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TIME SPENT IN A BALL BY A CRITICAL BRANCHING RANDOM WALK

by Amine Asselah & Bruno Schapira

Abstract. — We study a critical branching random walk on $\mathbb{Z}^d$. We focus on the tail of the time spent in a ball, and our study, in dimension four and higher, sheds new light on the recent result of Angel, Hutchcroft and Jarai [AHJ21], in particular on the special features of the critical dimension four. Finally, we analyze the number of walks transported by the branching random walk on the boundary of a distant ball.

Résumé (Temps passé dans une boule pour une marche aléatoire branchante critique)
Nous étudions la queue de distribution du temps passé dans une boule par une marche aléatoire branchante critique. Notre étude apporte un éclairage nouveau aux résultats récents de Angel, Hutcroft et Jarai, en particulier sur le cas de la dimension 4. Enfin nous étudions également le nombre de particules déposées sur la frontière d'une boule.

Contents

1. Introduction

In this paper we study a critical branching random walk (BRW) on $\mathbb{Z}^d$. Whereas the study of the volume of the range of random walks is a central object of probability theory, the range of branching random walks, in dimension larger than one, stayed in

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the shadows. Quite recently, Le Gall and Lin [LGL15, LGL16] proved limit theorems for the volume of the range, say $\mathcal{R}_n$, of a random walk indexed by a Galton-Watson tree conditioned on having n vertices, as n goes to infinity. In particular they discovered that in dimension five and larger, $\mathcal{R}_n$ scales linearly, whereas in dimension four it scales like $n/\log(n)$, and in dimension three and lower it scales like $n^{d/4}$. Thus with BRW one recovers the well-known trichotomy for the asymptotic behavior of the range of a simple random walk, going back to Dvoretzky and Erdős [DE51], except that the critical dimension is now equal to four instead of two. Later, Zhu in a series of works [Zhu16a, Zhu16b, Zhu19, Zhu21] extended part of Le Gall and Lin's analysis to general offspring and jump distributions, and most notably brought into light the notion of *branching capacity*, which is associated with BRW just as the *electrostatic capacity* is associated with random walk. Lalley and Zheng [LZ11], analyzed the occupation statistics at a fixed generation of the tree, and observed similar behaviors as for a simple random walk. Angel, Hutchcroft and Jarai in [AHJ21] studied the tail of the local times for the full tree, and discovered that the tail speed is exponential above the critical dimension (dimensions five and higher), but stretched exponential in the critical dimension four, a fact which does not have natural counterpart in the random walk setting. One of our motivation for the present paper is to bring some light on this remarkable observation, and in particular explain the tail behaviour in dimensions four and higher. When [AHJ21] follows a moment method, rooted in statistical mechanics, our approach is probabilistic, and aims at developing an analogue of excursion theory so useful to analyze random walks. Finally, to emphasize the recent vigor of BRW studies in high dimensions, let us mention [BC12, LSS24] dealing with recurrence and transience of a discrete snake, some recent results on the electrostatic capacity of the range of a BRW [BW22, BH22, BH23], and others on branching interlacements [PZ19, Zhu18], or on the range of tree-valued BRWs [DKLT22].

Here, we consider one Euclidean ball centered at the origin, and study two objects: (i) the tail of the time spent in this ball when the BRW starts at the origin; (ii) the tail of the (rescaled) number of walks hitting the ball, when the BRW starts from far away.

To state our results, let us introduce the needed notation. We let $\mathcal{T}$ be a critical Bienaymé-Galton-Watson tree (BGW tree for short), whose offspring distribution has a finite exponential moment. Consider $\{S_u, u \in \mathcal{T}\}$ an associated tree-indexed random walk, which we view alternatively as a branching random walk, where time is encoded by the tree $\mathcal{T}$ and whose jump distribution is the uniform measure θ on the neighbors of the origin. In other words, independent increments $\{X(e)\}$ are associated to the edges of the tree, and if $[\varnothing, u]$ is the sequence of edges between the root $\varnothing$ and vertex u, then

$$S_u = S_\varnothing + \sum_{e\in[\varnothing,u]} X(e).$$

When $z \in \mathbb{Z}^d$, we let $\mathbb{P}_z$ be the law of the BRW starting from z, i.e., conditioned on $\{S_\varnothing = z\}$, and simply write $\mathbb{P}$ when it starts from the origin. Given $\Lambda \subset \mathbb{Z}^d$, we define the time spent in Λ by the BRW as

$$\ell_{\mathcal{T}}(\Lambda) := \sum_{u\in\mathcal{T}} \mathbf{1}\{S_u \in \Lambda\}.$$

Let $B_r := \{z \in \mathbb{Z}^d : \|z\| < r\}$, where $\|\cdot\|$ denotes the Euclidean distance, and write $\Theta(f(t,r))$ for a function which is uniformly bounded from above and below by f, up to multiplicative positive constants (that may depend on the dimension). Our main result reads as follows.

Theorem 1.1. — *One has uniformly in $r \geqslant 1$, and $t \geqslant 1$,*

$$\mathbb{P}(\ell_{\mathcal{T}}(B_r) > t) = \Theta\big(\frac{1}{\sqrt{\min(t, r^4)}}\big) \times \begin{cases} \exp\big(-\Theta(t/r^4)\big) & \text{if } d \geqslant 5,\\ \exp\big(-\Theta(\sqrt{t/r^4})\big) & \text{if } d = 4,\\ (1 + t/r^4)^{-2/(4-d)} & \text{if } d = 1, 2, 3.\end{cases}$$

Note that when t is of order r^d, then in the exponential we obtain a factor which is of order r^{d-4}, in dimension 5 and higher, that is of the same order as the branching capacity of the ball B_r, as shown in [Zhu16a]. This is not merely a coincidence, and a more general result in this direction has been shown in the recent paper [ASS23]. On the other hand, it remains an interesting open problem to prove the existence of a limiting constant, in front of the t/r^4 in the exponential.

Remark 1.2. — In fact the result in dimension 1 and 2 holds under the weaker assumption that the offspring distribution of the BGW tree has only a finite second moment, and a finite third moment in dimension three, instead of a finite exponential moment. Moreover, the proof in dimension $1, 2, 3$ can be done the same way as in [AHJ21], using a simple moment method.

Heuristics. — Let us explain at a heuristic level the difference between dimension four on one hand and five and higher on the other hand. In the latter case, to occupy B_r, a good strategy for the BRW is to produce *waves*, which go from the boundary ∂B_r, up to the boundary of a larger concentric ball, say ∂B_{2r}, and back to B_r. Furthermore, (i) each wave starting with order r^2 BRWs hits the other boundary with order r^2 particles, at a constant cost; and (ii) when a single BRW starts on ∂B_r, it spends typically a time r^2 in B_r. Thus t/r^4 waves typically produce a local time t in B_r, and the cost is that of creating these t/r^4 waves.

In dimension four, the situation is drastically different: conditioned on coming from a distance R, a BRW typically brings $r^2 \cdot \log(R/r)$ particles on the boundary of B_r, so there is some advantage to coming from far away (the $\log(R/r)$ factor is absent in $d \geqslant 5$). This can be seen by saying that the process conditioned on hitting B_r has a single spine from which critical BRWs grow. We show that a *correct* scenario (see Figure 2 in Section 10) consists in producing $\log(R/r)$ spines, bringing a total number of particles of order $r^2 \cdot \log^2(R/r)$ and we equate this order with t/r^2 to get the desired stretched exponential cost.

We note that the idea of using waves to study the tail of the local time distribution was also found useful in the recent paper [BHJ23], which studies the thick points of branching Brownian motion and branching random walks.

Theorem 1.1 generalizes the analysis of [AHJ21], and its proof follows a probabilistic method. In this proof, we encounter many interesting objects whose exponential moments are studied, and they do present interest on their own. In order to present these additional results, we now define more objects. For a subset $\Lambda\subset\mathbb{Z}^d$, let $\mathcal{T}(\Lambda)$ be the set of vertices of $\mathcal{T}$ at which the BRW is in Λ, as well as at all its ancestral positions. In other words,

$$\mathcal{T}(\Lambda) := \big\{u\in\mathcal{T} : S_v\in\Lambda,\ \text{for all } v\leqslant u\big\}, \tag{1.1}$$

where we write $v\leqslant u$ if v is an ancestor of u (by which we include the case $v=u$). Also, the set of vertices corresponding to hitting times of Λ, called here frozen particles, is denoted as

$$\eta(\Lambda) := \big\{u\in\mathcal{T} : S_u\in\Lambda \text{ and } S_v\notin\Lambda \text{ for all } v<u\big\}, \tag{1.2}$$

where by $v<u$ we mean that v is an ancestor of u, which is different from u. Our main new estimates concern the exponential moments of the number of frozen particles during each wave. There are two distinct problems as whether we deal with starting points inside the ball, or outside it; the latter problem being the technical core of this paper.

Our first result concerns the case of a starting point lying inside the ball B_r, and we freeze walks as they reach its outer boundary ∂B_r. The result holds true irrespective of the dimension. For simplicity we write $\eta_r := \eta(\partial B_r)$.

Theorem 1.3. — *Assume $d\geqslant 1$. There exist positive constants c and λ_0 (only depending on the dimension), such that for any $0\leqslant\lambda\leqslant\lambda_0$, and any $r\geqslant 1$,*

$$\sup_{x\in B_r}\mathbb{E}_x\Big[\exp(\lambda|\eta_r|/r^2)\Big]\leqslant\exp(c\lambda/r^2). \tag{1.3}$$

Moreover,

$$\sup_{x\in B_r}\mathbb{E}_x\Big[\exp(\lambda|\eta_r|/r^2)\ \Big|\ \eta_r\neq\varnothing\Big]\leqslant\exp(c\lambda), \tag{1.4}$$

and

$$\sup_{x\in B_{r/2}}\mathbb{E}_x\Big[\exp(\lambda|\eta_r|/r^2)\Big]\leqslant\exp\big(\lambda+c\lambda^2/r^2\big). \tag{1.5}$$

Actually (1.4) easily follows from (1.3), once we know that the probability for the BRW to exit B_r starting from a point inside B_r, is *at least* of order $1/r^2$, see Lemma 3.8. On the other hand, (1.5) follows from (1.4), once we know that the former probability, starting from a point in $B_{r/2}$ is *at most* of order $1/r^2$, see Proposition 3.11.

Thus the heart of the matter is to prove (1.3). For this, we first show an analogous estimate for the total time spent inside the ball B_r by a BRW killed on its boundary, whose set of particles is given by $\mathcal{T}(B_r)$, see (1.1) and Proposition 1.4 below.

Then we observe that $|\eta_r|$ and $|\mathcal{T}(B_r)|$ are linked via some natural martingales whose exponential moments are controlled by those of $|\mathcal{T}(B_r)|$.

Proposition 1.4. — *Assume $d \geqslant 1$. There exist positive constants c and λ_0, such that for any $0 \leqslant \lambda \leqslant \lambda_0$, and any $r \geqslant 1$,*

$$\sup_{x\in B_r} \mathbb{E}_x\Big[\exp\big(\lambda|\mathcal{T}(B_r)|/r^4\big)\Big] \leqslant \exp(c\lambda/r^2).$$

In fact we prove a slightly stronger result in Proposition 9.1, which provides an important additional factor $1/r^2$ in the tail distribution, needed in the proof of Theorem 1.1.

To conclude the proof of Theorem 1.1, we need to control the exponential moments of $|\eta_r|$, when starting from a point outside B_r. This part is delicate, and this is where the role of the dimension comes into play. Roughly, the reason for this is that it could happen that many particles would freeze on ∂B_r, only after doing very large excursions away from it. In dimension five and higher the price to pay for these large excursions is too expensive for playing a significant role. As a consequence one can deduce a result which is similar to Theorem 1.3. Let us however emphasize the factor $(1-\varepsilon)$ appearing in (1.8), which comes from the transience of the random walk in dimension three and higher, and which guarantees that only a finite number of waves matter.

Theorem 1.5. — *Assume $d \geqslant 5$. There exist positive constants c, r_0 and λ_0 (only depending on the dimension), such that for any $0 \leqslant \lambda \leqslant \lambda_0$, any $r \geqslant r_0$, and $x \in B_{2r}^c$,*

$$\mathbb{E}_x\Big[\exp(\lambda|\eta_r|/r^2)\Big] \leqslant \exp(c\lambda/\|x\|^2). \tag{1.6}$$

As a consequence,

$$\sup_{x\in\partial B_{2r}} \mathbb{E}_x\Big[\exp(\lambda|\eta_r|/r^2)\ \Big|\ \eta_r \neq \varnothing\Big] \leqslant \exp(c\lambda), \tag{1.7}$$

and there exists $\varepsilon \in (0,1)$, such that for any $r \geqslant r_0$, and any $0 \leqslant \lambda \leqslant \lambda_0$,

$$\sup_{x\in\partial B_{2r}} \mathbb{E}_x\left[\exp\big(\lambda|\eta_r|/r^2\big)\right] \leqslant \exp\big(\lambda(1-\varepsilon)/r^2\big). \tag{1.8}$$

We note that the restriction $r \geqslant r_0$ in the above theorem could be dropped and replaced by $r \geqslant 1$, at the cost of some mild additional work, but since we shall not need it, we refrain from giving more details.

In dimension four, the situation is more subtle, and large excursions start to play a decisive role. In particular, one can show that all exponential moments of $|\eta_r|/r^2$ are infinite, when starting for instance from ∂B_{2r}. Thus one needs to consider instead a truncated version of η_r and renormalize it conveniently. We do this here, by killing the BRW once it reaches some large distance. To formulate our result, we define the deposition on B_r of trajectories which remain in B_R for $R > 2r$:

$$\eta_{r,R} = \eta_r \cap \mathcal{T}(B_R).$$

In other words $\eta_{r,R}$ is the set of vertices of $\mathcal{T}(B_R)$ corresponding to hitting times of ∂B_r. Then, we obtain the following.

Theorem 1.6. — *Assume $d=4$. There exist positive constants c, r_0 and λ_0, such that for any $0\leqslant\lambda\leqslant\lambda_0$, any $r\geqslant r_0$, $R\geqslant 2r$, and all $x\in B_R\smallsetminus B_{2r}$,*

$$\mathbb{E}_x\Big[\exp\Big(\frac{\lambda|\eta_{r,R}|}{r^2\log(R/r)}\Big)\Big]\leqslant\exp\Big(\frac{c\lambda}{\|x\|^2\log(R/r)}\Big). \tag{1.9}$$

Furthermore, if $R\geqslant 4r$,

$$\sup_{x\in\partial B_{2r}}\mathbb{E}_x\Big[\exp\Big(\frac{\lambda|\eta_{r,R}|}{r^2\log(R/r)}\Big)\;\Big|\;\eta_{r,R}\neq\varnothing\Big]\leqslant\exp(c\lambda), \tag{1.10}$$

and there exists $\varepsilon\in(0,1)$, such that

$$\sup_{x\in\partial B_{2r}}\mathbb{E}_x\left[\exp\Big(\frac{\lambda|\eta_{r,R}|}{r^2\log(R/r)}\Big)\right]\leqslant\exp\Big(\frac{\lambda(1-\varepsilon)}{r^2\log(R/r)}\Big). \tag{1.11}$$

Here as well, we note that the restriction $r\geqslant r_0$ could be dropped and replaced by $r\geqslant 1$.

These estimates allow to consider starting points which are not contained in the ball (equivalently balls not centered at the origin). For instance when $d\geqslant 5$, then uniformly in $r\geqslant 1$, $t\geqslant 1$, and $\|x\|\geqslant 2r$,

$$\mathbb{P}_x(\ell_{\mathcal{T}}(B_r)>t)=\Theta\big(r^{d-4}/\|x\|^{d-2}\big)\times\exp\big(-\Theta(t/r^4)\big).$$

Similar estimates could be proved when $\|x\|\leqslant 2r$, depending on the value of t, and the same could also be done in lower dimension.

The rest of the paper is organized as follows. Section 2 sets the notation and recall some basic results. Section 3 deals with some moment bounds for a Bienaymé-Galton-Watson process. We also recall there the spine decomposition of the BRW obtained by Zhu, and give bounds for the small moments of $|\eta_r|$ both when the starting point lies inside and outside the ball. Section 4 deals with Theorem 1.1 in low dimensions, and Section 5 deals with the exponential moments of the size of the localized BRW, as presented in Proposition 1.4. Theorem 1.3 is proved in Section 6. The technical heart of the paper spreads over three sections: Theorem 1.5 dealing with $d\geqslant 5$ is proved in Section 7, Theorem 1.6 dealing with $d=4$ is proved in Section 8, and the conclusion of the proof of the upper bounds in Theorem 1.1 for $d\geqslant 4$ is explained in Section 9. Finally, the lower bounds in high dimensions are given in Section 10.

