Journal de l'École polytechnique
Mathématiques

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Tome 11 (2024), p. 769-807.
https://doi.org/10.5802/jep.265

Publié avec le soutien
du Centre National de la Recherche Scientifique

Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 2270-518X

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ON UNIFORM POLYNOMIAL APPROXIMATION

by Anthony Poëls

Abstract. — Let n be a positive integer and ξ a transcendental real number. We are interested in bounding from above the uniform exponent of polynomial approximation $\widehat{\omega}_n(\xi)$. Davenport and Schmidt's original 1969 inequality $\widehat{\omega}_n(\xi) \leqslant 2n-1$ was improved recently, and the best upper bound known to date is $2n-2$ for each $n \geqslant 10$. In this paper, we develop new techniques leading us to the improved upper bound $2n - \frac{1}{3}n^{1/3} + \mathcal{O}(1)$.

Résumé (Sur l'approximation polynomiale uniforme). — Soient n un entier strictement positif et ξ un nombre réel transcendant. Nous cherchons à borner supérieurement l'exposant uniforme d'approximation polynomiale $\widehat{\omega}_n(\xi)$. Établie par Davenport et Schmidt en 1969, l'inégalité $\widehat{\omega}_n(\xi) \leqslant 2n-1$, a été améliorée pour la première fois récemment, et la meilleure borne supérieure connue à ce jour est $2n-2$ pour tout $n \geqslant 10$. Dans ce papier, nous développons de nouvelles techniques qui nous permettent d'obtenir la borne supérieure améliorée $2n - \frac{1}{3}n^{1/3} + \mathcal{O}(1)$.

Contents

Mathematical subject classification (2020). — 11J13, 11J82.
Keywords. — Exponent of Diophantine approximation, heights, uniform polynomial approximation.

1. Introduction

Let ξ be a non-zero real number and let n be a positive integer. Dirichlet's theorem (1842) is one of the most basic results of Diophantine approximation. It shows that for any real number $H > 1$, there exists a non-zero integer point $(x_0, \dots, x_n) \in \mathbb{Z}^{n+1}$ such that

$$\max\big\{|x_1|, \dots, |x_n|\big\} \leqslant H \quad \text{and} \quad |x_0 + x_1\xi + \cdots + x_n\xi^n| \leqslant H^{-n}. \tag{1.1}$$

It is natural to ask if we can improve the exponent n of H^{-n}, and this question gives rise to two Diophantine exponents. The so-called *uniform exponent of approximation* $\widehat{\omega}_n(\xi)$ (resp. the *ordinary* exponent $\omega_n(\xi)$), is the supremum of the real numbers $\omega > 0$ such that the system

$$\|P\| \leqslant H \quad \text{and} \quad 0 < |P(\xi)| \leqslant H^{-\omega}$$

admits a non-zero solution $P \in \mathbb{Z}[X]$ of degree at most n for each sufficiently large H (resp. for arbitrarily large H). Here, $\|P\|$ denotes the (naive) *height* of P, defined as the largest absolute value of its coefficients. These quantities have been extensively studied over the past half-century, see for example [5] for a nice overview of the subject. By Dirichlet's theorem, if ξ is not an algebraic number of degree $\leqslant n$, then we have

$$\omega_n(\xi) \geqslant \widehat{\omega}_n(\xi) \geqslant n,$$

and it is well known that those inequalities are equalities for almost all real numbers ξ (with respect to the Lebesgue measure). Note that if ξ is an algebraic number of degree d, then $\widehat{\omega}_n(\xi)$ and $\omega_n(\xi)$ are both equal to $\min\{n, d-1\}$ (it is a consequence of Schmidt's subspace theorem, see [5, Th. 2.10]). We can therefore restrict our study to the set of transcendental real numbers. The initial question "can we improve the exponent n in Dirichlet's theorem?" may be rephrased as follows: "does there exist a transcendental real number ξ satisfying $\widehat{\omega}_n(\xi) > n$?". For $n = 1$ the answer is negative and rather elementary to prove, so the first non-trivial case is $n = 2$. Before the early 2000s, it was conjectured that no such number existed. This belief was swept away by Roy's extremal numbers [20], [21], [1], whose exponent $\widehat{\omega}_2$ is equal to the maximal possible value $(3+\sqrt{5})/2 = 2.618\cdots$. Since then, several families of transcendental real numbers whose uniform exponent $\widehat{\omega}_2$ is greater than 2 have been discovered (see for example [22], [6], [15, 16]). However, for $n \geqslant 3$ the mystery remains, and it is still an open question whether or not there exists $\xi \in \mathbb{R} \smallsetminus \overline{\mathbb{Q}}$ with $\widehat{\omega}_n(\xi) > n$.

In this paper, we are interested in finding an upper bound for the uniform exponent $\widehat{\omega}_n(\xi)$, as this could provide clues to solving the initial problem. Brownawell's version of Gel'fond's criterion [3] implies that $\widehat{\omega}_n(\xi) \leqslant 3n$. In 1969, Davenport and Schmidt [10, Th. 2b] showed that for any transcendental real number ξ and any integer $n \geqslant 2$, we have

$$\widehat{\omega}_n(\xi) \leqslant 2n - 1. \tag{1.2}$$

Up to now, few improvements have been made. Bugeaud and Schleischitz [8, Th. 2.1] first got the upper bound

$$\widehat{\omega}_n(\xi) \leqslant n - \frac{1}{2} + \sqrt{n^2 - 2n + 1/4} = 2n - \frac{3}{2} + \varepsilon_n, \tag{1.3}$$

where $\varepsilon_n > 0$ tends to 0 as n tends to infinity. Recently, Marnat and Moshchevitin [13] proved an important conjecture of Schmidt and Summerer on the ratio $\widehat{\omega}_n(\xi)/\omega_n(\xi)$ (see also [19, Ch. 2] for an alternative proof based on parametric geometry of numbers). In [23], Schleischitz pointed out that we can use the aforementioned inequality in the proof of (1.3) to get

$$\widehat{\omega}_n(\xi) \leqslant 2n - 2,$$

for each $n \geqslant 10$. This is currently the best known upper bound. Let us also mention that using parametric geometry of numbers, Schleischitz [24, Th. 1.1] was able to replace the estimate (1.3) by

$$\widehat{\omega}_n(\xi) \leqslant \frac{3(n-1) + \sqrt{n^2 - 2n + 5}}{2} = 2n - 2 + \varepsilon_n'$$

where $\varepsilon_n' > 0$ tends to 0 as n tends to infinity. For $n = 3, \dots, 9$, bounds that are better than (1.2), but nevertheless (strictly) greater than $2n - 2$, are known. For example, it was proved in [8] that for each transcendental real number ξ, we have

$$\widehat{\omega}_3(\xi) \leqslant 3 + \sqrt{2} = 4.41\cdots,$$

see also the very recent work of Schleischitz [25]. In this paper, without relying on Marnat-Moshchevitin's inequality and with a different approach, we show in Section 7 that the upper bound $\widehat{\omega}_n(\xi) \leqslant 2n-2$ holds for any $n \geqslant 4$. We also improve the upper bound for $\widehat{\omega}_3$.

Theorem 1.1. — *Let $n \geqslant 3$ be an integer and $\xi \in \mathbb{R}$ be a transcendental real number. If $n \geqslant 4$, then*

$$\widehat{\omega}_n(\xi) \leqslant 2n - 2.$$

For $n = 3$, we have the weaker estimate $\widehat{\omega}_3(\xi) \leqslant 2 + \sqrt{5} = 4.23\cdots$.

We do not think that these upper bounds are optimal. It is interesting to note that Schleischitz, with a different method and under a technical condition, also found the estimates of Theorem 1.1, see [25]. Our main result below is a significant improvement on the previous results as n tends to infinity and does not require Marnat and Moshchevitin's inequality [13].

Theorem 1.2. — *Set $a = 1/3$. There exists a computable constant $N \geqslant 1$ such that, for each $n \geqslant N$ and any transcendental real number $\xi \in \mathbb{R}$, we have*

$$\widehat{\omega}_n(\xi) \leqslant 2n - an^{1/3}.$$

The constant $a = 1/3$ is not optimal. Numerical calculations based on the results from Section 11 suggest that we could take N rather "small" in Theorem 1.2 (maybe $N \leqslant 10^4$?). However, to keep the arguments and calculations as clear and simple as possible, we did not try to provide an explicit value of N.

Theorem 1.2 can be compared to [17, Th. 1.1], where we study $\widehat{\lambda}_n(\xi)$, the uniform exponent of rational simultaneous approximation to the successive powers $\Xi = (1, \xi, \xi^2, \dots, \xi^n)$ (which is known to be, in a sense, dual to $\widehat{\omega}_n(\xi)$), see Section 2 for the precise definition and more details. We were not able to deduce one result from the other, even though there are similarities in the arguments. For example, given a polynomial $P \in \mathbb{Z}[X]$ of degree at most n, which is a good approximation, we can associate the $k+1$ polynomials $P, XP, \dots, X^kP$ of degree at most $n+k$. They provide information on $\widehat{\omega}_{n+k}(\xi)$. On the other hand, if we consider $\boldsymbol{y} \in \mathbb{Z}^{n+1}$ which is a good approximation of Ξ (for simultaneous approximation), we can associate the $k+1$ blocks of successive $n+1-k$ coordinates of $\boldsymbol{y}$, which are rather good approximations of $(1, \xi, \dots, \xi^{n-k})$. They in turn provide information on $\widehat{\lambda}_{n-k}(\xi)$. Note that the difficulties in the proofs of both theorems are not in the same places. In particular, in this paper we have to work with *irreducible* polynomials, a rather heavy constraint. Also, one of the most delicate parts of our approach is to bound from above the ordinary exponent $\omega_n(\xi)$, whereas this is rather "simple" to do for the ordinary exponent $\lambda_n(\xi)$ in [17].

Before presenting our strategy, let us quickly explain Davenport and Schmidt's proof of the upper bound (1.2). Given a real number $\widehat{\omega} < \widehat{\omega}_n(\xi)$, they show, using elementary means and Gelfond's lemma, that there are infinitely many pairs of coprime polynomials $P, Q \in \mathbb{Z}[X]$ of degree at most n, such that

$$\|Q\| \leqslant \|P\| \quad \text{and} \quad \max\{|Q(\xi)|, |P(\xi)|\} \ll \|P\|^{-\widehat{\omega}},$$

(where the implicit constant only depends on n). It implies that the resultant $\operatorname{Res}(P,Q)$, which is a non-zero integer, satisfies

$$1 \leqslant |\operatorname{Res}(P,Q)| \ll \|P\|^{n-1}\|Q\|^{n-1} \max\big\{\|P\||Q(\xi)|, \|Q\||P(\xi)|\big\} \ll \|P\|^{2n-1-\widehat{\omega}}.$$

The first upper bound for $|\operatorname{Res}(P,Q)|$ is classical, see Lemma 4.1. Since $\|P\|$ can be arbitrarily large, they deduced that the exponent $2n-1-\widehat{\omega}$ is non-negative. Estimate (1.2) follows by letting $\widehat{\omega}$ tend to $\widehat{\omega}_n(\xi)$. Note that the term $2n$ in (1.2) is directly related to the size of the $2n \times 2n$ determinant defining $\operatorname{Res}(P,Q)$ (if we suppose that P and Q have degree exactly n).

The key idea in the proof of our main Theorem 1.2 is to work with a large number of "good" linearly independent polynomial approximations $Q_0, \dots Q_{j+1}$ rather than just two polynomials P and Q as above. By doing this, we can replace $\operatorname{Res}(P,Q)$ by a non-zero $(2n-j) \times (2n-j)$ determinant depending on the coefficients of $Q_0, \dots, Q_{j+1}$. Under the ideal and unlikely assumption that

$$\|Q_k\| \leqslant \|Q_0\| \quad \text{and} \quad |Q_k(\xi)| \ll \|Q_0\|^{-\widehat{\omega}} \qquad (\text{for } k = 0, \dots, j), \tag{1.4}$$

the aforementioned determinant would be bounded from above by $\|Q_0\|^{2n-j-1-\widehat{\omega}}$. So, together with an additional non-vanishing assumption, it would lead to $\widehat{\omega}_n(\xi) \leqslant 2n-j-1$. Several new difficulties arise when trying to make the above arguments work. We introduce the tools for the construction of the generalized resultant in Section 6. To ensure that this determinant does not vanish, we need the extra assumption that

$Q_0, \dots, Q_{j+1}$ are irreducible polynomials. The idea is to first fix a sequence of best approximations, that we call *minimal polynomials*, and then to consider their highest-degree irreducible factors (which also happen to be rather good approximations). We deal with this question in Section 5. Two obstacles remain. Firstly, note that it may be possible that the best polynomial approximations span a subspace of dimension 3, even when ξ is transcendental and n is large, see [14, Th. 1.3]. Therefore, as soon as $j > 1$ (we will later choose $j \asymp n^{1/3}$), we have to justify that we can find $j+2$ linearly independent polynomials as above. The second major problem is the control of the sequence $Q_0, \dots, Q_{j+1}$. Estimates (1.4) seem out of reach, instead we get upper bounds of the form

$$\|Q_k\| \leqslant \|Q_0\| \quad \text{and} \quad |Q_k(\xi)| \ll \|Q_0\|^{-\widehat{\omega}\theta} \qquad (\text{for } k = 0, \dots, j), \tag{1.5}$$

where $\theta < 1$ depends only on n and j, and is "close" to 1 if j is not too large compared to n. The main ingredients for showing this are related to *twisted heights*, see Sections 8.2 and the appendix, and an important inequality on the height of subspaces due to Schmidt. The parameter θ in (1.5) is a function of the exponent of best approximation $\omega_n(\xi)$. We show in Section 10 that if the uniform exponent satisfies $\widehat{\omega}_n(\xi) \geqslant 2n - d$ (with $d \ll n^{1/3}$), then the ordinary exponent $\omega_n(\xi)$ is bounded from above by $2n + 2d^2$, and the ratio $\widehat{\omega}_n(\xi)/\omega_n(\xi)$ is therefore close to 1. This part, which is essentially independent from the others, is rather delicate, because we work with the polynomials Q_i. They are certainly irreducible, but not as good approximations as the minimal polynomials. More precisely, there could be large gaps between the height of two successive Q_i. If we could drop the irreducibility condition and directly work with the sequence of minimal polynomials, we could possibly replace the upper bound $2n - \mathcal{O}(n^{1/3})$ with $2n - \mathcal{O}(n^{1/2})$ in Theorem 1.2. Section 11 is devoted to the proof of Theorem 1.2.

Acknowledgements. — I would like to thank Damien Roy for his attentive reading of this work and his many comments, which helped to improve its overall presentation and clarity. I also thank the referee for their work and valuable comments.

2. Notation

Throughout this paper, ξ denotes a transcendental real number. The floor (resp. ceiling) function is denoted by $\lfloor \cdot \rfloor$ (resp. $\lceil \cdot \rceil$). If $f, g : I \to [0, +\infty)$ are two functions on a set I, we write $f = \mathcal{O}(g)$ or $f \ll g$ or $g \gg f$ to mean that there is a positive constant c such that $f(x) \leqslant c g(x)$ for each $x \in I$. We write $f \asymp g$ when both $f \ll g$ and $g \ll f$ hold.

Let K be a field. If $\mathcal{A}$ is a subset of a K-vector space V, we denote by $\langle \mathcal{A} \rangle_K \subseteq V$ the K-vector space spanned by $\mathcal{A}$, with the convention that $\langle \varnothing \rangle_K = \{0\}$.

Given a ring A (typically $A = \mathbb{R}$ or $\mathbb{Z}$) and an integer $n \geqslant 0$, we denote by $A[X]$ the ring of polynomials in X with coefficients in A, and by $A[X]_{\leqslant n} \subseteq A[X]$ the subgroup of polynomials of degree at most n. We say that $P \in \mathbb{Z}[X]$ is *primitive* if it non-zero and the greatest common divisor of its coefficients is 1. Given $P=\sum_{k=0}^{n}a_kX^k\in\mathbb{R}[X]$, we set

$$\|P\|=\max_{0\leqslant k\leqslant n}|a_k|.$$

Gelfond's lemma is the following statement (see e.g. [4, Lem. A.3] as well as [3]). For any non-zero polynomials $P_1,\dots,P_r\in\mathbb{R}[X]$ with product $P=P_1\cdots P_r$ of degree at most n, we have

$$e^{-n}\|P_1\|\cdots\|P_r\|<\|P\|<e^{n}\|P_1\|\cdots\|P_r\|. \tag{2.1}$$

In particular, for each non-zero polynomial $P\in\mathbb{Z}[X]_{\leqslant n}$ and each factor $Q\in\mathbb{Z}[X]$ of P, we have $e^{-n}\|Q\|<\|P\|$. We will often use (2.1) as follows. If $Q\in\mathbb{Z}[X]_{\leqslant n}$ is irreducible and if $P\in\mathbb{Z}[X]_{\leqslant n}$ is a non-zero polynomial which satisfies $\|P\|\leqslant e^{-n}\|Q\|$, then Q cannot divide P. They are thus coprime polynomials.

We recall the definition of the resultant, which, as explained in the introduction, is useful for estimating the exponent $\widehat{\omega}_n(\xi)$ (see also Section 4). Let $P,Q\in\mathbb{Z}[X]$ be non-constant polynomials of degree p and q respectively, and let $a_i,b_j\in\mathbb{Z}$ such that $P(X)=\sum_{k=0}^{p}a_kX^k$ and $Q(X)=\sum_{k=0}^{q}b_kX^k$. Their *resultant* $\operatorname{Res}(P,Q)$ is defined as the $(q+p)$-dimensional determinant

$$\operatorname{Res}(P,Q)=\underbrace{\begin{vmatrix} a_p & 0 & \cdots & & b_q & 0 & \cdots & \\ a_{p-1} & a_p & & & b_{q-1} & b_q & & \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & \\ a_0 & & & & b_0 & & & \\ 0 & a_0 & & & 0 & b_0 & & \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & \\ & & & a_0 & & & & b_p \end{vmatrix}}_{\displaystyle q \qquad\qquad\qquad p}. \tag{2.2}$$

Besides the exponents of linear approximation ω_n and $\widehat{\omega}_n$, we will also need the following exponents of simultaneous rational approximation. For each positive integer n, the exponent $\widehat{\lambda}_n(\xi)$ (resp. $\lambda_n(\xi)$) is the supremum of the real numbers $\lambda\geqslant 0$ such that the system

$$|y_0|\leqslant Y \quad\text{and}\quad L(\boldsymbol{y})\leqslant Y^{-\lambda} \quad\text{where } L(\boldsymbol{y}):=\max_{1\leqslant k\leqslant n}|y_0\xi^k-y_k|,$$

admits a non-zero integer solution $\boldsymbol{y}=(y_0,\dots,y_n)\in\mathbb{Z}^{n+1}$ for each sufficiently large $Y\geqslant 1$ (resp. for arbitrarily large Y). Dirichlet's theorem [26, §II.1, Th. 1A] implies that $\widehat{\lambda}_n(\xi)\geqslant 1/n$. The best upper bounds known to date for $\widehat{\lambda}_n(\xi)$ when $n\geqslant 4$ are established in joint work with Roy in [17]. In particular, there is an explicit positive constant a such that

$$\widehat{\lambda}_n(\xi)\leqslant\frac{1}{n/2+an^{1/2}+1/3},$$

and sharper results are also obtained when n is small.

