Journal de l'École polytechnique
Mathématiques

Ning Guo & Fei Liu

Purity and quasi-split torsors over Prüfer bases

Tome 11 (2024), p. 187-246.

https://doi.org/10.5802/jep.253

Publié avec le soutien
du Centre National de la Recherche Scientifique

Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 2270-518X

Journal de l'École polytechnique
Mathématiques
Tome 11, 2024, p. 187–246 DOI: 10.5802/jep.253

PURITY AND QUASI-SPLIT TORSORS OVER PRÜFER BASES

by Ning Guo & Fei Liu

Abstract. — We establish an analogue of the Zariski–Nagata purity theorem for finite étale covers on smooth schemes over Prüfer rings by demonstrating Auslander's flatness criterion in this non-Noetherian context. We derive an Auslander–Buchsbaum formula for general local rings, which provides a useful tool for studying the algebraic structures involved in our work. Through the analysis of reflexive sheaves, we prove various purity theorems for torsors under certain group algebraic spaces, such as the reductive ones. Specifically, using results from [EGA IV_4] on parafactoriality on smooth schemes over normal bases, we prove the purity for cohomology groups of multiplicative type groups at this level of generality. Subsequently, we leverage the aforementioned purity results to resolve the Grothendieck–Serre conjecture for torsors under a quasi-split reductive group scheme over schemes smooth over Prüfer rings. Along the way, we also prove a version of the Nisnevich purity conjecture for quasi-split reductive group schemes in our Prüferian context, inspired by the recent work of Česnavičius [Čes22b].

Résumé (Pureté et torseurs quasi-déployés sur les bases de Prüfer). — Nous établissons un analogue du théorème de pureté de Zariski–Nagata pour les revêtements étales sur les schémas lisses sur les anneaux de Prüfer en démontrant le critère de platitude d'Auslander dans ce contexte non noethérien. Nous dérivons une formule d'Auslander–Buchsbaum pour les anneaux locaux généraux, qui fournit un outil utile pour étudier les structures algébriques impliquées dans notre travail. Grâce à l'analyse des faisceaux réflexifs, nous prouvons divers théorèmes de pureté pour les torseurs sous certains espaces algébriques en groupes, notamment ceux qui sont réductifs. En particulier, en utilisant des résultats de [EGA IV_4] sur la parafactorialité sur les schémas lisses sur des bases normales, nous prouvons la pureté pour les groupes de cohomologie des groupes de type multiplicatif à ce niveau de généralité. Ensuite, nous utilisons les résultats de pureté susmentionnés pour résoudre la conjecture de Grothendieck–Serre pour les torseurs sous un schéma en groupes réductifs quasi-déployés sur des schémas lisses sur des anneaux de Prüfer. Nous prouvons également une version de la conjecture de pureté de Nisnevich pour les schémas en groupes réductifs quasi-déployés dans notre contexte prüferien, inspirée par les travaux récents de Česnavičius [Čes22b].

Mathematical subject classification (2020). — 14F22, 14F20, 14G22, 16K50.
Keywords. — Purity, Zariski–Nagata, Auslander–Buchsbaum, Grothendieck–Serre, vector bundles, principal bundles, Prüfer rings, torsors, homogeneous spaces, group schemes, valuation rings.

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 851146). This work was done under support of the grant №075-15-2022-289 for creation and development of Euler International Mathematical Institute.

e-ISSN: 2270-518X http://jep.centre-mersenne.org/

Contents

1. Introduction

1.1. Purity and regularity. — In algebraic geometry, purity refers to a diverse range of phenomena in which certain invariants or categories associated to geometric objects are insensitive to the removal of closed subsets of large codimensions. In the classical Noetherian world, purities, say, for vector bundles (and even torsors), or for finite étale covers, are intimately related to the regularities measured by lengths of regular sequences of geometric objects. For a concrete instance, the Auslander–Buchsbaum formula [AB57, Th. 3.7]

$$\operatorname{depth}_R M + \operatorname{proj.dim}_R M = \operatorname{depth}_R R$$

controls the projective dimension of the finite type module M over the Noetherian local ring R via depths, leading to the purity for vector bundles on regular local rings of dimension 2 ([Sam64, Prop. 2]). Granted this, Colliot-Thélène and Sansuc [CTS79, Th. 6.13] established the purity for reductive torsors over arbitrary regular local ring R of dimension 2 by bootstrapping from the vector bundle case: for every reductive R-group scheme G, the restriction map

$$H^1_{\text{ét}}(\operatorname{Spec} R, G) \xrightarrow{\sim} H^1_{\text{ét}}(\operatorname{Spec} R \smallsetminus \{\mathfrak{m}_R\}, G)$$

is bijective. Nevertheless, not only does the term 'regularity' make sense for Noetherian rings, its non-Noetherian generalization can still enlighten us to contemplate purity problems.

1.2. Regularity and its avatar over Prüfer bases. — The concept of regularity for non-Noetherian rings was first introduced by Bertin, as found in references [Ber71] and [Ber72, Def. 3.5], specifically for coherent local rings. A ring R is termed *regular* if every finitely generated ideal of R possesses a finite projective dimension. According to Serre's homological characterization [Ser56, Th. 3], this definition aligns with the traditional understanding of regularity in the context of Noetherian rings. A typical non-Noetherian example can be found in *Prüfer domains*. By definition, these are domains whose all local rings as valuation rings. Recall that an integral domain V is a *valuation ring* if every pair $a, b \in V \smallsetminus \{0\}$ satisfies either $a \in (b)$ or $b \in (a)$. As a case in point, Noetherian valuation rings are precisely either discrete valuation rings or fields. The regularity of Prüfer domains is a direct consequence of the characteristic that all finitely generated ideals of valuation rings are principal.(1) As a further instance, every smooth algebra over a Prüfer domain is coherent regular (Lemma 3.7). For recent research on the regularity of schemes over Prüfer domains, see [Kna03], [Kna08]. In addition to their regularity and other properties (see e.g. Lemma 3.2), Prüfer domains are ubiquitous in the study of nonarchimedean geometry, Zariski–Riemann spaces, and other fields, which motivates further investigation of their algebro-geometric properties.

1.3. Auslander–Buchsbaum for general local rings. — Let A be a local ring with quasi-compact punctured spectrum and M an A-module having a finite resolution by finite free A-modules. We have the following Auslander–Buchsbaum formula

$$\operatorname{proj.dim}_A(M) + \operatorname{depth}_A(M) = \operatorname{depth}_A(A).$$

Here $\operatorname{proj.dim}_A(0) = -\infty$ and $\operatorname{depth}_A M$ is the smallest i such that the i-th local cohomology of M is nonzero (Section 2.5). Our proof is significantly different from the classical case [AB57, Th. 3.7]. Specifically, we bypass the interpretation of projective dimensions in terms of Tor functors, which is a crucial ingredient in Auslander–Buchsbaum's argument.

1.4. Basic setup I. — The purity part of the present article focuses on a semilocal Prüfer domain R with $\dim R > 0$ (and with $\dim R < \infty$ if necessary), an R-flat finite type algebraic space X with regular R-fibres, and a closed subspace $Z \subset X$ such that $j\colon X \smallsetminus Z \hookrightarrow X$ is quasi-compact. For a point $x \in X$ lying in an open subscheme, the local ring of X at x makes sense and we denote $A := \mathscr{O}_{X,x}$. When involving torsors on X, we let G be an X-group algebraic space that étale locally permits an embedding $G \hookrightarrow \mathrm{GL}_n$ such that GL_n / G is X-affine. This condition is fulfilled if G is X-reductive,(2) or finite locally free.

1.5. Purity for torsors on smooth relative curves over Prüfer bases. — Once the projective dimensions of reflexive sheaves on X are controlled, by imposing codimensional constraints on Z, we may extend vector bundles on $X \smallsetminus Z$ to X, as in Noetherian scenarios. Subsequently, this allows us to obtain the purity Theorem 6.3 for G-torsors (where G is as in the Basic setup I): if X is an S-curve and Z satisfies

$$Z_\eta = \varnothing \quad \text{for each generic point } \eta \in S \quad \text{and} \quad \operatorname{codim}(Z_s, X_s) \geqslant 1 \text{ for all } s \in S,$$

(1) To elucidate, given a Prüfer domain R (which is coherent), any partial resolution $0 \to N \to R^n \to I \to 0$ of a finitely generated ideal $I \subset R$ results in N being finitely presented. As N is finite free over each local ring of R, it is consequently finite projective over R.

(2) Namely, it is a smooth affine X-group algebraic space G whose geometric X-fibres are (connected) reductive algebraic groups.

then restriction induces the following equivalence of categories of G-torsors

$$\mathbf{Tors}(X_{\mathrm{fppf}}, G) \xrightarrow{\sim} \mathbf{Tors}((X \smallsetminus Z)_{\mathrm{fppf}}, G).$$

In particular, passing to isomorphism classes of objects, we have the bijection

$$H^1_{\mathrm{fppf}}(X, G) \simeq H^1_{\mathrm{fppf}}(X \smallsetminus Z, G)$$

of nonabelian cohomology pointed sets. Meanwhile, a local version Theorem 6.5 allows us to loose constraints on the relative dimension of X: if either

$$\begin{aligned} &x \in X_\eta \quad \text{with } \dim \mathscr{O}_{X_\eta, x} = 2, \text{ or} \\ &x \in X_s \quad \text{with } s \neq \eta \text{ and } \dim \mathscr{O}_{X_s, x} = 1, \end{aligned}$$

then every G-torsor over $\operatorname{Spec} \mathscr{O}_{X,x} \smallsetminus \{x\}$ extends uniquely to a G-torsor over $\mathscr{O}_{X,x}$. This permits us to iteratively extend reductive torsors beyond a closed subset of higher fiberwise codimensions.

1.6. Zariski–Nagata over Prüfer bases. — The Zariski–Nagata purity, known as "purity of branch locus", states that every finite extension $A \subset B$ of rings with A regular Noetherian and B normal is unramified if and only if so it is in codimension one on $\operatorname{Spec} B$. This purity was settled by Zariski [Zar58] in a geometric context, and more algebraically by Nagata [Nag59] based on Chow's local Bertini theorem. In contrast to them, Auslander gave an alternative proof [Aus62, Th. 1.4] by skillful homological methods leading to a criterion for flatness. In [SGA 2_{new}, Exp. X, §3], Grothendieck reformulated their results into a purity concerning finite étale covers and proved this purity on a Noetherian local ring that is a complete intersection of dimension $\geqslant 3$ by reducing the assertion to hypersurfaces via several passages involving formal completions. Nevertheless, a practical deficiency of the latter argument is that, even over a rank-one valuation ring V with pseudo-uniformizer ϖ, the coherence of the ϖ-adic completion $\widehat{A}$ of a certain local V-algebra A is unknown to us, not to mention the crucial primary decomposition on it (that will guarantee a certain finiteness result). To circumvent this technical obstacle, we revert to Auslander's argument by establishing a Prüferian counterpart Theorem 4.1 of the criterion for flatness [Aus62, Th. 1.3]. Granted this, we acquire the Prüferian Zariski–Nagata Theorem 6.9: the pullback

$$\mathrm{F\acute{E}t}_X \xrightarrow{\sim} \mathrm{F\acute{E}t}_{X \smallsetminus Z} \quad \text{is an equivalence}$$

for every closed subset $Z \subset X$ in the Basic setup I Section 1.4 that satisfies the conditions

$$\begin{cases} \operatorname{codim}(Z_\eta, X_\eta) \geqslant 2 & \text{for each generic point } \eta \in S, \text{ and} \\ \operatorname{codim}(Z_s, X_s) \geqslant 1 & \text{for all } s \in S. \end{cases}$$

In particular, if X is connected and $\overline{x}\colon \operatorname{Spec} \Omega \to X \smallsetminus Z$ is a geometric point, then

$$\text{the map} \quad \pi_1^{\text{ét}}(X \smallsetminus Z, \overline{x}) \longrightarrow \pi_1^{\text{ét}}(X, \overline{x}) \quad \text{is bijective.}$$

1.7. Basic setup II. — The rest of this section deals mainly with the following. For a semilocal Prüfer domain R with fraction field K, an integral R-smooth scheme X, the semilocalization $A := \mathscr{O}_{X,\boldsymbol{x}}$ of X at a finite subset $\boldsymbol{x} \subset X$ contained in a single affine open of X, and a *quasi-split* reductive A-group scheme G, we study the trivialization behaviour of G-torsors.

1.8. Grothendieck–Serre for quasi-split groups. — The Grothendieck–Serre conjecture predicts that every torsor under a reductive group scheme G over a regular local ring A is trivial if it becomes trivial over $\operatorname{Frac} A$. This conjecture was settled in the affirmative when A contains a field (that is, A is of equicharacteristic), but in the mixed characteristic case, except for several sporadic or low dimensional cases, the conjecture remains open beyond quasi-split groups [Česká22a]. For a detailed review of the state of the art in this area, see [Pan18, §5] , as well as [GL23, §1.2] for a summary of recent developments. In this paper, we prove Theorem 8.1(ii), thereby generalizing the main result of [Čes22a] to the Prüferian context: in the basic setup Section 1.7, assume in addition that the Prüfer ring R is of Krull dimension 1, then every generically trivial G-torsor is trivial, that is, we have

$$\operatorname{Ker}\left(H^1(A,G) \longrightarrow H^1(\operatorname{Frac} A, G)\right) = \{*\}.$$

The proof follows a similar strategy of [Čes22a] (with its earlier version given by Fedorov [Fed22]), and the key input is our toral version of purity Theorem 7.9 and Grothendieck–Serre type Proposition 7.11 in this context. More precisely, by the valuative criterion of properness, a generically trivial torsor on X, say, reduces to a generically trivial torsor under a Borel B away from a closed subset Z of X that has codimension $\geqslant 2$ (resp. $\geqslant 1$) in the generic (resp. non-generic) R-fiber. Further, thanks to the aforementioned toral purity and Grothendieck–Serre type results, the above B-torsor even reduces to a $\operatorname{rad}^u(B)$-torsor on $X \smallsetminus Z$. Then, with the help of the geometric Lemma 8.2 (unfortunately, whose validity imposes the dimension-1 constraint on R), we can reduce to studying torsors over the relative affine line via excision and patchings, and we then conclude by [GL23, Th. 5.1].

1.9. A version of Nisnevich's purity conjecture for quasi-split groups

Now, we turn to Nisnevich's purity conjecture, where we require the total isotropicity of group schemes. A reductive group scheme G over a scheme S is *totally isotropic* at a point $s \in S$ if in the following canonical decomposition

$$G^{\mathrm{ad}}_{\mathscr{O}_{S,s}} \cong \textstyle\prod_i \operatorname{Res}_{A_i/\mathscr{O}_{S,s}}(G_i)$$

(cf. [SGA $3_{\mathrm{III\ new}}$, Exp. XXIV, Prop. 5.10(i)]) every G_i contains a $\mathbb{G}_{m,A_i}$, where $\mathscr{O}_{S,s} \to A_i$ is finite étale, and G_i is an adjoint semisimple A_i-group scheme whose geometric A_i-fibres have connected Dynkin diagram of fixed type i. If this holds for all $s \in S$, then G is *totally isotropic.* For instance, tori and quasi-split group schemes are totally isotropic.

Proposed by Nisnevich [Nis89, Conj. 1.3] and modified due to the anisotropic counterexamples of Fedorov [Fed22, Prop. 4.1], the Nisnevich conjecture predicts that, for a regular semilocal ring R, a regular parameter r (i.e., $r \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ for every maximal ideal $\mathfrak{m} \subset R$), and a reductive R-group scheme G such that $G_{R/rR}$ is totally isotropic, every generically trivial G-torsor on $R[\frac{1}{r}]$ is trivial, namely,

$$\mathrm{Ker}(H^1(R[1/r], G) \to H^1(\mathrm{Frac}\, R, G)) = \{*\}.$$

The case when R is a local ring of a regular affine variety over a field and $G = \mathrm{GL}_n$ was settled by Bhatwadekar–Rao [BR83] and was subsequently extended to arbitrary regular local rings containing fields by Popescu [Pop02, Th. 1]. Nisnevich in [Nis89] proved the conjecture in dimension two, assuming that R is a local ring with infinite residue field and that G is quasi-split. For the state of the art, the conjecture was settled in equicharacteristic case and in several mixed characteristic case by Česnavičius in [Čes22b, Th. 1.3] (previously, Fedorov [Fed21] proved the case when R contains an infinite field). Besides, the toral case and some low dimensional cases are known and surveyed in [Čes22b, §3.4.2(1)] including Gabber's result [Gab81, Chap. 1, Th. 1] for the local case $\dim R \leqslant 3$ when G is either GL_n or PGL_n. We prove Theorem 8.1(i): in the Basic Setup II Section 1.7,

$$\mathrm{Ker}\left(H^1(A \otimes_R K, G) \longrightarrow H^1(\mathrm{Frac}\, A, G)\right) = \{*\}.$$

1.10. Notations and conventions. — All rings in this paper are commutative with units, unless stated otherwise. Also, we adopt the notion of *normal schemes* as in [EGA I, 4.1.4], that is, they are schemes whose all local rings are integrally closed domains. A *Prüfer scheme* is a scheme that is covered by spectra of Prüfer domains. For a point s of a scheme (resp. for a prime ideal $\mathfrak{p}$ of a ring), we let $\kappa(s)$ (resp. $\kappa(\mathfrak{p})$) denote its residue field. For a global section s of a scheme S, we write $S[1/s]$ for the open locus where s does not vanish. For a ring A, we let $\mathrm{Frac}\, A$ denote its total ring of fractions. For a morphism of algebraic spaces $S' \to S$, we let $(-)_{S'}$ denote the base change functor from S to S'; if $S = \mathrm{Spec}\, R$ and $S' = \mathrm{Spec}\, R'$ are both affine schemes, we will often write $(-)_{R'}$ for $(-)_{S'}$.

Let S be an algebraic space, and let G be an S-group algebraic space. For an S-algebraic space T, by a G-torsor over T we shall mean a $G_T := G \times_R T$-torsor (see Definition 5.2). Denote by $\mathbf{Tors}(S_{\mathrm{fppf}}, G)$ (resp. $\mathbf{Tors}(S_{\text{ét}}, G)$) the groupoid of G-torsors on S that are fppf locally (resp. étale locally) trivial; specifically, if G is S-smooth (e.g. G is S-reductive, see below), then every fppf locally trivial G-torsor is étale locally trivial, so we have

$$\mathbf{Tors}(S_{\mathrm{fppf}}, G) = \mathbf{Tors}(S_{\text{ét}}, G).$$

For a scheme X, let $\mathbf{Pic}(X)$ denote the category of invertible $\mathscr{O}_X$-modules. When X is locally coherent (Section 2.1), let $\mathscr{O}_X$-$\mathbf{Rflx}$ denote the category of reflexive $\mathscr{O}_X$-modules.

Let S be an algebraic space. By a reductive S-group algebraic space we mean a smooth affine S-group algebraic space whose geometric S-fibres are (connected) reductive algebraic groups. When S is a scheme, this aligns with the definition of reductive S-group schemes presented in [SGA $3_{\mathrm{III\ new}}$].

Acknowledgements. — The authors wish to express their gratitude to Kęstutis Česnavičius and Ivan Panin for their unwavering support throughout the development of this work. Additionally, the authors extend their appreciation to Matthew Morrow and Colliot-Thélène for proposing the Grothendieck–Serre conjecture on smooth schemes over semilocal Prüfer rings during the defense of the first author. The authors also thank Kęstutis Česnavičius, Arnab Kundu, Shang Li, Ivan Panin, and Jiandi Zou for insightful discussions on various aspects of the article over the past few months. The authors are especially grateful to Laurent Moret-Bailly for providing useful remarks and correspondence, which have helped to improve the quality of the work. The authors would like to thank Enlin Yang from Peking University and Weizhe Zheng from the Morningside Center of Mathematics (Chinese Academy of Sciences) for their hospitality. Finally, the authors would like to express their gratitude to the referees for their careful reading, insightful remarks and suggestions, which significantly enhanced the quality of this paper.

2. Reflexive sheaves and depths

2.1. Locally coherent schemes. — A module M finitely generated over a ring A is *coherent* if its finitely generated A-submodules are all finitely presented. A ring A is *coherent* if it is coherent as an A-module. It is worth noting that one can determine the coherence of either a module or a ring Zariski locally. Coherent rings characteristically encompass Noetherian rings, but of primary significance to our discussion are finitely generated flat algebras over Prüfer domains, as referenced in Lemma 3.7.

On a scheme X, a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is *coherent* if there exists an affine open cover $X=\cup_i U_i$ such that $\mathscr{F}(U_i)$ is a coherent $\mathscr{O}_X(U_i)$-module for every i. In such instances, this property holds true for all affine open covers of X. A scheme X is *locally coherent* if $\mathscr{O}_X$ is coherent as an $\mathscr{O}_X$-module. A locally coherent scheme is *coherent* if it is quasi-compact quasi-separated.

Given a scheme X, the *dual* of an $\mathscr{O}_X$-module $\mathscr{F}$ is defined as

$$\mathscr{F}^\vee := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{O}_X).$$

Lemma 2.2. — *Let X be a scheme and let $\mathscr{F}$ and $\mathscr{G}$ be coherent $\mathscr{O}_X$-modules.*

(i) *If $\mathscr{F}\xrightarrow{f}\mathscr{G}$ is a morphism of $\mathscr{O}_X$-modules, then* $\operatorname{Ker} f$ *and* $\operatorname{Coker} f$ *are coherent.*

(ii) *Assume that X is integral. If $\mathscr{G}$ is $\mathscr{O}_X$-torsion-free, so is $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$. In particular, $\mathscr{F}^\vee$ is $\mathscr{O}_X$-torsion-free.*

Now, assume that X is locally coherent.

(iii) *An $\mathscr{O}_X$-module is coherent if and only if it is Zariski locally finitely presented.*

(iv) *$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$ is coherent. In particular, $\mathscr{F}^\vee$ is coherent.*

Proof. — We will argue Zariski locally on X. For (i), see [Stacks, 01BY]. For (ii), we apply $\mathscr{H}om_{\mathscr{O}_X}(-,\mathscr{G})$ to $\mathscr{O}_X^I\twoheadrightarrow\mathscr{F}$ and get an embedding $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})\hookrightarrow\prod_I\mathscr{G}$.

As $\prod_I \mathscr{G}$ is $\mathscr{O}_X$-torsion-free, so is $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$. For (iii), see [Stacks, 05CX]. For (iv), we apply (iii) to choose a finite presentation $\mathscr{O}_X^{\oplus m} \to \mathscr{O}_X^{\oplus n} \to \mathscr{F} \to 0$ and take $\mathscr{H}om_{\mathscr{O}_X}(-, \mathscr{G})$ of it. Then $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is a kernel of a map of coherent modules, so by (i) is coherent. $\square$

2.2.1. *Reflexive sheaves.* — For a locally coherent scheme X and a coherent $\mathscr{O}_X$-module $\mathscr{F}$, consider the canonical map

$$\phi_{\mathscr{F}} \colon \mathscr{F} \longrightarrow \mathscr{F}^{\vee\vee}.$$

If $\phi_{\mathscr{F}}$ is an isomorphism, then $\mathscr{F}$ is *reflexive*. We let $\mathscr{O}_X$-**Rflx** denote the category of reflexive $\mathscr{O}_X$-modules. For instance, every vector bundle on X is reflexive. An $\mathscr{O}_X$-module $\mathscr{G}$ is *Zariski locally finitely copresented* if it Zariski locally fits into an exact sequence $0 \to \mathscr{G} \to \mathscr{O}_X^{\oplus m} \to \mathscr{O}_X^{\oplus n}$ for some integers m and n. If X is further assumed to be integral, then the following Lemma 2.3(ii) shows that, for every coherent $\mathscr{O}_X$-module $\mathscr{G}$, the double dual $\mathscr{G}^{\vee\vee}$ is $\mathscr{O}_X$-reflexive, hence $\mathscr{G}^{\vee\vee}$ is called the *reflexive hull* of $\mathscr{G}$.

Lemma 2.3. — *Let X be an integral locally coherent scheme and let $\mathscr{F}$ and $\mathscr{G}$ be coherent $\mathscr{O}_X$-modules.*

(i) *The double dual $\phi_{\mathscr{F}} \colon \mathscr{F} \to \mathscr{F}^{\vee\vee}$ is injective if and only if $\mathscr{F}$ is $\mathscr{O}_X$-torsion-free. In particular, all reflexive $\mathscr{O}_X$-modules are $\mathscr{O}_X$-torsion-free.*

(ii) *If $\mathscr{G}$ is reflexive, then so is $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$. Therefore, the dual of any coherent $\mathscr{O}_X$-module is reflexive.*

(iii) *$\mathscr{F}$ is reflexive if and only if it is Zariski locally finitely copresented, if and only if there are a finite locally free $\mathscr{O}_X$-module $\mathscr{L}$ and a torsion-free $\mathscr{O}_X$-module $\mathscr{N}$ fitting into the short exact sequence*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{L} \longrightarrow \mathscr{N} \longrightarrow 0.$$

(iv) *If $\mathscr{G}$ is reflexive, then the following natural map $\mathscr{H}om_{\mathscr{O}_X}(\phi_{\mathscr{F}}, \mathscr{G})$*

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}^{\vee\vee}, \mathscr{G}) \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$$

is an isomorphism.

Proof. — We may prove all the assertions Zariski locally.

(i) By Lemma 2.2(ii), the injectivity of $\phi_{\mathscr{F}}$ implies that $\mathscr{F}$ is $\mathscr{O}_X$-torsion-free. Conversely, if $\mathscr{F}$ is torsion-free, then locally $\mathscr{F} \subset \mathscr{O}_X^{\oplus n}$ for some integer n; taking double dual, we find that the composite $\mathscr{F} \to \mathscr{F}^{\vee\vee} \to \mathscr{O}_X^{\oplus n}$ is injective, so $\phi_{\mathscr{F}}$ is also injective.

(ii) We apply Lemma 2.2(iii) to choose a finite presentation and we take its $\mathscr{H}om_{\mathscr{O}_X}(-, \mathscr{G})$. So we get an exact sequence $0 \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \xrightarrow{u} \mathscr{G}^{\oplus m} \to \mathscr{G}^{\oplus n}$ for some integers m and n. Taking double dual of this exact sequence, we obtain the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) \arrow[r, "u"] \arrow[d] & \mathscr{G}^{\oplus m} \arrow[r] \arrow[d] & \mathscr{G}^{\oplus n} \arrow[d] \\ & \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})^{\vee\vee} \arrow[r, "u^{\vee\vee}"] & (\mathscr{G}^{\vee\vee})^{\oplus m} \arrow[r] & (\mathscr{G}^{\vee\vee})^{\oplus n} \end{tikzcd}$$

the reflexivity of $\mathscr{G}$ and diagram chase reduces us to showing that $u^{\vee\vee}$ is injective. Take $(-)\otimes_{\mathscr{O}_X} K(X)$ of the exact sequence

$$(\mathscr{G}^{\vee})^{\oplus m} \xrightarrow{u^{\vee}} \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})^{\vee} \longrightarrow \mathrm{Coker}(u^{\vee}) \longrightarrow 0.$$

Since $\mathscr{F}$ is finitely presented, by [Stacks, 0583], we have $\mathrm{Coker}(u^{\vee})_{K(X)} = 0$, namely, $\mathrm{Coker}(u^{\vee})$ is torsion. Thus $\mathrm{Ker}(u^{\vee\vee}) = \mathrm{Coker}(u^{\vee})^{\vee} = 0$, as desired.

(iii) If $\mathscr{F}$ is reflexive, we choose a finite presentation of $\mathscr{F}^{\vee}$ (which is coherent by Lemma 2.2(iv)) and then taking its dual yields a finite copresentation of $\mathscr{F}^{\vee\vee}\simeq\mathscr{F}$. Conversely, if $\mathscr{F}$ is finitely copresented as $0\to\mathscr{F}\to\mathscr{O}_X^{\oplus m}\xrightarrow{v}\mathscr{O}_X^{\oplus n}$, then $\mathscr{F}\simeq \mathrm{Coker}(v^{\vee})^{\vee}$, which is reflexive by (ii). Taking $\mathscr{L} := \mathscr{O}_X^{\oplus m}$ and $\mathscr{N} := \mathrm{Im}(v)$, we obtain the desired short exact sequence.

(iv) Pick a finite copresentation $\mathscr{O}_X^{\oplus n}\to\mathscr{O}_X^{\oplus m}\to\mathscr{G}\to 0$ and take $\mathscr{H}om(\phi_{\mathscr{F}},-)$ on it. We have the commutative diagram of $\mathscr{O}_X$-modules with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}^{\vee\vee},\mathscr{G}) \arrow[r] \arrow[d, "\mathscr{H}om_{\mathscr{O}_X}(\phi_{\mathscr{F}},\mathscr{G})"'] & \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}^{\vee\vee},\mathscr{O}_X^{\oplus m}) \arrow[r] \arrow[d] & \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}^{\vee\vee},\mathscr{O}_X^{\oplus n}) \arrow[d] \\ 0 \arrow[r] & \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) \arrow[r] & \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{O}_X^{\oplus m}) \arrow[r] & \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{O}_X^{\oplus n}) \end{tikzcd}$$

By the assertion (ii), $\mathscr{F}^{\vee}$ is reflexive, hence $\phi_{\mathscr{F}}^{\vee,m},\phi_{\mathscr{F}}^{\vee,n}$ are bijective and so is $\mathscr{H}om_{\mathscr{O}_X}(\phi_{\mathscr{F}},\mathscr{G})$. □

By reflexive hull, reflexive sheaves extend from quasi-compact open (cf. [GR18, Prop. 11.3.8(i)]).

Corollary 2.4. — *For a quasi-compact open U in a coherent integral scheme X, the restriction $\mathscr{O}_X\text{-}\mathbf{Rflx}\to\mathscr{O}_U\text{-}\mathbf{Rflx}$ is essentially surjective.*

Proof. — A reflexive $\mathscr{O}_U$-module $\mathscr{F}$, by [Stacks, 0G41] and Lemma 2.2(iii), extends to a coherent $\mathscr{O}_X$-module $\widetilde{\mathscr{F}}$. By Lemma 2.3(ii), the reflexive $\widetilde{\mathscr{F}}^{\vee\vee}$ extends $\mathscr{F}$. □

Now, we recall the notion of depth in terms of local cohomology and use it to describe reflexive sheaves.