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2. Notation and basic tools

We let $\mathcal{T}$ be a Bienaymé-Galton-Watson tree (BGW for short), with offspring distribution some measure μ on the set of integers. Throughout the paper we assume that μ is critical, in the sense that its mean is equal to one, and that it has a finite variance, which we denote by σ^2. When dimension is three, we assume furthermore that it has a finite third moment, and in higher dimension we assume that it has some finite exponential moment. For $u \in \mathcal{T}$ we let ξ_u be its number of children, so that

$$\mathbb{E}[\xi_u] = 1, \quad \mathbb{V}\mathrm{ar}(\xi_u) = \sigma^2, \quad \text{for all } u \in \mathcal{T}.$$

We denote the root of the tree by $\varnothing$. We write $|u|$ the generation of a vertex $u \in \mathcal{T}$, i.e., its distance to the root of the tree. We let $u \wedge v$ be the least common ancestor of u and v, i.e., the vertex at maximal distance from the root, among the ancestors of both u and v. For $n \geqslant 0$, we let Z_n be the number of vertices at generation n; in particular by definition $Z_0 = 1$ (the process $\{Z_n\}_{n\geqslant 0}$ is often called a BGW process, or sometimes just a Galton-Watson process in the literature). We also let $\mathcal{T}_n := \{u \in \mathcal{T} : |u| \leqslant n\}$. It follows from our hypotheses on μ, that for any $n \geqslant 1$, one has

$$\mathbb{E}[Z_n] = 1, \quad \text{and} \quad \mathbb{V}\mathrm{ar}(Z_n) = n\sigma^2. \tag{2.1}$$

We also recall Kolmogorov's estimate (see [AN04, Th. 1 p. 19]):

$$\mathbb{P}(Z_n \neq 0) \sim \frac{2}{\sigma^2 n}, \quad \text{as } n \longrightarrow \infty. \tag{2.2}$$

Recall the definition (1.1) of $\mathcal{T}(\Lambda)$, for $\Lambda \subset \mathbb{Z}^d$, and for $n \geqslant 1$, set

$$\mathcal{Z}_n(\Lambda) = \{u \in \mathcal{T}(\Lambda) : \ |u| = n\}. \tag{2.3}$$

We define the outer boundary of a subset $\Lambda \subset \mathbb{Z}^d$ as

$$\partial\Lambda := \{z \in \Lambda^c : \exists y \in \Lambda \text{ with } \|y - z\| = 1\}.$$

We let $(X_e)_e$ be a collection of independent and identically distributed random variables indexed by the edges of the tree, with joint law the uniform measure on the neighbors of the origin in $\mathbb{Z}^d$ (for a formal construction, see for instance [Shi15]). Then we define the branching random walk $\{S_u,\ u \in \mathcal{T}\}$, as the tree-indexed random walk, which means that for any vertex u, $S_u - S_\varnothing$ is the sum of the random variables X_e along the unique geodesic path from u to the root. We write $\mathbb{E}$ the expectation with respect to the BRW. We let $\mathbf{P}$ be the law of the standard random walk (which we shall also abbreviate as SRW) $\{S_n\}_{n\geqslant 0}$ on $\mathbb{Z}^d$, starting from the origin. For $x \in \mathbb{Z}^d$, we let $\mathbf{P}_x$ be the law of the SRW starting from x. For $r > 0$, we denote by H_r the hitting time of ∂B_r, for the SRW:

$$H_r := \inf\{n \geqslant 0 : S_n \in \partial B_r\}. \tag{2.4}$$

If $d \geqslant 3$, we let G be the Green's function, which is defined for any $z \in \mathbb{Z}^d$, by

$$G(z) = \sum_{n=0}^{\infty} \mathbf{P}(S_n = z).$$

We recall that under our assumption on the jump distribution, there exists a constant $c_G > 0$ (only depending on the dimension), such that (see [LL10, Th. 4.3.1]):

$$G(z) = c_G \cdot \|z\|^{2-d} + \mathcal{O}(\|z\|^{-d}). \tag{2.5}$$

Furthermore, the function G is harmonic on $\mathbb{Z}^d \setminus \{0\}$, in the sense that for all x different from the origin, $G(x) = \mathbf{E}_x[G(S_1)]$. As a consequence, using the optional stopping time theorem, we deduce that for some positive constants c and C, one has for any $r \geqslant 1$ and any $x \notin B_r$,

$$\frac{cr^{d-2}}{\|x\|^{d-2}} \leqslant \frac{G(x)}{\sup_{z\in\partial B_r} G(z)} \leqslant \mathbf{P}_x(H_r < \infty) \leqslant \frac{G(x)}{\inf_{z\in\partial B_r} G(z)} \leqslant \frac{Cr^{d-2}}{\|x\|^{d-2}}. \tag{2.6}$$

As in [AHJ21], we shall also make use of Paley-Zygmund's inequality, which asserts that for any nonnegative random variable X having finite second moment, and for any $\varepsilon \in [0,1)$,

$$\mathbb{P}(X \geqslant \varepsilon\mathbb{E}[X]) \geqslant \frac{(1-\varepsilon)^2 \cdot \mathbb{E}[X]^2}{\mathbb{E}[X^2]}. \tag{2.7}$$

A useful variant of (2.7), which comes after a little algebra reads

$$\mathbb{P}\big(X \geqslant \varepsilon \cdot \mathbb{E}[X \mid X \neq 0]\big) \geqslant \frac{(1-\varepsilon)^2 \cdot \mathbb{E}[X]^2}{\mathbb{E}[X^2]}. \tag{2.8}$$

Finally, given two functions f and g, we write $f \lesssim g$, if there exists a constant $C > 0$, such that $f \leqslant Cg$, and similarly for $f \gtrsim g$.

3. Preliminary results

3.1. Exponential moments for the BGW process. — In this subsection, we prove two elementary facts on the BGW process. Recall that we assume the offspring distribution μ to have mean one, and some finite exponential moment. Our first result shows that some exponential moment of Z_n/n is finite, conditionally on Z_n being nonzero (which is also known to converge in law to an exponential random variable with mean one as n goes to infinity, see [AN04, Th. 2 p. 20]).

Lemma 3.1. — *There exist $\lambda > 0$, such that*

$$\sup_{n\geqslant 1} \mathbb{E}\big[\exp(\lambda Z_n/n) \bigm| Z_n \neq 0\big] < \infty.$$

Note that the result is not new, and much stronger results are known, see for instance [NV75, NV03], but for reader's convenience we shall provide a direct and short proof here.

Proof. — Note that by (2.2), there exists $c > 0$, such that for all $n \geqslant 1$,

$$\begin{aligned}\mathbb{E}\big[\exp(\lambda Z_n/n) \bigm| Z_n \neq 0\big] &= 1 + \frac{\mathbb{E}\big[\exp(\lambda Z_n/n)\big] - 1}{\mathbb{P}(Z_n \neq 0)} \\ &\leqslant 1 + cn\left(\mathbb{E}\big[\exp(\lambda Z_n/n)\big] - 1\right).\end{aligned}$$

Thus it suffices to show that for some positive constants c and λ_0, one has for all $\lambda \leqslant \lambda_0$, and all $n \geqslant 1$,

$$\varphi_n(\lambda) := \mathbb{E}\big[\exp(\lambda Z_n/n)\big] \leqslant \exp\Big(\frac{\lambda + c\lambda^2}{n}\Big). \tag{3.1}$$

We prove this by induction over n. Note that the result for $n=1$ follows from the fact that Z_1 is distributed as μ, which has a finite exponential moment by hypothesis. Indeed, this implies that for some $c_0>0$, and all λ small enough,

$$\varphi_1(\lambda)=\mathbb{E}\big[\exp(\lambda Z_1)\big]\leqslant 1+\lambda+\lambda^2\mathbb{E}\big[Z_1^2\exp(\lambda Z_1)\big]\leqslant 1+\lambda+c_0\lambda^2\leqslant \exp(\lambda+c_0\lambda^2).$$

Now assume that (3.1) holds true for some n, and let us show it for $n+1$. Recall that conditionally on Z_n, Z_{n+1} is distributed as a sum of Z_n i.i.d. random variables with the same law as Z_1. Therefore, plugging the above computation yields

$$\varphi_{n+1}(\lambda)=\mathbb{E}\Big[\exp\Big(\frac{\lambda Z_{n+1}}{n+1}\Big)\Big]=\mathbb{E}\Big[\varphi_1\Big(\frac{\lambda}{n+1}\Big)^{Z_n}\Big]\leqslant \varphi_n\Big(\frac{\lambda n}{n+1}+c_0\frac{\lambda^2 n}{(n+1)^2}\Big).$$

Note that the induction hypothesis reads also as $n\log\varphi_n(\lambda)\leqslant\lambda+c\lambda^2$. Then

$$\begin{aligned}(n+1)\log\varphi_{n+1}(\lambda)&\leqslant\frac{n+1}{n}\Big(\frac{n}{n+1}\lambda+c_0\frac{n}{(n+1)^2}\lambda^2+c\Big(\frac{\lambda n}{n+1}+c_0\frac{\lambda^2 n}{(n+1)^2}\Big)^2\Big)\\&\leqslant\lambda+\frac{c_0}{n+1}\lambda^2+c\frac{n}{n+1}\lambda^2+2cc_0\frac{\lambda^3}{n}+cc_0^2\frac{\lambda^4}{n^2}\\&\leqslant\lambda+c\lambda^2-\frac{c-c_0}{n+1}\lambda^2+2cc_0\frac{\lambda^3}{n}+cc_0^2\frac{\lambda^4}{n^2}.\end{aligned}$$

Thus if we choose $c=2c_0$, and λ small enough so that for any $n\geqslant 1$,

$$4c_0\frac{n+1}{n}\lambda+2c_0^2\frac{n+1}{n^2}\lambda^2<1,$$

one obtains

$$\varphi_{n+1}(\lambda)\leqslant\exp\Big(\frac{\lambda+c\lambda^2}{n+1}\Big).$$

This establishes the induction step, and concludes the proof of the lemma. □

Our second result concerns the exponential moments for the total size of the BGW tree, up to some fixed generation, and is proved along the same lines.

Lemma 3.2. — *There exist positive constants c and λ_0, such that for any $\lambda\in[0,\lambda_0]$, and any $n\geqslant 1$,*

$$\mathbb{E}\big[\exp(\lambda|\mathcal{T}_n|/n^2)\big]\leqslant\exp\Big(\frac{\lambda+c\lambda^2}{n}\Big).$$

Proof. — We prove the result by induction on $n\geqslant 1$. The case $n=1$ has already been seen in the proof of Lemma 3.1, and only relies on the fact that Z_1 has a finite exponential moment by assumption. Assume now that it holds for some n, and let us prove it for $n+1$. Since conditionally on Z_1, $|\mathcal{T}_{n+1}|$ is the sum of Z_1 i.i.d. random variables distributed as $|\mathcal{T}_n|$, we deduce from the induction hypothesis, that for some c and λ_0 one has for all $\lambda\leqslant\lambda_0$,

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(\frac{\lambda|\mathcal{T}_{n+1}|}{(n+1)^2}\Big)\mid Z_1\Big]&\leqslant\exp\Big(\frac{\lambda n/(n+1)+c\lambda^2(n/(n+1))^3}{n+1}Z_1\Big)\\&\leqslant\exp\Big(\frac{\lambda+c\lambda^2-\lambda/(n+1)}{n+1}Z_1\Big).\end{aligned}$$

Integrating now both sides over Z_1, and using the result for $n=1$, gives

$$\mathbb{E}\Big[\exp\Big(\frac{\lambda|\mathcal{T}_{n+1}|}{(n+1)^2}\Big)\Big] \leqslant \exp\Big(\frac{\lambda + c\lambda^2 - \lambda/(n+1) + c(\lambda+c\lambda^2)^2/(n+1) + c\lambda^2/(n+1)^3}{n+1}\Big) \leqslant \exp\Big(\frac{\lambda + c\lambda^2 - (\lambda - 2c\lambda^2 + \mathcal{O}(\lambda^3))/(n+1)}{n+1}\Big),$$

and the right-hand side is well smaller than $\exp((\lambda+c\lambda^2)/(n+1))$, provided λ_0 is small enough. This concludes the proofs of the induction step, and of the lemma. □

3.2. Spine decomposition for the BRW. — We present here a spine decomposition of the BRW conditioned to hit a set, which was introduced by Zhu to derive upper bounds on hitting probabilities, see [Zhu16a, Zhu21]. We shall use it also later for proving the lower bound in Theorem 1.1 in dimension four.

Following the terminology of [Zhu16a, Zhu21], an *adjoint BGW tree* is a BGW tree in which only the law of the number of children of the root has been modified, and follows the law $\widetilde{\mu}$, given by $\widetilde{\mu}(i) := \sum_{j\geqslant i+1}\mu(j)$, for $i \geqslant 0$. The associated tree-indexed random walk is the *adjoint BRW*. Then we define $k_\Lambda(x)$, for $x\in\mathbb{Z}^d$, as the probability for an adjoint BRW starting from x to hit Λ. Now, given an integer n and a path $\gamma:\{0,\dots,n\}\to\mathbb{Z}^d$, we define, with $|\gamma|=n$,

$$p_\Lambda(\gamma) := \prod_{i=0}^{|\gamma|-1} \theta\big(\gamma(i+1)-\gamma(i)\big)\cdot\big(1-k_\Lambda(\gamma(i))\big), \tag{3.2}$$

where we recall that θ is the uniform measure on the neighbors of the origin. In other words, $p_\Lambda(\gamma)$ is the probability that a SRW starting from $\gamma(0)$ follows the path γ during its first n steps, when it is killed at each step with probability given by the function k_Λ at its current position.

We next define the probability measures $\{\mu_\Lambda^z\}_{z\in\Lambda^c}$ on the integers by

$$\mu_\Lambda^z(m) := \sum_{\ell\geqslant 0}\mu(\ell+m+1)r_\Lambda(z)^\ell/\big(1-k_\Lambda(z)\big),\quad \text{for all } m\geqslant 0,$$

where $r_\Lambda(z)$ is the probability that a BRW starting from z does not visit Λ, conditionally on the root having only one child. We call biased BRW starting from z, a BRW starting from z, conditioned on the number of children of the root having law μ_Λ^z.

Furthermore, a finite path γ is said to go from x to Λ, which we denote as $\gamma\colon x\to\Lambda$, if for $n=|\gamma|$, $\gamma(n)\in\Lambda$, and $\gamma(i)\notin\Lambda$ for all $i<n$. In other words this simply means that the path γ is defined up to its hitting time of Λ (note that it includes the possibility that $n=0$ and $\gamma(0)\in\Lambda$). We shall also later write for simplicity $\gamma: x\to y$, when Λ is reduced to a single point y. Moreover, if $\Lambda\subset A$, we write $\gamma: A\to\Lambda$ when the path γ is such that $\gamma(0)\in A$, and γ goes from $\gamma(0)$ to Λ. Then given $x\in\mathbb{Z}^d$, and $\gamma: x\to\Lambda$, we call γ-biased BRW, the union of γ, together with for each $i\in\{0,\dots,|\gamma|-1\}$, a biased BRW starting from $\gamma(i)$, independently for each i, and starting from $\gamma(n)$ some usual independent BRW.

We are now in position to describe the law of the BRW starting at some $x \in \mathbb{Z}^d$, and conditioned on hitting Λ. For simplicity, we restrict ourselves to the part which intersects Λ, since this is the only one that is needed here. On the event $\{\ell_{\mathcal{T}}(\Lambda) > 0\}$, one defines the first entry vertex, as the smallest vertex $u \in \mathcal{T}$ in the lexicographical order, for which $S_u \in \Lambda$. Then, denoting by $\overleftarrow{u}$ the unique geodesic path in the BGW tree going from the root to the vertex u, we let $\Gamma = \Gamma(\mathcal{T})$ be the path in $\mathbb{Z}^d$, made of the successive positions of the BRW along that path. Thus if $\overleftarrow{u} = (u_0 = \varnothing, u_1, \dots, u_n = u)$, with $n = |u|$, then

$$\Gamma = \big(S_\varnothing = x, S_{u_1}, \dots, S_{u_n}\big).$$

Note that by definition Γ is a path which goes from x to Λ in our terminology. The following result comes from [Zhu21, Prop. 2.4].

Proposition 3.3 ([Zhu21]). — *Assume $d \geqslant 1$. Let $\Lambda \subset \mathbb{Z}^d$, and $x \in \mathbb{Z}^d$ be given.*

(1) *For any path $\gamma : x \to \Lambda$, one has*

$$\mathbb{P}_x(\Gamma = \gamma, \ell_{\mathcal{T}}(\Lambda) > 0) = p_\Lambda(\gamma).$$

(2) *Furthermore, conditionally on $\{\Gamma = \gamma,\ \ell_{\mathcal{T}}(\Lambda) > 0\}$, the trace of the BRW on Λ has the same law as the trace of a γ-biased BRW.*

The product formula (3.2) defining p_Λ implies that Γ satisfies the (strong) Markov property, in the following sense. Given $x \in \mathbb{Z}^d$, we define a probability measure $\mathbb{P}_\Lambda^x$ on the set of paths $\gamma : x \to \Lambda$, by

$$\mathbb{P}_\Lambda^x(\gamma) := \frac{p_\Lambda(\gamma)}{\sum_{\gamma' : x \to \Lambda} p_\Lambda(\gamma')},$$

which is nothing else than the law of Γ, conditionally on the event $\{\ell_{\mathcal{T}}(\Lambda) > 0\}$. For convenience, we also set $\mathbb{P}_\Lambda^x(\gamma) = 0$, if γ is not a path that goes from x to Λ. Then we can state the Markov property as follows (we only state a particular case, but the same would hold for any stopping time).

Corollary 3.4. — *Let $\Lambda \subset A \subset \mathbb{Z}^d$, and $x \in \mathbb{Z}^d$ be given. Let $\tau_A := \inf\{i \geqslant 0 : \Gamma(i) \in A\}$. Then for any path $\gamma : A \to \Lambda$, one has*

$$\mathbb{P}_\Lambda^x\Big((\Gamma(\tau_A), \dots) = \gamma \mid \Gamma(0), \dots, \Gamma(\tau_A)\Big) = \mathbb{P}_\Lambda^{\Gamma(\tau_A)}(\gamma).$$

Proof. — It suffices to notice that by (3.2), for any path $\gamma_0 : x \to \gamma(0)$, writing $\gamma \circ \gamma_0$ for the concatenation of γ_0 and γ, one has

$$\begin{aligned} \mathbb{P}_\Lambda^x\Big((\Gamma(\tau_A), \dots) = \gamma \mid (\Gamma(0), \dots, \Gamma(\tau_A)) = \gamma_0\Big) \\ &= \frac{p_\Lambda(\gamma \circ \gamma_0)}{\sum_{\gamma' : x \to \Lambda} p_\Lambda(\gamma')} \times \frac{\sum_{\gamma' : x \to \Lambda} p_\Lambda(\gamma')}{p_\Lambda(\gamma_0) \sum_{\gamma'' : \gamma(0) \to \Lambda} p_\Lambda(\gamma'')} \\ &= \frac{p_\Lambda(\gamma) \cdot p_\Lambda(\gamma_0)}{p_\Lambda(\gamma_0) \sum_{\gamma'' : \gamma(0) \to \Lambda} p_\Lambda(\gamma'')} = \mathbb{P}_\Lambda^{\gamma(0)}(\gamma). \end{aligned}$$

□

3.3. Hitting Probability Lower Bounds. — Here we derive rough lower bounds for the hitting probabilities of balls, using a second moment method. The result is as follows (recall (1.2)):

Lemma 3.5. — *There exists a constant $c > 0$ (only depending on the dimension), such that for any $r \geqslant 1$ and $x \notin B_r$,*

$$\mathbb{P}_x(|\eta_r| > 0) \geqslant c \cdot \begin{cases} \dfrac{r^{d-4}}{\|x\|^{d-2}} & \text{if } d \geqslant 5, \\ \dfrac{1}{\|x\|^2 \log(1+\|x\|/r)} & \text{if } d = 4. \end{cases}$$

Remark 3.6. — Note that in dimension five and higher the result follows from [Zhu16a], which proves as well an upper bound of the same order. We include a short proof here, for the reader's convenience. In dimension four, a more precise asymptotic is proved in [Zhu21] in case $r = 1$ (see also [Zhu19] for a rough upper bound still in the case $r = 1$).