3. Minimal polynomials

A *sequence of minimal polynomials* (associated to n and ξ) is a sequence $(P_i)_{i\geqslant 0}$ of non-zero polynomials in $\mathbb{Z}[X]_{\leqslant n}$ satisfying the following properties:

(i) the sequence $\big(\|P_i\|\big)_{i\geqslant 0}$ is strictly increasing;

(ii) the sequence $\big(|P_i(\xi)|\big)_{i\geqslant 0}$ is strictly decreasing;

(iii) if $|P(\xi)| < |P_i(\xi)|$ for some index $i \geqslant 0$ and a non-zero $P \in \mathbb{Z}[X]_{\leqslant n}$, then $\|P\| \geqslant \|P_{i+1}\|$.

Note that if we require the dominant coefficient of P_i to be positive (and since ξ is transcendental), then the above sequence is unique up to its first terms. Let $(P_i)_{i\geqslant 0}$ be a sequence as above. We have the classical formulas:

$$\widehat{\omega}_n(\xi) = \liminf_{i\to\infty} \frac{-\log|P_i(\xi)|}{\log\|P_{i+1}\|} \quad \text{and} \quad \omega_n(\xi) = \limsup_{i\to\infty} \frac{-\log|P_i(\xi)|}{\log\|P_i\|}. \tag{3.1}$$

In particular, given a positive real number $\widehat{\omega}$ with $\widehat{\omega} < \widehat{\omega}_n(\xi)$, then we have, for each sufficiently large index i,

$$|P_i(\xi)| \leqslant \|P_{i+1}\|^{-\widehat{\omega}} \quad \text{and} \quad \|P_{i+1}\|^{\tau} \leqslant \|P_i\|, \quad \text{where } \tau := \frac{\widehat{\omega}}{\omega_n(\xi)}, \tag{3.2}$$

(with the convention $\tau = 0$ if $\omega_n(\xi) = \infty$). The second inequality in (3.2) asks for an upper bound on $\omega_n(\xi)$. Given a non-zero $P \in \mathbb{Z}[X]$, we set $\omega(P) = 0$ if $\|P\| = 1$. Otherwise, we denote by $\omega(P)$ the real number satisfying

$$|P(\xi)| = \|P\|^{-\omega(P)}.$$

With this notation, we have

$$\omega_n(\xi) = \limsup_{\substack{\|P\|\to\infty \\ P\in\mathbb{Z}[X]_{\leqslant n}}} \omega(P) = \limsup_{i\to\infty} \omega(P_i) \quad \text{and} \quad \liminf_{i\to\infty} \omega(P_i) \geqslant \widehat{\omega}_n(\xi). \tag{3.3}$$

The following results are well-known. We prove them for the sake of completion. The first one follows from the arguments of the proof of [9, Lem. 2] (see also [21, Lem. 4.1]).

Lemma 3.1. — *Let $i \geqslant 0$ and write $V_i = \langle P_i, P_{i+1}\rangle_{\mathbb{R}} \subseteq \mathbb{R}[X]_{\leqslant n}$. Then $\{P_i, P_{i+1}\}$ forms a $\mathbb{Z}$-basis of the lattice $V_i \cap \mathbb{Z}[X]_{\leqslant n}$.*

Proof. — By contradiction, suppose that $\{P_i, P_{i+1}\}$ is not a $\mathbb{Z}$-basis of $V_i \cap \mathbb{Z}[X]_{\leqslant n}$. Then there exists a non-zero $Q \in \mathbb{Z}[X]_{\leqslant n}$ which may be written as $Q = rP_i + sP_{i+1}$, where $r, s \in \mathbb{Q}$ satisfy $|r|, |s| \leqslant 1/2$. In particular, we have

$$\|Q\| \leqslant |r|\|P_i\| + |s|\|P_{i+1}\| < \|P_{i+1}\|,$$
$$|Q(\xi)| \leqslant |r||P_i(\xi)| + |s||P_{i+1}(\xi)| < |P_i(\xi)|.$$

This contradicts the minimality property of P_i. □

The next result is analogous to the second part of [21, Lem. 4.1]. The construction of S_i is due to Davenport and Schmidt [9].

Lemma 3.2. — *For each $i \geqslant 0$, define*

$$S_i = P_i(\xi)P_{i+1} - P_{i+1}(\xi)P_i \in \mathbb{R}[X]_{\leqslant n}.$$

Then

$$\frac{1}{2}\|S_i\| \leqslant \|P_{i+1}\||P_i(\xi)| \leqslant 2\|S_i\|.$$

Moreover, if for integers $0 \leqslant i < j$ the space spanned by $P_i, P_{i+1}, \cdots, P_j$ has dimension 2, then $S_{j-1} = \pm S_i$. In particular

$$\|P_{i+1}\||P_i(\xi)| \asymp \|P_j\||P_{j-1}(\xi)|.$$

Remark 3.3. — Note that the quantity $\|S_i\|$ satisfies $\|S_i\| \asymp \mathcal{D}_\xi(V_i)$, where $\mathcal{D}_\xi$ is defined in Section 8.2 and $V_i = \langle P_i, P_{i+1}\rangle_{\mathbb{R}}$. We will study the function $\mathcal{D}_\xi$ more deeply later.

Proof. — We easily get $\|S_i\| \leqslant 2\|P_{i+1}\||P_i(\xi)|$. Define $R_+, R_- \in \mathbb{Z}[X]_{\leqslant n}$ by

$$R_\pm = P_{i+1} \pm P_i.$$

Suppose that there exists $\varepsilon \in \{+,-\}$ such that $|R_\varepsilon(\xi)| \leqslant |P_i(\xi)|/2$. Then by minimality of P_i, we must have $\|R_\varepsilon\| \geqslant \|P_{i+1}\|$. Since $S_i = P_i(\xi)R_\varepsilon - R_\varepsilon(\xi)P_i$, we find

$$\|S_i\| \geqslant |P_i(\xi)|\|R_\varepsilon\| - |R_\varepsilon(\xi)|\|P_i\| \geqslant \frac{1}{2}\|P_{i+1}\||P_i(\xi)|.$$

Assume that $|R_+(\xi)|, |R_-(\xi)| \geqslant |P_i(\xi)|/2$. This is equivalent to

$$|P_{i+1}(\xi)| \leqslant \frac{1}{2}|P_i(\xi)|.$$

Again, this yields $\|S_i\| \geqslant |P_i(\xi)|\|P_{i+1}\| - |P_{i+1}(\xi)|\|P_i\| \geqslant \|P_{i+1}\||P_i(\xi)|/2$.

Now, let us write $V_i = \langle P_i, \dots, P_j\rangle_{\mathbb{R}}$, with $j > i$, and suppose that V_i has dimension 2. We need to prove that $S_{j-1} = \pm S_i$. If $j = i+1$ it is automatic, we may therefore assume that $j \geqslant i+2$. By Lemma 3.1, there exist $a, b \in \mathbb{Z}$ such that $P_i = aP_{i+1} + bP_{i+2}$. Since $\{P_i, P_{i+1}\}$ is also a $\mathbb{Z}$-basis of V_i, we have $b = \pm 1$, and we deduce that

$$S_i = \big(aP_{i+1}(\xi) + bP_{i+2}(\xi)\big)P_{i+1} - P_{i+1}(\xi)\big(aP_{i+1} + bP_{i+2}\big) = -bS_{i+1} = \pm S_{i+1}.$$

By induction, we get $S_i = \pm S_{i+1} = \cdots = \pm S_{j-1}$. □

The proof of [9, Lem. 3] (which deals with the case $n = 2$) yields the classical following result.

Lemma 3.4. — *Suppose $n \geqslant 2$. Then, there are infinitely many indices $i \geqslant 1$ for which P_{i-1}, P_i and P_{i+1} are linearly independent.*

Proof. — By contradiction, suppose that there exists $i \geqslant 0$ such that $V = \langle P_i, P_{i+1}, \dots\rangle_{\mathbb{R}}$ has dimension 2. By Lemma 3.2 there exists $c > 0$ such that for each $j > i$ we have

$$0 < \|P_{i+1}\||P_i(\xi)| \leqslant c\|P_j\||P_{j-1}|.$$

This leads to a contradiction since $\|P_j\||P_{j-1}| \leqslant \|P_j\|^{1-\widehat{\omega}_n(\xi)+o(1)}$ tends to 0 as j tends to infinity. □

Remark 3.5. — As mentioned in the introduction, it is however possible that all polynomials P_i with i large enough lie in a subspace of dimension 3 , see [14, Th. 1.3].

4. Resultant and first estimates

The following useful result can easily be derived from the proof of [10, §5] (see also of [3, Lem. 1]). We recall the arguments since they illustrate (in a simpler situation) how we will deal with generalized determinants.

Lemma 4.1. — *Let p, q be positive integers with $p, q \leqslant n$. There exists a constant $c > 0$ depending on ξ and n only, with the following property. For any polynomials $P, Q \in \mathbb{Z}[X]$ of degree p and q respectively, we have*

$$|\operatorname{Res}(P, Q)| \leqslant c\|P\|^{q-1}\|Q\|^{p-1} \max\big\{\|P\||Q(\xi)|, \|Q\||P(\xi)|\big\}.$$

Proof. — Let $a_i, b_j \in \mathbb{Z}$ such that $P(X) = \sum_{k=0}^{p} a_k X^k$ and $Q(X) = \sum_{k=0}^{q} b_k X^k$. For $i = 1, \dots, p+q-1$, we add to the last row of the determinant (2.2) its i-th row multiplied by ξ^{p+q-i}. This last row now becomes

$$\Big(\xi^{q-1}P(\xi), \dots, \xi P(\xi), P(\xi), \xi^{p-1}Q(\xi), \dots, \xi Q(\xi), Q(\xi)\Big).$$

Using the upper bounds $|a_i| \leqslant \|P\|$ and $|b_j| \leqslant \|Q\|$ for the other entries of (2.2), we obtain

$$|\operatorname{Res}(P, Q)| \ll \|P\|^{q-1}|P(\xi)|\|Q\|^{p} + \|P\|^{q}\|Q\|^{p-1}|Q(\xi)|,$$

where the implicit constant only depends on p, q and ξ. □

The next result, which is also based on inequalities involving resultants, will be used in Section 10. It ensures that if $R \in \mathbb{Z}[X]$ is a "good" approximation, in the sense that $R(\xi)$ is very small compared to $\|R\|$, and if we write R as a product of coprime polynomials $B_1 \cdots B_k$, then one of those factors is also a "good" approximation, while the product of the others is not.

Lemma 4.2. — *Let m, k be positive integers. There exists a constant $c > 0$ depending on m and ξ only, with the following property. Let $B_1, \dots, B_k \in \mathbb{Z}[X]$ be non-constant, pairwise coprime polynomials, and suppose that $R := B_1 \cdots B_k$ has degree at most m. Then, there exists $j \in \{1, \dots, k\}$ such that*

$$|B_j(\xi)| \leqslant c\|R\|^{m-1}|R(\xi)| \quad \textit{and} \quad \prod_{\substack{i=1\\ i\neq j}}^{k} |B_i(\xi)| \geqslant c^{-1}\|R\|^{-(m-1)}.$$

Proof. — If $k = 1$ this is trivial. We now suppose that $k \geqslant 2$ and write $d_i = \deg(B_i)$ for $i = 1, \dots, k$. By hypothesis, we have $\deg(R) = d_1 + \cdots + d_k \leqslant m$. Note that

$$|R(\xi)| = \prod_{i=1}^{k} |B_i(\xi)| \quad \text{and} \quad \|R\| \asymp \prod_{i=1}^{k} \|B_i\|, \tag{4.1}$$

the second inequality coming from Gelfond's lemma (with an implicit constant depending only on m). Choose $j \in \{1,\dots,k\}$ such that $|B_j(\xi)|$ is minimal and fix $i \in \{1,\dots,k\}$ with $i \neq j$. Since B_i and B_j are coprime, their resultant $\mathrm{Res}(B_i,B_j)$ is a non-zero integer. Using Lemma 4.1, we find

$$\begin{aligned}1 \leqslant |\operatorname{Res}(B_i,B_j)| &\ll \|B_i\|^{d_j-1}\|B_j\|^{d_i-1}\big(\|B_j\||B_i(\xi)| + \|B_i\||B_j(\xi)|\big)\\ &\ll \|B_i\|^{d_j}\|B_j\|^{d_i}|B_i(\xi)|,\end{aligned}$$

with an implicit constant depending only on ξ and m, hence

$$-\log|B_i(\xi)| \leqslant d_j \log\|B_i\| + d_i\log\|B_j\| + \mathcal{O}(1).$$

On the other hand, by summing the above inequalities for $i \neq j$, and by using (4.1), we obtain

$$\begin{aligned}\sum_{\substack{i=1\\ i\neq j}}^{k} -\log|B_i(\xi)| &\leqslant d_j\sum_{\substack{i=1\\ i\neq j}}^{k}\log\|B_i\| + (m-d_j)\log\|B_j\| + \mathcal{O}(1)\\ &\leqslant (m-1)\log\|R\| + \mathcal{O}(1).\end{aligned}$$

We easily deduce that

$$\prod_{\substack{i=1\\ i\neq j}}^{k}|B_i(\xi)| \gg \|R\|^{-(m-1)} \quad\text{and}\quad |R(\xi)| = \prod_{i=1}^{k}|B_i(\xi)| \gg |B_j(\xi)|\|R\|^{-(m-1)}. \qquad \square$$

5. A sequence of irreducible polynomials

As explained in the introduction, to get the upper bound $\widehat{\omega}_n(\xi) \leqslant 2n-1$, the strategy of Davenport and Schmidt [10] consists in considering the resultant $\mathrm{Res}(P,Q)$ of two "good" polynomial approximations $P,Q \in \mathbb{Z}[X]_{\leqslant n}$. To ensure that $\mathrm{Res}(P,Q)$ does not vanish, they need a polynomial P which is irreducible (for it is then easy to find Q so that P and Q are coprime). The same difficulty appears in [8]. Similarly, we will not work directly with a sequence of minimal polynomials. Instead, we will considerer the largest irreducible factors of the minimal polynomials. Now, let n,d be integers with

$$2 \leqslant d < 1 + \frac{n}{2}.$$

In this section, we assume that the transcendental real number ξ satisfies $\widehat{\omega}_n(\xi) > 2n-d$ and we fix a real number $\widehat{\omega}$ (arbitrarily close to $\widehat{\omega}_n(\xi)$) such that

$$\widehat{\omega}_n(\xi) > \widehat{\omega} > 2n-d. \tag{5.1}$$

We denote by $(P_i)_{i\geqslant 0}$ a sequence of minimal polynomials associated to n and ξ. Our goal is to prove the existence of a sequence $(Q_i)_{i\geqslant 0}$ as below.

Proposition 5.1. — *Suppose that* (5.1) *holds. Then, there exist a sequence* $(Q_i)_{i\geqslant 0}$ *of pairwise distinct polynomials in* $\mathbb{Z}[X]_{\leqslant n}$ *and an index* $j_0 \geqslant 0$ *with the following properties. The sequence* $(\|Q_i\|)_{i\geqslant 0}$ *is bounded below by* 2*, unbounded and non-decreasing, and for any* $i \geqslant 0$

(i) *Q_i is irreducible (over $\mathbb{Z}$) and has degree at least $n-d+2$;*

(ii) *Q_i divides P_j for some index $j \geqslant j_0$ (not necessarily unique), and for each $j \geqslant j_0$ there exists $k \geqslant 0$ such that Q_k divides P_j;*

(iii) *$|Q_i(\xi)| = \|Q_i\|^{-\omega(Q_i)} \leqslant \|Q_i\|^{-\widehat{\omega}}$, and we further have*

$$\omega_n(\xi) = \limsup_{k\to\infty} \omega(Q_k) \quad \textit{and} \quad \liminf_{k\to\infty} \omega(Q_k) \geqslant \widehat{\omega}_n(\xi). \tag{5.2}$$

(iv) *if Q_i divides a minimal polynomial P_j with $j \geqslant j_0$, then*

$$\|P_j\| \leqslant \|Q_i\|^{1+\theta_i}, \quad \textit{where } \theta_i = \frac{\omega(Q_i) - 2n + d}{n - 2d + 3}; \tag{5.3}$$

(v) *we have*

$$\|Q_{i+1}\|^{\tau} \leqslant \|Q_i\| \quad \textit{where } \tau = \frac{\widehat{\omega}\big(\widehat{\omega} - n - d + 3\big)}{\omega_n(\xi)\big(\omega_n(\xi) - n - d + 3\big)}, \tag{5.4}$$

with the convention $\tau = 0$ if $\omega_n(\xi) = \infty$.

The above proposition is essentially a consequence of Lemma 5.3 below. Assertion (iii) ensures that the polynomials Q_i are quite good approximations, and they can be used to compute the exponent of best approximation $\omega_n(\xi)$. Estimate (5.4) is the analog of the second inequality of (3.2) but is way more difficult to prove. The main reason behind this difficulty is that there may be many polynomials $P \in \mathbb{Z}[X]_{\leqslant n}$ with $\|Q_i\| < \|P\| < \|Q_{i+1}\|$ and $|P(\xi)| < Q_i(\xi)$.

In order to prove Proposition 5.1, we need the two technical lemmas below. Essentially, they will be used to prove that the factors of P_i of small degree are bad approximations. This will lead to the existence of a factor of large degree which is necessarily a rather good approximation.

Lemma 5.2. — *Suppose that (5.1) holds. Then, there exists a constant $c \in (0,1)$ depending only on ξ and n such that for any non-zero polynomial $R \in \mathbb{Z}[X]_{\leqslant n-d+1}$ we have*

$$|R(\xi)| \geqslant c\|R\|^{-(n+\deg(R)-1)} \geqslant c\|R\|^{-(2n-d)}. \tag{5.5}$$

In particular (5.5) holds for any $R \in \mathbb{Z}[X]_{\leqslant d-2}$.

Proof. — If R is constant we have $|R(\xi)| = \|R\|$ and the result is trivial. Now, suppose that R is irreducible and not constant. We adapt the arguments of Davenport and Schmidt [10, §5–6]. Set $H = e^{-n}\|R\|$. By definition of $\widehat{\omega}_n(\xi)$ and $\widehat{\omega}$, if H is sufficiently large, there exists a non-zero $P \in \mathbb{Z}[X]_{\leqslant n}$ such that

$$\|P\| \leqslant H \quad \text{and} \quad |P(\xi)| \leqslant H^{-\widehat{\omega}}.$$

By (2.1), the (irreducible) polynomial R is not a factor of P, they are thus coprime polynomials. Their resultant is a non-zero integer, and using Lemma 4.1, we obtain

$$\begin{aligned} 1 &\ll \|P\|^{\deg(R)-1}\|R\|^n|P(\xi)| + \|P\|^{\deg(R)}\|R\|^{n-1}|R(\xi)| \\ &\ll H^{n+\deg(R)-1-\widehat{\omega}} + H^{n+\deg(R)-1}|R(\xi)|. \end{aligned}$$

Since $\widehat{\omega} > 2n-d$ and $\deg(R) \leqslant n-d+1$, the first term tends to 0 as H tends to infinity. Hence $1 \ll H^{n+\deg(R)-1}|R(\xi)|$, which implies (5.5).

