Journal de l'École polytechnique
Mathématiques

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Tome 10 (2023), p. 837-866.

https://doi.org/10.5802/jep.234

Publié avec le soutien
du Centre National de la Recherche Scientifique

Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 2270-518X

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A FINITE DIMENSIONAL PROOF OF A RESULT OF HUTCHINGS ABOUT IRRATIONAL PSEUDO-ROTATIONS

by Patrice Le Calvez

Abstract. — We prove that the Calabi invariant of a C^1 pseudo-rotation of the unit disk, that coincides with a rotation on the unit circle, is equal to its rotation number. This result has been shown some years ago by Michael Hutchings (under very slightly stronger hypothesis). While the original proof used Embedded Contact Homology techniques, the proof of this article uses generating functions and the dynamics of the induced gradient flow.

Résumé (Une preuve en dimension finie d'un résultat de Hutchings sur les pseudo-rotations irrationnelles)

Nous montrons que l'invariant de Calabi d'une pseudo-rotation irrationnelle de classe C^1 qui coïncide avec une rotation sur le bord, est égal au nombre de rotation. Ce résultat a été démontré il y a quelques années par Michael Hutchings (sous des hypothèses légèrement plus fortes). Alors que la démonstration originale s'inscrit dans le formalisme de l'« Embedded Contact Homology », la preuve que nous donnons utilise les fonctions génératrices et les propriétés dynamiques du flot de gradient associé.

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1. Introduction

1.1. Statement of the main theorem. — We denote by $\mathbb{D}$ the closed unit disk of the Euclidean plane and by $\mathbb{S}$ the unit circle. We will furnish $\mathbb{D}$ with the standard area form $\omega = dx\wedge dy$ and will denote by $\mathrm{Diff}^1_\omega(\mathbb{D})$ the group of diffeomorphisms of class C^1 that preserve ω (we will say that f is *symplectic*). Note that every element of $\mathrm{Diff}^1_\omega(\mathbb{D})$ preserves the orientation and therefore is isotopic to the identity. More precisely, the set $\mathrm{Diff}^1_\omega(\mathbb{D})$ is path-connected when furnished with the C^1-topology. We will

Mathematical subject classification (2020). — 37E30, 37E45, 37J11.
Keywords. — Irrational pseudo-rotation, Calabi invariant, generating function, rotation number, linking number.

also denote by $\mathrm{Homeo}_*(\mathbb{D})$ the group of orientation preserving homeomorphisms of $\mathbb{D}$ (which is also path-connected when furnished with the C^0-topology).

If $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$ fixes every point in a neighborhood of $\mathbb{S}$, its *Calabi invariant* $\mathrm{Cal}(f) \in \mathbb{R}$ is a well-studied object that has several interpretations (see [Cal70], [Fat80], [GG97] for instance). It admits a natural extension to $\mathrm{Diff}^1_\omega(\mathbb{D})$. We will explain this extension, as described by Benoit Joly in his thesis [Jol21]. If κ is a primitive of ω, and $A : \mathbb{D} \to \mathbb{R}$ a primitive of $f^*(\kappa) - \kappa$, then $\int_{\mathbb{S}} A\, d\mu$ does not depend on μ, where μ is a Borel probability measure invariant by $f_{|\mathbb{S}}$.(1) Consequently, there exists a unique primitive $A_{f,\kappa}$ of $f^*(\kappa) - \kappa$ satisfying $\int_{\mathbb{S}} A_{f,\kappa}\, d\mu = 0$ for such μ. Moreover, the quantity $\mathrm{Cal}(f) = \int_{\mathbb{D}} A_{f,\kappa}\, \omega$ does not depend on the choice of κ. For example we have $\mathrm{Cal}(f) = 0$ if f is an Euclidean rotation. Note that $\mathrm{Cal}(f)$ is the usual Calabi invariant in case f fixes every point in a neighborhood of $\mathbb{S}$ (in that case, $A_{f,\kappa}$ vanishes on a neighborhood of $\mathbb{S}$).

We can also define a real function $\widetilde{\mathrm{Cal}}$ on the universal covering space $\widetilde{\mathrm{Diff}^1_\omega}(\mathbb{D})$ of $\mathrm{Diff}^1_\omega(\mathbb{D})$ as follows. Every identity isotopy of f in $\mathrm{Diff}^1_\omega(\mathbb{D})$, meaning every continuous path $I = (f_s)_{s\in[0,1]}$ in $\mathrm{Diff}^1_\omega(\mathbb{D})$ joining the identity map of $\mathbb{D}$ to f, is homotopic, relative to its endpoints, to a Hamiltonian isotopy $I' = (f'_s)_{s\in[0,1]}$. It means that there exists a time dependent divergence free vector field $(X_s)_{s\in[0,1]}$ of class C^1 such that for every $z \in \mathbb{D}$ it holds that $\frac{d}{ds} f'_s(z) = X_s(f_s(z))$. There exists a (uniquely defined) family $(H_s)_{s\in[0,1]}$ of functions of class C^2 vanishing on $\mathbb{S}$, such that for every $s \in [0,1]$, every $z \in \mathbb{D}$ and every $v \in \mathbb{R}^2$ it holds that $dH_s(z).v = \omega(v, X_s(z))$. The quantity $\widetilde{\mathrm{Cal}}(I) = \int_0^1 \left(\int_{\mathbb{D}} H_s\, \omega\right) ds$ depends only on the homotopy class $[I]$ of I and we can also denote it $\widetilde{\mathrm{Cal}}([I])$. Let us give a simple example. For every $\alpha \in \mathbb{R}$, denote R_α the rotation of angle $2\pi\alpha$ and T_α the isotopy $(R_{s\alpha})_{s\in[0,1]}$. It holds that $\widetilde{\mathrm{Cal}}([T_\alpha]) = \pi^2\alpha$. As stated in [Jol21], there is a link between Cal and $\widetilde{\mathrm{Cal}}$ that we will explain now. Every identity isotopy $I = (f_s)_{s\in[0,1]}$ of f in $\mathrm{Diff}^1_\omega(\mathbb{D})$ defines by restriction an identity isotopy $I_{|\mathbb{S}} = (f_{s|\mathbb{S}})_{s\in[0,1]}$ of $f_{|\mathbb{S}}$ in $\mathrm{Diff}^1(\mathbb{S})$. Moreover, every homotopy class $[I] \in \widetilde{\mathrm{Diff}^1_\omega}(\mathbb{D})$ defines by restriction a homotopy class $[I]_{|\mathbb{S}} \in \widetilde{\mathrm{Diff}^1_*}(\mathbb{S})$, where $\widetilde{\mathrm{Diff}^1_*}(\mathbb{S})$ is the universal covering space of the group $\mathrm{Diff}^1_*(\mathbb{S})$ of orientation preserving diffeomorphisms of the circle. We have the following equation,

$$\widetilde{\mathrm{Cal}}([I]) = \mathrm{Cal}(f) + \pi^2 \operatorname{rot}([I]_{|\mathbb{S}}),$$

where $\operatorname{rot}([I]_{|\mathbb{S}})$ is the *Poincaré rotation number* of $[I]_{|\mathbb{S}}$. In particular, if μ is a Borel probability measure invariant by $f_{|\mathbb{S}}$, then the asymptotic cycle of μ defined by an isotopy I (see Schwartzman [Sch57]) is equal to $\operatorname{rot}([I])\varpi \in H_1(\mathbb{S}, \mathbb{R})$, where ϖ is the fundamental class of $\mathbb{S}$ defining its usual orientation.

There is a more dynamical interpretation of the Calabi invariant in the case of a compactly supported symplectic diffeomorphism of the open disk, due to Fathi [Fat80]

(1)Such a measure is unique when the rotation number of $f_{|\mathbb{S}}$ is irrational and supported in the periodic point set otherwise.

and developed by Gambaudo and Ghys [GG97]. This interpretation is still valid in this more general situation. Consider the usual angular form

$$d\theta = \frac{xdy - ydx}{x^2 + y^2}$$

and define

$$W = \{(z, z') \in \mathbb{D} \times \mathbb{D} \mid z \neq z'\}.$$

Consider $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$. For every identity isotopy $I = (f_s)_{s\in[0,1]}$ of f in $\mathrm{Homeo}_*(\mathbb{D})$ and every $(z, z') \in W$, one gets a path

$$\begin{aligned} I_{z,z'} : [0,1] &\longrightarrow \mathbb{R}^2 \setminus \{0\} \\ s &\longmapsto f_s(z') - f_s(z). \end{aligned}$$

The function

$$\begin{aligned} \mathrm{ang}_I : W &\longrightarrow \mathbb{R} \\ (z, z') &\longmapsto \frac{1}{2\pi} \int_{I_{z,z'}} d\theta \end{aligned}$$

depends only on the homotopy class of I in $\mathrm{Homeo}_*(\mathbb{D})$ and is bounded because f is a diffeomorphism of class C^1. In particular, one naturally gets a function $\mathrm{ang}_{[I]}$ for every $[I] \in \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$.

If $I = (f_s)_{s\in[0,1])}$ and $I' = (f'_s)_{s\in[0,1]}$ are two identity isotopies in $\mathrm{Diff}^1_\omega(\mathbb{D})$, one can define an identity isotopy $II' = (f''_s)_{s\in[0,1]}$ in $\mathrm{Diff}^1_\omega(\mathbb{D})$ writing:

$$f''_s = \begin{cases} f'_{2s} & \text{if } 0 \leqslant s \leqslant 1/2, \\ f_{2s-1} \circ f'_1 & \text{if } 1/2 \leqslant s \leqslant 1. \end{cases}$$

The homotopy class of II' depends only on $[I]$ and $[I']$ and one gets a group structure on $\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ by setting $[I][I'] = [II']$. Note that for every $n \geqslant 1$ we have

$$\mathrm{ang}_{[I]^n} = \sum_{k=0}^{n-1} \mathrm{ang}_{[I]} \circ (f^k \times f^k).$$

Generalizing a proof due to Shelukhin [She15] the following equality

$$\widetilde{\mathrm{Cal}}([I]) = \int_W \mathrm{ang}_{[I]}(z, z')\, \omega(z)\, \omega(z').$$

is proved in [Jol21]. Let us add that this interpretation of the Calabi invariant has permitted to Gambaudo and Ghys [GG97] to prove that two elements of $\mathrm{Diff}^1_\omega(\mathbb{D})$ fixing every point in a neighborhood of $\mathbb{S}$ and conjugate by an orientation and area preserving homeomorphism have the same Calabi invariant. To conclude, just note the following:

– the map $[I] \mapsto \widetilde{\mathrm{Cal}}([I])$ defined on $\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ is a morphism;

– the map $f \mapsto \mathrm{Cal}(f)$ defined on $\mathrm{Diff}^1_\omega(\mathbb{D})$is a homogeneous quasi-morphism;

– for every $[I]$, $[I']$ in $\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$, there is a unique $k \in \mathbb{Z}$ such that $[I'] = [I][T_k]$ and we have:

$$\mathrm{ang}_{[I']} = \mathrm{ang}_{[I]} + k, \quad \widetilde{\mathrm{Cal}}([I']) = \widetilde{\mathrm{Cal}}([I]) + \pi^2 k.$$

– for every $[I] \in \widetilde{\mathrm{Diff}^1_\omega}(\mathbb{D})$ and every p, q in $\mathbb{Z}$ we have

$$\widetilde{\mathrm{Cal}}([I]^q[T_p]) = \widetilde{\mathrm{Cal}}([I])^q + \pi^2 p.$$

We will say that $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$ is an *irrational pseudo-rotation* if it fixes 0 and does not possess any other periodic point. Using an extension of Poincaré-Birkhoff theorem due to Franks [Fra88], we know that for every lift $[I] \in \widetilde{\mathrm{Diff}^1_\omega}(\mathbb{D})$ of f, there exists $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$ such that the sequence of maps $z \mapsto n^{-1} \mathrm{ang}_{[I]^n}(0, z)$ converges uniformly to the function $z \mapsto \alpha$. Of course it holds that $\alpha = \widetilde{\mathrm{rot}}([I]_{|\mathbb{S}})$, by definition of $\widetilde{\mathrm{rot}}([I]_{|\mathbb{S}})$, and that $\overline{\alpha} = \alpha + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$ is the Poincaré rotation number of $f_{|\mathbb{S}}$. We will say that f is a pseudo-rotation of rotation number $\overline{\alpha}$. The goal of the article is to prove the following:

Theorem 1.1. — *Let $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$ be an irrational pseudo-rotation such that $f_{|\mathbb{S}}$ is C^1-conjugate to a rotation. If $[I] \in \widetilde{\mathrm{Diff}^1_\omega}(\mathbb{D})$ is a lift of f, then we have*

$$\widetilde{\mathrm{Cal}}([I]) = \pi^2\, \widetilde{\mathrm{rot}}([I]_{|\mathbb{S}}),$$

or equivalently, we have $\mathrm{Cal}(f) = 0$.

The fact that the equalities $\widetilde{\mathrm{Cal}}([I]) = \pi^2\, \widetilde{\mathrm{rot}}([I]_{|\mathbb{S}})$ and $\mathrm{Cal}(f) = 0$ are equivalent is due to the equation

$$\widetilde{\mathrm{Cal}}([I]) = \mathrm{Cal}(f) + \pi^2\, \mathrm{rot}([I]_{|\mathbb{S}}).$$

It is not difficult to prove that a C^1 diffeomorphism of $\mathbb{S}$ can be extended to a C^1 symplectic diffeomorphism of $\mathbb{D}$. So, it is sufficient to prove the theorem in case f coincides with a rotation on the unit circle. This theorem was already known and due to Michael Hutchings (with a very slightly stronger hypothesis), appearing as an easy consequence of a theorem we will recall now. Suppose that $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$ coincides with a rotation R_α in a neighborhood of $\mathbb{S}$. Fix a primitive κ of ω and note that $A_{f,\kappa}$ vanishes on $\mathbb{S}$ whether α is rational or irrational. If O is a periodic orbit of f, one proves easily that $\sum_{z\in O} A_{f,\kappa}(z)$ does not depend on the choice of κ. So, one can define the *mean action* of O as being

$$\mathrm{act}(O) = \frac{1}{\# O} \sum_{z\in O} A_{f,\kappa}(z).$$

One has the following ([Hut16]).(2)

Theorem 1.2. — *Suppose that $f \in \mathrm{Diff}^\infty_\omega(\mathbb{D})$ coincides with a rotation in a neighborhood of $\mathbb{S}$ and denote $\mathcal{O}$ the set of periodic orbits of f. Then it holds that:*

– *if* $\mathrm{Cal}(f) < 0$, *then* $\inf_{O\in\mathcal{O}} \mathrm{act}(O) \leqslant \mathrm{Cal}(f)$,
– *if* $\mathrm{Cal}(f) > 0$, *then* $\sup_{O\in\mathcal{O}} \mathrm{act}(O) \geqslant \mathrm{Cal}(f)$.

(2)The proof is stated for smooth diffeomorphisms but should be possibly extended supposing a low differentiability condition, as Michael Hutchings explained to us.

It is easy to prove that if $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$ coincides with a rotation on $\mathbb{S}$, then we have $\mathrm{act}(\{0\}) = 0$. Consequently, by Theorem 1.2, it holds that $\mathrm{Cal}(f) = 0$ if f is an irrational pseudo-rotation. It must be noticed that Hutchings' theorem has been recently improved by Pirnapasov [Pir21] (still in the smooth category) that uses a preliminary extension result. A much weaker hypothesis on the boundary is needed, even weaker than being a rotation on the boundary. A nice corollary, is the fact that $\mathrm{Cal}(f) = 0$ for every smooth irrational pseudo-rotation f, regardless of any condition on the boundary.