Remark 3.7. — We shall also use this lower bound in dimension four for a biased BRW. In this case the result follows immediately from the lemma, and the fact that a biased BRW has a probability bounded from below to have at least one child, from where starts a fresh usual BRW.

Proof. — Recall that we denote by H_r the hitting time of ∂B_r for a standard random walk. One has, using (2.6) for the last inequality,

$$\begin{aligned}\mathbb{E}_x\big[|\eta_r|\big] &= \sum_{n=0}^{\infty} \mathbb{E}_x\Big[\textstyle\sum_{|u|=n} \mathbf{1}\{u \in \eta_r\}\Big] \\ &= \sum_{n=0}^{\infty} \mathbb{E}[Z_n] \cdot \mathbf{P}_x(H_r = n) = \mathbf{P}_x(H_r < \infty) \gtrsim (r/\|x\|)^{d-2}.\end{aligned}$$

Now we bound the second moment. Recall that by definition, if $u \in \eta_r$, then none of its descendant can be in η_r. Thus summing first over all possible $w \in \mathcal{T}$ and then integrating over all possible $u \neq v$, with $u \wedge v = w$ (and necessarily both u and v different from w), gives (recall the notation (2.3))

$$\begin{aligned}\mathbb{E}_x\big[|\eta_r|^2\big] &= \mathbf{P}_x(H_r < \infty) + \sum_{k=0}^{\infty} \mathbb{E}_x\Big[\textstyle\sum_{w \in \mathcal{Z}_k((\partial B_r)^c)} \xi_w(\xi_w - 1) \mathbf{P}_{S_w}(H_r < \infty)^2\Big] \\ &= \mathbf{P}_x(H_r < \infty) + \sigma^2 \sum_{k=0}^{\infty} \mathbb{E}_x\Big[\textstyle\sum_{w \in \mathcal{Z}_k((\partial B_r)^c)} \mathbf{P}_{S_w}(H_r < \infty)^2\Big] \\ &\leqslant \mathbf{P}_x(H_r < \infty) + \sigma^2 \sum_{k=0}^{\infty} \mathbb{E}_x\Big[\textstyle\sum_{|w|=k} \mathbf{1}\{S_w \notin \partial B_r\} \cdot \mathbf{P}_{S_w}(H_r < \infty)^2\Big] \\ &\lesssim (r/\|x\|)^{d-2} + r^{2(d-2)} \sum_{k=0}^{\infty} \mathbf{E}_x\Big[\frac{\mathbf{1}\{S_k \notin \partial B_r\}}{\|S_k\|^{2(d-2)}}\Big] \\ &= (r/\|x\|)^{d-2} + r^{2(d-2)} \sum_{z \notin B_r} \frac{G(z-x)}{\|z\|^{2(d-2)}} \lesssim \begin{cases} \dfrac{r^d}{\|x\|^{d-2}} & \text{if } d \geqslant 5, \\ \dfrac{r^4}{\|x\|^2} \cdot \log(1+\|x\|/r) & \text{if } d = 4. \end{cases}\end{aligned}$$

The result follows using that by Cauchy-Schwarz inequality,

$$\mathbb{P}_x(|\eta_r| > 0) \geqslant \frac{\mathbb{E}_x\big[|\eta_r|\big]^2}{\mathbb{E}_x\big[|\eta_r|^2\big]}. \qquad \square$$

The next result holds in any dimension.

Lemma 3.8. — *Assume $d \geqslant 1$. There exists a constant $c > 0$ (only depending on the dimension), such that for any $r \geqslant 1$,*

$$\inf_{x\in B_r} \mathbb{P}_x(|\eta_r| > 0) \geqslant c/r^2.$$

Proof. — The proof is similar to the previous lemma. Note first that for any $x \in B_r$,

$$\mathbb{E}_x[|\eta_r|] = \mathbf{P}_x(H_r < \infty) = 1,$$

and as before,

$$\mathbb{E}_x\big[|\eta_r|^2\big] \leqslant 1 + \sigma^2 \sum_{k=0}^{\infty} \mathbf{P}_x(H_r > k) = 1 + \sigma^2\, \mathbf{E}_x[H_r] \lesssim r^2.$$

The result follows. $\square$

Finally we state a result concerning the first and third moments of $|\eta_r|$, when starting from ∂B_{2r}.

Lemma 3.9. — *Assume $d \geqslant 1$.*

(1) *For any $x \in B_r^c$, one has $\mathbb{E}_x\big[|\eta_r|\big] = \mathbf{P}_x(H_r < \infty)$. In particular, when $d \geqslant 3$, there exists $\varepsilon > 0$, such that for any $r \geqslant 1$,*

$$\sup_{x\in\partial B_{2r}} \mathbb{E}_x\big[|\eta_r|\big] \leqslant 1 - \varepsilon.$$

(2) *Assume $d \geqslant 4$. There exists $C > 0$, such that for any $r \geqslant 1$,*

$$\sup_{x\in\partial B_{2r}} \mathbb{E}_x\big[|\eta_r|^3\big] \leqslant Cr^4.$$

Remark 3.10. — The last point of the lemma can be understood, by considering that starting from ∂B_{2r}, the probability to hit ∂B_r is of order $1/r^2$, and conditionally on hitting it, the number of frozen particles $|\eta_r|$ is typically of order r^2. This reasoning would apply also to all other moments of fixed order, but the third moment will suffice for our purpose.

Proof. — We start with the first point. The equality $\mathbb{E}_x\big[|\eta_r|\big] = \mathbf{P}_x(H_r < \infty)$ has already been seen in the proof of Lemma 3.5. Then the fact that if $\|x\| \geqslant 2r$, this quantity is bounded from above by some constant smaller than one, in dimension three and higher, follows from (2.5) and (2.6).

Let us prove the second point now. When we sum over triples, say $(u, v, w) \in \eta_r$, we distinguish two cases. Either at least two of them are equal, and we just bound the corresponding sum by three times the second moment of $|\eta_r|$, or the three points are distinct. In the latter case, we again distinguish between two possible situations:

either, $u\wedge v = u\wedge w$, or $u\wedge v \neq u\wedge w$. In both cases, by summing first over $(u\wedge v)\wedge(u\wedge w)$ yields, for any $x\in\partial B_{2r}$, (recall the notation (2.3)),

$$\begin{aligned}(3.3)\quad \mathbb{E}_x\big[|\eta_r|^3\big] \leqslant 3\mathbb{E}_x\big[|\eta_r|^2\big] \\ +\sum_{k=0}^{\infty}\mathbb{E}_x\Big[\textstyle\sum_{u\in\mathcal{Z}_k((\partial B_r)^c)}\xi_u(\xi_u-1)(\xi_u-2)\mathbf{P}_{S_u}(H_r<\infty)^3\Big] \\ +\sum_{k=0}^{\infty}\mathbb{E}_x\Big[\textstyle\sum_{u\in\mathcal{Z}_k((\partial B_r)^c)}\xi_u(\xi_u-1)\mathbf{P}_{S_u}(H_r<\infty)\cdot\mathbb{E}_{S_u}\big[|\eta_r|^2\big]\Big].\end{aligned}$$

Next, using (2.6) we get

$$\begin{aligned}\sum_{k=0}^{\infty}\mathbb{E}_x\Big[\textstyle\sum_{u\in\mathcal{Z}_k((\partial B_r)^c)}\xi_u(\xi_u-1)(\xi_u-2)\mathbf{P}_{S_u}(H_r<\infty)^3\Big] \\ \leqslant Cr^{3(d-2)}\sum_{k=0}^{\infty}\mathbf{E}_x\Big[\frac{\mathbf{1}\{S_k\notin B_r\}}{\|S_k\|^{3(d-2)}}\Big]\leqslant Cr^{3(d-2)}\sum_{z\notin B_r}\frac{G(z-x)}{\|z\|^{3(d-2)}}\leqslant Cr^2.\end{aligned}$$

Finally for the last sum in (3.3), we use the computation from the proof of Lemma 3.5, in particular the fact that the second moment of $|\eta_r|$ is $\mathcal{O}(r^2)$, when starting from $x\notin B_r$. This yields the upper bound

$$\begin{aligned}Cr^{2d-2}\sum_{k=0}^{\infty}\mathbf{E}_x\Big[\frac{\mathbf{1}\{S_k\notin B_r\}}{\|S_k\|^{2(d-2)}}\log(1+\|S_k\|/r)\Big]\leqslant Cr^{2d-2}\sum_{z\notin B_r}\frac{G(z-x)}{\|z\|^{2(d-2)}}\cdot\log(1+\|z\|/r) \\ \leqslant Cr^4,\end{aligned}$$

concluding the proof of the lemma. □

3.4. Hitting probability upper bounds. — In this section we provide upper bounds for hitting probabilities of spheres, which are of the same order as the lower bounds obtained previously.

The first result considers hitting probabilities of a sphere, starting from inside the ball. The result is not new, in particular it was already proved in [Zhu16a, Prop. 10.3] under only a first moment hypothesis on the offspring distribution (and mild condition on the jump distribution of the walk). For completeness, we provide here an alternative proof, which however requires a finite second moment of the offspring distribution. We also mention [Kes95] which proves similar results.

Proposition 3.11 ([Zhu16a]). — *Assume $d\geqslant 1$. There exists $C>0$, such that for any $r\geqslant 1$,*

$$\sup_{x\in B_{r/2}}\mathbb{P}_x(|\eta_r|>0)\leqslant\frac{C}{r^2}.$$

Proof. — Assume without loss of generality that r^2 is an integer. We first write for $x \in B_{r/2}$, using (2.2), that for some constant $C>0$,

$$\begin{aligned}\mathbb{P}_x(|\eta_r|>0) &\leqslant \sum_{k=0}^{r^2} \mathbb{P}_x(Z_k \neq 0, Z_{k+1}=0, |\eta_r|>0) + \mathbb{P}(Z_{r^2}\neq 0)\\ &\leqslant \sum_{k=0}^{r^2} \mathbb{P}_x(Z_k \neq 0, Z_{k+1}=0, |\eta_r|>0) + \frac{C}{r^2}.\end{aligned}$$

Now fix some $k \leqslant r^2$, and note that conditionally on Z_k, the probability for Z_{k+1} to be zero is equal to $\mu(0)^{Z_k}$. Then by using a first moment bound, and the fact that for any vertex at generation k, the probability that the BRW reaches ∂B_r along the line of its ancestors is given exactly by the probability for a SRW to reach ∂B_r, we get for some constant $c>0$,

$$\begin{aligned}\mathbb{P}_x(Z_k \neq 0, Z_{k+1}=0, |\eta_r|>0) &\leqslant \mathbb{E}\big[Z_k\mu(0)^{Z_k}\big]\cdot \mathbf{P}_x(H_r\leqslant k)\\ &\lesssim \mathbb{E}\big[Z_k\mu(0)^{Z_k}\big]\cdot \exp(-cr^2/k),\end{aligned} \tag{3.4}$$

where the last inequality is well-known, see e.g. [LL10, Prop. 2.1.2]. Letting $f_k(s) = \sum_{n=0}^{\infty}\mathbb{P}(Z_k=n)s^n$, the generating function of Z_k, one has

$$\mathbb{E}\big[Z_k\mu(0)^{Z_k}\big] = \sum_{n=0}^{\infty} n\mu(0)^n\mathbb{P}(Z_k=n) = \mu(0)f_k'(\mu(0)).$$

We claim that for some constant $C>0$ (depending only on μ), one has

$$f_k'(\mu(0)) \leqslant \frac{C}{k^2}. \tag{3.5}$$

Indeed, this follows from the results in [AN04]. First, note that it suffices to prove (3.5) when k is an even integer, since for any $s\leqslant 1$, and $k\geqslant 1$, $f_k'(s) = f'(f_{k-1}(s))f_{k-1}'(s) \leqslant f_{k-1}'(s)$. Now the process $(Z_{2k})_{k\geqslant 0}$ is a critical branching process, whose offspring distribution μ' satisfies $\mu'(1)>0$. Hence [AN04, Lem. 2 p. 12] shows that $\mathbb{P}(Z_k=n)\leqslant \mathbb{P}(Z_k=1)\cdot \pi_n$, for all $n\geqslant 1$, and some constants $(\pi_n)_n$ in $[0,\infty]$. Then [AN04, Th. 2 p. 13] shows that $\sum_{n\geqslant 1} n\pi_n\mu(0)^n < \infty$, and [AN04, Cor. 2 p. 23] gives that $\mathbb{P}(Z_k=1)\leqslant C/k^2$, for some constant $C>0$, which altogether proves (3.5). Injecting this estimate in (3.4) and summing over k concludes the proof of the lemma. □

The second result concerns hitting probabilities of a ball starting from a point outside it.

Proposition 3.12 ([Zhu16a, Zhu19]). — *There exists $C>0$, such that for any $r\geqslant 1$, and any $x\notin B_{2r}$,*

$$\mathbb{P}_x(|\eta_r|>0) \leqslant C\cdot \begin{cases} \dfrac{r^{d-4}}{\|x\|^{d-2}} & \text{if } d\geqslant 5,\\[2mm] \dfrac{1}{\|x\|^2\log(\|x\|/r)} & \text{if } d=4.\end{cases}$$

Proof. — The result in dimensions at least five is given in [Zhu16a], and the proof in dimension four is identical to the case $r=1$, which is done in [Zhu19]. We leave the details to the reader. □

Remark 3.13. — We shall also need the result of the proposition for an adjoint BRW. For this, one can use a union bound, by summing over all the children of the root. Since the expected degree of the root in an adjoint BGW tree is finite, under the hypothesis that the offspring distribution has finite second moment, the result for the adjoint BRW follows from Proposition 3.12.

3.5. Proof of Theorem 1.1 for short times. — We deal in this section with $t\leqslant r^4$.

The upper bound. — We use that $\ell_{\mathcal{T}}(B_r)$ coincides essentially with the total size of the BGW tree $\mathcal{T}$. Indeed, using (2.2) and Markov's inequality, yields

$$\begin{aligned}\mathbb{P}(\ell_{\mathcal{T}}(B_r)>t)\leqslant \mathbb{P}(|\mathcal{T}|>t)&\leqslant \mathbb{P}(Z_{\sqrt{t}}\neq 0)+\mathbb{P}(1+Z_1+\cdots+Z_{\sqrt{t}}>t)\\ &\lesssim \frac{1}{\sqrt{t}}+\frac{1}{t}\cdot\mathbb{E}[Z_1+\cdots+Z_{\sqrt{t}}]\lesssim\frac{1}{\sqrt{t}},\end{aligned}$$

giving the desired upper bound.

The lower bound. — We divide the proof in two steps. We first show that we can bring of the order of $\sqrt{t}$ particles in $B_{r/2}$ at a cost of order $1/\sqrt{t}$, and then we show that conditionally on this event, the union of the BRWs emanating from these particles are likely to spend a time t in the ball B_r.

Let us start with the first step. Let $r_0:=t^{1/4}/2$. Note that by hypothesis $r_0\leqslant r/2$. By Proposition 3.11, one has for some constant $C>0$,

$$\mathbb{P}(|\eta_{r_0}|>0)\leqslant C/\sqrt{t},$$

and since $\mathbb{E}[|\eta_{r_0}|]=1$, we have, for some constant $c>0$,

$$\mathbb{E}\big[|\eta_{r_0}|\ \big|\ |\eta_{r_0}|>0\big]\geqslant c\sqrt{t}.$$

Then, by using Paley-Zygmund's inequality (2.8) (with $\varepsilon=1/2$), and the second moment estimates in the proof of Lemma 3.8, we deduce that for some constant $\rho_1>0$,

$$\mathbb{P}(|\eta_{r_0}|\geqslant\rho_1\sqrt{t})\geqslant\frac{\rho_1}{\sqrt{t}}, \tag{3.6}$$

concluding the first step.

Now we move to the second step. Let $X:=\sum_{\sqrt{t}/4\leqslant k\leqslant\sqrt{t}}Z_k$. We first claim that for some $c_0>0$ (independent of t and r) we have

$$\mathbb{P}\big(X\geqslant c_0t\big)\geqslant\frac{c_0}{\sqrt{t}}. \tag{3.7}$$

Indeed, observe that $\mathbb{E}[X]\geqslant\frac{1}{2}\sqrt{t}$, and thus by Kolmogorov's estimate (2.2), one has for some $c>0$,

$$\mathbb{E}[X\mid X\neq 0]\geqslant ct.$$

On the other hand, using (2.1), we get

$$\mathbb{E}[X^2] \leqslant \sqrt{t} \cdot \sum_{\sqrt{t}/4 \leqslant i \leqslant \sqrt{t}} \mathbb{E}[Z_i^2] \leqslant t^{3/2},$$

and (3.7) follows using Paley-Zygmund's inequality (2.8).

Now, define

$$Y = \sum_{\sqrt{t}/4 \leqslant |u| \leqslant \sqrt{t}} \mathbf{1}\{S_u \in B_r\}.$$

One has for any $z \in \partial B_{r_0}$,

$$\mathbb{E}_z[Y \mid \mathcal{T}] = \sum_{\sqrt{t}/4 \leqslant k \leqslant \sqrt{t}} Z_k \cdot \mathbf{P}_z(S_k \in B_r) \geqslant \delta \cdot X,$$

with

$$\delta := \inf_{x \in B_r} \inf_{k \leqslant r^2} \mathbf{P}_x(S_k \in B_r) > 0.$$

On the other hand, by definition one has $Y \leqslant X$, and thus (2.7) gives

$$\mathbb{P}_z(Y \geqslant c_0 \delta t/2 \mid \mathcal{T}) \cdot \mathbf{1}\{X \geqslant c_0 t\} \geqslant \frac{\frac{1}{4}\mathbb{E}_z\big[Y \mid \mathcal{T}\big]^2}{\mathbb{E}_z\big[Y^2 \mid \mathcal{T}\big]} \cdot \mathbf{1}\{X \geqslant c_0 t\} \geqslant \frac{\delta^2}{4} \cdot \mathbf{1}\{X \geqslant c_0 t\}.$$

Note also that by definition $Y \leqslant \ell_{\mathcal{T}}(B_r)$. Hence, taking expectation on both sides of the last inequality and plugging (3.7), gives with $\rho_2 := c_0 \delta^2/4$,

$$\inf_{z \in \partial B_{r_0}} \mathbb{P}_z(\ell_{\mathcal{T}}(B_r) \geqslant \rho_2 t) \geqslant \frac{\rho_2}{\sqrt{t}}. \tag{3.8}$$

To conclude, notice that, as mentioned at the beginning of the proof, the time spent in B_r is larger than the time spent by the union of the BRWs emanating from the particles in η_{r_0} (which by definition are on ∂B_{r_0}). Hence, denoting by $\mathcal{N}$ a binomial random variable with number of trials $\lfloor \rho_1 \sqrt{t} \rfloor$ and probability of success $\rho_2/\sqrt{t}$, we get using (3.6) together with (3.8), for some constant $\rho > 0$ (independent of t and r),

$$\mathbb{P}(\ell_{\mathcal{T}}(B_r) \geqslant t) \geqslant \mathbb{P}(\ell_{\mathcal{T}}(B_r) \geqslant t, |\eta_{r_0}| \geqslant \rho_1 \sqrt{t}) \geqslant \mathbb{P}(\mathcal{N} \geqslant 1/\rho_2) \cdot \mathbb{P}(|\eta_{r_0}| \geqslant \rho_1 \sqrt{t}) \geqslant \frac{\rho}{\sqrt{t}},$$

as wanted.