2.5. Depths. — For a scheme X and an open immersion $j\colon U\hookrightarrow X$ with closed complement $i\colon Z := X\smallsetminus U\hookrightarrow X$, consider the functor

$$\underline{\Gamma}_Z\colon \mathbf{Z}_X\text{-}\mathbf{Mod}\longrightarrow \mathbf{Z}_X\text{-}\mathbf{Mod}\qquad \mathscr{F}\longmapsto \mathrm{Ker}(\mathscr{F}\longrightarrow j_*j^*\mathscr{F})$$

which sends an abelian sheaf $\mathscr{F}$ on X to its largest subsheaf supported on Z. The functor $\underline{\Gamma}_Z$ is left exact, giving rise to a right derived functor

$$R\underline{\Gamma}_Z : D^+(\mathbf{Z}_X\text{-}\mathbf{Mod}) \longrightarrow D^+(\mathbf{Z}_X\text{-}\mathbf{Mod}).$$

Explicitly, for an object $K \in D^+(\mathbf{Z}_X\text{-}\mathbf{Mod})$, taking an injective resolution $K \to I^\bullet$, one computes $R\underline{\Gamma}_Z(K)$ via $R\underline{\Gamma}_Z(I^\bullet)$. Thus, $R\underline{\Gamma}_Z$ actually factors over the canonical functor $D^+(\mathbf{Z}_Z\text{-}\mathbf{Mod}) \to D^+(\mathbf{Z}_X\text{-}\mathbf{Mod})$ and satisfies $\underline{\Gamma}_Z \simeq R^0\underline{\Gamma}_Z$. Moreover, we have the following exact sequence $0 \to \underline{\Gamma}_Z(I^\bullet) \to I^\bullet \to j_*j^*(I^\bullet) \to 0$, giving rise to the functorial distinguished triangle

$$R\underline{\Gamma}_Z(K) \longrightarrow K \longrightarrow Rj_*j^*(K) \longrightarrow R\underline{\Gamma}_Z(K)[1].$$

Assume now that $j\colon U \hookrightarrow X$ is *quasi-compact*. Then, for any quasi-coherent sheaf $\mathscr{F}$ on X, from the above triangle we see that $R\underline{\Gamma}_Z(\mathscr{F})$ has quasi-coherent cohomology sheaves as $Rj_*j^*(\mathscr{F})$ does so. Consider the functor $\Gamma_Z := \Gamma(X,-) \circ \underline{\Gamma}_Z$. Since the functor $\underline{\Gamma}_Z$ sends injective sheaves to injective sheaves ($\underline{\Gamma}_Z$ admits an exact left adjoint i_*i^*), the derived functor $R\Gamma_Z$ of Γ_Z is canonically isomorphic to $R\Gamma(X,-) \circ R\underline{\Gamma}_Z$. Therefore, if $X = \operatorname{Spec} A$ is affine and M is an A-module, by the quasi-coherence of $R^i\underline{\Gamma}_Z(\widetilde{M})$ we get $R^i\underline{\Gamma}_Z(\widetilde{M}) \simeq (R^i\Gamma_Z(\widetilde{M}))^\sim$ for all $i \geqslant 0$; in particular, all the A-modules $R^i\Gamma_Z(\widetilde{M})$ are supported on Z. Moreover, we have the following distinguished triangle

$$R\Gamma_Z(\widetilde{M}) \longrightarrow M \longrightarrow R\Gamma(U,\widetilde{M}) \longrightarrow R\Gamma_Z(\widetilde{M})[1],$$

which gives an exact sequence $0 \to \Gamma_Z(\widetilde{M}) \to M \to \Gamma(U,\widetilde{M}) \to R^1\Gamma_Z(\widetilde{M}) \to 0$ and isomorphisms $R^i\Gamma_Z(\widetilde{M}) \simeq H^{i-1}(U,\widetilde{M})$ for all $i > 1$.

In what follows, we will mainly consider the case where $X = \operatorname{Spec} A$ for a local ring $(A, x := \mathfrak{m}_A)$ such that the punctured spectrum $U_A := \operatorname{Spec} A \smallsetminus \{x\}$ is *quasi-compact*. Denote by $j\colon U_A \hookrightarrow X$ the open immersion. In this case, we have seen that the cohomology modules $R^i\Gamma_x(\widetilde{M})$ are all supported on $\{x\}$, where M is an A-module. For simplicity, we shall often write $R^i\Gamma_x(M)$ for $R^i\Gamma_x(\widetilde{M})$. We warn readers that the derived functor $M \mapsto R^i\Gamma_x(M)$ here is obtained from an injective resolution of the associated quasi-coherent sheaf $\widetilde{M}$, not from an injective resolution of the A-module M. These two approaches do not, in general, produce equivalent theories, unless A is Noetherian (see [Stacks, 0A6P]).

As defined in [GR18, Def. 10.4.14], the *depth* of an A-module M is given by

$$\operatorname{depth}_A(M) := \sup\{n \in \boldsymbol{Z} \mid R^i\Gamma_x M = 0 \text{ for all } i < n\} \in \boldsymbol{Z}_{\geqslant 0} \cup \{+\infty\}.$$

For a *finitely generated* A-module N supported on $\{x\}$, consider a closely related quantity

$$\tau_N(M) := \sup\{n \in \boldsymbol{Z} \mid \operatorname{Ext}^i_A(N, M) = 0 \ \text{ for all } i < n\} \in \boldsymbol{Z}_{\geqslant 0} \cup \{+\infty\}.$$

We note that by the finite generation of N, all the A-modules $\operatorname{Ext}^i_A(N,M)$ are supported on $\{x\}$.

Lemma 2.6. — *Let $(A, \mathfrak{m}_A)$ be a local ring with quasi-compact punctured spectrum. Let M be an A-module and N a finitely generated A-module supported on $\{\mathfrak{m}_A\}$. Let $(f_1, \cdots, f_r)$ be an M-regular sequence in $\mathfrak{m}_A$. We have (notations as in Section 2.5)*

$$\operatorname{depth}_A(M) = \operatorname{depth}_A(M/\textstyle\sum_{i=1}^r f_i M) + r \quad \textit{and} \quad \tau_N(M) = \tau_N(M/\textstyle\sum_{i=1}^r f_i M) + r.$$

Proof. — Denote $x := \mathfrak{m}_A$ as the closed point of $\operatorname{Spec} A$. The two equalities are proved similarly, so we only treat the one concerning depths. By induction on r, it suffices to consider for a nonzero $f \in \mathfrak{m}_A$ the following short exact sequence

$$0 \longrightarrow M \xrightarrow{f} M \longrightarrow M/fM \longrightarrow 0$$

to show that $\operatorname{depth}_A(M) = \operatorname{depth}_A(M/fM) + 1$. We have a long exact sequence

$$\cdots \to R^{i-1}\Gamma_x M \xrightarrow{f} R^{i-1}\Gamma_x M \to R^{i-1}\Gamma_x(M/fM) \to R^i\Gamma_x M \xrightarrow{f} R^i\Gamma_x M \to \cdots.$$

Assume that for an integer n we have $\operatorname{depth}_A(M) \geqslant n+1$, so $R^i\Gamma_x M = 0$ for $0 \leqslant i \leqslant n$. Then the displayed exact sequence implies that $R^i\Gamma_x(M/fM) = 0$ for $0 \leqslant i \leqslant n-1$, so $\operatorname{depth}_A(M/fM) \geqslant n$. Conversely, if $\operatorname{depth}_A(M/fM) \geqslant n$, then $R^i\Gamma_x(M/fM) = 0$ for $0 \leqslant i \leqslant n-1$. The displayed exact sequence implies that $R^i\Gamma_x M = 0$ for $0 \leqslant i \leqslant n-1$, and there is an injection $R^n\Gamma_x M \xhookrightarrow{\times f} R^n\Gamma_x M$. However, since the A-module $R^n\Gamma_x M$ is supported on $\{x\}$ and $f \in \mathfrak{m}_A$, we deduce that $R^n\Gamma_x M = 0$, that is, $\operatorname{depth}_A(M) \geqslant n+1$. $\square$

Examples

(2.7) Consider a valuation ring V that is not a field. If the punctured spectrum U_V is quasi-compact (e.g. when $0 < \dim V < \infty$), then there exists an $f \in \mathfrak{m}_V$ such that $\dim(V/fV) = 0$, which implies $\operatorname{depth}_V(V/fV) = 0$. From the formula in Lemma 2.6, we deduce that $\operatorname{depth}_V(V) = 1$. Conversely, if U_V is not quasi-compact, then one can show that $\operatorname{depth}_V(V) \geqslant 2$.

(2.8) Assume that A is a Noetherian local ring and $N = A/I$ for an ideal $I \subsetneq A$ (e.g. $N = A/\mathfrak{m}_A$). Then for any finitely generated A-module M, we have

$$\operatorname{depth}_A(M) = \tau_N(M).$$

Indeed, utilizing Lemma 2.6, one can check that both of them equal the length of any maximal M-regular sequence in $\mathfrak{m}_A$ (so the length is independent of all choices). However, this may be false when A is non-Noetherian. For instance, we let $A := V$ be a *non-discrete* valuation ring of finite rank and let $N := V/\mathfrak{m}_V$ be its residue field. Take $M = V/fV$ for a nonzero $f \in \mathfrak{m}_V$. Then,

– $\operatorname{depth}_V(V/fV) = 0$. Let $\mathfrak{p} \subsetneq \mathfrak{m}_V$ be the second largest prime ideal of V. The non-discreteness of V implies that $\mathfrak{m}_V \neq fV$, so we can pick $g \in \mathfrak{m}_V \smallsetminus (\mathfrak{p} \cup fV)$. Then $f_1 := f/g$ is in $\mathfrak{m}_V$, and its image $\overline{f}_1 \in V/fV$ has annihilator gV which strictly contains $\mathfrak{p}$, that is, $0 \neq \overline{f}_1 \in \Gamma_{\mathfrak{m}_V}(V/fV)$.

– For any nonzero element $\overline{h} \in V/fV$, we have $\mathfrak{m}_V\overline{h} \neq 0$, that is, $\operatorname{Hom}_V(V/\mathfrak{m}_V, V/fV) = 0$. Indeed, any such non-invertible $\overline{h}$ is represented by some $h \in \mathfrak{m}_V$ with $fV \subsetneq hV$. Then $\overline{h}$ has annihilator $(f/h)V$, which is strictly contained in $\mathfrak{m}_V$ because V is non-discrete valued.

Therefore, we have $\tau_{V/\mathfrak{m}_V}(V/fV) \geqslant 1 > 0 = \mathrm{depth}_V(V/fV)$.

Lemma 2.9. — *Let A be a local ring with quasi-compact punctured spectrum, and let M, N be A-modules.*

(i) *If* $\mathrm{depth}_A(N) \geqslant 1$, *then* $\mathrm{depth}_A(\mathrm{Hom}_A(M,N)) \geqslant 1$.

(ii) *If* $\mathrm{depth}_A(N) \geqslant 2$ *and M is finite over A, then* $\mathrm{depth}_A(\mathrm{Hom}_A(M,N)) \geqslant 2$.

(iii) *If A is coherent and M is a reflexive A-module, then*

$$\mathrm{depth}_A(M) \geqslant \min(2, \mathrm{depth}_A(A)).$$

Proof. — Pick a presentation $A^{\oplus J} \to A^{\oplus I} \to M \to 0$ and take $\mathrm{Hom}_A(-,N)$, we get a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M,N) \longrightarrow \prod_I N \longrightarrow N' \longrightarrow 0,$$

where $N' \subset \prod_J N$ for some index set J. Since $\mathrm{depth}_A(N) \geqslant 1$, by definition, we have $\mathrm{depth}_A(\prod_J N) \geqslant 1$ and $\mathrm{depth}_A(N') \geqslant 1$. For (ii), take $R\Gamma_x$ of this short exact sequence, where x is the closed point of $\mathrm{Spec}\, A$. Since $\mathrm{depth}_A(N) \geqslant 2$ and I is finite, we have $R\Gamma_x^i(\prod_I N) = \bigoplus_I R\Gamma_x^i N = 0$. Combine this with $R\Gamma_x^0 N' = 0$, both $R^0\Gamma_x$ and $R^1\Gamma_x$ of $\mathrm{Hom}_A(M,N)$ vanish, so we get $\mathrm{depth}_A(\mathrm{Hom}_A(M,N)) \geqslant 2$. The last assertion follows from (i) and (ii). $\square$

Corollary 2.10. — *Let X be an integral, coherent, and topologically locally Noetherian scheme and $j\colon U \hookrightarrow X$ an open immersion such that every $z \in X \smallsetminus U$ satisfies $\mathrm{depth}_{\mathscr{O}_{X,z}}(\mathscr{O}_{X,z}) \geqslant 2$. Then, every reflexive $\mathscr{O}_X$-module $\mathscr{F}$ satisfies*

$$\mathscr{F} \xrightarrow{\sim} j_* j^* \mathscr{F}.$$

Proof. — As U contains the generic point of X, the injectivity follows because $\mathscr{F}$ and $j_*j^*\mathscr{F}$ are subsheaves of their common generic stalk. To show the surjectivity, we show that every section $s \in \mathscr{F}(U)$ extends over X. Let $U_1 \subset X$ be the domain of definition of s; it is open and contains U. If $U_1 \neq X$, then every maximal point $z \in X \smallsetminus U_1$ is contained in $X \smallsetminus U$. In particular, we have $\mathrm{depth}_{\mathscr{O}_{X,z}}(\mathscr{O}_{X,z}) \geqslant 2$ by assumption. By Lemma 2.9(iii), the localization $\mathscr{F}_z := \mathscr{F}|_{\mathscr{O}_{X,z}}$ satisfies $\mathrm{depth}_{\mathscr{O}_{X,z}}(\mathscr{F}_z) \geqslant 2$, hence the restriction of s to $\mathrm{Spec}\,\mathscr{O}_{X,z} \cap U_1 = \mathrm{Spec}\,\mathscr{O}_{X,z} \smallsetminus \{z\}$ extends to a section $s_1 \in \Gamma(V_1, \mathscr{F})$ for an affine open neighborhood V_1 of z in X (by the quasi-compactness of the open immersion $U_1 \hookrightarrow X$). Since X is integral and $\mathscr{F}$ is $\mathscr{O}_X$-torsion-free, the two local sections s_1 and s agree on $V_1 \cap U$, and thus can be glued to a section over $V_1 \cup U$. This implies that z is already in U_1, a contradiction. $\square$

Corollary 2.11. — *Let X be an integral, coherent, and topologically locally Noetherian scheme. Let $j : U \hookrightarrow X$ be an open subscheme such that every $z \in X \smallsetminus U$ satisfies $\mathrm{depth}_{\mathscr{O}_{X,z}}(\mathscr{O}_{X,z}) \geqslant 2$. Then, taking j^* and j_* induce an equivalence of categories*

$$\mathscr{O}_X\text{-}\mathbf{Rflx} \xrightarrow{\sim} \mathscr{O}_U\text{-}\mathbf{Rflx}.$$

Proof. — By Corollary 2.4, the restriction functor is essentially surjective. To show the full faithfulness, we pick two reflexive $\mathscr{O}_X$-modules $\mathscr{F},\mathscr{G}$ and consider $\mathscr{H} := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$. By Lemma 2.3(ii), $\mathscr{H}$ is a reflexive $\mathscr{O}_X$-module, so Corollary 2.10 implies that $\mathscr{H}(X) \simeq \mathscr{H}(U)$, as desired. □

2.12. (Weakly) associated primes. — Let R be a ring and M an R-module. We define the set of *associated primes* of M, denoted as $\mathrm{Ass}_R(M)$, to be the collection of prime ideals $\mathfrak{p} \subset R$ for which $R^0\Gamma_{\mathfrak{p}} M_{\mathfrak{p}} \neq 0$. Note that in the literature, the associated primes referred to in this context are often termed as "weakly associated primes" (see [Stacks, 0546] and [Bou98, Chap. IV, §1, Ex. 17]). It then follows that $\mathrm{Ass}_R(M) \neq \varnothing$ whenever $M \neq 0$: for any nonzero $m \in M$, $\mathrm{Ass}_R(M)$ contains all the minimal elements of

$$\mathrm{Supp}(m) := \{\mathfrak{p} \in \mathrm{Spec}\, R \colon m_{\mathfrak{p}} \neq 0 \in M_{\mathfrak{p}}\}.$$

Now, we present Proposition 2.13, whose argument is pointed out by L. Moret-Bailly, as a generalization of its Noetherian case [SGA 2_{new}, Exp. III, Prop. 1.3]. In addition to its generality, this result leads to an alternative proof of Lemma A.6 concerning the "matrix of direct sums of modules".

Proposition 2.13. — *Let A be a ring, M a finitely presented A-module, and N an A-module. We have*

$$\mathrm{Ass}_A(\mathrm{Hom}_A(M,N)) = \mathrm{Supp}(M) \cap \mathrm{Ass}_A(N).$$

Proof. — It is clear that $\mathrm{Ass}_A(\mathrm{Hom}_A(M,N)) \subset \mathrm{Supp}(M) \cap \mathrm{Ass}_A(N)$, so it suffices to prove the converse inclusion. For a prime ideal $\mathfrak{p} \in \mathrm{Supp}(M) \cap \mathrm{Ass}_A(N)$, we have $\mathfrak{p} \in \mathrm{Supp}(M_{\mathfrak{p}}) \cap \mathrm{Ass}_{A_{\mathfrak{p}}}(N_{\mathfrak{p}})$. As M is finitely presented, it suffices to show that there is an $f \in \mathrm{Hom}_A(M,N)$ such that $\mathfrak{p}$ the minimal one in

$$\mathrm{Supp}(f) = \{\mathfrak{q} \in \mathrm{Spec}\, A \mid 0 \neq f_{\mathfrak{q}} \in \mathrm{Hom}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}}, N_{\mathfrak{q}})\}.$$

Hence we are reduced to showing the following local case Lemma 2.14. □

Lemma 2.14. — *Let $(A,\mathfrak{m})$ be a local ring, M a finitely presented nonzero A-module, and N an A-module with* $\mathrm{Supp}(N) = \{\mathfrak{m}\}$. *Then we have* $\mathrm{Hom}_A(M,N) \neq 0$.

Proof. — As N has an A-submodule of the form A/J for an ideal $J \subset A$ with $\mathrm{Supp}(J) = \{\mathfrak{m}\}$, we may replace both A and N by A/J and replace M by C/JC iteratively to assume that A is a local ring of dimension zero. It suffices to prove that $M^\vee \neq 0$. As M has a nonzero quotient of the form A/I for a finitely generated ideal $I \subset A$, it suffices to note that $(A/I)^\vee = \mathrm{Ann}(I) \neq 0$, because I is nilpotent. □

Lemma 2.15. — *Let R be a domain with topologically Noetherian spectrum and $\alpha \colon M \to N$ a morphism of R-modules. Then α is an isomorphism, provided that N is torsion-free and every prime $\mathfrak{p} \subset R$ satisfies:*

– *either $\alpha_{\mathfrak{p}} \colon M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is an isomorphism, or* $\mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geqslant 2$.

Proof. — Take $K := \mathrm{Ker}(\alpha)$. If $K \neq 0$, we choose an associated prime $\mathfrak{p} \in \mathrm{Ass}_R(K)$, then $0 \neq R^0\Gamma_\mathfrak{p} K_\mathfrak{p} \subset R^0\Gamma_\mathfrak{p} M_\mathfrak{p} = 0$ (since $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geqslant 2 > 0$), a contradiction. This proves that α is injective. Take $Q := \mathrm{Coker}(\alpha)$. If $Q \neq 0$, we choose an associated prime $\mathfrak{p} \in \mathrm{Ass}_R(Q)$. Since $R^0\Gamma_\mathfrak{p} M_\mathfrak{p} = R^1\Gamma_\mathfrak{p} M_\mathfrak{p} = R^0\Gamma_\mathfrak{p} N_\mathfrak{p} = 0$, taking $R\Gamma_\mathfrak{p}$ of the short exact sequence $0 \to M_\mathfrak{p} \to N_\mathfrak{p} \to Q_\mathfrak{p} \to 0$ yields that $R^0\Gamma_\mathfrak{p} Q_\mathfrak{p} = 0$, a contradiction. Thus α is also surjective. $\square$

2.15.1. *Weak dimensions*. — Recall the *weak dimension* of a ring B, denoted by $\mathrm{wdim}(B)$, is defined as

$$\mathrm{wdim}(B) = \sup\{\mathrm{fl.dim}_B(M) \mid M \text{ is a } B\text{-module}\},$$

the supremum of flat dimensions of all B-modules. For example, we know from [Stacks, 092S] that $\mathrm{wdim}(B) \leqslant 1$ if and only if all local rings of B are valuation rings. Note that if a B-module M has a (possibly infinite) resolution by finite free modules, then $\mathrm{fl.dim}_B(M) = \mathrm{proj.dim}_B(M)$. This holds in particular if B is coherent and M is finitely presented. Since any such M is an iterated extension of cyclic, finitely presented B-modules, we conclude that (for B coherent)

$$\mathrm{wdim}(B) = \sup\{\mathrm{proj.dim}(B/J) \mid J \subset B \text{ is a finitely generated ideal}\}.$$

2.16. Coherent regular rings. — A coherent ring R is termed *regular* if every finitely generated ideal of R has finite projective dimension. By Serre's homological characterization, one recovers the classical regularity for Noetherian rings. However, the primary focus of this paper is on the class of non-Noetherian regular rings (e.g. a flat finite type ring over a valuation ring with regular fibres, see Lemma 3.7). For readers' convenience, we briefly summarize the key properties of general regular coherent rings, which closely resemble their Noetherian counterparts. Other useful properties can be found in the appendix.

Theorem 2.17. — *For a coherent regular local ring A, the following assertions hold.*

(i) *A is a normal domain; more precisely*

(ii) *A is the intersection (in* $\mathrm{Frac}\, A$*) of its local rings which are valuation rings;*

(iii) *Any two nonzero elements of A have a greatest common divisor and a least common multiple;*

(iv) *If the punctured spectrum U_A is quasi-compact, then*

$$\mathrm{depth}_A(A) = \mathrm{wdim}(A) < \infty.$$

So, for every coherent local ring R with quasi-compact punctured spectrum,

$$\mathrm{wdim}(R) < \infty \quad \textit{if and only if} \quad R \textit{ is regular.}$$

Proof

(i) This follows from [Ber72, Cor. 4.3].

(ii) The reference for this is [Que71, Prop. 2.4]. Assuming the topological Noetherianness of $\mathrm{Spec}\, A$, this also follows from (i) and the more general Proposition 2.22 (or Theorem 2.21).

(iii) See [Que71, Prop. 3.2]. Assuming the topological Noetherianness of $X :=$ Spec A, this result can also be deduced from Theorem 2.20 as follows: consider two nonzero elements, f and g, in A. Now, examine the two (finitely generated) ideals, $fA + gA$ and $fA \cap gA$, of A. These two ideals define two coherent sheaves on X, which are invertible around each point of X whose local ring is a valuation ring, or, equivalently, has a weak dimension $\leqslant 1$ ([Stacks, 092S]). By Theorem 2.20, there exist two invertible sheaves on X that coincide with the two ideal sheaves at all local rings of X having weak dimension $\leqslant 1$. Since X is local, these two invertible sheaves are free, and, in view of (ii), any generators provide a greatest common divisor and a least common multiple of f and g.

(iv) The finiteness of $\mathrm{wdim}(A)$ under the quasi-compactness assumption on U_A was originally established by Quentel [Que71, Cor. 1.1]. We refer to Corollary A.8 for additional details. $\square$

Lemma 2.18. — *Let R be a coherent ring and $n \geqslant 2$ an integer. Denote $(-)^\vee := \mathrm{Hom}_R(-, R)$. Then*

$$\mathrm{wdim}(R) \leqslant n \quad \text{if and only if} \quad \mathrm{fl.dim}_R(M^\vee) \leqslant n-2 \text{ for every } R\text{-module } M.$$

Proof. — If $\mathrm{wdim}(R) \leqslant n$, then for any R-module M one takes a presentation $F_2 \to F_1 \to M \to 0$ for free R-modules F_1 and F_2. As R is coherent, by [Stacks, 05CZ] and Lazard's theorem [Stacks, 058G], $F_1^\vee$ and $F_2^\vee$ are flat, so the exact sequence $0 \to M^\vee \to F_1^\vee \to F_2^\vee \to C \to 0$ for $C := \mathrm{Coker}(F_1^\vee \to F_2^\vee)$ yields $\mathrm{fl.dim}_R(M^\vee) \leqslant n-2$. Conversely, to estimate $\mathrm{wdim}(R)$, it suffices to consider finitely presented R-modules and their flat dimensions. Every finitely presented R-module M fits into an exact sequence

$$0 \longrightarrow K \longrightarrow R^{\oplus m} \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0,$$

where $K := \mathrm{Ker}(R^{\oplus m} \to R^{\oplus n})$, so by Lemma 2.3(iii), K is a reflexive R-module. Thus, by assumption, we have $\mathrm{fl.dim}_R(M) = \mathrm{fl.dim}(K^{\vee\vee}) + 2 \leqslant n$. This gives the desired inequality $\mathrm{wdim}(R) \leqslant n$. $\square$

A locally coherent scheme is *regular* if it is covered by spectra of coherent regular rings.

Theorem 2.19. — *Let X be a locally coherent, integral scheme and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module.*

(i) *If $\mathscr{F}$ is reflexive at a point $x \in X$ and $\mathrm{wdim}\,\mathscr{O}_{X,x}$ is finite,*[3] *then we have*

$$\mathrm{proj.dim}_{\mathscr{O}_{X,x}} \mathscr{F}_x \leqslant \max(0, \mathrm{wdim}\,\mathscr{O}_{X,x} - 2).$$

(ii) *If $\mathrm{wdim}\,\mathscr{O}_{X,x} \leqslant 2$ for all $x \in X$, every reflexive $\mathscr{O}_X$-module is locally free.*

[3] For instance, when $\mathscr{O}_{X,x}$ is regular with a quasi-compact punctured spectrum; see Theorem 2.17(iv).

Proof. — We first note that $\operatorname{wdim}\mathscr{O}_{X,x}<\infty$ implies that $\mathscr{O}_{X,x}$ is regular, and the regularity of $\mathscr{O}_{X,x}$ guarantees that every coherent $\mathscr{O}_{X,x}$-module is a perfect complex. The assertion (ii) follows from (i). The assertion (i) follows from Lemma 2.18 because the perfectness of $\mathscr{F}_x$ leads to the following desired inequality

$$\operatorname{proj.dim}_{\mathscr{O}_{X,x}}(\mathscr{F}_x)=\operatorname{fl.dim}_{\mathscr{O}_{X,x}}(\mathscr{F}_x^{\vee\vee})\leqslant\max(0,\operatorname{wdim}\mathscr{O}_{X,x}-2).\qquad\square$$

Theorem 2.20. — *Let X be a topologically locally Noetherian, integral, locally coherent, regular scheme. For a closed subset $Z\subset X$ with $\operatorname{wdim}\mathscr{O}_{X,z}\geqslant 2$ for all $z\in Z$ and the canonical open immersion $j\colon X\smallsetminus Z\hookrightarrow X$, the restriction j^* and pushforward j_* induce the following equivalences of categories:*

$$\mathscr{O}_X\text{-}\mathbf{Rflx}\xrightarrow{\sim}\mathscr{O}_{X\smallsetminus Z}\text{-}\mathbf{Rflx},\qquad \mathbf{Pic}\,X\xrightarrow{\sim}\mathbf{Pic}\,X\smallsetminus Z.$$

In particular, for every scheme Y affine over X, we have a bijection of sets

$$Y(X)\simeq Y(X\smallsetminus Z).$$

Moreover, if $\operatorname{wdim}\mathscr{O}_{X,x}\leqslant 2$ for all $x\in X$, then we have an equivalence of categories:

$$\mathbf{Vect}(X)\xrightarrow{\sim}\mathbf{Vect}(X\smallsetminus Z).$$

Proof. — We first note that the regularity of X guarantees that every coherent sheaf on X is a perfect complex. The equivalence for vector bundles follows from that for reflexive sheaves and Theorem 2.19(ii). By Theorem 2.17(iv), the depth and weak dimension agree for each local ring of X, so the equivalence of reflexive sheaves follows from Corollary 2.11. For the assertion concerning **Pic**, by the result for reflexive modules, it is enough to show that, for every invertible $\mathscr{O}_{X\smallsetminus Z}$-module $\mathscr{I}$, its unique $\mathscr{O}_X$-reflexive extension $\mathscr{F}:=j_*\mathscr{I}$ is actually invertible. Consider the *category* $\mathbf{gr}.\mathbf{Pic}(X)$ *of graded invertible $\mathscr{O}_X$-modules*, whose objects are pairs $(\mathscr{L},\alpha)$ consisting of an invertible $\mathscr{O}_X$-module $\mathscr{L}$ and a locally constant function $\alpha\colon X\to\boldsymbol{Z}$, and morphisms $h\colon(\mathscr{L},\alpha)\to(\mathscr{M},\beta)$ satisfies $h_x=0$ if $\alpha(x)\neq\beta(x)$ for all $x\in X$. By [KM76, Th. 1], there is a unique determinant functor from $D(\mathscr{O}_X\text{-}\mathbf{Mod})^*_{\mathrm{perf}}$ the groupoid of perfect complexes of $\mathscr{O}_X$-modules

$$\mathbf{det}\colon D(\mathscr{O}_X\text{-}\mathbf{Mod})^*_{\mathrm{perf}}\longrightarrow\mathbf{gr}.\mathbf{Pic}(X).$$

Also, this functor **det** commutes with arbitrary base change. Thus, since the complex $\mathscr{F}[0]$ is perfect, by Theorem 2.19(i), the invertible $\mathscr{O}_X$-module $\mathbf{det}(\mathscr{F}[0])$ is well-defined, and we have isomorphisms

$$\mathbf{det}(\mathscr{F}[0])\xrightarrow{\sim}j_*j^*\mathbf{det}(\mathscr{F}[0])\xrightarrow{\sim}j_*\mathbf{det}(j^*\mathscr{F}[0])\xleftarrow{\sim}j_*\mathbf{det}(\mathscr{I}[0])\xleftarrow{\sim}j_*\mathscr{I}.$$

Here, the first isomorphism follows from the assertion concerning **Rflx**, and the second isomorphism used the base-change property of **det**. This shows that

$$\mathscr{F}\simeq\mathbf{det}(\mathscr{F}[0])\quad\text{ is invertible.}$$

Finally, since $X\smallsetminus Z$ is schematically-dense in X, the injectivity of the restriction map $Y(X)\to Y(X\smallsetminus Z)$ follows from [Stacks, 084N]. To prove the surjectivity, we (locally on X) write Y as the relative spectrum of a quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{A}$. It remains to observe that every morphism $\mathscr{A}_U \to \mathscr{O}_U$ gives rise to a morphism

$$\mathscr{A} \longrightarrow j_*\mathscr{A}_U \longrightarrow j_*\mathscr{O}_U \simeq \mathscr{O}_X. \qquad \square$$

The following Theorem 2.21 generalizes Serre's conditions $(R_1)+(S_2)$ to the coherent case. The key ingredient can be traced back to the research of the "multiplicative theory of ideals" by M. Zafrullah, cf. [Zaf78, Lem. 7].