Let us continue with a nice consequence of Theorem 1.1. Let $D \subset \mathbb{D}$ be a closed disk that does not contain 0 and does not meet $\mathbb{S}$. One constructs easily $f' \in \mathrm{Diff}^\infty_\omega(\mathbb{D})$ arbitrarily close to the identity map for the C^∞-topology, fixing every point outside D and such that $\mathrm{Cal}(f') \neq 0$. If $[I]$ is a lift of f and $[I']$ is the lift of f' such that $[I']_{|\mathbb{S}}$ is the trivial homotopy, then it holds that

$$\widetilde{\mathrm{Cal}}([I][I']) = \widetilde{\mathrm{Cal}}([I]) + \widetilde{\mathrm{Cal}}([I']) = \widetilde{\mathrm{rot}}([I]_{|\mathbb{S}}) + \mathrm{Cal}(f') \neq \widetilde{\mathrm{rot}}([I]_{|\mathbb{S}}) = \widetilde{\mathrm{rot}}(([I][I'])_{|\mathbb{S}}).$$

This implies that $f \circ f'$ is not an irrational pseudo-rotation, which means that it has at least one periodic orbit different from $\{0\}$. Such a periodic orbit must meet D.

The proof of Hutchings is based on a reduction to a problem of contact geometry and then the applications of methods of Embedded Contact Homology theory. The previous perturbation result is true in a more general situation. Asaoka and Irie [AI16], using Embedded Contact Homology as well, have proved that it remains true provided $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$ has no periodic point inside D. Other striking applications of contact or symplectic geometry to dynamics on surfaces have appeared recently that use elaborated tools of contact or symplectic geometry (for instance see Cristofaro-Gardiner, Humilière, Seyfaddini [CGHS20], Cristofaro-Gardiner, Prasad, Zhang [CGPZ21], or Edtmair, Hutchings [EH21]). The proof of Theorem 1.1 that will be given in the present article does not use Floer homology but only generating functions. Is there any hope to find proofs of these deep results using such classical tools?

1.2. Idea of the proof. — Let us state first the following result of Bramham [Bra15]:

Theorem 1.3. — *Every irrational pseudo-rotation $f \in \mathrm{Diff}^\infty_\omega(\mathbb{D})$ is the limit, for the C^0 topology, of a sequence of finite order C^∞ diffeomorphisms.*

Write $\overline{\alpha} = \mathrm{rot}(f)$. The proof says something more precise: if $(q_n)_{n\geqslant 0}$ is a sequence of positive integers such that $(q_n\overline{\alpha})_{n\geqslant 0}$ converges to 0 in $\mathbb{R}/\mathbb{Z}$, then there exists a sequence of homeomorphisms $(f_n)_{n\geqslant 0}$ fixing 0 and satisfying $(f_n)^{q_n} = \mathrm{Id}$, that converges to f for the C^0 topology (to obtain a sequence of smooth approximations one needs a simple additional argument of approximation). Theorem 1.1 would have been an easy consequence of Theorem 1.3 if the stronger following properties were true:

– the f_n are symplectic diffeomorphisms of class C^1;
– the sequence $(f_n)_{n\geqslant 0}$ converges to f in the C^1 topology.

Unfortunately, there is no reason why the f_n appearing in the construction of the sequence satisfy these properties.

The original proof of Theorem 1.3 uses pseudoholomorphic curve techniques from symplectic geometry. In [LC16] we succeeded to find a finite dimensional proof by using generating functions, like in Chaperon's broken geodesics method [Cha84]. Let us remind the ideas of this last proof. The hypothesis were slightly different, the result was stated for an irrational pseudo-rotation $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$ coinciding with a rotation on $\mathbb{S}$. This last property permits us to extend our map, also denoted f, to a piecewise C^1 diffeomorphism of the whole plane, being an integrable polar twist map with increasing rotation number in an annulus $\{z\in\mathbb{R}^2 \mid 1\leqslant |z|\leqslant r_0\}$ and equal to an irrational rotation in $\{z\in\mathbb{R}^2 \mid |z|\geqslant r_0\}$. Let I be an identity isotopy of $f_{|\mathbb{D}}$ in $\mathrm{Diff}^1_\omega(\mathbb{D})$ that fixes 0. We write $\alpha=\mathrm{rot}([I]_{|\mathbb{S}})$. One can extend I to an identity isotopy $(f_s)_{s\in[0,1]}$ of f, also denoted I, where each f_s is an integrable polar map in $\{z\in\mathbb{R}^2 \mid 1\leqslant |z|\leqslant r_0\}$ and a rotation in $\{z\in\mathbb{R}^2 \mid |z|\geqslant r_0\}$. The circle $\mathbb{S}$ is accumulated from outside by invariant circles $S_{a/b}$, such that $[I]^b[T_{-a}]_{|S_{a/b}}\in \widetilde{\mathrm{Diff}^1}(S_{p/q})$ is the trivial homotopy. Moreover, one can find a sequence $(a_n,b_n)\in\mathbb{Z}\times(\mathbb{N}\setminus\{0\})$ such that $|b_n\alpha-a_n|\leqslant b_n^{-1}$. The map f being piecewise C^1, one can write it as the composition of m maps "close to $\mathrm{Id}_{\mathbb{R}^2}$", where $m\geqslant 1$, and then construct a m-periodic family $(h_i)_{i\in\mathbb{Z}}$ of generating real functions that are C^1 with Lipschitz derivatives. One knows that for every $b\geqslant 1$, the fixed point set of f^b corresponds to the singular point set of a A-Lipschitz vector field ζ defined on a $2mb$-dimensional space E_b, furnished with a natural scalar product. This vector field is the gradient flow of a function $\boldsymbol{h}$ defined in terms of the h_i, $i\in\mathbb{Z}$, and the constant A is independent of b. In particular each circle $S_{a/b}\subset\mathbb{R}^2$ corresponds to a curve $\Sigma_a\subset E_b$ of singularities of ζ. A fundamental result is the fact that Σ_a bounds a disk $\Delta_a\subset E_b$ that contains the singular point corresponding to the fixed point 0 and that is invariant by the flow and by a natural $\mathbb{Z}/b\mathbb{Z}$ action on E_b. Moreover, the dynamics of the flow of ζ on Δ_a is north-south and the non trivial orbits have the same energy. This energy can be explicitly computed and is small if a/b is a convergent of α. Using the independence of the Lipschitz constant A, one can deduce that ζ is "uniformly small" on Δ_a. The disk Δ_a projects homeomorphically on the disk bounded by $S_{a/b}$, denoted $D_{a/b}$, by an explicit map q_1. The $\mathbb{Z}/b\mathbb{Z}$ action on Δ_a defines, by projection on $D_{a/b}$, a homeomorphism $\widehat{f}$ of order b that coincides with f on $S_{a/b}$. Using what has been said above, in particular the fact that ζ is small on Δ_a, one can prove that $\widehat{f}$ is uniformly close to f on the disk $D_{a/b}$ if a/b is a convergent of α. This is the way we prove the approximation result.

The orbits of $\zeta_{|\Delta_a}$ define a radial foliation on Δ_a that projects by q_1 onto a topological radial foliation $\mathcal{F}_1$ of $D_{a/b}$ that is invariant by $\widehat{f}$. It is natural to ask whether one can compute the Calabi invariant of f by using the fact that $\widehat{f}$ is arbitrarily close to f. The main problem is that $\widehat{f}$ and $\mathcal{F}_1$ are continuous. There is no differentiability anymore, and differentiability is crucial while dealing with Calabi invariants, regardless of the approach we choose. Nevertheless, the fact that we are dealing with Lipschitz maps close to Id (the maps $f_1,\dots,f_m$ appearing in the decomposition of f) permit us to state some quantitative results. The three key-points that will be used to get the proof are related to the foliation $\mathcal{F}_1$:

(1) the energy of the non trivial orbits of $\zeta_{|\Delta_a}$ is bounded by $K|b\alpha - a|$, where K does not depend on a and b;

(2) the leaves of $\mathcal{F}_1$ are *Brouwer lines* of $I^b T_{-a|D_{a/b}}$;

(3) the *winding distance* between $\mathcal{F}_1$ and $f^{-b}(\mathcal{F}_1)$ is bounded by $4mb$.

Let us clarify these three points.

The non trivial orbits of $\zeta_{|\Delta_a}$ have the same energy. This quantity measures the area swept by a leaf of $\mathcal{F}_1$ along the isotopy $I^b T_{-a|D_{a/b}}$.

By saying that the leaves of $\mathcal{F}_1$ are Brouwer lines, we mean that they are pushed on the right along the isotopy $I^b T_{-a|D_{a/b}}$. Equivalently, we can say the following: let $\widetilde{D}_{a/b}$ be the universal covering space of $D^*_{a/b} = D_{a/b} \setminus (S_{a/b} \cup \{0\})$, let $\widetilde{I} = (\widetilde{f}_s)_{s\in[0,1]}$ be the identity isotopy on $\widetilde{D}_{a/b}$ that lifts $I_{|D^*_{a/b}}$ and set $\widetilde{f} = \widetilde{f}_1$. Denote $\widetilde{\mathcal{F}_1}$ the (non singular) foliation of $\widetilde{D}_{a/b}$ that lifts $\mathcal{F}_{1|D^*_{a/b}}$. Every leaf $\widetilde{\phi}$ of $\widetilde{\mathcal{F}_1}$ is an oriented topological line of $\widetilde{D}_{a/b}$ that separates its complement into two components, one on the left and one on the right. The time one map T of the identity isotopy of $\widetilde{D}_{a/b}$ that lifts $T_{1|D^*_{a/b}}$ generates the group of covering automorphisms. The assertion (2) states that $\widetilde{f}^b \circ T^{-a}(\widetilde{\phi})$ is on the right of $\widetilde{\phi}$ for every leaf $\widetilde{\phi}$. The assertion (1) states that the area of the domain between $\widetilde{\phi}$ and $\widetilde{f}^b \circ T^{-a}(\widetilde{\phi})$, which is equal to the energy of the corresponding orbit of ζ, is bounded by $K|b\alpha - a|$.

The assertion (3) says that the foliation $f^{-b}(\mathcal{F}_1)$ winds relative to $\mathcal{F}_1$ no more than $4mq$. A precise definition will be given in the next section. Just say that if $\mathcal{F}$ is a radial foliation of class C^1, there exists $K > 0$ such that for every $b > 0$ the winding distance between the foliations $\mathcal{F}$ and $f^{-b}(\mathcal{F})$ is bounded by Kb. The foliation that appears in our construction depends on a and b, is not differentiable, possesses interesting dynamical properties stated in (1) and (2) and nevertheless satisfies a "differential-like" property.

The properties (1), (2) and (3) will permit us to bound the Calabi invariant of each map $f_{|D_{a/b}}$. Note that we will use the third definition, as the map is only piecewise C^1 on $D_{a/b}$. The Calabi invariant will be small if a/b is a convergent that is close to α. By a limit process, we will prove that $\mathrm{Cal}(f_{|\mathbb{D}}) = 0$.

The rest of the article is divided into three sections.

Section 2 will be dedicated to the study of topological radial foliations defined on a disk. We will introduce different objects on the set of such foliations, in particular the winding distance between two foliations. We will see how to compute the Calabi invariant of a symplectic diffeomorphism by using such a foliation.

In Section 3 we will recall the construction done in [LC16] and quickly explained in the introduction. In particular we will define $\mathcal{F}_1$ and prove the properties (1), (2) and (3) stated above.

We will prove Theorem 1.1 in the short section 4.

Acknowledgements. — I would like to thank Benoit Joly, Frédéric Le Roux and Sobhan Seyfaddini for so useful talks. I am particularly indebted to John Franks for the many conversations we had on this subject some years ago and would like to thank him warmly. Finally, I would like to thank the referee, the many comments have permit to improve the redaction of the article.

2. Radial foliations and Calabi invariant

We will introduce a formal setting to study radial foliations on a disk, based on a "discretization of the angles", that seems pertinent for studying topological foliations. Classical objects related to disk homeomorphisms, like rotation numbers or linking numbers, will be investigated in this formalism. One can look at Bechara [Bec20] for similar questions in a differentiable context.

2.1. The space of radial foliations. — Let D be an open disk of the form $D = \{z \in \mathbb{R}^2 \mid |z| < r\}$, where $r > 0$. We will set $D^* = D \smallsetminus \{0\}$ and will denote $\widetilde{D}$ the universal covering space of D^*. We will consider the sets

$$W = \{(z, z') \in D^* \times D^* \mid z \neq z'\} \quad \text{and} \quad \widetilde{W} = \{(\widetilde{z}, \widetilde{z}') \in \widetilde{D} \times \widetilde{D} \mid \widetilde{z} \neq \widetilde{z}'\}.$$

A *radial foliation* is an oriented topological foliation on D^* such that every leaf ϕ is a *ray*, meaning that it satisfies

$$\alpha(\phi) = \{0\}, \quad \omega(\phi) \subset \partial D = \{z \in \mathbb{C} \mid |z| = r\}.$$

The sets $\alpha(\phi)$ and $\omega(\phi)$ are defined as follows: if $t \mapsto \phi(t)$ is a real parametrization of ϕ compatible with the orientation of the leaf, then we have

$$\alpha(\phi) = \bigcap_{t\in\mathbb{R}} \overline{\phi((-\infty, t])}, \quad \omega(\phi) = \bigcap_{t\in\mathbb{R}} \overline{\phi([t, +\infty))},$$

where the notation $\overline{Y}$ means the closure of Y in $\mathbb{R}^2$. In other words, ϕ tends to 0 in the past and to ∂D in the future. We denote $\mathfrak{F}$ the set of radial foliations. The group $\mathrm{Homeo}_*(D^*)$ of homeomorphisms of D^* that are isotopic to the identity coincides with the group of orientation preserving homeomorphisms that fix the two ends of D^*. It acts naturally on $\mathfrak{F}$: if $\mathcal{F} \in \mathfrak{F}$ and $f \in \mathrm{Homeo}_*(D^*)$, then the foliation $f(\mathcal{F})$, whose leaves are the images by f of the leaves of $\mathcal{F}$, is a radial foliation. Moreover, $\mathrm{Homeo}_*(D^*)$ acts transitively on $\mathfrak{F}$. If $\mathrm{Homeo}_*(D^*)$ is endowed with the C^0 topology, meaning the compact-open topology applied to maps and the inverse maps, the stabilizer $\mathrm{Homeo}_{*,\mathcal{F}}(D^*)$ of $\mathcal{F} \in \mathfrak{F}$ is a closed subgroup of $\mathrm{Homeo}_*(D^*)$. Consequently, the map

$$\begin{aligned} \mathrm{Homeo}_*(D^*)/\,\mathrm{Homeo}_{*,\mathcal{F}}(D^*) &\longrightarrow \mathfrak{F} \\ f\ \mathrm{Homeo}_{*,\mathcal{F}}(D^*) &\longmapsto f(\mathcal{F}) \end{aligned}$$

is bijective and $\mathfrak{F}$ can be furnished with a natural C^0 topology. It is the topology, induced from the quotient topology defined on $\mathrm{Homeo}_*(D^*)/\,\mathrm{Homeo}_{*,\mathcal{F}}(D^*)$ by this identification map. Note that this topology does not depend on $\mathcal{F}$. It is well known that the fundamental groups of $\mathrm{Homeo}_*(D^*)$ and $\mathrm{Homeo}_{*,\mathcal{F}}(D^*)$ are infinite cyclic and that the morphism $i_* : \pi_1(\mathrm{Homeo}_{*,\mathcal{F}}(D^*), \mathrm{Id}) \to \pi_1(\mathrm{Homeo}_*(D^*), \mathrm{Id})$ induced by the inclusion map $i : \mathrm{Homeo}_{*,\mathcal{F}}(D^*) \to \mathrm{Homeo}_*(D^*)$ is bijective. This implies that $\mathfrak{F}$ is simply connected. In the whole article, when $\mathcal{F}$ is a radial foliation, the notation $\widetilde{\mathcal{F}}$

means the lift of $\mathcal{F}$ to the universal covering space $\widetilde{D}$. Note that $\widetilde{\mathcal{F}}$ is a trivial foliation. In particular, every leaf $\widetilde{\phi}$ of $\widetilde{\mathcal{F}}$ is an oriented line of $\widetilde{D}$ that separates $\widetilde{D}$ into two connected open sets, the component of $\widetilde{D} \setminus \widetilde{\phi}$ lying on the left of $\widetilde{\phi}$ will be denoted $L(\widetilde{\phi})$ and the component lying on the right will be denoted $R(\widetilde{\phi})$. We get a total order $\preceq_{\mathcal{F}}$ on the set of leaves of $\widetilde{\mathcal{F}}$ as follows:

$$\widetilde{\phi} \preceq_{\mathcal{F}} \widetilde{\phi}' \iff R(\widetilde{\phi}) \subset R(\widetilde{\phi}').$$