This concludes the proof of Theorem 1.1 in case $t \leqslant r^4$. □

4. Proof of Theorem 1.1 in low dimension

In dimension one, two and three, the proofs are easier, and only require small moment estimates, which enables us to use the results from [AHJ21]. More precisely, in dimension one and two both the lower and upper bounds only require estimates of the first and second moment, and in dimension three we need an additional third moment. We define for $d \in \{1, 2, 3\}$,

$$R = (t/r^d)^{1/(4-d)}, \quad \text{and} \quad N = R^2.$$

Recall that one can assume here that $t \geqslant r^4$, in which case it amounts to show that

$$\mathbb{P}(\ell_{\mathcal{T}}(B_r) > t) = \Theta(R^{-2}).$$

The proof is based on the following result, which is given by [AHJ21, Lem. 4.3]. For $n \geqslant 0$, and $x \in \mathbb{Z}^d$, write

$$\ell_n(x) = \sum_{|u| \leqslant n} \mathbf{1}\{S_u = x\}.$$

Lemma 4.1 ([AHJ21]). — *If μ is critical and has a finite second moment, then*

$$\sup_{x \in \mathbb{Z}^2} \mathbb{E}[\ell_n(x)^2] \lesssim n.$$

If additionally μ has a finite third moment, then

$$\sup_{x \in \mathbb{Z}^3} \mathbb{E}[\ell_n(x)^3] \lesssim \sqrt{n}.$$

As a consequence, writing $\ell_N(B_r) := \sum_{x \in B_r} \ell_N(x)$, we get

$$\begin{cases} \mathbb{E}[\ell_N(B_r)^2] \lesssim r^4 N & \text{if } d = 2, \\ \mathbb{E}[\ell_N(B_r)^3] \lesssim r^9 \sqrt{N} & \text{if } d = 3. \end{cases}$$

Note also that by linearity, one has when $d = 1$,

$$\mathbb{E}[\ell_N(B_r)] \lesssim r\sqrt{N}.$$

Therefore, for any $d \in \{1, 2, 3\}$,

$$\mathbb{P}(\ell_{\mathcal{T}}(B_r) > t) \leqslant \mathbb{P}(Z_N \neq 0) + \mathbb{P}(\ell_N(B_r) > t) \leqslant \frac{1}{N} + \frac{\mathbb{E}[\ell_N(B_r)^d]}{t^d} \lesssim R^{-2}.$$

For the lower bounds we use Paley-Zygmund's inequality (2.8), which we apply with

$$X := \ell_{2MN}(B_r) - \ell_{MN}(B_r) = \sum_{k=MN+1}^{2MN} \sum_{|u|=k} \mathbf{1}\{S_u \in B_r\},$$

where M is some well chosen integer to be fixed later. We need the following first moment bounds: if $t \geqslant r^4$ (equivalently $N \geqslant r^2$),

$$\begin{aligned} \mathbb{E}[X] &= \sum_{k=MN+1}^{2MN} \mathbb{E}\Big[\sum_{|u|=k} \mathbf{1}\{S_u \in B_r\}\Big] \\ &= \sum_{k=MN+1}^{2MN} \mathbb{E}[Z_k] \cdot \mathbf{P}(S_k \in B_r) = \sum_{k=MN+1}^{2MN} \mathbf{P}(S_k \in B_r) \\ &\gtrsim \begin{cases} r\sqrt{N} & \text{if } d = 1, \\ r^2 & \text{if } d = 2, \\ r^3/\sqrt{N} & \text{if } d = 3. \end{cases} \end{aligned}$$

Note also that $\mathbb{P}(X > 0) \leqslant \mathbb{P}(Z_{MN} > 0) \sim 2/\sigma^2 MN$. Thus if M is chosen large enough, in any dimension $d \in \{1, 2, 3\}$, for $t \geqslant r^4$,

$$\mathbb{E}[X \mid X > 0] \geqslant t.$$

It follows that in dimensions one and two,

$$\mathbb{P}(\ell_{\mathcal{T}}(B_r) \geqslant t) \geqslant \mathbb{P}(X \geqslant t) \gtrsim R^{-2}.$$

In dimension three we need to use a third moment asymptotic, due to the presence of a log term in the second moment. As noticed in [AHJ21, Lem. 4.4], for a nonnegative random variable with a finite third moment:

$$\mathbb{P}(X \geqslant \varepsilon \mathbb{E}[X \mid X > 0]) \geqslant \frac{(1-\varepsilon)^{3/2}\mathbb{E}[X]^{3/2}}{\mathbb{E}[X^3]^{1/2}}.$$

Applying this with X as above, we get as well $\mathbb{P}(\ell_{\mathcal{T}}(B_r) > t) \gtrsim R^{-2}$, concluding the proof of Theorem 1.1 in dimensions one, two and three.

5. Proof of Proposition 1.4

In this section, we deal with the exponential moments of $|\mathcal{T}(B_r)|/r^4$. It amounts to show that there are positive constants c and λ_0, such that for any $\lambda < \lambda_0$, and any $r \geqslant 1$,

$$\sup_{x \in B_r} \mathbb{E}_x\Big[\exp\big(\lambda |\mathcal{T}(B_r)|/r^4\big)\Big] \leqslant \exp(c\lambda/r^2).$$

Assume without loss of generality that $r^2 \in \mathbb{N}$. Recall the notation (2.3) for $\mathcal{Z}_n(B_r)$, and let $Z_n(B_r) := |\mathcal{Z}_n(B_r)|$. Then for $0 \leqslant j < r^2$, we define

$$\Upsilon_j = \sum_{i=0}^{\infty} Z_{ir^2+j}(B_r).$$

Note that $|\mathcal{T}(B_r)| = \sum_{j<r^2} \Upsilon_j$, and by Holder's inequality

$$\begin{aligned} \mathbb{E}_x\Big[\exp(\lambda|\mathcal{T}(B_r)|/r^4)\Big] &= \mathbb{E}_x\Big[\textstyle\prod_{j<r^2}\big(\exp(\lambda\Upsilon_j/r^2)\big)^{1/r^2}\Big] \\ &\leqslant \sup_{j<r^2} \mathbb{E}_x\big[\exp(\lambda\Upsilon_j/r^2)\big]. \end{aligned} \tag{5.1}$$

Thus, we need a uniform exponential moment on the family $\{\Upsilon_j/r^2\}_{j<r^2}$. Recall (2.4) and note that for any $x \in B_r$, and $k \geqslant 0$, $\mathbb{E}_x[Z_k(B_r)] \leqslant \mathbf{P}_x(H_{2r} \geqslant k)$. Furthermore, it is well-known that there is $\rho < 1$, such that

$$\sup_{r \geqslant 1} \sup_{x \in B_r} \mathbf{P}_x(H_{2r} \geqslant r^2) \leqslant \rho.$$

It follows from these last two observations that for any $r \geqslant 1$,

$$\sup_{j<r^2} \sup_{x \in B_r} \mathbb{E}_x[Z_{j+r^2}(B_r)] \leqslant \sup_{x \in B_r} \mathbf{P}_x(H_{2r} \geqslant r^2) \leqslant \rho.$$

Let λ_0 be such that the conclusion of Lemma 3.1 holds. Then, using that $Z_{j+r^2}(B_r) \leqslant Z_{j+r^2}$, we get for some constant $c > 0$ (that might change from line to line, and depend on λ_0), that for any $\lambda \leqslant \lambda_0$, any $j < r^2$ and $x \in B_r$

$$\begin{aligned}\mathbb{E}_x\big[\exp(\lambda Z_{j+r^2}(B_r)/r^2)\big] &\leqslant 1+\frac{\rho\lambda}{r^2}+\lambda^2\mathbb{E}\big[(Z_{j+r^2}^2/r^4)\exp(\lambda Z_{j+r^2}/r^2)\big]\\ &\leqslant 1+\frac{\rho\lambda}{r^2}+c\lambda^2\mathbb{E}\big[\exp(\lambda_0 Z_{j+r^2}/r^2)\mid Z_{j+r^2}\neq 0\big]\cdot\mathbb{P}(Z_{j+r^2}\neq 0)\\ &\leqslant 1+\frac{\rho\lambda}{r^2}+\frac{c\lambda^2}{r^2},\end{aligned}$$

where for the last inequality we used (2.2) and Lemma 3.1. We deduce that there exists (some possibly smaller) λ_0, and $\gamma < 1$, such that for all $\lambda \in [0, \lambda_0]$, and all $r \geqslant 1$,

$$\sup_{x\in B_r}\sup_{j<r^2}\mathbb{E}_x\big[\exp(\lambda Z_{j+r^2}(B_r)/r^2)\big]\leqslant \exp((1-\gamma)\lambda/r^2). \tag{5.2}$$

Now for $u \in \mathcal{T}$, and $n \geqslant 0$, we write $Z_n^u(B_r)$ for the random variable with the same law as $Z_n(B_r)$ but translated in the subtree emanating from vertex u. Note that for $i \geqslant 1$,

$$Z_{ir^2+j}(B_r)=\sum_{u\in\mathcal{Z}_{(i-1)r^2+j}(B_r)} Z_{r^2}^u(B_r).$$

Then (5.2) implies (after successive conditioning) that for $\lambda \leqslant \lambda_0$,

$$\mathbb{E}_x\Big[\exp\Big(\frac{\lambda}{r^2}\textstyle\sum_{i=1}^\infty Z_{ir^2+j}(B_r)-\dfrac{(1-\gamma)\lambda}{r^2}\textstyle\sum_{i=1}^\infty Z_{(i-1)r^2+j}(B_r)\Big)\Big]\leqslant 1. \tag{5.3}$$

The next step is to choose p and q such that $q^2(1-\gamma) = 1$ and $1/p + 1/q = 1$. By using (5.3) and applying Holder's inequality twice, we get that for $\lambda \leqslant \lambda_0/pq$,

$$\begin{aligned}\mathbb{E}_x\big[\exp(\lambda/r^2\Upsilon_j)\big]&=\mathbb{E}_x\Big[\exp(\lambda/r^2 Z_j(B_r))\cdot\exp\Big(\frac{\lambda}{r^2}\textstyle\sum_{i=1}^\infty Z_{ir^2+j}(B_r)\Big)\Big]\\ &\leqslant\Big(\mathbb{E}_x\Big[\exp(\frac{p\lambda}{r^2}Z_j)\Big]\Big)^{1/p}\Big(\mathbb{E}_x\Big[\exp\Big(\frac{q^2(1-\gamma)\lambda}{r^2}\Upsilon_j\Big)\Big]\Big)^{1/q^2}.\end{aligned}$$

Then (3.1) gives for some constant $C > 0$, and λ small enough,

$$\mathbb{E}_x\Big[\exp\Big(\frac{\lambda}{r^2}\Upsilon_j\Big)\Big]\leqslant\Big(\mathbb{E}_x\Big[\exp\Big(\frac{p\lambda}{r^2}Z_j\Big)\Big]\Big)^{1/\gamma p}\leqslant e^{C\lambda/r^2}.$$

Now, using (5.1), the result follows. □

6. Proof of Theorem 1.3

First note that as mentioned in the introduction, (1.4) follows from (1.3) and Lemma 3.8. Indeed, these imply that for some positive constants c and c', for all $x \in B_r$, and all $\lambda < \lambda_0$,

$$\begin{aligned}\mathbb{E}_x\Big[\exp(\lambda|\eta_r|/r^2)\;\Big|\;\eta_r\neq\varnothing\Big]&=1+\frac{\mathbb{E}_x\big[\exp(\lambda|\eta_r|/r^2)\big]-1}{\mathbb{P}_x(|\eta_r|>0)}\\ &\leqslant 1+cr^2\big(\exp(c\lambda/r^2)-1\big)\leqslant 1+c'\lambda\leqslant\exp(c'\lambda).\end{aligned}$$

We now move to the proof of (1.3) and (1.5). For $n \geqslant 0$, we use the notation

$$\mathcal{T}_n(\Lambda):=\{u\in\mathcal{T}(\Lambda):|u|\leqslant n\},\quad\text{ and }\quad \eta_r^n:=\{u\in\eta_r:|u|\leqslant n\}.$$

Recall also the notation (2.3). The main point is to observe that η_r^n and $\mathcal{T}_n(B_r)$ are linked via some martingale $\{M_n\}_{n\geqslant 0}$, defined for $n\geqslant 0$, by

$$M_n := \sum_{u\in\mathcal{Z}_n(B_r)} \|S_u\|^2 + \sum_{u\in\eta_r^n} \|S_u\|^2 - \big(|\mathcal{T}_n(B_r)| + |\eta_r^n|\big).$$

The next lemma gathers the results needed about this process.

Lemma 6.1. — *The following hold for the process $\{M_n, n\geqslant 0\}$:*

(1) *it is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\geqslant 0}$ defined by $\mathcal{F}_n = \sigma(\mathcal{T}_n, \{S_u\}_{|u|\leqslant n})$;*

(2) *furthermore, it converges almost surely towards*

$$M_\infty := \sum_{u\in\eta_r} \|S_u\|^2 - \big(|\mathcal{T}(B_r)| + |\eta_r|\big).$$

Proof. — The second part of the lemma is immediate since almost surely the tree $\mathcal{T}$ is finite, and thus $\mathcal{Z}_n(B_r)=\varnothing$, for all n large enough. So let us prove the first point now. Set $\nabla M_n := M_{n+1}-M_n$, for $n\geqslant 0$, and for $u\in\mathcal{T}$, let $\mathcal{N}(u)$ be the set of children of u. Recall that $\xi_u = |\mathcal{N}(u)|$ by definition. Then note that

$$\sum_{u\in\mathcal{Z}_{n+1}(B_r)} \|S_u\|^2 = \sum_{u\in\mathcal{Z}_n(B_r)} \sum_{v\in\mathcal{N}(u)} \|S_v\|^2 - \sum_{u\in\eta_r^{n+1}\smallsetminus\eta_r^n} \|S_u\|^2,$$
$$|\mathcal{T}_{n+1}(B_r)| - |\mathcal{T}_n(B_r)| = |\mathcal{Z}_{n+1}(B_r)|,$$
$$\sum_{u\in\mathcal{Z}_n(B_r)} \xi_u = |\mathcal{Z}_{n+1}(B_r)| + |\eta_r^{n+1}| - |\eta_r^n|,$$

which altogether yield

$$\nabla M_n = \sum_{u\in\mathcal{Z}_n(B_r)} \Big((\xi_u-1)\|S_u\|^2 + \sum_{v\in\mathcal{N}(u)} (\|S_v\|^2 - \|S_u\|^2 - 1)\Big). \tag{6.1}$$

The result follows since for $u\in\mathcal{Z}_n(B_r)$, ξ_u is independent of $\mathcal{F}_n$, hence of $\|S_u\|$, and moreover, since the jump distribution of S is centered and supported on the set of neighbors of the origin, one has for any $v\in\mathcal{N}(u)$, by Pythagoras,

$$\mathbb{E}_x\Big[\|S_v\|^2 \mid \mathcal{F}_n, \mathcal{N}(u)\Big] = \|S_u\|^2 + 1. \qquad \square$$

We can now finish the proof of the theorem, by showing (1.3) and (1.5).