Theorem 2.21. — *For a coherent domain R, the following are equivalent:*

(i) *R is normal;*

(ii) *R is the intersection (in $\operatorname{Frac} R$) of its local rings which are valuation rings;*

(iii) *every local ring A of R is either a valuation ring, or for any $f \in \mathfrak{m}_A$ we have*

$$\mathfrak{m}_A \notin \operatorname{Ass}(A/fA).$$

If local rings of R have topologically Noetherian spectra, then these are equivalent to

(†) *every local ring of R is either a valuation ring or of* depth $\geqslant 2$.

Proof. — First, for (iii) $\Leftrightarrow$ (†) under the Noetherian assumption, note that for every R-module M we have $\operatorname{Ass}(M) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = 0\}$, so it suffices to apply Lemma 2.6, since R is a domain.

The implication (ii) $\Rightarrow$ (i) is clear, since an intersection of integrally closed subrings is integrally closed.

For (iii) $\Rightarrow$ (ii), we need to show that, for nonzero elements $f, g \in R$, if $g \in fR_\mathfrak{p}$ whenever $R_\mathfrak{p}$ is a valuation ring, $\mathfrak{p} \in \operatorname{Spec} R$, then already $g \in fR$. If not, we consider the annihilator $\operatorname{Ann} \subsetneq R$ of $\overline{g} \in R/fR$ in R. By assumption on f, g, we have $\operatorname{Ann}_\mathfrak{p} = R_\mathfrak{p}$ whenever $R_\mathfrak{p}$ is a valuation ring. For any minimal element $\mathfrak{p}$ in $V(\operatorname{Ann}) \subset \operatorname{Spec} R$, since $\operatorname{Ann}_\mathfrak{p} \neq R_\mathfrak{p}$, we deduce that $R_\mathfrak{p}$ is not a valuation ring. But since $\mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass}(R_\mathfrak{p}/fR_\mathfrak{p})$, we have obtained a contradiction.

Finally, for (i) $\Rightarrow$ (iii), we may assume that R is a normal, coherent local domain. It suffices to show that, if there exists an $f \in \mathfrak{m}_R$ such that $\mathfrak{m}_R \in \operatorname{Ass}(R/fR)$, then R is a valuation ring.

Note that since R is coherent, the inverse J^{-1} of every finitely generated ideal $J \subset R$ is again finitely generated; under the normality assumption on R, we also have $J \subset (J^{-1})^{-1} \subset R$. Indeed, the first inclusion is clear, and for the second, any $x \in (J^{-1})^{-1}$ satisfies $xJ^{-1} \subset R \subset J^{-1}$ which, by the Cayley–Hamilton, implies that x is integral over R. Thus, $x \in R$, by normality.

For every ideal $I \subset R$ define I^t as the union of $(J^{-1})^{-1}$ for all finitely generated subideals $J \subset I$, which is again an ideal of R (unless it is equal to R, but this is shown to be impossible below). It is straightforward to check that $I_1^t \subset I_2^t$ whenever $I_1 \subset I_2 \subset R$, $(aI)^t = aI^t$ for every $a \in \operatorname{Frac} R \smallsetminus \{0\}$, and $I \subset I^t = (I^t)^t$ (e.g. using that any finitely generated ideal of R equals its triple inverse).

Claim 2.1. — *We have $\mathfrak{m}_R = \mathfrak{m}_R^t$. In particular, every subideal $J \subset \mathfrak{m}_R$ satisfies $J^t \subset \mathfrak{m}_R$.*

Proof of the claim. — It suffices to prove the first assertion. Consider the set of all ideals $I \subset R$ satisfying $I^t = I$. This set is stable under taking increasing unions (of ideals), so Zorn's lemma implies that every such I is contained in a maximal one (ordered by inclusion); denote by $\mathcal{M}$ the subset of such maximal elements. Then we have the following:

– every ideal $I \subset R$ not contained in any ideal in $\mathcal{M}$ satisfies $I^t = R$: if not, since the ideal $I^t \subset R$ satisfies $I \subset I^t = (I^t)^t$, it would be contained in some element of $\mathcal{M}$, a contradiction;

– every $\mathfrak{q} \in \mathcal{M}$ is a prime ideal: if $x, y \in R \smallsetminus \mathfrak{q}$ but $xy \in \mathfrak{q}$, then the inclusion $y(\mathfrak{q} + (x)) \subset \mathfrak{q}$ yields $y(\mathfrak{q} + (x))^t \subset \mathfrak{q}^t = \mathfrak{q}$, but $(\mathfrak{q} + (x))^t = R$ by the above, hence $y \in \mathfrak{q}$, a contradiction;

– $R = \bigcap_{\mathfrak{q}\in\mathcal{M}} R_{\mathfrak{q}}$. It suffices to show that each $a \in \bigcap_{\mathfrak{q}\in\mathcal{M}} R_{\mathfrak{q}}$ is contained in R. By assumption, we have $a^{-1}R \cap R \not\subset \mathfrak{q}$ for all $\mathfrak{q} \in \mathcal{M}$, so by the above we have $R = ((a^{-1}R \cap R)^{-1})^{-1}$. Taking a further inverse, we get $R = (a^{-1}R \cap R)^{-1}$, but the latter contains a, as desired.

Now we will use the crucial assumption that there exists an $f \in \mathfrak{m}_R$ such that $\mathfrak{m}_R \in \mathrm{Ass}(R/fR)$. This is equivalent to that, there exist $f, g \in R$ such that $g \notin fR$ and $\mathfrak{m}_R$ is the unique prime ideal of R containing Ann, the annihilator of $0 \neq \overline{g} \in R/fR$ in R. If $\mathfrak{m}_R \notin \mathcal{M}$ (i.e., $\mathfrak{m}_R \neq \mathfrak{m}_R^t$), then none of $\mathfrak{q} \in \mathcal{M}$ (which are primes) contains Ann, so that $\mathrm{Ann}_{\mathfrak{q}} = R_{\mathfrak{q}}$, i.e., $g \in fR_{\mathfrak{q}}$, for all $\mathfrak{q} \in \mathcal{M}$. But by the intersection formula $R = \bigcap_{\mathfrak{q}\in\mathcal{M}} R_{\mathfrak{q}}$, we would have $g \in fR$, a contradiction. □

Now we show that R is a valuation ring. By [Stacks, 090Q], it suffices to show that $I := (x, y)$ is principal for arbitrary nonzero elements $x, y \in R$. Observe that $(I \cdot I^{-1})^{-1} = R$: indeed, every $c \in (I \cdot I^{-1})^{-1}$ satisfies that $c \cdot I \cdot I^{-1} \subset R$, hence $c \cdot I^{-1} \subset I^{-1}$. By Cayley–Hamilton, c is integral over R, but R is normal, thus $c \in R$, as desired. Consequently, we have $(I \cdot I^{-1})^t = ((I \cdot I^{-1})^{-1})^{-1} = R$, which, by Claim 2.1, implies that $I \cdot I^{-1} = R$. Thus, I is invertible and so principal because R is local. □

Proposition 2.22. — *For an integral, locally coherent, and topologically locally Noetherian scheme X, a coherent $\mathscr{O}_X$-module $\mathscr{F}$, if X is normal, then the following conditions are equivalent:*

(i) *$\mathscr{F}$ is reflexive;*

(ii) *$\mathscr{F}$ is torsion-free and* $\mathrm{depth}_{\mathscr{O}_x}(\mathscr{F}_x) \geqslant \min(2, \mathrm{depth}_{\mathscr{O}_{X,x}}(\mathscr{O}_x))$ *for all $x \in X$;*

(iii) *$\mathscr{F}$ is torsion-free and we have the following equality*

$$\mathscr{F} = \textstyle\bigcap_{x\in X_0} \mathscr{F}_x \quad \textit{in the generic stalk } \mathscr{F}_K := \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}_X,$$

where $X_0 := \{x \in X \mid \mathrm{wdim}\, \mathscr{O}_{X,x} \leqslant 1\} = \{x \in X \mid \mathscr{O}_{X,x}$ *is a valuation ring*$\}$.

Proof. — The assertions (i)–(iii) are local, we may assume that $X = \mathrm{Spec}\, A$ is affine and $\mathscr{F} = \widetilde{M}$ for a coherent A-module M. Thus, A has quasi-compact punctured spectrum. By Theorem 2.21, every local ring of A is either of depth $\geqslant 2$ or a valuation ring. It follows from Lemmas 2.3(i) and 2.9 that (i) $\Rightarrow$ (ii). Assume that (ii) holds, then the inclusion map $M \to \bigcap_{\mathfrak{p}\in X_0} M_{\mathfrak{p}}$ is an isomorphism at every prime $\mathfrak{p} \in X_0$. For any prime ideal $\mathfrak{p} \notin X_0$, the assumption on depths implies that $\operatorname{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geqslant 2$, hence Lemma 2.15 gives that $M \to \bigcap_{\mathfrak{p}\in X_0} M_{\mathfrak{p}}$ is an isomorphism, namely, (iii) holds. Finally, assume that (iii) holds. By the assumption on X (or A), for every prime $\mathfrak{p} \in X_0$, the local ring $A_{\mathfrak{p}}$ is a valuation ring with topologically Noetherian spectrum. As a torsion-free finite $A_{\mathfrak{p}}$-module, $M_{\mathfrak{p}}$ is free, so it is reflexive. Thus, the reflexive hull $\phi_M : M \to M^{\vee\vee}$ is an isomorphism at all $\mathfrak{p} \in X_0$. Moreover, since $M^{\vee\vee}$ is reflexive (Lemma 2.3(ii)) and (i) $\Rightarrow$ (iii) is already proved, we have $M^{\vee\vee} = \bigcap_{\mathfrak{p}\in X_0}(M^{\vee\vee})_{\mathfrak{p}}$. Combining this with the assumption $M = \bigcap_{\mathfrak{p}\in X_0} M_{\mathfrak{p}}$, we see that $M^{\vee\vee} = \bigcap_{\mathfrak{p}\in X_0}(M^{\vee\vee})_{\mathfrak{p}} = \bigcap_{\mathfrak{p}\in X_0} M_{\mathfrak{p}} = M$ and ϕ_M is an isomorphism, that is, M is reflexive. □

Corollary 2.23. — *Let A be a topologically locally Noetherian normal coherent domain with fraction field K. Let L/K be an extension of fields and B the integral closure of A in L. If B is a coherent A-module, then B is reflexive over A.*

Proof. — By Proposition 2.22, it suffices to prove that $B = \bigcap_{\mathfrak{p}} B_{\mathfrak{p}}$ where $\mathfrak{p} \subset A$ ranges over prime ideals such that $A_{\mathfrak{p}}$ is a valuation ring. As B is normal, it remains to show that $B \supset \bigcap_{\mathfrak{p}} B_{\mathfrak{p}}$. Take $b \in \bigcap_{\mathfrak{p}} B_{\mathfrak{p}}$ and its minimal polynomial $F(x) = x^r + c_{r-1}x^{r-1} + \cdots + c_0 \in K[x]$. For each $\mathfrak{p} \subset A$, the extension of domains $B_{\mathfrak{p}}/A_{\mathfrak{p}}$ is finite, so $b \in B_{\mathfrak{p}}$ satisfies $F(x) \in \operatorname{Frac}(A_{\mathfrak{p}})[x] = K[x]$ and $c_i \in A_{\mathfrak{p}}$. Consequently, by Proposition 2.22 again, we have $c_i \in A$ for each i. In particular, $F(x) \in A[x]$ so $b \in B$, as desired. □

By using the criterion Proposition 2.22, we easily obtain the following result for deducing Theorem 6.8.

Corollary 2.24. — *Let $f : X \to Y$ be a finite, finitely presented, surjective morphism of topologically locally Noetherian, integral, normal, coherent schemes. Let $\mathscr{F}$ be a reflexive $\mathscr{O}_X$-module.*

(i) *$f_*\mathscr{F}$ is a reflexive $\mathscr{O}_Y$-module.*

(ii) *Let $j : Y \hookrightarrow \overline{Y}$ be an open immersion of integral coherent topologically locally Noetherian schemes. If* $\operatorname{depth} \mathscr{O}_{\overline{Y},y} \geqslant 2$ *for all $y \in \overline{Y} \smallsetminus Y$, then $j_*f_*\mathscr{O}_X$ is a reflexive $\mathscr{O}_{\overline{Y}}$-module, and the morphism*

$$\overline{f} : \underline{\operatorname{Spec}}_{\overline{Y}}(j_*f_*\mathscr{O}_X) \longrightarrow \overline{Y} \quad \text{is finite.}$$

In particular, $\underline{\operatorname{Spec}}_{\overline{Y}}(j_*f_*\mathscr{O}_X)$ is the relative normalization of $\overline{Y}$ in X.

3. Geometry of schemes over Prüfer bases

In this section, we recollect useful geometric properties of scheme over Prüfer bases.

3.1. Geometric properties and reduction methods

Lemma 3.2. — *For a Prüfer domain R with spectrum S, a finite type irreducible S-scheme X, a point $x \in X$ and its image $s \in S$, the following assertions hold*

(i) *all nonempty S-fibres have the same dimension;*

(ii) *if X is S-flat and X_s is generically reduced, then for any maximal point $\xi \in X_s$, we have an extension of valuation rings*

$$\mathscr{O}_{S,s} \hookrightarrow \mathscr{O}_{X,\xi}$$

inducing an isomorphism of value groups;

(iii) *for $x' \in X$ with $x' \neq x$ whose image is denoted by s', if $x \in \overline{\{x'\}}$, then*
 - *either $s = s'$, and we have $\dim(\mathscr{O}_{X_{s'},x'}) < \dim(\mathscr{O}_{X_s,x})$;*
 - *or $s \in \overline{\{s'\}}, s \neq s'$, and we have $\dim(\mathscr{O}_{X_{s'},x'}) \leqslant \dim(\mathscr{O}_{X_s,x})$.*

Proof. — For (i), see [EGA IV_3, Lem. 14.3.10]. For (ii), see [MB22, Th. A]. Now, to prove (iii), we may assume that X is affine and has a pure relative dimension, say, n, over R. By assumption, we have $s \in \overline{\{s'\}}$. Assume that we are not in the first case. The schematic closure $\overline{\{x'\}}$ is a dominant scheme of finite type over $\overline{\{s'\}}$ (which corresponds to the spectrum of a valuation ring). Therefore, by (i), all its non-empty fibres have the same dimension. Thus, we deduce from $\overline{\{x'\}} \supset \overline{\{x\}}$ that

$$\dim(\overline{\{x'\}}_{s'}) = \dim(\overline{\{x'\}}_s) \geqslant \dim(\overline{\{x\}}_s).$$

Hence, we have

$$\dim(\mathscr{O}_{X_{s'},x'}) = n - \dim(\overline{\{x'\}}_{s'}) \leqslant n - \dim(\overline{\{x\}}_s) = \dim(\mathscr{O}_{X_s,x}). \qquad \square$$

The following Lemma 3.3 provides us a passage to the case when there is a section.

Lemma 3.3. — *For a valuation ring V, an essentially finitely presented (resp essentially smooth[(4)]) V-local algebra A, there are an extension of valuation rings V'/V with trivial extension of value groups, and an essentially finitely presented (resp. essentially smooth) V-map $V' \to A$ with finite extension of residue fields.*

Proof. — Assume $A = \mathscr{O}_{X,x}$ for an affine scheme X finitely presented over V and a point $x \in X$ lying over the closed point $s \in \operatorname{Spec} V$. Denote $t = \operatorname{tr.deg}(\kappa(x)/\kappa(s))$. As $\kappa(x)$ is a finite extension of $\ell := \kappa(s)(a_1, \cdots, a_t)$ for a transcendence basis $(a_i)_1^t$ of $\kappa(x)/\kappa(s)$, we have $t = \dim_\ell \Omega^1_{\ell/\kappa(s)} \leqslant \dim_{\kappa(x)} \Omega^1_{\kappa(x)/\kappa(s)}$. Choose sections $b_1, \cdots, b_t \in \Gamma(X, \mathscr{O}_X)$ such that $d\overline{b_1}, \cdots, d\overline{b_t} \in \Omega^1_{\kappa(x)/\kappa(s)}$ are linearly independent over $\kappa(x)$, where the bar stands for their images in $\kappa(x)$. Define $p : X \to \mathbb{A}^t_V$ by sending the standard coordinates $T_1, \cdots, T_t$ of $\mathbb{A}^t_V$ to $b_1, \cdots, b_t$, respectively. Since $d\overline{b_1}, \cdots, d\overline{b_t} \in \Omega^1_{\kappa(x)/\kappa(s)}$ are linearly independent, the image $\eta := p(x)$ is the generic point of $\mathbb{A}^t_{\kappa(s)}$, so $V' := \mathscr{O}_{\mathbb{A}^t_V,\eta}$ is a valuation ring whose value group is $\Gamma_{V'} \simeq \Gamma_V$. Note that $\kappa(x)/\kappa(\eta)$ is finite, the map $V' \to A$ induces a finite residue fields extension.

When $V \to A$ is essentially smooth, the images of $db_1, \cdots, db_t$ under the map $\Omega^1_{X/V} \otimes \kappa(x) \to \Omega^1_{\kappa(x)/\kappa(s)}$ are linearly independent, so are their images in $\Omega^1_{X/V} \otimes \kappa(x)$. Hence, p is essentially smooth at x. $\square$

In the sequel, Lemma 3.4 (cf. [GL23, Lem. 2.2]), combined with limit arguments, often allows us to only consider Prüfer rings of finite Krull dimension.

(4)By definition, this means that A is a local ring of a smooth V-algebra.

Lemma 3.4. — *Every semilocal Prüfer domain R is a filtered union of its subrings R_i such that:*

(i) *for every i, R_i is a semilocal Prüfer domain of finite Krull dimension; and*

(ii) *for i large enough, $R_i \to R$ induces a bijection on the sets of maximal ideals hence is fpqc.*

3.4.1. *Geometric presentation for the Grothendieck–Serre.* — In both Fedorov's and Česnavičius' works on mixed characteristic Grothendieck–Serre, significant emphasis is placed on geometric results of a certain type, reminiscent of Gabber–Quillen, as demonstrated in [Fed22, Prop. 3.18] and [Čes22a, Var. 3.7], respectively. Similarly, in our context, we observe an analogous result to loc. cit. and record it below.

Lemma 3.5. — *Let*

(i) *R be a semilocal Prüfer ring;*

(ii) *$\overline{X}$ be a projective, flat R-scheme with fibres of pure dimension $d > 0$;*

(iii) *$X \subset \overline{X}$ be an open subscheme, smooth over R;*

(iv) *$\boldsymbol{x} \subset X$ be a finite subset;*

(v) *$Y \subset X$ be a closed subscheme which is R-fiberwise of codimension $\geqslant 1$; assume also that $\overline{Y} \smallsetminus Y$ is R-fiberwise of codimension $\geqslant 2$.*

Then, there are

(i) *an affine open $S \subset \mathbb{A}_R^{d-1}$ and an affine open neighbourhood $U \subset X$ of $\boldsymbol{x}$, and*

(ii) *a smooth morphism $\pi : U \to S$ of relative dimension 1*

such that $Y \cap U$ is S-finite.

Proof. — This can be proved similarly as [Čes22a, Var. 3.7]. □

3.6. Regularity and Reflexive sheaves over Prüfer bases. — Now, we turn to the special case of schemes over Prüfer bases. To begin with, we consider the coherence and calculation of depths over Prüfer bases, as the following Lemmas 3.7 and 3.8.

Lemma 3.7. — *Let X be a scheme that is flat and locally of finite type over a Prüfer domain R.*

(i) *X is locally of finite presentation and locally coherent.*

(ii) *For every point $x \in X$, the local ring $\mathscr{O}_{X,x}$ is coherent.*

(iii) *If $x \in X$ lies over $s \in \operatorname{Spec} R$ and $\mathscr{O}_{X_s,x}$ is regular, then every finitely presented $\mathscr{O}_{X,x}$-module has a finite resolution by finite free $\mathscr{O}_{X,x}$-modules of length $\leqslant \dim \mathscr{O}_{X_s,x} + 1$. In particular, $\mathscr{O}_{X,x}$ is coherent regular.*

(iv) *The local ring $\mathscr{O}_{X,x}$ in* (iii) *is a normal domain.*

Proof. — We may assume that $X = \operatorname{Spec} A$ is affine for a finite type flat R-algebra A.

For (i), by [RG71, Cor. 3.4.7] (or [Nag66, Th. 3'] when R is a valuation ring), every finitely generated, flat algebra over a domain is finitely presented. So A is finitely presented over R. The coherence of A thus follows from the following facts: 1) any polynomial ring over a Prüfer domain R is coherent ([Gla89, Cor. 7.3.4]); 2) the quotient of a coherent ring by a finitely generated ideal is again coherent.

The assertion (ii) follows because each local ring of a coherent ring is coherent. For (iii), see [GR18, Prop. 11.4.1]. Finally, (iv) follows from a more general result Theorem 2.17(i). $\square$

By the above Lemma 3.7, a flat, locally of finite type scheme over a valuation ring is locally coherent. Further, when the fibres are Cohen-Macaulay, the depth of its local ring can be computed as follows.

Lemma 3.8. — *Let R be a Prüfer domain and let X be an R-flat finite type scheme. Let $x \in X$ be a point with image $s \in \operatorname{Spec} R$ and local ring $A := \mathscr{O}_{X,x}$. If the following conditions hold*

(a) *s is not the generic point of* $\operatorname{Spec} R$*;*

(b) *$\mathscr{O}_{X_s,x}$ is Cohen-Macaulay; and*

(c) *A has quasi-compact punctured spectrum, or equivalently, so does $\mathscr{O}_{\operatorname{Spec} R,s}$,*

then, A has a regular sequence in $\mathfrak{m}_A$ of length $d+1$ for $d = \dim \mathscr{O}_{X_s,x}$. In particular,

(i) $\operatorname{depth}_A(A) = d+1$*; and*

(ii) *$\tau_N(A) \geqslant d+1$ for any finitely generated A-module N supported on $\{x\}$, therefore,*

$$\operatorname{Ext}^i_A(N, A) = 0 \quad \text{for all } i \leqslant d.$$

Proof. — We may assume that R is a valuation ring V with quasi-compact punctured spectrum U_V such that x is in the closed fiber. In view of Lemma 2.6, it suffices to show that $\mathfrak{m}_A$ contains a regular sequence of A whose common vanishing locus is zero-dimensional. As U_V is quasi-compact, we can pick an $f \in \mathfrak{m}_V$ such that $\dim(V/fV) = 0$. Let $(g_1, \cdots, g_d)$ be a sequence in $\mathfrak{m}_A$ such that their images in the local ring $A/\mathfrak{m}_V A$ forms a regular sequence. By the flatness criterion [EGA IV_3, Th. 11.3.8], $(g_1, \cdots, g_d)$ is a regular sequence of A, and the quotient ring $\overline{A} := A/(g_1, \cdots, g_d)$ is V-flat with maximal ideal $\mathfrak{m}_V \overline{A} = \mathfrak{m}_{\overline{A}}$. Therefore, $(g_1, \cdots, g_d, f)$ is a regular sequence of A for which $\dim A/(fA + \sum_{i=1}^d g_i A) = 0$, as desired. $\square$

Let R be a Prüfer domain, X a scheme flat and of finite type over R, and A the local ring of X at a point $x \in X$. So A is coherent. Knaf proved [Kna08, Th. 1.1] that A is coherent regular if and only if $\operatorname{wdim}(A)$ is finite; moreover, in this coherent regular case we have (cf. Lemma A.3(ii))

$$\operatorname{wdim}(A) = \dim(A \otimes_R \kappa(\mathfrak{q})) + \operatorname{wdim}(R_{\mathfrak{q}}),$$

where $\mathfrak{q} \subset R$ lies below x and we have $\operatorname{wdim}(R_{\mathfrak{q}}) = 0$ whenever x lies over $\operatorname{Frac} R$ otherwise $\operatorname{wdim}(R_{\mathfrak{q}}) = 1$. Besides, if A is coherent regular, then $A \otimes_R \kappa(\mathfrak{q})$ is Cohen-Macaulay and even Noetherian regular if $R_{\mathfrak{q}}$, or equivalently, A is non-Noetherian ([Kna08, Th. 1.3]). Conversely, we see that the regularity of $A \otimes_R \kappa(\mathfrak{q})$ implies the regularity of A (cf. Lemma 3.7(iii)). We formulate these into the following.

Proposition 3.9. — *Let R be a Prüfer domain and $(A, \mathfrak{m}_A)$ a local ring that is R-flat and essentially finitely presented. Denote $\mathfrak{q} \subset R$ the prime underlies $\mathfrak{m}_A$.*

(i) *If $\mathfrak{m}_A$ is a minimal prime of $A \otimes_{R_\mathfrak{q}} \kappa(\mathfrak{q})$, then A is regular if and only if A is a valuation ring.*

(ii) *If A is non-Noetherian, then A is regular if and only if $A \otimes_{R_\mathfrak{q}} \kappa(\mathfrak{q})$ is regular, in which case we have $\mathrm{wdim}(A) = \dim(A \otimes_R \kappa(\mathfrak{q})) + \mathrm{wdim}(R_\mathfrak{q})$.*

(iii) *Assume that A is regular with Noetherian spectrum. Define $X_0 \subset \mathrm{Spec}\, A$ as*

$$X_0 := \{\textit{height-one primes of } A \otimes \mathrm{Frac}\, R\} \cup \{\textit{minimal primes of } R\textit{-fibres of } A\}.$$

Then $X_0 = \{x \in \mathrm{Spec}\, A \mid \mathscr{O}_{\mathrm{Spec}\, A, x} \text{ is a valuation ring}\}$, and all local rings at $x \in (\mathrm{Spec}\, A) \smallsetminus X_0$ have depth $\geqslant 2$, or equivalently, have weak dimensions $\geqslant 2$.

Proof. — It suffices to deal with (iii). By Lemma 3.2(ii), X_0 is the set of all points of X where the local rings are valuation rings. The regularity of A and Theorem 2.17(i) imply that A is normal, thus by Theorem 2.21, all local rings at $x \in (\mathrm{Spec}\, A) \smallsetminus X_0$ have depth $\geqslant 2$. Finally, by Corollary A.8, the regularity of A yields depth $=$ wdim locally on $\mathrm{Spec}\, A$, thus all local rings at $x \in (\mathrm{Spec}\, A) \smallsetminus X_0$ have wdim $\geqslant 2$. □

Finally, we present a Prüferian variant of Theorem 2.20, the purities of reflexive modules, of line bundles, and of vector bundles. Note that our base now is a general Prüfer domain so the local rings under consideration may have non-quasi-compact punctured spectra. The crux is to carefully carry out a limit argument, which preserves the fiberwise codimensions of closed subsets.

Lemma 3.10. — *Let S be a semilocal, affine, integral Prüfer scheme and η its generic point. Given*

(i) *a flat, surjective, finitely presented morphism $f\colon X \to S$ with regular fibres (resp. f is smooth);*

(ii) *a coherent $\mathscr{O}_X$-module $\mathscr{F}$ that is reflexive at a point $x \in X$; and*

(iii) *a constructible closed $Z \subset X$ such that*

$$\mathrm{codim}(Z_s, X_s) \geqslant 1 \textit{ for every } s \in S \quad \textit{and} \quad \mathrm{codim}(Z_\eta, X_\eta) \geqslant 2.$$

Then, there are

(1) *a semilocal, affine, integral Prüfer scheme S_0 of finite dimension with generic point η_0;*

(2) *a flat, surjective, finite type morphism $f_0\colon X_0 \to S_0$ with regular fibres (resp. f_0 is smooth) such that $X_0 \times_{S_0} S \simeq X$;*

(3) *a coherent $\mathscr{O}_{X_0}$-module $\mathscr{F}_0$ as the inverse image of $\mathscr{F}$ and is reflexive at the image x_0 of x; and*

(4) *a constructible closed subset $Z_0 \subset X_0$ such that $Z_0 \times_{S_0} S \simeq Z$ and*

$$\mathrm{codim}((Z_0)_s, (X_0)_s) \geqslant 1 \textit{ for every } s \in S_0 \quad \textit{and} \quad \mathrm{codim}((Z_0)_{\eta_0}, (X_0)_{\eta_0}) \geqslant 2.$$

Proof. — We apply Lemma 3.4 to S then use limit arguments in [EGA IV_3, §8]. The condition that X has regular S-fibres descends to X_{λ_0} by [EGA IV_2, Prop. 6.5.3] (resp. the smoothness of f descends by [EGA IV$_4$, Prop. 17.7.8]). The reflexive $\mathscr{O}_X$-module $\mathscr{F}$ descends thanks to [EGA IV$_3$, Th. 8.5.2] and by applying [EGA IV$_3$, Cor. 8.5.2.5] to $\mathscr{F} \xrightarrow{\sim} \mathscr{F}^{\vee\vee}$. Because Z is constructible closed, by [EGA IV$_3$, Th. 8.3.11], it descends to Z_λ such that $p_\lambda^{-1}(Z_\lambda) = Z$. For $f_\lambda\colon X_\lambda \to S_\lambda$, by the transversality of fibres and [EGA IV$_2$, Cor. 4.2.6], Z_λ does not contain any irreducible components of $f_\lambda^{-1}(s_\lambda)$ for any $s_\lambda \in S_\lambda$. Finally, the image of the generic point $\eta \in S$ is the generic point $\eta_\lambda \in S_\lambda$. By [EGA IV$_2$, Cor. 6.1.4], we have $\operatorname{codim}((Z_\lambda)_{\eta_\lambda}, (X_\lambda)_{\eta_\lambda}) = \operatorname{codim}(Z_\eta, X_\eta) \geqslant 2$. ☐

Theorem 3.11. — *Let R be a semilocal Prüfer domain with spectrum S and $f\colon X \to S$ a flat, locally of finite type morphism of schemes with regular fibres. Let $Z \subset X$ be a constructible closed subset such that*

$$\begin{cases} \operatorname{codim}(Z_s, X_s) \geqslant 1 & \textit{for all } s \in S, \textit{ and} \\ \operatorname{codim}(Z_\eta, X_\eta) \geqslant 2 & \textit{for the generic point } \eta \in S. \end{cases}$$

For the open immersion $j\colon X \smallsetminus Z \hookrightarrow X$, the restriction j^* and pushforward j_* induce the equivalences

$$\mathscr{O}_X\text{-}\mathbf{Rflx} \xrightarrow{\sim} \mathscr{O}_{X\smallsetminus Z}\text{-}\mathbf{Rflx} \quad \textit{and} \quad \mathbf{Pic}\, X \xrightarrow{\sim} \mathbf{Pic}\, X \smallsetminus Z.$$

In particular, for every scheme Y affine over X, we have a bijection of sets

$$Y(X) \xrightarrow{\sim} Y(X \smallsetminus Z).$$

Proof. — The problem is Zariski-local on X, so we may assume that f is of finite type. By a limit argument involving Lemma 3.10, we may further assume that $|S|$ is finite. Then, $|X|$ is the finite union of its S-fibres $|X_s|$, which are Noetherian spaces, so X is topologically Noetherian. By Lemma 3.7(iii), X is coherent regular (see also Corollary A.4). Since the local rings of X are normal (Theorem 2.17(i)) and the generic fibre X_η is regular and schematic dense in X, X is Zariski locally an integral scheme. The assumption on Z implies that $X \smallsetminus Z$ contains $X_0 \subset X$ defined in Proposition 3.9(iii), so for every $z \in Z$, we have $\operatorname{wdim} \mathscr{O}_{X,z} \geqslant 2$. Hence, the assertion follows from Theorem 2.20. ☐

4. Auslander's flatness criterion

The goal is to establish Theorem 4.1 as a counterpart of Auslander's flatness criterion [Aus62, Th. 1.3] on schemes smooth over valuation rings. As expected, our criterion leads to the analog of Zariski–Nagata purity (Theorem 6.9).

