{j=0}^4 \int_{\mathbb{R}^3} \rho^{2j-2s_c} |\nabla^j (\chi_s \Psi_{\ell,m}^D(\rho) Y^{(\ell,m)}(\theta, \varphi))|^2 dy \\ &\lesssim \sum_{j=0}^4 \int_{|y| \geq 1} \rho^{2j-2s_c} \xi(e^{-s}\rho)^{2n_p+1} |\nabla^j (\Psi_{\ell,m}^D(\rho) Y^{(\ell,m)}(\theta, \varphi))|^2 dy, \end{aligned}$$

where in the last inequality we have used that for all  $e^s \leq \rho \leq e^{2s}$ ,

$$|\partial_\rho^j \chi_s(\rho)| \lesssim_j e^{-js} \lesssim \rho^{-j}.$$

Since the universal constant here does not depend on  $m$ , we sum over  $\ell$  and  $m$  to infer

$$\sum_{\ell, m} \sum_{j=0}^4 \lambda_m^{4-j} \int_2^{e^s} \rho^{2j+2\alpha-1} |\partial_\rho^j \Psi_{\ell, m}^D|^2 d\rho \lesssim \sum_{j=0}^4 I_j(s).$$

Note the universal  $L^\infty$ -bound for spherical harmonics which one can find in [30] states that

$$\|Y^{(\ell, m)}(\theta, \varphi)\|_{L^\infty(\mathbb{S}^2)} \lesssim \lambda_m^{1/4}.$$

Thus, we infer for  $2 \leq |y| \leq e^s$ ,

$$\begin{aligned} \left| \frac{\rho^{-\kappa} \Psi^D(y)}{u_n^D} \right| &\lesssim \|\Psi^D\|_{L^\infty(|y|=2)} + \sum_{\ell, m} \|Y^{(\ell, m)}\|_{L^\infty(\mathbb{S}^2)} \int_2^{e^s} |\partial_\rho(\rho^{\alpha-\kappa} \Psi_{\ell, m}^D)| d\rho \\ &\lesssim \|\Psi^D\|_{L^\infty(|y|=2)} + \sum_{\ell, m} \lambda_m^{1/4} \left( \int_2^{e^s} \frac{d\rho}{\rho^{1+2\kappa}} \right)^{1/2} \left( \int_2^{e^s} \rho^{2\alpha-1} (|\Psi_{\ell, m}^D|^2 + \rho^2 |\partial_\rho \Psi_{\ell, m}^D|^2) d\rho \right)^{1/2} \\ &\leq \|\Psi^D\|_{L^\infty(|y|=2)} + \left( \sum_{\ell, m} \lambda_m^{-3/2} \right)^{1/2} \left( \sum_{\ell, m} \lambda_m^2 \int_2^{e^s} \rho^{2\alpha-1} (|\Psi_{\ell, m}^D|^2 + \rho^2 |\partial_\rho \Psi_{\ell, m}^D|^2) d\rho \right)^{1/2} \\ &\lesssim \|\Psi^D\|_{L^\infty(|y|=2)} + \left( \sum_{\ell=0}^4 I_\ell(s) \right)^{1/2}. \end{aligned}$$

Next, we bound the derivatives of  $\Psi^D$ . Explicit calculation of the derivatives of  $Y^{(\ell, m)}$  yields

$$\|\partial_\theta Y^{(\ell, m)}\|_{L^\infty(\mathbb{S}^2)} + \|\partial_\varphi Y^{(\ell, m)}\|_{L^\infty(\mathbb{S}^2)} \lesssim \lambda_m^{3/4}.$$

Then, for  $2 \leq |y| \leq e^s$ , by writing  $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) = (\rho, \theta, \varphi)$  and  $(n_1, n_2, n_3) = (0, -1, -1)$ , we infer

$$\begin{aligned} \left| \frac{\rho^{1-\kappa} \nabla \Psi^D(y)}{u_n^D} \right| &\lesssim \|\nabla \Psi^D\|_{L^\infty(|y|=2)} + \sum_{i=1}^3 \int_2^{e^s} \sup_{\mathbb{S}^2} |\partial_\rho(\rho^{\alpha+1+n_i-\kappa} \partial_{\tilde{y}_i}(\Psi_{\ell, m}^D(\rho) Y^{(\ell, m)}))| d\rho \\ &\lesssim \|\nabla \Psi^D\|_{L^\infty(|y|=2)} + \sum_{\ell, m} \int_2^{e^s} \left[ \underbrace{\lambda_m^{1/4} |\partial_\rho(\rho^{\alpha+1-\kappa} \partial_\rho \Psi_{\ell, m}^D)|}_{=\partial_\rho \text{ term}} + \underbrace{\lambda_m^{3/4} |\partial_\rho(\rho^{\alpha-\kappa} \Psi_{\ell, m}^D)|}_{=\partial_\theta, \partial_\varphi \text{ terms}} \right] d\rho. \end{aligned}$$

Then, as before,

$$\begin{aligned} (\partial_\rho \text{ term}) &\lesssim \sum_{\ell, m} \lambda_m^{1/4} \left( \int_2^{e^s} \rho^{-1-2\kappa} d\rho \right)^{1/2} \left( \int_2^{e^s} \rho^{2\alpha+1} (|\partial_\rho \Psi_{\ell, m}^D|^2 + \rho^2 |\partial_\rho^2 \Psi_{\ell, m}^D|^2) d\rho \right)^{1/2} \\ &\leq \left( \sum_{\ell, m} \lambda_m^{-3/2} \right)^{1/2} \left( \sum_{\ell, m} \lambda_m^2 \int_2^{e^s} \rho^{2\alpha+1} (|\partial_\rho \Psi_{\ell, m}^D|^2 + \rho^2 |\partial_\rho^2 \Psi_{\ell, m}^D|^2) d\rho \right)^{1/2} \\ &\lesssim \left( \sum_{\ell=0}^4 I_\ell(s) \right)^{1/2} \end{aligned}$$

and similarly,

$$\begin{aligned} (\partial_\theta, \partial_\varphi \text{ terms}) &\lesssim \left( \sum_{\ell,m} \lambda_m^{-3/2} \right)^{1/2} \left( \sum_{\ell,m} \lambda_m^3 \int_2^{e^s} \rho^{2\alpha-1} (|\Psi_{\ell,m}^D|^2 + \rho^2 |\partial_\rho \Psi_{\ell,m}^D|^2) d\rho \right)^{1/2} \\ &\lesssim \left( \sum_{\ell=0}^4 I_\ell(s) \right)^{1/2}. \end{aligned}$$

Thus, we infer for all  $2 \leq |y| \leq e^s$  that

$$\left| \frac{\rho^{1-\kappa} \nabla \Psi^D(y)}{u_n^D} \right| \lesssim \|\nabla \Psi^D\|_{L^\infty(|y|=2)} + \left( \sum_{\ell=0}^4 I_\ell(s) \right)^{1/2}.$$

*Step 3 (Exterior Bound).* — We now propagate the  $L^\infty$ -bound to the region outside of the self-similar scale. From the claim in Step 1, we infer the bound

$$\sum_{j=0}^i \lambda_m^{i-j} \int_0^\infty \rho^{2j+2\alpha+2n_p} |f^{(j)}|^2 d\rho \lesssim_i \sum_{j=0}^i \int_{\mathbb{R}^3} \rho^{2j+2\alpha+2n_p-2} |\nabla^j f(\rho) Y^{(\ell,m)}|^2 dy$$

with some universal constant independent of  $m$ . Using the same  $\eta$  and decomposition of  $\Psi^D$  as in Step 2 and apply the above bound with  $f(\rho) = \tilde{\chi}_s \Psi_{\ell,m}^D(\rho)$  for a cut-off  $\tilde{\chi}_s(y) = \varphi(2e^{-s}|y|)$ , we infer

$$\begin{aligned} \sum_{j=0}^4 \lambda_m^{4-j} \int_{e^s}^\infty r^{2j+2\alpha-1} \xi(e^{-s}\rho)^{2n_p+1} |\partial_\rho^j \Psi_{\ell,m}^D|^2 d\rho \\ \lesssim \sum_{j=0}^4 \int_{|y| \geq 1} \rho^{2j-2s_c} \xi(e^{-s}\rho)^{2n_p+1} |\nabla^j (\Psi_{\ell,m}^D(\rho) Y^{(\ell,m)}(\theta, \varphi))|^2 dy. \end{aligned}$$

Thus, as in Step 2, we infer

$$\sum_{\ell,m} \sum_{j=0}^4 \lambda_m^{4-j} \int_{e^s}^\infty r^{2j+2\alpha-1} \xi(e^{-s}\rho)^{2n_p+1} |\partial_\rho^j \Psi_{\ell,m}^D|^2 d\rho \lesssim \sum_{j=0}^4 I_j(s).$$

Thus, we infer for  $|y| \geq e^s$ ,

$$\left| \frac{\rho^{-\kappa} \Psi^D(y)}{u_n^D} \right| \lesssim \left\| \frac{\rho^{-\kappa} \Psi^D}{u_n^D} \right\|_{L^\infty(|y|=e^s)} + \sum_{\ell,m} \lambda_m^{1/4} \int_{e^s}^\infty e^{-n_p s} |\partial_\rho (\rho^{\alpha-\kappa} \Psi_{\ell,m}^D)| d\rho.$$