We can also define a partial order $\leqslant_{\mathcal{F}}$ on $\widetilde{D}$: denote $\widetilde{\phi}_{\widetilde{z}}$ the leaf of $\widetilde{\mathcal{F}}$ that contains $\widetilde{z}$ and write $\widetilde{z} <_{\mathcal{F}} \widetilde{z}'$ if $\widetilde{z} \neq \widetilde{z}'$, if $\widetilde{\phi}_{\widetilde{z}} = \widetilde{\phi}_{\widetilde{z}'}$ and if the segment of $\widetilde{\phi}_{\widetilde{z}}$ beginning at $\widetilde{z}$ and ending at $\widetilde{z}'$ inherits the orientation of $\widetilde{\phi}_{\widetilde{z}}$.

2.2. Topological angles in the universal covering space. — Consider the natural projection

$$\begin{aligned} \pi : \mathbb{Z} &\longrightarrow \mathbb{Z}/4\mathbb{Z} \\ k &\longmapsto \dot{k} = k + 4\mathbb{Z}. \end{aligned}$$

If $\mathbb{Z}/4\mathbb{Z}$ is endowed with the topology whose open sets are

$$\varnothing, \{\dot{1}\}, \{\dot{3}\}, \{\dot{1}, \dot{3}\}, \{\dot{1}, \dot{2}, \dot{3}\}, \{\dot{3}, \dot{0}, \dot{1}\}, \{\dot{0}, \dot{1}, \dot{2}, \dot{3}\},$$

and $\mathbb{Z}$ with the topology generated by the sets $\{2k+1\}$ and $\{2k-1, 2k, 2k+1\}$, $k \in \mathbb{Z}$, then π is a covering map.(3) Note that both sets $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}$ are non Hausdorff but path connected.

If k, ℓ are two integers, we will define

$$\lambda(k,\ell) = \begin{cases} 0 & \text{if } k = \ell, \\ \#(k,\ell) \cap 4\mathbb{Z} + (\#\{k,\ell\} \cap 4\mathbb{Z})/2 & \text{if } k < \ell, \\ -\#(\ell,k) \cap 4\mathbb{Z} - (\#\{\ell,k\} \cap 4\mathbb{Z})/2 & \text{if } k > \ell. \end{cases}$$

Note that for every integers k, ℓ, m we have

$$\lambda(k,\ell) + \lambda(\ell,m) = \lambda(k,m).$$

The quantity $\lambda(k,\ell)$ measures the "algebraic intersection number" of a continuous path $\gamma : [s_0, s_1] \to \mathbb{Z}/4\mathbb{Z}$ with $\{\dot{0}\}$, if γ is lifted to a path $\widehat{\gamma} : [s_0, s_1] \to \mathbb{Z}$ joining k to ℓ.

For every $(\widetilde{z}, \widetilde{z}') \in \widetilde{W}$, we can define a continuous function $\theta_{\widetilde{z},\widetilde{z}'} : \mathfrak{F} \to \mathbb{Z}/4\mathbb{Z}$ as follows:

$$\theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F}) = \begin{cases} \dot{0} & \text{if } \widetilde{z}' >_{\mathcal{F}} \widetilde{z}, \\ \dot{1} & \text{if } \phi_{\widetilde{z}'} \succ_{\mathcal{F}} \phi_{\widetilde{z}}, \\ \dot{2} & \text{if } \widetilde{z}' <_{\mathcal{F}} \widetilde{z}', \\ \dot{3} & \text{if } \phi_{\widetilde{z}'} \prec_{\mathcal{F}} \phi_{\widetilde{z}}. \end{cases}$$

The space $\mathfrak{F}$ being simply connected, the Lifting Theorem asserts that there exists a continuous function, $\widehat{\theta}_{\widetilde{z},\widetilde{z}'} : \mathfrak{F} \to \mathbb{Z}$, uniquely defined up to an additive constant in $4\mathbb{Z}$, such that $\pi \circ \widehat{\theta}_{\widetilde{z},\widetilde{z}'} = \theta_{\widetilde{z},\widetilde{z}'}$.

(3) This topology on $\mathbb{Z}$ is usually called the *digital line topology* or *Khalimsky topology* on $\mathbb{Z}$.

Note that

– $\theta_{\widetilde{z}',\widetilde{z}}(\mathcal{F}) = \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F}) + \dot{2}$, for every $\mathcal{F} \in \mathfrak{F}$,
– $\widehat{\theta}_{\widetilde{z},\widetilde{z}'}$ and $\widehat{\theta}_{\widetilde{z}',\widetilde{z}}$ can be chosen such that $\widehat{\theta}_{\widetilde{z}',\widetilde{z}}(\mathcal{F}) = \widehat{\theta}_{\widetilde{z},\widetilde{z}'}(\mathcal{F}) + 2$ for every $\mathcal{F} \in \mathfrak{F}$,
– $\theta_{\widetilde{f}(\widetilde{z}),\widetilde{f}(\widetilde{z}')}(f(\mathcal{F})) = \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F})$, for every $\mathcal{F} \in \mathfrak{F}$ and every $f \in \mathrm{Homeo}_*(D^*)$,
– $\widehat{\theta}_{\widetilde{f}(\widetilde{z}),\widetilde{f}(\widetilde{z}')}$ and $\widehat{\theta}_{\widetilde{z},\widetilde{z}'}$ can be chosen such that $\widehat{\theta}_{\widetilde{f}(\widetilde{z}),\widetilde{f}(\widetilde{z}')}(f(\mathcal{F})) = \widehat{\theta}_{\widetilde{z},\widetilde{z}'}(\mathcal{F})$, for every $\mathcal{F} \in \mathfrak{F}$.

In particular, if $\mathcal{F}$ and $\mathcal{F}'$ are two radial foliations, the numbers

$$\widehat{\tau}(\widetilde{z},\widetilde{z}',\mathcal{F},\mathcal{F}') = \widehat{\theta}_{\widetilde{z},\widetilde{z}'}(\mathcal{F}') - \widehat{\theta}_{\widetilde{z},\widetilde{z}'}(\mathcal{F}) \quad \text{and} \quad \lambda(\widetilde{z},\widetilde{z}',\mathcal{F},\mathcal{F}') = \lambda(\widehat{\theta}_{\widetilde{z},\widetilde{z}'}(\mathcal{F}), \widehat{\theta}_{\widetilde{z},\widetilde{z}'}(\mathcal{F}'))$$

do not depend on the choice of the lift $\widehat{\theta}$.

Suppose that $(\widetilde{z},\widetilde{z}') \in \widetilde{W}$, that $f \in \mathrm{Homeo}_*(D^*)$, that $\widetilde{f}$ is a lift of f and that $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{F}''$ belong to $\mathfrak{F}$. The following results are immediate:

– $|\lambda(\widetilde{z},\widetilde{z}',\mathcal{F},\mathcal{F}')| \leqslant |\widehat{\tau}(\widetilde{z},\widetilde{z}',\mathcal{F},\mathcal{F}')|$,
– $\widehat{\tau}(\widetilde{z}',\widetilde{z},\mathcal{F},\mathcal{F}') = \widehat{\tau}(\widetilde{z},\widetilde{z}',\mathcal{F},\mathcal{F}')$,
– $\widehat{\tau}(\widetilde{z},\widetilde{z}',\mathcal{F},\mathcal{F}') + \widehat{\tau}(\widetilde{z},\widetilde{z}',\mathcal{F}',\mathcal{F}'') = \widehat{\tau}(\widetilde{z},\widetilde{z}',\mathcal{F},\mathcal{F}'')$,
– $\lambda(\widetilde{z},\widetilde{z}',\mathcal{F},\mathcal{F}') + \lambda(\widetilde{z},\widetilde{z}',\mathcal{F}',\mathcal{F}'') = \lambda(\widetilde{z},\widetilde{z}',\mathcal{F},\mathcal{F}'')$,
– $\widehat{\tau}(\widetilde{z},\widetilde{z}',\mathcal{F}',\mathcal{F}) = -\widehat{\tau}(\widetilde{z},\widetilde{z}',\mathcal{F},\mathcal{F}')$,
– $\lambda(\widetilde{z},\widetilde{z}',\mathcal{F}',\mathcal{F}) = -\lambda(\widetilde{z},\widetilde{z}',\mathcal{F},\mathcal{F}')$,
– $\widehat{\tau}(\widetilde{f}(\widetilde{z}),\widetilde{f}(\widetilde{z}'),f(\mathcal{F}),f(\mathcal{F}')) = \widehat{\tau}(\widetilde{z},\widetilde{z}',\mathcal{F},\mathcal{F}')$,
– $\lambda(\widetilde{f}(\widetilde{z}),\widetilde{f}(\widetilde{z}'),f(\mathcal{F}),f(\mathcal{F}')) = \lambda(\widetilde{z},\widetilde{z}',\mathcal{F},\mathcal{F}')$.

The second assertion indicates that $(\widetilde{z},\widetilde{z}') \mapsto \widehat{\tau}(\widetilde{z}',\widetilde{z},\mathcal{F},\mathcal{F}')$ is symmetric. The next result clarifies the lack of symmetry of $(\widetilde{z},\widetilde{z}') \mapsto \lambda(\widetilde{z}',\widetilde{z},\mathcal{F},\mathcal{F}')$. For every $(\widetilde{z},\widetilde{z}') \in \widetilde{W}$, define a function $\delta_{\widetilde{z},\widetilde{z}'} : \mathfrak{F} \to \{-1/2, 0, 1/2\}$ as follows:

$$\delta_{\widetilde{z},\widetilde{z}'}(\mathcal{F}) = \begin{cases} 0 & \text{if } \widetilde{\phi}_{\widetilde{z}'} = \widetilde{\phi}_{\widetilde{z}}, \\ 1/2 & \text{if } \widetilde{\phi}_{\widetilde{z}'} \succ_{\mathcal{F}} \phi_{\widetilde{z}}, \\ -1/2 & \text{if } \widetilde{\phi}_{\widetilde{z}'} \prec_{\mathcal{F}} \phi_{\widetilde{z}}. \end{cases}$$

The next result express that $\lambda(\widetilde{z},\widetilde{z}',\mathcal{F},\mathcal{F}') - \lambda(\widetilde{z}',\widetilde{z},\mathcal{F},\mathcal{F}')$ is the "algebraic intersection number" of a "well-oriented" continuous path $\gamma : [s_0, s_1] \to \mathbb{Z}/4\mathbb{Z}$ joining $\widehat{\theta}_{\widetilde{z}',\widetilde{z}}(\mathcal{F})$ to $\widehat{\theta}_{\widetilde{z},\widetilde{z}'}(\mathcal{F}')$ with $\{\dot{0}\} - \{\dot{2}\}$.

Lemma 2.1. — *For every $(\widetilde{z},\widetilde{z}') \in \widetilde{W}$ and every $\mathcal{F}$, $\mathcal{F}'$ in $\mathfrak{F}$, it holds that*

$$\lambda(\widetilde{z},\widetilde{z}',\mathcal{F},\mathcal{F}') - \lambda(\widetilde{z}',\widetilde{z},\mathcal{F},\mathcal{F}') = \delta_{\widetilde{z},\widetilde{z}'}(\mathcal{F}') - \delta_{\widetilde{z},\widetilde{z}'}(\mathcal{F}).$$

Proof. — Recall that

$$\lambda(\widetilde{z},\widetilde{z}',\mathcal{F},\mathcal{F}') = \lambda(\widehat{\theta}_{\widetilde{z},\widetilde{z}'}(\mathcal{F}), \widehat{\theta}_{\widetilde{z},\widetilde{z}'}(\mathcal{F}'))$$

and that one can suppose that

$$\widehat{\theta}_{\widetilde{z}',\widetilde{z}}(\mathcal{F}) = \widehat{\theta}_{\widetilde{z},\widetilde{z}'}(\mathcal{F}) + 2, \quad \widehat{\theta}_{\widetilde{z}',\widetilde{z}}(\mathcal{F}') = \widehat{\theta}_{\widetilde{z},\widetilde{z}'}(\mathcal{F}') + 2.$$

Now observe that

$$\lambda(\widetilde{z},\widetilde{z}',\mathcal{F},\mathcal{F}')-\lambda(\widetilde{z}',\widetilde{z},\mathcal{F},\mathcal{F}')=\begin{cases}0 & \text{if } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F})=\theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F}'),\\ 0 & \text{if } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F})=\dot{0} \text{ and } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F}')=\dot{2},\\ 0 & \text{if } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F})=\dot{2} \text{ and } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F}')=\dot{0},\\ -1 & \text{if } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F})=\dot{1} \text{ and } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F}')=\dot{3},\\ 1 & \text{if } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F})=\dot{3} \text{ and } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F}')=\dot{1},\\ -1/2 & \text{if } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F})=\dot{1} \text{ and } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F}')=\dot{0},\\ 1/2 & \text{if } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F})=\dot{0} \text{ and } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F}')=\dot{1},\\ -1/2 & \text{if } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F})=\dot{0} \text{ and } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F}')=\dot{3},\\ 1/2 & \text{if } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F})=\dot{3} \text{ and } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F}')=\dot{0},\\ -1/2 & \text{if } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F})=\dot{1} \text{ and } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F}')=\dot{2},\\ 1/2 & \text{if } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F})=\dot{2} \text{ and } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F}')=\dot{1},\\ -1/2 & \text{if } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F})=\dot{2} \text{ and } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F}')=\dot{3},\\ 1/2 & \text{if } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F})=\dot{3} \text{ and } \theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F}')=\dot{2}.\end{cases}$$

□

The next result will be useful later:

Lemma 2.2. — *For every compact set $\widetilde{K}\subset\widetilde{W}$ and every $\mathcal{F},\mathcal{F}'$ in $\mathfrak{F}$, there exists $M>0$ such that for every $(\widetilde{z},\widetilde{z}')\in\widetilde{K}$ it holds that*

$$|\widehat{\tau}(\widetilde{z},\widetilde{z}',\mathcal{F},\mathcal{F}')|\leqslant M.$$

Proof. — For every $(\widetilde{z},\widetilde{z}')\in\widetilde{W}$, there exists a neighborhood $\widetilde{d}_{\widetilde{z},\widetilde{z}'}\times\widetilde{d}'_{\widetilde{z},\widetilde{z}'}\subset\widetilde{W}$ of $(\widetilde{z},\widetilde{z}')$, where $\widetilde{d}_{\widetilde{z},\widetilde{z}'}$ and $\widetilde{d}'_{\widetilde{z},\widetilde{z}'}$ are topological closed disks. One can cover $\widetilde{K}$ by a finite family of such neighborhoods. So it is sufficient to prove the result when $\widetilde{K}=\widetilde{d}\times\widetilde{d}'$ is the product of two topological closed disks. The map

$$\begin{aligned}\widetilde{d}\times\widetilde{d}'\times\mathfrak{F}&\longrightarrow\mathbb{Z}/4\mathbb{Z}\\ (\widetilde{z},\widetilde{z}',\mathcal{F})&\longmapsto\theta_{\widetilde{z},\widetilde{z}'}(\mathcal{F})\end{aligned}$$

being continuous and the space $\widetilde{d}\times\widetilde{d}'\times\mathfrak{F}$ being simply connected, one can lift this last map to a continuous map

$$\begin{aligned}\widetilde{d}\times\widetilde{d}'\times\mathfrak{F}&\longrightarrow\mathbb{Z}\\ (\widetilde{z},\widetilde{z}',\mathcal{F})&\longmapsto\widehat{\theta}_{\widetilde{z},\widetilde{z}'}(\mathcal{F}).\end{aligned}$$

For every $\mathcal{F}$ in $\mathfrak{F}$, the function $(\widetilde{z},\widetilde{z}')\mapsto\widehat{\theta}_{\widetilde{z},\widetilde{z}'}(\mathcal{F})$ is continuous on $\widetilde{d}\times\widetilde{d}'$ and so is bounded by compactness of $\widetilde{d}\times\widetilde{d}'$. This last affirmation comes from the fact that for every $\ell\geqslant 1$, the set $\{(\widetilde{z},\widetilde{z}')\in\widetilde{d}\times\widetilde{d}'\mid|\widehat{\theta}_{\widetilde{z},\widetilde{z}'}(\mathcal{F})|<2\ell\}$ is an open subset of $\widetilde{d}\times\widetilde{d}'$ as the preimage of an open set by a continuous map. So, for every $\mathcal{F},\mathcal{F}'$ in $\mathfrak{F}$, the function $(\widetilde{z},\widetilde{z}')\mapsto\widehat{\theta}_{\widetilde{z},\widetilde{z}'}(\mathcal{F}')-\widehat{\theta}_{\widetilde{z},\widetilde{z}'}(\mathcal{F})=\tau(\widetilde{z},\widetilde{z}',\mathcal{F},\mathcal{F}')$ is bounded on $\widetilde{d}\times\widetilde{d}'$. □

Let us conclude with the following:

Lemma 2.3. — *The function $d : \mathfrak{F}\times\mathfrak{F}\to\mathbb{N}\cup\{+\infty\}$ is an extended distance, where*
$$d(\mathcal{F},\mathcal{F}') = \sup_{(\widetilde z,\widetilde z')\in\widetilde W} |\widehat\tau(\widetilde z,\widetilde z',\mathcal{F},\mathcal{F}')|.$$
Moreover, the equality $d(f(\mathcal{F}), f(\mathcal{F}')) = d(\mathcal{F},\mathcal{F}')$ holds for every $f\in\mathrm{Homeo}_*(D^*)$.

Proof. — One proves immediately that d is symmetric and satisfies the triangular inequality. The fact that $d(\mathcal{F},\mathcal{F})=0$ for every $\mathcal{F}\in\mathfrak{F}$ is also obvious. It remains to prove that $d(\mathcal{F},\mathcal{F}')\neq 0$ if $\mathcal{F}$ and $\mathcal{F}'$ are distinct radial foliations. In that case, one can find $(\widetilde z,\widetilde z')\in\widetilde W$ such that $\widetilde z$ and $\widetilde z'$ belong to the same leaf of $\widetilde{\mathcal{F}}$ and to different leaves of $\widetilde{\mathcal{F}}'$. Consequently, $\theta(\widetilde z,\widetilde z',\mathcal{F})\in\{\dot 0,\dot 2\}$ and $\theta(\widetilde z,\widetilde z',\mathcal{F}')\in\{\dot 1,\dot 3\}$. This implies that $\widehat\theta(\widetilde z,\widetilde z',\mathcal{F})\neq\widehat\theta(\widetilde z,\widetilde z',\mathcal{F}')$ if $\widehat\theta$ is a lift of θ.

The equality $d(\widetilde f(\mathcal{F}),\widetilde f(\mathcal{F}')) = d(\mathcal{F},\mathcal{F}')$ if $f\in\mathrm{Homeo}_*(D^*)$ is a direct consequence of the equality
$$\widehat\tau(\widetilde f(\widetilde z),\widetilde f(\widetilde z'), f(\mathcal{F}), f(\mathcal{F}')) = \widehat\tau(\widetilde z,\widetilde z',\mathcal{F},\mathcal{F}'). \qquad\square$$

We will call d the *winding distance* between two radial foliations.

2.3. Topological angles in the annulus. — Let $\mathcal{F}$ be a radial foliation and f a homeomorphism of D^* isotopic to the identity. Let $I=(f_s)_{s\in[0,1]}$ be an identity isotopy of f in $\mathrm{Homeo}_*(D^*)$ and $\widetilde I = (\widetilde f_s)_{s\in[0,1]}$ the lifted identity isotopy to $\widetilde D$. The function
$$s\in[0,1]\longmapsto \theta_{\widetilde f_s(\widetilde z),\widetilde f_s(\widetilde z')}(\mathcal{F}) = \theta_{\widetilde z,\widetilde z'}(f_s^{-1}(\mathcal{F}))\in\mathbb{Z}/4\mathbb{Z}$$
can be lifted to a function
$$s\in[0,1]\longmapsto \widehat\theta_{\widetilde z,\widetilde z'}(f_s^{-1}(\mathcal{F}))\in\mathbb{Z}$$
and the difference between the value at 1 and the value at 0 of this last map is nothing but $\widehat\tau(\widetilde z,\widetilde z',\mathcal{F}, f^{-1}(\mathcal{F}))$. Of course it depends neither on the choice of I, nor on the choice of the lift $\widetilde I$ of I. Suppose now that $\widetilde z$ and $\widetilde z'$ project onto two different points of D^*. Note that if k is large enough, then for every $s\in[0,1]$, it holds that
$$\theta_{\widetilde f_s(\widetilde z),\widetilde f_s(T^k(\widetilde z'))}(\mathcal{F}) = \dot 1,\quad \theta_{\widetilde f_s(\widetilde z),\widetilde f_s(T^{-k}(\widetilde z'))}(\mathcal{F}) = \dot 3,$$
and so the functions
$$s\longmapsto \widehat\theta_{\widetilde z,T^k(\widetilde z')}(f_s^{-1}(\mathcal{F})),\quad s\longmapsto \widehat\theta_{\widetilde z,T^{-k}(\widetilde z')}(f_s^{-1}(\mathcal{F}))$$
are constant, which means that
$$\widehat\tau(\widetilde z, T^k(\widetilde z'),\mathcal{F}, f^{-1}(\mathcal{F})) = \widehat\tau(\widetilde z, T^{-k}(\widetilde z'),\mathcal{F}, f^{-1}(\mathcal{F})) = 0.$$
Consequently, if $\mathcal{F}$ and $\mathcal{F}'$ are two radial foliations, then for every $(z,z')\in W$, one can define
$$\overline{\widehat\tau}(z,z',\mathcal{F},\mathcal{F}') = \sum_{k\in\mathbb{Z}} |\widehat\tau(\widetilde z, T^k(\widetilde z'),\mathcal{F},\mathcal{F}')|,\qquad \widehat\tau(z,z',\mathcal{F},\mathcal{F}') = \sum_{k\in\mathbb{Z}}\widehat\tau(\widetilde z, T^k(\widetilde z'),\mathcal{F},\mathcal{F}'),$$
$$\lambda(z,z',\mathcal{F},\mathcal{F}') = \sum_{k\in\mathbb{Z}}\lambda(\widetilde z, T^k(\widetilde z'),\mathcal{F},\mathcal{F}'),$$
each sum being independent of the choice of the lifts $\widetilde z,\widetilde z'$ of z,z'.

Suppose that $(z,z')\in W$, that $f\in \mathrm{Homeo}_*(D^*)$ and that $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{F}''$ belong to $\mathfrak{F}$. The following results are immediate

- $|\widehat{\tau}(z,z',\mathcal{F}',\mathcal{F})|\leqslant \overline{\overline{\tau}}(z,z',\mathcal{F},\mathcal{F}')$,
- $|\lambda(z,z',\mathcal{F}',\mathcal{F})|\leqslant \overline{\overline{\tau}}(z,z',\mathcal{F},\mathcal{F}')$,
- $\overline{\overline{\tau}}(z',z,\mathcal{F},\mathcal{F}')=\overline{\overline{\tau}}(z,z',\mathcal{F},\mathcal{F}')$,
- $\widehat{\tau}(z',z,\mathcal{F},\mathcal{F}')=\widehat{\tau}(z,z',\mathcal{F},\mathcal{F}')$,
- $\overline{\overline{\tau}}(z,z',\mathcal{F}',\mathcal{F})=\overline{\overline{\tau}}(z,z',\mathcal{F},\mathcal{F}')$,
- $\widehat{\tau}(z,z',\mathcal{F},\mathcal{F}')+\widehat{\tau}(z,z',\mathcal{F}',\mathcal{F}'')=\widehat{\tau}(z,z',\mathcal{F},\mathcal{F}'')$,
- $\lambda(z,z',\mathcal{F},\mathcal{F}')+\lambda(z,z',\mathcal{F}',\mathcal{F}'')=\lambda(z,z',\mathcal{F},\mathcal{F}'')$,
- $\widehat{\tau}(z,z',\mathcal{F}',\mathcal{F})=-\widehat{\tau}(z,z',\mathcal{F},\mathcal{F}')$,
- $\lambda(z,z',\mathcal{F}',\mathcal{F})=-\lambda(z,z',\mathcal{F},\mathcal{F}')$,
- $\widehat{\tau}(f(z),f(z'),f(\mathcal{F}),f(\mathcal{F}'))=\widehat{\tau}(z,z',\mathcal{F},\mathcal{F}')$,
- $\lambda(f(z),f(z'),f(\mathcal{F}),f(\mathcal{F}'))=\lambda(z,z',\mathcal{F},\mathcal{F}')$.

Note that we have the following:

Lemma 2.4. — *For every compact subset $K\subset W$, for every $\mathcal{F},\mathcal{F}'$ in $\mathfrak{F}$, there exists $M>0$ such that for every $(z,z')\in K$ it holds that*
$$\overline{\overline{\tau}}(z,z',\mathcal{F},\mathcal{F}')\leqslant M.$$

Proof. — Like in the proof of Lemma 2.2, it is sufficient to study the case where $K=d\times d'$ is the product of two topological closed disks. Choose a lift $\widetilde{d}$ of d and a lift $\widetilde{d}'$ of d' in $\widetilde{D}$. Choose $f\in\mathrm{Homeo}_*(D^*)$ such that $\mathcal{F}'=f^{-1}(\mathcal{F})$ and an identity isotopy $(f_s)_{s\in[0,1]}$ of f. Lift this isotopy to an identity isotopy $(\widetilde{f}_s)_{s\in[0,1]}$ on $\widetilde{D}$. There exists $k_0>0$ such that if $k\geqslant k_0$, then for every $s\in[0,1]$, every $\widetilde{z}\in\widetilde{d}$ and every $\widetilde{z}'\in\widetilde{d}'$ it holds that
$$\theta_{\widetilde{f}_s(\widetilde{z}),\widetilde{f}_s(T^k(\widetilde{z}'))}(\mathcal{F})=\dot{1},\quad \theta_{\widetilde{f}_s(\widetilde{z}),\widetilde{f}_s(T^{-k}(\widetilde{z}'))}(\mathcal{F})=\dot{3},$$
and this implies that
$$\widehat{\tau}(\widetilde{z},T^k(\widetilde{z}'),\mathcal{F},\mathcal{F}')=\widehat{\tau}(\widetilde{z},T^{-k}(\widetilde{z}'),\mathcal{F},\mathcal{F}')=0.$$
To conclude, it remains to apply Lemma 2.2 to the compact sets $\widetilde{d}\times T^k(\widetilde{d}')$, $|k|<k_0$. $\square$

Let us conclude this section by stating an analogous of Lemma 2.1. Define first what is a *displacement function*. For every $f\in\mathrm{Homeo}_*(D^*)$, every lift $\widetilde{f}$ of f to $\widetilde{D}$ and every ray ϕ we can define a function $m_{\widetilde{f},\phi}:D^*\to\mathbb{Z}$ as follows: we choose a lift $\widetilde{\phi}$ of ϕ, then for every $z\in\mathbb{D}$, we consider the lift $\widetilde{z}$ of z such that $\widetilde{z}\in\overline{L(\widetilde{\phi})\cap R(T(\widetilde{\phi}))}$ and we denote $m_{\widetilde{f},\phi}(z)$ the integer m such that $\widetilde{f}(\widetilde{z})\in\overline{L(T^m(\widetilde{\phi}))\cap R(T^{m+1}(\widetilde{\phi}))}$, noting that it does not depend on the choice of $\widetilde{\phi}$. Observe that $m_{T^k\circ\widetilde{f},\phi}=m_{\widetilde{f},\phi}+k$, for every $k\in\mathbb{Z}$. One proves easily the following result:

Lemma 2.5. — *For every compact subset $K\subset D^*$ and every rays ϕ,ϕ', there exists $M>0$ such that for every $f\in\mathrm{Homeo}_*(D^*)$ and every lift $\widetilde f$ of f to $\widetilde D$, it holds that*

$$z\in K \text{ and } f(z)\in K \implies |m_{\widetilde f,\phi}(z)-m_{\widetilde f,\phi'}(z)|\leqslant M.$$

The following result clarifies the lack of symmetry of $(z,z')\mapsto\lambda(z,z',\mathcal F,\mathcal F')$.