Theorem 4.1. — *Let V be a valuation ring, X a V-smooth scheme, and $x \in X$ a point. Set $A := \mathscr{O}_{X,x}$. Let M be a reflexive A-module. If $\operatorname{End}_A(M)$ is isomorphic to a direct sum of copies of M, then M is A-free.*

Similar to Auslander's proof, our strategy relies on an estimate of the length of cohomology groups of M. To begin with, we introduce the length function on torsion modules over valuation rings.

4.2. Lengths of torsion modules. — Let V be a valuation ring, distinct from a field, with fraction field K, and a valuation map $\nu\colon K \to \Gamma$. Every finitely presented torsion V-module M can be expressed as

$$M \simeq \bigoplus_i V/a_iV \quad \text{for finitely many } a_i \in V \smallsetminus \{0\}.$$

Define the *length* of M as $\delta(M) = \sum_i \nu(a_i) \in \Gamma_{\geqslant 0}$. The element $\delta(M)$ is well defined, and $\delta(M) = 0$ if and only if $M = 0$. Every acyclic, bounded complex $M^\bullet$ of torsion, finitely presented V-modules satisfies

$$\textstyle\sum_j (-1)^j \delta(M^j) = 0.$$

Lemma 4.3. — *Let V be a valuation ring that is not a field. Let $(V, \mathfrak{m}_V) \to (A, \mathfrak{m}_A)$ be an essentially smooth, local map of local rings. Denote by A-$\mathbf{Mod}_{\mathrm{tor,fp}}$ the collection of all finitely presented A-modules M supported on $\{\mathfrak{m}_A\}$. Then there exist a totally ordered abelian group Γ and a map $\ell\colon A$-$\mathbf{Mod}_{\mathrm{tor,fp}} \to \Gamma_{\geqslant 0}$ satisfying the following properties:*

– *for A-module $M \in A$-$\mathbf{Mod}_{\mathrm{tor,fp}}$, we have $\ell(M) = 0$ if and only if $M = 0$;*

– *for every acyclic, bounded complex $M^\bullet$ such that $M^j \in A$-$\mathbf{Mod}_{\mathrm{tor,fp}}$ for each j, one has*

$$\textstyle\sum_j (-1)^j \ell(M^j) = 0.$$

It is worth mentioning that the set A-$\mathbf{Mod}_{\mathrm{tor,fp}}$ might be trivial, consisting only of the zero module. In fact, this occurs precisely when $\operatorname{Spec} V \smallsetminus \{\mathfrak{m}_V\}$ is not quasi-compact. This totally ordered abelian group Γ is a value group, and in the sequel, we will only use its property of being partially ordered.

Proof. — First we assume that the structural map $V \to A$ admits a retraction $A \twoheadrightarrow V$. In this case we claim that M is finitely presented over V and is V-torsion. So we can simply let Γ be the valuation group of V and set $\ell(M) := \delta(M)$, where δ is delivered from Section 4.2. Indeed, it is clear that M is V-torsion. Any section $\operatorname{Spec} V \to \operatorname{Spec} A$ is a regular immersion [Stacks, 067R], so there is a finitely generated ideal $J \subset A$ such that $V \simeq A/J$. Hence, since $M \in A$-$\mathbf{Mod}_{\mathrm{tor,fp}}$, we see that $J^nM = 0$ for a large n. On the other hand, the essential smoothness of A over V implies that J/J^2 is a free $V \simeq A/J$-module whose rank equals the rank of the free A-module $\Omega^1_{A/V}$, and there is a natural isomorphism of graded $V \simeq A/J$-algebras

$$\textstyle\bigoplus_{n\geqslant 0} J^n/J^{n+1} \simeq \operatorname{Sym}^\bullet_{A/J}(J/J^2).$$

In particular, A/J^n is a finite free V-module for every $n \geqslant 1$. Therefore, by tensoring a presentation

$$A^N \longrightarrow A^{N'} \longrightarrow M \longrightarrow 0$$

of M with A/J^n for a large enough n, we get a desired finite presentation of M over V.

In the general case, we first use Lemma 3.3 to reduce to the case when the residue fields extension of $V \to A$ is finite. Then, if B is the integral closure of V in an algebraic closure of $\operatorname{Frac} V$, we let V' be a valuation ring of $\operatorname{Frac}(B)$ centered at a maximal ideal of B. It is clear that V' is absolutely integral closed, so it is strictly

Henselian and there exists a V-map $\phi\colon A/\mathfrak{m}_A \to V'/\mathfrak{m}_{V'}$. Let $A' := A \otimes_V V'$. Then ϕ induces a V'-map $\phi'\colon A' \to V'/\mathfrak{m}_{V'}$; let $\mathfrak{p} \subset A'$ be its kernel. Then $A'_{\mathfrak{p}}$ is essentially smooth over V' and ϕ' induces a V'-map $A'_{\mathfrak{p}} \to V'/\mathfrak{m}_{V'}$, which, by the Henselianity of V', lifts to a V'-map $A'_{\mathfrak{p}} \to V'$. By the previous paragraph, the lemma is true for $A'_{\mathfrak{p}}$, say, with corresponding map ℓ' valued in Γ, where Γ is the valuation group of V'. Since $A \to A'_{\mathfrak{p}}$ is faithfully flat, it suffices to define

$$\ell(M) := \ell'(M \otimes_A A'_{\mathfrak{p}}). \qquad \square$$

4.4. Homological algebra lemmas

Lemma 4.5. — *Let $(A, \mathfrak{m}_A)$ be a coherent local ring with a regular sequence of length $d \geqslant 1$. Let $M \xrightarrow{\phi} N$ be a morphism of coherent A-modules that induces an isomorphism over $\operatorname{Spec} A \smallsetminus \{\mathfrak{m}_A\}$. We have isomorphisms $\operatorname{Ext}^i_A(N, A) \xrightarrow{\sim} \operatorname{Ext}^i_A(M, A)$ for all $i < d-1$ and a monomorphism $\operatorname{Ext}^{d-1}_A(N, A) \hookrightarrow \operatorname{Ext}^{d-1}_A(M, A)$.*

This will be applied to Theorem 4.1 with $A := \mathscr{O}_{X,x}$, $d = \dim(\mathscr{O}_{X_s,x}) + 1$, and s is not the generic point.

Proof. — By Lemma 2.2(i), $\operatorname{Ker}\phi$ and $\operatorname{Coker}\phi$ are coherent A-modules supported on $\{\mathfrak{m}_A\}$. By assumption and Lemma 2.6, we have $\tau_{\operatorname{Ker}\phi}(A) \geqslant d$ and $\tau_{\operatorname{Coker}\phi}(A) \geqslant d$ (see Section 2.5 for the definition of $\tau_{-}(-)$). Consider the following short exact sequences

$$0 \longrightarrow \operatorname{Ker}\phi \longrightarrow M \longrightarrow \operatorname{Im}\phi \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \operatorname{Im}\phi \longrightarrow N \longrightarrow \operatorname{Coker}\phi \longrightarrow 0.$$

By applying $\operatorname{Hom}_A(-, A)$, we get two long exact sequences concerning Ext's, and the lemma follows from $\operatorname{Ext}^i_A(\operatorname{Ker}\phi, A) = 0$ and $\operatorname{Ext}^i_A(\operatorname{Coker}\phi, A) = 0$ for $i < d$. $\square$

Lemma 4.6. — *Let V be a valuation ring, X a V-smooth finite type scheme, and $x \in X$ a point that lies over a non-generic point $s \in \operatorname{Spec} V$. For finitely presented $A := \mathscr{O}_{X,x}$-modules M and N, $\operatorname{Ext}^i_A(M, N)$ and $\operatorname{Tor}^A_i(M, N)$ are finitely presented over A for all $i \geqslant 0$ and are zero for $i > d+1$, where $d = \dim \mathscr{O}_{X_s,x}$.*

Proof. — By Lemma 3.7(iii), M has a resolution by finite free A-modules of length $\leqslant d+1$: $F_\bullet \to M$, $F_i = 0$ for $i > d+1$. Then the A-modules

$$\operatorname{Ext}^i_A(M, N) = H^i(\operatorname{Hom}(F_\bullet, N)) \quad \text{and} \quad \operatorname{Tor}^A_i(M, N) = H_i(F_\bullet \otimes N)$$

are all coherent, i.e., finitely presented A-modules, and are zero for $i > d+1$. $\square$

Lemma 4.7. — *Let A be a coherent domain with a topological Noetherian spectrum. If A is normal, then for every finitely presented A-module M, we have a natural isomorphism*

$$\operatorname{End}_A(M)^{\vee\vee} \xrightarrow{\sim} \operatorname{End}_A(M^{\vee\vee}).$$

Proof. — By the functoriality of $(-)^{\vee\vee}$, there is a natural homomorphism

$$\operatorname{End}_A(M) \longrightarrow \operatorname{End}_A(M^{\vee\vee}).$$

As the target is reflexive by Lemma 2.3(ii), this map factors through $\operatorname{End}_A(M)^{\vee\vee}$, thus yielding a natural map $\operatorname{End}_A(M)^{\vee\vee} \to \operatorname{End}_A(M^{\vee\vee})$. Since both the source and target of this map are reflexive, by Theorem 2.21 and Proposition 2.22, it is enough to check this map is an isomorphism when A is a valuation ring. But if this is the case, then there are an $N \in \boldsymbol{Z}_{\geqslant 0}$ and finitely many $a_i \in \mathfrak{m}_A \smallsetminus \{0\}$ for which

$$M \simeq A^{\oplus N} \oplus (\textstyle\bigoplus_i A/a_i A).$$

Consequently, we conclude by the following isomorphisms

$$\mathrm{End}_A(M)^{\vee\vee} \simeq \mathrm{End}_A(A^{\oplus N}) \simeq \mathrm{End}_A(M^{\vee\vee}). \qquad \square$$

4.8. Proof of Theorem 4.1. — The proof proceeds as the following steps.

Preliminary cases and reductions. — Firstly, since X is locally of finite presentation over S and M is finitely presented over A, by a standard limit argument involving Lemma 3.4, we are reduced to the case when V is a finite-rank valuation ring. Secondly, if V' is a valuation ring of an algebraic closure of $\mathrm{Frac}(V)$ that dominates V and if $x' \in X' := X \times_V V'$ is a point lying over $x \in X$, then $M_{A'} := M \otimes_A A'$ is a finitely presented reflexive A'-module and $\mathrm{End}_{A'}(M_{A'}) \simeq \mathrm{End}_A(M) \otimes_A A'$ is isomorphic to a direct sum of copies of $M_{A'}$, where $A' := \mathscr{O}_{X',x'}$ (because A' is faithfully flat over A). By faithfully flat descent [Stacks, 08XD, 00NX], the freeness of M over A is equivalent to the freeness of $M_{A'}$ over A'. Therefore, by replacing V by V', A by A', and M by $M_{A'}$, we are reduced to the case where $\mathrm{Frac}(V)$ is algebraically closed. It is important to emphasize that the smoothness of X plays a crucial role in this reduction, as the regularity of fibres is not maintained under base changes. Furthermore, while the assumption that $\mathrm{Frac}\, V$ is algebraically closed will be invoked only towards the conclusion of the proof, the true necessity lies in ensuring the perfectness of all residue fields of V.

Let $s \in \mathrm{Spec}\, V$ be the image of x. Set $d_x := \dim \mathscr{O}_{X_s,x}$ and $r := \dim V$. The case $r = 0$ and d_x arbitrary is classical. The case r arbitrary and $d_x = 0$ is trivial, where A is a valuation ring (Lemma 3.2(ii)). The case r arbitrary and $d_x = 1$ follows from Theorem 2.19(ii). Thus, we may assume $d_x \geqslant 2$ in the sequel.

Case 1: r is arbitrary and $d_x = 2$. — We will proceed by induction on r. The induction hypothesis is that the assertion holds for $d_x = 2$ and $r' \leqslant r - 1$. Notice that, for any proper generalization $x' \in X$ of x that lies over, say, $s' \in \mathrm{Spec}\, V$, by Lemma 3.2(iii), we have either $s' = s$ and $d_{x'} < 2$, or the height of s' is less than r and $d_{x'} \leqslant 2$. Hence, by induction hypothesis and the preliminary cases above, the assertion holds for $\mathscr{O}_{X,x'}$. As $M_{x'}$ is a finitely presented reflexive $\mathscr{O}_{X,x'}$-module and

$$\mathrm{End}_{\mathscr{O}_{X,x'}}(M_{x'}) = \mathrm{End}_{\mathscr{O}_{X,x}}(M) \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x'} \simeq (\textstyle\bigoplus M) \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x'} = \textstyle\bigoplus M_{x'},$$

the induction hypothesis applies to the $\mathscr{O}_{X,x'}$-module $M_{x'}$, implying that $M_{x'}$ is $\mathscr{O}_{X,x'}$-free. In other words, $\widetilde{M}$ is locally free over $\mathrm{Spec}\, A \smallsetminus \{x\}$. Consider the following evaluation map

$$M^{\vee} \otimes_A M \longrightarrow \mathrm{Hom}_A(M, M), \quad f \otimes m \longmapsto [m' \mapsto f(m')m].$$

By the local freeness of $\widetilde{M}$ over $\operatorname{Spec} A \smallsetminus \{x\}$, it is an isomorphism over $\operatorname{Spec} A \smallsetminus \{x\}$. Since $d_x = 2 > 1$, by Lemma 4.5, we apply $\operatorname{Ext}^1_A(-, A)$ to the above map to obtain

$$\operatorname{Ext}^1_A(M^\vee \otimes M, A) \simeq \operatorname{Ext}^1_A(\operatorname{End}_A(M), A) \simeq \operatorname{Ext}^1_A(M, A)^{\oplus \mathrm{rk}_M} \tag{4.8.1}$$

as isomorphisms of A-modules that are supported on $\{x\}$ by the local freeness of $\widetilde{M}$ over $\operatorname{Spec} A \smallsetminus \{x\}$, where $\mathrm{rk}_M = \dim_{\operatorname{Frac} A} M \otimes_A \operatorname{Frac} A$. By Lemma 4.6, the modules in (4.8.1) are also finitely presented over A.

For the adjunction $\operatorname{Hom}_A(M, \operatorname{Hom}_A(M^\vee, -)) \simeq \operatorname{Hom}_A(M \otimes M^\vee, -)$, we take their derived functors valued at A, so the E_2-page of the associated Grothendieck spectral sequence yields a monomorphism

$$\operatorname{Ext}^1_A(M, M) \hookrightarrow \operatorname{Ext}^1_A(M \otimes M^\vee, A) \overset{(4.8.1)}{\simeq} \operatorname{Ext}^1_A(M, A)^{\oplus \mathrm{rk}_M},$$

where we have used $M^{\vee\vee} \simeq M$; again, by the local freeness of $\widetilde{M}$ over $\operatorname{Spec} A \smallsetminus \{x\}$ and Lemma 4.6, they are finitely presented supported on $\{x\}$. In particular, the map ℓ from Lemma 4.3 applies so we have

$$\ell(\operatorname{Ext}^1_A(M, M)) \leqslant \mathrm{rk}_M \cdot \ell(\operatorname{Ext}^1_A(M, A)). \tag{4.8.2}$$

Since M is reflexive, by Theorem 2.19(i), we have $\operatorname{proj.dim}_A M \leqslant d_x - 1 = 1$. We prove $\operatorname{proj.dim}(M) = 0$ by contradiction. If $\operatorname{proj.dim}(M) = 1$, then M has a free resolution $0 \to F_1 \to F_0 \to M \to 0$ by finite A-modules. This sequence is nonsplit, corresponding to a nontrivial extension class in

$$\operatorname{Ext}^1_A(M, F_1) \simeq \operatorname{Ext}^1_A(M, A)^{\operatorname{rank}(F_1)}.$$

In particular, $C := \operatorname{Ext}^1_A(M, A) \neq 0$. Applying $\operatorname{Hom}_A(-, A)$ to $F_\bullet \to M$ yields

$$\text{an exact sequence} \quad 0 \longrightarrow M^\vee \longrightarrow F_0^\vee \longrightarrow F_1^\vee \longrightarrow \operatorname{Ext}^1_A(M, A) \longrightarrow 0.$$

Tensoring it with M, we get an exact sequence

$$F_0^\vee \otimes_A M \longrightarrow F_1^\vee \otimes_A M \longrightarrow \operatorname{Ext}^1_A(M, A) \otimes_A M \longrightarrow 0.$$

Since

$$\begin{aligned} \operatorname{Coker}(F_0^\vee \otimes M \longrightarrow F_1^\vee \otimes M) &\simeq \operatorname{Coker}(\operatorname{Hom}_A(F_0, M) \longrightarrow \operatorname{Hom}_A(F_1, M)) \\ &= \operatorname{Ext}^1_A(M, M), \end{aligned}$$

we deduce that $\operatorname{Ext}^1_A(M, M) \simeq \operatorname{Ext}^1_A(M, A) \otimes_A M = C \otimes_A M$.

By tensoring $0 \to F_1 \to F_0 \to M \to 0$ with $C = \operatorname{Ext}^1_A(M, A)$ (which is nonzero, finitely presented, and supported at $\{x\}$, by the local freeness of $\widetilde{M}$ over $\operatorname{Spec} A \smallsetminus \{x\}$), we get an exact sequence of finitely presented A-modules supported on $\{x\}$:

$$0 \longrightarrow \operatorname{Tor}^A_1(C, M) \longrightarrow C \otimes_A F_1 \longrightarrow C \otimes_A F_0 \longrightarrow C \otimes_A M \longrightarrow 0.$$

Denote $\mathrm{rk}_M := \operatorname{rank} F_0 - \operatorname{rank} F_1 > 0$. Applying the map ℓ from Lemma 4.3, we get

$$\begin{aligned} \ell(C \otimes_A M) &= \ell(C \otimes_A F_0) - \ell(C \otimes_A F_1) + \ell(\operatorname{Tor}^A_1(C, M) \\ &= \mathrm{rk}_M \cdot \ell(C) + \ell(\operatorname{Tor}^A_1(C, M)). \end{aligned} \tag{4.8.3}$$

On the other hand, since $C\otimes_A M\simeq \operatorname{Ext}^1_A(M,M)$, we deduce the following inequality

$$\ell(C\otimes_A M)\overset{(4.8.2)}{\leqslant}\operatorname{rk}_M\cdot\ell(C). \tag{4.8.4}$$

The combination of (4.8.3) and (4.8.4) leads to $\ell(\operatorname{Tor}_1^A(C,M))=0$. So, we obtain a short exact sequence

$$0\longrightarrow C\otimes_A F_1\longrightarrow C\otimes_A F_0\longrightarrow C\otimes_A M\longrightarrow 0,$$

which combined with Lemma A.6 implies that the map $F_1\to F_0$ splits, that is, M is A-free, contradicting our assumption that $\operatorname{proj.dim}_A(M)=1$. This completes the case when r is arbitrary and $d_x=2$.

Case 2: r is arbitrary and $d_x>2$. — We will proceed by double induction on the pair $(r=\operatorname{ht}(s),d_x)$. By induction hypothesis, the assertion holds for all smooth V-schemes X' and all points $x'\in X'$ such that $\operatorname{ht}(s')\leqslant\operatorname{ht}(s)$ and $d_{x'}\leqslant d_x$, where $s'\in\operatorname{Spec}V$ lies below x', and at least one of equalities is strict. In particular, by Lemma 3.2(iii), the induction hypothesis applies to $\mathscr{O}_{X,x'}$ for all proper generalization $x'\in X$ of x. As $M_{x'}$ is finitely presented reflexive over $\mathscr{O}_{X,x'}$ and

$$\operatorname{End}_{\mathscr{O}_{X,x'}}(M_{x'})=\operatorname{End}_{\mathscr{O}_{X,x}}(M)_{x'}\simeq\bigoplus M_{x'},$$

the induction hypothesis gives that $M_{x'}$ is $\mathscr{O}_{X,x'}$-free. In other words, $\widetilde{M}$ is locally free over $\operatorname{Spec}A\smallsetminus\{x\}$.

Claim 4.8.5 ([Stacks, 057F]). — *Assume that the residue field extension of $V\to A$ is separable (e.g. this holds if $\kappa(s):=V/\mathfrak{m}_V$ is perfect), then there exists an $a\in A$ such that $\overline{A}:=A/(a)$ is essentially V-smooth and*

$$\dim(\overline{A}/\mathfrak{m}_V\overline{A})=d_x-1.$$

Since our V has algebraically closed fraction field (by the first paragraph), all of its primes have algebraically closed residue fields, so we can choose $a\in A$ as in the above claim. Since a is a nonzerodivisor in A and $M=\operatorname{Hom}_A(M^\vee,A)$, we see that a is M-regular. Set $\overline{M}:=M/aM$. Applying $\operatorname{Hom}_A(M,-)$ to the short exact sequence $0\to M\xrightarrow{a}M\to\overline{M}\to 0$, we get an exact sequence

$$0\longrightarrow\operatorname{Hom}_A(M,M)\xrightarrow{\ a\ }\operatorname{Hom}_A(M,M)\longrightarrow\operatorname{Hom}_A(M,\overline{M})\longrightarrow\operatorname{Ext}^1_A(M,M).$$

Substituting our assumption $\operatorname{Hom}_A(M,M)\cong M^{\oplus\operatorname{rk}_M}$ into it yields an exact sequence

$$0\longrightarrow\overline{M}^{\oplus\operatorname{rk}_M}\longrightarrow\operatorname{Hom}_{\overline{A}}(\overline{M},\overline{M})\longrightarrow T\longrightarrow 0$$

of $\overline{A}$-modules, where $T\subset\operatorname{Ext}^1_A(M,M)$ is a finitely presented $\overline{A}$-submodule (see Lemma 4.6), which, by the locally freeness of $\widetilde{M}$ over $\operatorname{Spec}A\smallsetminus\{x\}$, is supported on $\{x\}$. Since $\dim(\overline{A}/\mathfrak{m}_V\overline{A})=d_x-1\geqslant 2$, taking dual (as $\overline{A}$-modules) of the above short exact sequence and using Lemma 4.5, we see that

$$(\overline{M}^\vee)^{\oplus\operatorname{rk}_M}\simeq\operatorname{Hom}_{\overline{A}}(\overline{M},\overline{M})^\vee.$$

Taking dual further and invoking Lemma 4.7, we get the following isomorphism

$$(\overline{M}^{\vee\vee})^{\oplus \mathrm{rk}_M} \simeq \mathrm{Hom}_{\overline{A}}(\overline{M}^{\vee\vee}, \overline{M}^{\vee\vee}).$$

Since the double dual $\overline{M}^{\vee\vee}$ is finitely presented and reflexive over $\overline{A}$ (Lemma 2.3(ii)), we can apply the induction hypothesis to the $\overline{A}$-module $\overline{M}^{\vee\vee}$ and conclude that it is $\overline{A}$-free. The same lemma also implies that $\overline{M}^{\vee}$ is $\overline{A}$-reflexive, so that $\overline{M}^{\vee} \simeq \overline{M}^{\vee\vee\vee}$ is $\overline{A}$-free of rank rk_M.

Finally, we show that M is A-free. Since $\widetilde{M}$ is locally free over $\operatorname{Spec} A \smallsetminus \{x\}$, the natural map $\overline{M} \to \overline{M}^{\vee\vee}$ is an isomorphism over $\operatorname{Spec} A \smallsetminus \{x\}$. Since $\dim(\overline{A}/\mathfrak{m}_V\overline{A}) = d_x - 1 > 1$, we may apply Lemma 4.5 to see that $\mathrm{Ext}^1_{\overline{A}}(\overline{M}, \overline{A}) \simeq \mathrm{Ext}^1_{\overline{A}}(\overline{M}^{\vee\vee}, \overline{A}) = 0$. Since a is M-regular, we have

$$\mathrm{Ext}^1_A(M, \overline{A}) \simeq \mathrm{Ext}^1_{\overline{A}}(M \otimes^{\mathbf{L}}_A \overline{A}, \overline{A}) \simeq \mathrm{Ext}^1_{\overline{A}}(\overline{M}, \overline{A}) = 0.$$

Applying $\mathrm{Hom}_A(M, -)$ to the short exact sequence $0 \to A \xrightarrow{a} A \to \overline{A} \to 0$, we get an exact sequence

$$0 \to M^{\vee} \xrightarrow{a} M^{\vee} \to \mathrm{Hom}_A(M, \overline{A}) \to \mathrm{Ext}^1_A(M, A) \xrightarrow{a} \mathrm{Ext}^1_A(M, A) \to \mathrm{Ext}^1_A(M, \overline{A}).$$

All modules are finitely presented over A. Since $\mathrm{Ext}^1_A(M, \overline{A}) = 0$, Nakayama's lemma gives that $\mathrm{Ext}^1_A(M, A) = 0$. Therefore, $M^{\vee}/aM^{\vee} \simeq \mathrm{Hom}_A(M, \overline{A}) = \overline{M}^{\vee}$ is $\overline{A}$-free of rank rk_M, implying that $\dim_{\kappa(x)}(M^{\vee} \otimes_A \kappa(x)) = \mathrm{rk}_M = \mathrm{rk}_{M^{\vee}}$. It follows that $M^{\vee}$, and equivalently $M \simeq M^{\vee\vee}$, is A-free.

5. Generalities on torsors over algebraic spaces

5.1. Setup. — Throughout this section, we let S denote a base scheme, X an algebraic space over S, and G an X-group algebraic space.

Definition 5.2

(1) A (right) *G-torsor* (for the fppf topology) is an X-algebraic space $\mathcal{P}$ equipped with a G-action $a : \mathcal{P} \times_X G \to \mathcal{P}$ such that the following conditions hold:

(i) the induced morphism $\mathcal{P} \times_X G \to \mathcal{P} \times_X \mathcal{P}$, $(p, g) \mapsto (p, a(p, g))$, is an isomorphism; and

(ii) there exists a fppf covering $\{X_i \to X\}_{i \in I}$ of algebraic spaces [Stacks, 03Y8] such that $\mathcal{P}(X_i) \neq \varnothing$ for every $i \in I$.

(2) For G-torsors $\mathcal{P}_1$ and $\mathcal{P}_2$, a *morphism* $\mathcal{P}_1 \to \mathcal{P}_2$ is a G-equivariant morphism $\mathcal{P}_1 \to \mathcal{P}_2$ of X-algebraic spaces.

(3) By a trivialization of a G-torsor $\mathcal{P}$ we mean a G-equivariant isomorphism $t : G \xrightarrow{\sim} \mathcal{P}$, where G acts on itself via right multiplication; this amounts to the choice of a section $t(1_G) \in \mathcal{P}(X)$. A G-torsor $\mathcal{P}$ is *trivial* if there exists a trivialization, or, equivalently, if $\mathcal{P}(X) \neq \varnothing$.

Note that every morphism of two G-torsors is an isomorphism. To see this, one may pass to a fppf covering of X to reduce to the case when both torsors are trivial; in this case the assertion is trivial.

Remark 5.3. — One can also define a sheaf torsor for an X-group algebraic space G. It is a sheaf

$$\mathcal{P} : (\mathbf{Sch}_{/S})^{\mathrm{opp}}_{\mathrm{fppf}} \longrightarrow \mathbf{Set}$$

equipped with a map $\mathcal{P} \to X$ of sheaves and a G-action $a : \mathcal{P} \times_X G \to \mathcal{P}$ such that the above two conditions (i) and (ii) in (1) hold. However, it turns out that such a sheaf torsor is necessarily representable by an algebraic space, so working with sheaf torsors adds no more generality. To see this, let $\{X_i \to X\}_{i\in I}$ be a fppf covering as in (ii) that trivializes $\mathcal{P}$. Then every $\mathcal{P} \times_X X_i \simeq G \times_X X_i$ is an algebraic space, and the map

$$\textstyle\bigsqcup_i \mathcal{P} \times_X X_i \to \mathcal{P}$$

is representable by algebraic spaces and is a fppf covering, because it is the base change of the fppf covering $\bigsqcup_i X_i \to X$ of algebraic spaces via $\mathcal{P} \to X$. Here, all coproducts are taken in the category of sheaves on $(\mathbf{Sch}\,/S)_{\mathrm{fppf}}$. It follows from (3) of [Stacks, 04S6] that $\mathcal{P}$ is an algebraic space, as desired.

Let $\mathcal{P}_1, \mathcal{P}_2$ be two G-torsors. Define a functor

$$\underline{\mathrm{Isom}}_X(\mathcal{P}_1, \mathcal{P}_2) : (\mathbf{Sch}_{/X})^{\mathrm{opp}} \longrightarrow \mathbf{Set}$$

which associates to any scheme T over X the set of G_T-equivariant isomorphisms $\mathcal{P}_{1,T} \to \mathcal{P}_{2,T}$ over T.

Lemma 5.4. — *For two G-torsors $\mathcal{P}_1$ and $\mathcal{P}_2$, $\underline{\mathrm{Isom}}_X(\mathcal{P}_1, \mathcal{P}_2)$ is an algebraic space over S. Further, $G \to X$ is quasi-compact (resp. étale, smooth, flat, separated, (locally) of finite type, (locally) of finite presentation, quasi-affine, affine, or finite) if and only if $\underline{\mathrm{Isom}}_X(\mathcal{P}_1, \mathcal{P}_2) \to X$ is so.*

Proof. — Since $\underline{\mathrm{Isom}}_X(\mathcal{P}_1, \mathcal{P}_2)$ is fppf locally on X isomorphic to G, it admits a representable fppf covering by algebraic spaces, hence it is an algebraic space by [Stacks, 04S6].