If R is not irreducible, we write $R = \prod_{i=1}^{s} R_i$ with integer $s \geqslant 1$ and $R_1, \ldots, R_s \in \mathbb{Z}[X]$ irreducible of degree $\leqslant \deg(R)$ (possibly constant). Combining $\|R\| \asymp \prod_{i=1}^{s} \|R_i\|$ together with (5.5) applied to the irreducible factors R_i, we find

$$|R(\xi)| = \prod_{i=1}^{s} |R_i(\xi)| \gg \prod_{i=1}^{s} \|R_i\|^{-(n+\deg(R)-1)} \gg \|R\|^{-(n+\deg(R)-1)}.$$

Finally, the last assertion comes from the fact that $d-1 \leqslant n+d-1$ (since $d \leqslant 1+n/2$). □

Lemma 5.3. — *Suppose that* (5.1) *holds. There exist $i_0 \geqslant 0$ and a constant $c > 0$ such that for each $i \geqslant i_0$ the polynomial P_i has a unique irreducible factor $\widetilde{P}_i \in \mathbb{Z}[X]$ of degree $\geqslant n-d+2$ and positive leading coefficient. It satisfies*

$$|P_i(\xi)|\|P_i\|^{n+d-3} \geqslant c|\widetilde{P}_i(\xi)|\|\widetilde{P}_i\|^{n+d-3}, \tag{5.6}$$

moreover $\big(\|\widetilde{P}_i\|\big)_{i\geqslant i_0}$ tends to infinity and as i tends to infinity. For each i large enough, we have $\|\widetilde{P}_i\| > 1$, and writing $|\widetilde{P}_i(\xi)| = \|\widetilde{P}_i\|^{-\omega(\widetilde{P}_i)}$, we furthermore have

$$\omega_n(\xi) = \limsup_{i\to\infty} \omega(\widetilde{P}_i) \quad \text{and} \quad \liminf_{i\to\infty} \omega(\widetilde{P}_i) \geqslant \widehat{\omega}_n(\xi). \tag{5.7}$$

Proof. — First, note that since $d < 1+n/2$, if we decompose P_i as a product of irreducibles, there is at most one factor of degree $\geqslant n-d+2$. Fix $i \geqslant 0$ large enough so that $\omega(P_i) \geqslant \widehat{\omega}$, and write

$$P := P_i = \prod_{k=1}^{s} R_k,$$

where $R_1, \ldots, R_s \in \mathbb{Z}[X]$ are irreducible polynomials (and s is a positive integer). Suppose that $\deg(R_k) \leqslant n-d+1$ for each $k = 1, \ldots, s$. Then, by Lemma 5.2 together with $\|P\| \asymp \prod_k \|R_k\|$, we find

$$\|P\|^{-\widehat{\omega}} \geqslant |P(\xi)| = \prod_{k=1}^{s} |R_k(\xi)| \gg \prod_{k=1}^{s} \|R_k\|^{-(2n-d)} \asymp \|P\|^{-(2n-d)}.$$

This is impossible if i is sufficiently large since $\widehat{\omega} > 2n-d$. Therefore, if i is large enough, one of the factors R_k has degree at least $n-d+2$. Without loss of generality, we may suppose that it is $R := R_1$. Write $S := \prod_{k=2}^{s} R_k$, so that $P = RS$. We have $\deg(S) \leqslant d-2$, and (5.5) of Lemma 5.2 yields

$$|S(\xi)| \gg \|S\|^{-(n+d-3)}.$$

Together with $\|P\| \asymp \|R\|\|S\|$, this leads to

$$|P(\xi)| = |R(\xi)||S(\xi)| \gg |R(\xi)|\|S\|^{-(n+d-3)} \asymp |R(\xi)|\|R\|^{n+d-3}\|P\|^{-(n+d-3)},$$

and (5.6) follows easily by setting $\widetilde{P}_i := R$. The rest of the proof is based only on (5.6) and the inequality $\|\widetilde{P}_i\| \ll \|P_i\|$. Note that $|P_i(\xi)|\|P_i\|^{n+d-3} \ll \|P_i\|^{n+d-3-\widehat{\omega}}$ tends to 0 as i tends to infinity (using $d < 1+n/2$ together with $\widehat{\omega} > 2n-d$). We deduce that

$|\widetilde{P}_i(\xi)|\|\widetilde{P}_i\|^{n+d-3}$ also tends to 0 as i tends to infinity, which is possible only if $\|\widetilde{P}_i\|$ tends to infinity. In particular, if i is large enough we must have $\|\widetilde{P}_i\| > 1$. Writing $|\widetilde{P}_i(\xi)| = \|\widetilde{P}_i\|^{-\omega(\widetilde{P}_i)}$, we also have $\omega(\widetilde{P}_i) > n + d - 3$. Now, using $\|\widetilde{P}_i\| \ll \|P_i\|$, and taking the logarithms of the two sides of (5.6), we get

$$\begin{aligned}\big(\omega(P_i) - (n + d - 3)\big) \log \|P_i\| &\leqslant \big(\omega(\widetilde{P}_i) - (n + d - 3)\big) \log \|\widetilde{P}_i\| + \mathcal{O}(1)\\ &\leqslant \big(\omega(\widetilde{P}_i) - (n + d - 3)\big)\big(\log \|P_i\| + \mathcal{O}(1)\big) + \mathcal{O}(1).\end{aligned}$$

By dividing by $\log \|P_i\|$ and by simplifying, we deduce that $\omega(\widetilde{P}_i) \geqslant \omega(P_i)\big(1 - o(1)\big)$ and (5.7) follows easily from (3.3). □

Proof of Proposition 5.1. — Let $i_0 \geqslant 0$ and $(\widetilde{P}_i)_{i\geqslant i_0}$ given by Lemma 5.3. Let $(Q_i)_{i\geqslant 0}$ be the (infinite) sequence of factors $(\widetilde{P}_j)_{j\geqslant i_0}$ reordered by increasing height, without repetition. By Lemma 5.3, we may assume i_0 large enough so that $\|Q_i\| > 1$ for each i, as well as $|Q_i(\xi)| \leqslant \|Q_i\|^{-\widehat{\omega}}$. This sequence clearly satisfies the first assertions (i) to (iii), the third one coming from (5.7) together with (3.3).

Now, let $i \geqslant 0$ and let $j \geqslant i_0$ be an index such that Q_i divides P_j. Since we have $\|Q_i\| \ll \|P_j\|$ by Gelfond's lemma, the index j tends to infinity as i tends to infinity. Then, estimate (5.6) can be rewritten as

$$|P_j(\xi)|^{-1}\|P_j\|^{-n-d+3} \ll |Q_i(\xi)|^{-1}\|Q_i\|^{-n-d+3} = \|Q_i\|^{\omega(Q_i)-n-d+3}. \tag{5.8}$$

Using $|P_j(\xi)|^{-1} \gg \|P_j\|^{\widehat{\omega}}$ and $\widehat{\omega} > 2n - d$, we get, for each large enough i,

$$\|P_j\|^{n-2d+3} \leqslant \|Q_i\|^{\omega(Q_i)-n-d+3},$$

which is equivalent to (5.3). So, assertion (iv) holds assuming i_0 large enough.

It remains to prove assertion (v). Note that this is trivial if $\omega_n(\xi) = \infty$. Let us assume that $\omega_n(\xi) < \infty$ and fix a small $\varepsilon > 0$ to be chosen later. For each pair (i, j) as above with $j \geqslant i_0$ large enough as a function of ε, we have $\omega(P_j) > \widehat{\omega}_n(\xi) - \varepsilon/2$ and $\omega(Q_i) < \omega_n(\xi) + \varepsilon/2$, and thus (5.8) yields

$$\|P_j\|^{\widehat{\omega}_n(\xi)-\varepsilon-n-d+3} \leqslant \|Q_i\|^{\omega_n(\xi)+\varepsilon-n-d+3},$$

for each $i \geqslant 0$ and each $j \geqslant i_0$ such that Q_i divides P_j. We define k as the largest index such that

$$\|P_k\| \leqslant \|Q_i\|^{\theta(\varepsilon)}, \quad \text{where } \theta(\varepsilon) = \frac{\omega_n(\xi) + \varepsilon - n - d + 3}{\widehat{\omega}_n(\xi) - \varepsilon - n - d + 3}.$$

Since $\|P_j\| \leqslant \|Q_i\|^{\theta(\varepsilon)}$, by maximality of k we have $i_0 \leqslant j \leqslant k$. Let ℓ be such that Q_ℓ divides P_{k+1}. We find

$$\|P_k\| \leqslant \|Q_i\|^{\theta(\varepsilon)} < \|P_{k+1}\| \leqslant \|Q_\ell\|^{\theta(\varepsilon)},$$

and therefore $\ell \geqslant i + 1$. On the other hand, since by Gelfond's lemma we have $\|Q_\ell\| \ll \|P_{k+1}\|$, we deduce from (3.2) that

$$\|Q_{i+1}\| \leqslant \|Q_\ell\| \ll \|P_{k+1}\| \ll \|P_k\|^{\omega_n(\xi)/\widehat{\omega}} \leqslant \|Q_i\|^{\omega_n(\xi)\theta(\varepsilon)/\widehat{\omega}}.$$

We now choose $\varepsilon > 0$ small enough so that

$$\theta(\varepsilon) < \frac{\omega_n(\xi) - n - d + 3}{\widehat{\omega} - n - d + 3}.$$

This is possible since $\widehat{\omega} < \widehat{\omega}_n(\xi)$, and it yields (5.4) for each $i \geqslant 0$, assuming that i_0 is large enough. □

6. On the dimension of some polynomial subspaces

We start by introducing some families of vector spaces spanned by polynomials, and we study their dimensions.

Definition 6.1. — Let $k \geqslant n$ be an integer and let $\mathcal{A}$ be a subset of $\mathbb{R}[X]_{\leqslant n}$. We define

$$\begin{aligned} \mathcal{B}_k(\mathcal{A}) &= \{Q, XQ, \dots, X^{k-\deg(Q)}Q\,;\, Q \in \mathcal{A} \smallsetminus \{0\}\} \subseteq \mathbb{R}[X]_{\leqslant k}, \\ V_k(\mathcal{A}) &= \langle \mathcal{B}_k(\mathcal{A}) \rangle_{\mathbb{R}}\,, \\ g_{\mathcal{A}}(k) &= \dim V_k(\mathcal{A}). \end{aligned}$$

The spaces $V_k(\mathcal{A})$ play the role of the spaces $\mathcal{U}^k(\mathcal{A})$ in [17, §3] (for simultaneous approximation). We obtain analog properties. Note that if $\mathcal{A}$ contains at least one non-zero polynomial, then

$$V_n(\mathcal{A}) \subsetneq V_{n+1}(\mathcal{A}) \subsetneq \cdots . \tag{6.1}$$

The goal of this section is to prove the following result. We could not find a reference for the proposition below.

Proposition 6.2. — *Let k be an integer with $0 \leqslant k \leqslant n$, and let $\mathcal{A}$ be a set of $k+1$ linearly polynomials of $\mathbb{R}[X]_{\leqslant n}$. Suppose that the* gcd *of the elements of $\mathcal{A}$ is* 1 *(in other words, the ideal spanned by $\mathcal{A}$ is $\mathbb{R}[X]$). Then*

$$V_{2n-k}(\mathcal{A}) = \mathbb{R}[X]_{\leqslant 2n-k}. \tag{6.2}$$

The case $k = 1$ is a classical result (it is implied by the fact that the resultant of two coprime polynomials is non-zero). The proof of Proposition 6.2 is given at the end of the section. Recall that a function $f : \{n, n+1, \dots\} \to \mathbb{R}$ is *concave* if for any $i > n$, it satisfies

$$f(i) - f(i-1) \geqslant f(i+1) - f(i).$$

The next result is a dual version of [17, Prop. 3.1] (where we deal with simultaneous approximation to the successive powers of ξ).

Lemma 6.3. — *Let $\mathcal{A} \neq \{0\}$ be a non-empty subset of $\mathbb{R}[X]_{\leqslant n}$. The function $g_{\mathcal{A}}$ is concave and (strictly) increasing on $\{n, n+1, \dots\}$.*

Proof. — The series of inclusions (6.1) shows that the function $g_{\mathcal{A}}$ is increasing on $\{n, n+1, \dots\}$. For simplicity, we write $V_i = V_i(\mathcal{A})$ and $\mathcal{B}_i = \mathcal{B}_i(\mathcal{A})$ for each $i \geqslant n$. Given an integer $i \geqslant n$ we have $XV_i \subseteq V_{i+1}$, and we set

$$h(i) := \dim \big(V_{i+1}/XV_i\big) = g_{\mathcal{A}}(i+1) - g_{\mathcal{A}}(i).$$

We have to prove that h is decreasing on $\{n,n+1,\cdots\}$. Fix $i \geqslant n+1$ and consider the linear map $\pi : V_i \to V_{i+1}/XV_i$ defined by $\pi(P) = P + XV_i$. Since $\mathcal{B}_i \cup X\mathcal{B}_i = \mathcal{B}_{i+1}$, we have $V_i + XV_i = V_{i+1}$. So π is surjective, and consequently $\operatorname{Im}\pi = V_{i+1}/XV_i$ is isomorphic to $V_i/\ker\pi$. On the other hand, $XV_{i-1} \subseteq V_i \cap XV_i \subseteq \ker\pi$, so XV_{i-1} is subspace of $\ker\pi$. Hence

$$h(i-1) = \dim\big(V_i/XV_{i-1}\big) \geqslant \dim\big(V_i/\ker\pi\big) = \dim\big(V_{i+1}/XV_i\big) = h(i). \qquad \square$$

Lemma 6.4. — *Let $P, Q \in \mathbb{R}[X]_{\leqslant n}$ be two coprime polynomials. Then, we have*

$$\dim V_{n+j}(P,Q) \geqslant 2(j+1),$$

for each $j \in \{0,\dots,n-1\}$. In particular $V_{2n-1}(P,Q) = \mathbb{R}[X]_{\leqslant 2n-1}$.

Proof. — Let p (resp. q) denote the degree of P (resp. of Q). There exist $\alpha,\beta \in \mathbb{R}$ such that the polynomial $\widetilde{P} := P(X)(X-\alpha)^{n-p}$ and $\widetilde{Q} := Q(X)(X-\beta)^{n-q}$ are coprime (and of degree exactly n). Fix $j \in \{0,\dots,n-1\}$. Since $\widetilde{P}$ and $\widetilde{Q}$ are coprime and $j < n$, the linear map

$$\begin{aligned} \mathbb{R}[X]_{\leqslant j} \times \mathbb{R}[X]_{\leqslant j} &\longrightarrow \mathbb{R}[X]_{\leqslant n+j} \\ (R,S) &\longmapsto R\widetilde{P} + S\widetilde{Q} \end{aligned}$$

is injective, so its image $V_{n+j}(\widetilde{P},\widetilde{Q}) \subseteq V_{n+j}(P,Q)$ has dimension $2(j+1)$. $\square$

Proof of Proposition 6.2. — For simplicity, we write $g = g_{\mathcal{A}}$. Recall that $\mathcal{A}$ has cardinality $k+1$, so that $g(n) \geqslant \operatorname{card}(\mathcal{A}) = k+1$. If $k = n$, then (6.2) is automatic (since in that case $\mathcal{A}$ contains a basis of $\mathbb{R}[X]_{\leqslant n}$). So, we may assume that $k < n$. We first prove that for each sufficiently large m, we have

$$V_m(\mathcal{A}) = \mathbb{R}[X]_{\leqslant m}. \tag{6.3}$$

Indeed, since the ideal spanned by $\mathcal{A}$ is $\mathbb{R}[X]$, there exists an integer $\ell \geqslant n$ such that $1 \in V_\ell(\mathcal{A})$. Let P be a non-zero element in $\mathcal{A}$ of degree d, and set $m = \ell + d$. Then $V_m(\mathcal{A})$ contains $\mathbb{R}[X]_{\leqslant d}$, as well as the polynomials $P, XP, \dots, X^\ell P$. We easily deduce (6.3).

By contradiction, suppose that (6.2) does not hold, i.e.,

$$g(2n-k) \leqslant 2n-k. \tag{6.4}$$

We distinguish two cases. Suppose first that $g(2n-k) - g(2n-k-1) \geqslant 2$. By concavity, then $g(j) - g(j-1) \geqslant 2$ for each j with $n < j \leqslant 2n-k$, and we deduce that

$$g(2n-k) \geqslant g(n) + 2(n-k) \geqslant k+1+2(n-k) = 2n-k+1,$$

since $g(n) \geqslant \operatorname{card}(\mathcal{A}) = k+1$. This contradicts (6.4), so $g(2n-k) - g(2n-k-1) \leqslant 1$. Since the function g is increasing and concave, it is linear with slope 1 on

$$\{2n-k, 2n-k+1, \dots\}.$$

Choosing $m > 2n-k$ such that (6.3) holds, we obtain by (6.4)

$$m+1 = g(m) = g(2n-k) + m - (2n-k) \leqslant m,$$

a contradiction. Hence (6.2) holds. □

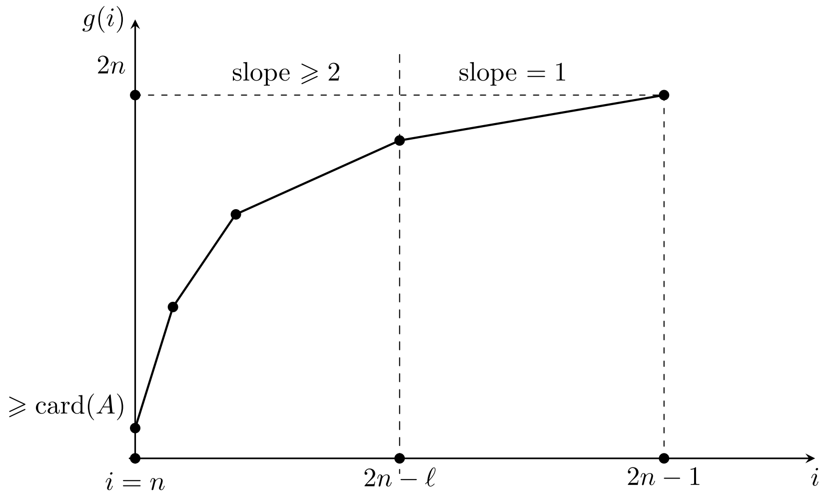

Figure 1. Graph of the piecewise linear function interpolating the values $g(i) = \dim V_i(\mathcal{A})$ at integers $i \in \{n, \ldots, 2n-1\}$.