Since

$$\begin{aligned} \sum_{\ell,m} \lambda_m^{1/4} \int_{e^s}^\infty e^{-n_p s} |\partial_\rho (\rho^{\alpha-\kappa} \Psi_{\ell,m}^D)| d\rho \\ \lesssim \sum_{\ell,m} \lambda_m^{1/4} \left( \int_{e^s}^\infty e^s \rho^{-2-2\kappa} d\rho \right)^{1/2} \left( \int_{e^s}^\infty \rho^{2\alpha-1} \xi(e^{-s}\rho)^{2n_p+1} (|\Psi_{\ell,m}^D|^2 + \rho^2 |\partial_\rho \Psi_{\ell,m}^D|^2) d\rho \right)^{1/2} \\ \leq \left( \sum_{\ell,m} \lambda_m^{-3/2} \right)^{1/2} \left( \sum_{\ell,m} \lambda_m^2 \int_{e^s}^\infty \rho^{2\alpha-1} \xi(e^{-s}\rho)^{2n_p+1} (|\Psi_{\ell,m}^D|^2 + \rho^2 |\partial_\rho \Psi_{\ell,m}^D|^2) d\rho \right)^{1/2} \\ \lesssim \left( \sum_{\ell=0}^4 I_\ell(s) \right)^{1/2}, \end{aligned}$$

combining with the interior bound, we infer (8.12) for  $\Psi^D$ . As in Step 2, we can bound the derivatives of  $\Psi^D$  in the region  $|y| \geq e^s$ . This concludes the proof of (8.12).  $\square$

8.3. PROOF OF PROPOSITION 8.1. — We are in position to prove Proposition 8.1.

*Step 1 (Energy estimates).* — We claim the energy estimate

$$(8.15) \quad \frac{dI_j}{ds} \lesssim e^{-\varepsilon s}$$

holds for some  $\varepsilon > 0$  for all  $0 \leq j \leq 4$  so in particular, by the choice of initial value (8.6),

$$I_j(s) \leq I_j(s_0) + Ce^{-\varepsilon s_0}$$

is arbitrarily small for  $s_0$  sufficiently large.

*Case 1 ( $1 \leq j \leq 4$ ).* — Suppose claim holds for  $< j$  cases. Denote by  $I_j^\Psi, I_j^\Omega$  the weighted  $L^2$ -norm of  $\Psi^D$  and  $\Omega^D$  in  $I_j$ . For the  $\Psi^D$  component, we infer

$$\begin{aligned} & \frac{dI_j^\Psi}{ds} \\ &= \int_{\mathcal{D}_j} \rho^{2j-2s_c} \left[ -\rho \frac{\partial}{\partial \rho} \xi(e^{-s}\rho)^{2n_p+1} |\nabla^j \Psi^D|^2 + 2\xi(e^{-s}\rho)^{2n_p+1} \nabla^j \Psi^D \cdot \partial_s \nabla^j \Psi^D \right] dy \\ &\leq 2 \int_{\mathcal{D}_j} \rho^{2j-2s_c} \xi(e^{-s}\rho)^{2n_p+1} \left[ (j + \Lambda + \partial_s) \nabla^j \Psi^D \right] \cdot \nabla^j \Psi^D dy \\ &\quad + \mathbb{1}_{j \leq 2} \int_{\{|y|=1\}} |\nabla^j \Psi^D|^2 d\sigma(y), \end{aligned}$$

where we integrate by parts for the last inequality and note that the boundary term at infinity is non-positive. Here,  $\sigma$  is the standard measure on the unit sphere. We bound the boundary term using the Sobolev embedding  $H^2(|y| \leq 1) \hookrightarrow L^\infty(|y| \leq 1)$ :

$$(8.16) \quad \int_{\{|y|=1\}} |\nabla^j \Psi^D|^2 d\sigma(y) \lesssim \|\nabla^j \Psi^D\|_{L^\infty(|y|=1)}^2 \lesssim \|\Psi\|_{H^4(|y| \leq 1)}^2 \lesssim \|X\|_{\mathbb{H}}^2 \leq e^{-(\delta_0/2)s}$$

for  $j \leq 2$ , where in the final inequality, we have used (8.10). By the commutation relations

$$[\nabla^k, \Lambda] = k\nabla^k$$

and (8.3), we infer for some  $\varepsilon > 0$ ,

$$\begin{aligned} \frac{dI_j^\Psi}{ds} &\leq 2 \int_{\mathcal{D}_j} \rho^{2j-2s_c} \xi(e^{-s}\rho)^{2n_p+1} \nabla^j (\Lambda + \partial_s) \Psi^D \cdot \nabla^j \Psi^D dy + e^{-\varepsilon s} \\ &= -2 \int_{\mathcal{D}_j} \rho^{2j-2s_c} \xi(e^{-s}\rho)^{2n_p+1} \nabla^j \Omega^D \cdot \nabla^j \Psi^D dy + e^{-\varepsilon s}. \end{aligned}$$

Similarly, for  $\Omega^D$  component, it follows from the above commutation relation and (8.3) that

$$(8.17) \quad \begin{aligned} \frac{dI_j^\Omega}{ds} &\leq 2 \int_{\mathcal{D}_j} \rho^{2j-2s_c} \xi(e^{-s}\rho)^{2n_p+1} \left[ (j + \Lambda + \partial_s) \nabla^{j-1} \Omega^D \right] \cdot \nabla^{j-1} \Omega^D dy \\ &= 2 \int_{\mathcal{D}_j} \rho^{2j-2s_c} \xi(e^{-s}\rho)^{2n_p+1} \nabla^{j-1} (-\Delta \Psi^D - \tilde{\Psi}^p + \mathcal{E}) \cdot \nabla^{j-1} \Omega^D dy, \end{aligned}$$

where we recall the definition (8.4) of  $\mathcal{E}$ . Integrate by parts the first term we infer

$$(8.18) \quad \begin{aligned} &2 \int_{\mathcal{D}_j} \rho^{2j-2s_c} \xi(e^{-s}\rho)^{2n_p+1} (-\nabla^{j+1} \Psi^D) \cdot \nabla^{j-1} \Omega^D dy \\ &\leq 2 \int_{\mathcal{D}_j} \rho^{2j-2s_c} \xi(e^{-s}\rho)^{2n_p+1} \nabla^j \Psi^D \cdot \nabla^j \Omega^D dy \\ &\quad + 2 \int_{\mathcal{D}_j} \nabla \left[ \rho^{2j-2s_c} \xi(e^{-s}\rho)^{2n_p+1} \right] \cdot \nabla^j \Psi^D \nabla^{j-1} \Omega^D dy. \end{aligned}$$

From the bootstrap bound (8.10) and (8.9), we infer for  $2\varepsilon < \delta_0/(2k+1-2s_c) = \delta_0/(7-2s_c)$ , the bound for the last term above

$$\begin{aligned} &\int_{\mathcal{D}_j} \rho^{2j-2s_c-1} \xi(e^{-s}\rho)^{2n_p+1} |\nabla^j \Psi^D| |\nabla^{j-1} \Omega^D| dy \\ &\leq e^{-\varepsilon s} \int_{|y| \geq e^{\varepsilon s}} \rho^{2j-2s_c} \xi(e^{-s}\rho)^{2n_p+1} |\nabla^j \Psi^D| |\nabla^{j-1} \Omega^D| dy \\ &\quad + e^{\varepsilon(2k+1-2s_c)s} \int_{|y| \leq e^{\varepsilon s}} \langle \rho \rangle^{-2(k+1-j)} |\nabla^j \Psi| |\nabla^{j-1} \Omega| dy \\ &\leq e^{-\varepsilon s} (I_j^\Psi I_j^\Omega)^{1/2} + e^{(\delta_0/2)s} \int_{|y| \leq e^{\varepsilon s}} (|\nabla^j \Psi|^2 + |\nabla^{j-1} \Omega|^2) \langle \rho \rangle^{-2(4-j)} dy \\ &\leq e^{-\varepsilon s} I_j + e^{(\delta_0/2)s} \|X\|_{\mathbb{H}}^2 \leq e^{-\varepsilon s} I_j + e^{-(\delta_0/2)s} \leq e^{-\varepsilon s} \end{aligned}$$

for some  $\varepsilon > 0$ . Note that we have used Hardy's inequality from Lemma 6.2:

$$(8.19) \quad \int_{|y| \leq e^{\varepsilon s}} |\nabla^j \Psi|^2 \langle \rho \rangle^{-2(4-j)} dy \lesssim \|\Psi\|_{H_4}^2 \leq \|X\|_{\mathbb{H}}^2$$

and similarly for  $\Omega$ . Thus, we infer the bound for (8.18):

$$\begin{aligned} &2 \int_{\mathcal{D}_j} \rho^{2j-2s_c} \xi(e^{-s}\rho)^{2n_p+1} (-\nabla^{j+1} \tilde{\Psi}) \cdot \nabla^{j-1} \Omega^D dy \\ &\leq 2 \int_{\mathcal{D}_j} \rho^{2j-2s_c} \xi(e^{-s}\rho)^{2n_p+1} \nabla^j \Psi^D \cdot \nabla^j \Omega^D dy + C e^{-\varepsilon s}. \end{aligned}$$

Next, we prove the bound for the term with  $\tilde{\Psi}^p = (u_n + \Psi)^p$  and  $\mathcal{E}$ . By the bootstrap bound (8.8) together with the asymptotic behaviour of  $u_n^D$ , it holds for  $\ell = 0, 1$  that

$$\left\| \frac{\langle \rho \rangle^{\ell-\kappa} \nabla^\ell \tilde{\Psi}(s)}{u_n^D} \right\|_{L^\infty(\mathbb{R}^3)} \lesssim 1,$$

where we have used the Sobolev bound (8.16) in the region  $|y| \leq 1$ . Thus, we infer for  $j \leq 4$ ,

$$\begin{aligned} \left| \nabla^{j-1} (|\tilde{\Psi}|^{p-1} \tilde{\Psi}) \right| &\lesssim \sum_{i=1}^{j-1} |\tilde{\Psi}|^{p-i} \sum_{|\beta|=j-1, \beta>0} |\nabla^{\beta_1} \tilde{\Psi}| \dots |\nabla^{\beta_i} \tilde{\Psi}| \\ &\lesssim \sum_{\ell=1}^{j-1} |\nabla^\ell \tilde{\Psi}| \sum_{i=1}^{j-1} |\tilde{\Psi}|^{p-i} \sum_{\substack{|\beta|=j-\ell-1, \\ \|\beta\|_\infty \leq 1}} |\nabla^{\beta_1} \tilde{\Psi}| \dots |\nabla^{\beta_{i-1}} \tilde{\Psi}| \\ &\lesssim \sum_{\ell=0}^{j-1} \langle \rho \rangle^{-j+\ell+1+(-\alpha+\kappa)(p-1)} |\nabla^\ell \tilde{\Psi}| \leq \sum_{\ell=0}^{j-1} \langle \rho \rangle^{-j+\ell-3/4} |\nabla^\ell \tilde{\Psi}|, \end{aligned}$$

where we have used that  $\kappa < 1/4(p-1)$  and that  $p > 5$  to bound  $|\tilde{\Psi}|^{p-i}$ .