The list properties of morphisms of algebraic spaces are all stable under base changes and are fppf local on the target, see [Stacks, 03KG] (resp. [Stacks, 03XT, 03ZF, 03MM, 03KM, 040Y, 0410, 03WM, 03WG, 03ZQ]). Consequently, since the functor $\underline{\mathrm{Isom}}_X(\mathcal{P}_1, \mathcal{P}_2)$ is fppf locally on X isomorphic to G, the properties of G are inherited by and can be detected from $\underline{\mathrm{Isom}}_X(\mathcal{P}_1, \mathcal{P}_2)$. □

Since every G-torsor $\mathcal{P} \to X$ trivializes over a fppf covering $\{X_i \to X\}$, one may try to obtain $\mathcal{P}$ by gluing the trivial G_{X_i}-torsors $\mathcal{P}_{X_i}$ using the canonical isomorphisms

$$\phi_{ij} : (\mathcal{P}_{X_i})_{X_{ij}} \simeq \mathcal{P}_{X_{ij}} \simeq \mathcal{P}_{X_j})_{X_{ji}}, \qquad \text{where } X_{ij} = X_i \times_X X_j.$$

It turns out that, unlike the case of schemes, this is always possible in the framework of algebraic spaces, see Lemma 5.6. Note that, by taking $U := \bigsqcup_i X_i$, we may assume that $\mathcal{P}_U$ is trivial for a fppf covering $U \to X$ with U an algebraic space.

Definition 5.5 (Descent data for torsors). — Let S, X and G be as in Section 5.1. Let $U \to X$ be a fppf covering of algebraic spaces over S. For every integer $n \geqslant 0$, denote by $U^{(n)}$ the n-fold fibre product of U over X. The *category of descent data* for G-torsors relative to $U \to X$, denoted

$$\mathbf{Tors}\big((U^{(2)} \rightrightarrows U)_{\mathrm{fppf}}, G\big),$$

has pairs $(\mathcal{Q}, \phi)$ as objects, where

– $\mathcal{Q} \to U$ is a G_U-torsor; and

– $\phi : \mathrm{pr}_1^*\mathcal{Q} \to \mathrm{pr}_2^*\mathcal{Q}$ is an isomorphism of $G_{U^{(2)}}$-torsors such that the following diagram commutes (i.e., the cocycle condition holds)

$$\begin{tikzcd} \mathrm{pr}_{12}^*\mathrm{pr}_1^*\mathcal{Q} \arrow[r, "\mathrm{pr}_{12}^*(\phi)"] \arrow[d, "\wr"'] & \mathrm{pr}_{12}^*\mathrm{pr}_2^*\mathcal{Q} \arrow[r, "\simeq"] & \mathrm{pr}_{23}^*\mathrm{pr}_1^*\mathcal{Q} \arrow[r, "\mathrm{pr}_{23}^*(\phi)"] & \mathrm{pr}_{23}^*\mathrm{pr}_2^*\mathcal{Q} \arrow[d, "\wr"] \\ \mathrm{pr}_{13}^*\mathrm{pr}_1^*\mathcal{Q} \arrow[rrr, "\mathrm{pr}_{13}^*(\phi)"] & & & \mathrm{pr}_{13}^*\mathrm{pr}_2^*\mathcal{Q}. \end{tikzcd}$$

A *morphism* from a pair $(\mathcal{Q}, \phi)$ to another pair $(\mathcal{Q}', \phi')$ is a morphism $\theta : \mathcal{Q} \to \mathcal{Q}'$ of G_U-torsors compatible with ϕ and ϕ', that is, $\mathrm{pr}_2^*(\theta)\phi = \phi'\mathrm{pr}_1^*(\theta)$.

To every G-torsor $\mathcal{P}$ one can associate a pair $\Psi(\mathcal{P}) := (\mathcal{P}_U, \mathrm{can})$ via base changes, where can denotes the canonical isomorphism $\mathrm{pr}_1^*(\mathcal{P}_U) \simeq \mathcal{P}_{U^{(2)}} \simeq \mathrm{pr}_2^*(\mathcal{P}_U)$. Thus we obtain a functor

$$\Psi : \mathbf{Tors}(X_{\mathrm{fppf}}, G) \longrightarrow \mathbf{Tors}((U^{(2)} \rightrightarrows U)_{\mathrm{fppf}}, G).$$

Lemma 5.6 (Descent G-torsors). — Ψ *is an equivalence of categories.*

In other words, every descent data $(\mathcal{Q}, \phi)$ for G-torsors are effective in the sense that there exists a G-torsor $\mathcal{P}$ and an isomorphism $\mathcal{Q} \simeq \mathcal{P}_U$ compatible with θ and the canonical descent data for $\mathcal{P}_U$.

Proof. — The full faithfulness of Ψ follows from the sheaf property of the functor $\underline{\mathrm{Isom}}_X(\mathcal{P}_1, \mathcal{P}_2)$ for any G-torsors $\mathcal{P}_1$ and $\mathcal{P}_2$. For the essential surjectivity, we pick a descent data $(\mathcal{Q}, \phi)$, and need to show that there exists a G-torsor $\mathcal{P}$ for which $(\mathcal{P}_U, \mathrm{can}) \simeq (\mathcal{Q}, \phi)$.

When both X and U are schemes, this is proved in [Stacks, 04U1]. The case of algebraic spaces can be proved similarly, and we repeat the argument for convenience. First we view $\mathcal{Q}$ as a sheaf on the site $(\mathbf{AS}/U)_{\mathrm{fppf}}$ (by the natural equivalence of the topoi associated to $(\mathbf{AS}/U)_{\mathrm{fppf}}$ and $(\mathbf{Sch}/U)_{\mathrm{fppf}}$). Since descent data for sheaves on any site are always effective [Stacks, 04TR], we may find a sheaf $\mathcal{P}$ on the site $(\mathbf{AS}/X)_{\mathrm{fppf}}$ and an isomorphism of sheaves $\mathcal{P}_U \simeq \mathcal{Q}$ compatible with the descent data. Further, since maps of sheaves on any site can be glued [Stacks, 04TQ], the G_U-action on $\mathcal{Q}$ descends to a G-action on $\mathcal{P}$. All the assumptions (i) and (ii) of Definition 5.2 hold, because they can be checked on the fppf covering $U \to X$. It remains to see that $\mathcal{P}$ is representable by an algebraic space over X. However, this follows from (3) of [Stacks, 04S6], in view of the fact that the map $\mathcal{Q} \to \mathcal{P}$ is representable by algebraic spaces and is a fppf covering (being a base change of the fppf covering $U \to X$). □

We end this section with the following result, which is used repeatedly in the sequel.

Lemma 5.7. — *Let S be a scheme, X an algebraic space over S, and G an X-group algebraic space. Let $f\colon Y \to X$ be a morphism of algebraic spaces over S. Assume that the following conditions hold:*

(i) *for every fppf covering $T \to X$ with T a scheme, the map $G(T) \to G(Y_T)$ is bijective, where $Y_T := Y \times_X T$; and*

(ii) *for every G_Y-torsor $\mathcal{P}$, there is an fppf covering $T \to X$ with T a scheme such that $\mathcal{P}_{Y_T}$ lies in the essential image of f_T^*, where $f_T := f \times_X T$.*

Then pullback induces an equivalence $f^*\colon \mathbf{Tors}(X_{\mathrm{fppf}}, G) \xrightarrow{\sim} \mathbf{Tors}(Y_{\mathrm{fppf}}, G_Y)$.

Similarly, if $G \to X$ is smooth, then we have an equivalence

$$f^* : \mathbf{Tors}(X_{\text{ét}}, G) \xrightarrow{\sim} \mathbf{Tors}(Y_{\text{ét}}, G_Y),$$

provided that one replaces 'fppf' by 'étale' everywhere in the above assumptions.

Proof. — We prove the lemma for fppf torsors. The assumption (i) implies that the functor f^* is fully faithful. It remains to check the essential surjectivity. Let $\mathcal{P}$ be a G_Y-torsor. By assumption (ii) there is a fppf covering $T \to X$ with T a scheme and a G_T-torsor $\mathcal{Q}$ such that $f_T^*\mathcal{Q} \simeq \mathcal{P}_{Y_T}$. Using this isomorphism we can transform the canonical descent data on $\mathcal{P}_{Y_T}$ to a descent data

$$\theta : \mathrm{pr}_1^* f_T^*\mathcal{Q} \xrightarrow{\sim} \mathrm{pr}_2^* f_T^*\mathcal{Q}$$

on $f_T^*\mathcal{Q}$ (relative to the covering $Y_T \to Y$). For every integer $n \geqslant 0$, denote by $T^{(n)}$ the n-fold fibre product of T over X. Using the canonical identifications

$$\mathrm{pr}_1^* f_T^*\mathcal{Q} = f_{T^{(2)}}^*\mathrm{pr}_1^*\mathcal{Q} \quad \text{and} \quad \mathrm{pr}_2^* f_T^*\mathcal{Q} = f_{T^{(2)}}^*\mathrm{pr}_2^*\mathcal{Q},$$

the full faithfulness of $f_{T^{(2)}}$ implies that there is a unique isomorphism

$$\tau : \mathrm{pr}_1^*\mathcal{Q} \xrightarrow{\sim} \mathrm{pr}_2^*\mathcal{Q}$$

of $G_{T^{(2)}}$-torsors such that $f_{T^{(2)}}^*(\tau) = \theta$. Since

$$\mathrm{pr}_{13}^*(\theta) = \mathrm{pr}_{13}^*(f_{T^{(2)}}^*(\tau)) = f_{T^{(3)}}^*\mathrm{pr}_{13}^*(\tau)$$

and

$$\begin{aligned}
\mathrm{pr}_{13}^*(\theta) &= \mathrm{pr}_{23}^*(\theta)\mathrm{pr}_{12}^*(\theta) \\
&= \mathrm{pr}_{23}^*\left(f_{T^{(2)}}^*(\tau)\right)\mathrm{pr}_{12}^*\left(f_{T^{(2)}}^*(\tau)\right) \\
&= f_{T^{(3)}}^*\left(\mathrm{pr}_{23}^*(\tau)\right) f_{T^{(3)}}^*\left(\mathrm{pr}_{12}^*(\tau)\right) \\
&= f_{T^{(3)}}^*\left(\mathrm{pr}_{23}^*(\tau)\mathrm{pr}_{12}^*(\tau)\right),
\end{aligned}$$

the full faithfulness of $f_{T^{(3)}}^*$ implies that $\mathrm{pr}_{13}^*(\tau) = \mathrm{pr}_{23}^*(\tau)\mathrm{pr}_{12}^*(\tau)$, that is, τ is a descent data on $\mathcal{Q}$ relative to $T \to X$. By Lemma 5.6, there is a G-torsor $\mathcal{R}$ and an isomorphism $(\mathcal{Q},\phi)\simeq(\mathcal{R}_T,\mathrm{can})$ of descent data. Pulling back to Y_T, we get an isomorphism of descent data

$$(\mathcal{P}_{Y_T},\mathrm{can})\simeq f_T^*(\mathcal{Q},\tau)\simeq(\mathcal{R}_{Y_T},\mathrm{can}).$$

By Lemma 5.6 again (applied to the covering $Y_T\to Y$), we see that $f^*(\mathcal{R})=\mathcal{R}_Y\simeq\mathcal{P}$, as desired. □

6. Purity for torsors and finite étale covers

We begin with discussing generalities on linear groups that will be fundamental in multiple types of purities for torsors. The overall argument is bootstrapped from that for vector bundles, and controlling on the projective dimensions of extended reflexive sheaves leads to relative-dimensional constraints. In particular, we obtain the purity for torsors on relative curves and its local variants in Section 6.1, where the constraints on dimensions in the local case are more flexible. This allows us to shrink complements of domains of reductive torsors (cf. [GL23, Prop. 2.9 & Cor. 2.10]), which is crucial for our proof of the Grothendieck–Serre for constant reductive group schemes given in [GL23]. Finally, utilizing our Prüferian analog of Auslander's flatness criterion (cf. Theorem 4.1), we establish in Section 6.7 the Purity Theorem 6.8 for torsors under finite locally free groups, and, consequently, we obtain a Prüferian counterpart of the Zariski–Nagata Purity Theorem 6.9.

6.0.1. *Coaffine locally linear groups*. — Let X be an algebraic space. An X-group algebraic space G is *linear* if there exists a group monomorphism $G\hookrightarrow\mathrm{GL}(\mathscr{V})$ for a locally free $\mathscr{O}_X$-module $\mathscr{V}$ of finite rank.

An X-group algebraic space G is *fppf locally coaffine* (resp. *étale locally coaffine*), if, fppf locally (resp. étale locally) on X, there is a monomorphism $G\hookrightarrow\mathrm{GL}(\mathscr{V})$ such that the sheaf quotient $\mathrm{GL}(\mathscr{V})/G$ is representable by an X-affine algebraic space. Such a G is necessarily affine over X and the monomorphism $G\hookrightarrow\mathrm{GL}(\mathscr{V})$ is automatically a closed immersion. For instance, if G is X-reductive (resp. X-finite locally free), then G is étale locally coaffine[(5)] (resp. fppf locally coaffine). In the sequel, we will mainly consider (fppf or étale) locally coaffine X-group algebraic spaces G.

6.1. Purity for torsors on relative curves. — Now we study the extension behavior of torsors over relative curves. Motivated by [EGA IV$_4$, Prop. 21.9.4] that every invertible sheaf on a curve over a field extends across finitely many closed points, Proposition 6.2 concerns relative curves over Prüfer rings and generalizes [Guo24, Lem. 7.3].

(5)If G is X-reductive, then étale locally G splits, so (by e.g. [Gil22, Th. 1.1 & Cor. 4.3]) there exists a closed immersion $G\hookrightarrow\mathrm{GL}_{n,X}$ for some integer n, and [Alp14, 9.4.1] ensures that the quotient $\mathrm{GL}_{n,X}/G$ is X-affine of finite type.

6.1.1. *Torsors on relative curves*. — Let R be a semilocal Prüfer domain with spectrum S and X an S-flat scheme of finite type with regular one-dimensional fibres. Let $D \subset X$ be an S-quasi-finite closed subscheme contained in an affine open $\operatorname{Spec} A \subset X$, and cut out by a nonzerodivisor $t \in A$. Assume that

(i) $|S|$ is finite; or

(ii) D is S-finite.

Note that these imply that D is semilocal, and (ii) holds for instance when X is S-proper. Consider

– $B_D := \operatorname{Spec} \widehat{A}$, the formal neighborhood of D, where $\widehat{A} := \varprojlim_n A/t^n A$;

– $U_D := B_D \smallsetminus D = \operatorname{Spec} \widehat{A}[\frac{1}{t}]$, the punctured formal neighborhood.

Indeed, B_D is semilocal, which follows from the semilocality of D. As D is S-quasi-finite, it is S-flat. By [Stacks, 0B9D], D is a relative effective Cartier divisor, so its each nonempty fibre has codimension one. By Hensel's lemma, $(\widehat{A}, t\widehat{A})$ is a Henselian pair, in particular, $t\widehat{A}$ is contained in all maximal ideals of $\widehat{A}$. Combining this with the fiberwise codimension-one property of D, we conclude that B_D is semilocal.

The following proposition specializes to [Guo24, Lem. 7.3] when $A = V[t]$ and $D = \operatorname{Spec} A/tA$ for a valuation ring V.

Proposition 6.2. — *With the setup in Section* 6.1.1, *the restriction functor for vector bundles*

$$\mathbf{Vect}_X \longrightarrow \mathbf{Vect}_{X \smallsetminus D}$$

is essentially surjective. Furthermore, we have

$$H^1_{\mathrm{Zar}}(U_D, \mathrm{GL}_n) = H^1_{\text{ét}}(U_D, \mathrm{GL}_n) = \{*\}.$$

Proof. — Note that by Lemma 3.7(iii) or Proposition 3.9(ii), all local rings of X have weak dimension $\leqslant 2$. By Corollary 2.4, every vector bundle on $X \smallsetminus D$ extends to a reflexive sheaf on X, which, by Theorem 2.19(ii), is actually a vector bundle. This proves the first assertion.

Next, let $\mathscr{V}$ be a vector bundle on $U_D = \operatorname{Spec} \widehat{A}[\frac{1}{t}]$ and denote the Henselization of the pair (X, D) by (B_D^{h}, D). Write $B_D^{\mathrm{h}} = \operatorname{Spec} A^{\mathrm{h}}$ and set $U_D^{\mathrm{h}} := B_D^{\mathrm{h}} \smallsetminus D = \operatorname{Spec} A^{\mathrm{h}}[1/t]$. Then, since the t-adic completion of A^{h} is isomorphic to $\widehat{A}$, [BČ22, Cor. 2.1.22(c)] implies that $\mathscr{V}$ descends to a vector bundle $\mathscr{V}^{\mathrm{h}}$ on U_D^{h}. Since B_D^{h} is the limit of elementary étale neighborhoods $D \subset X'$ of $D \subset X$, by a limit argument, $\mathscr{V}^{\mathrm{h}}$ descends to a vector bundle $\mathscr{V}'$ on $X' \smallsetminus D$ for some $D \subset X'$. Since $\mathbf{Vect}_{X'} \to \mathbf{Vect}_{X' \smallsetminus D}$ is essentially surjective, $\mathscr{V}'$ extends to a vector bundle $\widetilde{\mathscr{V}}'$ on X' whose pullback gives a vector bundle $\widetilde{\mathscr{V}}$ on B_D extending $\mathscr{V}$. Finally, as in Section 6.1.1, $\widehat{A}$ is semilocal, so over which the bundle $\widetilde{\mathscr{V}}$ and thus $\mathscr{V}$ is trivial. ☐

The following purity result in the case when G is a reductive X-group scheme was proved in [GL23, Th. 2.7]. It turns out that the same argument works for any X-group algebraic space G that is étale locally coaffine (cf. Section 6.0.1) in the setting of algebraic spaces.

Theorem 6.3. — *Let S be a semilocal affine Prüfer scheme and X an S-flat, locally of finite type algebraic space with regular one-dimensional S-fibres. Let G be an X-affine group algebraic space that is étale locally coaffine.*(6) *Given a closed subspace Z of X such that the inclusion $j\colon X \smallsetminus Z \hookrightarrow X$ is quasi-compact, and*

$$Z_\eta = \varnothing \quad \text{for each generic point } \eta \in S \quad \text{and} \quad \operatorname{codim}(Z_s, X_s) \geqslant 1 \quad \text{for all } s \in S.$$

Then, the restriction gives an equivalence of categories of G-torsors

$$\mathbf{Tors}(X_{\mathrm{fppf}}, G) \xrightarrow{\sim} \mathbf{Tors}((X \smallsetminus Z)_{\mathrm{fppf}}, G). \tag{6.3.1}$$

Consequently, when considering isomorphism classes, there exists a bijection:

$$H^1_{\mathrm{fppf}}(X, G) \simeq H^1_{\mathrm{fppf}}(X \smallsetminus Z, G).$$

Furthermore, if S is allowed to be a general Prüfer algebraic space,(7) *(not necessarily semilocal) the above conclusions remain valid as long as G is finitely presented over X (and étale locally coaffine).*

Proof. — First, consider the case when S is a semilocal affine Prüfer scheme. We show that (6.3.1) is an equivalence. Since G is X-affine, by checking étale locally and using Theorem 2.20, we see that $G(X) \simeq G(X \smallsetminus Z)$, which proves the full faithfulness (this uses the analogous bijection when we base change everything to a scheme étale over X and the condition on fibres remains valid).

For essential surjectivity, we choose a G-torsor $\mathcal{P}$ over $X \smallsetminus Z$ and wish to extend $\mathcal{P}$ to a G-torsor over X. In the special case where $G = \mathrm{GL}_n$, G-torsors correspond to vector bundles of rank n. Due to the condition on fibres imposed on Z, by Proposition 3.9(iii) and Theorem 2.20, the pushforward $j_*\mathcal{P}$ is reflexive. Now, we leverage the essential assumption that X has one-dimensional regular S-fibres: by Proposition 3.9(ii), every local ring of X is coherent regular with wdim $\leqslant 2$. Therefore, Theorem 2.19(ii) implies that the reflexive $\mathscr{O}_X$-module $j_*\mathcal{P}$ is locally free. For a general G, by gluing in the étale topology, it suffices to demonstrate that $\mathcal{P}$ extends, at least *étale locally* on X, to a G-torsor over X (see Lemma 5.7).

To prove this, we may assume that X is affine, $G \subset \mathrm{GL}_{n,X}$, and that $\mathrm{GL}_{n,X}/G$ is affine over X. We exploit the following commutative diagram of pointed sets, where the two rows are exact sequences:

$$\begin{tikzcd}
(\mathrm{GL}_{n,X}/G)(X) \arrow[r] \arrow[d, "\wr"'] & H^1_{\mathrm{fppf}}(X, G) \arrow[r] \arrow[d] & H^1_{\mathrm{fppf}}(X, \mathrm{GL}_{n,X}) \arrow[d] \\
(\mathrm{GL}_{n,X}/G)(X \smallsetminus Z) \arrow[r] & H^1_{\mathrm{fppf}}(X \smallsetminus Z, G) \arrow[r] & H^1_{\mathrm{fppf}}(X \smallsetminus Z, \mathrm{GL}_{n,X}).
\end{tikzcd}$$

The bijectivity of the left vertical arrow follows from Theorem 2.20. By the case of vector bundles, we may replace X by an affine open cover to assume that the induced

(6)For example, G could be X-reductive, or X-finite locally free.

(7)Namely, it admits an étale cover by a disjoint union of spectra of Prüfer domains.

$\mathrm{GL}_{n,X\smallsetminus Z}$-torsor $\mathcal{P}\times^{G_{X\smallsetminus Z}}\mathrm{GL}_{n,X\smallsetminus Z}$ is already trivial. Then, a diagram chase yields a G-torsor $\mathcal{Q}$ over X such that $\mathcal{Q}|_{X\smallsetminus Z}\simeq\mathcal{P}$.

Now, let us assume that S is a Prüfer algebraic space, and that G is finitely presented over X. We may assume that X is of finite type over S. By Lemma 3.7(i), the S-flatness of X ensures its finite presentation over S. So, G and hence all G-torsors are also finitely presented over S. At this point, we can use a standard argument to deduce the conclusion in the present case from the previous semilocal case. For example, to establish full faithfulness, we prove that $G(X)\simeq G(X\smallsetminus Z)$. The question is étale local on X, allowing us to assume that S is an affine Prüfer scheme and X is an affine scheme. We consider both sides as Zariski sheaves on S, given by $T\mapsto G(X_T)$ and $T\mapsto G(X_T\smallsetminus Z_T)$, respectively. The previous semilocal case implies that $G(X(s))\simeq G(X(s)\smallsetminus Z(s))$ for each point $s\in S$, where $X(s):=X\times_S\mathrm{Spec}(\mathscr{O}_{S,s})$, etc. Since j is quasi-compact and $G\to X$ is finitely presented, both sides of the last bijection can be identified with the stalks of the two Zariski sheaves at s. This implies $G(X)\simeq G(X\smallsetminus Z)$, as desired. □

Remark 6.4. — In higher relative dimensions, even in the classical Noetherian setting, the purity Theorem 6.3 is inapplicable, even for the simplest group $G=\mathrm{GL}_n$. For instance, there exists a vector bundle over $\mathrm{Spec}(R)\smallsetminus\{\mathfrak{m}_R\}$ that cannot be extended to $\mathrm{Spec}\,R$, where $(R,\mathfrak{m}_R)$ is any Noetherian regular local ring of Krull dimension at least three.

The following is a local version of Theorem 6.3. Its proof closely mirrors that of Theorem 6.3 (also refer to [GL23, Th. 2.8], or can be straightforwardly deduced from it.

Theorem 6.5 (Local variant of purity). — *Let V be a valuation ring of finite rank, with spectrum S, and let $\eta\in S$ be the generic point. Let X be an S-flat, finite type scheme with regular fibres. Let G be an X-group scheme that is coaffine étale locally. If $x\in X$ satisfies one of the following*

(i) *$x\in X_\eta$ with $\dim\mathscr{O}_{X_\eta,x}=2$, or*

(ii) *$x\in X_s$ (where $s\neq\eta$) with $\dim\mathscr{O}_{X_s,x}=1$,*

then every G-torsor over $\mathrm{Spec}\,\mathscr{O}_{X,x}\smallsetminus\{x\}$ extends uniquely to a G-torsor over $\mathscr{O}_{X,x}$.

It is worth noting that the stipulation of V possessing finite rank guarantees that any finite type V-scheme will be topologically Noetherian. This ensures that the punctured spectrum $\mathrm{Spec}\,\mathscr{O}_{X,x}\smallsetminus\{x\}$ remains quasi-compact. This property is fundamental when establishing results concerning vector bundles.

As a corollary of Theorem 6.5, one proves the following result about extending generically trivial torsors.

Corollary 6.6 (Extending generically trivial torsors, [GL23, Cor. 2.10])
Fix

(i) *R a semilocal Prüfer domain with spectrum S;*

(ii) *X an S-flat finite type quasi-separated scheme with regular fibres;*
(iii) *Y the spectrum of a local ring of an affine open subset of X;*
(iv) *$r \in R$ a nonzero element; and*
(v) *G a reductive X-group scheme.*

Then, every generically trivial G-torsor over Y (resp. over an open subset of $X[1/r] := X_{R[1/r]}$) extends to a G-torsor over an open subset $U \subset X$. Here, the complementary closed $Z := X \smallsetminus U$ satisfies the condition:

$$\begin{cases} \operatorname{codim}(Z_\eta, X_\eta) \geqslant 3 & \textit{for each generic point } \eta \in S, \\ \operatorname{codim}(Z_s, X_s) \geqslant 2 & \textit{for all } s \in S. \end{cases}$$

6.7. Purity for finite locally free torsors and the Zariski–Nagata. — By combining Theorem 2.20 on the purity of reflexive sheaves and Auslander's flatness criterion Theorem 4.1, we are able to establish the following Prüferian analog of a result of Moret-Bailly, detailed in [Mar16].

Theorem 6.8 (Purity for torsors under finite locally free groups)

(i) *Let S be a Prüfer algebraic space and X an S-smooth algebraic space. Let G be an X-finite, locally free group algebraic space. Given a closed $Z \subset X$ such that $j\colon X \smallsetminus Z \hookrightarrow X$ is quasi-compact, and*

$$\begin{cases} \operatorname{codim}(Z_\eta, X_\eta) \geqslant 2 & \textit{for each generic point } \eta \in S, \\ \operatorname{codim}(Z_s, X_s) \geqslant 1 & \textit{for all } s \in S, \end{cases}$$

then the restriction induces an equivalence of categories of G-torsors:

$$\mathbf{Tors}(X_{\mathrm{fppf}}, G) \xrightarrow{\sim} \mathbf{Tors}((X \smallsetminus Z)_{\mathrm{fppf}}, G).$$

Consequently, when considering isomorphism classes, there exists a bijection:

$$H^1_{\mathrm{fppf}}(X, G) \simeq H^1_{\mathrm{fppf}}(X \smallsetminus Z, G).$$

(ii) *Let V be a finite-rank valuation ring, X a V-smooth scheme, and G an X-finite locally free group scheme. Let $x \in X$ be a point such that $\dim \mathscr{O}_{X,x} \geqslant 2$. If x is not a maximal point in the S-fibres of X, then the restriction functor establishes an equivalence of categories of G-torsors:*

$$\mathbf{Tors}((\operatorname{Spec} \mathscr{O}_{X,x})_{\mathrm{fppf}}, G) \xrightarrow{\sim} \mathbf{Tors}((\operatorname{Spec} \mathscr{O}_{X,x} \smallsetminus \{x\})_{\mathrm{fppf}}, G).$$

Consequently, when considering isomorphism classes, there exists a bijection:

$$H^1_{\mathrm{fppf}}(\operatorname{Spec} \mathscr{O}_{X,x}, G) \simeq H^1_{\mathrm{fppf}}(\operatorname{Spec} \mathscr{O}_{X,x} \smallsetminus \{x\}, G).$$

We anticipate that the theorem is valid for all X which are flat, locally finitely presented over S (or over V), with regular fibres. However, our approach does not confirm this due to its reliance on Auslander's criterion for flatness (see Theorem 4.1), which necessitates the smoothness of X over S (or over V).

Proof

(i) We may assume that $X \to S$ is quasi-compact. The assumption implies that X and G are both finitely presented over S. Thus, similar to the proof of Theorem 6.3, we are reduced to the case where S is the spectrum of a valuation ring.

It remains to verify the assumptions of Lemma 5.7. All assumptions of Lemma 5.7 are étale local on X, so we may assume that X is a scheme, finitely presented over S. Employing the limit argument Lemma 3.10 involving Lemma 3.4, we can further restrict our scenario to instances where S has a finite Krull dimension. Furthermore, since $|S|$ is finite and each R-fibre of X is Noetherian, $|X|$ is Noetherian.

The condition (i) of Lemma 5.7 can be deduced from Proposition 3.9(iii) and Theorem 2.20, which needs the condition on fibres of Z.