7. Proof of Theorem 1.1 (case $d = 2$)

In this section, we deal with the case $d = 2$ to prove Theorem 1.1, namely that $\widehat{\omega}_3(\xi) \leqslant 2 + \sqrt{5} = 4.23\cdots$ and $\widehat{\omega}_n(\xi) \leqslant 2n-2$ for each $n \geqslant 4$. The estimate $\widehat{\omega}_n(\xi) \leqslant 2n-2$ was already known for $n \geqslant 10$, however for $n = 4, \ldots, 9$ it is a new result. For $n = 3$, our bound improves on the bound $\widehat{\omega}_3(\xi) \leqslant 3 + \sqrt{2} = 4.41\cdots$ due to Bugeaud and Schleischitz [8]. Moreover, our proof does not require Marnat-Moshchevitin's inequality [13].

Proof of Theorem 1.1. — Suppose that $\widehat{\omega}_n(\xi) > 2n-2$, and fix a real number $\widehat{\omega}$ such that

$$\widehat{\omega}_n(\xi) > \widehat{\omega} > 2n-2.$$

Let $(P_i)_{i\geqslant 0}$ be a sequence of minimal polynomials associated to n and ξ as in Section 3. According to Lemma 5.3 (with $d = 2$) there exists an index $i_0 \geqslant 0$ such that P_i has degree n and is irreducible for each $i \geqslant i_0$. Consequently, up to a finite number of terms, the sequence $(P_i)_{i\geqslant 0}$ coincides with the sequence $(Q_i)_{i\geqslant 0}$ of Proposition 5.1. Let I denotes the set of indices $i \geqslant i_0 + 1$ such that P_{i-1}, P_i and P_{i+1} are linearly independent. By Lemmas 3.2 and 3.4, the set I is infinite, and for any consecutive $i < j$ in I, we have

$$\|P_{i+1}\||P_i(\xi)| \asymp \|P_j\||P_{j-1}(\xi)|.$$

Furthermore, the irreducible polynomials P_i and P_{i+1} are also coprime since $\|P_i\| < \|P_{i+1}\|$ and $|P_i(\xi)| > |P_{i+1}(\xi)|$. Lemma 4.1 yields

$$1 \leqslant |\operatorname{Res}(P_i, P_{i+1})| \ll \|P_i\|^{n-1}\|P_{i+1}\|^n |P_i(\xi)| \ll \|P_i\|^{n-1}\|P_j\|^n |P_{j-1}(\xi)| \ll \|P_i\|^{n-1}\|P_j\|^{n-\widehat{\omega}}.$$

We deduce that

$$\|P_j\| \leqslant \|P_i\|^{\theta} \quad \text{where } \theta = \frac{n-1}{\widehat{\omega}-n}. \tag{7.1}$$

Let $h < i < j$ be consecutive indices in I. We have the following configuration

$$\langle P_h, P_{h+1}\rangle_{\mathbb{R}} = \langle P_{i-1}, P_i\rangle_{\mathbb{R}} \neq \langle P_i, P_{i+1}\rangle_{\mathbb{R}},$$

so P_h, P_{h+1}, P_{i+1} are linearly independent. Proposition 6.2 combined with Lemma 6.4 implies that

$$\Big(\mathbb{R}[X]_{\leqslant n-2}P_h \oplus \mathbb{R}[X]_{\leqslant n-2}P_{h+1}\Big) + \mathbb{R}[X]_{\leqslant n-2}P_{i+1} = \mathbb{R}[X]_{\leqslant 2n-2}.$$

Choose $k \in \{0, \dots, n-2\}$ such that $\big(P_h, \dots, X^{n-2}P_h, P_{h+1}, \dots, X^{n-2}P_{h+1}, X^k P_{i+1}\big)$ is a basis of $\mathbb{R}[X]_{\leqslant 2n-2}$. We denote by M the matrix of this basis expressed in the canonical basis $(1, X, \dots, X^{2n-2})$. Estimating $\det(M)$ as in the proof of Lemma 4.1 (in other words, for $\ell = 2, \dots, 2n-2$, we add to the first row of M the ℓ-th row multiplied by $\xi^{\ell-1}$), we get the estimates

$$1 \leqslant \big|\det\big(M\big)\big| \ll |P_h(\xi)|\|P_h\|^{n-2}\|P_{h+1}\|^{n-1}\|P_{i+1}\|.$$

Now, since $\|P_{h+1}\|^{n-1}|P_h(\xi)| \asymp \|P_{h+1}\|^{n-2}\|P_i\||P_{i-1}(\xi)| \ll \|P_i\|^{n-1-\widehat{\omega}}$, we deduce that

$$\|P_i\|^{\widehat{\omega}-n+1} \ll \|P_h\|^{n-2}\|P_j\|. \tag{7.2}$$

For consecutive $i < j$ in I, define $\tau_i \in (0,1)$ by

$$\|P_i\| = \|P_j\|^{\tau_i}$$

and set $\tau = \limsup_{i\in I, i\to\infty} \tau_i \in [0,1]$. Let $h < i < j$ be consecutive indices in I as previously. By (7.2), we obtain

$$\widehat{\omega} - n + 1 \leqslant (n-2)\tau_h + \frac{1}{\tau_i} + o(1) \leqslant (n-2)\tau + \frac{1}{\tau_i} + o(1).$$

We infer that

$$p(\tau) \geqslant 0, \quad \text{where } p(t) = (n-2)t^2 - (\widehat{\omega} - n + 1)t + 1. \tag{7.3}$$

Note that

$$p(0) = 1, \qquad p\Big(\frac{1}{n-2}\Big) = \frac{2n-2-\widehat{\omega}}{n-2} < 0 \quad \text{and} \quad p(1) = 2n-2-\widehat{\omega} < 0.$$

We deduce that p has one root $\alpha \in (0, 1/(n-2))$ and one root larger than 1. Since $\tau \in [0,1]$ and $p(\tau) \geqslant 0$, we obtain $\tau \leqslant \alpha$. Combined with the estimate $\|P_i\| = \|P_j\|^{\tau_i} \ll \|P_i\|^{\theta\tau_i}$ valid for any $i \in I$ (this is a consequence of (7.1)), this leads to

$$1 \leqslant \theta\tau \leqslant \theta\alpha < \frac{n-1}{(n-2)^2}. \tag{7.4}$$

We easily check that this is impossible when $n \geqslant 4$ (the right-hand side is strictly less than 1), thus $\widehat{\omega}_n(\xi) \leqslant 2n-2$ for each $n \geqslant 4$.

We now deal with the case $n=3$. Suppose by contradiction that $\widehat{\omega}_3(\xi) > 2+\sqrt{5}$ and choose $\widehat{\omega}$ such that

$$\widehat{\omega}_3(\xi) > \widehat{\omega} > 2+\sqrt{5}.$$

The polynomial p from (7.3) becomes $p(t) = t^2 - (\widehat{\omega}-2)t + 1$. Denote by α its smallest root, and by $\beta = (\sqrt{5}-1)/2$ the smallest root of the polynomial $t^2 - \sqrt{5}t + 1$. We find

$$0 = \beta^2 - \sqrt{5}\beta + 1 > \beta^2 - (\widehat{\omega}-2)\beta + 1 = p(\beta),$$

hence $\alpha < \beta$. Combined with $\theta = 2/(\widehat{\omega}-3) < 1/\beta$, this implies that $\theta\alpha < 1$, which contradicts (7.4). It follows that $\widehat{\omega}_3(\xi) \leqslant 2+\sqrt{5}$. □

8. Multilinear algebra and height of polynomial subspaces

This section is divided into two parts. We introduce and study a quantity $\mathcal{D}_\xi(V)$ associated to a subspace $V \subseteq \mathbb{R}^m$ defined over $\mathbb{Q}$ in Section 8.2. Intuitively, $\mathcal{D}_\xi(V)$ is small if V is spanned by good polynomials approximations of $\mathbb{Z}[X]$ (i.e., small when evaluated at ξ). This will be a key-point for estimating the height of the polynomials Q_i of Section 5. In order to define $\mathcal{D}_\xi$, we need some tools of multilinear algebra that we recall in Section 8.1. In the appendix we give another interpretation of $\mathcal{D}_\xi$ in term of twisted heights.

8.1. Multilinear algebra and Hodge duality.

— For each integer m, we view $\mathbb{R}^{m+1}$ as an Euclidean space for the usual scalar product $(\cdot \mid \cdot)$, and we denote by $\|\cdot\|_2$ the associated Euclidean norm. For each $k = 1, \cdots, m+1$, we identify $\bigwedge^k \mathbb{R}^{m+1}$ with $\mathbb{R}^N$, where $N = \binom{m+1}{k}$, via an ordering of the Plücker coordinates, and we denote by $\|\boldsymbol{y}\|_2$ the norm of a point $\boldsymbol{y} \in \bigwedge^k \mathbb{R}^{m+1} \cong \mathbb{R}^N$. This is independent of the ordering. Let V be a k-dimensional subspace of $\mathbb{R}^{m+1}$ defined over $\mathbb{Q}$, i.e., such that $\langle V \cap \mathbb{Q}^{m+1} \rangle_{\mathbb{R}} = V$. Its (standard) *height* $\mathcal{H}(V)$ is the covolume of the lattice $V \cap \mathbb{Z}^{m+1}$ inside V (with the convention that $\mathcal{H}(V) = 1$ if $V = \{0\}$). Explicitly, we have

$$\mathcal{H}(V) := \|\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_k\|_2\,,$$

for any $\mathbb{Z}$-basis $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)$ of the lattice $V \cap \mathbb{Z}^{m+1}$. Schmidt established the very nice inequality

$$\mathcal{H}(U \cap V)\mathcal{H}(U + V) \leqslant \mathcal{H}(U)\mathcal{H}(V),$$

valid for any subspaces U, V of $\mathbb{R}^{m+1}$ defined over $\mathbb{Q}$ (see [27, Ch. I, Lem. 8A]). In this paper, we need to work with a "twisted" height and the corresponding version of Schmidt's inequality (obtained by following Schmidt's original arguments).

Let $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{m+1})$ denotes the canonical basis of $\mathbb{R}^{m+1}$, and let k be an integer with $0 \leqslant k \leqslant m+1$. The Hodge star operator

$$* : \textstyle\bigwedge^k \mathbb{R}^{m+1} \xrightarrow{\sim} \bigwedge^{m+1-k} \mathbb{R}^{m+1}$$

is defined by

$$*(\boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_k}) = \varepsilon_{i_1,\ldots,i_k} \boldsymbol{e}_{j_1} \wedge \cdots \wedge \boldsymbol{e}_{j_{m+1-k}}$$

for any indices $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_{m+1-k}$ forming a partition of $\{1, \ldots, m{+}1\}$, where $\varepsilon_{i_1,\ldots,i_k}$ denotes the signature of the substitution $(1, \ldots, m{+}1) \mapsto (j_1, \ldots, j_{m+1-k}, i_1, \ldots, i_k)$. Given $\boldsymbol{X} \in \bigwedge^k \mathbb{R}^{m+1}$, the point $*\boldsymbol{X}$ is called the *Hodge dual* of $\boldsymbol{X}$.

We now collect some useful properties of the Hodge star operator, see for example [11], [2] and [7, §3] for more details. First,

$$\|{*\boldsymbol{X}}\|_2 = \|\boldsymbol{X}\|_2 \quad \text{and} \quad *\,(*\boldsymbol{X}) = (-1)^{k(m+1-k)}\boldsymbol{X}$$

for any $\boldsymbol{X} \in \bigwedge^k \mathbb{R}^{m+1}$. If $\boldsymbol{X} = \boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_k$ is a system of Plücker coordinates of a k-dimensional subspace $V \subseteq \mathbb{R}^{m+1}$, then $*\boldsymbol{X}$ is a system of Plücker coordinates of its orthogonal $V^\perp$. This implies the classical identity

$$\mathcal{H}(V) = \mathcal{H}(V^\perp).$$

If $k > 0$, then given $\boldsymbol{y} \in \mathbb{R}^{m+1}$ and a multivector $\boldsymbol{X} \in \bigwedge^k \mathbb{R}^{m+1}$, the point

$$\boldsymbol{y} \lrcorner \boldsymbol{X} = *\big(\boldsymbol{y} \wedge (*\boldsymbol{X})\big) \in \textstyle\bigwedge^{k-1} \mathbb{R}^{m+1}$$

is called the *contraction* of $\boldsymbol{X}$ by $\boldsymbol{y}$ (see [7, Lem. 2]). Explicitly, if $\boldsymbol{X} = \boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_k$ is a decomposable multivector, then

$$\boldsymbol{y} \lrcorner \boldsymbol{X} = \sum_{i=1}^{k} (-1)^{k-i}\, (\boldsymbol{x}_i \mid \boldsymbol{y})\; \boldsymbol{x}_1 \wedge \cdots \wedge \widehat{\boldsymbol{x}_i} \wedge \cdots \wedge \boldsymbol{x}_k, \tag{8.1}$$

where the hat on $\boldsymbol{x}_i$ means that this term is omitted from the wedge product (see [7, Eq. (3.3)]). In particular, if $k = 1$ and $\boldsymbol{X} = \boldsymbol{x} \in \mathbb{R}^{m+1}$, we simply have

$$\boldsymbol{y} \lrcorner \boldsymbol{x} = (\boldsymbol{y} \mid \boldsymbol{x})\,. \tag{8.2}$$

8.2. Schmidt's inequality. — Let m be a non-negative integer and set $\Xi_m = (1, \xi, \xi^2, \ldots, \xi^m)$. We keep the notation of Section 8.1.

Definition 8.1. — Let V be a k-dimensional subspace of $\mathbb{R}^{m+1}$ defined over $\mathbb{Q}$, with $k \geqslant 1$, and let $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)$ be a $\mathbb{Z}$-basis of the lattice $V \cap \mathbb{Z}^{m+1}$. We set

$$\mathcal{D}_\xi(V) = \|\Xi_m \lrcorner \boldsymbol{X}\|_2 = \|\Xi_m \wedge (*\boldsymbol{X})\|_2\,,$$

where $\boldsymbol{X} = \boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_k$. By convention, we set $\mathcal{D}_\xi(\{0\}) = 0$. Following the notation of [17, §11], we also set

$$L_\xi(V) = \|\Xi_m \wedge \boldsymbol{X}\|_2\,,$$

with the convention that $L_\xi(\{0\}) = \|\Xi_m\|_2$.

Remark. — If $(\boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_k)$ is another $\mathbb{Z}$-basis of $V \cap \mathbb{Z}^{m+1}$, then $\boldsymbol{x}'_1 \wedge \cdots \wedge \boldsymbol{x}'_k = \pm\boldsymbol{X}$. Consequently, $\mathcal{D}_\xi(V)$ and $L_\xi(V)$ do not depend on the choice of the basis. In [17], we considered $L_\xi(V)$ for spaces V spanned by good simultaneous approximations. The function $\mathcal{D}_\xi$ is connected to the quantity introduced in [18, Def. 7.1] (where we work in a number field K instead of $\mathbb{Q}$). Note that $\mathcal{D}_\xi(V) = 0$ if and only if $\Xi_m \in V^\perp$. Since ξ is transcendental, this is only possible when $V = \{0\}$. We have

$$\mathcal{D}_\xi(\mathbb{R}^{m+1}) = \|\Xi_m\|_2 \asymp 1,$$

where the implicit constants depend on ξ and m only. Moreover, (8.2) implies that

$$\mathcal{D}_\xi\big(\langle \boldsymbol{x}\rangle_{\mathbb{R}}\big) = |\,(\Xi_m \mid \boldsymbol{x})\,| \tag{8.3}$$

for any primitive integer point $\boldsymbol{x}\in\mathbb{Z}^{m+1}$. Equation (8.1) yields the explicit formula

$$\mathcal{D}_\xi(V) = \Big\| \sum_{i=1}^{k} (-1)^{k-i}\,(\boldsymbol{x}_i \mid \Xi_m)\; \boldsymbol{x}_1\wedge\cdots\wedge\widehat{\boldsymbol{x}}_i\wedge\cdots\wedge\boldsymbol{x}_k \Big\|_2. \tag{8.4}$$

On the other hand, if $(\boldsymbol{y}_1,\dots,\boldsymbol{y}_{m+1-k})$ is a $\mathbb{Z}$-basis of $V^\perp\cap\mathbb{Z}^{m+1}$, then $*\boldsymbol{X} = \pm\boldsymbol{y}_1\wedge\cdots\wedge\boldsymbol{y}_{m+1-k}$. Consequently, we can also write

$$\mathcal{D}_\xi(V) = \big\|\Xi_m\wedge\boldsymbol{y}_1\wedge\cdots\wedge\boldsymbol{y}_{m+1-k}\big\|_2 = L_\xi(V^\perp). \tag{8.5}$$

Both formulas for $\mathcal{D}_\xi(V)$ will be useful.

Proposition 8.2 (Schmidt's inequality). — *For any subspaces U,V of $\mathbb{R}^{m+1}$ defined over $\mathbb{Q}$, we have*

$$\mathcal{D}_\xi(U\cap V)\mathcal{D}_\xi(U+V) \leqslant \mathcal{D}_\xi(U)\mathcal{D}_\xi(V) \tag{8.6}$$

and

$$L_\xi(U\cap V)L_\xi(U+V) \leqslant L_\xi(U)L_\xi(V). \tag{8.7}$$

Proof. — In view of (8.5), we only need to prove that (8.7) holds for any pair (U,V) as in the statement of the proposition (for then, it suffices to apply (8.7) to the pair $(U^\perp,V^\perp)$). We follow Schmidt's arguments [27, Ch. I, Lem. 8A]. For any pure products $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}\in\bigwedge\mathbb{R}^{m+1}$, we have

$$\|\boldsymbol{X}\|_2\,\|\boldsymbol{X}\wedge\boldsymbol{Y}\wedge\boldsymbol{Z}\|_2 \leqslant \|\boldsymbol{X}\wedge\boldsymbol{Y}\|_2\,\|\boldsymbol{X}\wedge\boldsymbol{Z}\|_2\,. \tag{8.8}$$

Let U,V be subspaces of $\mathbb{R}^{m+1}$ defined over $\mathbb{Q}$. If $U=\{0\}$ or $V=\{0\}$, then (8.7) is trivial, so we may assume that U and V have dimension $\geqslant 1$. Let $\boldsymbol{x}_1,\dots,\boldsymbol{x}_r$ be a $\mathbb{Z}$-basis of $U\cap V\cap\mathbb{Z}^{m+1}$, which we complete to a $\mathbb{Z}$-basis $\boldsymbol{x}_1,\dots,\boldsymbol{x}_r,\boldsymbol{y}_1,\dots,\boldsymbol{y}_s$ of $U\cap\mathbb{Z}^{m+1}$ (resp. $\boldsymbol{x}_1,\dots,\boldsymbol{x}_r,\boldsymbol{z}_1,\dots,\boldsymbol{z}_t$ of $V\cap\mathbb{Z}^{m+1}$). Set

$$\boldsymbol{X} = \Xi_m\wedge\boldsymbol{x}_1\wedge\dots\boldsymbol{x}_r,\quad \boldsymbol{Y} = \boldsymbol{y}_1\wedge\cdots\wedge\boldsymbol{y}_s\quad\text{and}\quad \boldsymbol{Z} = \boldsymbol{z}_1\wedge\cdots\wedge\boldsymbol{z}_t.$$

We get (8.7) by applying (8.8) to the above pure products. □

We identify $\mathbb{R}[X]_{\leqslant m}$ to $\mathbb{R}^{m+1}$ and $\mathbb{R}^{m+1}$ to the space of $(m+1)\times 1$ column matrices with real coefficients via the isomorphism

$$\sum_{k=0}^{m} a_k X^k \longmapsto (a_0,\dots,a_m)\quad\text{and}\quad (a_0,\dots,a_m)\mapsto \begin{pmatrix} a_0\\ \vdots\\ a_m\end{pmatrix}. \tag{8.9}$$

Then, for any $P\in\mathbb{R}[X]_{\leqslant m}$, we have $P(\xi) = (\boldsymbol{z}\mid\Xi_m)$, where $\boldsymbol{z}\in\mathbb{R}^{m+1}$ corresponds to P. In particular, if $P\in\mathbb{Z}[X]_{\leqslant m}$ is primitive, then (8.3) may be rewritten as

$$\mathcal{D}_\xi\big(\langle P\rangle_{\mathbb{R}}\big) = |P(\xi)|. \tag{8.10}$$

We will repeatedly use the following "twisted" dual version of [17, Lem. 2.1].