Next, we bound  $\mathcal{E}$  where we recall the definition (8.4) of  $\mathcal{E}$ . Observe that

$$\partial_\rho^j \eta(e^{-s} \rho) = e^{-js} \eta^{(j)}(e^{-s} \rho) \lesssim \langle \rho \rangle^{-j} \eta(e^{-s} \rho).$$

In view of the asymptotic behaviors of  $u_n^D$  and its derivatives, we have that for  $j \leq 4$ ,

$$\left| \nabla^{j-1} \mathcal{E} \right| \lesssim \left| \nabla^{j-1} \left( \eta(e^{-s} \rho) u_n^p - (\Delta \eta(e^{-s} \rho)) u_n - 2e^{-s} \eta'(e^{-s} \rho) u_n' \right) \right| \lesssim \langle \rho \rangle^{-j-1} u_n^D.$$

Adding the two bounds obtained above, we infer

$$(8.20) \quad \left| \nabla^{j-1} (|\tilde{\Psi}|^{p-1} \tilde{\Psi} - \mathcal{E}) \right| \lesssim \sum_{\ell=0}^{j-1} \langle \rho \rangle^{-j+\ell-3/4} (|\nabla^\ell \Psi^D| + |\nabla^\ell u_n^D|).$$

We improve the above bound in the region  $\rho \leq e^s$ . Here,  $\eta(e^{-s} \rho) \equiv 1$  so  $\mathcal{E} = u_n^p$  and we infer for  $j \leq 4$ ,

$$\begin{aligned} \left| \nabla^{j-1} (|\tilde{\Psi}|^{p-1} \tilde{\Psi} - \mathcal{E}) \right| &\lesssim \left| \nabla^{j-1} \left( \Psi \int_0^1 |u_n + \tau \Psi|^{p-1} d\tau \right) \right| \\ &\lesssim \sup_{0 \leq \tau \leq 1} |u_n + \tau \Psi|^{p-4} \sum_{i=0}^{j-1} |\nabla^i \Psi| \sum_{\substack{|\beta|=j-1-i, \\ \beta_1 \geq \beta_2 \geq \beta_3}} \prod_{q=1}^3 \sup_{0 \leq \tau \leq 1} |\nabla^{\beta_q} (u_n + \tau \Psi)|. \end{aligned}$$

Since  $i + \beta_1 + \beta_2 + \beta_3 = j - 1 \leq 3$  in the sum above,  $\beta_2, \beta_3 \leq 1$  so the  $L^\infty$ -bound (8.8) applies. Then, we have for all  $\rho \leq e^s$  that

$$\begin{aligned} (8.21) \quad &\left| \nabla^{j-1} (|\tilde{\Psi}|^{p-1} \tilde{\Psi} - \mathcal{E}) \right| \\ &\lesssim \langle \rho \rangle^{(-\alpha+\kappa)(p-2)} \sum_{i=0}^{j-1} |\nabla^i \Psi| \left( |\nabla^{j-1-i} u_n| + |\nabla^{j-1-i} \Psi| \right) \\ &\lesssim \sum_{i=0}^{j-1} \langle \rho \rangle^{-j+i+1+(-\alpha+\kappa)(p-1)} |\nabla^i \Psi^D| \lesssim \sum_{i=0}^{j-1} \langle \rho \rangle^{-j+i-3/4} |\nabla^i \Psi^D|. \end{aligned}$$

Thus, using the bounds (8.20) and (8.21) above, we infer for the  $\tilde{\Psi}^p$  and  $\mathcal{E}$  terms in (8.17) that

$$\begin{aligned} & \int_{\mathcal{D}_j} \rho^{2j-2s_c} \xi(e^{-s}\rho)^{2n_p+1} |\nabla^{j-1}(\tilde{\Psi}^p - \mathcal{E})| |\nabla^{j-1}\Omega^D| dy \\ & \lesssim \sum_{\ell=0}^{j-1} \int_{\mathcal{D}_j} \langle \rho \rangle^{j+\ell-2s_c-1/2} \xi(e^{-s}\rho)^{2n_p+1} \left( |\nabla^\ell \Psi^D| + \mathbb{1}_{|y|\geq e^s} |\nabla^\ell u_n^D| \right) |\nabla^{j-1}\Omega^D| dy \\ & \leq \sum_{\ell=0}^{j-1} \int_{|y|\geq e^s} \rho^{j+\ell-2s_c-1/2} \xi(e^{-s}\rho)^{2n_p+1} |\nabla^\ell u_n^D| |\nabla^{j-1}\Omega^D| dy \\ & \quad + e^{-(\varepsilon/2)s} \sum_{\ell=0}^{j-1} \int_{|y|\geq e^{\varepsilon s}} \rho^{j+\ell-2s_c} \xi(e^{-s}\rho)^{2n_p+1} |\nabla^\ell \Psi^D| |\nabla^{j-1}\Omega^D| dy \\ & \quad + \sum_{\ell=0}^{j-1} \int_{|y|\leq e^{\varepsilon s}} \langle \rho \rangle^{j+\ell-2s_c-1/2} |\nabla^\ell \Psi| |\nabla^{j-1}\Omega| dy. \end{aligned}$$

Thus, from the bootstrap bound (8.10) and Hardy’s inequality (8.19), we infer for  $2\varepsilon < \delta_0/(2k + (3/2) - 2s_c) = \delta_0/((15/2) - 2s_c)$ , the bound

$$\begin{aligned} & \int_{\mathcal{D}_j} \rho^{2j-2s_c} \xi(e^{-s}\rho)^{2n_p+1} |\nabla^{j-1}(\tilde{\Psi}^p - \mathcal{E})| |\nabla^{j-1}\Omega^D| dy \\ & \leq \sum_{\ell=0}^{j-1} \left[ \left( \int_{e^s}^\infty \rho^{-5/2} \xi(e^{-s}\rho) d\rho \right)^{1/2} (I_j^\Omega)^{1/2} + e^{-(\varepsilon/2)s} (I_\ell^\Psi I_j^\Omega)^{1/2} \right. \\ & \quad \left. + e^{(\delta_0/2)s} \left( \int_{|y|\leq e^{\varepsilon s}} |\nabla^\ell \Psi|^2 \langle \rho \rangle^{-2(4-\ell)} dy \right)^{1/2} \left( \int_{|y|\leq e^{\varepsilon s}} |\nabla^{j-1}\Omega|^2 \langle \rho \rangle^{-2(4-j)} dy \right)^{1/2} \right] \\ & \leq e^{-(3/4)s} I_j + \sum_{\ell=0}^j e^{-(\varepsilon/2)s} I_\ell + e^{(\delta_0/2)s} \|X\|_{\mathbb{H}}^2 \lesssim e^{-(3/4)s} + e^{-(\varepsilon/2)s} + e^{-(\delta_0/2)s}. \end{aligned}$$

Take smaller  $\varepsilon$  if necessary, we infer

$$\frac{dI_j^\Omega}{ds} \leq 2 \int_{\mathcal{D}_j} \rho^{2j-2s_c} \xi(e^{-s}\rho)^{2n_p+1} \nabla^j \Psi^D \cdot \nabla^j \Omega^D dy + Ce^{-\varepsilon s}.$$

Hence, by adding the bounds for  $I_j^\Psi$  and  $I_j^\Omega$ , we obtain the overall bound

$$(8.22) \quad \frac{dI_j}{ds} \lesssim e^{-\varepsilon s},$$

i.e., the claim (8.15) holds.

*Case 2 ( $j = 0$ ).* — Note that  $I_0 = I_0^\Psi$ . As in Case 1, we have for some  $\varepsilon > 0$  that

$$\frac{dI_0}{ds} \leq -2 \int_{|y|\geq 1} \rho^{-2s_c} \xi(e^{-s}\rho)^{2n_p+1} \Omega^D \Psi^D dy + e^{-\varepsilon s}.$$

From the bootstrap bound (8.10) and (8.9), we infer for  $2\varepsilon < \delta_0/(2k + 1 - 2s_c) = \delta_0/(7 - 2s_c)$ , the bound:

$$\begin{aligned} & \int_{|y| \geq 1} \rho^{-2s_c} \xi(e^{-s}\rho)^{2n_p+1} |\Psi^D \Omega^D| dy \\ & \leq e^{-\varepsilon s} \int_{|y| \geq e^{\varepsilon s}} \rho^{1+2s_c} \xi(e^{-s}\rho)^{2n_p+1} |\Psi^D \Omega^D| dy \\ & \quad + e^{\varepsilon(2k+1-2s_c)s} \int_{1 \leq |y| \leq e^{\varepsilon s}} \rho^{-(2k+1)} |\Psi \Omega| dy \\ & \leq e^{-\varepsilon s} (I_0^\Psi I_1^\Omega)^{1/2} + e^{(\delta_0/2)s} \int_{|y| \leq e^{\varepsilon s}} \left( |\Psi|^2 \langle \rho \rangle^{-2(k+1)} + |\Omega|^2 \langle \rho \rangle^{-2k} \right) dy \\ & \leq e^{-\varepsilon s} + e^{-(\delta_0/2)s} \leq e^{-\varepsilon s} \end{aligned}$$

for some  $\varepsilon > 0$ . Hence, the claim follows.