To verify the condition (ii) of Lemma 5.7, we will check that, étale locally on X, every G-torsor over $X \smallsetminus Z$ extends to a G-torsor over X. Let $\mathcal{P}$ be a $G_{X\smallsetminus Z}$-torsor. By Proposition 3.9(iii) and Corollary 2.24, $j_*\mathscr{O}_{\mathcal{P}}$ is a reflexive $\mathscr{O}_X$-module. First, we prove the $\mathscr{O}_X$-flatness of $j_*\mathscr{O}_{\mathcal{P}}$. We can use Noetherian induction to reduce to the case where X is local, essentially smooth over R, and $Z = \{x\}$ is its closed point. Then, Auslander's criterion Theorem 4.1 reduces us to showing the isomorphism

$$\mathscr{H}om_{\mathscr{O}_X}(j_*\mathscr{O}_{\mathcal{P}}, j_*\mathscr{O}_{\mathcal{P}}) \simeq (j_*\mathscr{O}_{\mathcal{P}})^{\oplus r}, \quad \text{where } r = \operatorname{rank}_{\mathscr{O}_X}\mathscr{O}_G.$$

Note that in such local case, we have $\mathscr{O}_G \simeq \mathscr{O}_X^{\oplus r}$. Consider the following map

$$\begin{aligned} &\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_G, j_*\mathscr{O}_{\mathcal{P}}) \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(j_*\mathscr{O}_{\mathcal{P}}, j_*\mathscr{O}_{\mathcal{P}}),\\ &\qquad\qquad f \longmapsto \Big(j_*\mathscr{O}_{\mathcal{P}} \xrightarrow{j_*\rho} \mathscr{O}_G \otimes_{\mathscr{O}_X} j_*\mathscr{O}_{\mathcal{P}} \xrightarrow{(f,\mathrm{id})} j_*\mathscr{O}_{\mathcal{P}}\Big)\end{aligned}$$

of reflexive $\mathscr{O}_X$-modules. This is an isomorphism: by Theorem 2.20, it suffices to argue over $X \smallsetminus Z$, then its explicit inverse is

$$g \longmapsto \Big(\mathscr{O}_{G_{X\smallsetminus Z}} \xrightarrow{\mathrm{id}\otimes 1} \mathscr{O}_{G_{X\smallsetminus Z}} \otimes_{\mathscr{O}_{X\smallsetminus Z}} \mathscr{O}_{\mathcal{P}} \xrightarrow{(\rho,\mathrm{id})^{-1}} \mathscr{O}_{\mathcal{P}} \otimes_{\mathscr{O}_{X\smallsetminus Z}} \mathscr{O}_{\mathcal{P}} \xrightarrow{(g,\mathrm{id})} \mathscr{O}_{\mathcal{P}}\Big).$$

We now prove that the G-torsor structure of $\mathcal{P}$ extends uniquely to that of the scheme $\underline{\operatorname{Spec}}_X(j_*\mathscr{O}_{\mathcal{P}})$. As G is finite locally free, by projection formula [Stacks, 01E8], taking j_* of the co-action $\rho : \mathscr{O}_{\mathcal{P}} \to j^*\mathscr{O}_G \otimes_{\mathscr{O}_{X\smallsetminus Z}} \mathscr{O}_{\mathcal{P}}$ yields

$$j_*\rho\colon \quad j_*\mathscr{O}_{\mathcal{P}} \longrightarrow \mathscr{O}_G \otimes_{\mathscr{O}_X} j_*\mathscr{O}_{\mathcal{P}}.$$

To check that $j_*\rho$ is a co-action, we verify the associativity, the commutativity of the following diagram

$$\begin{tikzcd} j_*\mathscr{O}_{\mathcal{P}} \arrow[r, "j_*(\rho)"] \arrow[d, "j_*(\rho)"] & \mathscr{O}_G \otimes_{\mathscr{O}_X} j_*\mathscr{O}_{\mathcal{P}} \arrow[d, "\mathrm{id}\otimes j_*(\rho)"] \\ \mathscr{O}_G \otimes_{\mathscr{O}_X} j_*\mathscr{O}_{\mathcal{P}} \arrow[r, "\mu_G\otimes\mathrm{id}"] & \mathscr{O}_G \otimes_{\mathscr{O}_X} \mathscr{O}_G \otimes_{\mathscr{O}_X} j_*\mathscr{O}_{\mathcal{P}}, \end{tikzcd}$$

where $\mu_G : \mathscr{O}_G \to \mathscr{O}_G \otimes_{\mathscr{O}_X} \mathscr{O}_G$ is the co-multiplication of G. Since all sheaves involved are $\mathscr{O}_X$-reflexive, the commutativity over $X \smallsetminus Z$ by Theorem 2.20 extends over X.

Finally, the following map

$$(j_*\rho, 1\otimes \mathrm{id})\colon \quad j_*\mathscr{O}_{\mathcal{P}} \otimes_{\mathscr{O}_X} j_*\mathscr{O}_{\mathcal{P}} \longrightarrow \mathscr{O}_G \otimes_{\mathscr{O}_X} j_*\mathscr{O}_{\mathcal{P}},$$

by the $\mathscr{O}_X$-flatness of $j_*\mathscr{O}_{\mathcal{P}}$ and Theorem 2.20, is an isomorphism since so is its restriction on $X \smallsetminus Z$.

(ii) A similar argument can be employed for its proof. Specifically, to establish the essential surjectivity of the restriction functor, the finite-rank assumption on V ensures that $j : \mathrm{Spec}\,\mathscr{O}_{X,x} \smallsetminus \{x\} \hookrightarrow \mathrm{Spec}\,\mathscr{O}_{X,x}$ is quasi-compact quasi-separated. Consequently, for any G-torsor $\mathcal{P}$ over $\mathrm{Spec}\,\mathscr{O}_{X,x} \smallsetminus \{x\}$, $j_*\mathscr{O}_{\mathcal{P}}$ is a reflexive $\mathscr{O}_{X,x}$-module (by Theorem 2.20). By invoking Auslander's criterion (see Theorem 4.1), we deduce that $j_*\mathscr{O}_{\mathcal{P}}$ is free over $\mathscr{O}_{X,x}$ and retains the G-torsor structure from $\mathcal{P}$. This subsequently yields the sought-after extension of $\mathcal{P}$ to $\mathrm{Spec}\,\mathscr{O}_{X,x}$. □

From Theorem 6.8 one can now derive an analog of the classical Zariski–Nagata.

Theorem 6.9 (Zariski–Nagata purity for finite étale covers)

(i) *Let S be a Prüfer algebraic space, and X an S-smooth algebraic space. Given a closed subspace Z of X such that $j\colon X \smallsetminus Z \hookrightarrow X$ is quasi-compact, and that*

$$\begin{cases} \mathrm{codim}(Z_\eta, X_\eta) \geqslant 2 & \textit{for each generic point } \eta \in S, \\ \mathrm{codim}(Z_s, X_s) \geqslant 1 & \textit{for all } s \in S, \end{cases}$$

then the restriction gives an equivalence of categories of finite étale objects

$$\mathrm{FÉt}_X \xrightarrow{\sim} \mathrm{FÉt}_{X\smallsetminus Z}.$$

(ii) *Let R be a finite-rank valuation ring with spectrum S, and consider an S-smooth scheme X. Given a point $x \in X$ with properties such that $\dim \mathscr{O}_{X,x} \geqslant 2$ and x is not a maximal point in the S-fibres of X. Then the restriction functor establishes an equivalence of categories of finite étale covers*

$$\mathrm{FÉt}_{\mathrm{Spec}\,\mathscr{O}_{X,x}} \xrightarrow{\sim} \mathrm{FÉt}_{\mathrm{Spec}\,\mathscr{O}_{X,x}\smallsetminus\{x\}}.$$

Again, we anticipate that the theorem is valid for all X which are flat, locally finitely presented over S, with regular fibres.

Proof

(i) Full faithfulness. For finite étale covers $\pi_i : X_i \to X$, $i = 1, 2$, consider

$$\text{the } X\text{-functor} \quad Y := \underline{\mathrm{Hom}}_X(X_1, X_2)$$

that parameterizes X-morphisms from X_1 to X_2; it is a subfunctor of

$$\underline{\mathrm{Hom}}_X(\pi_{2,*}\mathscr{O}_{X_2}, \pi_{1,*}\mathscr{O}_{X_1})$$

consisting of sections compatible with algebraic structures of $\pi_{2,*}\mathscr{O}_{X_2}$ and $\pi_{1,*}\mathscr{O}_{X_1}$, which amount to the commutativity of a certain diagram of $\mathscr{O}_X$-modules. It follows that $Y \subset \underline{\mathrm{Hom}}_X(\pi_{2,*}\mathscr{O}_{X_2}, \pi_{1,*}\mathscr{O}_{X_1})$ is a closed subfunctor. Thus, Y is an algebraic space finite over X. (Using the infinitesimal criterion for formal smoothness, one can check that $Y \to X$ is even finite étale, but we will not need this in the sequel.) By Theorem 2.20, we have $Y(X) \simeq Y(X \smallsetminus Z)$, thereby proving the full faithfulness.

Essential surjectivity. Given the full faithfulness established in the preceding paragraph, we can first pass to an étale cover and then to a connected component, allowing us to assume that X is an integral affine scheme. To apply Theorem 6.8(i) and conclude, it suffices to observe that the category of finite étale covers of X of degree n (with isomorphisms) is equivalent to the category of $\mathfrak{S}_{n,X}$-torsors. (Here $\mathfrak{S}_n$ denotes the n-th symmetric group.)

(ii) This is proved in the same manner as (i), using Theorem 6.8(ii) instead of Theorem 6.8(i). □

7. Cohomology of groups of multiplicative type

Inspired by the purity results in [ČS24, Th. 7.2.8], we investigate the parafactoriality over Prüfer bases and then present the purity for cohomology of group algebraic spaces of multiplicative type.

7.1. Geometrically parafactorial pairs

7.1.1. *Parafactorial pairs.* — Let $(X,\mathscr{O}_X)$ be a ringed space with a closed subspace $Z\subset X$ and the canonical open immersion $j\colon V:=X\smallsetminus Z\hookrightarrow X$, if for every open subspace $U\subset X$ the following restriction is an equivalence of categories

$$\mathbf{Pic}\,U\xrightarrow{\sim}\mathbf{Pic}\,(U\cap V)\qquad \mathscr{L}\longmapsto \mathscr{L}|_{U\cap V},$$

then the pair (X,Z) is *parafactorial.* In particular, for an invertible $\mathscr{O}_X$-module $\mathscr{L}$,

$$\mathscr{L}(U)=\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{O}_U,\mathscr{L}|_U)\simeq \mathrm{Hom}_{\mathscr{O}_{U\cap V}}(\mathscr{O}_{U\cap V},\mathscr{L}|_{U\cap V})=\mathscr{L}(U\cap V)$$

for all open $U\subset X$; in other words, $\mathscr{L}\simeq j_*j^*\mathscr{L}$. A local ring A is *parafactorial* if the pair $(\operatorname{Spec}A,\{\mathfrak{m}_A\})$ is parafactorial. We list several parafactorial pairs (X,Z) and local rings.

(i) By [EGA IV$_4$, Prop. 21.13.8], a local ring A is parafactorial amounts to

$$\mathrm{Pic}(\mathrm{Spec}(A)\smallsetminus\{\mathfrak{m}_A\})=0\quad\text{and}\quad A\simeq\Gamma(\mathrm{Spec}(A)\smallsetminus\{\mathfrak{m}_A\},\widetilde{A});$$

(ii) By [EGA IV$_4$, Ex. 21.13.9(ii)], a Noetherian factorial local ring is parafactorial if and only if its Krull dimension is at least 2;

(iii) When X is locally Noetherian, Z satisfies $\mathrm{codim}(Z,X)\geqslant 4$, and $\mathscr{O}_{X,z}$ are locally complete intersection[(8)] for all $z\in Z$, by [SGA 2$_{\mathrm{new}}$, Exp. XI, Prop. 3.3 & Th. 3.13(ii)], the pair (X,Z) is parafactorial;

(iv) For a normal scheme S, an S-smooth scheme X and a closed $Z\subset X$ satisfying

$$\begin{cases}\mathrm{codim}(Z_\eta,X_\eta)\geqslant 2 & \text{for each generic point } \eta\in S,\\ \mathrm{codim}(Z_s,X_s)\geqslant 1 & \text{for every } s\in S,\end{cases}$$

by [EGA IV$_4$, Prop. 21.14.3], the pair (X,Z) is parafactorial.

(8)This means that its completion is the quotient of complete regular local ring by an ideal generated by a regular sequence.

7.1.2. *Geometrically parafactorial pairs*. — Below, we are mainly interested in the case when X is a scheme, and $Z \subset X$ is a closed subscheme such that the canonical open immersion $j\colon V := X \smallsetminus Z \hookrightarrow X$ is quasi-compact. By [EGA IV$_2$, Lem. 2.3.1], the quasi-compactness of j guarantees that the pushforward by j of a quasi-coherent $\mathscr{O}_V$-module is quasi-coherent and its formation commutes with arbitrary flat base changes (in particular, localizations).

A pair (X, Z) is *geometrically parafactorial* if $j : V = X \smallsetminus Z \hookrightarrow X$ is quasi-compact and if, for every X-étale X' with base change $Z' := Z \times_X X'$, the pair (X', Z') is parafactorial. A local ring A with a quasi-compact punctured spectrum is *geometrically parafactorial* if its strict Henselization A^{sh} is parafactorial. By the following Lemma 7.2, one can see that a local ring A with a quasi-compact punctured spectrum is geometrically parafactorial if and only if the pair $(\operatorname{Spec} A, \{\mathfrak{m}_A\})$ is geometrically parafactorial.

Lemma 7.2. — *Let A be a local ring with a quasi-compact punctured spectrum. Then A is geometrically parafactorial if and only if for any local and essentially étale*[9] *map $A \to B$ of local rings, B is parafactorial.*

Proof. — Assume that A is geometrically parafactorial, that is, A^{sh} is parafactorial. Let $A \to B$ be a local and essentially étale map; we will show that B is parafactorial. For any local ring C, denote by $j_C : U_C^{\circ} := \operatorname{Spec}(C) \smallsetminus \{\mathfrak{m}_C\} \hookrightarrow U_C := \operatorname{Spec}(C)$ the canonical open immersion. Choose an A-map $B \to A^{\mathrm{sh}}$. Let $\mathscr{L}$ be an invertible $\mathscr{O}_{U_B^{\circ}}$-module. By the quasi-compactness of j_B (inherited from that of $j = j_A$) and the faithful flatness of $U_{A^{\mathrm{sh}}} \to U_B$, the $\mathscr{O}_{U_B}$-module $j_{B,*}\mathscr{L}$ is quasi-coherent and its pullback to $U_{A^{\mathrm{sh}}}$ is isomorphic to $j_{A^{\mathrm{sh}},*}(\mathscr{L}|_{U^{\circ}_{A^{\mathrm{sh}}}})$. Since A^{sh} is parafactorial, we have $j_{A^{\mathrm{sh}},*}(\mathscr{L}|_{U^{\circ}_{A^{\mathrm{sh}}}}) \simeq \mathscr{O}_{U_{A^{\mathrm{sh}}}}$. Descent theory implies that $j_{B,*}\mathscr{L} \simeq \mathscr{O}_{U_B}$ and thus $\mathscr{L} \simeq \mathscr{O}_{U_B^{\circ}}$. Similarly, by considering the pullback to $U_{A^{\mathrm{sh}}}$, we see that the natural map $\mathscr{O}_{U_B} \to j_{B,*}\mathscr{O}_{U_B^{\circ}}$ is bijective, that is, $B \simeq \Gamma(U_B^{\circ}, \mathscr{O}_{U_B^{\circ}})$. This proves that B is parafactorial (cf. Section 7.1.1(i)).

For the other side, fixing a separable closure $\overline{\kappa}_A$ of $\kappa_A = A/\mathfrak{m}_A$ and a geometric point $\overline{t} : A \to \overline{\kappa}_A$, then A^{sh} is the filtered colimit of all B for essentially étale local ring maps $A \to B$ along with a geometric point $\overline{t}_B : B \to \overline{\kappa}_A$ lifting $\overline{t}$. Consequently, if all such B are parafactorial, then we have the following equivalences

$$\mathbf{Pic}\, U_{A^{\mathrm{sh}}} \xleftarrow{\sim} \text{2-colim}_{(B,\overline{t}_B)} \mathbf{Pic}\, U_B \xrightarrow{\sim} \text{2-colim}_{(B,\overline{t}_B)} \mathbf{Pic}\, U_B^{\circ} \xrightarrow{\sim} \mathbf{Pic}\, U^{\circ}_{A^{\mathrm{sh}}},$$

where the rightmost equivalence follows from [Stacks, 0B8W], because U_A° is quasi-compact. This proves that A is geometrically parafactorial. $\square$

The following is a generalization of [EGA IV$_4$, Prop. 21.13.10] to the case of topologically locally Noetherian schemes.[10]

[9] Recall that by definition an essentially étale ring map is a localization of an étale ring map.

[10] A scheme is topologically locally Noetherian if it admits a cover by open subschemes whose underlying topological spaces are Noetherian (i.e., any descending sequence of closed subsets is eventually constant); it is topologically Noetherian if its underlying topological space is Noetherian.

Lemma 7.3. — *For a topologically locally Noetherian scheme X and a closed subscheme $Z \subset X$,*

(i) *the pair (X, Z) is parafactorial if and only if $\mathscr{O}_{X,z}$ is parafactorial for every $z \in Z$;*

(ii) *the pair (X, Z) is geometrically parafactorial if and only if $\mathscr{O}^{\mathrm{sh}}_{X,\overline{z}}$ is parafactorial for every geometric point $\overline{z} \to Z$.*

Proof. — By [EGA IV$_4$, Cor. 21.13.6(i)], we can work Zariski locally on X, so we may assume throughout that X is topologically Noetherian. By [BS15, Lem. 6.6.10(3)], any quasi-compact étale cover of X is also topologically Noetherian. Therefore, taking into account of Lemma 7.2, we see that (ii) follows from (i) because Spec $\mathscr{O}^{\mathrm{sh}}_{X,\overline{z}}$ is the inverse limit of étale neighborhoods of $\overline{z} \to X$.

It remains to prove (i). Assume that (X, Z) is a parafactorial pair and denote by $j\colon V := X \smallsetminus Z \hookrightarrow X$ the canonical open immersion. For each $z \in Z$, denote $U_z :=$ Spec $\mathscr{O}_{X,z}$ and $U_z^\circ := U_z \smallsetminus \{z\}$. To show that $\mathscr{O}_{X,z}$ is parafactorial, we prove that every invertible $\mathscr{O}_{U_z^\circ}$-module $\mathscr{L}_0$ is isomorphic to $\mathscr{O}_{U_z^\circ}$. By [EGA IV$_3$, Prop. 8.2.13] and [EGA I, Prop. 2.4.2], U_z° is the inverse limit of $B^\circ := B \smallsetminus (B \cap \overline{\{z\}})$, where B ranges over all open affine neighborhoods of $z \in X$. Since $B^\circ \hookrightarrow B$ is quasi-compact (by topological Noetherianness), the limit argument [Stacks, 0B8W] implies that there exists such a B and an invertible $\mathscr{O}_{B^\circ}$-module $\mathscr{L}_{B^\circ}$ for which $\mathscr{L}_0 \simeq \mathscr{L}_{B^\circ}|_{U_z^\circ}$. By assumption and [EGA IV$_4$, Cor. 21.13.6(i)(ii)], the pair $(B, B \cap \overline{\{z\}})$ is parafactorial. Thus, there exists an invertible $\mathscr{O}_B$-module $\widetilde{\mathscr{L}_B}$ such that $\widetilde{\mathscr{L}_B}|_{B^\circ} \simeq \mathscr{L}_{B^\circ}$. Shrinking B if necessary, we have $\widetilde{\mathscr{L}_B} \simeq \mathscr{O}_B$ hence $\mathscr{L}_0 \simeq \mathscr{O}_{U_z^\circ}$.

Conversely, assuming that $\mathscr{O}_{X,z}$ are parafactorial for all $z \in Z$, we will prove that the pair (X, Z) is parafactorial. We first show that for any open $U \subset X$ with the open immersion $j_U : U \cap V \hookrightarrow U$, the canonical map $\mathscr{E} \to j_{U,*}(\mathscr{E}|_{U\cap V})$ is bijective for any finite-rank locally free $\mathscr{O}_U$-module $\mathscr{E}$. By [EGA IV$_4$, Cor. 21.13.3(b)], this is equivalent to saying that $\mathscr{O}_X \xrightarrow{\sim} j_*\mathscr{O}_V$ is bijective. The Noetherian assumption implies that j is quasi-compact, so $j_*\mathscr{O}_V$ is quasi-coherent and commutes with flat base changes. It suffices to prove that we have an isomorphism after localizing at points $z \in X$. This is trivial if $z \in V$, and if $z \in Z$ it follows from parafactoriality of $\mathscr{O}_{X,z}$ (see Section 7.1.1(i)).

To finish the proof, by [EGA IV$_4$, Prop. 21.13.5], it remains to show that for every invertible $\mathscr{O}_V$-module $\mathscr{L}$, the pushforward $j_*\mathscr{L}$ is an invertible $\mathscr{O}_X$-module. For this, we consider the subset

$$\Omega := \{x \in X \mid j_*\mathscr{L} \text{ is invertible on an open neighborhood of } x\};$$

it is open in X and contains $V = X \smallsetminus Z$. Denote $Y := X \smallsetminus \Omega \subset Z$. If $Y \neq \varnothing$, we choose a maximal point $y \in Y$ (every non-empty scheme has maximal points). Then $\Omega \cap U_y = U_y^\circ$, and so $\mathscr{L}_0 := (j_*\mathscr{L})|_{U_y^\circ}$ is an invertible $\mathscr{O}_{U_y^\circ}$-module. The parafactoriality of $\mathscr{O}_{X,y}$ yields an extension of $\mathscr{L}_0$ to an invertible $\mathscr{O}_{U_y}$-module $\widetilde{\mathscr{L}_0}$, which, by the limit argument [Stacks, 0B8W] again, descends to an invertible $\mathscr{O}_W$-module $\widetilde{\mathscr{L}_W}$ for an open affine neighborhood W of $y \in X$. As W shrinks, it becomes U_y and $\Omega \cap W$ becomes U_y°. Since $\Omega \cap W$ is quasi-compact, *loc. cit.* implies that we may shrink W to assume that the restrictions of $j_*\mathscr{L}$ and $\widetilde{\mathscr{L}_W}$ to $\Omega \cap W$ are equal. Set $\Omega' := \Omega \cup W$. By Zariski gluing, we obtain an invertible $\mathscr{O}_{\Omega'}$-module $\mathscr{L}'$ such that $\mathscr{L}'|_W = \widetilde{\mathscr{L}_W}$ and $\mathscr{L}'|_\Omega = (j_*\mathscr{L})|_\Omega$. Since $V \subset \Omega$, we have $\mathscr{L}'|_V = \mathscr{L}$, and so $\mathscr{L}' \xrightarrow{\sim} (j_*(\mathscr{L}'|_V))|_{\Omega'} = (j_*\mathscr{L})|_{\Omega'}$ (by the previous paragraph), which leads to a desired contradiction with the definition of Ω. □

Proposition 7.4. — *Let S be a normal scheme and X an S-scheme. Assume that one of the following holds:*

(i) *either $X \to S$ is a smooth morphism of topologically locally Noetherian schemes; or*

(ii) *S is semilocal Prüfer of finite dimension and X is S-flat locally of finite type with regular S-fibres.*

Then, if $x \in X$ is not a maximal point of S-fibres of X and satisfies $\dim \mathscr{O}_{X,x} \geqslant 2$, it holds that $\mathscr{O}_{X,x}$ is geometrically parafactorial, namely, $\mathscr{O}_{X,x}^{\mathrm{sh}}$ is parafactorial.

Proof. — Notice that, in both cases (i)–(ii), the scheme X is topologically locally Noetherian. By Lemma 7.2, the parafactoriality of $\mathscr{O}_{X,x}^{\mathrm{sh}}$ is equivalent to those of $\mathscr{O}_{X',x'}$ for all X-étale X' and $x' \in X'$ lying over x. Moreover, since all X' and x' satisfy the conditions in the statement above (in the case (i), the topologically locally Noetherianness of X' follows from [BS15, Lem. 6.6.10(3)]), thus it suffices to show that $\mathscr{O}_{X,x}$ is parafactorial. For the Zariski closure $Z := \overline{\{x\}}$, by Lemma 7.3, we are reduced to finding a small open neighborhood U of $x \in X$ such that $(U, Z \cap U)$ is a parafactorial pair. Now, take an arbitrary open neighborhood U of $x \in X$, by [EGA IV$_3$, Prop. 9.5.3] applied to $Z \subset X$, shrinking U if needed, we may assume that $U \cap Z$ does not contain any irreducible components of S-fibres of X. In addition, if some $z \in Z$ lies over a maximal point $\eta \in S$, since x specializes to z, then we have $\dim \mathscr{O}_{X_\eta,z} = \dim \mathscr{O}_{X,z} \geqslant 2$. Consequently, we have $\operatorname{codim}(X_\eta \cap Z, X_\eta) \geqslant 2$ and, by Section 7.1.1(iv) (in case (i)) and Theorem 2.20 (in case (ii)), the desired parafactoriality of $(U, Z \cap U)$ follows. □

7.5. Purity for groups of multiplicative type. — The goal of this subsection is to study purity for groups of multiplicative type over algebraic spaces that are *topologically locally Noetherian* in the following sense: they admit étale covers by topologically locally Noetherian schemes. By [BS15, Lem. 6.6.10(3)], we see that any scheme étale over a topologically locally Noetherian algebraic space is again topologically locally Noetherian.

7.5.1. *The fppf local cohomology*. — Let X be an algebraic space. Let $(\mathbf{Sch}_{/X})_{\mathrm{fppf}}$ denote the site of schemes over X with fppf covers. Recall that the sections of a sheaf on $(\mathbf{Sch}_{/X})_{\mathrm{fppf}}$ over an X-algebraic space U are defined as the set of morphisms from the fppf sheaf

$$\mathbf{Sch}_{/X} \ni V \longmapsto \mathrm{Hom}_X(V, U).$$

Given a closed subspace $Z \subset X$, denote by $j : X \smallsetminus Z \hookrightarrow X$ the open immersion. For an abelian sheaf $\mathscr{F}$ on $(\mathbf{Sch}_{/X})_{\mathrm{fppf}}$, we define

$$\mathcal{H}^0_Z(\mathscr{F})(V) := \mathrm{Ker}(\mathscr{F}(V) \longrightarrow \mathscr{F}(V \smallsetminus (V \times_X Z))), \qquad V \in \mathbf{Sch}_{/X};$$

it is the largest subsheaf of $\mathscr{F}$ supported on Z. For an algebraic space U over X, we denote

$$H^0_Z(U, \mathscr{F}) := \Gamma(U, \mathcal{H}^0_Z(\mathscr{F})).$$

Both the functors $\mathscr{F} \mapsto \mathcal{H}^0_Z(\mathscr{F})$ and $\mathscr{F} \mapsto H^0_Z(U, \mathscr{F})$ are left exact. Let $\mathcal{H}^i_Z$ and $H^i_Z(U, -)$ denote their i-th right derived functors. From the definition we see that $\mathcal{H}^i_Z(\mathscr{F})$ can be identified with the sheafification of the presheaf $V \mapsto H^i_Z(V, \mathscr{F})$. Moreover, as the functor $\mathcal{H}^0_Z$ sends injective sheaves to injective sheaves (because it admits an exact left adjoint), we have the local-to-global E_2-spectral sequence:

$$E_2^{pq} = H^p(U, \mathcal{H}^q_Z(\mathscr{F})) \Rightarrow H^{p+q}_Z(U, \mathscr{F}).$$

By employing an injective resolution, one also derives the long exact sequence for fppf local cohomology, analogous to the approach used for Zariski local cohomology as discussed in Section 2.5. The key lies in the surjectivity of the restriction map $\mathscr{F}(X) \to \mathscr{F}(X \smallsetminus Z)$ for an injective abelian sheaf $\mathscr{F}$, a result that naturally follows by applying $\mathrm{Hom}(-, \mathscr{F})$ to the monomorphism $j_!\mathbf{Z}_{X \smallsetminus Z} \to \mathbf{Z}_X$.

Although not directly pertinent to our discussion, let's briefly elucidate the connection with Zariski local cohomology presented in Section 2.5. Consider the natural morphism $\mu : (\mathbf{Sch}/X)_{\mathrm{fppf}} \to X_{\mathrm{Zar}}$ to the small Zariski site. For any abelian sheaf $\mathscr{F}$ on $(\mathbf{Sch}_{/X})_{\mathrm{fppf}}$, there exists a natural comparison map

$$R\Gamma_{\mathrm{Zar}}(X, \mu_*\mathscr{F}) \longrightarrow R\Gamma_{\mathrm{Zar}}(X, R\mu_*\mathscr{F}) = R\Gamma_{\mathrm{fppf}}(X, \mathscr{F}).$$

While this is not an isomorphism in general, it holds true if X is a scheme and $\mathscr{F}$ is quasi-coherent, as $\mu_*\mathscr{F} = R\mu_*\mathscr{F}$ (quasi-coherent sheaves have vanishing higher fppf-cohomology on affines). Considering the long exact sequence for local cohomology, a similar comparison result holds for local cohomology.

On occasion, we will use the étale local cohomology. Its definition mirrors its counterpart, using either the big or small étale site of X in lieu of the fppf site.

The following result concerning étale descent of fppf local cohomology will be needed.

Lemma 7.6 (cf. [ČS24, Lem. 7.1.1]). — *For an algebraic space X, a closed subspace $Z \subset X$, and an abelian sheaf $\mathscr{F}$ on $(\mathbf{Sch}_{/X})_{\mathrm{fppf}}$, if for any integer $q \geqslant 0$, $\widetilde{\mathcal{H}}^q_Z(\mathscr{F})$ denotes the étale-sheafification of the presheaf $(V \to X) \mapsto H^q_Z(V, \mathscr{F})$ where $V \to X$ is étale, then we have a convergent spectral sequence*

$$E_2^{pq} = H^p_{\text{ét}}(X, \widetilde{\mathcal{H}}^q_Z(\mathscr{F})) \Rightarrow H^{p+q}_Z(X, \mathscr{F}).$$

Proof. — There is a trouble with recursive references, so we give a proof for the convenience of the readers.

Let $\mathbf{Ab}(-)$ denote the category of abelian sheaves on a site. Consider the following sequence of functors:

$$\mathbf{Ab}((\mathbf{Sch}_{/X})_{\mathrm{fppf}}) \xrightarrow{\mathcal{H}^0_Z} \mathbf{Ab}((\mathbf{Sch}_{/X})_{\mathrm{fppf}}) \xrightarrow{\nu_*} \mathbf{Ab}((\mathbf{\acute{E}tSch}_{/X})_{\text{ét}}) \xrightarrow{\Gamma(X,-)} \mathbf{Ab},$$

where $(\mathbf{\acute{E}tSch}_{/X})_{\text{ét}}$ is the site of étale schemes over X with étale covers, and $\nu : (\mathbf{Sch}_{/X})_{\mathrm{fppf}} \to (\mathbf{\acute{E}tSch}_{/X})_{\text{ét}}$ denotes the natural morphism of sites. The first two functors send injectives to injectives because they admit exact left adjoints. For any abelian sheaf $\mathscr{F}$ on $(\mathbf{Sch}_{/X})_{\mathrm{fppf}}$, we have $(\nu_* \circ \mathcal{H}^0_Z)(\mathscr{F})(V) = H^0_Z(V, \mathscr{F})$ where V is a scheme étale over X. This implies that the q-th right derived functor of $\nu_* \circ \mathcal{H}^0_Z$ is given by the étale sheafification of the presheaf $(V \to X) \mapsto H^q_Z(V, \mathscr{F})$. Now the lemma follows from the Grothendieck spectral sequence applied to the functors $\nu_* \circ \mathcal{H}^0_Z$ and $\Gamma(X,-)$. $\square$

7.6.1. *Setup*. — From now on we assume the following, unless stated otherwise:

– let X be a topologically locally Noetherian algebraic space, $Z \subset X$ a closed subspace, and $j : X \smallsetminus Z \hookrightarrow X$ the canonical open immersion;

– for every geometric point $\overline{z} \to Z$, the strict local ring $\mathscr{O}^{\mathrm{sh}}_{X,\overline{z}}$[11] is (geometrically) parafactorial;[12]

– let M be an X-group algebraic space of multiplicative type, that is, its base change $M_{X'}$ is a group scheme of multiplicative type, where $X' \to X$ is some étale cover by a scheme X'.