Lemma 8.3. — *There is a positive constant c, which only depends on n and ξ, with the following property. For any linearly independent polynomials $P_1,\dots,P_k\in\mathbb{Z}[X]_{\leqslant m}$ (with $k\geqslant 1$), we have*

$$\mathcal{D}_\xi\big(\langle P_1,\dots,P_k\rangle_{\mathbb{R}}\big)\leqslant c\sum_{i=1}^{k}\frac{|P_i(\xi)|}{\|P_i\|}\prod_{j=1}^{k}\|P_j\|. \tag{8.11}$$

Note that for any $P\in\mathbb{Z}[X]_{\leqslant m}$, Eq. (8.10) implies that $\mathcal{D}_\xi(\langle P\rangle_{\mathbb{R}})\leqslant|P(\xi)|$.

Proof. — Let $Q_1,\dots,Q_k$ be a $\mathbb{Z}$-basis of $V\cap\mathbb{Z}[X]_{\leqslant m}$, where $V=\langle P_1,\dots,P_k\rangle_{\mathbb{R}}$. There exists a non-zero $\alpha\in\mathbb{Z}$ such that

$$P_1\wedge\cdots\wedge P_k=\alpha Q_1\wedge\cdots\wedge Q_k,$$

and so

$$\mathcal{D}_\xi(V)=\|\Xi_m\lrcorner(Q_1\wedge\cdots\wedge Q_k)\|_2\leqslant\|\Xi_m\lrcorner(P_1\wedge\cdots\wedge P_k)\|_2\,.$$

On the other hand, by (8.1) combined with Hadamard's inequality, we obtain

$$\begin{aligned}\|\Xi_m\lrcorner(P_1\wedge\cdots\wedge P_k)\|_2&=\Big\|\sum_{i=1}^{k}(-1)^{k-i}P_i(\xi)\cdot P_1\wedge\cdots\wedge\widehat{P_i}\wedge\cdots\wedge P_k\Big\|_2\\&\ll\sum_{i=1}^{k}|P_i(\xi)|\|P_1\|\cdots\widehat{\|P_i\|}\cdots\|P_k\|\end{aligned}$$

(recall that the norm $\|\cdot\|$ is defined in Section 2). □

9. Subfamilies of polynomials: dimension and height

Let d,n,ξ and $\widehat{\omega}$ be as in Section 5. In particular we have

$$2\leqslant d<1+\frac{n}{2},$$

and we suppose that (5.1) holds, namely

$$\widehat{\omega}_n(\xi)>\widehat{\omega}>2n-d.$$

Let us fix a sequence of minimal polynomials $(P_i)_{i\geqslant 0}$ associated to n and ξ as in Section 3. We denote by $(Q_i)_{i\geqslant 0}$ the sequence of irreducible factors given by Proposition 5.1. In particular, for each $i\geqslant 0$ we have

$$|Q_i(\xi)|\leqslant\|Q_i\|^{-\widehat{\omega}}, \tag{9.1}$$

as well as

$$\|Q_{i+1}\|^{\tau}\leqslant\|Q_i\|,\quad\text{where }\tau=\frac{\widehat{\omega}\big(\widehat{\omega}-n-d+3\big)}{\omega_n(\xi)\big(\omega_n(\xi)-n-d+3\big)}\in[0,1). \tag{9.2}$$

Assuming that d is not too large, we will prove in the next section that $\omega_n(\xi)<\infty$, and thus $\tau>0$. Here, we investigate the following question: can we find "large" subfamilies of $(Q_i)_{i\geqslant 0}$ which are linearly independent, and whose elements have "comparable" height? More precisely, given two indices $k<i$, can we find an exponent $\theta_j\in(0,1)$ which depends only on d,n and the dimension $j+1$ of the subspace

$\langle Q_k, Q_{k+1}, \dots, Q_i\rangle_{\mathbb{R}}$ (and not on the indices i and k), such that $\|Q_i\|^{\theta_j} \ll \|Q_k\|$? For $i = k+1$, we already have (9.2). With this goal in mind, let us introduce some notation.

Definition 9.1. — Let $m_n = m_n(\xi) \in [2, n+1]$ be the integer

$$m_n := \lim_{i\to\infty} \dim(\langle Q_i, Q_{i+1}, \dots\rangle_{\mathbb{R}}).$$

Remark. — Note that we might have $m_n < n+1$, since, unlike for simultaneous approximation (see [17, Eq. (5.3)]), it is possible that the sequence $(P_i)_{i\geqslant j}$ is contained in a proper subspace of $\mathbb{R}[X]_{\leqslant n}$, see e.g. [14]. However, we will show later that under the hypothesis $d \asymp n^{1/3}$, we have $m_n \gg n^{1/3}$. The next definition is somewhat dual to [17, Def. 5.2]. However, note that in [17, Def. 5.2], the sets $A_j[i]$ are constructed from the points $\boldsymbol{x}_i, \boldsymbol{x}_{i+1}, \dots$ coming *after* the good approximation $\boldsymbol{x}_i$, whereas in the present setting we need to consider the points $Q_i, Q_{i-1}, \dots$ coming *before* Q_i. It does not seem to work well the other way round.

Definition 9.2. — Let $j_1 > j_0 \geqslant 0$ be such that

$$\dim\langle Q_{j_0}, Q_{j_0+1}, \dots, Q_{j_1}\rangle_{\mathbb{R}} = \dim\langle Q_{j_0}, Q_{j_0+1}, \dots\rangle_{\mathbb{R}} = m_n.$$

For each $i \geqslant j_1$ and $j = 0, \dots, m_n - 2$, we define

$$\sigma_j(i) = k, \quad A_j[i] = \{Q_k, Q_{k+1}, \dots, Q_i\} \quad \text{and} \quad Y_j(i) = \|Q_{k-1}\|,$$

where $k \in \{j_0+1, \dots, i\}$ is the smallest index such that $\dim\langle Q_k, \dots, Q_i\rangle_{\mathbb{R}} = j+1$.

Proposition 6.2 implies that

$$V_{2n-j}\big(A_j[i]\big) = \mathbb{R}[X]_{\leqslant 2n-j} \qquad (j = 1, \dots, m_n - 2). \tag{9.3}$$

Definition 9.3. — Let $\tau \in (0,1)$. We associate to τ a sequence $(\tau_j)_{0\leqslant j\leqslant n/2}$ by setting $\tau_0 = \tau$, and for $j = 1, \dots, \lfloor n/2\rfloor$

$$\tau_j = \alpha_j\Big(\tau_{j-1} - \frac{2j-1}{2n-d}\Big), \quad \text{where } \alpha_j = \frac{(2n-d)\tau^2}{(n-2j)\tau + n - j + 1}.$$

The first main result of this section is the following.

Proposition 9.4. — *Let $\tau \in (0,1)$ and let $(\tau_j)_{0\leqslant j\leqslant n/2}$ be as in Definition 9.3. Suppose that*

$$\|Q_{i+1}\|^{\tau} \leqslant \|Q_i\| \qquad \textit{for each sufficiently large } i. \tag{9.4}$$

Then for each large enough i, we also have

$$\|Q_i\|^{\tau_j} \ll Y_j(i) \qquad \textit{for } j = 0, \dots, \min\big\{\lfloor n/2\rfloor, m_n - 2\big\}, \tag{9.5}$$

with implicit constants which do not depend on i and j.

Remark. — We will use the exponent τ given by (9.2). We will prove that under suitable conditions, the exponent of best approximation $\omega_n(\xi)$ is not "too large". This will ensure that τ is "close" to 1. This issue, which is one of the delicate parts of this paper, will be dealt with in Section 10.

In order to get (9.5), we will try to adopt a strategy similar to the one of [17, §5] in the setting of simultaneous approximation to the successive powers of ξ. New difficulties arise however, for example we need to work with $\mathcal{D}_\xi$ instead of the standard height of subspaces (see Section 8.1). Schmidt's inequality (8.6) will play a key-role in our proofs. We use the notation of Definition 6.1 for the sets $\mathcal{B}_k(\mathcal{A})$ and the subspaces $V_k(\mathcal{A}) \subseteq \mathbb{R}[X]_{\leqslant k}$.

Proof. — Without loss of generality, we may suppose that the index j_0 is large enough so that (9.4) holds for each $i \geqslant j_0 - 1$. Fix $i \geqslant j_1$, and for simplicity write $m = m_n$ and $Y_k := Y_k(i)$ for $k = 0, \dots, m-2$.

We prove (9.5) by induction on j. If $j = 0$, we have $Y_0 = \|Q_{i-1}\|$ since $\sigma_0(i) = i$. By (9.4) applied with $i-1$ instead of i, we get $\|Q_i\|^{\tau_0} \leqslant Y_0$. Now, let $j \in \{1, \dots, m-2\}$ with $j \leqslant n/2$ such that (9.5) holds for $j-1$. If $\tau_j \leqslant 0$, then (9.5) holds trivially for j. We assume that $\tau_j > 0$. Consequently, we also have $\tau_{j-1} > 0$. Write $P := Q_{\sigma_j(i)}$ and $Q := Q_{\sigma_j(i)+1}$. By (9.4), we have

$$\|Q\|^{\tau^2} \leqslant \|P\|^{\tau} \leqslant Y_j. \tag{9.6}$$

Since P and Q are coprime, Lemma 6.4 implies that $\dim V_{2n-j}(P,Q) \geqslant 2(n-j+1)$. Therefore, there exists a family of $2n-3j+1$ linearly independent polynomials

$$\mathcal{U}_j := \{U_0, \dots, U_{2n-3j}\} \subseteq \mathcal{B}_{2n-j}(P,Q)$$

such that $\langle A_j[i]\rangle_{\mathbb{R}} \cap \langle \mathcal{U}_j\rangle_{\mathbb{R}} = \{0\}$. Note that since $j \leqslant n/2$, we may choose $\mathcal{U}_j$ such that it contains at least $n-2j$ polynomials whose height is equal to $\|P\|$. The remaining $n-j+1$ ones have height $\leqslant \|Q\|$. By (9.3), we have $V_{2n-j}\big(A_j[i]\big) = \mathbb{R}[X]_{\leqslant 2n-j}$. Therefore, there exists

$$\mathcal{V}_j := \{V_1, \dots, V_{j-1}\} \subseteq \mathcal{B}_{2n-j}(A_j[i]) = \mathcal{B}_{2n-j}(Q_{\sigma_j(i)}, \dots, Q_i)$$

(with the convention $\mathcal{V}_j = \varnothing$ if $j = 1$) such that we have the direct sum

$$\langle A_j[i]\rangle_{\mathbb{R}} \oplus \langle \mathcal{U}_j\rangle_{\mathbb{R}} \oplus \langle \mathcal{V}_j\rangle_{\mathbb{R}} = \mathbb{R}[X]_{\leqslant 2n-j}.$$

All the polynomials of $\mathcal{V}_j$ have height at most $\|Q_i\|$. Let $k \in \{\sigma_j(i), \dots, i\}$ which maximizes $|Q_k(\xi)|/\|Q_k\|$ and define

$$A := \langle A_j[i]\rangle_{\mathbb{R}} \quad \text{and} \quad B := \langle \mathcal{U}_j \cup \mathcal{V}_j \cup \{Q_k\}\rangle_{\mathbb{R}},$$

so that $A + B = \mathbb{R}[X]_{\leqslant 2n-j}$ and $A \cap B = \langle Q_k\rangle_{\mathbb{R}}$. We will now make a crucial use of the function $\mathcal{D}_\xi$ introduced in Definition 8.1 (here, the ambient space is $\mathbb{R}[X]_{\leqslant 2n-j}$, identified to $\mathbb{R}^{2n-j+1}$ via (8.9)). Recall that

$$\mathcal{D}_\xi(A+B) = \mathcal{D}_\xi\big(\mathbb{R}[X]_{\leqslant 2n-j}\big) = \big\|(1, \xi, \dots, \xi^{2n-j})\big\|_2 \asymp 1,$$

and that according to (8.10) the primitive polynomial Q_k satisfies

$$\mathcal{D}_\xi(A \cap B) = \mathcal{D}_\xi\big(\langle Q_k\rangle_{\mathbb{R}}\big) = |Q_k(\xi)|.$$

Schmidt's inequality (8.6) applied with the subspaces A and B yields

$$|Q_k(\xi)| \asymp \mathcal{D}_\xi(A+B)\mathcal{D}_\xi(A \cap B) \leqslant \mathcal{D}_\xi(A)\mathcal{D}_\xi(B), \tag{9.7}$$

the implicit constants depending only on n and ξ (and not on the indices i,j). It remains to estimate $\mathcal{D}_\xi(A)$ and $\mathcal{D}_\xi(B)$. The subspace $B \subseteq \mathbb{R}[X]_{\leqslant 2n-j}$ is generated by the $2n-2j+1$ linearly independent polynomials $\mathcal{V} = \mathcal{U}_j \cup \mathcal{V}_j \cup \{Q_k\}$. Moreover (see the remarks after the constructions of $\mathcal{U}_j$ and $\mathcal{V}_j$), we have

$$\prod_{R\in\mathcal{V}} \|R\| \leqslant \|P\|^{n-2j}\|Q\|^{n-j+1}\|Q_i\|^{j-1}\|Q_k\|.$$

By choice of k, for each $R \in \mathcal{V}$ we also have $|R(\xi)|/\|R\| \ll |Q_k(\xi)|/\|Q_k\|$, and Lemma 8.3 combined with the above yields the upper bound

$$\mathcal{D}_\xi(B) \ll |Q_k(\xi)|\|P\|^{n-2j}\|Q\|^{n-j+1}\|Q_i\|^{j-1}.$$

The space $A = \langle A_j[i]\rangle_{\mathbb{R}} \subseteq \mathbb{R}[X]_{\leqslant 2n-j}$ is spanned by a set $\mathcal{U}$ of $j+1$ linearly polynomials that may be chosen among $Q_{\sigma_{j-1}(i)-1},\ldots,Q_{i-1}$, Q_i. For each $R \in \mathcal{U}$, we have $\|R\| \leqslant \|Q_i\|$ and $|R(\xi)| \leqslant \|R\|^{-\widehat{\omega}} \leqslant Y_{j-1}^{-\widehat{\omega}}$. Combined with Lemma 8.3, we obtain

$$\mathcal{D}_\xi(A) \ll \sum_{R\in\mathcal{U}} |R(\xi)| \prod_{\substack{S\in\mathcal{U}\\ S\neq R}} \|S\| \ll Y_{j-1}^{-\widehat{\omega}}\|Q_i\|^{j}.$$

Then, combining the above upper bounds for $\mathcal{D}_\xi(B)$ and $\mathcal{D}_\xi(A)$ with (9.7) and (9.6), we get

$$Y_{j-1}^{\widehat{\omega}} \ll \|P\|^{n-2j}\|Q\|^{n-j+1}\|Q_i\|^{2j-1} \ll Y_j^{(n-2j)/\tau+(n-j+1)/\tau^2}\|Q_i\|^{2j-1},$$

where the implicit constants depend on n and ξ only. Using the induction hypothesis, we also have $\|Q_i\|^{\widehat{\omega}\,\tau_{j-1}} \ll Y_{j-1}^{\widehat{\omega}}$, hence

$$\|Q_i\|^{\widehat{\omega}\,\tau_{j-1}-2j+1} \ll Y_j^{(n-2j)/\tau+(n-j+1)/\tau^2} = Y_j^{(2n-d)/\alpha_j}.$$

Rising each term to the power $\alpha_j/(2n-d)$ and using $\widehat{\omega} > 2n-d$, we easily deduce (9.5) for j. This concludes our induction step. ☐

Remark 9.5. — We could get a slightly greater exponent τ_j in the above proposition by using a more precise estimate for $\mathcal{D}_\xi(A)$. However, this improvement would at best lead to a larger constant a in Theorem 1.2 ; the term $n^{1/3}$ would remain the same, whereas we are expecting $n^{1/2}$. We preferred to keep the arguments simple.

The following result is inspired by Laurent's approach in [12, Lem. 5].

Proposition 9.6. — *Let the hypotheses be as in Proposition 9.4 and write $m = m_n$. For any $\lambda < \lambda_n(\xi)$, there are infinitely many indices i such that*

$$Y_{m-2}(i) < \|Q_i\|^{1/(\widehat{\omega}\lambda\tau)}.$$

In particular, there are infinitely many indices i such that

$$Y_{m-2}(i) \leqslant \|Q_i\|^{\mu}, \quad \text{where } \mu := \frac{n}{(2n-d)\tau}. \tag{9.8}$$

Proof. — By definition of m, the subspace

$$V = \big\langle Q_{\sigma_{m-2}(i)-1}, Q_{\sigma_{m-2}(i)}, \dots, Q_i \big\rangle_{\mathbb{R}} \tag{9.9}$$

of $\mathbb{R}[X]_{\leqslant n}$ is independent of i for $i \geqslant j_1$, where j_1 comes from Definition 9.2. It has dimension m since $\dim A_{m-2}[i] = m-1$ and $Q_{\sigma_{m-2}(i)-1} \notin A_{m-2}[i]$. Fix two positive real numbers α, λ with $\lambda < \alpha < \lambda_n(\xi)$, and suppose by contradiction that there exists an index $i_0 \geqslant j_1$ such that for each $i \geqslant i_0$

$$Y_{m-2}(i) \geqslant \|Q_i\|^{\theta}, \quad \text{where } \theta = \frac{1}{\widehat{\omega}\lambda\tau}. \tag{9.10}$$

By hypothesis, we can also assume that $\|Q_{i+1}\|^{\tau} \leqslant \|Q_i\|$ for each $i \geqslant i_0$. Identifying $\mathbb{R}[X]_{\leqslant n}$ with $\mathbb{R}^{n+1}$ via the isomorphism (8.9), we claim that the point $\Xi = (1, \xi, \xi^2, \dots, \xi^n)$ is orthogonal to V, with respect to the standard scalar product $(\cdot \mid \cdot)$ of $\mathbb{R}^{n+1}$.