*Step 2 (Improvement of (8.8) and (8.9)).* — Given  $d_0 \ll 1$ , we claim that these quantities can be bounded by  $d_0$  in  $s \in [s_0, s^*]$ .

*Improved bound for the weighted Sobolev norm.* — It follows from the energy estimate (8.15) and the choice of initial value (8.6) that given  $d_0 \ll 1$ , we have that for all  $s \in [s_0, s^*]$  and  $j$  such that  $0 \leq j \leq 4$ ,

$$(8.23) \quad I_j(s) \leq I_j(s_0) + Ce^{-\varepsilon s_0} \leq d_0$$

for  $s_0$  sufficiently large.

*Improved pointwise bound.* — Let  $j$  be such that  $0 \leq j \leq 1$ . By Sobolev embedding and (8.10), we infer for large  $s_0$  that

$$\|\nabla^j \Psi^D\|_{L^\infty(|y| \leq 2)} \ll d_0.$$

Then, by Lemma 8.2, we have that for  $0 \leq j \leq 1$ ,

$$(8.24) \quad \left\| \frac{\rho^{j-\kappa} \nabla^j \Psi^D}{u_n^D} \right\|_{L^\infty(|y| \geq 1)} \lesssim \|\nabla^j \Psi^D\|_{L^\infty(|y|=2)} + \left( \sum_{\ell=0}^4 I_\ell(s) \right)^{1/2} \leq d_0.$$

where the last inequality follows from (8.23).

*Step 3 (Improved  $\|\cdot\|_{\mathbb{H}}$  bound and non-linear bound).* — Recall that

$$(8.25) \quad \begin{aligned} G_\Omega &= -|\Psi + u_n|^{p-1}(\Psi + u_n) + u_n^p + pu_n^{p-1}\Psi \\ &= -p(p-1)\Psi^2 \int_0^1 (1-\tau)|u_n + \tau\Psi|^{p-3}(u_n + \tau\Psi) d\tau. \end{aligned}$$

We claim that by choosing  $s_0$  sufficiently large and  $c > 0$  small,

$$(8.26) \quad \forall s \in [s_0, s^*], \quad \|G(s)\|_{\mathbb{H}} \leq \|X(s)\|_{\mathbb{H}}^{1+c}.$$

For  $\rho > 0$ , we have

$$(8.27) \quad |\nabla^k G_\Omega| \lesssim \sum_{i+j+\ell=k} |\nabla^i \Psi| |\nabla^j \Psi| \left| \int_0^1 (1-\tau) \nabla^\ell (|u_n + \tau\Psi|^{p-3}(u_n + \tau\Psi)) d\tau \right|.$$



For  $m \leq 3$  and  $p > 5$ , we have the bound:

$$(8.28) \quad \left| \int_0^1 (1 - \tau) |u_n + \tau \Psi|^{p-m-3} (u_n + \tau \Psi) d\tau \right| \lesssim \sup_{0 \leq \tau \leq 1} |u_n + \tau \Psi|^{p-m-2}.$$

This, together with the  $L^\infty$ -bound (8.8) which implies  $|\Psi| \lesssim \langle \rho \rangle^{-\alpha+\kappa}$  and the asymptotic behaviour of  $u_n$ , we infer

$$(8.29) \quad \begin{aligned} & \left| \int_0^1 (1 - \tau) \nabla^\ell (|u_n + \tau \Psi|^{p-3} (u_n + \tau \Psi)) d\tau \right| \\ & \lesssim \sum_{m=0}^\ell \int_0^1 (1 - \tau) |u_n + \tau \Psi|^{p-m-3} (u_n + \tau \Psi) d\tau \sum_{|\beta|=\ell} \prod_{q=1}^m (|\nabla^{\beta_q} u_n| + |\nabla^{\beta_q} \Psi|) \\ & \lesssim \sum_{m=0}^\ell \rho^{(-\alpha+\kappa)(p-m-2)} \sum_{|\beta|=\ell} \prod_{q=1}^m (|\nabla^{\beta_q} u_n| + |\nabla^{\beta_q} \Psi|). \end{aligned}$$

Note that (8.28) applies since  $m \leq \ell \leq k = 3$ . Also, at most one of  $\beta_1, i, j$  is  $> 1$  i.e., we can apply the  $L^\infty$ -bound (8.8) for at least two of  $\nabla^{\beta_1} \Psi, \nabla^i \Psi, \nabla^j \Psi$  factors. Thus, we infer

$$\begin{aligned} |\nabla^k G_\Omega| & \lesssim \sum_{i+j+\ell=k} |\nabla^i \Psi| |\nabla^j \Psi| \sum_{m=0}^\ell \rho^{(-\alpha+\kappa)(p-m-2)} \sum_{|\beta|=\ell} \prod_{q=1}^m (|\nabla^{\beta_q} u_n| + |\nabla^{\beta_q} \Psi|) \\ & \lesssim \sum_{i+j+\ell=k} \rho^{-\alpha-j+\kappa} |\nabla^i \Psi| \sum_{m=0}^\ell \rho^{(-\alpha+\kappa)(p-m-2)} \rho^{m(-\alpha+\kappa)-\ell} \\ & \lesssim \sum_{i=0}^k \rho^{(-\alpha+\kappa)(p-1)+i-k} |\nabla^i \Psi| \lesssim \sum_{i=0}^k \rho^{i-k-3/2} |\nabla^i \Psi|, \end{aligned}$$

where the final inequality follows from  $\kappa < 1/2(p+1)$ . Then for  $R \geq 1$ , by setting  $k = 3$ , we infer

$$(8.30) \quad \int_{|y| \geq R} |\nabla^3 G_\Omega|^2 dy \lesssim \sum_{i=0}^3 \int_{|y| \geq R} \rho^{-2i-3} |\nabla^{3-i} \Psi|^2 dy \lesssim R^{-1} \|\Psi\|_{H^3}^2 \leq R^{-1} \|X\|_{\mathbb{H}}^2,$$

where we have used the Hardy's inequality (8.19). Now we consider the region  $0 \leq \rho \leq R$ . Denote

$$H_R^3 := H^3(B_R(0)).$$

Then, there exists  $M_1 > 0$  such that

$$\|\phi\psi\|_{H_R^3}^2 \leq R^{M_1} \|\phi\|_{H_R^3}^2 \|\psi\|_{H_R^3}^2 \quad \forall \phi, \psi \in H_R^3,$$

since  $3 = k > d/2 = 3/2$  so that  $H^3(\mathbb{R}^3)$  is an algebra. From (8.28) and the assumption  $3 = k < p - 2$  we infer that,

$$\sum_{m=0}^3 \left\| \int_0^1 (1 - \tau) |u_n + \tau \Psi|^{p-m-3} (u_n + \tau \Psi) d\tau \right\|_{L^\infty(\mathbb{R}^3)} \lesssim 1.$$

Note also that the  $L^\infty$ -bound (8.8) implies  $|\nabla^j \Psi| \lesssim \langle \rho \rangle^{-j-\alpha+\kappa}$  for  $0 \leq j \leq 2$  and for all  $s \in [s_0, s^*]$ . Then it follows from (8.25) that

$$\begin{aligned}
 \int_{|y| \leq R} |\nabla^3 G_\Omega|^2 dy &\leq \|G_\Omega\|_{H_R^3}^2 \\
 &\lesssim R^{2M_1} \|\Psi\|_{H_R^3}^4 \left\| \int_0^1 (1-\tau) |u_n + \tau \Psi|^{p-3} (u_n + \tau \Psi) d\tau \right\|_{H_R^3}^2 \\
 (8.31) \quad &\lesssim R^{2M_1} \|\Psi\|_{H_R^3}^4 \sum_{|\beta| \leq 3} \left\| \prod_q (|\nabla^{\beta_q} u_n| + |\nabla^{\beta_q} \Psi|) \right\|_{L^2(B_R(0))}^2 \\
 &\lesssim R^{2M_1} \|\Psi\|_{H_R^3}^4 \sum_{m=0}^3 (\|u_n\|_{H_R^3}^2 + \|\Psi\|_{H_R^3}^2)^m \lesssim R^M \|X\|_{\mathbb{H}}^4
 \end{aligned}$$

for some  $M > 0$ . Set  $R = \|X\|_{\mathbb{H}}^{-2/(1+M)}$  and add (8.30) with (8.31) so the claim (8.26) follows by choosing  $c < 1/(1+M)$ .

By the decay estimate in Corollary 7.2,

$$\begin{aligned}
 (8.32) \quad \|(I - P)X(s)\|_{\mathbb{H}} &\lesssim e^{-(\delta/2)(s-s_0)} \|X(s_0)\|_{\mathbb{H}} + \int_{s_0}^s e^{-(\delta/2)(s-\tau)} \|G(\tau)\|_{\mathbb{H}} d\tau \\
 &\lesssim e^{-(\delta/2)s} \left[ e^{(\delta/2)s_0} \|X\|_{\mathbb{H}} + \int_{s_0}^s e^{((\delta/2)-(\delta_0/2)(1+c))\tau} d\tau \right] \lesssim e^{-(\delta/2)s}
 \end{aligned}$$

since  $\delta/(1+c) < \delta_0$ . This, together with (8.7), we infer

$$\|X(s)\|_{\mathbb{H}} \lesssim e^{-(\delta/2)s}.$$

This proves an improved bound for (8.10). Then, by (8.26), the non-linear bound (8.11) follows.  $\square$

### APPENDIX A. BOUND ON SELF-SIMILAR PROFILES

In this section, we derive some asymptotic properties for  $\rho \rightarrow \infty$  of the smooth profiles  $u_n$  constructed in Theorem 1.