Proof of (1.3). — Observe first that when $1\leqslant r\leqslant 2$, the result follows from Proposition 1.4, since $\eta_r\subset\mathcal{T}(B_{r+1})$, for any $r\geqslant 1$. Hence one can assume now that $r\geqslant 2$. By definition, one has

$$\sum_{u\in\eta_r} \|S_u\|^2 \geqslant r^2|\eta_r|.$$

Thus in view of Proposition 1.4 and Lemma 6.1, it just amounts to show that M_∞/r^4 has some finite exponential moment. To see this, first recall that by assumption the offspring distribution has some finite exponential moment. Therefore, for any $u \in \mathcal{T}$, and λ small enough,

$$\mathbb{E}\Big[\exp(\lambda(\xi_u - 1)/r^2)\Big] \leqslant 1 + c\lambda^2/r^4 \leqslant \exp(c\lambda^2/r^4),$$

for some constant $c > 0$. Likewise, if X has distribution θ, then for any $z \in B_r$, by Cauchy-Schwarz,

$$\Big|\|z+X\|^2 - \|z\|^2 - 1\Big| \leqslant 2r \quad \text{and} \quad \mathbb{E}\big[\|z+X\|^2\big] = \|z\|^2 + 1.$$

Therefore, there exists $c > 0$, such that for every $\lambda \leqslant 1$,

$$\mathbb{E}\Big[\exp\big(\lambda \cdot \frac{\|z+X\|^2 - \|z\|^2 - 1}{r^4}\big)\Big] \leqslant 1 + c\lambda^2/r^6 \leqslant \exp(c\lambda^2/r^6).$$

It follows, using (6.1) and bounding $\|S_u\|^2$ by r^2 in this formula, that for any $n \geqslant 0$, and λ small enough, for some constant $c > 0$,

$$\mathbb{E}_x\Big[\exp\Big(\lambda \frac{\nabla M_n}{r^4}\Big) \;\Big|\; \mathcal{F}_n\Big] \leqslant \exp\Big(c\lambda^2 \frac{|\mathcal{Z}_n(B_r)|}{r^4}\Big).$$

We deduce by successive conditioning, that for any $n \geqslant 1$,

$$\mathbb{E}_x\Big[\exp\Big(\frac{\lambda}{r^4}(M_n - M_0) - c\lambda^2 \textstyle\sum_{k=0}^{n-1} \frac{|\mathcal{Z}_k(B_r)|}{r^4}\Big)\Big] \leqslant 1.$$

Note also that by definition for any $x \in B_r$, under $\mathbb{P}_x$,

$$M_0 = (\|x\|^2 - 1) \leqslant r^2, \qquad \text{and} \qquad \sum_{k \geqslant 0} |\mathcal{Z}_k(B_r)| \leqslant |\mathcal{T}(B_r)|.$$

Hence, using Cauchy-Schwarz and Fatou's lemma, we obtain the existence of some positive constants c and λ_0, such that for any $\lambda \in [0, \lambda_0]$, any $r \geqslant 1$, and any $x \in B_r$,

$$\mathbb{E}_x\Big[\exp\big(\lambda M_\infty / r^4\big)\Big] \leqslant \liminf_{n \to \infty} \mathbb{E}_x\Big[\exp\big(\lambda M_n / r^4\big)\Big] \leqslant \mathbb{E}_x\Big[\exp\Big(\frac{\lambda}{r^2} + c\lambda^2 \frac{|\mathcal{T}(B_r)|}{r^4}\Big)\Big].$$

Then (1.3) follows from Proposition 1.4. □

Proof of (1.5). — Note that for any $x \in B_r$, $\mathbb{E}_x\big[|\eta_r|\big] = \mathbf{P}_x(H_r < \infty) = 1$. Therefore, by expanding the exponential, we find using Proposition 3.11 and (1.3) at the third line, that for any $x \in B_{r/2}$,

$$\begin{aligned}
\mathbb{E}_x\Big[\exp(\lambda|\eta_r|/r^2)\Big] &\leqslant 1 + \frac{\lambda}{r^2} + \frac{\lambda^2}{r^4}\mathbb{E}_x\Big[|\eta_r|^2 \exp(\lambda|\eta_r|/r^2)\Big] \\
&= 1 + \frac{\lambda}{r^2} + \frac{\lambda^2}{r^4}\mathbb{E}_x\Big[|\eta_r|^2 \exp(\lambda|\eta_r|/r^2) \;\Big|\; \eta_r \neq \varnothing\Big] \cdot \mathbb{P}_x(|\eta_r| > 0) \\
&\leqslant 1 + \frac{\lambda}{r^2} + c\frac{\lambda^2}{r^2}\mathbb{E}_x\Big[\exp(\lambda_0|\eta_r|/r^2) \;\Big|\; \eta_r \neq \varnothing\Big] \leqslant 1 + \frac{\lambda}{r^2} + c\,\frac{\lambda^2}{r^2} \\
&\leqslant \exp\Big(\frac{\lambda + c\lambda^2}{r^2}\Big),
\end{aligned}$$

where λ_0 is the constant appearing in the statement of (1.3), and c is another constant that might change from line to line (and depend on λ_0). □

7. Proof of Theorem 1.5

Proof of (1.6). — We assume here that $d \geqslant 5$. Let $r \geqslant 1$ be given and x satisfying $\|x\| \geqslant 2r$. We recall that $\eta_{r,R} = \eta_r \cap \mathcal{T}(B_R)$. We also define $\eta^u_{r,R}$, for $u \in \mathcal{T}$, as the random variable with the same law as $\eta_{r,R}$, but in the subtree emanating from u, and similarly for other variables with an additional upper script u.

Let i_0 be the smallest integer such that $\|x\| \leqslant r2^{i_0}$. Define $R_0 = r2^{i_0}$, $Z_{r,0} = |\eta_{r,R_0}|$. Let also for $i \geqslant 1$,

$$R_i := 2^i R_0, \quad \text{and} \quad Z_{r,i} := \sum_{u \in \eta_{R_{i-1}}} |\eta^u_{r,R_i}|. \tag{7.1}$$

Then by definition, under $\mathbb{P}_x$,

$$|\eta_r| \leqslant \sum_{i \geqslant 0} Z_{r,i}.$$

Thus, by monotone convergence, one has for any $\lambda \geqslant 0$, $r \geqslant 1$, and $\|x\| \geqslant 2r$

$$\mathbb{E}_x\Big[\exp\big(\lambda|\eta_r|/r^2\big)\Big] = \lim_{I\to\infty} \mathbb{E}_x\Big[\exp\Big(\lambda \frac{\sum_{i=0}^I Z_{r,i}}{r^2}\Big)\Big]. \tag{7.2}$$

Furthermore, if for $i \geqslant 0$, we let $\mathcal{G}_i$ denote the sigma-field generated by the BRW frozen on ∂B_{R_i}, then on one hand, $Z_{r,j}$ is $\mathcal{G}_i$-measurable, for all $j \leqslant i$, and conditionally on $\mathcal{G}_{i-1}$, $Z_{r,i}$ is a sum of $|\eta_{R_{i-1}}|$ independent random variables. It remains now to bound their exponential moment.

Proposition 7.1. — *Assume $d \geqslant 5$. For $r \geqslant 1$, $R \geqslant r$, and $\lambda \geqslant 0$, let*

$$\varphi_{r,R}(\lambda) := \sup_{R \leqslant \|x\| \leqslant 2R} \mathbb{E}_x\Big[\exp\big(\lambda|\eta_{r,2R}|/r^2\big)\Big].$$

There exist positive constants c, r_0 and λ_0, such that for all $r \geqslant r_0$, $R \geqslant r$, and $\lambda \in [0, \lambda_0]$,

$$\varphi_{r,R}(\lambda) \leqslant \exp\big(c\lambda/R^2\big).$$

We postpone the proof of this proposition to the end of this section, and continue the proof of (1.6). Note that the proposition implies with our previous notation, assuming $r \geqslant r_0$ and $\lambda \leqslant \lambda_0$,

$$\mathbb{E}_x\Big[\exp\big(\lambda Z_{r,0}/r^2\big)\Big] \leqslant \exp\big(c\lambda/\|x\|^2\big).$$

Furthermore, by combining Proposition 7.1 with Theorem 1.3, we deduce that for all $i \geqslant 1$, almost surely

$$\mathbb{E}_x\Big[\exp\Big(\lambda\frac{Z_{r,i}}{r^2} - c\lambda\frac{|\eta_{R_{i-1}}|}{R_{i-1}^2}\Big) \mid \mathcal{G}_{i-1}\Big] \leqslant 1.$$

It follows by induction that for any $I \geqslant 1$, and $\lambda \leqslant \lambda_0$,

$$\mathbb{E}_x\Big[\exp\Big(\lambda\textstyle\sum_{i=1}^I \frac{Z_{r,i}}{r^2} - c\lambda\sum_{i=0}^{I-1}\frac{|\eta_{R_i}|}{R_i^2}\Big) \mid \mathcal{G}_0\Big] \leqslant 1.$$

Using next Cauchy-Schwarz inequality, we get that for any $I \geqslant 1$, and $\lambda \leqslant \lambda_0/2$,

$$\mathbb{E}_x\Big[\exp\Big(\lambda\textstyle\sum_{i=0}^{I}\dfrac{Z_{r,i}}{r^2}\Big)\Big] \leqslant \exp\big(c\lambda/\|x\|^2\big)\cdot\mathbb{E}_x\Big[\exp\Big(2c\lambda\textstyle\sum_{i=0}^{I-1}\dfrac{|\eta_{R_i}|}{R_i^2}\Big)\Big]^{1/2}.$$

Now in order to compute the exponential moment in the right-hand side, we use Theorem 1.3. Indeed, note that for any $i \geqslant 1$,

$$\eta_{R_{i+1}} = \bigcup_{u\in\eta_{R_i}} \eta^u_{R_{i+1}}.$$

Since as we condition on η_{R_i} the subtrees emanating from the vertices in η_{R_i} are independent, (1.5) shows that for λ small enough, for any $i \geqslant 0$,

$$\mathbb{E}_x\Big[\exp\big(\lambda|\eta_{R_{i+1}}|/R_{i+1}^2\big)\;\Big|\;\eta_{R_i}\Big] \leqslant \exp\big(2\lambda|\eta_{R_i}|/R_{i+1}^2\big) = \exp\big(\lambda|\eta_{R_i}|/2R_i^2\big).$$

Since $R_0 \geqslant \|x\|$, it follows by induction, that for any $I \geqslant 1$, and all λ small enough,

$$\mathbb{E}_x\Big[\exp\Big(\lambda\sum_{i=0}^{I-1}\frac{|\eta_{R_i}|}{R_i^2}\Big)\Big] \leqslant \exp\Big(\frac{2\lambda(1+\cdots+2^{-I})}{\|x\|^2}\Big) \leqslant \exp(4\lambda/\|x\|^2).$$

Together with (7.2), this concludes the proof of (1.6).

Then the proofs of (1.7) and (1.8) follow exactly as for the corresponding estimates, respectively (1.4) and (1.5), from Theorem 1.3. For (1.8), it suffices to use in addition the first point of Lemma 3.9. □

Proof of Proposition 7.1. — For this we need two preliminary results, Lemmas 7.2 and 7.3 below. We note that the first one holds in fact in any dimension $d \geqslant 3$, and will be used also for the case $d = 4$, in the next section. Recall two handy notation: if $r < R$, we write

$$\eta_{r,R} = \eta_r \cap \mathcal{T}(B_R), \quad \text{and} \quad \eta_{R,r} = \eta_R \cap \mathcal{T}((\partial B_r)^c).$$

In other words, $\eta_{R,r}$ is the set of particles which freeze on ∂B_R before reaching ∂B_r.

Lemma 7.2. — *Assume $d \geqslant 3$. Define for $r \geqslant 1$, and $\lambda > 0$,*

$$\varphi_r(\lambda) := \sup_{r\leqslant\|x\|\leqslant 2r} \mathbb{E}_x\Big[\exp\big(\lambda|\eta_{r,2r}|/r^2\big)\Big].$$

There exist positive constants c and λ_0 (only depending on the dimension), such that for any $r \geqslant 1$, and $0 \leqslant \lambda \leqslant \lambda_0$,

$$\varphi_r(\lambda) \leqslant \exp\big(c\lambda/r^2\big).$$

Proof. — Consider the process $\{\widetilde{M}_n\}_{n\geqslant 0}$ defined for $n \geqslant 0$ by

$$\widetilde{M}_n := \sum_{u\in\mathcal{Z}_n(B_{2r}\setminus\partial B_r)} G(S_u) + \sum_{\substack{u\in\eta_{r,2r}\\|u|\leqslant n}} G(S_u) + \sum_{\substack{u\in\eta_{2r,r}\\|u|\leqslant n}} G(S_u).$$

Note that for each $n \geqslant 0$, one has

$$\nabla\widetilde{M}_n := \widetilde{M}_{n+1} - \widetilde{M}_n = \sum_{u\in\mathcal{Z}_n(B_{2r}\setminus\partial B_r)} \Big\{(\xi_u - 1)G(S_u) + \sum_{v\in\mathcal{N}(u)} (G(S_v) - G(S_u))\Big\},$$

where we recall that $\mathcal{N}(u)$ denotes the set of children of u. Therefore, since G is harmonic on $\mathbb{Z}^d \setminus \{0\}$, this process is a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n\geqslant 0}$, as defined in Lemma 6.1. Moreover, as $n\to\infty$, it converges almost surely toward $\widetilde{M}_\infty$ given by

$$\widetilde{M}_\infty = \sum_{u\in\eta_{r,2r}} G(S_u) + \sum_{u\in\eta_{2r,r}} G(S_u).$$

Letting $G(r) := \inf_{x\in\partial B_r} G(x)$, we thus have $\widetilde{M}_\infty \geqslant G(r)|\eta_{r,2r}|$. By Fatou's lemma, this yields

$$\varphi_r(\lambda) \leqslant \liminf_{n\to\infty} \sup_{r\leqslant\|x\|\leqslant 2r} \mathbb{E}_x\Big[\exp\Big(\frac{\lambda\widetilde{M}_n}{r^2 G(r)}\Big)\Big]. \tag{7.3}$$

Now, as in the proof of (1.3) one has for some constant $c>0$, for any $n\geqslant 0$, and any λ small enough,

$$\mathbb{E}_x\Big[\exp\Big(\frac{\lambda(\widetilde{M}_n-\widetilde{M}_0)}{r^2G(r)} - \frac{c\lambda^2|\mathcal{T}_n(B_{2r})|}{r^4}\Big)\Big]\leqslant 1,$$

from which we infer using Cauchy-Schwarz and the fact that under $\mathbb{P}_x$, $\widetilde{M}_0 = G(x)$,

$$\mathbb{E}_x\Big[\exp\Big(\frac{\lambda\widetilde{M}_n}{r^2G(r)}\Big)\Big] \leqslant \mathbb{E}_x\Big[\exp\Big(\frac{2c\lambda^2|\mathcal{T}_n(B_{2r})|}{r^4}\Big)\Big]^{1/2}\cdot\exp\Big(\frac{\lambda G(x)}{r^2G(r)}\Big).$$

Thus the lemma follows from Proposition 1.4 and (2.5), together with (7.3). □

We can now state the following result, which will be our main building block in the proof of Proposition 7.1.

Lemma 7.3. — *Assume $d\geqslant 5$. There exist positive constants $r_0\geqslant 1$ and λ_0, such that for any $r\geqslant r_0$, and $\lambda\in[0,\lambda_0]$,*

$$\sup_{x\in\partial B_{2r}} \mathbb{E}_x\Big[\exp\Big(\frac{\lambda|\eta_{r,4r}|}{r^2} + \frac{\lambda|\eta_{4r,r}|}{16r^2}\Big)\Big] \leqslant \exp\big(\lambda/4r^2\big).$$

Proof. — We note that for any $x\in\partial B_{2r}$,

$$\mathbb{E}_x\Big[\exp\big(\lambda|\eta_{r,4r}|/r^2\big)\Big] \leqslant 1 + \frac{\lambda\mathbb{E}_x\big[|\eta_{r,4r}|\big]}{r^2} + \frac{\lambda^2}{r^4}\mathbb{E}_x\Big[|\eta_{r,4r}|^2\exp\big(\lambda|\eta_{r,4r}|/r^2\big)\Big].$$

Now, similarly as in Lemma 3.9, one has

$$\mathbb{E}_x\big[|\eta_{r,4r}|\big] = \mathbf{P}_x(H_r<H_{4r}) \leqslant \frac{G(x)-G(4r)}{G(r)-G(4r)},$$

with $G(s) = \inf_{z\in\partial B_s} G(z)$, for $s\geqslant 1$. Therefore, using (2.5), we deduce that for r large enough,

$$\sup_{x\in\partial B_{2r}} \mathbb{E}_x\big[|\eta_{r,4r}|\big] \leqslant \frac{1}{8}.$$

Using next Proposition 3.12, and (the proof of) Lemma 7.2 we deduce, as for the proof of (1.5), that for λ small enough,

$$\sup_{x\in\partial B_{2r}} \mathbb{E}_x\Big[|\eta_{r,4r}|^2\exp\big(\lambda|\eta_{r,4r}|/r^2\big)\Big] \leqslant cr^2,$$

for some constant $c>0$. It follows that for λ small enough, and r large enough,

$$\sup_{x\in\partial B_{2r}}\mathbb{E}_x\Big[\exp\big(\lambda|\eta_{r,4r}|/r^2\big)\Big]\leqslant \exp\big(\lambda/6r^2\big).$$

The same argument leads to

$$\sup_{x\in\partial B_{2r}}\mathbb{E}_x\Big[\exp\big(\lambda|\eta_{4r,r}|/16r^2\big)\Big]\leqslant \exp\big(\lambda/16r^2\big),$$

and the lemma follows by using Cauchy-Schwarz inequality, since $1/6+1/16\leqslant 1/4$. □

Proof of Proposition 7.1. — Assume $r\geqslant r_0$, with r_0 given by Lemma 7.3, and for $i\geqslant 0$, set $R_i=r2^i$. Let also

$$\varphi_{r,i}(\lambda):=\sup_{x\in\partial B_{R_i}}\mathbb{E}_x\Big[\exp\Big(\frac{\lambda|\eta_{r,R_{i+1}}|}{r^2}+\frac{\lambda|\eta_{R_{i+1},r}|}{R_{i+1}^2}\Big)\Big].$$

We will prove by induction that, for all $r\geqslant r_0$, and all $\lambda\in[0,\lambda_0]$, with λ_0 as in Lemma 7.3, one has for all $i\geqslant 0$,

$$\varphi_{r,i}(\lambda)\leqslant \exp\big(\lambda/R_i^2\big). \tag{7.4}$$

We claim that this implies the proposition. Indeed, let $R\geqslant r\geqslant r_0$ be given and assume that $R_i\leqslant R<R_{i+1}$, for some $i\geqslant 0$. Let also x be such that $R\leqslant\|x\|\leqslant 2R$. If $R\leqslant\|x\|\leqslant R_{i+1}$, then under $\mathbb{P}_x$,

$$|\eta_{r,2R}|\leqslant\sum_{u\in\eta_{R_i,R_{i+1}}}|\eta^u_{r,R_{i+1}}|+\sum_{u\in\eta_{R_{i+1},r}}|\eta^u_{r,R_{i+2}}|,$$

showing that the desired result follows indeed from (7.4), Lemma 7.2, and (1.3). On the other hand, if $R_{i+1}\leqslant\|x\|\leqslant 2R$, then under $\mathbb{P}_x$,

$$|\eta_{r,2R}|\leqslant\sum_{u\in\eta_{R_{i+1},R_{i+2}}}|\eta^u_{r,R_{i+2}}|+\sum_{u\in\eta_{R_{i+2},r}}|\eta^u_{r,R_{i+3}}|,$$

from which the result follows as well.

Thus it only amounts to prove (7.4), which we now show by induction on $i\geqslant 0$. Note that when $i=0$, the result is immediate by definition, and when $i=1$, the result is given by Lemma 7.3.