By definition of $\lambda_n(\xi)$, there exist infinitely many non-zero $\boldsymbol{y} = (y_0, \dots, y_n) \in \mathbb{Z}^{n+1}$ satisfying

$$L(\boldsymbol{y}) = \max_{1\leqslant k\leqslant n} |y_0\xi^k - y_k| \leqslant Y^{-\alpha}, \quad \text{where } Y = \|\boldsymbol{y}\| = \max_{1\leqslant k\leqslant n} |y_k|.$$

Let $(\boldsymbol{y}_i)_{i\geqslant 0}$ be an unbounded sequence of such points ordered by increasing norm. This sequence converges projectively to $\Xi = (1, \xi, \xi^2, \dots, \xi^n)$. Without loss of generality, we may assume that $\|\boldsymbol{y}_0\|^{\alpha} > 2(n+1)\|Q_{i_0}\|$. Fix an index j arbitrarily large. For simplicity, set $\boldsymbol{y} := \boldsymbol{y}_j$ and $Y = \|\boldsymbol{y}_j\|$. There exists an index $i \geqslant i_0$ such that

$$\|Q_i\| < \frac{Y^{\alpha}}{2(n+1)} \leqslant \|Q_{i+1}\| \leqslant \|Q_i\|^{1/\tau}. \tag{9.11}$$

Note that i tends to infinity with j. Let $k \in \{\sigma_{m-2}(i)-1, \dots, i\}$. The polynomial $Q := Q_k$ is identified with an integer point $\boldsymbol{z} \in \mathbb{Z}^{n+1}$ such that $Q(\xi) = (\boldsymbol{z} \mid \Xi)$. Since $(\boldsymbol{z} \mid \boldsymbol{y}) = (\boldsymbol{z} \mid \boldsymbol{y} - y_0\Xi) + y_0\,(\boldsymbol{z} \mid \Xi)$, we get

$$|\,(\boldsymbol{z} \mid \boldsymbol{y})\,| \leqslant (n+1)\|Q\|L(\boldsymbol{y}) + Y|Q(\xi)|$$

(cf. [12, Lem. 5]). Our hypothesis (9.10) yields

$$\|Q_i\|^{\theta} \leqslant Y_{m-2}(i) \leqslant \|Q\| \leqslant \|Q_i\|.$$

Using (9.11) together with $L(\boldsymbol{y}) \leqslant Y^{-\alpha}$, we get

$$(n+1)\|Q\|L(\boldsymbol{y}) < \frac{1}{2}.$$

Moreover, (9.11) also yields $Y^{1/\widehat{\omega}} \ll \|Q_i\|^{1/(\widehat{\omega}\alpha\tau)} = \|Q_i\|^{\theta\lambda/\alpha}$, where the implicit constant only depends on n. Since $\lambda < \alpha$, we may choose j so large that $(2Y)^{1/\widehat{\omega}} < \|Q_i\|^{\theta}$. Combining this with the estimate $|Q(\xi)| \leqslant \|Q\|^{-\widehat{\omega}}$ from (9.1), we also get

$$Y|Q(\xi)| \leqslant Y\|Q\|^{-\widehat{\omega}} \leqslant Y\|Q_i\|^{-\theta\,\widehat{\omega}} < \frac{1}{2}.$$

We conclude that the integer $|\,(\boldsymbol{z} \mid \boldsymbol{y})\,|$ is (strictly) less that 1. It is thus equal to 0, and so $\boldsymbol{y}$ and $\boldsymbol{z}$ are orthogonal. By letting k vary, this implies that $\boldsymbol{y} = \boldsymbol{y}_j$ is orthogonal to the subspace V. Since this is true for all sufficiently large j, it follows that the (projective) limit Ξ is also orthogonal to V. This proves our claim and provides the required contradiction since no Q_i vanishes at the transcendental number ξ. Thus, (9.10) does not hold for arbitrarily large indices i. Estimate (9.8) follows by noticing that $\lambda_n(\xi) \geqslant 1/n$ by Dirichlet's theorem, and recalling that $\widehat{\omega} > 2n-d$. We may therefore choose $\lambda < \lambda_n(\xi)$ so that $\lambda\widehat{\omega} > (2n-d)/n$. □

Corollary 9.7. — *Under the same hypotheses, suppose moreover that $m = m_n$ satisfies $m-2 \leqslant n/2$, and let $(\tau_j)_{0\leqslant j\leqslant n/2}$ be as in Definition 9.3. Then, we have*

$$\tau_{m-2} \leqslant \mu = \frac{n}{(2n-d)\tau}.$$

Proof. — By Propositions 9.4 and 9.6 there are infinitely many indices i for which $\|Q_i\|^{\tau_{m-2}} \ll Y_{m-2}(i) \leqslant \|Q_i\|^{\mu}$. Since $\|Q_i\|$ tends to infinity with i, we deduce that $\tau_{m-2} \leqslant \mu$. □

10. Upper bound on the exponent of best approximation

This section is devoted to the proof of the following upper bound for $\omega_n(\xi)$.

Proposition 10.1. — *Suppose that $\widehat{\omega}_n(\xi) > 2n-d$, with an integer $d \in \mathbb{N}$ satisfying $2 \leqslant d \leqslant \sqrt[3]{n/4}$. Then, we have the upper bound*

$$\omega_n(\xi) \leqslant 2n + P(n,d), \quad \text{where } P(n,d) = \frac{n(4d^2-d-5)+8d^2-2d-15}{2n-8d^2+2d+15}.$$

If moreover we have $d \leqslant \left\lceil \sqrt[3]{n/16} \right\rceil$ and $n > 16$, then

$$\omega_n(\xi) \leqslant 2n + 2d^2.$$

Let d, n, ξ and $\widehat{\omega}$ be as in Sections 5 and 9. We suppose thus that $2 \leqslant d < 1 + n/2$ and that (5.1) holds, namely

$$\widehat{\omega}_n(\xi) > \widehat{\omega} > 2n-d.$$

Fix a sequence of minimal polynomials $(P_i)_{i\geqslant 0}$ associated to n and ξ as in Section 3. We denote by $(Q_i)_{i\geqslant 0}$ the sequence of irreducible factors given by Proposition 5.1. Unless otherwise stated, all the constants implicit in the symbols $\ll, \gg, \asymp$ and $\mathcal{O}(\cdot)$ depend only on n, d, ξ and $\widehat{\omega}$.

According to Proposition 5.1, we have $\omega_n(\xi) = \limsup_{i\to\infty} \omega(Q_i)$. By (5.3), we also have

$$\|P_j\| \leqslant \|Q_i\|^{1+\theta_i} \quad \text{with } \theta_i = \frac{\omega(Q_i)-2n+d}{n-2d+3}, \tag{10.1}$$

for each $i \geqslant 0$ and each j such that Q_i divides P_j. Proposition 10.1 implies that if d^3 is small compared to n, then $\theta_i = \mathcal{O}(d^2/n)$ is small, and Q_i has "almost" the same norm as P_j.

In order to bound from above $\omega_n(\xi)$, it suffices to do so for $\omega(Q_i)$. We could try to use (10.1), which implies that any minimal polynomial of height greater than $\|Q_i\|^{1+\theta_i}$ is not divisible by Q_i. They are thus coprime and we may consider their (non-zero) resultant. However we cannot conclude, as θ_i is too large. To resolve this problem, we need several lemmas. We first start by a few simple observations. A quick computation yields

$$(1+\theta_i)(2n-d)=\omega(Q_i)+(n+d-3)\theta_i. \tag{10.2}$$

More generally, for each $\eta\geqslant 0$, we have

$$\big(1+\theta_i(1-\eta)\big)(2n-d)=\omega(Q_i)+\big(n+d-3-\eta(2n-d)\big)\theta_i. \tag{10.3}$$

Under the condition $\eta<(n+d-3)/(2n-d)$, which holds as soon as $\eta<1/2$, this implies that for each $i\geqslant 0$, we have

$$|Q_i(\xi)|=\|Q_i\|^{-\omega(Q_i)}>\|Q_i\|^{-\big(1+\theta_i(1-\eta)\big)(2n-d)}. \tag{10.4}$$

Lemma 10.2. — *Let $i\geqslant 0$ and $\eta\in[0,1/2)$, and suppose that $R\in\mathbb{Z}[X]_{\leqslant d-2}$ is a non-zero polynomial such that $P:=Q_iR$ has degree at most n and satisfies*

$$\|P\|\leqslant H:=\|Q_i\|^{1+\theta_i(1-\eta)}\quad\textit{and}\quad|P(\xi)|\leqslant H^{-2n+d}. \tag{10.5}$$

Define

$$\eta'=\frac{(2n-d)\eta}{n+d-3}\quad\textit{and}\quad\eta''=\frac{(2n-2d+3)\eta+d-3}{n+d-3}.$$

Then, we have the following properties.

(i) *The polynomial R is non-constant. We have $d\geqslant 3$ and*

$$\|R\|^{-(n+d-3)}\ll|R(\xi)|\leqslant\|Q\|^{-(n+d-3)(1-\eta')\theta_i}. \tag{10.6}$$

(ii) *There exist a non-constant irreducible polynomial $A\in\mathbb{Z}[X]_{\leqslant n}$ and an integer $e\in[1,d-2]$ such that A^e divides R,*

$$\|A^e\|\gg\|Q\|^{\theta(1-\eta'')}\quad\textit{and}\quad\|A^e\|^{-(n+d-3)}\ll|A^e(\xi)|. \tag{10.7}$$

(iii) *Let A and e be as in* (ii). *If $S\in\mathbb{Z}[X]_{\leqslant d-2}$ is a non-zero polynomial such that A and S are coprime and $\|S\|\leqslant\|A^e\|$, then*

$$|S(\xi)|\gg\|A^e\|^{-(2d-5)}. \tag{10.8}$$

Proof. — Fix $i\geqslant 0$. For simplicity, write $Q:=Q_i$ and $\theta=\theta_i$. By Gelfond's lemma, we have

$$\|Q\|\|R\|\asymp\|QR\|=\|P\|\leqslant\|Q\|^{1+\theta(1-\eta)},$$

so that

$$\|R\|\ll\|Q\|^{\theta(1-\eta)}. \tag{10.9}$$

The first inequality of (10.6) and the second of (10.7) are consequences of Lemma 5.2 (using $\deg(A^e)\leqslant\deg(R)\leqslant d-2$). Using (10.3) together with (10.5) and $\|Q\|^{-\omega(Q)}=|Q(\xi)|$, we find

$$|Q(\xi)R(\xi)|=|P(\xi)|\leqslant\|Q\|^{-\big(1+\theta(1-\eta)\big)(2n-d)}=|Q(\xi)|\|Q\|^{-\big(n+d-3-\eta(2n-d)\big)\theta}.$$

Simplifying by $|Q(\xi)|$ yields the second inequality of (10.6). In particular we have $|R(\xi)|<1$ since $\|Q\|>1$ (and $\theta>0$ as well as $\eta'\leqslant 2\eta<1$). Consequently $R\in\mathbb{Z}[X]_{\leqslant d-2}$ cannot be constant, and thus $d\geqslant 3$.

Without loss of generality, we may suppose that P (and thus R) is primitive. Let us consider the factorization of R over $\mathbb{Z}$. There exist an integer $k \geqslant 1$, irreducible (non-constant) pairwise distinct polynomials $A_1, \dots, A_k \in \mathbb{Z}[X]$ and positive integers $\alpha_1, \dots, \alpha_k$ such that

$$R = \prod_{j=1}^{k} A_j^{\alpha_j} = \prod_{j=1}^{k} B_j \qquad \text{with } B_j := A_j^{\alpha_j} \text{ for each } j = 1, \dots, k.$$

According to Lemma 4.2, there exists $j \in \{1, \dots, k\}$ such that $B = B_j$ satisfies

$$|B(\xi)| \ll \|R\|^{d-3} |R(\xi)|.$$

Set $A = A_j$ and $e = \alpha_j$. We use (10.9) to bound $\|R\|$ from above, and the second inequality of (10.6) to bound $|R(\xi)|$ from above. Then, Lemma 5.2 applied to the polynomial $B \in \mathbb{Z}[X]_{\leqslant d-2}$ together with the above yields

$$\|B\|^{-(n+d-3)} \ll |B(\xi)| \ll \|R\|^{d-3} |R(\xi)| \ll \|Q\|^{(d-3)(1-\eta)\theta - (n+d-3)(1-\eta')\theta},$$

Since by definition of η' and η'' we have

$$1 - \eta' - \frac{d-3}{n+d-3}(1-\eta) = 1 - \eta'',$$

we deduce that

$$\|B\|^{-(n+d-3)} \ll |B(\xi)| \ll \|Q\|^{-(n+d-3)\theta(1-\eta'')}. \tag{10.10}$$

and (10.7) follows easily upon recalling that $A^e = B$. Now, suppose that $S \in \mathbb{Z}[X]_{\leqslant d-2}$ is a non-zero polynomial coprime to A with $\|S\| \leqslant \|B\|$. If S is constant, then (10.8) is trivial. We may therefore assume that S has degree at least 1. Then, the estimate of Lemma 4.1 yields

$$\begin{aligned} 1 \leqslant |\operatorname{Res}(B, S)| &\ll \|B\|^{d-3} \|S\|^{d-2} |B(\xi)| + \|B\|^{d-2} \|S\|^{d-3} |S(\xi)| \\ &\ll \|B\|^{2d-5} \big(|B(\xi)| + |S(\xi)|\big) \end{aligned} \tag{10.11}$$

(where the implicit constants depend on ξ, n and c). As B divides R, we have $\|B\| \ll \|R\|$. Together with (10.9), this gives $\|B\| \ll \|Q\|^{\theta(1-\eta)}$. Combining the above with (10.10), we obtain

$$\|B\|^{2d-5} |B(\xi)| \ll \|Q\|^{(2d-5)\theta(1-\eta) - (n+d-3)\theta(1-\eta'')}.$$

On the other hand, using $\eta \leqslant 1/2$ we get

$$(2d-5)(1-\eta) - (n+d-3)(1-\eta'') = (2n-4d+8)\eta - (n-2d+5) \leqslant -1.$$

Since for each large enough i, the number $\theta = \theta_i$ is bounded from below by

$$\rho = \frac{\widehat{\omega} - 2n + d}{n - 2d + 3} > 0,$$

it follows that $\|B\|^{2d-5} |B(\xi)| \ll \|Q\|^{-\rho}$ tends to 0 as i tends to infinity. Consequently, (10.11) becomes

$$1 \ll \|B\|^{2d-5} |S(\xi)|,$$

hence (10.8). □

Lemma 10.3. — *Let $\eta\in[0,1/2)$. As in Lemma 10.2, we set*
$$\eta''=\frac{(2n-2d+3)\eta+d-3}{n+d-3}.$$
Suppose either that we have $d=2$, or that we have $d\geqslant 3$, $\eta''\in[0,1/2)$ and
$$\frac{1-2\eta''}{1-\eta''}\geqslant 1-\frac{1}{d-2}+\frac{2d}{n}. \tag{10.12}$$
Then for each large enough $i\geqslant 0$, there exist $Z\in\mathbb{R}$ with $\|Q_i\|\leqslant Z\leqslant \|Q_i\|^{1+\theta_i(1-\eta)}$ and a non-zero $P\in\mathbb{Z}[X]_{\leqslant n}$, coprime to Q_i, which satisfies $|P(\xi)|<|Q_i(\xi)|$ and
$$\|P\|\leqslant Z \quad\text{and}\quad |P(\xi)|\leqslant Z^{-(2n-d)}. \tag{10.13}$$

Proof. — Since $\widehat{\omega}_n(\xi)>2n-d$, there exists $X_0\geqslant 0$ such that for each $X\geqslant X_0$ the system
$$\|P\|\leqslant X \quad\text{and}\quad |P(\xi)|\leqslant X^{-(2n-d)}$$
has a non-zero solution P in $\mathbb{Z}[X]_{\leqslant n}$. Fix $i\geqslant 0$ such that $\|Q_i\|\geqslant X_0$, and choose a non-zero solution $P\in\mathbb{Z}[X]_{\leqslant n}$ of the above system with $X:=\|Q_i\|^{1+\theta_i(1-\eta)}$. For simplicity, write $Q=Q_i$ and $\theta=\theta_i$. We have $|P(\xi)|\leqslant X^{-(2n-d)}<|Q(\xi)|$ thanks to (10.4). If P and Q are coprime, then the conclusion holds with $Z=X$. We may therefore assume that P and Q are not coprime. Then Q divides P, and assertion (i) of Lemma 10.2 implies that $d\geqslant 3$. Let $A\in\mathbb{Z}[X]_{\leqslant d-2}$ and $e\in[1,d-2]$ be the non-constant irreducible polynomial and the integer given by Lemma 10.2 (ii). In particular we have $\deg(A^e)\leqslant d-2$ and (10.7) holds. Set $Z:=e^{-2n}\|QA^e\|$, and define ν by the relation
$$Z=\|Q\|^{1+\theta(1-\nu)}.$$
By Gelfond's lemma and by definition of Z and ν, we have
$$\|Q\|^{\theta(1-\nu)}\asymp\|A^e\|\gg\|Q\|^{\theta(1-\eta'')},$$
the last inequality coming from (10.7). We deduce that $\nu\leqslant\eta''+\mathcal{O}\big(1/\log\|Q\|\big)$. Since $\eta''<1/2$ we may assume i large enough so that $\nu<1/2$. On the other hand, since QA^e divides P, by (2.1), we have
$$Z<e^{-n}\|QA^e\|\leqslant\|P\|\leqslant X=\|Q\|^{1+\theta(1-\eta)},$$
hence $\nu\geqslant\eta$. We now consider a non-zero solution $\widetilde{P}\in\mathbb{Z}[X]_{\leqslant n}$ of the system
$$\|\widetilde{P}\|\leqslant Z \quad\text{and}\quad |\widetilde{P}(\xi)|\leqslant Z^{-(2n-d)}. \tag{10.14}$$
We claim that $\widetilde{P}$ and Q are coprime. Suppose by contradiction that Q divides $\widetilde{P}$. There exists $\widetilde{R}\in\mathbb{Z}[X]$ such that $\widetilde{P}=Q\widetilde{R}$. Write $\widetilde{R}=A^f\widetilde{S}$, with $f\in\mathbb{N}$ and $\widetilde{S}\in\mathbb{Z}[X]_{\leqslant d-2}$ coprime to A. By (2.1) and by definition of Z, and since Q and $\widetilde{S}$ divide $\widetilde{P}$, we obtain
$$\|Q\|\|\widetilde{S}\|<e^n\|\widetilde{P}\|\leqslant e^nZ=e^{-n}\|QA^e\|<\|Q\|\|A^e\|.$$
We deduce that $\|\widetilde{S}\|\leqslant\|A^e\|$. Similarly,
$$\|QA^f\|<e^n\|\widetilde{P}\|\leqslant e^nZ=e^{-n}\|QA^e\|.$$