LEMMA A.1. — *Let  $u_n$  be the self-similar profiles constructed in Proposition 5.1. For all  $k \in \mathbb{N}$ , as  $\rho \rightarrow \infty$ ,*

$$(A.1) \quad \partial_\rho^k u_n = \mathcal{O}(\rho^{-\alpha-k}), \quad \partial_\rho^k (u_n^{p-1}) = \mathcal{O}(\rho^{-2-k}).$$

*Proof.* — In view of (3.15), taking  $\varepsilon \ll 1$  we infer

$$u_n = \mathcal{O}(\rho^{-\alpha}), \quad u'_n = \mathcal{O}(\rho^{-\alpha-1})$$

and  $u_n \geq 0$  for all  $\rho$  sufficiently large. It follows immediately that

$$u_n^{p-1} = \mathcal{O}(\rho^{-2}), \quad (u_n^{p-1})' = (p-1)u_n^{p-2}u'_n = \mathcal{O}(\rho^{-3}).$$

In view of (1.4), we infer

$$|u_n^{(k)}| \lesssim_{k,\rho} \partial^{k-2} \left[ \frac{1}{\rho^2} (\rho u'_n + u_n + u_n^p) \right]$$

for all  $\rho > \rho_0$  and  $k \geq 2$ . Suppose lemma holds for some  $k \geq 2$ . Then by hypothesis, for all  $\rho > \rho_0$ ,

$$|u_n^{(k+1)}| \lesssim \sum_{j=0}^k \rho^{-j-2} u_n^{(k-j-1)} + \sum_{j=0}^{k-1} \rho^{-j-2} \sum_{i=0}^{k-j-1} u_n^{(i)} (u_n^{p-1})^{(k-j-i-1)} \lesssim \rho^{-\alpha-k-1}.$$

Furthermore, by hypothesis and bound on  $u_n^{(k+1)}$ , we infer

$$|(u_n^{p-1})^{(k+1)}| \lesssim \sum_{j=0}^{k+1} u_n^{p-k+j-2} \sum_{\substack{|\beta|=k+1 \\ \beta > 0}} u_n^{(\beta_1)} \dots u_n^{(\beta_j)} \lesssim \rho^{-3-k},$$

and this concludes the proof by induction. □

### APPENDIX B. MAXIMALITY OF $\tilde{\mathcal{M}}$

In this section, we consider the problem (6.21). Given  $H$  such that  $H(\rho)Y^{(\ell,m)} \in C_c^\infty(\mathbb{R}^3)$ , we seek solution to

$$(B.1) \quad [\mathcal{L} - \rho^{-2}\lambda_m - (\Lambda + R + 1)(\Lambda + R) + pu_n^{p-1}]\Psi = H.$$

LEMMA B.1. — *Let  $H \in C^\infty([0, \infty))$ . Then for  $R$  sufficiently large, there exists a unique solution  $\Psi \in C^1([0, \infty))$  to (B.1). Furthermore, if  $H(\rho)Y^{(\ell,m)} \in C_c^\infty(\mathbb{R}^3)$ , then  $\Psi(\rho)Y^{(\ell,m)}$  is smooth on  $\mathbb{R}^3$ .*

*Proof*

*Step 1 (Solutions at  $\rho = 0$ ).* — Set  $(\Psi_1, \Psi_2) = (\rho^{m+1}\Psi, \partial_\rho(\rho^{m+1}\Psi))$ . Writing (B.1) in the form required in Proposition 3.2,

$$(B.2) \quad \begin{cases} \rho \partial_\rho \Psi_1 = \rho \Psi_2, \\ \rho \partial_\rho \Psi_2 = \frac{\rho}{1-\rho^2} [\xi - pu_n^{p-1}] \Psi_1 + \frac{\rho}{1-\rho^2} \left[ \frac{2m}{\rho} + \eta \rho \right] \Psi_2 + \frac{\rho^{m+2}}{1-\rho^2} H, \end{cases}$$

where

$$\xi = (m - \alpha - R + 1)(m - \alpha - R), \quad \eta = -2(m - \alpha - R).$$

Hence,

$$\rho \partial_\rho \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = A(\rho) \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} + \frac{\rho^{m+2}}{1-\rho^2} \begin{pmatrix} 0 \\ H \end{pmatrix},$$

where  $A$  is smooth in  $[0, 1)$ ,

$$A(0) = \begin{pmatrix} 0 & 0 \\ 0 & 2m \end{pmatrix}$$

with  $\sigma(A(0)) = \{0, 2m\}$ . Thus, by Proposition 3.2 with  $\ell = 2m + 1$ , we infer for all  $a, b \in \mathbb{R}$ , there exists a unique smooth solution to the homogeneous problem for (B.2) such that

$$(\Psi_1, \Psi'_1, \dots, \Psi_1^{(2m)}, \Psi_1^{(2m+1)})(0) = (a, 0, \dots, 0, b).$$

Since  $H(\rho)Y^{(\ell,m)}$  is smooth radial,  $H = \mathcal{O}_{\rho \rightarrow 0}(\rho^m)$  so from Proposition 3.2 we can write the solution  $\Psi_{a,b}$  to (B.2) with the boundary condition

$$(\Psi_{a,b}, \Psi'_{a,b}, \dots, \Psi_{a,b}^{(2m)}, \Psi_{a,b}^{(2m+1)})(0) = (a, 0, \dots, 0, b)$$

as

$$\Psi_{a,b} = \Psi_0 + a\psi_1 + b\psi_2, \quad \begin{cases} \psi_1(\rho) \propto 1 + \mathcal{O}_{\rho \rightarrow 0}(\rho^{2m+2}), \\ \psi_2(\rho) \propto \rho^{2m+1} + \mathcal{O}_{\rho \rightarrow 0}(\rho^{2m+2}), \end{cases}$$

where  $\psi_1, \psi_2$  are the linearly independent solutions to the homogeneous problem for (B.2) in  $[0, 1)$  with appropriate initial values.

*Step 2 (Solutions at  $\rho = 1$ ).* — For  $(\tilde{\Psi}_1, \tilde{\Psi}_2) = (\Psi, \partial_\rho \Psi)$ , we write (B.1) as

$$\begin{cases} (\rho - 1)\partial_\rho \tilde{\Psi}_1 = (\rho - 1)\tilde{\Psi}_2, \\ (\rho - 1)\partial_\rho \tilde{\Psi}_2 = \frac{-(\alpha + R)(\alpha + R + 1) + pu_n^{p-1} - \lambda_m/\rho^2}{1 + \rho} \tilde{\Psi}_1 \\ \qquad \qquad \qquad + \frac{(2/\rho) - 2(\alpha + R + 1)\rho}{1 + \rho} \tilde{\Psi}_2 - \frac{H}{1 + \rho}. \end{cases}$$

Hence,

$$(\rho - 1)\partial_\rho \begin{pmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{pmatrix} = B(\rho) \begin{pmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{pmatrix} + \frac{1}{\rho + 1} \begin{pmatrix} 0 \\ H \end{pmatrix},$$

where  $B$  is smooth in  $(0, \infty)$ ,

$$B(1) = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ -(\alpha + R)(\alpha + R + 1) - \lambda_m + pu_n^{p-1}(1) & 2s_c - 2R - 3 \end{pmatrix}$$

with  $\sigma(B(1)) = \{s_c - R - 3/2, 0\}$ . Thus, by Proposition 3.2, for all  $b \in \mathbb{R}$ , there exists a unique smooth solution  $\tilde{\Psi}_b \in C^\infty((0, \infty))$  to (B.1) with

$$(\tilde{\Psi}_c(1), \tilde{\Psi}'_c(1)) = \left( 2c, -\left[ \alpha + R + 1 + \frac{pu_n^{p-1}(1) - \lambda_m}{-s_c + R + 3/2} \right]c + \frac{H(1)}{-s_c + R + 3/2} \right).$$

We can write

$$\tilde{\Psi}_c = \tilde{\Psi}_0 + c\tilde{\psi}, \quad (\tilde{\psi}(1), \tilde{\psi}'(1)) = \left( 2, -(\alpha + R + 1) - \frac{pu_n^{p-1}(1) - \lambda_m}{-s_c + R + 3/2} \right),$$

where  $\tilde{\psi}$  is the unique solution to the homogeneous problem for (B.1) in  $(0, \infty)$  with the given initial values.

*Step 3 (Matching).* — Next, we claim that for  $R$  sufficiently large and for all  $m \geq 0$ , the homogeneous problem for (B.1) with  $H = 0$  has a unique  $C^1$  solution  $\Psi \equiv 0$  on  $[0, 1]$ . Suppose otherwise i.e., there is  $R$  arbitrarily large and  $m \geq 0$  such that there exists  $\Psi_{\ell,m} \not\equiv 0$  smooth in  $[0, 1]$  such that (B.1) holds with  $H = 0$  and  $\Psi = \Psi_{\ell,m}(\rho)Y^{(\ell,m)}$  is smooth at the origin. Extend uniquely the homogeneous solution  $\Psi_{\ell,m}$  to  $[1, \infty)$ . Then, using the fixed point argument as in the proof of Lemma B.2 we infer

$$\sum_{j=0}^{k+3} \sup_{\rho \geq 1} \rho^{\alpha+R+j} |\partial_\rho^j \Psi_{\ell,m}| < \infty$$

and therefore,  $(\Psi, -(\Lambda + R)\Psi) \in \mathcal{D}_R$  where we recall the definition (6.17) of  $\mathcal{D}_R$  and

$$\langle \mathcal{M}X, X \rangle = R\langle X, X \rangle.$$

By dissipativity of  $\tilde{\mathcal{M}}$  for  $X \in \mathcal{D}_R$  proved in Step 1 of the proof of Proposition 6.4, we infer for all  $X \in \mathcal{D}_R$

$$\langle \mathcal{M}X, X \rangle \leq C \langle X, X \rangle$$

for some  $C$  independent of  $R$  and this is a contradiction, so we have our claim. This yields the uniqueness result.