Remark 7.7. — In the case when X is a scheme, Lemma 7.3(ii) implies that the first two assumptions above are equivalent to saying that the pair (X, Z) is geometrically parafactorial in the sense of Section 7.1.2. Moreover, Proposition 7.4(i)–(ii) gives examples of such pairs (X, Z), where Z satisfies the conditions

$$\begin{cases} \operatorname{codim}(Z_\eta, X_\eta) \geqslant 2 & \text{for every generic point } \eta \in S, \\ \operatorname{codim}(Z_s, X_s) \geqslant 1 & \text{for all } s \in S. \end{cases}$$

Proposition 7.8. — *In the Setup* 7.6.1*, assume that X is a scheme. For a point $z \in Z$ and an $\mathscr{O}_{X,z}$-torus T, we have*

$$H^i_{\{z\},\mathrm{fppf}}(\operatorname{Spec}(\mathscr{O}_{X,z}), T) \simeq H^i_{\{z\},\text{ét}}(\operatorname{Spec}(\mathscr{O}_{X,z}), T) = 0 \quad \textit{for} \quad 0 \leqslant i \leqslant 2.$$

(11) By a geometric point of an algebraic space we refer to a map from the spectrum of a separably closed field. For a given geometric point $\overline{t} : \operatorname{Spec}(\Omega) \to X$, the strict local ring $\mathscr{O}^{\mathrm{sh}}_{X,\overline{t}}$ of X at $\overline{t}$ is defined as the filtered colimit of all B, where $\operatorname{Spec}(B) \to X$ is an étale map, along with a geometric point $\overline{t}_B : \operatorname{Spec}(\Omega) \to \operatorname{Spec}(B)$ lifting $\overline{t}$. The strict local ring $\mathscr{O}^{\mathrm{sh}}_{X,\overline{t}}$, up to isomorphism, depends only on the equivalence class of the geometric point $\overline{t}$: indeed, if $\overline{t}' : \operatorname{Spec}(\Omega') \to \operatorname{Spec}(\Omega) \xrightarrow{\overline{t}} X$ is another geometric point and $\operatorname{Spec}(B) \to X$ is an étale map, there exists a natural bijection between liftings of $\overline{t}$ and $\overline{t}'$ to $\operatorname{Spec}(B)$. Clearly, this definition aligns with the classical definition of strict local rings for schemes.

(12) Note that the topologically locally Noetherian assumption on X implies that $\mathscr{O}^{\mathrm{sh}}_{X,\overline{z}}$ has a quasi-compact punctured spectrum, thus it is geometrically parafactorial if and only if it is parafactorial.

Proof. — The fppf and étale cohomology of a smooth group scheme coincides. So we may work with the étale site. By the local-to-global E_2-spectral sequence [SGA 4_{II}, Exp. V, Prop. 6.4],

$$H^p_{\text{ét}}(\operatorname{Spec}(\mathscr{O}_{X,z}), \mathcal{H}^q_{\{z\}}(T)) \Rightarrow H^{p+q}_{\{z\}}(\operatorname{Spec}(\mathscr{O}_{X,z}), T).$$

Therefore, it suffices to prove that $\mathcal{H}^q_{\{z\}}(T) = 0$ for $0 \leqslant q \leqslant 2$. Since $\mathscr{O}_{X,z}$ has a quasi-compact punctured spectrum, we can identify their stalks at a geometric point $\overline{z}$ lying over z:

$$\mathcal{H}^q_{\{z\}}(T)_{\overline{z}} = H^q_{\{\overline{z}\}}(\operatorname{Spec}(\mathscr{O}^{\mathrm{sh}}_{X,\overline{z}}), T).$$

Now, since $T_{\mathscr{O}^{\mathrm{sh}}_{X,\overline{z}}} \simeq \mathbb{G}^{\dim T}_{m,\mathscr{O}^{\mathrm{sh}}_{X,\overline{z}}}$ and $\mathscr{O}^{\mathrm{sh}}_{X,\overline{z}}$ is parafactorial, we have

$$H^q_{\text{ét}}(\operatorname{Spec}(\mathscr{O}^{\mathrm{sh}}_{X,\overline{z}}), T) \simeq H^q_{\text{ét}}(\operatorname{Spec}(\mathscr{O}^{\mathrm{sh}}_{X,\overline{z}}) \smallsetminus \{\overline{z}\}, T) \qquad \text{for } 0 \leqslant q \leqslant 1.$$

Moreover, as $\mathscr{O}^{\mathrm{sh}}_{X,\overline{z}}$ is strictly Henselian, we have

$$H^2_{\text{ét}}(\operatorname{Spec}(\mathscr{O}^{\mathrm{sh}}_{X,\overline{z}}), T) = 0.$$

Looking at the local cohomology exact sequence for the pair $(\operatorname{Spec}(\mathscr{O}^{\mathrm{sh}}_{X,\overline{z}}), \overline{z})$ and T, we see that

$$H^q_{\{\overline{z}\}}(\operatorname{Spec}(\mathscr{O}^{\mathrm{sh}}_{X,\overline{z}}), T) = 0 \qquad \text{for } 0 \leqslant q \leqslant 2,$$

giving that $\mathcal{H}^q_{\{z\}}(T) = 0$ for $0 \leqslant q \leqslant 2$, as desired. □

The following result is a variant of [ČS24, Th. 7.2.8(a)], where X is assumed topologically locally Noetherian but the local rings $\mathscr{O}_{X,z}$ are not supposed to be Noetherian for $z \in Z$.

Theorem 7.9. — *In the Setup* 7.6.1*, we have*

$$H^i_{\mathrm{fppf}}(X, M) \xrightarrow{\sim} H^i_{\mathrm{fppf}}(X \smallsetminus Z, M) \quad \textit{for } i = 0, 1, \textit{ and}$$
$$H^2_{\mathrm{fppf}}(X, M) \hookrightarrow H^2_{\mathrm{fppf}}(X \smallsetminus Z, M).$$

Proof. — By the local cohomology exact sequence for the pair (X, Z) and the sheaf M, everything reduces to showing the vanishings $H^q_Z(X, M) = 0$ for $0 \leqslant q \leqslant 2$. By the spectral sequence in Lemma 7.6, it suffices to show the vanishings of $\widetilde{\mathcal{H}}^q_Z(M)$, the étale-sheafification of the presheaf

$$(V \to X) \longmapsto H^q_Z(V, M), \qquad \text{where } V \longrightarrow X \text{ is étale}.$$

In particular, the problem is étale-local on X, so we may pass to an étale cover to assume that X is a scheme (noting that the assumptions of the Setup Section 7.6.1 still hold) and that M splits as μ_n or $\mathbb{G}_m$. Now, since $\mu_n = \ker(\mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m)$, it suffices to show that $\widetilde{\mathcal{H}}^q_Z(\mathbb{G}_m) = 0$ for $0 \leqslant q \leqslant 2$.

For $q = 0, 1$ this follows from the fact that the pair (X, Z) is parafactorial (using Lemma 7.3). (This is where the Noetherian hypothesis is used.)

For $q=2$, by the case $q=0,1$ already proved and the long exact sequence for local cohomology, we have

$$\begin{aligned} H^2_Z(X,\mathbb{G}_m) &\simeq \mathrm{Ker}(H^2_{\mathrm{fppf}}(X,\mathbb{G}_m)\longrightarrow H^2_{\mathrm{fppf}}(X\smallsetminus Z,\mathbb{G}_m)) \\ &\simeq \mathrm{Ker}(H^2_{\text{ét}}(X,\mathbb{G}_m)\longrightarrow H^2_{\text{ét}}(X\smallsetminus Z,\mathbb{G}_m)). \end{aligned}$$

The same is true for every scheme that is étale over X. Consequently, every class in $H^2_Z(X,\mathbb{G}_m)$ vanishes in an étale cover of X, since this property holds for $H^2_{\text{ét}}(X,\mathbb{G}_m)$. This implies that $\widetilde{\mathcal{H}}^2_Z(\mathbb{G}_m)=0$. □

7.10. Grothendieck–Serre type results for groups of multiplicative type

We record the following result from [GL23, Prop. 3.6] for later use.

Proposition 7.11. — *Let R be a Prüfer domain. Consider an irreducible scheme X essentially smooth over R having function field $K(X)$, and an X-group scheme M of multiplicative type. If there exists a connected finite étale Galois covering $X'\to X$ that splits M,*[(13)] *then the restriction maps*

$$H^1_{\mathrm{fppf}}(X,M)\longrightarrow H^1_{\mathrm{fppf}}(K(X),M)\quad \textit{and}\quad H^2_{\mathrm{fppf}}(X,M)\longrightarrow H^2_{\mathrm{fppf}}(K(X),M)$$

are injective in the following scenarios:

(i) *$X=\mathrm{Spec}(A)$ where A is a semilocal ring that's essentially smooth over R;*

(ii) *There exists an essentially smooth semilocal R-algebra A such that X embeds into $\overline{X}$ via a quasi-compact open immersion, with $\overline{X}$ being a smooth projective A-scheme having geometrically integral fibres. Furthermore, $\mathrm{Pic}(X_L)=0$ for any finite separable fields extension $L/\mathrm{Frac}(A)$, and $M=N_X$ where N is an A-group of multiplicative type (e.g. X could be a quasi-compact open subscheme of $\mathbb{P}^N_A$);*

(iii) *For any étale covering $X''\to X$ dominating $X'\to X$, we have $\mathrm{Pic}(X'')=0$.*

Moreover, if M is a flasque X-torus, then in all cases from (i)–(iii), the restriction

$$H^1_{\text{ét}}(X,M)\xrightarrow{\sim} H^1_{\text{ét}}(K(X),M)$$

is bijective.

8. Generically trivial torsors under a quasi-split group

In this section, we study generically trivial torsors under quasi-split reductive group schemes. The main result is Theorem 8.1, which consists of (i) a version of Nisnevich conjecture inspired by the recent preprint of Česnavičius [Čes22b, Th. 1.3(2)], who proved it in the case when R is a Dedekind domain, and (ii) the Grothendieck–Serre conjecture over one-dimensional Prüfer bases. The proof follows the strategy of [Čes22a] (with its earlier version given by Fedorov [Fed22]), which is possible due to the availability of the main tools in our Prüferian context, such as the toral version of purity (cf. [GL23, Th. 3.3] and Theorem 7.9) and the toral version of the Grothendieck–Serre conjecture (cf. Proposition 7.11(i)).

(13) Such a covering always exists, because X is normal and so M is isotrivial.

Theorem 8.1. — *For a semilocal Prüfer domain R with fraction field K, an integral, semilocal, and essentially smooth R-algebra A, and a quasi-split reductive A-group scheme G,*

(i) *every generically trivial G-torsor over $A \otimes_R K$ is trivial, that is,*

$$\operatorname{Ker}\big(H^1_{\text{ét}}(A \otimes_R K, G) \longrightarrow H^1_{\text{ét}}(\operatorname{Frac} A, G)\big) = \{*\};$$

(ii) *if R has Krull dimension 1, then every generically trivial G-torsor is trivial:*

$$\operatorname{Ker}\big(H^1_{\text{ét}}(A, G) \longrightarrow H^1_{\text{ét}}(\operatorname{Frac} A, G)\big) = \{*\}.$$

We start with the following consequence of Lemma 3.5, which is the key geometric input permitting a series of reductions that eventually lead to Theorem 8.1.

Lemma 8.2 (cf. [Čes22a, Prop. 4.1]). — *For*

(i) *a semilocal Prüfer domain R;*

(ii) *a smooth, faithfully flat R-algebra A of pure relative dimension $d \geqslant 1$;*

(iii) *a finite subset $\boldsymbol{x} \subset X := \operatorname{Spec} A$;*

(iv) *a closed subscheme $Y \subset X$ that satisfies*

$$\begin{cases} \operatorname{codim}(Y_s, X_s) \geqslant 1 & \textit{for all closed points } s \in \operatorname{Spec} R, \textit{ and} \\ \operatorname{codim}(Y_s, X_s) \geqslant 2 & \textit{otherwise.} \end{cases}$$

there are an affine open $U \subset \operatorname{Spec} A$ containing $\boldsymbol{x}$, an affine open $S \subset \mathbb{A}^{d-1}_R$, and a smooth R-morphism $\pi : U \to S$ of relative dimension 1 such that $Y \cap U$ is S-finite.

Proof. — Choosing an embedding of X into some affine space over R and taking schematic closure in the corresponding projective space, we get a projective compactification $\overline{X}$ of X. Since $\overline{X}$ is flat and projective over R, by Lemma 3.2(i), all its R-fibres have the same dimension d. Denoting by $\overline{Y} \subset \overline{X}$ the schematic closure of Y, to apply Lemma 3.5 and conclude (in which X is $\overline{X}$ here, W is X here, and Y is $\overline{Y}$ here), we need to check that the boundary $\overline{Y} \smallsetminus Y$ is R-fiberwisely of codimension $\geqslant 2$ in $\overline{X}$.

By [Stacks, 01RG], for a quasi-compact immersion of schemes, the schematic closure has the underlying space the topological closure. Thus, set-theoretically we have $\overline{Y} = \bigcup_y \overline{\{y\}}$, where y runs through the (finitely many) generic points of Y.

Let $y \in Y$ be a generic point, lying over $s \in \operatorname{Spec} R$. By Lemma 3.2(i), $\overline{X}$ has equal R-fibre dimension d and all non-empty R-fibres of $\overline{\{y\}}$ have the same dimension. If s is not a closed point, then for any specialization $s \rightsquigarrow s' \in \operatorname{Spec} R$, we have

$$\operatorname{codim}(\overline{\{y\}}_{s'}, \overline{X}_{s'}) = \operatorname{codim}(\overline{\{y\}}_s, \overline{X}_s) \geqslant 2;$$

a fortiori, the contribution of such a generic point y to the R-fibre codimension of $\overline{Y} \smallsetminus Y$ in $\overline{X}$ is $\geqslant 2$.

Otherwise, s is a closed point, then $\overline{\{y\}}_s = \overline{\{y\}} \subset \overline{Y_s}$. As assumed, $\operatorname{codim}(\overline{Y_s}, \overline{X}_s) = \operatorname{codim}(Y_s, X_s) \geqslant 1$, so we have $\operatorname{codim}(\overline{\{y\}}_s, \overline{X}_s) \geqslant 1$. But since the generic point y of $\overline{\{y\}}_s$ is not contained in $\overline{Y} \smallsetminus Y$, we deduce that the contribution of such a generic point y to the s-fibre codimension of $\overline{Y} \smallsetminus Y$ in $\overline{X}$ is again $\geqslant 2$. $\square$

Lemma 8.3 (Lifting the torsor to a smooth relative curve; cf. [Čes22a, Prop. 4.2])

Let R be a semilocal Prüfer domain with fraction field K. Let A be the semilocalization of an irreducible, R-smooth algebra A' at a finite subset $\boldsymbol{x} \subset \operatorname{Spec} A'$ and let G be a quasi-split reductive A-group scheme with a Borel subgroup B.

(1) *Given a generically trivial G-torsor P_K over $A_K := A \otimes_R K$, there are*

(i) *a smooth, affine relative A-curve C with a section $s \in C(A)$;*

(ii) *an A-finite closed subscheme $Z \subset C$;*

(iii) *a quasi-split reductive C-group scheme $\mathscr{G}$ with a Borel subgroup $\mathscr{B} \subset \mathscr{G}$ whose s-pullback is $B \subset G$, compatible with the quasi-pinnings;*

(iv) *a $\mathscr{G}$-torsor $\mathcal{P}_K$ over $C_K := C \times_R K$ whose s_{A_K}-pullback is P_K such that $\mathcal{P}_K$ reduces to a $\operatorname{rad}^u(\mathscr{B})$-torsor over $C_K \smallsetminus Z_K$ (here s_{A_K} denotes the image of s in $C(A_K)$).*

(2) *If R has Krull dimension 1 and P is a generically trivial G-torsor, there are*

(i) *a smooth, affine relative A-curve C with a section $s \in C(A)$;*

(ii) *an A-finite closed subscheme $Z \subset C$;*

(iii) *a quasi-split reductive C-group scheme $\mathscr{G}$ with a Borel subgroup $\mathscr{B} \subset \mathscr{G}$ whose s-pullback is $B \subset G$, compatible with the quasi-pinnings;*

(iv) *a $\mathscr{G}$-torsor $\mathcal{P}$ whose s-pullback is P such that $\mathcal{P}$ reduces to a $\operatorname{rad}^u(\mathscr{B})$-torsor over $C \smallsetminus Z$.*

Proof. — In the case (1) we can first use a limit argument involving Lemma 3.4 to reduce to the case when R has finite Krull dimension.

If A' is of relative dimension 0 over R, then $A_K = \operatorname{Frac}(A)$ and A is a semilocal Prüfer domain. Thus, P_K is trivial, and, by the Grothendieck–Serre conjecture on semilocal Prüfer schemes (cf. [GL23, Th. A.0.1]), P is also trivial. In this case we simply take $C = \mathbb{A}^1_A$, $s = 0 \in \mathbb{A}^1_A(A)$, $Z = \varnothing$, $(\mathscr{G}, \mathscr{B}) = (G_{\mathbb{A}^1_A}, B_{\mathbb{A}^1_A})$, and $\mathcal{P}_K = (P_K)_{\mathbb{A}^1_{A_K}}$ (resp. $\mathcal{P} = P_{\mathbb{A}^1_A}$). Thus, for what follows, we can assume that the relative dimension of A' over R is $d > 0$.

By spreading out and localizing A', we may assume that our quasi-split G (in particular, the Borel B) and torsor P all live over A', and P_K lives over A'_K. By [SGA $3_{\text{III new}}$, Exp. XXVI, Cor. 3.6 & Lem. 3.20], the quotient P_K/B_K (resp. P/B) is representable by a smooth projective scheme over A'_K (resp. over A'). Now we treat the cases (1)–(2) separately.

(1) Since P_K is generically trivial, the valuative criterion of properness applies to $P_K/B_K \to \operatorname{Spec} A'_K$, we find a closed subset $Y_K \subset \operatorname{Spec} A'_K$ of codimension $\geqslant 2$ such that $P_K/B_K \to \operatorname{Spec} A'_K$ has a section over $\operatorname{Spec} A'_K \smallsetminus Y_K$ that lifts to a generic section of P_K. In other words, $(P_K)_{\operatorname{Spec} A'_K \smallsetminus Y_K}$ reduces to a generically trivial $B_{\operatorname{Spec} A'_K \smallsetminus Y_K}$-torsor P_K^B. Consider the A'-torus $T := B/\operatorname{rad}^u(B)$ and the induced T-torsor

$$P_K^T := P_K^B/\operatorname{rad}^u(B)_K \quad \text{over } \operatorname{Spec}(A'_K) \smallsetminus Y_K.$$

Since P_K^T is generically trivial, by Corollary 6.6, it extends to a T-torsor $\widetilde{P_K^T}$ over $\operatorname{Spec} A' \smallsetminus F$ for a closed subscheme $F \subset \operatorname{Spec} A'$ satisfying

$$\operatorname{codim}(F_K, \operatorname{Spec} A'_K) \geqslant 3 \quad \text{and} \quad \operatorname{codim}(F_s, \operatorname{Spec} A'_s) \geqslant 2 \text{ for all } s \in \operatorname{Spec}(R);$$

by purity for tori (cf. [GL23, Th. 3.3] or Theorem 7.9), this torsor further extends to the whole $\operatorname{Spec} A'$. As $\widetilde{P_K^T}$ is generically trivial, by the Grothendieck–Serre conjecture for tori (Proposition 7.11(i)), we may localize A' around $\mathbf{x}$ to assume that $\widetilde{P_K^T}$, and hence also P_K^T, is already trivial. In other words, $(P_K)_{\operatorname{Spec}(A'_K)\smallsetminus Y_K}$ reduces to a $\operatorname{rad}^u(B)$-torsor over $\operatorname{Spec}(A'_K)\smallsetminus Y_K$.

Denote by Y the schematic closure of Y_K in $\operatorname{Spec} A'$; by Lemma 3.2(i), it is R-fiberwisely of codimension $\geqslant 2$ in $\operatorname{Spec} A'$. Applying Lemma 8.2 to the R-smooth algebra A' and the closed subscheme $Y \subset \operatorname{Spec} A'$, we obtain an affine open $U \subset \operatorname{Spec} A'$ containing $\boldsymbol{x}$, an affine open $S \subset \mathbb{A}_R^{d-1}$, and a smooth R-morphism $\pi : U \to S$ of relative dimension 1 such that $Y \cap U$ is S-finite.

Recall that A is the semilocal ring of U at $\mathbf{x}$. Denote

$$C := U \times_S \operatorname{Spec} A \quad \text{and} \quad Z := (Y \cap U) \times_S \operatorname{Spec} A.$$

Then C is a smooth affine relative A-curve, the diagonal in C induces a section $s \in C(A)$, and the closed subscheme $Z \subset C$ is A-finite. Thus (1i) and (1ii) hold. Let $\mathscr{B} \subset \mathscr{G}$ be the pullback of $B_U \subset G_U$ under the first projection $\mathrm{pr}_1 : C \to U$, and let $\mathcal{P}_K$ be the pullback of $(P_K)_{U_K}$ under the first projection $\mathrm{pr}_1 : C_K \to U_K$. Then, $\mathcal{P}_K$ is a $\mathscr{G}$-torsor over C_K, and, by construction, the s-pullback (resp. s_{A_K}-pullback) of $\mathscr{B} \subset \mathscr{G}$ (resp. of $\mathcal{P}_K$) is $B \subset G$ (resp. P_K). Finally, since P_K reduces to a $\operatorname{rad}^u(B)$-torsor over $\operatorname{Spec}(A'_K)\smallsetminus Y_K$, $\mathcal{P}_K$ reduces to a $\operatorname{rad}^u(\mathscr{B})$-torsor over $C_K \smallsetminus Z_K$. Thus (1iii) and (1iv) also hold.

(2) Recall that, by Lemma 3.2(ii), the local rings of all maximal points of R-fibres of $\operatorname{Spec} A'$ are valuation rings. As P is generically trivial, applying the valuative criterion of properness to $P/B \to \operatorname{Spec} A'$ yields a closed subscheme $Y \subset \operatorname{Spec} A'$, which avoids all the codimension 1 points of the generic fibre $\operatorname{Spec} A'_K$ and all the maximal points of R-fibres of $\operatorname{Spec} A'$. Moreover, $P/B \to \operatorname{Spec} A'$ has a section over $\operatorname{Spec} A' \smallsetminus Y$ that lifts to a generic section of P. In other words, Y satisfies

$$\operatorname{codim}(Y_K, \operatorname{Spec} A'_K) \geqslant 2 \quad \text{and} \quad \operatorname{codim}(Y_s, \operatorname{Spec} A'_s) \geqslant 1 \text{ for all } s \in \operatorname{Spec}(R).$$

Therefore, $P_{\operatorname{Spec} A'\smallsetminus Y}$ reduces to a generically trivial $B_{\operatorname{Spec} A'\smallsetminus Y}$-torsor P^B. Consider the A'-torus $T := B/\operatorname{rad}^u(B)$ and the induced T-torsor

$$P^T := P^B/\operatorname{rad}^u(B) \quad \text{over } \operatorname{Spec} A' \smallsetminus Y.$$

By purity for tori (cf. [GL23, Th. 3.3] or Theorem 7.9), P^T extends to a T-torsor $\widetilde{P^T}$ over $\operatorname{Spec} A'$. As $\widetilde{P^T}$ is generically trivial, by the Grothendieck–Serre conjecture for tori (Proposition 7.11(i)), we may localize A' around $\mathbf{x}$ to assume that $\widetilde{P^T}$, and hence also P^T, is already trivial. In other words, $P_{\operatorname{Spec} A'\smallsetminus Y}$ reduces to a $\operatorname{rad}^u(B)_{\operatorname{Spec} A'\smallsetminus Y}$-torsor.

Now, applying Lemma 8.2 to the R-smooth algebra A' and the closed subscheme $Y \subset \operatorname{Spec} A'$, we obtain an affine open $U \subset \operatorname{Spec} A'$ containing $\boldsymbol{x}$, an affine open $S \subset \mathbb{A}_R^{d-1}$, and a smooth R-morphism $\pi : U \to S$ of relative dimension 1 such that $Y \cap U$ is S-finite.

Recall that A is the semilocal ring of U at $\mathbf{x}$. Denote

$$C := U \times_S \operatorname{Spec} A \quad \text{and} \quad Z := (Y \cap U) \times_S \operatorname{Spec} A.$$

Then C is a smooth affine relative A-curve, the diagonal in C induces a section $s \in C(A)$, and the closed subscheme $Z \subset C$ is A-finite. So (2i) and (2ii) hold. Let $\mathscr{B} \subset \mathscr{G}$ and $\mathcal{P}$ be the pullback of $B_U \subset G_U$ and P_U under the first projection $\mathrm{pr}_1 : C \to U$, respectively. Then, $\mathcal{P}$ is a $\mathscr{G}$-torsor over C, and, by construction, the s-pullback of $\mathscr{B} \subset \mathscr{G}$ and $\mathcal{P}$ are $B \subset G$ and P, respectively. Finally, since P reduces to a $\operatorname{rad}^u(B)$-torsor over $\operatorname{Spec} A' \smallsetminus Y$, $\mathcal{P}$ reduces to a $\operatorname{rad}^u(\mathscr{B})$-torsor over $C \smallsetminus Z$. Thus, (2iii) and (2iv) also hold. ☐

Lemma 8.4 ([Čess22a, Lem. 5.1]). — *For a semilocal ring A whose local rings are geometrically unibranch, an ideal $I \subset A$, reductive A-groups G and G' that on geometric A-fibres have the same type, fixed quasi-pinnings of G and G' extending Borel A-subgroups $B \subset G$ and $B' \subset G'$ and an A/I-group isomorphism*

$$\iota\colon G_{A/I} \xrightarrow{\sim} G'_{A/I}$$

respecting the quasi-pinnings; in particular, $\iota(B_{A/I}) = B'_{A/I}$, there are

(i) *a faithfully flat, finite, étale A-algebra $\widetilde{A}$ equipped with an A/I-point*

$$a\colon \widetilde{A} \twoheadrightarrow A/I;$$

(ii) *an $\widetilde{A}$-group isomorphism $\widetilde{\iota}\colon G_{\widetilde{A}} \xrightarrow{\sim} G'_{\widetilde{A}}$ respecting the quasi-pinnings such that $a^*(\widetilde{\iota}) = \iota$.*

Note that the original formulation [Čess22a, Prop. 5.1] assumed that A is Noetherian, though the Noetherianness of A was not used in the proof.