Lemma 2.6. — *Fix $\mathcal F$, $\mathcal F'$ in $\mathfrak F$. Choose a leaf ϕ of $\mathcal F$, a lift $\widetilde\phi$ of ϕ to $\widetilde D$ and a map $f\in\mathrm{Homeo}_*(D^*)$ such that $\mathcal F'=f^{-1}(\mathcal F)$. For every $z\in D^*$, denote $\widetilde z^*$ the unique lift of z such that $\widetilde z\in\overline{L(\widetilde\phi)}\cap R(T(\widetilde\phi))$. Then, for every $(z,z')\in W$, we have*

$$\lambda(z,z',\mathcal F,\mathcal F')-\lambda(z',z,\mathcal F,\mathcal F')=m_{\widetilde f,\phi}(z')-m_{\widetilde f,\phi}(z)+\delta_{\widetilde{f(z)}^*,\widetilde{f(z')}^*}(\mathcal F)-\delta_{\widetilde z^*,\widetilde z^*}(\mathcal F).$$

Proof. — Fix $(z,z')\in W$. During the proof, we will lighten the notations by writing

$$\widetilde z=\widetilde z^*,\quad \widetilde z'=\widetilde z'^*,\quad m=m_{\widetilde f,\phi}(z),\quad m'=m_{\widetilde f,\phi}(z').$$

By using Lemma 2.1 we get

$$\begin{aligned}
&\lambda(z,z',\mathcal F,f^{-1}(\mathcal F))-\lambda(z',z,\mathcal F,f^{-1}(\mathcal F))\\
&\qquad=\sum_{k\in\mathbb Z}\lambda(\widetilde z,T^k(\widetilde z'),\mathcal F,f^{-1}(\mathcal F))-\sum_{k\in\mathbb Z}\lambda(\widetilde z',T^k(\widetilde z),\mathcal F,f^{-1}(\mathcal F))\\
&\qquad=\sum_{k\in\mathbb Z}\lambda(\widetilde z,T^k(\widetilde z'),\mathcal F,f^{-1}(\mathcal F))-\sum_{k\in\mathbb Z}\lambda(T^k(\widetilde z'),\widetilde z,\mathcal F,f^{-1}(\mathcal F))\\
&\qquad=\sum_{k\in\mathbb Z}\lambda(\widetilde z,T^k(\widetilde z'),\mathcal F,f^{-1}(\mathcal F))-\lambda(T^k(\widetilde z'),\widetilde z,\mathcal F,f^{-1}(\mathcal F))\\
&\qquad=\sum_{k\in\mathbb Z}\delta_{\widetilde z,T^k(\widetilde z')}(f^{-1}(\mathcal F))-\delta_{\widetilde z,T^k(\widetilde z')}(\mathcal F)\\
&\qquad=\sum_{k\in\mathbb Z}\delta_{\widetilde f(\widetilde z),T^k(\widetilde f(\widetilde z'))}(\mathcal F)-\delta_{\widetilde z,T^k(\widetilde z')}(\mathcal F)\\
&\qquad=\sum_{k\in\mathbb Z}\delta_{\widetilde f(T^{-m}(\widetilde z)),T^k(\widetilde f(T^{-m}(\widetilde z')))}(\mathcal F)-\delta_{\widetilde z,T^k(\widetilde z')}(\mathcal F)\\
&\qquad=\sum_{k\in\mathbb Z}\delta_{\widetilde f(T^{-m}(\widetilde z)),T^{k+m'-m}(\widetilde f(T^{-m'}(\widetilde z')))}(\mathcal F)-\delta_{\widetilde z,T^k(\widetilde z')}(\mathcal F).
\end{aligned}$$

Recalling that $\widetilde z=\widetilde z^*$ and $\widetilde z'=\widetilde z'^*$, we have

$$\delta_{\widetilde z,T^k(\widetilde z')}(\mathcal F)=\begin{cases}-1/2 & \text{if } k<0,\\ 1/2 & \text{if } k>0,\end{cases}$$

and

$$\delta_{\widetilde f(T^{-m}(\widetilde z)),T^{k+m'-m}(\widetilde f(T^{-m'}(\widetilde z')))}(\mathcal F)=\begin{cases}-1/2 & \text{if } k+m'-m<0,\\ 1/2 & \text{if } k+m'-m>0.\end{cases}$$

So we deduce that

$$\delta_{\widetilde{f}(T^{-m}(\widetilde{z})),T^{k+m'-m}(\widetilde{f}(T^{-m'}(\widetilde{z}')))}(\mathcal{F})-\delta_{\widetilde{z},T^k(\widetilde{z}')}(\mathcal{F})$$

$$=\begin{cases}0 & \text{if } k<\min(0,m-m')\\ 0 & \text{if } k>\max(0,m-m')\\ 1 & \text{if } m-m'<k<0\\ -1 & \text{if } 0<k<m-m'\\ -\delta_{\widetilde{z}^*,\widetilde{z}'^*}(\mathcal{F})-1/2 & \text{if } k=0<m-m'\\ -\delta_{\widetilde{z}^*,\widetilde{z}'^*}(\mathcal{F})+1/2 & \text{if } k=0>m-m'\\ \delta_{\widetilde{f(\widetilde{z})}^*,\widetilde{f(\widetilde{z}')}^*}(\mathcal{F})+1/2 & \text{if } k=m-m'<0\\ \delta_{\widetilde{f(\widetilde{z})}^*,\widetilde{f(\widetilde{z}')}^*}(\mathcal{F})-1/2 & \text{if } k=m-m'>0\\ \delta_{\widetilde{f(\widetilde{z})}^*,\widetilde{f(\widetilde{z}')}^*}(\mathcal{F})-\check{\theta}_{\widetilde{z}^*,\widetilde{z}'^*}(\mathcal{F}) & \text{if } k=0=m-m'\end{cases}$$

and consequently, that

$$\sum_{k\in\mathbb{Z}}\delta_{\widetilde{f}(T^{-m}(\widetilde{z})),T^{k+m'-m}(\widetilde{f}(T^{-m'}(\widetilde{z}')))}(\mathcal{F})-\delta_{\widetilde{z},T^k(\widetilde{z}')}(\mathcal{F})$$
$$=m'-m+\delta_{\widetilde{f(z)}^*,\widetilde{f(z')}^*}(\mathcal{F})-\delta_{\widetilde{z}^*,\widetilde{z}'^*}(\mathcal{F}).\qquad(2.1)\quad\square$$

Remarks

(1) The function $\lambda_{f,\mathcal{F}}:(z,z')\mapsto\lambda(z,z',\mathcal{F},f^{-1}(\mathcal{F}))$ depends only on the foliations $\mathcal{F}$ and $f^{-1}(\mathcal{F})$ (and not explicitly in f) and the function $z\mapsto m_{\widetilde{f},\phi}(z)$ depends only on the ray ϕ and on $\widetilde{f}$.

(2) The equality proved in Lemma 2.6 can be written

$$\Lambda_{\widetilde{f},\mathcal{F},\phi}(z,z')-\Lambda_{\widetilde{f},\mathcal{F},\phi}(z',z)=\delta_{\widetilde{f(z)}^*,\widetilde{f(z')}^*}(\mathcal{F})-\delta_{\widetilde{z}^*,\widetilde{z}^*}(\mathcal{F}),$$

where

$$\Lambda_{\widetilde{f},\mathcal{F},\phi}(z,z')=\lambda_{f,\mathcal{F}}(z,z')+m_{\widetilde{f},\phi}(z).$$

(3) For every $k\in\mathbb{Z}$ it holds that $\Lambda_{T^k\circ\widetilde{f},\mathcal{F},\phi}=\Lambda_{\widetilde{f},\mathcal{F},\phi}+k$.

2.4. Rotation number and linking number. — We will see how to define rotation numbers and self linking numbers within this formalism.

Definition 2.7. — Fix $f\in\mathrm{Homeo}_*(D^*)$ and a lift $\widetilde{f}$ to $\widetilde{D}$. Say that $z\in D^*$ has a *rotation number* $\mathrm{rot}_{\widetilde{f}}(z)\in\mathbb{R}$ if:

(1) there exists a compact set $K\subset D^*$ such that $\#\{n\geqslant 0\mid f^n(z)\in K\}=+\infty$;

(2) if ϕ is a ray and if $K\subset D^*$ is a compact set containing z, then for every $\varepsilon>0$, there exists $n_0\geqslant 0$ such that for every $n\geqslant n_0$ it holds that

$$f^n(z)\in K\implies\left|\frac{1}{n}\sum_{i=0}^{n-1}m_{\widetilde{f},\phi}(f^i(z))-\mathrm{rot}_{\widetilde{f}}(z)\right|\leqslant\varepsilon.$$

Remark. — Note that $\sum_{i=0}^{n-1}m_{\widetilde{f},\phi}(f^i(z))=m_{\widetilde{f}^n,\phi}(z)$. Using Lemma 2.5, one deduces that if the second assertion is true for a ray ϕ, it is true for every ray ϕ'.

Definition 2.8. — Fix $f\in\mathrm{Homeo}^*(D^*)$ and a lift $\widetilde f$ to $\widetilde D$. Say that $(z,z')\in W$ *has a linking number* $\mathrm{link}_{\widetilde f}(z,z')\in\mathbb R$ if

(1) the points z and z' have a rotation number;

(2) there exists a compact set $K\subset W$ such that $\#\{n\geqslant 0\mid (f^n(z),f^n(z'))\in K\}=+\infty$;

(3) if $\mathcal F$ is a radial foliation, if ϕ is leaf of $\mathcal F$ and if $K\subset W$ is a compact set containing (z,z'), then for every $\varepsilon>0$, there exists $n_0\geqslant 0$ such that for every $n\geqslant n_0$ it holds that

$$(f^n(z),f^n(z'))\in K \implies \left|\frac1n\sum_{i=0}^{n-1}\Lambda_{\widetilde f,\mathcal F,\phi}(f^i(z),f^i(z'))-\mathrm{link}_{\widetilde f}(z,z')\right|\leqslant\varepsilon.$$

Remarks

(1) Note that

$$\sum_{i=0}^{n-1}\Lambda_{\widetilde f,\mathcal F,\phi}(f^i(z),f^i(z'))=\Lambda_{\widetilde f^n,\mathcal F,\phi}(z,z').$$

(2) As explained in the remark following the definition of the rotation number, we know that if the third assertion is true for a leaf ϕ, it is true for every other leaf. Let us explain now, why if the third assertion is true for a foliation $\mathcal F\in\mathfrak F$, it is true for every other foliation $\mathcal F'\in\mathfrak F$.[(4)] We have

$$\begin{aligned}\big|\lambda(z,z',\mathcal F,f^{-n}(\mathcal F))-\lambda(z,z',\mathcal F',f^{-n}(\mathcal F'))\big|&\\
&=\big|\lambda(z,z',\mathcal F,\mathcal F')+\lambda(z,z',\mathcal F',f^{-n}(\mathcal F))-\lambda(z,z',\mathcal F',f^{-n}(\mathcal F'))\big|\\
&=\big|\lambda(z,z',\mathcal F,\mathcal F')-\lambda(z,z',f^{-n}(\mathcal F),f^{-n}(\mathcal F'))\big|\\
&=|\lambda(z,z',\mathcal F,\mathcal F')-\lambda(f^n(z),f^n(z'),\mathcal F,\mathcal F')|\\
&\leqslant\overline{\overline{\tau}}(z,z',\mathcal F,\mathcal F')+\overline{\overline{\tau}}(f^n(z),f^n(z'),\mathcal F,\mathcal F').\end{aligned}$$

It remains to apply Lemma 2.4: for every compact set $K\subset D^*$, there exists $M>0$ such that for every point (z,z') satisfying $(z,z')\in K$ and $(f^n(z),f^n(z'))\in K$, we have $\overline{\overline{\tau}}(z,z',\mathcal F,\mathcal F')\leqslant M$ and $\overline{\overline{\tau}}(f^n(z),f^n(z'),\mathcal F,\mathcal F')\leqslant M$.

(3) By the remark following Lemma 2.6, if $\mathrm{link}_{\widetilde f}(z,z')$ exists, then $\mathrm{link}_{\widetilde f}(z',z)$ exists and $\mathrm{link}_{\widetilde f}(z',z)=\mathrm{link}_{\widetilde f}(z,z')$. Indeed, we have

$$\big|\Lambda_{\widetilde f^n,\mathcal F,\phi}(z,z')-\Lambda_{\widetilde f^n,\mathcal F,\phi}(z',z)\big|=\big|\delta_{\widetilde{f^n(z)}^*,\widetilde{f(^nz')}^*}(\mathcal F)-\delta_{\widetilde z^*,\widetilde z^*}(\mathcal F)\big|\leqslant 1.$$

Definition 2.9. — Fix $f\in\mathrm{Homeo}_*(D^*)$ and a lift $\widetilde f$ to $\widetilde D$. Suppose that f lets invariant a finite Borel measure μ.

[(4)]This assertion is more or less what is used by Gambaudo and Ghys to prove in [GG97] that two symplectic diffeomorphisms of $\mathbb D$ that are conjugate by an orientation and area preserving homeomorphism have the same Calabi invariant.

(1) Say that μ has a rotation number $\mathrm{rot}_{\widetilde{f}}(\mu)\in\mathbb{R}$ if μ-almost every point z has a rotation number and if the function $\mathrm{rot}_{\widetilde{f}}$ is μ-integrable, and in that case set

$$\mathrm{rot}_{\widetilde{f}}(\mu)=\int_{\mathbb{D}^*}\mathrm{rot}_{\widetilde{f}}\,d\mu.$$

(2) Say that μ has a *self-linking number* $\mathrm{link}_{\widetilde{f}}(\mu)\in\mathbb{R}$, if

– μ is non atomic,

– μ has a rotation number,

– $\mu\times\mu$-almost every pair $(z,z')\in W$ has a linking number and the function $\mathrm{link}_{\widetilde{f}}$ is $\mu\times\mu$-integrable, and in that case set

$$\mathrm{link}_{\widetilde{f}}(\mu)=\int_W \mathrm{link}_{\widetilde{f}}\,d\mu\times d\mu.$$

Remarks

(1) If there exists a ray ϕ such that $m_{\widetilde{f},\phi}$ is μ-integrable, then by Birkhoff Ergodic Theorem, one knows that μ has a rotation number and it holds that

$$\mathrm{rot}_{\widetilde{f}}(\mu)=\int_{D^*} m_{\widetilde{f},\phi}\,d\mu.$$

(2) If there exists a radial foliation $\mathcal{F}$ and a leaf ϕ of $\mathcal{F}$ such that $m_{\widetilde{f},\phi}$ is μ-integrable and $\lambda_{f,\mathcal{F}}$ is $\mu\times\mu$-integrable, then μ has a self-linking number and it holds that

$$\mathrm{link}_{\widetilde{f}}(\mu)=\int_W \Lambda_{\widetilde{f},\mathcal{F},\phi}\,d\mu\times d\mu.$$

Suppose that $f\in\mathrm{Diff}^1_\omega(\mathbb{D})$. Define $\mathbb{D}^*=\mathbb{D}\smallsetminus(\mathbb{S}\cup\{0\})$. Choose an identity isotopy $I=(f_s)_{s\in[0,1]}$ of f in $\mathrm{Homeo}_*(\mathbb{D})$ that fixes 0 and write $\widetilde{f}$ for the lift of $f_{|\mathbb{D}^*}$ naturally defined by the restriction of I to $\mathbb{D}^*$. Extend the isotopy to a family $(f_s)_{s\in\mathbb{R}}$ such that $f_{s+1}=f_s\circ f$ for every $s\in[0,1]$. Denote μ_ω the finite measure naturally defined by ω. Consider the Euclidean radial foliation $\mathcal{F}_*$ on $\mathbb{D}^*$, whose leaves are the paths $(0,1)\ni t\mapsto te^{2i\pi\alpha}$, $\alpha\in[0,1)$. Note now that for every leaf ϕ of $\mathcal{F}_*$, every $n\geqslant 1$ and every $z\in D^*$ it holds that

$$\bigl|m_{\widetilde{f}^n,\phi}(z)-\mathrm{ang}_{I^n}(0,z)\bigr|\leqslant 1.$$

Moreover, the function $z\mapsto\mathrm{ang}_I(0,z)$ is bounded because f is a C^1 diffeomorphism. This implies that $m_{\widetilde{f},\phi}$ is bounded and so, μ_{ω_ω} has a rotation number (according to Definition 2.9) and we have

$$\mathrm{rot}_{\widetilde{f}}(\mu)=\int_{D^*} m_{\widetilde{f},\phi}\,d\mu_\omega=\int_{D^*}\mathrm{ang}_I(0,z)\,d\mu_\omega.$$

Note now that for every $n\geqslant 1$ and every $(z,z')\in W$ it holds that

$$\bigl|\Lambda_{\widetilde{f}^n,\mathcal{F}_*,\phi}(z,z')-\mathrm{ang}^n_I(z,z')\bigr|\leqslant 1+1=2.$$

To get this inequality, one must consider a frame at z moving with time: at time n it is the image of the original frame at z by the rotation $R_{\mathrm{ang}_{I^n}(0,z)}$. Indeed, we have the two following properties:

– $|m_{\widetilde{f}^n,\phi}(z) - \mathrm{ang}_{I^n}(0,z)| \leqslant 1$;
– the difference between the variation of angle of the vector $f_s(z')-f_s(z)$, $s\in[0,n]$, in the moving frame and $\lambda_{f^n,\mathcal{F}_*}(z,z')$ is smaller than 1.