Choose  $R$  sufficiently large so the claim holds. Since  $\{\rho^{-m-1}\psi_1, \rho^{-m-1}\psi_2\}$  is a basis of solutions to the homogeneous problem in  $(0, 1)$ , there exist  $A, B \in \mathbb{R}$  such that

$$\tilde{\psi} = \rho^{-m-1}(A\psi_1 + B\psi_2)$$

in  $(0, 1)$ . If  $A = 0$ , then  $\tilde{\psi} \in C^\infty([0, 1])$ , contradicting the claim above. Since  $\{\rho^{-m-1}\psi_1, \rho^{-m-1}\psi_2\}$  is a basis of solutions to the homogeneous problem in  $(0, 1)$ , there exist  $a, b \in \mathbb{R}$  such that

$$\rho^{-m-1}\Psi_{a,b} = \tilde{\Psi}_0.$$

Then,

$$\Psi = \tilde{\Psi}_0 - \frac{a}{A}\tilde{\psi} = \rho^{-m-1}\left(\Psi_{a,b} - a\psi_1 - \frac{aB}{A}\psi_2\right)$$

is smooth at  $\rho = 0$  by the first equality and is smooth at  $\rho = 1$  by the second equality. Thus, we have the existence and uniqueness of a  $C^1([0, \infty))$  solution. Furthermore, if  $H(\rho)Y^{(\ell,m)}$  is smooth i.e.,  $H = \mathcal{O}_{\rho \rightarrow 0}(\rho^m)$  and  $H^{(m+2k+1)}(0) = 0$  for  $k \in \mathbb{N}_{\geq 0}$ , then it follows that  $\Psi^{(m+2k+1)}(0) = 0$  for  $k \in \mathbb{N}_{\geq 0}$ . Thus,  $\Psi(\rho)Y^{(\ell,m)}$  is smooth.  $\square$

LEMMA B.2. — For  $H$  such that  $H(\rho)Y^{(\ell,m)} \in C_c^\infty(\mathbb{R}^3)$ , let  $\Psi$  be the unique  $C^1$  solution to (B.1) found in Lemma B.1. Then for  $R$  sufficiently large,  $\Psi(\rho)Y^{(\ell,m)} \in H^{k+1}(\mathbb{R}^3)$ .

*Proof.* — Using the fixed point argument, we prove the existence of  $C^{k+1}$  solution  $\Psi$  to (B.1) in  $\{\rho \geq \rho_0\}$  for  $\rho_0$  sufficiently large with sufficiently rapid decay as  $\rho \rightarrow \infty$  so that  $\Psi \in H_{\text{rad}}^{k+1}(\{\rho \geq \rho_0\})$ . Then by uniqueness of solution, we argue that this solution is indeed what we found in Lemma B.1.

Consider the homogeneous problem for (B.1) without the  $pu_n^{\rho-1}$  potential term:

$$(B.3) \quad \underbrace{\left\{ (1-\rho^2)\partial_\rho^2 + [2\rho^{-1} - 2(\alpha+R+1)\rho]\partial_\rho - \lambda_m\rho^{-2} - (\alpha+R)(\alpha+R+1) \right\}}_{:=\mathcal{L}_R} \varphi = 0$$

in  $[1, \infty)$ . A computation similar to Lemma 3.1 yields a pair of linearly independent solutions

$$(B.4) \quad \begin{aligned} \varphi_1 &= \rho^{-\alpha-R-1} {}_2F_1\left(\frac{\alpha+R+m+1}{2}, \frac{\alpha+R-m}{2}, \frac{3}{2}, \rho^{-2}\right), \\ \varphi_2 &= \rho^{-\alpha-R} {}_2F_1\left(\frac{\alpha+R+m}{2}, \frac{\alpha+R-m-1}{2}, \frac{1}{2}, \rho^{-2}\right), \end{aligned}$$

with the Wronskian

$$W := \varphi_1'\varphi_2 - \varphi_2'\varphi_1 \propto \rho^{-2}|1 - \rho^2|^{s_c-R-3/2}.$$

Define the spaces

$$\begin{aligned} \overline{X}_{\rho_0} &= \left\{ w \in C^{k+1}((\rho_0, \infty)) \mid \|w\|_{\overline{X}_{\rho_0}} := \sum_{j=0}^{k+1} \sup_{\rho \geq \rho_0} \rho^{\alpha+R+j} |\partial_\rho^j w| \right\}, \\ \overline{Y}_{\rho_0} &= \left\{ w \in C^{k+1}((\rho_0, \infty)) \mid \|w\|_{\overline{Y}_{\rho_0}} := \sum_{j=0}^{k+1} \sup_{\rho \geq \rho_0} \rho^{\alpha+R+j+2} |\partial_\rho^j w| \right\}. \end{aligned}$$

We claim that for  $\rho_0 > 1$ , the resolvent map  $\mathcal{T}_R : \overline{Y}_{\rho_0} \rightarrow \overline{X}_{\rho_0}$  given by

$$\mathcal{T}_R(f) = \varphi_1 \int_{\rho_0}^\rho \frac{f \varphi_2}{(1-r^2)W} dr - \varphi_2 \int_{\rho_0}^\rho \frac{f \varphi_1}{(1-r^2)W} dr$$

is well-defined and bounded with  $\mathcal{L}_R \circ \mathcal{T}_R = \text{id}_{\overline{Y}_{\rho_0}}$ . Note that

$$\begin{aligned} \partial_\rho^j \mathcal{T}_R(f) &= \varphi_1^{(j)} \int_{\rho_0}^\rho \frac{f \varphi_2}{(1-r^2)W} dr - \varphi_2^{(j)} \int_{\rho_0}^\rho \frac{f \varphi_1}{(1-r^2)W} dr \\ &\quad + \sum_{i=0}^{j-2} \partial_\rho^i \left[ \frac{f(\varphi_1^{(j-i-1)} \varphi_2 - \varphi_2^{(j-i-1)} \varphi_1)}{(1-\rho^2)W} \right]. \end{aligned}$$

In view of (B.4) and the asymptotic expansion of the fundamental solutions, we infer

$$\partial_\rho^\ell \left[ \frac{\varphi_1^{(j-i-1)} \varphi_2 - \varphi_2^{(j-i-1)} \varphi_1}{(1-\rho^2)W} \right] = \mathcal{O}_{\rho \rightarrow \infty}(\rho^{i-j-\ell}).$$

Then for all  $\rho \geq \rho_0$  and  $0 \leq j \leq k+1$ ,

$$\begin{aligned} \rho^{\alpha+R+j} |\partial_\rho^j \mathcal{T}_R(f)| &\lesssim \left( \rho^{-1} \int_{\rho_0}^\rho \rho^{-2} d\rho \right) \sup_{r \geq \rho_0} r^{\alpha+R+2} |f| + \left( \int_{\rho_0}^\rho \rho^{-3} d\rho \right) \sup_{r \geq \rho_0} r^{\alpha+R+2} |f| \\ &\quad + \sum_{i=0}^{j-2} \rho^{i-j} \left( \rho^{j-i-2} \sup_{r \geq \rho_0} r^{\alpha+R+i+2} |\partial_\rho^i f| \right) \lesssim \rho_0^{-2} \|f\|_{\overline{Y}_{\rho_0}}. \end{aligned}$$

Thus,  $\mathcal{T}_R$  is a bounded map with operator norm  $\|\mathcal{T}_R\| \lesssim \rho_0^{-2}$  as claimed. Now, we solve the fixed point problem:

$$(B.5) \quad \Psi = \underbrace{c_1 \varphi_1 + c_2 \varphi_2 + \mathcal{T}_R[H - u_n^{p-1} \Psi]}_{:=G_R(\Psi)}$$

for  $c_1, c_2$  such that the  $\Psi(\rho_0), \Psi'(\rho_0)$  agree with the corresponding values of the unique solution in Lemma B.1. Note that  $\varphi_1, \varphi_2 \in \overline{X}_{\rho_0}, H \in C_c^\infty([0, \infty))$ . By Lemma A.1,  $\partial_\rho^j(u_n^{p-1}) = \mathcal{O}(\rho^{-m-2})$  as  $\rho \rightarrow \infty$ , so we infer

$$\|u_n^{p-1} \Psi\|_{\overline{Y}_{\rho_0}} \lesssim \|\Psi\|_{\overline{X}_{\rho_0}}$$

and hence,  $H - u_n^{p-1} \Psi \in \overline{Y}_{\rho_0}$ , so indeed  $G_R : \overline{X}_{\rho_0} \rightarrow \overline{X}_{\rho_0}$ . For  $\rho_0$  sufficiently large,  $G_R$  is a contraction map since for all  $\Psi_1, \Psi_2 \in \overline{X}_{\rho_0}$ ,

$$\|G_R(\Psi_1) - G_R(\Psi_2)\|_{\overline{X}_{\rho_0}} \lesssim \|\mathcal{T}_R\| \|u_n^{p-1}(\Psi_1 - \Psi_2)\|_{\overline{Y}_{\rho_0}} \lesssim \rho_0^{-2} \|\Psi_1 - \Psi_2\|_{\overline{X}_{\rho_0}}.$$

Thus, it follows from the Banach fixed point theorem that there exists a unique  $\Psi \in \overline{X}_{\rho_0}$  such that (B.5) holds. Taking  $R > s_c, \overline{X}_{\rho_0}$  continuously embeds in  $H_{\text{rad}}^{k+1}(\{\rho \geq \rho_0\})$ , so  $\Psi \in H_{\text{rad}}^{k+1}(\{\rho \geq \rho_0\})$ . Also, by uniqueness of solution to an ODE at ordinary point, this is indeed the solution we found in Lemma B.1.  $\square$

APPENDIX C. BEHAVIOUR OF THE SOBOLEV NORM

In this section, we prove the asymptotic behaviors (1.7), (1.8) and (1.9) of the Sobolev norms of the blow up solutions. In this section we denote by  $\tau$  the self-similar time in order to distinguish from the Sobolev exponent  $s$ .