Assume next that it holds up to some integer $i\geqslant 1$, and let us prove it for $i+1$. For $x\in\partial B_{R_{i+1}}$, we define inductively four sequences $\{\zeta^j_k\}_{k\geqslant 0}$, for $j\in\{0,1,2,3\}$, of vertices of $\mathcal{T}$ as follows. Let $\zeta^2_0:=\{\varnothing\}$ be the root of $\mathcal{T}$. Next, we can first define for any $k\geqslant 0$,

$$\zeta^1_k=\bigcup_{u\in\zeta^2_k}\eta^u_{R_i,R_{i+2}},\quad\text{and}\quad \zeta^2_{k+1}=\bigcup_{u\in\zeta^1_k}\eta^u_{R_{i+1},r}.$$

Then we let

$$\zeta^0_k:=\bigcup_{u\in\zeta^1_k}\eta^u_{r,R_{i+1}},\quad\text{and}\quad \zeta^3_k=\bigcup_{u\in\zeta^2_k}\eta^u_{R_{i+2},R_i}.$$

In particular under $\mathbb{P}_x$, with $x\in\partial B_{R_{i+1}}$, one has for any $k\geqslant 0$,

$$S_u\in\partial B_{R_{i+j-1}},\ \text{if } u\in\zeta^j_k,\ \text{for } j\in\{1,2,3\},\quad\text{and}\quad S_u\in\partial B_r,\ \text{if } u\in\zeta^0_k. \tag{7.5}$$

See Figure 1 below where an illustration of $\{\zeta_k^0\}$ is drawn. Moreover,

$$\eta_{r,R_{i+2}} = \bigcup_{k=0}^{\infty} \zeta_k^0, \quad \text{and} \quad \eta_{R_{i+2},r} = \bigcup_{k=0}^{\infty} \zeta_k^3.$$

Indeed, concerning the first equality, note that any particle reaching ∂B_r, before hitting $\partial B_{R_{i+2}}$, will make a number of excursions between ∂B_{R_i} and $\partial B_{R_{i+1}}$ back and forth, before at some point reaching ∂B_{R_i}, and then ∂B_r without hitting $\partial B_{R_{i+1}}$, and a similar argument leads to the second equality.

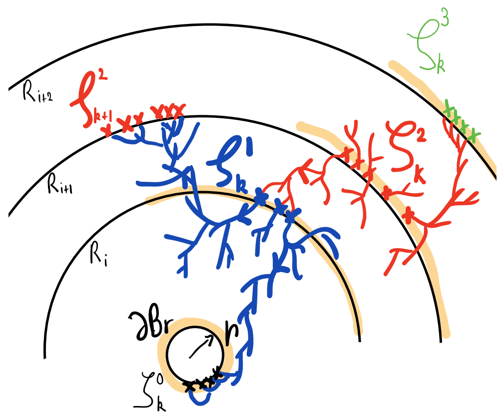

Figure 1. Waves

By monotone convergence, we deduce that

$$\varphi_{r,i+1}(\lambda) \leqslant \lim_{n\to\infty} \sup_{x\in\partial B_{R_{i+1}}} \mathbb{E}_x\Big[\exp\Big(\frac{\lambda\sum_{k=0}^n |\zeta_k^0|}{r^2} + \frac{\lambda\sum_{k=0}^n |\zeta_k^3|}{R_{i+2}^2}\Big)\Big].$$

For $k \geqslant 0$, we let $\mathcal{G}_k$ be the sigma-field generated by the tree $\mathcal{T}$ cut at vertices in $\zeta_k^1 \cup \zeta_k^3$, together with the positions of the BRW at the vertices on this subtree. We also let $\mathcal{H}_k$ be the sigma-field generated by the tree cut at vertices in ζ_k^2, together with the positions of the BRW on the corresponding subtree.

The induction hypothesis implies that almost surely, one has

$$\mathbb{E}_x\Big[\exp\big(\lambda|\zeta_n^0|/r^2\big) \mid \mathcal{G}_n\Big] \leqslant \exp\big(\lambda|\zeta_n^1|/R_i^2\big).$$

Then Lemma 7.3 ensures, that for $0 \leqslant \lambda \leqslant \lambda_0$, and $r \geqslant r_0$, almost surely (recall (7.5)),

$$\mathbb{E}_x\Big[\exp\Big(\frac{\lambda|\zeta_n^1|}{R_i^2} + \frac{\lambda|\zeta_n^3|}{R_{i+2}^2}\Big) \mid \mathcal{H}_n\Big] \leqslant \exp\big(\lambda|\zeta_n^2|/R_{i+1}^2\big).$$

Applying again the induction hypothesis, we get that almost surely,

$$\mathbb{E}_x\Big[\exp\Big(\frac{\lambda|\zeta_{n-1}^0|}{r^2} + \frac{\lambda|\zeta_n^2|}{R_{i+1}^2}\Big) \mid \mathcal{G}_{n-1}\Big] \leqslant \exp\big(\lambda|\zeta_{n-1}^1|/R_i^2\big).$$

Then an elementary induction shows that for all $n \geqslant 1$,

$$\mathbb{E}_x\Big[\exp\Big(\frac{\lambda\sum_{k=0}^n|\zeta_k^0|}{r^2}+\frac{\lambda\sum_{k=0}^n|\zeta_k^3|}{R_{i+2}^2}\Big)\Big]\leqslant \exp\big(\lambda/R_{i+1}^2\big),$$

proving (7.4) for $i+1$, which concludes the proof of Proposition 7.1. □

8. Proof of Theorem 1.6

The proof is similar to the proof of Theorem 1.5, but one has to be slightly more careful. The main difference comes from the following modified version of Proposition 7.1.

Proposition 8.1. — *Assume $d=4$. There exist positive constants c, r_0 and λ_0, such that for any $r\geqslant r_0$, $R\geqslant r$, and $\lambda\in[0,\lambda_0]$,*

$$\sup_{R\leqslant\|z\|\leqslant 2R}\mathbb{E}_z\Big[\exp\Big(\lambda\frac{|\eta_{r,2R}|}{r^2\log(R/r)}\Big)\Big]\leqslant\exp\Big(\frac{c\lambda}{R^2\log(R/r)}\Big).$$

Once this proposition is established, the end of the proof of Theorem 1.6 is almost the same as in dimension five and higher. Indeed, let us postpone the proof of Proposition 8.1 for a moment, and conclude the proof of Theorem 1.6 first.

Proof of Theorem 1.6. — Let $r\geqslant r_0$, with r_0 as in Proposition 8.1, and $R\geqslant 2r$ be given, and let also x be such that $2r\leqslant\|x\|\leqslant R$. Let i_0 be the smallest integer, such that $\|x\|\leqslant r2^{i_0}$, and let I be the smallest integer such that $R\leqslant r2^{i_0+I}$. Recall the definition (7.1) of R_i and $Z_{r,i}$, and notice that

$$|\eta_{r,R}|\leqslant\sum_{i=0}^I Z_{r,i}.$$

Moreover, as in the proof of Theorem 1.5, on one hand Proposition 8.1 shows that (with the same constant c as in its statement),

$$\mathbb{E}_x\Big[\exp\Big(\frac{\lambda}{r^2\log(R/r)}\textstyle\sum_{i=0}^I Z_{r,i}\Big)\Big]$$
$$\leqslant\exp\Big(\frac{c\lambda}{\|x\|^2\log(R/r)}\Big)\cdot\mathbb{E}_x\Big[\exp\Big(\frac{2c\lambda}{\log(R/r)}\cdot\textstyle\sum_{i=0}^{I-1}\frac{|\eta_{R_i}|}{R_i^2}\Big)\Big]^{1/2},$$

and on the other hand, using (1.5), we get by induction

$$\mathbb{E}_x\Big[\exp\Big(\frac{2c\lambda}{\log(R/r)}\cdot\textstyle\sum_{i=0}^{I-1}\frac{|\eta_{R_i}|}{R_i^2}\Big)\Big]\leqslant\exp\Big(\frac{8c\lambda}{\|x\|^2\log(R/r)}\Big),$$

which concludes the proof of (1.9). The proof of (1.10) follows exactly as for the corresponding statement in Theorem 1.3, namely (1.4). Indeed, using a similar argument as in the proof of Lemma 3.5, one can see that uniformly over $R\geqslant 4r$, one has $\inf_{x\in\partial B_{2r}}\mathbb{P}(\eta_{r,R}\neq\varnothing)\geqslant c/r^2$, for some constant $c>0$. Finally concerning the proof of (1.11), one can argue as follows. First using the first point of Lemma 3.9, one has for some constant $\varepsilon > 0$,

$$\mathbb{E}_x\Big[\exp\Big(\frac{\lambda|\eta_{r,R}|}{r^2\log(R/r)}\Big)\Big] \leqslant 1+\frac{(1-\varepsilon)\lambda}{r^2\log(R/r)}+\frac{\lambda^2}{r^4\log(R/r)^2}\mathbb{E}_x\Big[|\eta_{r,R}|^2\exp\Big(\frac{2\lambda|\eta_{r,R}|}{r^2\log(R/r)}\Big)\Big].$$

Then, using Proposition 3.12 and Hölder's inequality at the third line, and (1.10) and the second point of Lemma 3.9 at the last one, we get

$$\begin{aligned}
\mathbb{E}_x\Big[\exp\Big(\frac{\lambda|\eta_{r,R}|}{r^2\log(R/r)}\Big)\Big]
&\leqslant 1+\frac{(1-\varepsilon)\lambda}{r^2\log(R/r)}\\
&\quad+\frac{\lambda^2}{r^4\log(R/r)^2}\mathbb{E}_x\Big[|\eta_{r,R}|^2\exp\Big(\frac{\lambda|\eta_{r,R}|}{r^2\log(R/r)}\Big)\;\Big|\;\eta_{r,R}\neq\varnothing\Big]\cdot\mathbb{P}_x(|\eta_{r,R}|>0)\\
&\leqslant 1+\frac{(1-\varepsilon)\lambda}{r^2\log(R/r)}\\
&\quad+\frac{c\lambda^2}{r^{4+2/3}\log(R/r)^2}\mathbb{E}_x\Big[|\eta_r|^3\Big]^{2/3}\cdot\mathbb{E}_x\Big[\exp\Big(\frac{3\lambda|\eta_{r,R}|}{r^2\log(R/r)}\Big)\;\Big|\;\eta_{r,R}\neq\varnothing\Big]^{1/3}\\
&\leqslant 1+\frac{(1-\varepsilon)\lambda}{r^2\log(R/r)}+\frac{c\lambda^2}{r^2\log(R/r)^2}\leqslant\exp\Big(\frac{(1-\varepsilon')\lambda}{r^2\log(R/r)}\Big),
\end{aligned}$$

for some constant $\varepsilon'\in(0,1)$, proving well (1.11). This concludes the proof of Theorem 1.6. □

It remains now to prove Proposition 8.1, which requires some more care than for the proof of Proposition 7.1. In particular one needs first an improved version of Lemma 7.3.

Lemma 8.2. — *Assume $d=4$. There exists positive constants c and λ_0, such that for any $\lambda,\lambda'\in[0,\lambda_0]$, and any $r\geqslant 1$,*

$$\sup_{x\in\partial B_{2r}}\mathbb{E}_x\Big[\exp\Big(\lambda\frac{|\eta_{r,4r}|}{r^2}+\lambda'\frac{|\eta_{4r,r}|}{4r^2}\Big)\Big]\leqslant\exp\Big(\frac{\lambda(1+c/r+c\lambda)+\lambda'(1+c/r+c\lambda')}{5r^2}\Big).$$

Proof. — The proof is similar to the proof of Lemma 7.3. First we write for any $x\in\partial B_{2r}$,

$$\mathbb{E}_x\Big[\exp\big(\lambda|\eta_{r,4r}|/r^2\big)\Big]\leqslant 1+\frac{\lambda\mathbb{E}_x\big[|\eta_{r,4r}|\big]}{r^2}+\frac{\lambda^2}{r^4}\mathbb{E}_x\Big[|\eta_{r,4r}|^2\exp\big(\lambda|\eta_{r,4r}|/r^2\big)\Big],$$

and

$$\mathbb{E}_x\big[|\eta_{r,4r}|\big]=\mathbf{P}_x(H_r<H_{4r})\leqslant\frac{G(x)-G(4r)}{G(r)-G(4r)},$$

with $G(s)=\inf_{z\in\partial B_s} G(z)$, for $s\geqslant 1$. Using (2.5), we deduce that in dimension four, for some constant $c>0$,

$$\sup_{x\in\partial B_{2r}} \mathbb{E}_x\big[|\eta_{r,4r}|\big] \leqslant \frac{1}{5}(1+c/r).$$

We conclude as in the proof of Lemma 7.3 that for λ small enough,

$$\sup_{x\in\partial B_{2r}} \mathbb{E}_x\Big[\exp\big(\lambda|\eta_{r,4r}|/r^2\big)\Big] \leqslant \exp\Big(\frac{\lambda(1+c/r+c\lambda)}{5r^2}\Big).$$

The same argument leads to

$$\sup_{x\in\partial B_{2r}} \mathbb{E}_x\Big[\exp\big(\lambda'|\eta_{4r,r}|/4r^2\big)\Big] \leqslant \exp\Big(\frac{\lambda'(1+c/r+c\lambda')}{5r^2}\Big),$$

and the lemma follows using Cauchy-Schwarz inequality. □

Proof of Proposition 8.1. — Given $r>0$, let $R_i=r2^i$, for $i\geqslant 1$. We will prove the existence of positive constants c, r_0, and $\lambda_1>0$, such that for all $r\geqslant r_0$, $\lambda\in[0,\lambda_1]$, and $i\geqslant 1$,

$$\sup_{x\in\partial B_{R_i}} \mathbb{E}_x\Big[\exp\big(\lambda|\eta_{r,R_{i+1}}|/ir^2\big)\Big] \leqslant \exp\big(c\lambda/iR_i^2\big). \tag{8.1}$$

Note that exactly as in the proof of Proposition 7.1, the desired result would follow.

Now consider the functions $\psi_{r,i}$, defined for $\lambda,\lambda'\geqslant 0$ by

$$\psi_{r,i}(\lambda,\lambda') := \sup_{x\in\partial B_{R_i}} \mathbb{E}_x\Big[\exp\Big(\frac{\lambda|\eta_{r,R_{i+1}}|}{ir^2}+\frac{\lambda'|\eta_{R_{i+1},r}|}{iR_i^2}\Big)\Big].$$

We claim that there exist positive constants r_0, κ and $\lambda_1\leqslant 1$, such that for all $r\geqslant r_0$, all $i\geqslant 1$, and all $\lambda,\lambda'\in[0,\lambda_1]$,

$$\psi_{r,i}(\lambda,\lambda') \leqslant \exp\Big(\frac{\alpha_i(\lambda+\lambda')\cdot\lambda+\beta_i(\lambda+\lambda')\cdot\lambda'}{iR_i^2}\Big), \tag{8.2}$$

writing for $t\geqslant 0$,

$$\alpha_i(t) := \prod_{j=1}^{i}\Big(1+\frac{\kappa}{R_j}+\frac{\kappa t}{i}\Big), \quad \text{and} \quad \beta_i(t)=1+\frac{\kappa}{R_i}+\frac{\kappa t}{i}.$$

Note that once κ is fixed, one has $\sup_i \alpha_i(1)<\infty$, hence taking $\lambda'=0$ in (8.2) gives (8.1), and thus concludes the proof of the proposition.

We prove (8.2) by induction, and start by fixing the constants κ, r_0 and λ_1. For this define for $i\geqslant 0$, with c the constant appearing in Lemma 8.2,

$$\rho_i := 1+\frac{c}{R_i}+\frac{10c(\lambda+\lambda')}{i+1}, \quad \text{and} \quad \nu_i := 1+\frac{\kappa}{R_i}+\frac{\kappa(\lambda+\lambda')}{i+1}.$$

Then fix κ and r_0 large enough, and $\lambda_1\leqslant 1$ small enough, such that for all $r\geqslant r_0$, and all $\lambda,\lambda'\in[0,\lambda_1]$,

$$\rho_0\leqslant 5/3, \qquad \nu_0\leqslant 2, \qquad \sup_i \alpha_i(2\lambda_1)\leqslant 2, \qquad \lambda_1\leqslant\frac{\lambda_0}{10}, \tag{8.3}$$

where λ_0 is the constant given by Lemma 8.2, and

$$\frac{5\rho_i}{5-\rho_i\nu_i} \leqslant \frac{5}{4}\cdot\Big(1+\frac{\kappa}{R_{i+1}}+\frac{\kappa(\lambda+\lambda')}{i+1}\Big). \tag{8.4}$$

To be more precise, one may first choose $\kappa = 100c$, and then take r_0 large enough, and λ_1 small enough so that (8.3) and (8.4) are satisfied, for all $r \geqslant r_0$, $i \geqslant 0$ and $\lambda, \lambda' \in [0, \lambda_1]$. Now (8.2) for $i = 1$ is given by Lemma 8.2, at least provided $c \leqslant \kappa/2$, which we can always assume. Assume next that (8.2) holds for some $i \geqslant 1$, and let us prove it for $i+1$. Define $\{\zeta_k^j\}_{k\geqslant 0}$, for $j \in \{0,1,2,3\}$, as well as $\{\mathcal{G}_k\}_{k\geqslant 0}$ and $\{\mathcal{H}_k\}_{k\geqslant 0}$, as in the proof of Proposition 7.1. Recall that by monotone convergence, one has

$$\psi_{r,i+1}(\lambda,\lambda') \leqslant \lim_{n\to\infty}\sup_{x\in\partial B_{R_{i+1}}} \mathbb{E}_x\Big[\exp\Big(\frac{\lambda\sum_{k=0}^n|\zeta_k^0|}{(i+1)r^2}+\frac{\lambda'\sum_{k=0}^n|\zeta_k^3|}{(i+1)R_{i+1}^2}\Big)\Big].$$

The induction hypothesis implies that almost surely,

$$\mathbb{E}_x\Big[\exp\Big(\frac{\lambda|\zeta_n^0|}{(i+1)r^2}\Big)\;\Big|\;\mathcal{G}_n\Big] \leqslant \exp\Big(\frac{\widetilde{\alpha}_i\lambda|\zeta_n^1|}{(i+1)R_i^2}\Big), \qquad \text{with } \widetilde{\alpha}_i := \alpha_i\Big(\frac{i(\lambda+\lambda')}{i+1}\Big).$$