Consequently, the polynomial QA^e cannot be a factor of QA^f (by (2.1) once again). Thus $f \leqslant e-1$. Since $\|\widetilde{S}\| \leqslant \|A^e\|$, the last assertion of Lemma 10.2 yields

$$|\widetilde{S}(\xi)| \gg \|A^e\|^{-(2d-5)}. \tag{10.15}$$

By hypothesis $\nu < 1/2$, and Lemma 10.2 (i) applied to the solution $\widetilde{P} = Q\widetilde{R}$ of the system (10.14) gives the estimate

$$|\widetilde{R}(\xi)| \leqslant \|Q\|^{-(n+d-3)(1-\nu')\theta}, \quad \text{where } \nu' = \frac{(2n-d)\nu}{n+d-3}. \tag{10.16}$$

We now use (10.15) and $|A^e(\xi)| \gg \|A^e\|^{-(n+d-3)}$ (coming from (10.7)) together with $f \leqslant e-1 \leqslant d-3$. We get the lower bound

$$\begin{aligned}\log|\widetilde{R}(\xi)| &= \frac{f}{e}\log|A^e(\xi)| + \log|\widetilde{S}(\xi)|\\ &\geqslant -\Big[\Big(1-\frac{1}{e}\Big)(n+d-3) + 2d-5\Big]\log\|A^e\| + \mathcal{O}(1)\\ &\geqslant -\Big[\Big(1-\frac{1}{d-2}\Big)(n+d-3)+2d-5\Big]\theta(1-\nu)\log\|Q\| + \mathcal{O}(1),\end{aligned}$$

the last inequality following from $\|A^e\| \asymp \|Q\|^{\theta(1-\nu)}$. Comparing this with (10.16) and noting that $\nu' \leqslant 2\nu$, we obtain

$$\frac{1-2\nu}{1-\nu} \leqslant \frac{1-\nu'}{1-\nu} \leqslant 1 - \frac{1}{d-2} + \frac{2d-5}{n+d-3} + \mathcal{O}\big(1/\log\|Q\|\big).$$

The function $\nu \mapsto (1-2\nu)/(1-\nu)$ is decreasing on $[0,1/2]$. Using the estimate $\nu \leqslant \eta'' + \mathcal{O}\big(1/\log\|Q\|\big)$, we obtain

$$\frac{1-2\eta''}{1-\eta''} \leqslant 1 - \frac{1}{d-2} + \frac{2d-5}{n+d-3} + \mathcal{O}\big(1/\log\|Q\|\big).$$

Since $(2d-5)/(n+d-3) < 2d/n$, this contradicts our hypothesis (10.12) when i is sufficiently large. So, if i is large enough, then $\widetilde{P}$ and Q are coprime. Finally, the lower bound $|\widetilde{P}(\xi)| \leqslant Z^{-(2n-d)} < |Q(\xi)|$ follows from (10.4) with η replaced by ν (since $\nu < 1/2$), by a similar argument as in the beginning of the proof. □

Proof of Proposition 10.1. — The condition $d \leqslant \sqrt[3]{n/4}$ implies that $d \leqslant 1 + n/2$. Define

$$\eta = \frac{1}{2d+5/2}, \quad \eta'' = \frac{(2n-2d+3)\eta + d - 3}{n+d-3} \quad \text{and} \quad \nu = \frac{1}{d+1}.$$

We claim that the hypotheses of Lemma 10.3 are satisfied for this choice of parameters. For $d = 2$, this is automatic since $\eta < 1/2$. If $d \geqslant 3$, a direct computation yields

$$\eta'' - \nu = \frac{-n + 4d^3 - 11d^2 - 13d + 6}{(4d+5)(n+d-3)(d+1)} < 0,$$

so that $\eta'' < \nu \leqslant 1/3$. Since $x \mapsto (1-2x)/(1-x)$ is decreasing on $[0,1/2]$, we deduce that $(1-2\eta'')/(1-\eta'') \geqslant (1-2\nu)/(1-\nu)$. On the other hand, we have

$$\frac{1-2\nu}{1-\nu} - \Big(1 - \frac{1}{d-2} + \frac{2d}{n}\Big) = \frac{2(n-d^3+2d^2)}{nd(d-2)} \geqslant 0,$$

hence our claim. Consequently, for each large enough i there exists a non-zero polynomial $P\in\mathbb{Z}[X]_{\leqslant n}$ coprime with Q_i, satisfying

$$|P(\xi)|\leqslant|Q_i(\xi)|<1\quad\text{and}\quad\|P\|\leqslant\|Q_i\|^{1+\theta(1-\eta)}.$$

Such a polynomial is non-constant, and Lemma 4.1 yields

$$\begin{aligned}1\leqslant|\operatorname{Res}(Q_i,P)|&\ll\|Q_i\|^{n-1}\|P\|^{n}|Q_i(\xi)|+\|Q_i\|^{n}\|P\|^{n-1}|P(\xi)|\\&\ll\|Q_i\|^{n-1+n(1+\theta(1-\eta))-\omega(Q_i)}.\end{aligned}$$

As $\|Q_i\|$ tends to infinity, it follows that

$$n-1+n(1+\theta(1-\eta))-\omega(Q_i)\geqslant\mathcal{O}\big(1/\log\|Q_i\|\big).$$

Using the definition (10.1) of θ_i, this can be rewritten as

$$(n\eta-2d+3)\omega(Q_i)\leqslant 2\eta n^2-(3d+\eta d-5)n+2d-3+\mathcal{O}\big(1/\log\|Q_i\|\big).$$

The hypothesis $d\leqslant\sqrt[3]{n/4}$ implies $n\eta-2d+3>0$. Thus, after simplification

$$\begin{aligned}\omega(Q_i)+\mathcal{O}\Big(1/\log\|Q_i\|\Big)&\leqslant\frac{2\eta n^2-(3d+\eta d-5)n+2d-3}{n\eta-2d+3}\\&=2n+\frac{n(d-1-\eta d)+2d-3}{n\eta-2d+3}=2n+P(n,d),\end{aligned}$$

where $P(n,d)$ is as in the statement of Proposition 10.1 (and $\eta=1/(2d+5/2)$). We conclude that

$$\omega_n(\xi)=\limsup_{i\to\infty}\omega(Q_i)\leqslant 2n+P(n,d).$$

Set $Q(n,d)=(2n-8d^2+2d+15)(P(n,d)-2d^2)$. A direct computation yields

$$Q(n,d)=-n(d+5)+16d^4-4d^3-22d^2-2d-15.$$

If $d\leqslant\sqrt[3]{n/16}$ we have $16d^4\leqslant nd$, and therefore $Q(n,d)\leqslant 0$. We obtain $P(n,d)\leqslant 2d^2$, and consequently $\widehat{\omega}_n(\xi)\leqslant 2n+2d^2$. It remains to show that in the case $n\geqslant 17$ and $d=\lceil\rho\rceil$ with

$$\rho=\sqrt[3]{n/16},$$

we still have $Q(n,d)\leqslant 0$. If $17\leqslant n\leqslant 128$, or equivalently if $1<\rho\leqslant 2$, then we have $d=2$ and $Q(n,2)=-7n+117\leqslant 0$. The same reasoning leads to $Q(n,d)\leqslant 0$ for $2<\rho\leqslant 3$ and $3<\rho\leqslant 4$. We now suppose that $\rho>4$. Writing $d=\rho+t$, with $t\in[0,1]$, and using the fact that $16\rho^3=n$, we find

$$\begin{aligned}Q(n,d)\leqslant-n(d+5)+16d^4&=-16\rho^3(\rho+t+5)+16\big(\rho^4+4t\rho^3+6t^2\rho^2+4t^3\rho+t^4\big)\\&=16\rho^3(3t-5)+16\big(6t^2\rho^2+4t^3\rho+t^4\big)\leqslant 16R(\rho),\end{aligned}$$

where $R(x)=-2x^3+6x^2+4x+1$. As the coefficients of $R(x+4)$ are all negative, we have $R(x)\leqslant 0$ for each $x\geqslant 4$. In particular, $R(\rho)\leqslant 0$, and once again we obtain $Q(n,d)\leqslant 0$. ∎

Note that the upper bound $2n+P(n,d)$ is not optimal in Proposition 10.1 (and could be slightly improved by choosing the parameter η closer to $1/(2d)$).

11. Proof of the main theorem

In this last section we prove our main Theorem 1.2 in the following stronger form.

Theorem 11.1. — *Let $\varepsilon = 0.3748\cdots$ be the unique (positive) solution of the equation $(1+x)e^x = 2$ and set $a = \big(2\varepsilon(2-e^\varepsilon)/9\big)^{1/3} = 0.3567\cdots$. There exists an explicit constant $C>0$ such that, for each $n \geqslant 1$ and any transcendental real number $\xi \in \mathbb{R}$, we have*

$$\widehat{\omega}_n(\xi) \leqslant 2n - an^{1/3} + C.$$

Since $1/3 < a$, it implies Theorem 1.2. We first establish a preliminary result which uses the following notation. Let n, d be integers with $2 \leqslant d \leqslant \sqrt{n/4}$. In particular $d \leqslant 1 + n/2$. We define

$$\omega(d,n) := 2n + P(n,d), \quad \text{where } P(n,d) = \frac{n(4d^2-d-5)+8d^2-2d-15}{2n-8d^2+2d+15},$$

as well as

$$\tau(d,n) = \frac{(2n-d)(n-2d+3)}{\omega(d,n)\big(\omega(d,n)-n-d+3\big)} \quad \text{and} \quad \mu(d,n) := \frac{n}{(2n-d)\tau}.$$

Let $\big(\tau_i(d,n)\big)_{0\leqslant i\leqslant n/2}$ be the sequence associated to $\tau = \tau(d,n) \in (0,1)$ by Definition 9.3.

Theorem 11.2. — *Let n, d, j be non-negative integers with $2 \leqslant d \leqslant \sqrt{n/4}$ and $1 \leqslant j \leqslant n/2$. Suppose that*

$$\tau_k(d,n) > \mu(d,n) \quad \textit{for } k = 0,\dots,j. \tag{11.1}$$

Then for any transcendental real number ξ we have

$$\widehat{\omega}_n(\xi) \leqslant 2n - \min\big\{d, d_j\big\}, \quad \textit{where } d_j = 2j-1-\frac{j-1}{\tau_j(d,n)}. \tag{11.2}$$

Proof. — Fix a transcendental real number ξ. If $\widehat{\omega}_n(\xi) \leqslant 2n-d$, then (11.2) holds. We now assume that $\widehat{\omega}_n(\xi) > 2n-d$, and we choose a real number $\widehat{\omega}$ such that

$$\widehat{\omega}_n(\xi) > \widehat{\omega} > 2n-d.$$

Let $(P_i)_{i\geqslant 0}$ denote a sequence of minimal polynomials associated to n and ξ as in Section 3. We denote by $(Q_i)_{i\geqslant 0}$ the sequence of irreducible factors given by Proposition 5.1, and denote by

$$m := m_n(\xi)$$

the dimension of the spaces $\langle Q_i, Q_{i+1}, \dots\rangle_{\mathbb{R}}$ for each large enough i (as in Definition 9.1). Proposition 10.1 yields $\omega_n(\xi) \leqslant \omega(d,n)$, and by Proposition 5.1 (v), we get, for each large enough i,

$$\|Q_{i+1}\|^{\tau(d,n)} \leqslant \|Q_i\|.$$

For simplicity, we write $\tau = \tau(d,n)$ and $\tau_k = \tau_k(d,n)$ for each $k \in \mathbb{N}$ with $k \leqslant n/2$. If $m-2 \leqslant j \leqslant n/2$, then Corollary 9.7 yields $\tau_{m-2} \leqslant \mu(d,n)$, which contradicts the hypothesis (11.1). Hence, we must have $j < m-2$. Let $i \geqslant 0$. Set $Q = Q_{\sigma_j(i)}$. If i is large enough, there exists a non-zero $P \in \mathbb{Z}[X]_{\leqslant n}$ such that

$$\|P\| \leqslant e^{-n}\|Q\| =: X \quad \text{and} \quad |P(\xi)| \leqslant X^{-\widehat{\omega}}.$$

By (2.1) the (irreducible) polynomial Q does not divide P, they are thus coprime. Lemma 6.4 implies that $\dim V_{2n-j}(P,Q) \geqslant 2n-2j+2$. Choose a linearly independent subset

$$\mathcal{U}_j := \{U_1, \dots, U_{2n-2j+2}\} \subseteq \mathcal{B}_{2n-j}(P,Q)$$

of cardinality $2n-2j+2$. According to (9.3), we have $V_{2n-j}(A_j[i]) = \mathbb{R}[X]_{\leqslant 2n-j}$. So there exists

$$\mathcal{V}_j := \{V_1, \dots, V_{j-1}\} \subseteq \mathcal{B}_{2n-j}(A_j[i]) = \mathcal{B}_{2n-j}(Q_{\sigma_j(i)}, \dots, Q_i)$$

such that

$$\langle \mathcal{U}_j \rangle_{\mathbb{R}} \oplus \langle \mathcal{V}_j \rangle_{\mathbb{R}} = \mathbb{R}[X]_{\leqslant 2n-j}.$$

Then, identifying $\mathbb{R}[X]_{\leqslant 2n-j}$ with $\mathbb{R}^{2n-j+1}$ via (8.9), we form the determinant

$$1 \leqslant \big|\det(U_1, \dots, U_{2n-2j+2}, V_1, \dots, V_{j-1})\big|. \tag{11.3}$$

For $k = 1, \dots, 2n-2j+2$, we have

$$\|U_k\| \ll \|Q\| \quad \text{and} \quad |U_k(\xi)| \ll \|Q\|^{-\widehat{\omega}}.$$

On the other hand, for $k = 1, \dots, j-1$, we have by Equation (9.5) from Proposition 9.4

$$\|Q\| \ll \|V_k\| \ll \|Q_i\| \ll \|Q\|^{1/\tau_j} \quad \text{and} \quad |V_k(\xi)| \ll \|V_k\|^{-\widehat{\omega}} \ll \|Q\|^{-\widehat{\omega}}.$$

For $i = 2, \dots, 2n-j+1$, we add to the first row of the determinant (11.3) the i-th row multiplied by ξ^{i-1}. This first row now becomes

$$\big(U_1(\xi), \dots, U_{2n-2j+2}(\xi), V_1(\xi), \dots, V_{j-1}(\xi)\big).$$

By the above, the absolute value of each of its elements is $\ll \|Q\|^{-\widehat{\omega}}$. By expanding the determinant, we obtain

$$1 \ll \|Q\|^{2n-2j+1}\|Q_i\|^{j-1}\|Q\|^{-\widehat{\omega}} \ll \|Q\|^{2n-2j+1+(j-1)/\tau_j-\widehat{\omega}}.$$

By letting i tend to infinity, we deduce that

$$\widehat{\omega} \leqslant 2n-2j+1+(j-1)\tau_j^{-1} = 2n-d_j.$$

Since $\widehat{\omega}$ may be chosen arbitrarily close to $\widehat{\omega}_n(\xi)$, we finally get (11.2). □

In view of (11.2), the idea is now to choose d and j so that d is maximal and $d \approx d_j$. The next two results aim at simplifying condition (11.1) of Theorem 11.2. The second one also provides a simple lower bound for the exponent τ_j.

Lemma 11.1. — *Let n, d, j be non-negative integers with $2 \leqslant d \leqslant \sqrt{n/4}$ and $1 \leqslant j \leqslant n/2$. Suppose that j satisfies*

$$\frac{(2n-d)\tau(d,n)^2}{(n-2j)\tau(d,n)+n-j+1} \leqslant 1 \quad \text{and} \quad \tau_j(d,n) \geqslant 0. \tag{11.4}$$

Then, the sequence $\big(\tau_k(d,n)\big)_{0\leqslant k\leqslant j}$ *is (strictly) decreasing. In particular, condition* (11.1) *is fulfilled if moreover*

$$\tau_j(d,n) > \mu(d,n).$$

Proof. — Let $\alpha_1 \leqslant \cdots \leqslant \alpha_j$ be as in Definition 9.3. Condition (11.4) is equivalent to $\alpha_j \leqslant 1$ and $\tau_j(d,n) \geqslant 0$. By definition, we have

$$\tau_{k-1}(d,n) = \alpha_k^{-1}\tau_k(d,n) + \frac{2k-1}{2n-d} \qquad (\text{for } k = 1,\dots,j).$$

Since $\alpha_k^{-1} \geqslant \alpha_j^{-1} \geqslant 1$, this yields $\tau_{k-1}(d,n) > \tau_k(d,n)$. This proves the first assertion of our lemma. The second one follows easily. □

Lemma 11.2. — *Let* n,d,j *be non-negative integers with* $2 \leqslant d \leqslant \sqrt{n/4}$ *and* $1 \leqslant j \leqslant n/2$. *Define*

$$\alpha = \alpha(d,n) := \frac{(2n-d)\tau(d,n)^2}{(n-2)\tau(d,n)+n},$$

and suppose that

$$\alpha^j > \frac{j(2j-1)\tau(d,n)}{(n-2)\tau(d,n)+n} = \frac{j(2j-1)\alpha}{(2n-d)\tau(d,n)}. \tag{11.5}$$

Then, $\alpha \in (0,1)$ *and for* $k = 0,\dots,j$, *we have*

$$\tau_k(d,n) \geqslant \alpha^j\tau(d,n) - \frac{j(2j-1)\tau(d,n)^2}{(n-2)\tau(d,n)+n} > 0.$$

Proof. — We have $\alpha \in (0,1)$ since $\tau(d,n) < 1$ and $d \geqslant 2$. For simplicity, we write $\tau = \tau(d,n)$. Let $(\sigma_k)_{k\geqslant 0}$ be the sequence defined by $\sigma_0 = \tau$, and

$$\sigma_k = \alpha\Big(\sigma_{k-1} - \frac{2k-1}{2n-d}\Big) \qquad \text{for } k \geqslant 1.$$

Using (11.5), we find

$$\frac{\sigma_j}{\alpha^j} = \frac{\sigma_{j-1}}{\alpha^{j-1}} - \frac{2j-1}{(2n-d)\alpha^{j-1}} = \sigma_0 - \frac{1}{2n-d}\sum_{k=1}^{j}\frac{2k-1}{\alpha^{k-1}} \geqslant \tau - \frac{j(2j-1)}{(2n-d)\alpha^{j-1}} > 0. \tag{11.6}$$