*Proof of (1.7), (1.8), (1.9).* — Suppose that  $(\Phi, \Phi_t)$  is a blow up solution as in the statement of Theorem 2. Then, the bootstrap bounds in Proposition 8.1 are satisfied in the region  $\tau \in [s_0, \infty)$  in the self-similar time. In particular, from (8.23), we have that

$$(C.1) \quad \int_{|y| < e^\tau} \langle \rho \rangle^{2j-2s_c} (|\nabla^j \Psi|^2 + \mathbb{1}_{j \geq 1} |\nabla^{j-1} \Omega|^2) dy < d_0, \quad 0 \leq j \leq 4,$$

and from (8.10),

$$(C.2) \quad \int_{\mathbb{R}^3} (|\nabla^4 \Psi|^2 + |\nabla^3 \Omega|^2) dy < e^{-(\delta_0/2)\tau}.$$

Recall the definition of dampened profile  $u_n^D$  and perturbation  $\Psi^D$  from Section 8. From (8.23) with  $j = 0$ , we infer

$$\begin{aligned} \|\Phi\|_{L^2(|x|>1)}^2 &= \left\| \frac{1}{(T-t)^\alpha} \tilde{\Psi} \left( \frac{r}{T-t} \right) \right\|_{L^2(|x|>1)}^2 \\ &\lesssim \int_{|y| \geq e^\tau} \rho^{-2s_c} \xi(e^{-\tau} \rho)^{2n_p+1} |\tilde{\Psi}|^2 dy < d_0, \end{aligned}$$

where we have used that  $\xi(r) \gtrsim r$  for  $r \geq 1$  and that  $s_c < n_p$ . Similarly, setting  $j = 2$  in (8.23),

$$\begin{aligned} \|\Phi\|_{\dot{H}^2(|x|>1)}^2 &= \left\| \frac{1}{(T-t)^\alpha} \tilde{\Psi} \left( \frac{r}{T-t} \right) \right\|_{\dot{H}^2(|x|>1)}^2 \\ &\lesssim \int_{|y| \geq e^\tau} \rho^{-2(s_c-2)} \xi(e^{-\tau} \rho)^{2n_p+1} |\Delta \tilde{\Psi}|^2 dy < d_0. \end{aligned}$$

We interpolate the above two bounds and infer

$$(C.3) \quad \|\Phi\|_{\dot{H}^s(|x|>1)}^2 \lesssim d_0, \quad 0 \leq s \leq 2.$$

*Step 1 (H<sup>s<sub>c</sub></sup> Bound).* — In view of the Gagliardo-Nirenberg inequality (see [10]), we infer the  $H^{s_c}$  bound on  $\Psi^D$ :

$$\begin{aligned} \left\| \frac{1}{(T-t)^\alpha} \nabla_r^{s_c} \Psi^D \left( \frac{r}{T-t} \right) \right\|_{L^2(\mathbb{R}^3)}^2 &= \int_{\mathbb{R}^3} |\nabla^{s_c} \Psi^D|^2 dy \\ &\lesssim \left( \int_{\mathbb{R}^3} \langle \rho \rangle^{2(1-s_c)} |\nabla \Psi^D|^2 dy \right)^\theta \left( \int_{\mathbb{R}^3} \langle \rho \rangle^{2(2-s_c)} |\Delta \Psi^D|^2 dy \right)^{1-\theta}, \end{aligned}$$

where

$$s_c = \theta + 2(1 - \theta), \quad \theta \in (0, 1).$$

Thus, from (8.23), we infer

$$(C.4) \quad \left\| \frac{1}{(T-t)^\alpha} \nabla_r^{s_c} \Psi^D \left( \frac{r}{T-t} \right) \right\|_{L^2(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} |\nabla^{s_c} \Psi^D|^2 dy \lesssim d_0.$$

Also, note that for  $s \leq s_c$ ,

$$(C.5) \quad \left\| \frac{1}{(T-t)^\alpha} \nabla_r^s u_n \left( \frac{r}{T-t} \right) \right\|_{L^2(|x|<1)}^2 = e^{-2(s_c-s)\tau} \int_{|y| \leq e^\tau} |\nabla^s u_n|^2 dy \\ \sim c_{n,s} e^{-2(s_c-s)\tau} \int_1^{e^\tau} \rho^{2(s_c-s)-1} d\rho \sim c_{n,s} \begin{cases} 1 & s < s_c, \\ \tau & s = s_c. \end{cases}$$

From the above inequalities, together with (C.3) with  $s = s_c$  we infer

$$\|\Phi\|_{H^{s_c}}^2 = c_n(1 + o_{t \rightarrow T}(1)) |\log(T-t)|.$$

Similarly for  $\Phi_t$ . Hence, we infer (1.8).

*Step 2 (Subcritical Bound).* — Set  $j = 0$  in (C.1), we have the  $L^2$  bound on  $\Psi$ :

$$\left\| \frac{1}{(T-t)^\alpha} \Psi \left( \frac{r}{T-t} \right) \right\|_{L^2(|x|<1)}^2 \leq \int_{|x|<1} e^{2s_c\tau} \langle e^\tau r \rangle^{-2s_c} \left| \frac{1}{(T-t)^\alpha} \Psi \left( \frac{r}{T-t} \right) \right|^2 dx \\ = \int_{|y|<e^\tau} \langle \rho \rangle^{-2s_c} |\Psi|^2 dy < d_0.$$

From this, together with (C.3), we infer

$$\left\| \frac{1}{(T-t)^\alpha} \Psi^D \left( \frac{r}{T-t} \right) \right\|_{L^2(\mathbb{R}^3)}^2 \lesssim d_0.$$

Interpolating with the critical norm (C.4) above, we have for  $0 \leq s < s_c$ ,

$$\left\| \frac{1}{(T-t)^\alpha} \nabla_r^s \Psi^D \left( \frac{r}{T-t} \right) \right\|_{L^2(\mathbb{R}^3)}^2 \lesssim d_0.$$

Adding with the norm of the damped profile (C.5), we infer

$$\limsup_{t \rightarrow T} \|\Phi\|_{H^s}^2 < \infty.$$

Similarly for  $\Phi_t$ . Hence, we infer (1.7).

*Step 3 (Supercritical Bound).* — Since

$$\int_{|y| \geq e^\tau} |\nabla^4(u_n - u_n^D)|^2 dy \lesssim e^{-2(4-s_c)\tau},$$

it follows from (C.2) that

$$\int_{\mathbb{R}^3} |\nabla^4 \Psi^D|^2 dy \lesssim e^{-(\delta_0/2)\tau} + e^{-2(4-s_c)\tau}.$$

We interpolate this with (C.4) and infer, for  $s_c < s \leq 2$ ,

$$\|\Psi\|_{H^s}^2 \leq \int_{|y| \geq e^\tau} |\nabla^s(u_n - u_n^D)|^2 dy + \int_{\mathbb{R}^3} |\nabla^s \Psi^D|^2 dy \\ \lesssim_s e^{-2(s-s_c)\tau} + e^{-cs\tau} \rightarrow 0.$$

Similarly for  $\Omega$ . Hence, we infer (1.9). □



APPENDIX D. LIPSCHITZ DEPENDENCE OF INITIAL DATA

Recall from Section 8 the definition of the projection operator  $P$  onto  $V$  the subspace of unstable directions under semigroup action of the maximally dissipative operator  $\mathcal{M} - \mathcal{P}$ . In the proof of Proposition 8.1 and Corollary 7.3, it is proved that for any small initial perturbation in the stable direction:

$$\|(I - P)X(s_0)\|_{\mathbb{H}} \leq e^{-(\delta/2)s_0},$$

there exists a choice of  $PX(s_0)$  so that the solution is global in self-similar time with

$$\|PX(s)\|_{\mathbb{H}} \leq e^{-(\delta/2)(1+c/2)s} \quad s \geq s_0.$$

In this section, we prove that the choice of  $PX(s_0)$  is unique and is Lipschitz dependent on  $(I - P)X(s_0)$ . In particular, we show that for any two global solutions  $X$  and  $\bar{X}$ , if the initial difference in the unstable direction is too big compared with the initial differences in the stable direction, the unstable linear dynamics wins and expels the differences of unstable parameters away from 0. Hence one of the two solutions cannot blow up according to our scenario, yielding a contradiction. In particular, we claim the following:

LEMMA D.1. — *Let us assume  $X$  and  $\bar{X}$  are two global solutions as in Proposition 8.1 i.e., there hold the initial condition (8.5) and the bootstrap bounds (8.8), (8.10) for  $s \geq s_0$ . Denote by*

$$X_s = (I - P)X, \quad X_u = PX,$$

*the stable and unstable part of the perturbation and similarly  $\bar{X}_u, \bar{X}_s$ . Then, for  $s_0 \gg 1$  sufficiently large,*

$$(D.1) \quad \|\Delta X_u(s_0)\|_{\mathbb{H}} \leq c_{s_0} \|\Delta X_s(s_0)\|_{\mathbb{H}},$$

*where  $\Delta X_u = X_u - \bar{X}_u, \Delta X_s = X_s - \bar{X}_s$ .*

*Proof*

*Step 1 (Difference of nonlinear term).* — Recall (6.3) and define  $\Delta G = G - \bar{G}$ . Then,

$$\begin{aligned} \Delta G_{\Omega} &= -|\Psi + u_n|^{p-1}(\Psi + u_n) + |\bar{\Psi} + u_n|^{p-1}(\bar{\Psi} + u_n) + pu_n^{p-1}\Delta\Psi \\ &= p\Delta\Psi \left( u_n^{p-1} - \int_0^1 |u_n + \bar{\Psi} + \tau\Delta\Psi|^{p-1} d\tau \right). \end{aligned}$$

We claim the following nonlinear bound: there exists  $c > 0$  such that

$$\|\Delta G(s)\|_{\mathbb{H}} \lesssim e^{-(c\delta/2)s} \|\Delta X(s)\|_{\mathbb{H}}.$$

This is an analogue of (8.26) for the difference  $\Delta X$ .