Note in particular that $(\sup_i\widetilde{\alpha}_i)\lambda \leqslant 2\lambda \leqslant \lambda_0$, by (8.3). Hence, an application of Lemma 8.2 yields that almost surely

$$\mathbb{E}_x\Big[\exp\Big(\frac{\widetilde{\alpha}_i\lambda|\zeta_n^1|}{(i+1)R_i^2}+\frac{\lambda'|\zeta_n^3|}{(i+1)R_{i+1}^2}\Big)\;\Big|\;\mathcal{H}_n\Big] \leqslant \exp\Big(\frac{\rho_i(\widetilde{\alpha}_i\lambda+\lambda')|\zeta_n^2|}{5(i+1)R_i^2}\Big).$$

Observe next that by (8.3), one has

$$\frac{\rho_i}{5}\,(\widetilde{\alpha}_i\lambda+\lambda') \leqslant \lambda_1.$$

Thus applying again the induction hypothesis, we get that almost surely,

$$\begin{aligned}\mathbb{E}_x\Big[\exp\Big(\frac{\lambda|\zeta_{n-1}^0|}{(i+1)r^2}+\frac{\rho_i(\widetilde{\alpha}_i\lambda+\lambda')|\zeta_n^2|}{5(i+1)R_i^2}\Big)\;\Big|\;\mathcal{G}_{n-1}\Big]&\\ \leqslant \exp\Big(\Big\{\widetilde{\alpha}_i\lambda(1+\rho_i\nu_i/5)+\frac{\rho_i\nu_i\lambda'}{5}\Big\}\cdot\frac{|\zeta_{n-1}^1|}{(i+1)R_i^2}\Big).&\end{aligned}$$

Now, one has by (8.3) again

$$\widetilde{\alpha}_i(1+\rho_i\nu_i/5)\lambda+\frac{\rho_i\nu_i}{5}\lambda' \leqslant \big(2(1+4/5)+4/5\big)\lambda_1 \leqslant 10\cdot\lambda_1 \leqslant \lambda_0.$$

Hence one may apply Lemma 8.2, which gives (note the factor 10 appearing in the definition of ρ_i),

$$\begin{aligned}\mathbb{E}_x\Big[\exp\Big(\Big\{\widetilde{\alpha}_i\lambda(1+\rho_i\nu_i/5)+\frac{\rho_i\nu_i\lambda'}{5}\Big\}\cdot\frac{|\zeta_{n-1}^1|}{(i+1)R_i^2}+\frac{\lambda'|\zeta_{n-1}^3|}{(i+1)R_{i+1}^2}\Big)\;\Big|\;\mathcal{G}_{n-1}\Big]&\\ \leqslant \exp\Big((\widetilde{\alpha}_i\lambda+\lambda')\cdot(1+\rho_i\nu_i/5)\cdot\frac{\rho_i|\zeta_n^2|}{5(i+1)R_i^2}\Big).&\end{aligned}$$

Then an elementary induction shows that for all $n \geqslant 1$,

$$\mathbb{E}_x\Big[\exp\Big(\frac{\lambda\sum_{k=0}^n|\zeta_k^0|}{(i+1)r^2}+\frac{\lambda'\sum_{k=0}^n|\zeta_k^3|}{(i+1)R_{i+1}^2}\Big)\Big] \leqslant \exp\Big(\frac{4\rho_i}{5}\cdot\frac{\widetilde{\alpha}_i\lambda+\lambda'}{(i+1)R_{i+1}^2}\cdot\textstyle\sum_{k=0}^n(\rho_i\nu_i/5)^k\Big).$$

To conclude, note that by (8.4),

$$\rho_i \sum_{k=0}^{\infty} (\rho_i \nu_i/5)^k = \frac{5\rho_i}{5-\rho_i\nu_i} \leqslant \frac{5}{4}\cdot\Big(1+\frac{\kappa}{R_{i+1}}+\frac{\kappa(\lambda+\lambda')}{i+1}\Big) = \frac{5}{4}\beta_{i+1}(\lambda+\lambda').$$

Finally notice that $\widetilde{\alpha}_i\beta_{i+1}(\lambda+\lambda') = \alpha_{i+1}(\lambda+\lambda')$, thereby finishing the proof of the induction step for (8.2). This concludes the proof of Proposition 8.1. □

9. Proof of Theorem 1.1: upper bounds in $d \geqslant 4$

The proof in dimension four is slightly different from the higher dimensional case, but first one needs to strengthen the result of Proposition 1.4 as follows. Recall that we denote by $\mathcal{T}(B_r)$ the time spent in the ball B_r by a BRW, for which we freeze the particles reaching the boundary of the ball, see (1.1).

Proposition 9.1. — *Assume $d \geqslant 1$. There exist positive constants c and C, such that for all $r \geqslant 1$, and all $t \geqslant r^4$,*

$$\sup_{x\in B_r} \mathbb{P}_x(|\mathcal{T}(B_r)| > t) \leqslant \frac{C}{r^2}\exp(-ct/r^4).$$

Proof. — Given $u \in \mathcal{T}$, and $\Lambda \subset \mathbb{Z}^d$, we call $\mathcal{T}^u(\Lambda)$ the random variable with the same law as $\mathcal{T}(\Lambda)$ shifted in the subtree emanating from u. Recall also that $\mathcal{T}_n = \{u \in \mathcal{T} : |u| \leqslant n\}$, and assume without loss of generality that r^2 is an integer. One has

$$\mathbb{P}_x(|\mathcal{T}(B_r)| > t) \leqslant \mathbb{P}_x\big(\textstyle\sum_{|u|=r^2}|\mathcal{T}^u(B_r)| \geqslant t/2\big) + \mathbb{P}\big(|\mathcal{T}_{r^2}| > t/2\big). \tag{9.1}$$

For the first term on the right-hand side we use first Proposition 1.4 and Chebyshev's exponential inequality, which give for any λ small enough,

$$\mathbb{P}_x\big(\textstyle\sum_{|u|=r^2}|\mathcal{T}^u(B_r)| \geqslant t/2\big) \leqslant e^{-\lambda t/2r^4}\cdot \mathbb{E}\Big[\exp\big(c\lambda Z_{r^2}/r^2\big)\;\Big|\; Z_{r^2}\neq 0\Big]\cdot \mathbb{P}(Z_{r^2}\neq 0).$$

Then Lemma 3.1 and (2.2) yield for some constant $C > 0$,

$$\mathbb{P}_x\big(\textstyle\sum_{|u|=r^2}|\mathcal{T}^u(B_r)| \geqslant t/2\big) \leqslant \frac{C}{r^2}\exp(-\lambda t/2r^4).$$

Concerning now the second term on the right-hand side of (9.1), we use a similar idea. Let I be the smallest integer, such that $r^2 \leqslant 2^I$. Using this time both Lemmas 3.1 and 3.2 shows that for some λ small enough, and positive constants c, and C,

$$\begin{aligned}
\mathbb{P}\big(|\mathcal{T}_{r^2}| > t/2\big) &\leqslant \sum_{i=0}^{I-1}\mathbb{P}\Big(\textstyle\sum_{k=2^i}^{2^{i+1}} Z_k > 2^{i-I-1}t\Big)\\
&\leqslant \sum_{i=0}^{I-1}\exp\big(-\lambda 2^{i-I-1}t/2^{2i}\big)\cdot\mathbb{E}\Big[\exp\big(c\lambda Z_{2^i}/2^i\big)\;\Big|\; Z_{2^i}\neq 0\Big]\cdot\mathbb{P}(Z_{2^i}\neq 0)\\
&\leqslant \frac{C}{r^2}\sum_{i=0}^{I-1} 2^{I-i}\exp\big(-\lambda 2^{I-i}t/8r^4\big) \leqslant \frac{C}{r^2}\exp(-\lambda t/8r^4),
\end{aligned}$$

using $t \geqslant r^4$, for the last inequality. This concludes the proof of the proposition. □

We can now finish the proof of the upper bounds in Theorem 1.1 in dimensions four and higher.

Assume first that $d \geqslant 5$. Note that it suffices to prove the result for r large enough. In particular we assume now that $r \geqslant r_0$, with r_0 given by Theorem 1.5. Define two sequences $\{\zeta_k^j\}_{k\geqslant 0}$, for $j=1,2$, by $\zeta_0^1=\{\varnothing\}$ is the root of the tree, and for $k\geqslant 0$,

$$\zeta_k^2 := \bigcup_{u\in\zeta_k^1} \eta_{2r}^u, \quad \text{and} \quad \zeta_{k+1}^1 := \bigcup_{u\in\zeta_k^2} \eta_r^u.$$

Let also for $k \geqslant 0$,

$$L_k(B_r) := \sum_{u\in\zeta_k^1} |\mathcal{T}^u(B_{2r})|,$$

with the notation from the proof of the previous proposition. Then by definition, one has

$$\ell_{\mathcal{T}}(B_r) \leqslant \sum_{k\geqslant 0} L_k(B_r).$$

Therefore by Proposition 3.11, for any $\lambda > 0$,

$$\begin{aligned}\mathbb{P}\big(\ell_{\mathcal{T}}(B_r) > t\big) \\ \leqslant \mathbb{P}\big(|\mathcal{T}(B_{2r})| > t/2\big) + \frac{C}{r^2}\exp(-\lambda t/2r^4)\cdot \mathbb{E}\Big[\exp\Big(\frac{\lambda}{r^4}\textstyle\sum_{k\geqslant 1} L_k(B_r)\Big) \Bigm| |\eta_r|>0\Big].\end{aligned}$$

The first term on the right-hand side is handled by Proposition 9.1. Hence only the last expectation is at stake, and it just amounts to show that it is bounded. By monotone convergence one has

$$\mathbb{E}\Big[\exp\Big(\frac{\lambda}{r^4}\textstyle\sum_{k\geqslant 1} L_k(B_r)\Big) \Bigm| |\eta_r|>0\Big] = \displaystyle\lim_{n\to\infty} \mathbb{E}\Big[\exp\Big(\frac{\lambda}{r^4}\textstyle\sum_{k=1}^n L_k(B_r)\Big) \Bigm| |\eta_r|>0\Big].$$

Note also that if $\mathcal{G}_n$ denotes the sigma-field generated by the BGW tree cut at vertices in ζ_n^1, together with the positions of the BRW on this subtree, then by Proposition 1.4, almost surely for all λ small enough,

$$\mathbb{E}\Big[\exp\Big(\frac{\lambda L_n(B_r)}{r^4}\Big) \Bigm| \mathcal{G}_n\Big] \leqslant \exp\Big(\frac{c\lambda|\zeta_n^1|}{r^2}\Big),$$

for some constant $c>0$. It follows that for any $n\geqslant 0$,

$$\mathbb{E}\Big[\exp\Big(\frac{\lambda}{r^4}\textstyle\sum_{k=1}^n L_k(B_r) - \frac{c\lambda}{r^2}\sum_{k=1}^n |\zeta_k^1|\Big) \Bigm| |\eta_r|>0\Big] \leqslant 1.$$

Hence by Cauchy-Schwarz, it just amounts to show that for λ small enough, the sequence $\{u_n(\lambda)\}_{n\geqslant 1}$ defined by

$$u_n(\lambda) := \mathbb{E}\Big[\exp\Big(\frac{c\lambda}{r^2}\textstyle\sum_{k=1}^n |\zeta_k^1|\Big) \Bigm| |\eta_r|>0\Big],$$

is bounded. By combining (1.5) and (1.8), we get that for λ small enough, and some $\varepsilon>0$, one has for any $k\geqslant 2$, almost surely

$$\mathbb{E}\Big[\exp\Big(\frac{\lambda}{r^2}|\zeta_k^1|\Big) \Bigm| \mathcal{G}_{k-1}\Big] \leqslant \exp\Big(\frac{(1-\varepsilon)\lambda}{r^2}|\zeta_{k-1}^1|\Big),$$

and using also (1.4), we get for some constant $c'>0$,

$$\mathbb{E}\Big[\exp\Big(\frac{\lambda}{r^2}|\zeta_1^1|\Big)\ \Big|\ |\eta_r|>0\Big]\leqslant \exp(c'\lambda).$$

We deduce that for any $n\geqslant 1$, and λ small, that $\{u_n(\lambda)\}_{n\geqslant 1}$ is bounded through

$$u_n(\lambda)\leqslant \exp\big(c'\lambda\textstyle\sum_{k=0}^n(1-\varepsilon)^k\big)\leqslant \exp(c'\lambda/\varepsilon),$$

This concludes the proof of the upper bound in Theorem 1.1, in case $d\geqslant 5$.

In the case when $d=4$, the proof follows a similar pattern, but we need to consider a truncated deposition process on B_r. Instead of $\{\zeta_k^j\}_{k\geqslant 0}$, set $\widetilde{\zeta}_0^1=\{\varnothing\}$, and with $R=r2^I$, and $I:=\sqrt{t/r^4}$, we define for $k\geqslant 0$,

$$\widetilde{\zeta}_k^2:=\bigcup_{u\in\widetilde{\zeta}_k^1}\eta_{2r}^u,\quad\text{and}\quad \widetilde{\zeta}_{k+1}^1:=\bigcup_{u\in\widetilde{\zeta}_k^2}\eta_{r,R}^u.$$

Then we define similarly for $k\geqslant 0$,

$$L_k(B_r):=\sum_{u\in\widetilde{\zeta}_k^1}|\mathcal{T}^u(B_{2r})|,$$

and Chebyshev's inequality reads

$$\begin{aligned}\mathbb{P}\big(\ell_{\mathcal{T}}(B_r)>t\big)\leqslant \mathbb{P}\big(|\mathcal{T}(B_{2r})|>t/2\big)+\mathbb{P}(|\eta_R|>0)\\ +\frac{C}{r^2}e^{-\lambda t/2Ir^4}\cdot\mathbb{E}\Big[\exp\Big(\frac{\lambda}{Ir^4}\textstyle\sum_{k\geqslant 1}L_k(B_r)\Big)\ \Big|\ |\eta_r|>0\Big].\end{aligned}$$

The second term is $\mathcal{O}(1/R^2)$ by Proposition 3.11, and the other terms are handled exactly as in higher dimension, using the estimates from Theorem 1.6, instead as from Theorem 1.5. □

10. Proof of Theorem 1.1: lower bounds in $d\geqslant 4$

We start with the case of dimension five and higher, and consider the subtle case of dimension four separately.

10.1. Dimension five and higher.

— Assume $d\geqslant 5$, and recall that we may also assume here that $t\geqslant r^4$. The strategy we will use is to ask the BRW to reach ∂B_r (at a cost of order $1/r^2$), and then make order t/r^4 excursions (or waves) between ∂B_r and $\partial B_{r/2}$, at a cost of order $\exp(-\Theta(t/r^4))$.

More precisely, the proof relies on the following lemma, which holds in fact in any dimension (recall (1.2)).

Lemma 10.1. — *Assume $d\geqslant 1$. There exists a constant $c>0$, such that for any $r\geqslant 1$,*

$$\inf_{x\in B_{r/2}\cup\partial B_{2r}}\mathbb{P}_x(|\eta_r|\geqslant cr^2)\geqslant c/r^2.$$

Proof. — Let

$$\overline{\eta}_r:=\{u\in\eta_r: r^2\leqslant |u|\leqslant 2r^2\}.$$

Note that uniformly over $x \in B_{r/2} \cup \partial B_{2r}$,

$$\mathbb{E}_x[|\overline{\eta}_r|] = \sum_{n=r^2}^{2r^2} \mathbb{E}[Z_n] \cdot \mathbf{P}_x(H_r = n) = \mathbf{P}_x(r^2 \leqslant H_r \leqslant 2r^2) \gtrsim 1,$$

and using a similar computation as in Section 3.3,

$$\mathbb{E}_x\left[|\overline{\eta}_r|^2\right] = \mathbb{E}_x[|\overline{\eta}_r|] + \sigma^2 \sum_{k=0}^{r^2-1} \mathbf{E}_x\Big[\mathbf{1}\{H_r > k\} \cdot \mathbf{P}_{S_k}\big(r^2 - k \leqslant H_r \leqslant 2r^2 - k\big)^2\Big] \leqslant 1 + \sigma^2 r^2.$$

Now by (2.2),

$$\mathbb{P}_x(|\overline{\eta}_r| > 0) \leqslant \mathbb{P}(Z_{r^2} \neq 0) \lesssim 1/r^2.$$

Therefore,

$$\mathbb{E}_x\left[|\overline{\eta}_r| \mid |\overline{\eta}_r| > 0\right] \gtrsim r^2,$$

and Paley-Zygmund's inequality (2.8) gives, for some constant $c > 0$,

$$\mathbb{P}_x\left(|\overline{\eta}_r| > cr^2\right) \gtrsim 1/r^2.$$

Since $\overline{\eta}_r \subset \eta_r$, this proves the lemma. □

Consequently, if we start with order r^2 particles on ∂B_r, then with probability of order 1, there will be order r^2 particles reaching $\partial B_{r/2}$. Repeating this argument we see that with probability at least $\Theta(1/r^2) \cdot \exp(-\Theta(t/r^4))$, the BRW will make at least $\Theta(t/r^4)$ waves between ∂B_r and $\partial B_{r/2}$, with an implicit constant as large as wanted. The expected time spent on B_r by each of these waves is of order r^4, simply because for any starting point on $\partial B_{r/2}$, the expected time spent on B_r by the BRW killed on ∂B_r is of order r^2. Since all the waves are independent of each other (at least conditionally on the positions of the frozen particles), we shall deduce that the total time spent on B_r will exceed t.