One deduces that μ_ω has a self-linking number (according to Definition 2.9) and we have

$$\mathrm{link}_{\widetilde{f}}(\mu_\omega) = \int_W \Lambda_{\widetilde{f},\mathcal{F},\phi}\, d\mu_\omega \times d\mu_\omega = \int_W \mathrm{ang}_I^n(z,z')\, d\mu_\omega \times d\mu_\omega = \widetilde{\mathrm{Cal}}([I]).$$

Let us conclude with a proposition, that will useful to prove Theorem 1.1, and that summarizes what has been done in this section.

Proposition 2.10. — *Suppose that $f\in \mathrm{Diff}^1_\omega(\mathbb{D})$ fixes 0. Let I be an identity isotopy of f and $\widetilde{f}$ be the lift of $f_{|\mathbb{D}^*}$ naturally defined by $[I]$. If $\mathcal{F}\in\mathfrak{F}$ is a radial foliation and ϕ is a leaf of $\mathcal{F}$ such that $m_{\widetilde{f},\phi}$ is μ_ω-integrable and $\lambda_{f,\mathcal{F}}$ is $\mu_\omega\times\mu_\omega$-integrable, then it holds that*

$$\widetilde{\mathrm{Cal}}([I]) = \int_W \Lambda_{\widetilde{f},\mathcal{F},\phi}\, d\mu_\omega \times d\mu_\omega.$$

3. Construction of a good radial foliation for an irrational pseudo-rotation.

The three first sections of this chapter come from [LC16]. All proofs can be found there. The fourth one, concerning properties of projected foliations is mainly new. The last proposition of the fourth section and the isotopy defined in the last section already appeared in [LC99] but in a slightly different context (twist maps instead of untwisted maps).

3.1. Generating functions.

— Let us denote $\pi_1 : (x,y)\mapsto x$ and $\pi_2 : (x,y)\mapsto y$ the two projections defined on the Euclidean plane $\mathbb{R}^2$. An orientation preserving homeomorphism f of $\mathbb{R}^2$ will be called *untwisted* if the map

$$(x,y)\longmapsto (\pi_1(f(x,y)),y)$$

is a homeomorphism, which means that there exist two continuous functions g,g' on $\mathbb{R}^2$ such that

$$f(x,y)=(X,Y) \iff \begin{cases} x=g(X,y),\\ Y=g'(X,y).\end{cases}$$

In this case, the maps $X\mapsto g(X,y)$ and $y\mapsto g'(X,y)$ are orientation preserving homeomorphisms of $\mathbb{R}$. If moreover, f is area preserving, the continuous form $xdy+YdX$ is exact: there exists a C^1 function $h:\mathbb{R}^2\to\mathbb{R}$ such that

$$g=\frac{\partial h}{\partial y},\quad g'=\frac{\partial h}{\partial X}.$$

The function h, defined up to an additive constant, is a *generating function* of f.

We can make precise the definition by saying that f is a K *Lipschitz untwisted homeomorphism*, where $K\geqslant 1$, if

(i) f is untwisted;

(ii) f is K bi-Lipschitz;
(iii) the maps $X \mapsto g(X,y)$ and $y \mapsto g'(X,y)$ are K bi-Lipschitz;
(iv) the maps $y \mapsto g(X,y)$ and $X \mapsto g'(X,y)$ are K Lipschitz.

If f is a diffeomorphism of $\mathbb{R}^2$, denote $\mathrm{Jac}(f)(z)$ the Jacobian matrix at a point z. One proves easily, that for every $K>1$, there exists a neighborhood $\mathcal{U}$ of the identity matrix in the space of square matrices of order 2, such that every C^1 diffeomorphism satisfying $\mathrm{Jac}(f)(z)\in\mathcal{U}$, for every $z\in\mathbb{R}^2$, is a K Lipschitz untwisted homeomorphism.

Suppose that f is a C^1 orientation preserving diffeomorphism of $\mathbb{D}$ that fixes 0 and coincides with a rotation R_α on $\mathbb{S}$. Fix $\beta>\alpha$. We can extend our map to a homeomorphism of the whole plane (also denoted f) such that:

$$f(z)=\begin{cases} R_{\alpha+|z|-1}(z) & \text{if } 1\leqslant |z|\leqslant \beta-\alpha,\\ R_\beta(z) & \text{if } |z|\geqslant 1+\beta-\alpha.\end{cases}$$

Using the fact that the group of orientation preserving C^1 diffeomorphisms of $\mathbb{D}$ (and the group of symplectic diffeomorphisms of $\mathbb{D}$) that fix 0 and every point of $\mathbb{S}$, when furnished with the C^1 topology, is path connected, one can prove the following:

Proposition 3.1. — *For every $K>1$, one can find a decomposition $f=f_m\circ\cdots\circ f_1$, where each f_i is a K Lipschitz untwisted homeomorphism that fixes 0 and induces a rotation on every circle of origin 0 and radius $r\geqslant 1$. Moreover, if f is area preserving, one can suppose that each f_i preserves the area.*

We fix $K>1$ and a decomposition $f=f_m\circ\cdots\circ f_1$ given by Proposition 3.1. We define two families $(g_i)_{1\leqslant i\leqslant m}$, $(g'_i)_{1\leqslant i\leqslant m}$ of continuous maps as follows

$$f_i(x,y)=(X,Y) \iff \begin{cases} x=g_i(X,y),\\ Y=g'_i(X,y).\end{cases}$$

For every $i\in\{1,\dots,m\}$ one can find an identity isotopy $I_i=(f_{i,s})_{s\in[0,1]}$, where each $f_{i,s}$ is an untwisted map such that

$$f_{i,s}(x,y)=(X,Y) \iff \begin{cases} x=(1-s)X+sg_i(X,y),\\ Y=(1-s)y+sg'_i(X,y),\end{cases}$$

and a natural isotopy $I=I_m\circ\cdots\circ I_1$ of f. Note that if each f_i is area preserving, one can find a family $(h_i)_{1\leqslant i\leqslant m}$ of C^1 maps, such that

$$g_i=\frac{\partial h_i}{\partial y},\quad g'_i=\frac{\partial h_i}{\partial X}.$$

In that case every map $f_{i,s}$, $1\leqslant i\leqslant m$, $s\in[0,1]$, is area preserving and it holds that

$$(X,y)\longmapsto (1-s)Xy+sh_i(X,y)$$

is a generating function of $f_{i,s}$.

3.2. The vector field associated to a decomposition. — We consider in this section a C^1 orientation preserving diffeomorphism f of $\mathbb{D}$ that fixes 0 and coincides with a rotation R_α on $\mathbb{S}$. We suppose that it is extended and then decomposed into untwisted maps as in Section 3.1. We keep the same notations. We extend the families

$$(f_i)_{1\leqslant i\leqslant m}, \quad (g_i)_{1\leqslant i\leqslant m}, \quad (g'_i)_{1\leqslant i\leqslant m},$$

to m periodic families

$$(f_i)_{i\in\mathbb{Z}}, \quad (g_i)_{i\in\mathbb{Z}}, \quad (g'_i)_{i\in\mathbb{Z}},$$

and the family $(h_i)_{1\leqslant i\leqslant m}$ to a m periodic family $(h_i)_{i\in\mathbb{Z}}$ in case the f_i are area preserving.

We fix an integer $b\geqslant 1$ and consider the finite dimensional vector space

$$E_b = \bigl\{\boldsymbol{z} = (z_i)_{i\in\mathbb{Z}} \in (\mathbb{R}^2)^{\mathbb{Z}} \mid z_{i+mb} = z_i, \text{ for all } i \in \mathbb{Z}\bigr\},$$

furnished with the scalar product

$$\langle (z_i)_{i\in\mathbb{Z}}, (z'_i)_{i\in\mathbb{Z}}\rangle = \sum_{0<i\leqslant mb} x_i x'_i + y_i y'_i,$$

where $z_i = (x_i, y_i)$ and $z'_i = (x'_i, y'_i)$. We define on E_b a vector field $\zeta = (\zeta_i)_{i\in\mathbb{Z}}$ by writing

$$\zeta_i(\boldsymbol{z}) = (\xi_i(\boldsymbol{z}), \eta_i(\boldsymbol{z})) = \bigl(y_i - g'_{i-1}(x_i, y_{i-1}),\, x_i - g_i(x_{i+1}, y_i)\bigr).$$

Observe that ζ is invariant by the (b periodic) shift

$$\begin{aligned} \varphi : E_b &\longrightarrow E_b,\\ (z_i)_{i\in\mathbb{Z}} &\longmapsto (z_{i+m})_{i\in\mathbb{Z}}. \end{aligned}$$

Let us state some facts about ζ.

Lemma 3.2. — *The vector field ζ is A Lipschitz, where $A = \sqrt{6K^2+3}$.*

One deduces that the associated differential system

$$\begin{cases} \dot{x}_i = y_i - g'_{i-1}(x_i, y_{i-1}),\\ \dot{y}_i = x_i - g_i(x_{i+1}, y_i), \end{cases}$$

defines a flow on E_b. We will denote by $\boldsymbol{z}^t$ the image at time t of a point $\boldsymbol{z}\in E_b$ by this flow. As an application of Gronwall's lemma, one gets:

Lemma 3.3. — *For every $(\boldsymbol{z}, \boldsymbol{z}')\in E_b$ and every $t\in\mathbb{R}$, one has*

$$e^{-A|t|}\|\boldsymbol{z}-\boldsymbol{z}'\| \leqslant \|\boldsymbol{z}^t - \boldsymbol{z}'^t\| \leqslant e^{A|t|}\|\boldsymbol{z}-\boldsymbol{z}'\|$$

and

$$e^{-A|t|}\|\zeta(\boldsymbol{z})\| \leqslant \|\zeta(\boldsymbol{z}^t)\| \leqslant e^{A|t|}\|\zeta(\boldsymbol{z})\|.$$

In the case where the f_i are area preserving, observe that ζ is the gradient vector field of the function

$$\boldsymbol{h} : \boldsymbol{z} \longmapsto \sum_{0<i\leqslant mb} x_i y_i - h_{i-1}(x_i, y_{i-1})$$

and that $\boldsymbol{h}$ is invariant by φ. One can define the *energy* of an orbit $(\boldsymbol{z}^t)_{t\in\mathbb{R}}$ to be

$$\int_{-\infty}^{+\infty} \|\zeta(\boldsymbol{z}^t)\|^2\, dt = \lim_{t\to+\infty} \boldsymbol{h}(\boldsymbol{z}^t) - \lim_{t\to-\infty} \boldsymbol{h}(\boldsymbol{z}^t).$$

As a consequence of Lemma 3.2 it holds that

Lemma 3.4. — *For every $\boldsymbol{z} \in E_b$, one has*

$$\|\zeta(\boldsymbol{z})\|^2 \leqslant A \int_{-\infty}^{+\infty} \|\zeta(\boldsymbol{z}^t)\|^2\, dt = A\big(\lim_{t\to+\infty} \boldsymbol{h}(\boldsymbol{z}^t) - \lim_{t\to-\infty} \boldsymbol{h}(\boldsymbol{z}^t)\big).$$

For every $i \in \mathbb{Z}$, define the maps Q_i, P_i, $Q'_i : E_b \to \mathbb{R}^2$, where

$$Q_i(\boldsymbol{z}) = (g_i(x_{i+1}, y_i), y_i), \quad P_i(\boldsymbol{z}) = (x_i, y_i), \quad Q'_i(\boldsymbol{z}) = (x_i, g'_{i-1}(x_i, y_{i-1})).$$

Let us state the main properties of these maps:

– $f_i \circ Q_i = Q'_{i+1}$,
– $\zeta_i = J \circ (Q'_i - Q_i)$, where $J(x, y) = (-y, x)$,
– $\boldsymbol{z} \in E_b$ is a singularity of ζ if and only if $Q_i(\boldsymbol{z}) = Q'_i(\boldsymbol{z})$ for every $i \in \mathbb{Z}$,
– if $\boldsymbol{z} \in E_b$ is a singularity of ζ then $Q_i(\boldsymbol{z}) = P_i(\boldsymbol{z}) = Q'_i(\boldsymbol{z})$ for every $i \in \mathbb{Z}$,
– Q_1 induces a bijection between the singular set of ζ and the fixed point set of f^b,
– the sequence $\mathbf{0} = (0)_{i\in\mathbb{Z}}$ is a singular point of ζ that is sent onto 0 by each Q_i, P_i or Q'_i,
– ζ is C^1 in a neighborhood of $\mathbf{0}$.

3.3. The case of an irrational pseudo-rotation. — In this section we keep the notation of Section 3.2 but we suppose than f is an irrational pseudo-rotation. We suppose moreover that $\beta \notin \mathbb{Q}$ and that $(\alpha, \beta) \cap \mathbb{Z} = \varnothing$. The extension f is a piecewise C^1 area preserving transformation that satisfies the following properties:

– 0 is the unique fixed point of f;
– there is no periodic point of period b if $(b\alpha, b\beta) \cap \mathbb{Z} = \varnothing$;
– if $(b\alpha, b\beta) \cap \mathbb{Z} \neq \varnothing$, the set of periodic points of period b can be written $\bigcup_{\alpha<a/b<\beta} S_{a/b}$, where $S_{a/b}$ is the circle of center 0 and radius $1 + a/b - \alpha$.

In that case, we have the additional following properties for the vector ζ defined on E_b:

– the singular set consists of the constant sequence $\mathbf{0}$ and of finitely many smooth closed curves $(\Sigma_a)_{a\in(b\alpha,b\beta)\cap\mathbb{Z}}$;
– the curve Σ_a is sent homeomorphically onto $S_{a/b}$ by each Q_i, P_i or Q'_i;
– ξ is C^∞ in a neighborhood of Σ_a.