Lemma 8.5 (Changing the relative curve C to equate $\mathscr{G}$ and G_C; cf. [Čess22a, Prop. 5.2])
In the setting of Lemma 8.3, for both cases (1) and (2), we may replace C by an étale neighborhood of $\operatorname{Im}(s)$ to further achieve that

$$(\mathscr{G}, \mathscr{B}) = (G_C, B_C).$$

Proof. — Consider the semilocalization $\operatorname{Spec}(D)$ of C at the closed points of $\operatorname{Im}(s) \cup Z$. Since C is normal, all the local rings of D are geometrically unibranch. The image of the section $s : \operatorname{Spec} A \to \operatorname{Spec}(D)$ gives rise to a closed subscheme $\operatorname{Spec}(D/I) \subset \operatorname{Spec}(D)$. By the conclusion of Lemma 8.3, the restrictions of $\mathscr{B}_D \subset \mathscr{G}_D$ and $B_D \subset G_D$ to $\operatorname{Spec}(D/I)$ agree with each other in a way respecting their quasi-pinnings. Thus, by Lemma 8.4, there is a faithfully flat, finite, étale D-algebra $\widetilde{D}$, a point $\widetilde{s} : \widetilde{D} \twoheadrightarrow D/I \simeq A$ lifting $s : D \twoheadrightarrow D/I \simeq A$ such that $\mathscr{B}_{\widetilde{D}} \subset \mathscr{G}_{\widetilde{D}}$ is isomorphic to $B_{\widetilde{D}} \subset G_{\widetilde{D}}$ compatibly with the fixed identification of $\widetilde{s}$-pullbacks. We then spread out the finite étale morphism $\operatorname{Spec}(\widetilde{D}) \to \operatorname{Spec}(D)$ to obtain a finite étale morphism $\widetilde{C} \to C'$ for an open $C' \subset C$ that contains $\operatorname{Im}(s) \cup Z$, while preserving an $\widetilde{s} \in \widetilde{C}(A)$, and an isomorphism between $\mathscr{B}_{\widetilde{C}} \subset \mathscr{G}_{\widetilde{C}}$ and $B_{\widetilde{C}} \subset G_{\widetilde{C}}$. It remains to replace C, s, Z and $\mathcal{P}_K$ (resp. $\mathcal{P}$) by $\widetilde{C}$, $\widetilde{s}$, $Z \times_C \widetilde{C}$ and $(\mathcal{P}_K)_{\widetilde{C}_K}$ (resp. $\mathcal{P}_{\widetilde{C}}$). ☐

Lemma 8.6 (Changing the smooth relative curve C for descending to $\mathbb{A}^1_A$; [Čes22a, Prop. 6.5])

In the setting of Lemma 8.3*, for both cases* (1) *and* (2)*, in addition to* $(\mathscr{G},\mathscr{B}) = (G_C, B_C)$*, we may change* C *to further achieve that there is a flat* A*-map* $C \to \mathbb{A}^1_A$ *that maps* Z *isomorphically onto a closed* $Z' \subset \mathbb{A}^1_A$ *such that*

$$Z \simeq Z' \times_{\mathbb{A}^1_A} C.$$

Proof. — Assume that, in both cases (1) and (2) of Lemma 8.3, we have achieved the conclusion of Lemma 8.5. We have the data of a smooth affine relative A-curve C, a section $s \in C(A)$, and an A-finite closed subscheme $Z \subset C$. Replacing Z by $Z \cup \mathrm{im}(s)$, we may assume that s factors through Z. Unfortunately, in general, the A-finite scheme Z may be too large to embed into $\mathbb{A}^1_A$. (For instance, if $R = k$ is a finite field, then Z cannot be embedded into $\mathbb{A}^1_k$ as soon as $\# Z(k) > \# k$.) To overcome this difficulty, we first apply Panin's 'finite fields tricks' [Čes22a, Lem. 6.1] to obtain a finite morphism $\widetilde{C} \to C$ that is étale at the points in $\widetilde{Z} := \widetilde{C} \times_C Z$ such that s lifts to $\widetilde{s} \in \widetilde{C}(A)$, and there are no finite fields obstruction to embedding $\widetilde{Z}$ into $\mathbb{A}^1_A$ in the following sense: for every maximal ideal $\mathfrak{m} \subset A$ and every $d \geqslant 1$,

$$\#\big\{z \in \widetilde{Z}_{\kappa(\mathfrak{m})} \colon [\kappa(z)\colon \kappa(\mathfrak{m})] = d\big\} < \#\big\{z \in \mathbb{A}^1_{\kappa(\mathfrak{m})} \colon [\kappa(z)\colon \kappa(\mathfrak{m})] = d\big\}.$$

Then, by [Čes22a, Lem. 6.3], there are an affine open $C' \subset \widetilde{C}$ containing $\mathrm{im}(\widetilde{s})$, a quasi-finite, flat A-map $C' \to \mathbb{A}^1_A$ that maps Z isomorphically onto a closed subscheme $Z' \subset \mathbb{A}^1_A$ with

$$Z \simeq Z' \times_{\mathbb{A}^1_A} C'.$$

It remains to replace C by C', Z by $\widetilde{Z}$, s by $\widetilde{s}$, and $\mathcal{P}_K$ by $(\mathcal{P}_K)_{C'_K}$ (resp. $\mathcal{P}$ by $\mathcal{P}_{C'}$). $\square$

Lemma 8.7 (Descend to $\mathbb{A}^1_A$ via patching; cf. [Čes22a, Prop. 7.4]). — *In the setting of Lemma* 8.3*, for both cases* (1) *and* (2)*, we may further achieve that*

$$(\mathscr{G},\mathscr{B}) = (G_C, B_C), \quad C = \mathbb{A}^1_A, \quad \textit{and } s = 0 \in \mathbb{A}^1_A(A).$$

Proof. — By the reduction given in Lemma 8.6, we have a flat A-curve C, a section $s \in C(A)$, an A-finite closed subscheme $Z \subset C$, a quasi-finite, affine, flat A-map $C \to \mathbb{A}^1_A$ that maps Z isomorphically onto a closed subscheme $Z' \subset \mathbb{A}^1_A$ such that $Z = Z' \times_{\mathbb{A}^1_A} C$, and a G-torsor $\mathcal{P}_K$ over C_K whose s_{A_K}-pullback is P_K (resp. a G-torsor $\mathcal{P}$ over C whose s-pullback is P) and whose restriction to $C_K \smallsetminus Z_K$ (resp. $C \smallsetminus Z$) reduces to a $\mathrm{rad}^u(B)$-torsor. Now, since $Z = Z' \times_{\mathbb{A}^1_A} C \simeq Z'$, [Čes22a, Lem. 7.2] (the Noetherian hypothesis is not needed) implies the pullback maps

$$H^1_{\text{ét}}(\mathbb{A}^1_A \smallsetminus Z', \mathrm{rad}^u(G)) \longrightarrow H^1_{\text{ét}}(C \smallsetminus Z, \mathrm{rad}^u(G))$$

and

$$H^1_{\text{ét}}(\mathbb{A}^1_{A_K} \smallsetminus Z'_K, \mathrm{rad}^u(G)) \longrightarrow H^1_{\text{ét}}(C_K \smallsetminus Z_K, \mathrm{rad}^u(G))$$

are surjective. Combining these, we see that $\mathcal{P}_K|_{C_K \smallsetminus Z_K}$ (resp. $\mathcal{P}|_{C \smallsetminus Z}$) descends to a G-torsor $\mathcal{Q}_K$ (resp. $\mathcal{Q}$) over $\mathbb{A}^1_{A_K} \smallsetminus Z'_K$ (resp. $\mathbb{A}^1_A \smallsetminus Z'$) that reduces to a $\mathrm{rad}^u(B)$-torsor. By [Čes22a, Lem. 7.1], we may (non-canonically) glue $\mathcal{P}_K$ with $\mathcal{Q}_K$ (resp. $\mathcal{P}$ with $\mathcal{Q}$)

to descend $\mathcal{P}_K$ (resp. $\mathcal{P}$) to a G-torsor $\widetilde{\mathcal{P}_K}$ (resp. $\widetilde{\mathcal{P}}$) over $\mathbb{A}^1_{A_K}$ (resp. over $\mathbb{A}^1_A$) that reduces to a $\mathrm{rad}^u(B)$-torsor over $\mathbb{A}^1_{A_K} \smallsetminus Z'_K$ (resp. over $\mathbb{A}^1_A \smallsetminus Z'$). It remains to replace C by $\mathbb{A}^1_A$, Z by Z', $s \in C(A)$ by its image in $\mathbb{A}^1_A(A)$, and $\mathcal{P}_K$ by $\widetilde{\mathcal{P}_K}$ (resp. $\mathcal{P}$ by $\widetilde{\mathcal{P}}$). Finally, by shifting, we may assume that $s = 0 \in \mathbb{A}^1_A(A)$. $\square$

Proof of Theorem 8.1. — Let P_K (resp. P) be a generically trivial G_{A_K}-torsor (resp. G-torsor). By the reduction Lemma 8.7, we get an A-finite closed subscheme $Z \subset \mathbb{A}^1_A$, and a $G_{\mathbb{A}^1_{A_K}}$-torsor $\mathcal{P}_K$ (resp. $G_{\mathbb{A}^1_A}$-torsor $\mathcal{P}$) whose pullback along the zero section is P_K (resp. P) such that $(\mathcal{P}_K)|_{\mathbb{A}^1_{A_K} \smallsetminus Z_K}$ (resp. $\mathcal{P}|_{\mathbb{A}^1_A \smallsetminus Z}$) reduces to a $\mathrm{rad}^u(B)$-torsor. Since any A-finite closed subscheme of $\mathbb{A}^1_A$ is contained in $\{f = 0\}$ for some monic polynomial $f \in A[t]$, we may enlarge Z to assume that $\mathbb{A}^1_A \smallsetminus Z$ is affine, to the effect that any $\mathrm{rad}^u(B)$-torsor over $\mathbb{A}^1_{A_K} \smallsetminus Z_K$ (resp. over $\mathbb{A}^1_A \smallsetminus Z$), such as $(\mathcal{P}_K)|_{\mathbb{A}^1_{A_K} \smallsetminus Z_K}$ (resp. $\mathcal{P}|_{\mathbb{A}^1_A \smallsetminus Z}$), is trivial. By section theorem [GL23, Th. 5.1], the pullback of $\mathcal{P}_K$ (resp. of $\mathcal{P}$) along the zero section is trivial, that is, P_K (resp. P) is trivial, as desired. $\square$

Appendix. Regular coherent rings

In this appendix, we delve into the homological properties of coherent regular rings. A coherent ring is *regular* if its every finitely generated ideal has finite projective dimension. Localizations of a coherent regular ring remain coherent regular. This is due to the stability of coherence under localization, combined with the fact that every finitely generated ideal in the localization is the localization of a finitely generated ideal. Moreover, one can verify coherent regularity Zariski locally, as can be deduced from the following more general result.

Lemma A.1 (Faithfully flat descent for coherent regularity). — *For a faithfully flat ring map $A \to B$, if B is coherent regular, then so is A.*

Proof. — The coherence of A follows since the property of being finitely presented satisfies faithfully flat descent. To see the regularity of A, we let $I \subset A$ be a finitely generated ideal and pick a resolution $P_\bullet \to I$ with each P_i finite free over A (using the coherence of A). Since $A \to B$ is flat, $P_\bullet \otimes_A B \to I \otimes_A B \simeq IB$ is a resolution of the B-module IB. As B is regular, $\mathrm{Im}(P_{n+1} \otimes_A B \to P_n \otimes_A B) \simeq \mathrm{Im}(P_{n+1} \to P_n) \otimes_A B$ is finite projective over B for some $n \geqslant 0$, and so $\mathrm{Im}(P_{n+1} \to P_n)$ is finite projective over A by faithfully flat descent. $\square$

Lemma A.2 (Étale-local nature of coherent regularity). — *Let $A \to B$ be an étale ring map. If A is coherent regular, then so is B. The converse holds if $A \to B$ is faithfully flat and étale.*

Proof. — In light of Lemma A.1, it is enough to show the coherent regularity of B. By Zariski descent, we may work Zariski locally to assume that B is a (principal) localization of a finite free A-algebra C (using the local structure of étale algebras). In this scenario, every finitely generated ideal of C is finitely presented over A and so is it over C. Thus, C is coherent, and so is its localization B. Now, to show the regularity of B, our goal is to prove that $\mathrm{fl.dim}_B\, I < \infty$ for every finitely generated ideal $I \subset B$.

There exists a finitely generated ideal $J \subset C$ such that $J \otimes_C B \simeq JB = I$. As A is coherent regular, we have $\mathrm{fl.dim}_A\, J < \infty$. But I is a filtered colimit of copies of J, so $\mathrm{fl.dim}_A\, I < \infty$. Since B is A-flat, choosing a partial flat resolution of the B-module I, we are reduced to the following claim (cf. [Stacks, 05B9]). □

Claim A.2.1. — *If $A \to B$ is an étale ring map and M is a B-module which is A-flat, then M is B-flat.*

Proof of the claim. — We will argue by induction on the supremum of the cardinality of the geometric points in the fibres of $\operatorname{Spec} B \to \operatorname{Spec} A$. Since flatness can be checked Zariski locally, we may localize to assume that $\operatorname{Spec} B \to \operatorname{Spec} A$ is étale surjective. By the faithfully flat descent of flatness, we can replace A with B, B with $B \otimes_A B$, and M with $M \otimes_A B$, thus assuming that $A \to B$ possesses a retraction $B \to A$. In this scenario, we have $B \simeq A \times B_1$, where B_1 is an étale A-algebra. Consequently, $M \simeq M_0 \times M_1$, with $M_0 = M \otimes_B A$ and $M_1 = M \otimes_B B_1$. By assumption, both M_0 and M_1 are A-flat.

Now, since the supremum of the cardinality of the geometric points in the fibres of $\operatorname{Spec} B_1 \to \operatorname{Spec} A$ is strictly less than that of $\operatorname{Spec} B \to \operatorname{Spec} A$, our induction hypothesis gives the B_1-flatness of M_1, and consequently, the B-flatness of M. □

Lemma A.3. — *Let $f\colon A \to B$ be a flat local map of local rings and κ_A the residue field of A. Assume that B is coherent. Let M be a finitely presented B-module.*

(i) *If M is A-flat, then we have* $\mathrm{proj.dim}_B(M) \leqslant \mathrm{fl.dim}_{B\otimes_A \kappa_A}(M \otimes_A \kappa_A)$.

(ii) *In general, we have* $\mathrm{proj.dim}_B(M) \leqslant \mathrm{fl.dim}_A(M) + \mathrm{wdim}(B \otimes_A \kappa_A)$. *Consequently, we have*

$$\mathrm{wdim}(B) \leqslant \mathrm{wdim}(A) + \mathrm{wdim}(B \otimes_A \kappa_A).$$

(iii) *If every finitely presented B-module, considered as an A-module, and every finitely presented $B \otimes_A \kappa_A$-module have finite flat dimensions, then B is coherent regular.*

Proof. — For (i), set $\ell := \mathrm{fl.dim}_{B\otimes_A\kappa_A}(M \otimes_A \kappa_A)$. If $\ell = \infty$, there is nothing to show. Otherwise, we choose a partial resolution over B

$$0 \longrightarrow M' \longrightarrow P_{\ell-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with each P_i finite free and M' finitely presented (Lemma 2.2(i)). By assumption, M and B are A-flat, so by [Stacks, 03EY], this is an A-flat resolution of M. Tensoring it with $\kappa_A \otimes_A (-)$ gives a partial flat resolution of the $B \otimes_A \kappa_A$-module $M \otimes_A \kappa_A$. The latter has flat dimension ℓ, so that $M' \otimes_A \kappa_A$ is flat hence free over $B \otimes_A \kappa_A$. Since M' is A-flat, by [EGA IV$_3$, Prop. 11.3.7] and Nakayama's lemma, M' is free over B of the same rank. This proves that $\mathrm{proj.dim}_B(M) \leqslant \ell$, as desired.

In general, although M may not be A-flat, we can first choose a partial resolution

$$0 \longrightarrow M'' \longrightarrow F_{s-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with each F_i finite free over B and $s := \mathrm{fl.dim}_A(M)$ (if it is finite). Then, the finitely presented B-module M'' is A-flat, so we can apply the part (i) to M'' and obtain

$$\mathrm{proj.dim}_B(M) \leqslant s + \mathrm{proj.dim}_B(M'') \leqslant s + \mathrm{fl.dim}_{B\otimes_A \kappa_A}(M'' \otimes_A \kappa_A).$$

This simultaneously proves the assertions (ii)–(iii). □

Combining Lemma A.3 and Theorem 2.17(iv), we obtain the following.

Corollary A.4. — *Let $f\colon A \to B$ be a local map of local rings and κ_A the residue field of A. Assume that*

(i) *f is flat,*
(ii) *A has finite weak dimension,*
(iii) *$B \otimes_A \kappa_A$ is coherent regular, and*
(iv) *B is coherent,*

then B is regular. The same conclusion holds if (ii) *is replaced by*

(ii)′ *A is coherent regular with a quasi-compact punctured spectrum.*

Example A.5

Let $A \to B$ be a flat, regular ring map of coherent rings. If $\mathrm{wdim}(A) < \infty$ (this implies that A is regular; e.g. when A is a valuation ring), then B is also coherent regular.

Finally, we aim to prove Theorem A.7, which serves as an extension of the classical Auslander–Buchsbaum formula to general non-Noetherian rings, drawing parallels with the conventional regular scenario established in [AB57, Th. 3.7].

The following simple lemma from linear algebra will be used in the proof of Theorem A.7.

Lemma A.6. — *For a local ring $(A, \mathfrak{m}_A)$, a nonzero A-module M supported on $\{\mathfrak{m}_A\}$, and a matrix $H \in M_{m\times n}(A)$, if the A-linear map $H_M : M^{\oplus n} \to M^{\oplus m}$ induced by H (via left multiplication) is injective, then H admits a left inverse, or, equivalently, H exhibits $A^{\oplus n}$ as a direct summand of $A^{\oplus m}$.*

Proof. — Recall [Stacks, 0953] that the assumption on the support of M means that, for any $w \in M$ and any finitely generated ideal $I \subset A$, we have $I^N \cdot w = 0$ for large enough N. Denote $H = (h_{ij})$. We observe that at least one of h_{ij} is invertible. Otherwise, the entries h_{ij} generate a proper ideal I of A; pick a nonzero element $w \in M$ and let $N \geqslant 0$ be the smallest integer such that $I^N \cdot w \neq 0$, then $H_M((I^N \cdot w)^{\oplus n}) = 0$, contradicting our assumption that H_M is injective. Without loss of generality, we may assume that $h_{11} \in A^\times$. By subtracting suitable multiples of the first row of H to other rows (resp. the first column of H to other columns), we may also assume that $h_{1j} = 0$ for $1 < j \leqslant n$ and $h_{i1} = 0$ for $1 < i \leqslant m$ (the assumption and conclusion of the lemma are preserved if we replace H by $H_1 H H_2$, where $H_1 \in M_{m\times m}(A)$ and $H_2 \in$

$M_{n\times n}(A)$). In other words, we have $H=(h_{11})\oplus H'$, where $H'\in M_{(m-1)\times(n-1)}(A)$. Then the map $H'_M : M^{\oplus(n-1)}\to M^{\oplus(m-1)}$ induced by H' is also injective. So we may assume by induction that H' admits a left inverse $H''\in M_{(n-1)\times(m-1)}(A)$. Then $(h_{11}^{-1})\oplus H''$ is a left inverse of H. □

As L. Moret-Bailly pointed out, since $\mathrm{Ker}(H_M)=\mathrm{Hom}_A(\mathrm{Coker}(H^t),M)$, where H^t is the transpose of H, the above lemma is also a direct consequence of Lemma 2.14.

Now, we acquire the the Auslander–Buchsbaum formula (cf. [AB57, Th. 3.7]) for general local rings. Since our depth is not well-behaved over non-quasicompact punctured spectra, we need to assume this condition in the sequel.

Theorem A.7 (Auslander–Buchsbaum formula)

(i) *Let $(A,\mathfrak{m}_A)$ be a local ring with a quasi-compact punctured spectrum. Let M be an A-module having a finite resolution by finite free A-modules. Then we have*

$$\mathrm{proj.\,dim}_A(M)+\mathrm{depth}_A(M)=\mathrm{depth}_A(A).$$

(ii) *Let V be a valuation ring such that $s:=\mathfrak{m}_V$ is the radical of a finitely generated ideal. Let X be a V-flat finite type scheme and $x\in X$ a point lying over $s\in\mathrm{Spec}\,V$. Assume that the local ring $\mathscr{O}_{X_s,x}$ is regular. Denote $A:=\mathscr{O}_{X,x}$. Then, for every finitely presented A-module M,*

$$\mathrm{proj.\,dim}_A(M)+\mathrm{depth}_A(M)=\mathrm{depth}_A(A)=\dim(\mathscr{O}_{X_s,x})+1.$$

(By convention, $\mathrm{proj.\,dim}_A(0)=-\infty$.*)*

Proof. — By Lemma 3.7(ii)–(iii), the local ring $\mathscr{O}_{X,x}$ in (ii) is coherent regular. The assumption on V implies that $\mathrm{Spec}\,\mathscr{O}_{X,x}\smallsetminus\{x\}$ is quasi-compact, so, by Lemma 3.8, $\mathscr{O}_{X,x}$ has depth $\dim(\mathscr{O}_{X_s,x})+1$. Therefore, it is enough to prove part (i).

For (i), consider first the case where $\mathrm{depth}_A(A)=0$. We claim that every A-module M having a finite resolution by finite free A-modules is free; thus in this case the formula in (i) holds. Clearly, it suffices to prove that every short exact sequence of the form $0\to A^{\oplus m}\xrightarrow{s}A^{\oplus n}\to N\to 0$ splits. Indeed, since $R^0\Gamma_xA\neq 0$ and the map $R^0\Gamma_x(s)$ is injective, the last statement follows from Lemma A.6. Here and in what follows, we redefine $x:=\mathfrak{m}_A$.

Assume now that $\mathrm{depth}_A(A)\geqslant 1$. Set $d:=\mathrm{depth}_A(A)-1$. We will induct on $\mathrm{proj.\,dim}_A(M)$ to verify the formula in (i). If $\mathrm{proj.\,dim}_A(M)=0$, that is, M is A-free, then it is clear that the formula holds.

Next, assume that $\mathrm{proj.\,dim}_A(M)\geqslant 1$, so every partial resolution $0\to M'\xrightarrow{\iota}A^{\oplus n}\to M\to 0$ is non-split and satisfies $\mathrm{proj.\,dim}_A(M')=\mathrm{proj.\,dim}_A(M)-1$. It is a standard fact that M' also has a finite resolution by finite free A-modules. We exploit the associated long exact sequence

$$\cdots\to R^{i-1}\Gamma_xM'\longrightarrow R^{i-1}\Gamma_xA^{\oplus n}\longrightarrow R^{i-1}\Gamma_xM\longrightarrow R^i\Gamma_xM'\longrightarrow R^i\Gamma_xA^{\oplus n}\to\cdots.$$

If $\mathrm{proj.\,dim}_A(M)=1$, then $M'\simeq A^{\oplus m}$ for some $m\geqslant 1$. We have seen that $\mathrm{depth}_A(M')=d+1$, and so $R^i\Gamma_xM'=0$ for all $i\leqslant d$. It follows from the above long exact sequence that $R^i\Gamma_x M = 0$ for all $i \leqslant d-1$. If $R^d\Gamma_x M$ were zero, then

$$R^{d+1}\Gamma_x(\iota) : \bigl(R^{d+1}\Gamma_x A\bigr)^{\oplus m} \simeq R^{d+1}\Gamma_x M' \longrightarrow \bigl(R^{d+1}\Gamma_x A\bigr)^{\oplus n}$$

would be injective. Since $R^{d+1}\Gamma_x A$ is nonzero and supported on $\{x\}$, we deduce from Lemma A.6 that ι splits, and so M is A-free. This contradicts our assumption that $\operatorname{proj.dim}_A(M) = 1$. Therefore, $\operatorname{depth}_A(M) = d$, leading to the desired formula

$$\operatorname{proj.dim}_A(M) + \operatorname{depth}_A(M) = d + 1.$$

If $\operatorname{proj.dim}_A(M) > 1$, then $\operatorname{proj.dim}_A(M') = \operatorname{proj.dim}_A(M) - 1$. Applying the induction hypothesis to M', we obtain that

$$\begin{aligned} \operatorname{depth}_A(M') &= d + 1 - (\operatorname{proj.dim}_A(M) - 1) \\ &= d + 2 - \operatorname{proj.dim}_A(M), \quad \text{which is} \quad \leqslant d. \end{aligned}$$

It follows from the above long exact sequence that $R^{i-1}\Gamma_x M \simeq R^i\Gamma_x M'$ for all $i \leqslant d$. Combining this with the bound $\operatorname{depth}_A(M') \leqslant d$, we deduce that $\operatorname{depth}_A(M) = \operatorname{depth}_A(M') - 1$. Therefore, by induction hypothesis, we have

$$\operatorname{proj.dim}_A(M) + \operatorname{depth}_A(M) = (\operatorname{proj.dim}_A(M') + 1) + (\operatorname{depth}_A(M') - 1) = d + 1.$$

This finishes the induction step. $\square$

The following corollary fails without the quasi-compactness assumption, see Example 2.7.

Corollary A.8. — *If A is a regular coherent local ring with a quasi-compact punctured spectrum, then*

$$\operatorname{wdim}(A) = \operatorname{depth}_A(A) < \infty.$$

Proof. — The Auslander–Buchsbaum formula implies that every finitely presented A-module has projective dimension at most $\operatorname{depth}_A(A)$. Since for any ring R, we have

$$\operatorname{wdim}(R) = \sup\{\operatorname{fl.dim}(R/J) \mid \text{finitely generated ideal } J \subset R\},$$

we obtain that $\operatorname{wdim}(A) \leqslant \operatorname{depth}_A(A)$. On the other hand, the quasi-compactness assumption implies that there is a finitely generated ideal $I \subset A$ contained in the maximal ideal $\mathfrak{m}_A$ such that $\sqrt{I} = \mathfrak{m}_A$. Now we let $M := A/I$; it is a coherent A-module and therefore perfect with depth zero. Applying the Auslander–Buchsbaum formula, we have $\operatorname{proj.dim}_A(A/I) = \operatorname{depth}_A(A)$, which implies:

$$\operatorname{wdim}(A) = \operatorname{depth}_A(A) = \operatorname{proj.dim}_A(A/I) < \infty. \qquad \square$$

References

[Alp14] J. Alper – "Adequate moduli spaces and geometrically reductive group schemes", *Algebraic Geom.* **1** (2014), no. 4, p. 489–531.

[Aus62] M. Auslander – "On the purity of the branch locus", *Amer. J. Math.* **84** (1962), p. 116–125.

[AB57] M. Auslander & D. A. Buchsbaum – "Homological dimension in local rings", *Trans. Amer. Math. Soc.* **85** (1957), p. 390–405.

[Ber71] J. Bertin – "Anneaux cohérents réguliers", *C. R. Acad. Sci. Paris Sér. A-B* **273** (1971), p. 590–591.

[Ber72] ———, "Anneaux cohérents de dimension homologique finie", in *Colloque Algebre commut. (Rennes 1972)*, Publ. Sem. Math., vol. 16, Univ. Rennes, 1972, 12 p.

[BS15] B. Bhatt & P. Scholze – "The pro-étale topology for schemes", in *De la géométrie algébrique aux formes automorphes (I)*, Astérisque, vol. 369, Société Mathématique de France, Paris, 2015, p. 99–201.

[BR83] S. M. Bhatwadekar & R. A. Rao – "On a question of Quillen", *Trans. Amer. Math. Soc.* **279** (1983), p. 801–810.

[Bou98] N. Bourbaki – *Commutative algebra. Chapters 1–7*, Elements of Mathematics, Springer-Verlag, Berlin, 1998.

[BČ22] A. Bouthier & K. Česnavičius – "Torsors on loop groups and the Hitchin fibration", *Ann. Sci. École Norm. Sup. (4)* **55** (2022), no. 3, p. 791–864.

[Čes22a] K. Česnavičius – "Grothendieck–Serre in the quasi-split unramified case", *Forum Math. Pi* **10** (2022), article no. e9 (30 pages).

[Čes22b] ———, "Torsors on the complement of a smooth divisor", 2022, Available at `https://www.imo.universite-paris-saclay.fr/~cesnavicius/torsors-complement.pdf`.

[ČS24] K. Česnavičius & P. Scholze – "Purity for flat cohomology", *Ann. of Math. (2)* **199** (2024), no. 1, p. 51–180.

[CTS79] J.-L. Colliot-Thélène & J.-J. Sansuc – "Fibrés quadratiques et composantes connexes réelles", *Math. Ann.* **244** (1979), no. 2, p. 105–134.

[Fed21] R. Fedorov – "On the purity conjecture of Nisnevich for torsors under reductive group schemes", 2021, `arXiv:2109.10332`.

[Fed22] ———, "On the Grothendieck-Serre conjecture on principal bundles in mixed characteristic", *Trans. Amer. Math. Soc.* **375** (2022), no. 1, p. 559–586.

[Gab81] O. Gabber – "Some theorems on Azumaya algebras", in *Groupe de Brauer, (Les Plans-sur-Bex 1980)*, Lect. Notes in Math., vol. 844, Springer, 1981, p. 129–209.

[GR18] O. Gabber & L. Ramero – "Foundations for almost ring theory", 2018, `arXiv:math/0409584`.

[Gil22] P. Gille – "When is a reductive group scheme linear?", *Michigan Math. J.* **72** (2022), p. 439–448.

[Gla89] S. Glaz – *Commutative coherent rings*, Lect. Notes in Math., vol. 1371, Springer-Verlag, Berlin, 1989.

[EGA I] A. Grothendieck & J. Dieudonné – "Éléments de géométrie algébrique. I. Le langage des schémas", *Publ. Math. Inst. Hautes Études Sci.* **4** (1960), p. 5–228.

[EGA IV_2] ———, "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II", *Publ. Math. Inst. Hautes Études Sci.* **24** (1965), p. 5–231.

[EGA IV_3] ———, "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III", *Publ. Math. Inst. Hautes Études Sci.* **28** (1966), p. 5–255.

[EGA IV_4] ———, "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV", *Publ. Math. Inst. Hautes Études Sci.* **32** (1967), p. 5–361.

[Guo24] N. Guo – "The Grothendieck–Serre conjecture over valuation rings", *Compositio Math.* **160** (2024), no. 2, p. 317–355.

[GL23] N. Guo & F. Liu – "Grothendieck–Serre for constant reductive group schemes", 2023, `arXiv:2301.12460`.

[Kna03] H. Knaf – "Regular curves over Prüfer domains", in *Valuation theory and its applications. Vol. II (Saskatoon, 1999)*, American Mathematical Society, Providence, RI, 2003, p. 89–106.

[Kna08] ———, "Regular local algebras over valuation domains: weak dimension and regular sequences", *J. Algebra Appl.* **7** (2008), no. 5, p. 575–591.

[KM76] F. Knudsen & D. Mumford – "The projectivity of the moduli space of stable curves. I: Preliminaries on "det" and "Div"", *Math. Scand.* **39** (1976), p. 19–55.

[Mar16] A. Marrama – "A purity theorem for torsors", Algant master thesis, 2016.

[MB22] L. Moret-Bailly – "A construction of weakly unramified extensions of a valuation ring", *Rend. Sem. Mat. Univ. Padova* **147** (2022), p. 139–151.

[Nag59] M. Nagata – "On the purity of branch loci in regular local rings", *Illinois J. Math.* **3** (1959), p. 328–333.

[Nag66] ———, "Finitely generated rings over a valuation ring", *J. Math. Kyoto Univ.* **5** (1966), p. 163–169.

[Nis89] Y. A. Nisnevich – "Rationally trivial principal homogeneous spaces, purity and arithmetic of reductive group schemes over extensions of two-dimensional regular local rings", *C. R. Acad. Sci. Paris Sér. I Math.* **309** (1989), no. 10, p. 651–655.

[Pan18] I. Panin – "On Grothendieck-Serre conjecture concerning principal bundles", in *Proc. I.C.M. (Rio de Janeiro, 2018). Vol. II*, World Sci. Publ., Hackensack, NJ, 2018, p. 201–221.

[Pop02] D. Popescu – "On a question of Quillen", *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **45** (2002), no. 3-4, p. 209–212.

[Que71] Y. Quentel – "Sur le théorème d'Auslander-Buchsbaum", *C. R. Acad. Sci. Paris Sér. A-B* **273** (1971), p. 880–881.

[RG71] M. Raynaud & L. Gruson – "Critères de platitude et de projectivité. Techniques de "platification" d'un module", *Invent. Math.* **13** (1971), p. 1–89.

[Sam64] P. Samuel – "Anneaux gradués factoriels et modules réflexifs", *Bull. Soc. math. France* **92** (1964), p. 237–249.

[Ser56] J.-P. Serre – "Sur la dimension homologique des anneaux et des modules noethériens", 1956, p. 175–189.

[SGA 2_{new}] A. Grothendieck – *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*, Documents Math., vol. 4, Société Mathématique de France, Paris, 2005, Revised and annotated edition of the 1968 original.

[SGA $3_{\text{III new}}$] M. Demazure & A. Grothendieck – *Schémas en groupes (SGA 3). Tome III. Structure des schémas en groupes réductifs*, Documents Math., vol. 8, Société Mathématique de France, Paris, 2011, Revised and annotated edition of the 1970 original.

[SGA 4_{II}] M. Artin, A. Grothendieck & J.-L. Verdier – *Théorie des topos et cohomologie étale des schémas. Tome 2*, Lect. Notes in Math., vol. 270, Springer-Verlag, 1972.

[Stacks] The Stacks Project Authors – "The Stacks Project", 2019, `https://stacks.math.columbia.edu`.

[Zaf78] M. Zafrullah – "On finite conductor domains", *Manuscripta Math.* **24** (1978), p. 191–204.

[Zar58] O. Zariski – "On the purity of the branch locus of algebraic functions", *Proc. Nat. Acad. Sci. U.S.A.* **44** (1958), p. 791–796.

Manuscript received 12th March 2023
accepted 18th January 2024

Ning Guo, St. Petersburg branch of V. A. Steklov Mathematical Institute,
Fontanka 27, 191023 St. Petersburg, Russia
E-mail : `guo.ning@eimi.ru`

Fei Liu, Department of Mathematics, Southern University of Science and Technology,
Shenzhen, China
E-mail : `liufei54@pku.edu.cn`