On a formal level now, we define $\{\zeta_k^1\}_{k\geqslant 0}$, $\{\zeta_k^{1,1}\}_{k\geqslant 0}$ and $\{\zeta_k^2\}_{k\geqslant 0}$ inductively by $\zeta_0^{1,1} = \zeta_0^1 = \{\varnothing\}$ (the root of the tree), and then for $k \geqslant 0$,

$$\zeta_k^2 = \bigcup_{u \in \zeta_k^{1,1}} \eta_r^u, \qquad \zeta_{k+1}^1 = \bigcup_{u \in \zeta_k^2} \eta_{r/2}^u,$$

and for $\zeta_{k+1}^{1,1}$ we take any subset (chosen arbitrarily, for instance uniformly at random) of ζ_{k+1}^1 with $\lceil |\zeta_{k+1}^1|/2 \rceil$ points. We also let $\zeta_k^{1,2} := \zeta_k^1 \smallsetminus \zeta_k^{1,1}$, for $k \geqslant 0$. Next, set $N := \lceil Ct/r^4 \rceil$, with $C > 0$ some large constant to be fixed later, and define further for $k \geqslant 1$,

$$L_k(B_r) := \sum_{u \in \zeta_k^{1,2}} |\mathfrak{T}^u(B_r)|,$$

with the notation from the proof of Proposition 9.1. The reason why we partition the sets ζ_k^1 into two parts, is that we want to keep, for each $k \geqslant 1$, some independence between ζ_k^2 and $L_k(B_r)$, conditionally on ζ_k^1. Now, note that

$$\ell_{\mathfrak{T}}(B_r) \geqslant \sum_{k=1}^{N} L_k(B_r). \tag{10.1}$$

For $k \geqslant 1$, define $\mathcal{G}_k$ as the sigma-field generated by tree cut at vertices in ζ_k^1, together with the choice of $\zeta_k^{1,1}$ and the positions of the BRW on the vertices of this subtree. Let also $E_k := \{|\zeta_k^{1,1}| \geqslant cr^2\}$, with c chosen such that

$$\rho := \min\Big(\mathbb{P}(E_1), \inf_{k\geqslant 2} \mathbb{P}(E_k \mid E_{k-1})\Big) > 0. \tag{10.2}$$

Note that the existence of c is guaranteed by Lemma 10.1. Observe also that $E_k \in \mathcal{G}_k$, for any $k \geqslant 1$. Remember next that for some constant $c' > 0$, one has for all $r \geqslant 1$,

$$\inf_{x\in\partial B_{r/2}} \mathbb{E}_x\big[|\mathcal{T}(B_r)|\big] = \inf_{x\in\partial B_{r/2}} \mathbf{E}_x[H_r] \geqslant c'r^2.$$

As a consequence, for any $k \geqslant 1$, almost surely,

$$\mathbb{E}\big[L_k(B_r) \mid \mathcal{G}_k\big] \geqslant c'r^2|\zeta_k^{1,2}|.$$

On the other hand,

$$\sup_{x\in B_r} \mathbb{E}_x\big[|\mathcal{T}(B_r)|\big] = \sup_{x\in B_r} \mathbf{E}_x[H_r] = \mathcal{O}(r^2),$$

which entails that for some constant $K > 0$, for any $x \in B_r$,

$$\begin{aligned}\mathbb{E}_x\big[|\mathcal{T}(B_r)|^2\big] &\leqslant \mathbb{E}_x\big[|\mathcal{T}(B_r)|\big] + \sum_{k=0}^{\infty} \mathbb{E}_x\Big[\textstyle\sum_{\substack{u\in\mathcal{T}(B_r)\\|u|=k}} \xi_u(\xi_u - 1)\Big(\displaystyle\sup_{z\in B_r} \mathbb{E}_z\big[|\mathcal{T}(B_r)|\big]\Big)^2\Big]\\ &\leqslant Kr^6.\end{aligned}$$

As a consequence, one has for any $k \geqslant 0$,

$$\mathbb{E}\big[L_k(B_r)^2 \mid \mathcal{G}_k\big] \leqslant Kr^6|\zeta_k^{1,2}|.$$

Using then Paley-Zygmund's inequality (2.7) we get

$$\mathbb{P}(F_k \mid \mathcal{G}_k)\mathbf{1}_{E_k} \geqslant \frac{cc'^2}{4K}\,\mathbf{1}_{E_k}, \quad \text{with} \quad F_k := \Big\{L_k(B_r) \geqslant \frac{cc'r^4}{2}\Big\}. \tag{10.3}$$

Note now that on the event

$$\widetilde{E}_N := \bigcap_{k=1}^{N} E_k \cap F_k,$$

by (10.1) one has $\ell_{\mathcal{T}}(B_r) \geqslant Ncc'r^4/2$, which is larger than t, provided the constant C in the definition of N is chosen large enough. Moreover, for each $k \geqslant 0$, conditionally on $\mathcal{G}_k$, E_{k+1} and F_k are independent. Therefore by (10.2) and (10.3), one has also by induction

$$\mathbb{P}(\widetilde{E}_N) \geqslant \kappa^N,$$

with $\kappa = \rho cc'^2/4K$, concluding the proof of the lower bound in dimension five and higher. $\square$

10.2. Dimension Four. — We assume in this section that $d = 4$ and $t \geqslant r^4$. Since our scenario producing a lower bound is new, we first present heuristics, followed by the formal proof.

Heuristics. — Our strategy for producing a local time t in B_r is very much different than the scenario presented in $d \geqslant 5$. Recall two facts specific to dimension four, when the BRW starts from ∂B_R, with $R > r$: (i) first Theorem 1.6 shows that $\eta_{r,2R}$ is typically of order $r^2 \cdot \log(R/r)$, which is much larger than the corresponding number in $d \geqslant 5$; (ii) the probability of $\{\mathcal{T}(B_r) \neq \varnothing\}$ is smaller (by a factor log) than the corresponding probability in $d \geqslant 5$, and is of order $R^{-2} \cdot \log^{-1}(R/r)$. With these two facts in mind, let us start with the heuristics and set up notation. Set $I = 2\lfloor C\sqrt{t/r^4}\rfloor$, with $C > 0$ some large constant to be fixed later. Then for $i \leqslant I$, let $R_i := r2^i$, and

$$\mathcal{S}_i := \{z \in \mathbb{Z}^d : R_{i-1} \leqslant \|z\| \leqslant R_i\}.$$

Our first requirement will be for the BRW to reach distance R_I, and even more that $|\eta_{R_I}|$ be of order R_I^2. Lemma 10.1 ensures that this has probability $\Theta(1/R_I^2)$, which is the expected cost. Next, by Lemma 3.5 (or the second point (ii) recalled above), with probability $\Theta(1/I)$, one of the BRWs emanating from one of the vertices in η_{R_I} will reach $\partial B_{r/2}$. Furthermore, conditionally on this event, we know by Proposition 3.3 that one spine reaches $\partial B_{r/2}$, and brings there of the order of $r^2 \cdot I$ walks, in virtue of the point (i) recalled above. Since from any of the vertices of the spine start independent BRWs, and since, as we will show, the spine typically spends a time of order R_I^2 in the shell $\mathcal{S}_I$, we deduce that at least one of the BRWs starting from the spine in this shell will reach as well $\partial B_{r/2}$, with probability $\Theta(1/I)$ again. Conditionally on this, one has now two spines crossing the shell $\mathcal{S}_{I-1}$. The probability that one of the BRWs starting from one of these two spines in $\mathcal{S}_{I-1}$ reaches $\partial B_{r/2}$ is thus of order twice $\Theta(\frac{1}{I-1})$. Then by repeating this argument in all the shells $\mathcal{S}_I, \dots, \mathcal{S}_{I/2}$, one can make sure that $I/2$ spines reach $\partial B_{r/2}$, at a total cost of only $(1/R_I^2) \cdot \exp(-\Theta(I))$, which is still affordable. To conclude we know that the spines typically come with order $r^2(I/2 + \cdots + I) \geqslant r^2 I^2/4$ walks on $\partial B_{r/2}$. Since each of them leads afterward to a mean local time order r^2 in B_r, this concludes the heuristics. We show in Figure 2 the many spines originating from successive shells: the green spine gives birth to orange critical trees producing an orange spine which in turn gives birth to purple trees producing a purple spine, and so on and so forth.

Proof. — The formal proof follows very much the picture we just presented. Define the event $E_0 := \{|\eta_{R_I}| \geqslant cR_I^2\}$, with c as in Lemma 10.1. Let

$$E_1 := \big\{ \textstyle\sum_{u \in \eta_{R_I}} |\eta^u_{r/2}| > 0\big\}.$$

Conditionally on E_1, we know by Proposition 3.3 that there exists a path (or spine), which we denote by $\Gamma_1 = (\Gamma_1(i),\ 0 \leqslant i \leqslant |\Gamma_1|)$, emanating from one of the points in $\{S_u, u \in \eta_{R_I}\}$, and going up to $\partial B_{r/2}$. For $1 \leqslant j \leqslant I$, and a path γ, we let

$$\tau_j = \tau_j(\gamma) := \inf\big\{k \geqslant 0 : \gamma(k) \in \partial \mathcal{S}_j\big\}.$$

We then define E_2 as the event that one of the biased BRWs starting from the points in the path $\{\Gamma_1(i),\ 0 \leqslant i < \tau_{I-1}\}$, hits $\partial B_{r/2}$. Applying Proposition 3.3 again, it means that on the event $E_1 \cap E_2$ there exists a second spine Γ_2 emanating from one of the (neighbors of the) points in $\{\Gamma_1(i),\ 0 \leqslant i < \tau_{I-1}\}$, going up to $\partial B_{r/2}$. We can

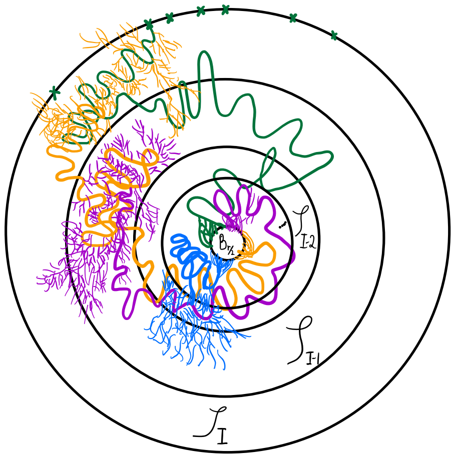

Figure 2. The Many Spines.

thus define E_3 as the event that one of the biased BRWs starting from the points in $\{\Gamma_1(i),\ \tau_{I-1} \leqslant i < \tau_{I-2}\} \cup \{\Gamma_2(i),\ 0 \leqslant i < \tau_{I-2}\}$, hits $\partial B_{r/2}$. Continuing like this in all the shells $\mathcal{S}_I, \dots, \mathcal{S}_{I/2}$, we define inductively for each $i = 1, \dots, I/2$, an event E_i, such that on $E_1 \cap \dots \cap E_i$, there are i spines $\Gamma_1, \dots, \Gamma_i$, starting respectively from a point in $\partial B_{R_I}, \partial\mathcal{S}_I, \dots, \partial\mathcal{S}_{I-i+2}$, and going up to $\partial B_{r/2}$. We claim that there exists a constant $\rho > 0$, such that for each $i \leqslant I/2$,

$$\mathbb{P}(E_{i+1} \mid E_1 \cap \dots \cap E_i) \geqslant \frac{\rho i}{I}. \tag{10.4}$$

Indeed, recall that by Corollary 3.4 the spines satisfy the strong Markov property. Therefore, for any i,

$$\mathbb{P}(E_{i+1} \mid E_1 \cap \dots \cap E_i) \geqslant 1 - \Big(1 - \inf_{x \in \partial B_{R_{I-i+1}}} r_i(x)\Big)^i,$$

where for any $x \in \partial B_{R_{I-i+1}}$, we denote by $r_i(x)$ the probability that on a path Γ starting from x, and sampled according to the measure $\mathbb{P}^x_{\partial B_{r/2}}$, one of the biased BRW starting from the points in $\{\Gamma(k),\ k \leqslant \tau_{I-i}\}$, hits $\partial B_{r/2}$. Since for any $z \in \mathcal{S}_{I-i+1}$ the probability to hit $\partial B_{r/2}$ is of order $I^{-1} \cdot R_{I-i}^{-2}$ (recall that we assume $i \leqslant I/2$), by Lemma 3.5 (which holds as well for a biased BRW, see Remark 3.7), one has for some constant $c > 0$, and any $\alpha > 0$

$$r_i(x) \geqslant \Big\{1 - \big(1 - c/IR_{I-i}^2\big)^{\alpha R_{I-i}^2}\Big\} \cdot \mathbb{P}^x_{\partial B_{r/2}}\big(\ell_\Gamma(\mathcal{S}_{I-i+1}) > \alpha R_{I-i}^2\big),$$

where for $j \geqslant 2$, we write $\ell_\gamma(\mathcal{S}_j)$ for the time spent on $\mathcal{S}_j$ by a path γ, before its hitting time of $B_{R_{j-1}}$. Thus, to conclude the proof of (10.4), it suffices to show that for some $\alpha > 0$, one has for any $j \geqslant 2$, and any $x \in \partial B_{R_j}$,

$$\mathbb{P}^x_{\partial B_{r/2}}\big(\ell_\Gamma(\mathcal{S}_j) > \alpha R_j^2\big) \geqslant \alpha. \tag{10.5}$$

Set now $\Lambda := \partial B_{r/2}$ to simplify notation. One has by definition of $\mathbb{P}_\Lambda^x$, and using also (3.2),

$$\begin{aligned}\mathbb{P}_\Lambda^x&\big(\ell_\Gamma(\mathcal{S}_j)\leqslant \alpha R_j^2\big)\\ &=\frac{\sum_{y\in\partial B_{R_{j-1}}}\big(\sum_{\gamma_1:x\to y}\mathbf{1}\{\ell_\gamma(\mathcal{S}_j)\leqslant \alpha R_j^2\}p_\Lambda(\gamma_1)\big)\cdot\big(\sum_{\gamma_2:y\to\Lambda}p_\Lambda(\gamma_2)\big)}{\sum_{y\in\partial B_{R_{j-1}}}\big(\sum_{\gamma_1:x\to y}p_\Lambda(\gamma_1)\big)\cdot\big(\sum_{\gamma_2:y\to\Lambda}p_\Lambda(\gamma_2)\big)}\\ &=\frac{\sum_{y\in\partial B_{R_{j-1}}}\big(\sum_{\gamma_1:x\to y}\mathbf{1}\{\ell_{\gamma_1}(\mathcal{S}_j)\leqslant \alpha R_j^2\}p_\Lambda(\gamma_1)\big)\cdot\mathbb{P}_y(|\eta_{r/2}|>0)}{\sum_{y\in\partial B_{R_{j-1}}}\big(\sum_{\gamma_1:x\to y}p_\Lambda(\gamma_1)\big)\cdot\mathbb{P}_y(|\eta_{r/2}|>0)}\\ &\leqslant C\frac{\sum_{y\in\partial B_{R_{j-1}}}\sum_{\gamma_1:x\to y}\mathbf{1}\{\ell_{\gamma_1}(\mathcal{S}_j)\leqslant \alpha R_j^2\}p_\Lambda(\gamma_1)}{\sum_{y\in\partial B_{R_{j-1}}}\sum_{\gamma_1:x\to y}p_\Lambda(\gamma_1)},\end{aligned}$$

for some constant $C>0$, using Lemma 3.5 and Proposition 3.12 for the last inequality. By (3.2), one has

$$\begin{aligned}\sum_{y\in\partial B_{R_{j-1}}}\sum_{\substack{\gamma_1:x\to y\\ \ell_{\gamma_1}(\mathcal{S}_j)\leqslant\alpha R_j^2}}p_\Lambda(\gamma_1)&\leqslant\sum_{y\in\partial B_{R_{j-1}}}\mathbf{P}_x\big(\ell_S(\mathcal{S}_j)\leqslant\alpha R_j^2,\, S_{\tau_{j-1}}=y\big)\\ &=\mathbf{P}_x\big(\ell_S(\mathcal{S}_j)\leqslant\alpha R_j^2,\,\tau_{j-1}<\infty\big),\end{aligned}$$

while using also Remark 3.13, we get that for some constant $c>0$,

$$\sum_{y\in\partial B_{R_{j-1}}}\sum_{\gamma_1:x\to y}p_\Lambda(\gamma_1)\geqslant\sum_{y\in\partial B_{R_{j-1}}}\sum_{\substack{\gamma_1:x\to y\\ |\gamma_1|\leqslant R_j^2}}p_\Lambda(\gamma_1)\geqslant c\mathbf{P}_x\big(\tau_{j-1}(S)\leqslant R_j^2\big)\geqslant c^2.$$

Now for any $\varepsilon>0$, one can find α small enough, such that

$$\mathbf{P}_x\big(\ell_S(\mathcal{S}_j)\leqslant\alpha R_j^2,\,\tau_{j-1}<\infty\big)\leqslant\mathbf{P}_x\big(\ell_S(\mathcal{S}_j)\leqslant\alpha R_j^2\big)\leqslant\varepsilon.$$

Altogether this proves (10.5), and thus also (10.4). Using also Lemma 10.1, it follows that for some positive constants c and κ,

$$\mathbb{P}(E_0\cap\cdots\cap E_{I/2})\geqslant\exp(-\kappa I)\times\mathbb{P}(E_0)\geqslant\frac{c}{R_I^2}\exp(-\kappa I).$$

We observe finally that on the event in the probability above, the number of particles which hit $\partial B_{r/2}$ dominates the sum of $I/2$ independent random variables $X_1,\dots,X_{I/2}$ distributed as $|\eta_{r/2}|$, under the conditional law $\mathbb{P}_z(\cdot\mid|\eta_{r/2}|>0)$, for some $z\in\partial B_{R_{I/2}}$. However, for any such starting point z, one has using the computation made in the proof of Lemma 3.5, together with Proposition 3.12,

$$\mathbb{E}_z\big[|\eta_{r/2}|\mid|\eta_{r/2}|>0\big]\gtrsim Ir^2,\quad\text{and}\quad\mathbb{E}_z\big[|\eta_{r/2}|^2\mid|\eta_{r/2}|>0\big]\lesssim r^4I^2.$$

Thus Paley-Zygmund's inequality (2.7) (applied to the law of $X_1+\cdots+X_{I/2}$) gives the existence of $\alpha>0$, such that

$$\mathbb{P}\big(X_1+\cdots+X_{I/2}\geqslant\alpha I^2r^2\big)\geqslant\alpha.$$

In other words, we just have proved that

$$\mathbb{P}\big(\textstyle\sum_{u\in\eta_{R_I}}|\eta_{r/2}^u|\geqslant\alpha r^2I^2\big)\geqslant\frac{c}{r^2}\exp(-\kappa' I),$$

for some positive constants c and κ'. The proof is now almost finished. To conclude, note that

$$\inf_{x\in B_{r/2}} \mathbb{E}_x\big[|\mathcal{T}(B_r)|\big] = \inf_{x\in B_{r/2}} \mathbf{E}_x[H_r] \gtrsim r^2,$$

and by definition,

$$\ell_{\mathcal{T}}(B_r) \geqslant \sum_{u\in\eta_{R_I}} \sum_{v\in\eta^u_{r/2}} |\mathcal{T}^v(B_r)|.$$

Thus by an application of the weak law of large numbers, and by taking the constant C in the definition of I large enough, we deduce

$$\mathbb{P}(\ell_{\mathcal{T}}(B_r) > t) \geqslant \frac{c}{r^2}\exp(-\kappa' I),$$

for some (possibly different) positive constants c and κ'. This concludes the proof of the lower bound in Theorem 1.1, in dimension four. □

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