We fix an integer $b \geqslant 2$ such that $(b\alpha, b\beta) \cap \mathbb{Z} \neq \varnothing$. If $\boldsymbol{z}$ and $\boldsymbol{z}'$ are two singular points of ζ, the quantity $\boldsymbol{h}(\boldsymbol{z}) - \boldsymbol{h}(\boldsymbol{0})$ is, up to the sign, the difference of *action* between the two corresponding fixed points of f^b. A computation gives us:

Lemma 3.5. — *For every $\boldsymbol{z} \in \Sigma_a$, one has*

$$\boldsymbol{h}(\boldsymbol{z}) - \boldsymbol{h}(\boldsymbol{0}) = \pi(a - b\alpha)\Big(1 + (a/b - \alpha) + \frac{(a/b-\alpha)^2}{3}\Big).$$

We will denote

$$C(a,b) = \pi(a - b\alpha)\Big(1 + (a/b - \alpha) + \frac{(a/b-\alpha)^2}{3}\Big).$$

Now let us state the fundamental result of [LC16]:

Proposition 3.6. — *The curve Σ_a bounds a topological disk $\Delta_a \subset E_b$ that satisfies the following:*

(i) *Δ_a contains the constant sequence $\mathbf{0}$;*
(ii) *Δ_a is invariant by φ;*
(iii) *each projection $\boldsymbol{z} \mapsto (x_i, y_{i-1})$, $i \in \mathbb{Z}$, is one to one on Δ_a;*
(iv) *each projection $\boldsymbol{z} \mapsto (x_i, y_i)$, $i \in \mathbb{Z}$, is one to one on Δ_a;*
(v) *Δ_a is invariant by the flow of ζ;*
(vi) *for every $\boldsymbol{z} \in \Delta_a^* = \Delta_a \setminus (\{\mathbf{0}\} \cup \Sigma_a)$, one has*

$$\lim_{t\to-\infty} \boldsymbol{z}^t = \mathbf{0} \quad \textit{and} \quad \lim_{t\to+\infty} d(\boldsymbol{z}^t, \Sigma_a) = 0.$$

Let us explain more precisely what is proved in [LC16]. In what follows, the function sign assigns $+1$ to a positive number and -1 to a negative number.

Let us consider the set

$$V = \{\boldsymbol{z} \in E_b \mid x_i \neq 0 \text{ and } y_i \neq 0 \text{ for all } i \in \mathbb{Z}\}$$

and the function L on V defined by the formula

$$\begin{aligned} L(\boldsymbol{z}) &= \frac{1}{4} \sum_{0<i\leqslant mb} \operatorname{sign}(x_i)\big(\operatorname{sign}(y_i) - \operatorname{sign}(y_{i-1})\big), \\ &= \frac{1}{4} \sum_{0<i\leqslant mb} \operatorname{sign}(y_i)\big(\operatorname{sign}(x_i) - \operatorname{sign}(x_{i+1})\big). \end{aligned}$$

It extends continuously to the open set

$$V' = \{\boldsymbol{z} \in E_b \mid x_i = 0 \Rightarrow y_{i-1}y_i > 0,\ y_i = 0 \Rightarrow x_i x_{i+1} > 0\}.$$

It is integer valued and takes its values in $\{-[mb/2], \dots, [mb/2]\}$, where the notation $[x]$ denotes the integer part of a real number x. The assertions (iii) and (iv) are immediate consequences of the following fact:

Proposition 3.7. — *If $\boldsymbol{z}$ and $\boldsymbol{z}'$ are two different points of Δ_a, then $\boldsymbol{z} - \boldsymbol{z}' \in V'$ and $L(\boldsymbol{z} - \boldsymbol{z}') = a$.*

The fundamental result that permits to construct Δ_a is the following:

Proposition 3.8. — *If $\boldsymbol{z}$, $\boldsymbol{z}$ are two distinct points of E_b satisfying $\boldsymbol{z}'-\boldsymbol{z}\notin V'$, then there exists $\varepsilon>0$ such that for every $t\in(0,\varepsilon]$, it holds that:*

$$\boldsymbol{z}'^{-t}-\boldsymbol{z}^{-t}\in V',\quad \boldsymbol{z}'^{t}-\boldsymbol{z}^{t}\in V',\quad L(\boldsymbol{z}'^{-t}-\boldsymbol{z}^{-t})<L(\boldsymbol{z}'^{t}-\boldsymbol{z}^{t}).$$

This result admits an infinitesimal version (see [LC99]):

Proposition 3.9. — *If $\boldsymbol{z}\in E_b$ is non singular and satisfies $\zeta(\boldsymbol{z})\notin W$, then there exists $\varepsilon>0$ such that for every $t\in(0,\varepsilon]$, it holds that:*

$$\zeta(\boldsymbol{z}'^{-t})\in V',\quad \zeta(\boldsymbol{z}'^{t})\in V',\quad L(\zeta(\boldsymbol{z}'^{-t}))<L(\zeta(\boldsymbol{z}'^{t}))$$

and

$$\boldsymbol{z}'^{-t}-\boldsymbol{z}\in V',\quad \boldsymbol{z}'^{t}-\boldsymbol{z}\in V',\quad L(\boldsymbol{z}'^{-t}-\boldsymbol{z})<L(\boldsymbol{z}'^{t}-\boldsymbol{z}).$$

As an immediate corollary of the second assertion, one gets the following result (not explicitly stated in [LC16])

Corollary 3.10. — *If $\boldsymbol{z}\in\Delta_a^*$, then $\zeta(\boldsymbol{z})\in V'$ and $L(\zeta(\boldsymbol{z}))=a$.*

The assertion (iii) tells us that the maps $Q_i|_{\Delta_a}$ and $Q'_i|_{\Delta_a}$, $i\in\mathbb{Z}$, induce homeomorphisms from Δ_a to $D_{a/b}$ denoted respectively q_i and q'_i [(5)]. The assertion (iv) tells us that the maps $P_i|_{\Delta_a}$, $i\in\mathbb{Z}$, induce homeomorphisms from Δ_a to $D_{a/b}$ denoted p_i. Note that $p_i\circ q_i^{-1}$ is a homeomorphism which let invariant every horizontal segment of $D_{a/b}$ and induces on this segment an increasing homeomorphism and similarly $p_i\circ {q'}_i^{-1}$ is a homeomorphism which let invariant every vertical segment of $D_{a/b}$ and induces on this segment an increasing homeomorphism. We denote $\mathcal{F}$ the radial foliation defined on Δ_a^* whose leaves are the gradient lines of ζ included in Δ_a^*.

For every $s\in[0,1]$, and every $i\in\mathbb{Z}$ we define

$$q_i^s=(1-s)q_i+sp_i,\quad {q'}_i^s=(1-s)p_i+sq'_i.$$

The maps q_i^s and ${q'}_i^s$ are homeomorphisms from Δ_a to $D_{a/b}$ (all coinciding on Σ_a) and send $\mathcal{F}$ onto a radial foliation of $D_{a/b}$.

3.4. Projected radial foliations. — Recall that d denotes the winding distance between two radial foliations. Let us begin with the following result:

Lemma 3.11. — *For every s_1,s_2 in $[0,1]$ and every $i\in\mathbb{Z}$, we have*

$$d(q_i^{s_1}(\mathcal{F}),q_i^{s_2}(\mathcal{F}))\leqslant 2,\quad d({q'}_i^{s_1}(\mathcal{F}),{q'}_i^{s_2}(\mathcal{F}))\leqslant 2.$$

Proof. — We fix $i\in\mathbb{Z}$ and will prove the first inequality, the proof of the second one being similar. The leaves of $q_i^s(\mathcal{F})$ are the paths $t\mapsto q_i^s(\boldsymbol{z}^t)$, $\boldsymbol{z}\in\Delta_a^*$. Note that the map $t\mapsto\pi_2(q_i^s(\boldsymbol{z}^t))$ is C^1 and that

$$\frac{d}{dt}\,\pi_2\circ q_i^s(\boldsymbol{z}^t)_{|t=0}=\eta_i(\boldsymbol{z})=x_i-g_i(x_{i+1},y_i).$$

(5)We do not refer to a to lighten the notations.

Say that $z\in D^*_{a/b}$ is *horizontal* if it can be written $z=q_i(\boldsymbol{z})$, where $x_i-g_i(x_{i+1},y_i)=0$. Denote H_i the set of horizontal points.[(6)] For every point $z\in D^*_{a/b}$, define $J(z)$ to be equal to $\{z\}$ if z is horizontal and to the largest horizontal interval contained in $D^*_{a/b}\smallsetminus H_i$ and containing z, if z is not horizontal. The sign of $\eta_i(q_i^{-1}(z'))$ does not depend on the choice of $z'\in J(z)$. We orient $J(z)$ with x increasing if this sign is positive and with x decreasing if this sign is negative. Note that the path $s\mapsto u_i^s(z)=q_i^s\circ q_i^{-1}(z)$ is constant if z is horizontal and is an oriented segment of $J(z)$ inheriting the same orientation if z is not horizontal. The fact that $\frac{d}{dt}\pi_2\circ q_i^s(\boldsymbol{z}^t)_{|t=0}=\eta_i(\boldsymbol{z})$, for every $s\in[0,1]$ and every $\boldsymbol{z}\in\Delta_a^*$, implies that if z is not horizontal, the oriented interval $J(z)$ is transverse to the foliations $q_i^s(\mathcal{F})$, $s\in[0,1]$, crossing locally every leaf from the left to the right.

Denote $\widetilde D_{a/b}$ the universal covering space of $D^*_{a/b}$ and for every $\widetilde z\in\widetilde D_{a/b}$ that lifts $z\in D^*_{a/b}$, denote $\widetilde J(\widetilde z)$ the lift of $J(z)$ containing $\widetilde z$, with the induced orientation in case $z\notin H_i$. Write $\widetilde H_i$ for the lift of H_i, write $\widetilde{q_i^s(\mathcal{F})}$ for the lift of $q_i^s(\mathcal{F})$ and denote $(\widetilde u_i^s)_{s\in[0,1]}$ the identity isotopy that lifts $(u_i^s)_{s\in[0,1]}$. Suppose that $\widetilde z\in\widetilde D_{a/b}$ is not in $\widetilde H_i$. For every $s_0\in[0,1]$, the oriented interval $\widetilde I(\widetilde z)$ is transverse to the foliation $\widetilde{q_i^{s_0}(\mathcal{F})}$, crossing locally every leaf from the right to the left. Moreover, if $0\leqslant s_1<s_2$, then $\widetilde u_i^{s_2}(\widetilde z)$ follows $\widetilde u_i^{s_1}(\widetilde z)$ on $\widetilde J(\widetilde z)$. Consequently it holds that:

$$0\leqslant s_0\leqslant 1 \text{ and } 0\leqslant s_1<s_2\leqslant 1 \implies \theta_{\widetilde u_i^{s_1}(\widetilde z),\widetilde u_i^{s_2}(\widetilde z)}(q_i^{s_0}(\mathcal{F}))=\dot 1.$$

Now, let us fix two different points $\widetilde z_0$ and $\widetilde z_1$ in $\widetilde D_{a/b}$ and $s\in[0,1]$. Observe that the three sets

$$\{(\widetilde z_0',\widetilde z_1')\in\widetilde J(\widetilde z_0)\times\widetilde J(\widetilde z_1)\mid \widetilde z_0'\neq\widetilde z_1',\,\theta_{\widetilde z_0',\widetilde z_1'}(q_i^s(\mathcal{F}))=\dot 1\},$$
$$\{(\widetilde z_0',\widetilde z_1')\in\widetilde J(\widetilde z_0)\times\widetilde J(\widetilde z_1)\mid \widetilde z_0'\neq\widetilde z_1',\,\theta_{\widetilde z_0',\widetilde z_1'}(q_i^s(\mathcal{F}))=\dot 3\},$$
$$\{(\widetilde z_0',\widetilde z_1')\in\widetilde J(\widetilde z_0)\times\widetilde J(\widetilde z_1)\mid \widetilde z_0'\neq\widetilde z_1',\,\theta_{\widetilde z_0',\widetilde z_1'}(q_i^s(\mathcal{F}))\in\{\dot 0,\dot 2\}\},$$

are connected, whether the points belong to $\widetilde H_i$ or not. Indeed, the paths $\widetilde J(\widetilde z_0)$ and $\widetilde J(\widetilde z_1)$ in $\widetilde D_{a/b}$ draw intervals of leaves of the foliation $\widetilde{q_i^s(\mathcal{F})}$ whose intersection is an interval of leaves if not empty. In particular the set of pairs $(\widetilde z_0',\widetilde z_1')\in\widetilde J(\widetilde z_0)\times\widetilde J(\widetilde z_1)$ such that $\widetilde z_0'$ and $\widetilde z_1'$ are distinct and belong to the same leaf of $\widetilde{q_i^s(\mathcal{F})}$ is connected: it is an interval (possibly empty) if $\widetilde J(\widetilde z_0)\cap\widetilde J(\widetilde z_1)=\varnothing$, it is empty if $\widetilde J(\widetilde z_0)\cap\widetilde J(\widetilde z_1)\neq\varnothing$ (because the intersection is a "horizontal" path in $\widetilde D_{a/b}$). The map $(\widetilde z_0',\widetilde z_1')\mapsto\theta_{\widetilde z_0',\widetilde z_1'}(q_i^s(\mathcal{F}))$ being continuous on $\widetilde W$, takes at most three values on $\widetilde J(\widetilde z_0)\times\widetilde J(\widetilde z_1)$ (either $\dot 0$ or $\dot 2$ is missing).

In particular, if $0\leqslant s_1<s_2\leqslant 1$, the map

$$s\in[s_1,s_2]\longmapsto\theta_{\widetilde u_i^s\circ(\widetilde u_i^{s_1})^{-1}(\widetilde z_0),\,\widetilde u_i^s\circ(\widetilde u_i^{s_1})^{-1}(\widetilde z_1)}(q_i^{s_2}(\mathcal{F}))=\theta_{\widetilde z_0,\widetilde z_1}(u_i^{s_1}\circ(u_i^s)^{-1}\circ q_i^{s_2}(\mathcal{F}))\in\mathbb{Z}/4\mathbb{Z}$$

(6)if $\eta_i(\boldsymbol{z})=0$, then $\xi_i(\boldsymbol{z})\neq 0$, so every leaf of $p_i(\mathcal{F})$ is a C^1 embedded line and H_i is nothing but the set of points where the foliation $p_i(\mathcal{F})$ is horizontal.

takes at most three values and is lifted to a map

$$s \in [s_1, s_2] \longmapsto \widehat{\theta}_{\widetilde{z}_0,\widetilde{z}_1}(u_i^{s_1} \circ (u_i^s)^{-1} \circ q_i^{s_2}(\mathcal{F})) \in \mathbb{Z}$$

taking at most three values, which implies that

$$|\widehat{\tau}(\widetilde{z}_0, \widetilde{z}_1, q_i^{s_2}(\mathcal{F}), q_i^{s_1}(\mathcal{F}))| \leqslant 2. \qquad \square$$

The homeomorphism $\widetilde{u}_i^s$ sends the foliation $\widetilde{q_i(\mathcal{F})}$ onto the foliation $\widetilde{q_i^s(\mathcal{F})}$. The leaves of $\widetilde{q_i(\mathcal{F})}$ are pushed on the right by the isotopy $(\widetilde{u}_i^s)_{s\in[0,1]}$. More precisely, for every $0 \leqslant s_1 < s_2 \leqslant 1$ and every leaf $\widetilde{\phi}$ of $\widetilde{q_i(\mathcal{F})}$, it holds that:

– $R(\widetilde{u}_i^{s_2}(\widetilde{\phi})) \subset R(\widetilde{u}_i^{s_1}(\widetilde{\phi}))$,
– if $\widetilde{z} \in \widetilde{\phi}$ belongs to $\widetilde{H}_i$, then $\widetilde{z} \in \widetilde{u}_i^{s_1}(\widetilde{\phi}) \cap \widetilde{u}_i^{s_2}(\widetilde{\phi})$,
– if $\widetilde{z} \in \widetilde{\phi}$ does not belong to $\widetilde{H}_i$, then $\widetilde{u}_i^{s_2}(\widetilde{z}) \in R(\widetilde{u}_i^{s_1}(\widetilde{\phi}))$.