Let  $\rho \geq 1$ . Note that for  $m \leq k = 3 < p - 1$ ,

$$(D.2) \quad \int_0^1 |u_n + \bar{\Psi} + \tau\Delta\Psi|^{p-m-1} d\tau \lesssim \sup_{\tau \in [0,1]} |u_n + \bar{\Psi} + \tau\Delta\Psi|^{p-m-1}.$$

Thus, using (8.8) and following the similar steps as in (8.29) we infer

$$\begin{aligned} |\nabla^k \Delta G_\Omega| &\lesssim \sum_{j+\ell=k} |\nabla^j \Delta \Psi| \sum_{m=0}^\ell \rho^{(p-m-1)(-\alpha+\kappa)} \sum_{|\beta|=\ell} \prod_{q=1}^m (|\nabla^{\beta_q}(u_n + \bar{\Psi})| + |\nabla^{\beta_q} \Delta \Psi|) \\ &\lesssim \sum_{j+\ell=k} |\nabla^j \Delta \Psi| \sum_{m=0}^\ell \rho^{(p-m-1)(-\alpha+\kappa)} \rho^{m(-\alpha+\kappa)-\ell} \lesssim \sum_{i=0}^k \rho^{i-k-3/2} |\nabla^i \Delta \Psi|, \end{aligned}$$

where in the last inequality, we have used that  $\kappa < 1/2(p + 1)$ . Then for  $R \geq 1$ , we infer

$$\begin{aligned} \text{(D.3)} \quad \int_{|y| \geq R} |\nabla^3 \Delta G_\Omega|^2 dy &\lesssim \sum_{i=0}^3 \int_{|y| \geq R} \rho^{-2i-3} |\Delta \nabla^{3-i} \Psi|^2 dy \\ &\lesssim R^{-1} \|\Delta \Psi\|_{H^4}^2 \leq R^{-1} \|\Delta X\|_{\mathbb{H}}^2, \end{aligned}$$

where we have used the Hardy’s inequality. We now bound  $\Delta G_\Omega$  in the region  $\rho \leq R$ . We rewrite

$$\Delta G_\Omega = -p(p-1) \Delta \Psi \int_0^1 \int_0^1 (\bar{\Psi} + \tau \Delta \Psi) |u_n + \tau'(\bar{\Psi} + \tau \Delta \Psi)|^{p-3} (u_n + \tau'(\bar{\Psi} + \tau \Delta \Psi)) d\tau d\tau'.$$

Note that for  $m \leq 3 < p - 2$ ,

$$\int_0^1 |u_n + \tau'(\bar{\Psi} + \tau \Delta \Psi)|^{p-m-2} d\tau' \lesssim \sup_{0 \leq \tau \leq 1} |u_n + \tau(\bar{\Psi} + \tau \Delta \Psi)|^{p-m-2}.$$

Thus, we infer from the assumption  $k = 3 < p - 1$  that

$$\sum_{m=0}^3 \left\| \int_0^1 |u_n + \tau'(\bar{\Psi} + \tau \Delta \Psi)|^{p-m-2} d\tau' \right\|_{L^\infty(\mathbb{R}^3)} \lesssim 1.$$

Then following the similar steps as in (8.31) by exploiting the algebra structure of the Sobolev space  $H_R^3$ , we bound the nonlinear difference in the region  $0 \leq \rho \leq R$ :

$$\begin{aligned} \int_{|y| \leq R} |\nabla^3 \Delta G_\Omega|^2 dy &\leq \|\Delta G_\Omega\|_{H_R^3}^2 \\ &\lesssim R^{2M_1} \|\Delta \Psi\|_{H_R^3}^2 (\|\Psi\|_{H_R^3}^2 + \|\bar{\Psi}\|_{H_R^3}^2) \sum_{m=0}^3 (\|u_n\|_{H_R^3}^2 + \|\Psi\|_{H_R^3}^2 + \|\bar{\Psi}\|_{H_R^3}^2)^m \\ &\lesssim R^M \|\Delta X\|_{\mathbb{H}}^2 (\|X\|_{\mathbb{H}}^2 + \|\bar{X}\|_{\mathbb{H}}^2) \lesssim R^M e^{-\delta s} \|\Delta X\|_{\mathbb{H}}^2 \end{aligned}$$

for some  $M > 0$ . Note that the final inequality follows from (8.10). Set  $R = e^{\delta s/(1+M)}$  and add (8.30) with (8.31) so the claim (8.26) follows by choosing  $c < 1/(1 + M)$ .

*Step 2 (Bound on initial perturbation).* — Recall that in the decomposition

$$\mathbb{H} = U \oplus V,$$

we have for all  $\lambda \in \sigma(\mathcal{M} - \mathcal{P})|_V$ , that  $\text{Re}(\lambda) \geq 0$ . Then, without loss of generality, restrict to an irreducible subspace so that for  $\text{Re}(\lambda) \geq 0$ , we write  $A := \mathcal{M} - \mathcal{P}$  as in (7.1). Then, from Duhamel’s formula, (6.3) implies

$$e^{(s_0-s)A} \Delta X_u(s) = \Delta X_u(s_0) + \int_{s_0}^\infty e^{(s_0-\tau)A} \Delta G_u(\tau) d\tau - \int_s^\infty e^{(s_0-\tau)A} \Delta G_u(\tau) d\tau,$$

where  $G_u = PG(s)$  and  $\Delta G_u = G_u - \overline{G}_u$ . Also, from (D.3), we bound

$$\begin{aligned} & \left\| e^{(s_0-s)A} \Delta X_u(s) + \int_s^\infty e^{(s_0-\tau)A} \Delta G_u(\tau) d\tau \right\|_{\mathbb{H}} \\ & \lesssim \frac{(s-s_0)^{m_\lambda-1}}{e^{\operatorname{Re}(\lambda)(s-s_0)}} \|\Delta X_u(s)\|_{\mathbb{H}} + \int_s^\infty (\tau-s_0)^{m_\lambda-1} e^{-\operatorname{Re}(\lambda)(\tau-s_0)} \|\Delta G_u\|_{\mathbb{H}} d\tau \rightarrow 0, \end{aligned}$$

since we have exponential decay of  $X, \overline{X}$  from (8.10) and of  $G, \overline{G}$  from (8.11). Thus, for all  $s \geq s_0$ ,

$$\begin{aligned} (D.4) \quad \|\Delta X_u(s)\|_{\mathbb{H}} &= \left\| \int_s^\infty e^{(s-\tau)A} \Delta G_u(\tau) d\tau \right\|_{\mathbb{H}} \\ &\leq \int_{s_0}^\infty \|\Delta G_u(\tau)\|_{\mathbb{H}} d\tau \leq \int_{s_0}^\infty e^{-(c\delta/2)\tau} \|\Delta X(\tau)\|_{\mathbb{H}} d\tau. \end{aligned}$$

Now, consider the evolution in the stable subspace  $U$  where  $A$  is dissipative, so Corollary 7.2 applies. Again, from Duhamel’s formula,

$$\Delta X_s(s) = e^{(s-s_0)A} \Delta X_s(s_0) + \int_{s_0}^s e^{(s-\tau)A} \Delta G_s(\tau) d\tau,$$

so we bound for all  $s \geq s_0$ :

$$\begin{aligned} \|\Delta X_s(s)\|_{\mathbb{H}} &\leq \|\Delta X_s(s_0)\|_{\mathbb{H}} + \int_{s_0}^s \|\Delta G_u(\tau)\|_{\mathbb{H}} d\tau \\ &\leq \|\Delta X_s(s_0)\|_{\mathbb{H}} + \int_{s_0}^s e^{-(c\delta/2)\tau} \|\Delta X(\tau)\|_{\mathbb{H}} d\tau. \end{aligned}$$

Taking supremum over  $s$ ,

$$\begin{aligned} \|\Delta X_s\|_{\mathbb{H}, L_s^\infty} &\leq \|\Delta X_s(s_0)\|_{\mathbb{H}} + (\|\Delta X_s\|_{\mathbb{H}, L_s^\infty} + \|\Delta X_u\|_{\mathbb{H}, L_s^\infty}) \int_{s_0}^\infty e^{-(c\delta/2)\tau} d\tau \\ &\lesssim \|\Delta X_s(s_0)\|_{\mathbb{H}} + \|\Delta X_u\|_{\mathbb{H}, L_s^\infty}, \end{aligned}$$

where in the last inequality, we absorb the  $\Delta X_s$  on the right-hand side by taking a large  $s_0$ . Thus, from (D.4),

$$\begin{aligned} \|\Delta X_u\|_{\mathbb{H}, L_s^\infty} &\leq \int_{s_0}^\infty e^{-(c\delta/2)\tau} (\|\Delta X_s\|_{\mathbb{H}, L_s^\infty} + \|\Delta X_u\|_{\mathbb{H}, L_s^\infty}) d\tau \\ &\lesssim e^{-(c\delta/2)s_0} (\|\Delta X_s(s_0)\|_{\mathbb{H}} + \|\Delta X_u\|_{\mathbb{H}, L_s^\infty}) \lesssim \|\Delta X_s(s_0)\|_{\mathbb{H}}. \end{aligned}$$

Again absorb the  $\Delta X_u$  term by taking a large  $s_0$ . Thus, we infer (D.1). □

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