In particular $\sigma_j \geqslant 0$. Since $\sigma_{k-1} \geqslant \alpha^{-1}\sigma_k$, by induction, we get $\sigma_j < \sigma_{j-1} < \cdots < \sigma_0$. Moreover, $\alpha = \alpha_1 \leqslant \alpha_k$, for each $k \in \mathbb{N}$ with $1 \leqslant k \leqslant n/2$, where α_k is as in Definition 9.3. Another quick induction yields $\sigma_k \leqslant \tau_k$ for $k = 0,\dots,j$. We conclude by combining $\sigma_j \leqslant \sigma_k \leqslant \tau_k$ with (11.6). □

Proof of Theorem 11.1. — Define a function $f : [0,\infty) \to \mathbb{R}$ by $f(x) = x(2-e^x)$. Let $\varepsilon = 0.3748\cdots$ be the unique solution of the equation $(1+x)e^x = 2$. It is the abscissa of the maximum of f. Set

$$a = \sqrt[3]{\frac{2\varepsilon(2-e^\varepsilon)}{9}} = 0.3567\cdots$$

Let $n \geqslant 1$ and define $d = d(n)$ and $j(n)$ by

$$d(n) = \lceil an^{1/3}\rceil \quad \text{and} \quad j := j(n) := \left\lceil \frac{2\varepsilon n}{9d^2}\right\rceil.$$

We suppose $n \geqslant 30$ so that $2 \leqslant d \leqslant 1+n/2$ and $1 \leqslant j \leqslant n/2$. Since $d^4/n^2 \asymp d/n = \mathcal{O}(n^{-2/3})$, we find $\omega(d,n) = 2n + 2d^2 + \mathcal{O}(d)$, and then

$$\tau(d,n) = 1 - \frac{3d^2}{n} + \mathcal{O}(1/n^{2/3}) \quad \text{and} \quad \alpha(d,n) = 1 - \frac{9d^2}{2n} + \mathcal{O}(1/n^{2/3}),$$

(where $\alpha(d,n)$ is defined in Lemma 11.2). In particular, by choice of j, we have

$$\begin{aligned} \alpha(d,n)^j = \exp\big(j\log(\alpha(d,n))\big) &= \exp\Big(-\frac{9jd^2}{2n} + \mathcal{O}(1/n^{1/3})\Big) \\ &= e^{-\varepsilon} + \mathcal{O}(1/n^{1/3}). \end{aligned} \tag{11.7}$$

Since

$$\frac{j(2j-1)\tau(d,n)}{(n-2)\tau(d,n)+n} = \mathcal{O}(1/n^{1/3}),$$

there exists $N_1 \geqslant 30$ such that condition (11.5) of Lemma 11.2 is fulfilled for each $n \geqslant N_1$. Thus, for $k = 0,\dots,j$, we have

$$\tau_k(d,n) \geqslant \alpha(d,n)^j\tau(d,n) - \frac{j(2j-1)\tau(d,n)^2}{2n-2} = e^{-\varepsilon} + \mathcal{O}(1/n^{1/3}).$$

In particular $d_j = 2j - 1 - (j-1)/\tau_j(d,n)$ satisfies

$$d_j \geqslant j\big(2 - e^{\varepsilon}\big) + \mathcal{O}(1) = \frac{2\varepsilon(2-e^{\varepsilon})n}{9d^2} + \mathcal{O}(1) = \frac{a^3 n}{d^2} + \mathcal{O}(1) = d + \mathcal{O}(1).$$

On the other hand, we have

$$\mu(d,n) = \frac{n}{(2n-d)\tau} = \frac{1}{2} + \mathcal{O}(1/n^{1/3}).$$

Since $e^{-\varepsilon} > 1/2$, by (11.7) there exists $N_2 \geqslant N_1$ such that condition (11.1) of Theorem 11.2 is fulfilled for each $n \geqslant N_2$. We conclude that for any $n \geqslant N_2$ and any transcendental real number ξ, we have

$$\widehat{\omega}_n(\xi) \leqslant 2n - \min\{d, d_i\} = 2n - d + \mathcal{O}(1). \qquad \square$$

Appendix. Twisted heights

The purpose of this appendix is to give another interpretation of the quantity $\mathcal{D}_\xi(V)$ defined in Section 8.2. Our first approach was actually to work with the heights $\mathcal{H}_T$ defined below. We are thankful to Damien Roy for pointed out the link with Hodge's duality.

Fix $A \in \mathrm{GL}(\mathbb{R}^{m+1})$ and let V be a k-dimensional subspace of $\mathbb{R}^{m+1}$ defined over $\mathbb{Q}$. Its (twisted) height $\mathcal{H}_A(V)$ is defined as the covolume of the lattice $A(V \cap \mathbb{Z}^{m+1})$ inside the subspace $A(V)$ (with the convention that $\mathcal{H}_A(V) = 1$ if $V = \{0\}$). Explicitly, we have

$$\mathcal{H}_A(V) := \|A\boldsymbol{x}_1 \wedge \cdots \wedge A\boldsymbol{x}_k\|_2\,, \tag{A.1}$$

where $(\boldsymbol{x}_1,\dots,\boldsymbol{x}_k)$ is any $\mathbb{Z}$-basis of the lattice $V \cap \mathbb{Z}^{m+1}$. Then Schmidt's inequality generalizes as follows

$$\mathcal{H}_A(U+V)\mathcal{H}_A(U\cap V) \leqslant \mathcal{H}_A(U)\mathcal{H}_A(V) \tag{A.2}$$

for any subspaces U, V of $\mathbb{R}^{m+1}$ defined over $\mathbb{Q}$. The proof is the same as for rational subspaces (see [27, Ch. I, Lem. 8A] and [13, §5]). Similarly to Marnat and Moshchevitin [13, §5], we consider twisted heights of the following form. Let $T > 1$ be a parameter. We define the matrix $A_{m,T} \in \mathrm{GL}(\mathbb{R}^{m+1})$ as

$$A_{m,T} = \begin{pmatrix} T^m & 0 & \dots & 0 \\ 0 & T^{-1} & & \\ \vdots & & \ddots & \\ 0 & & & T^{-1} \end{pmatrix} \begin{pmatrix} 1 & \xi & \cdots & & \xi^m \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ & & & & \\ 0 & \cdots & & 0 & 1 \end{pmatrix},$$

so that for each polynomial $P = a_0 + \cdots + a_m X^m \in \mathbb{Z}[X]_{\leqslant m}$ (identified to a point of $\mathbb{R}^{m+1}$ via (8.9)), we have

$$A_{m,T} \begin{pmatrix} a_0 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} T^m P(\xi) \\ T^{-1} a_1 \\ \vdots \\ T^{-1} a_m \end{pmatrix}. \tag{A.3}$$

We denote by $\mathcal{H}_{m,T}$ (or simply $\mathcal{H}_T$ if there is no ambiguity about the integer m) the twisted height $\mathcal{H}_A$ associated to the matrix $A = A_{m,T}$. Note that

$$\mathcal{H}_T\big(\mathbb{R}[X]_{\leqslant m}\big) = \mathcal{H}_T(\mathbb{R}^{m+1}) = \det(A) = 1.$$

Definition A.1. — Let V be a subspace of $\mathbb{R}[X]_{\leqslant m}$ defined over $\mathbb{Q}$. We set

$$\mathcal{D}'_\xi(V) = \lim_{T\to+\infty} T^{-\operatorname{codim}(V)} \mathcal{H}_{m,T}(V),$$

where $\operatorname{codim}(V) = m + 1 - \dim(V)$ denotes the codimension of the space V inside $\mathbb{R}[X]_{\leqslant m}$. In particular, $\mathcal{D}'_\xi(\mathbb{R}[X]_{\leqslant n}) = 1$, and for any primitive polynomial $P \in \mathbb{Z}[X]_{\leqslant m}$, we have

$$\mathcal{D}'_\xi\big(\langle P\rangle_{\mathbb{R}}\big) = |P(\xi)| = \mathcal{D}_\xi\big(\langle P\rangle_{\mathbb{R}}\big).$$

Our goal is now to prove that for any non-zero subspace $V \subseteq \mathbb{R}[X]_{\leqslant m} \simeq \mathbb{R}^{m+1}$ defined over $\mathbb{Q}$, we have

$$\mathcal{D}'_\xi(V) \asymp \mathcal{D}_\xi(V),$$

where $\mathcal{D}_\xi$ is as in Definition 8.1 (and the implicit constant depends on m and ξ only). First, note that since $\dim(U + V) + \dim(U \cap V) = \dim U + \dim V$ for any subspaces U, V of $\mathbb{R}[X]_{\leqslant m}$, we deduce from (A.2) (with $A = A_{m,T}$) the following version of Schmidt's inequality, which is the analog of Proposition 8.2

$$\mathcal{D}'_\xi(U + V)\mathcal{D}'_\xi(U \cap V) \leqslant \mathcal{D}'_\xi(U)\mathcal{D}'_\xi(V), \tag{A.4}$$

valid for any U, V of $\mathbb{R}[X]_{\leqslant m}$ defined over $\mathbb{Q}$.

Proposition A.2. — *Let V be a k-dimensional subspace of $\mathbb{R}^{m+1}$ defined over $\mathbb{Q}$, with $1 \leqslant k \leqslant m+1$, and set $\Xi_m = (1, \xi, \dots, \xi^m)$. We have*

$$\mathcal{D}_\xi(V) \ll \mathcal{D}'_\xi(V) \leqslant \mathcal{D}_\xi(V), \tag{A.5}$$

where the implicit constant depends on ξ and m only. Moreover, for any $\mathbb{Z}$-basis $(\boldsymbol{x}_1,\dots,\boldsymbol{x}_k)$ of $V\cap\mathbb{Z}^{m+1}$, we have

$$\mathcal{D}'_\xi(V)=\Big\|\sum_{i=1}^{k}(-1)^{k-i}\,(\Xi_m\mid\boldsymbol{x}_i)\;\boldsymbol{x}_1^+\wedge\cdots\wedge\widehat{\boldsymbol{x}_i^+}\wedge\cdots\wedge\boldsymbol{x}_k^+\Big\|_2,\tag{A.6}$$

where $\boldsymbol{x}_i^+\in\mathbb{Z}^m$ denotes the point $\boldsymbol{x}_i$ deprived of its first coordinate.

Before to prove this result, we introduce some notation that we will need in the proof. Given two positive integers p and q, we define $\mathcal{I}(p,q)$ as the set of p-tuples $(i_1,\dots,i_p)$ of integers with $1\leqslant i_1<\cdots<i_p\leqslant q$. Let $\boldsymbol{e}=(\boldsymbol{e}_1,\dots,\boldsymbol{e}_q)$ be the canonical basis of $\mathbb{R}^q$. For any $I\in\mathcal{I}(p,q)$ as above, set $\boldsymbol{e}_I=\boldsymbol{e}_{i_1}\wedge\cdots\wedge\boldsymbol{e}_{i_p}\in\bigwedge^p\mathbb{R}^q$. For any $\boldsymbol{X}\in\bigwedge^p\mathbb{R}^q$, we call *$I$-coordinate of $\boldsymbol{X}$* its $\boldsymbol{e}_I$-coordinate in the basis $(\boldsymbol{e}_J)_{J\in\mathcal{I}(p,q)}$. For any $\boldsymbol{x}_1,\dots,\boldsymbol{x}_p\in\mathbb{R}^q$, we denote by $M(\boldsymbol{x}_1,\dots,\boldsymbol{x}_p)$ the $q\times p$ matrix whose columns are $\boldsymbol{x}_1,\dots,\boldsymbol{x}_p$ written in the basis $\boldsymbol{e}$, and by $\mathfrak{D}_I(\boldsymbol{x}_1,\dots,\boldsymbol{x}_p)$ the minor formed by the rows of $M(\boldsymbol{x}_1,\dots,\boldsymbol{x}_p)$ of index i in I. Then, writing $\boldsymbol{X}=\boldsymbol{x}_1\wedge\cdots\wedge\boldsymbol{x}_k$, we have the classical formulas

$$\boldsymbol{X}=\sum_{I\in\mathcal{I}(p,q)}\mathfrak{D}_I(\boldsymbol{x}_1,\dots,\boldsymbol{x}_p)\boldsymbol{e}_I\quad\text{and}\quad\|\boldsymbol{X}\|_2^2=\sum_{I\in\mathcal{I}(p,q)}\mathfrak{D}_I(\boldsymbol{x}_1,\dots,\boldsymbol{x}_p)^2.\tag{A.7}$$

Therefore, for each $I\in\mathcal{I}(p,q)$, the I-coordinate of $\boldsymbol{X}$ is $\mathfrak{D}_I(\boldsymbol{x}_1,\dots,\boldsymbol{x}_p)$.

Proof of Proposition A.2. — Fix $T\geqslant 1$ and for $i=1,\dots,k$, set

$$\boldsymbol{Z}=\sum_{i=1}^{k}p_i\boldsymbol{x}_1\wedge\cdots\wedge\widehat{\boldsymbol{x}}_i\wedge\cdots\wedge\boldsymbol{x}_k,\quad\text{where } p_i=(-1)^{i+1}\,(\Xi_m\mid\boldsymbol{x}_i)\,,$$

$$\boldsymbol{Y}=\lim_{T\to+\infty}T^{-m+k-1}\boldsymbol{y}_1(T)\wedge\cdots\wedge\boldsymbol{y}_k(T),\quad\text{where } \boldsymbol{y}_i=\boldsymbol{y}_i(T)=A_{m,T}(\boldsymbol{x}_i)\in\mathbb{R}^{m+1}.$$

By (8.4) we have

$$\mathcal{D}_\xi(V)=\|\boldsymbol{Z}\|_2\quad\text{and}\quad\mathcal{D}'_\xi(V)=\|\boldsymbol{Y}\|_2\,.$$

We prove the following properties. For $i=1,\dots,k$ we set $\boldsymbol{z}_i=((\Xi_m\mid\boldsymbol{x}_i)\,,\boldsymbol{x}_i)\in\mathbb{R}^{m+2}$.

(i) For each $J=(1,j_2,\dots,j_k)\in\mathcal{I}(k,m+2)$, the J-coordinate of $\boldsymbol{z}_1\wedge\cdots\wedge\boldsymbol{z}_k$ is equal to the K-coordinate of $\boldsymbol{Z}$, where $K=(j_2-1,\dots,j_k-1)\in\mathcal{I}(k-1,m+1)$.

Fix $I=(i_1,\dots,i_k)\in\mathcal{I}(k,m+1)$.

(ii) If $i_1\geqslant 2$, then the I-coordinate of $\boldsymbol{Y}$ is equal to 0.

(iii) If $i_1=1$, then the I-coordinate of $\boldsymbol{Y}$ is equal to the J-coordinate of $\boldsymbol{z}_1\wedge\cdots\wedge\boldsymbol{z}_k$, where $J=(1,i_2+1,\dots,i_k+1)$. It is also equal to the K-coordinate of $\boldsymbol{Z}$, where $K=(i_2,\dots,i_k)$.

To prove the first assertion, it suffices to expand the determinant $\mathfrak{D}_J(\boldsymbol{z}_1,\dots,\boldsymbol{z}_k)$ along its first row. Let $I=(i_1,\dots,i_k)\in\mathcal{I}(k,m+1)$. Suppose first that $i_1\neq 1$. Then, by Hadamard's inequality, the I-coordinate of $\boldsymbol{y}_1(T)\wedge\cdots\wedge\boldsymbol{y}_k(T)$ satisfies

$$|D_I(\boldsymbol{y}_1,\dots,\boldsymbol{y}_k)|\ll\prod_{j=1}^{k}T^{-1}\|\boldsymbol{x}_j\|=\mathcal{O}(T^{-k}),$$

and we deduce that the I-coordinate of $\boldsymbol{Y}$ is equal to 0, which proves assertion (ii). Suppose now that $i_1 = 1$ and set $J = (1, i_2+1, \dots, i_k+1)$. Then

$$D_I(\boldsymbol{y}_1, \dots, \boldsymbol{y}_k) = T^{m+1-k}\mathfrak{D}_J(\boldsymbol{z}_1, \dots, \boldsymbol{z}_k),$$

hence the first part of (iii). The second part is obtained by combining the above with assertion (i).

We deduce from the last two assertions that all the non-zero coordinates of $\boldsymbol{Y}$ are coordinates of $\boldsymbol{Z}$, thus $\|\boldsymbol{Y}\|_2 \leqslant \|\boldsymbol{Z}\|_2$, which proves the second inequality in (A.5). For the first estimate, we need to estimate the K-coordinates of $\boldsymbol{Z}$ with $K \in \mathcal{I}(k-1, m+1)$ of the form $(1, i_2, \dots, i_{k-1})$. According to assertion (i), they are exactly the determinants $\mathfrak{D}_J(\boldsymbol{z}_1, \dots, \boldsymbol{z}_k)$ with $J = (1, 2, j_3, \dots, j_k)$ in $\mathcal{I}(k, m+2)$.

Fix a $J \in \mathcal{I}(k, m+2)$ as above. The second row of the matrix $M(\boldsymbol{z}_1, \dots, \boldsymbol{z}_k)$ is a linear combination of the remaining rows (with coefficients in absolute value between 1 and $|\xi|^m$). We deduce that $\mathfrak{D}_J(\boldsymbol{z}_1, \dots, \boldsymbol{z}_k)$ can be written as a linear combination of $\mathfrak{D}_{J'}(\boldsymbol{z}_1, \dots, \boldsymbol{z}_k)$, where J' belong to the subset of $\mathcal{I}(k, m+2)$ consisting in the k-tuples whose second element is $\geqslant 3$. By assertion (iii), they are all coordinates of $\boldsymbol{Y}$, hence $|\mathfrak{D}_J(\boldsymbol{z}_1, \dots, \boldsymbol{z}_k)| \ll \|\boldsymbol{Y}\|_2$. We conclude that $\|\boldsymbol{Z}\|_2 \ll \|\boldsymbol{Y}\|_2$.

Finally, fix $(i_2, \dots, i_k) \in \mathcal{I}(k-1, m)$ and set $K = (i_2+1, \dots, i_k+1)$. By definition of $\boldsymbol{Z}$, the K-coordinate of $\boldsymbol{Z}$ is equal to

$$\sum_{i=1}^{k} p_i \mathfrak{D}_I(\boldsymbol{x}_1, \cdots, \widehat{\boldsymbol{x}_i}, \cdots, \boldsymbol{x}_k) = \sum_{i=1}^{k} p_i \mathfrak{D}_J(\boldsymbol{x}_1^+, \cdots, \widehat{\boldsymbol{x}_i^+}, \cdots, \boldsymbol{x}_k^+).$$

By assertion (iii), this is also the $(1, K)$-coordinate of $\boldsymbol{Y}$. So, the set of non-zero coordinates of $\boldsymbol{Y}$ is exactly equal to the set of non-zero coordinates of the point

$$\sum_{i=1}^{k} p_i\, \boldsymbol{x}_1^+ \wedge \cdots \wedge \widehat{\boldsymbol{x}_i^+} \wedge \cdots \wedge \boldsymbol{x}_k^+.$$

Equation (A.6) follows from the second identity of (A.7). $\square$

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Manuscript received 16th May 2024
accepted 18th July 2024

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