Similarly, for every $\boldsymbol{z} \in \Delta_a^*$, the map $t \mapsto \pi_1(q'^s_i(\boldsymbol{z}^t))$ is C^1 and we have

$$\frac{d}{dt}\pi_1 \circ q'^s_i(\boldsymbol{z}^t)_{|t=0} = \xi_i(\boldsymbol{z}) = y_i - g_{i-1}(x_i, y_{i-1}).$$

Say that $z \in D^*_{a/b}$ is *vertical* if it can be written $z = q'_i(\boldsymbol{z})$, where $y_i - g_{i-1}(x_i, y_{i-1}) = 0$. Denote V_i the set of vertical points and $\widetilde{V}_i$ its lift in $\widetilde{D}_{a/b}$. Define $u'^s_i = q'^s_i \circ p_i^{-1}$ and lift the isotopy $(u'^s_i)_{s\in[0,1]}$ to an identity isotopy $(\widetilde{u}'^s_i)_{s\in[0,1]}$. Write $\widetilde{q'^s_i(\mathcal{F})}$ for the lift of $q'^s_i(\mathcal{F})$. Then, for every $0 \leqslant s_1 < s_2 \leqslant 1$ and every leaf $\widetilde{\phi}$ of $\widetilde{q'_i(\mathcal{F})}$, it holds that:

– $R(\widetilde{u}'^{s_2}_i(\widetilde{\phi})) \subset R(\widetilde{u}'^{s_1}_i(\widetilde{\phi}))$,
– if $\widetilde{z} \in \widetilde{\phi}$ belongs to $\widetilde{V}_i$, then $\widetilde{z} \in \widetilde{u}'^{s_1}_i(\widetilde{\phi}) \cap \widetilde{u}'^{s_2}_i(\widetilde{\phi})$,
– if $\widetilde{z} \in \widetilde{\phi}$ does not belong to $\widetilde{V}_i$, then $\widetilde{u}'^{s'}_i(\widetilde{z}) \in R(\widetilde{u}'^s_i(\widetilde{\phi}))$.

To conclude, consider the family $(v_i^s)_{s\in[0,2]}$, where

$$v_i^s = \begin{cases} u_i^s & \text{if } 0 \leqslant s \leqslant 1,\\ u'^{s-1}_i \circ u_i^1 = q'^{s-1}_i \circ q_i^{-1} & \text{if } 1 \leqslant s \leqslant 2.\end{cases}$$

It is an isotopy from Id to $q'_i \circ q_i^{-1}$. The isotopy $(v^s_{i\,|D^*_{a/b}})_{s\in[0,2]}$ can be lifted to an identity isotopy $(\widetilde{v}_i^s)_{s\in[0,2]}$ such that

$$\widetilde{v}_i^s = \begin{cases} \widetilde{u}_i^s & \text{if } 0 \leqslant s \leqslant 1,\\ \widetilde{u}'^{s-1}_i \circ \widetilde{u}_i^1 & \text{if } 1 \leqslant s \leqslant 2.\end{cases}$$

The map $\widetilde{v}_i^s$ sends the foliation $\widetilde{q_i(\mathcal{F})}$ onto the foliation $\widetilde{q_i^s(\mathcal{F})}$ if $0 \leqslant s \leqslant 1$ and onto the foliation $\widetilde{q'^{s-1}_i(\mathcal{F})}$ if $1 \leqslant s \leqslant 2$. Moreover:

– for every $0 \leqslant s_1 < 1 < s_2 \leqslant 2$ and every leaf $\widetilde{\phi}$ of $\widetilde{\mathcal{F}}_i^0$, it holds that $\overline{R(\widetilde{v}_i^{s_2}(\widetilde{\phi}))} \subset R(\widetilde{v}_i^{s_1}(\widetilde{\phi})$.

Indeed, the sets H_i and V_i are disjoint.

3.5. Construction of a good isotopy. — One gets a m-periodic family of homeomorphisms $(\widehat{f}_i)_{i\in\mathbb{Z}}$ of $D_{a/b}$ by writing:

$$\widehat{f}_i = q_{i+1} \circ q_i^{-1}.$$

Its periodicity comes from the equalities

$$\widehat{f}_{i+m} = q_{i+m+1} \circ q_{i+m}^{-1} = q_{i+1} \circ \varphi_{|\Delta_a} \circ (q_i \circ \varphi_{|\Delta_a})^{-1}.$$

Moreover, $\widehat{f} = \widehat{f}_m \circ \cdots \circ \widehat{f}_1$ has order q because

$$\widehat{f} = q_{m+1} \circ q_1^{-1} = q_1 \circ (\varphi_{|\Delta_a}) \circ q_1^{-1},$$

or equivalently because

$$\widehat{f}^b = \widehat{f}_{mb} \circ \cdots \circ \widehat{f}_1 = q_{mb+1} \circ q_1^{-1} = \mathrm{Id}_{D_{a/b}}.$$

Note also that $\widehat{f}_i$ fixes 0 and coincides with f_i on $S_{a/b}$ because

$$f_i|_{D_{a/b}} = q'_{i+1} \circ q_i^{-1}.$$

Let us define an isotopy $\check{I} = (\check{f}_s)_{s\in[0,mb]}$ from Id to f^b in the following way:

If $s \in [2k, 2k+1]$, $0 \leqslant k < mb$, then

$$\begin{aligned}\check{f}_s &= f_{mq} \circ \cdots \circ f_{mb-k+1} \circ q_{mb-k+1}^{s-2k} \circ q_1^{-1}\\ &= f_{mq} \circ \cdots \circ f_{mb-k+1} \circ q_{mb-k+1}^{s-2k} \circ q_{mq-k}^{-1} \circ \widehat{f}_{mb-k-1} \circ \cdots \circ \widehat{f}_1.\end{aligned}$$

If $s \in [2k+1, 2k+2]$, $0 \leqslant k < mq$, then

$$\begin{aligned}\check{f}_s &= f_{mb} \circ \cdots \circ f_{mb-k+1} \circ {q'}_{mb-k+1}^{s-2k-1} \circ q_1^{-1}\\ &= f_{mq} \circ \cdots \circ f_{mb-k+1} \circ {q'}_{mb-k+1}^{s-2k-1} \circ q_{mb-k}^{-1} \circ \widehat{f}_{mb-k-1} \circ \cdots \circ \widehat{f}_1.\end{aligned}$$

Note that

$$\begin{aligned}\check{f}_{2k} &= f_{mb} \circ \cdots \circ f_{mb-k+1} \circ \widehat{f}_{mb-k} \circ \cdots \circ \widehat{f}_1,\\ \check{f}_{2k+1} &= f_{mb} \circ \cdots \circ f_{mb-k+1} \circ p_{mb-k+1} \circ q_{mb-k}^{-1} \circ \widehat{f}_{mb-k-1} \circ \cdots \circ \widehat{f}_1.\end{aligned}$$

In particular, we have

$$\check{f}_s(q_1(\mathcal{F})) = \begin{cases} f_{mb} \circ \cdots \circ f_{mb-k+1}(q_i^{s-2k}(\mathcal{F})) & \text{if } s \in [2k, 2k+1],\\ f_{mb} \circ \cdots \circ f_{mb-k+1}({q'}_i^{s-2k-1}(\mathcal{F})) & \text{if } s \in [2k+1, 2k+2].\end{cases}$$

Proposition 3.12. — *For every s_1, s_2 in $[0, 2mb]$ we have*

$$d\left(\check{f}_{s_1}(q_1(\mathcal{F})), \check{f}_{s_2}(q_1(\mathcal{F}))\right) \leqslant 2|s_2 - s_1| + 4.$$

Moreover, if s_1 and s_2 are integers we have

$$d\left(\check{f}_{s_1}(q_1(\mathcal{F})), \check{f}_{s_2}(q_1(\mathcal{F}))\right) \leqslant 2|s_2 - s_1|.$$

Proof. — We will use Lemma 3.11. Note that for every s_1, s_2 in $[2k, 2k+1]$ we have

$$\begin{aligned} &d\big(\check{f}_{s_1}(q_1(\mathcal{F})), \check{f}_{s_2}(q_1(\mathcal{F})\big)\\ &\qquad = d\Big(f_{mb} \circ \cdots \circ f_{mq-k+1}(q_i^{s_1-2k}(\mathcal{F})), f_{mb} \circ \cdots \circ f_{mb-k+1}(q_i^{s_2-2k}(\mathcal{F}))\Big)\\ &\qquad = d\big(q_i^{s_1-2k}(\mathcal{F}), q_i^{s_2-2k}(\mathcal{F})\big) \leqslant 2,\end{aligned}$$

and similarly for every s_1, s_2 in $[2k+1, 2k+2]$ we have

$$\begin{aligned} d\big(\check{f}_{s_1}(q_1(\mathcal{F})), \check{f}_{s_2}(q_1(\mathcal{F}))\big) \\ &= d\Big(f_{mb}\circ\cdots\circ f_{mb-k+1}({q'}_i^{s_1-2k-1}(\mathcal{F})), f_{mb}\circ\cdots\circ f_{mb-k+1}({q'}_i^{s_2-2k-1}(\mathcal{F})\Big) \\ &= d\big({q'}_i^{s_1-2k-1}(\mathcal{F}), {q'}_i^{s_2-2k-1}(\mathcal{F}))\big) \leqslant 2. \end{aligned}$$

We easily deduce Proposition 3.12. □

We immediately deduce

Corollary 3.13. — *It holds that*

$$d\left(f^b(q_1(\mathcal{F})), q_1(\mathcal{F})\right) \leqslant 4mb.$$

Let us conclude this section by the following result

Proposition 3.14

If $\widetilde{\check{I}} = (\widetilde{\check{f}}_s)_{s\in[0,2mb]}$ is the identity isotopy that lifts $(\check{f}_{s|D^*_{a/b}})_{s\in[0,2mb]}$, then every leaf $\widetilde{\phi}$ of $\widetilde{q_1(\mathcal{F})}$ is a Brouwer line of $\widetilde{\check{f}}_{2mb}$.

Proof. — Let ϕ be a leaf of $\mathcal{F}$, then for every $s\in[0,2mq]$, we have

$$\check{f}_s(q_1(\phi)) = \begin{cases} f_{mq}\circ\cdots\circ f_{mb-k+1}\circ q_{mb-k+1}^{s-2k}(\phi) & \text{if } s\in[2k,2k+1],\ 0\leqslant k<mb,\\ f_{mq}\circ\cdots\circ f_{mb-k+1}\circ {q'}_{mb-k+1}^{s-2k}(\phi) & \text{if } s\in[2k+1,2k+2],\ 0\leqslant k<mq.\end{cases}$$

Equivalently, it holds that

$$\check{f}_s(q_1(\phi)) = f_{mq}\circ\cdots\circ f_{mb-k+1}\circ v_{mb-k+1}^{s-2k}(q_{mb-k+1}(\phi))$$

if $s\in[2k,2k+2]$, $0\leqslant k<mb$.

We deduce that for every leaf $\widetilde{\phi}_1$ of $\widetilde{q_1(\mathcal{F})}$, it holds that $R(\widetilde{\check{f}}_{s_2}(\widetilde{\phi}_1))\subset R(\widetilde{\check{f}}_{s_1}(\widetilde{\phi}_1))$ if $0\leqslant s_1<s_2\leqslant 2mb$ and that $R(\widetilde{\check{f}}_{2k+2}(\widetilde{\phi}_1))\subset R(\widetilde{\check{f}}_{2k}(\widetilde{\phi}_1))$ if $0\leqslant k<mb$. In particular we have $R(\widetilde{\check{f}}_{2mb}(\widetilde{\phi}_1))\subset R(\widetilde{\phi}_1)$. □

4. Proof of Theorem 1.1

Proof of Theorem 1.1. — Let $f\in\mathrm{Diff}^1_\omega(\mathbb{D})$ be an irrational pseudo-rotation that coincides with a rotation on $\mathbb{S}$. We want to prove that $\mathrm{Cal}(f)=0$. We extended our map to the whole plane and decompose it in untwisted maps as explained in Section 3. In particular we get a natural identity isotopy I such that $\mathrm{rot}(I_{|\mathbb{S}})=\alpha$, where $\alpha\in\mathbb{R}\smallsetminus\mathbb{Q}$ and we want to prove that $\widetilde{\mathrm{Cal}}(I)=\pi^2\alpha$. We use the results of Section 3, keeping the same notations. If $(a,b)\in\mathbb{Z}\times\mathbb{N}\smallsetminus\{0\}$, satisfies $a\in(b\alpha,b\beta)$, then $f_{|D_{a/b}}$ is a piecewise diffeomorphism of class C^1. It is easy to see that $\widetilde{\mathrm{Cal}}(I_{|D_{a/b}})$ is well-defined, if one refers to the third definition given in the introduction. It can be explicitly computed, we have

$$\widetilde{\mathrm{Cal}}(I_{|D_{a/b}}) = \widetilde{\mathrm{Cal}}(I) + 2\int_1^{1+a/b-\alpha} (\pi r^2)(\alpha - r + 1)2\pi r\, dr$$
$$= \widetilde{\mathrm{Cal}}(I) + 4\pi^2 \int_1^{1+a/b-\alpha} r^3(\alpha - r + 1)dr.$$

Of course it holds that

$$\lim_{a/b\to\alpha} \widetilde{\mathrm{Cal}}(I_{|D_{a/b}}) = \widetilde{\mathrm{Cal}}(I_{|\mathbb{D}}).$$

We have constructed a radial foliation $\mathcal{F}$ on $\Delta_a^* \subset E_b$. We define $\mathcal{F}_1 = q_1(\mathcal{F})$ and denote $\widetilde{\mathcal{F}}_1$ the lift of $\mathcal{F}_1$ to $\widetilde{D}_{a/b}$. We define $W_{a/b} = \{(z, z') \in D^*_{a/b} \mid z \neq z'\}$. Note that the area of $D_{a/b}$ is

$$A(a, b) = \pi(1 + a/b - \alpha)^2$$

and recall that the difference of the value of $\boldsymbol{h}$ between the points of Σ_a and $\{0\}$ is

$$C(a, b) = \pi(a - b\alpha)\Big(1 + (a/b - \alpha) + \frac{(a/b - \alpha)^2}{3}\Big).$$

Lemma 4.1. — *The following bound holds*

$$\mu_\omega \times \mu_\omega \Big(\{(z, z') \in W_{a/b} \mid \overline{\widehat{\tau}}(z, z', \mathcal{F}_1, f^{-b}(\mathcal{F}_1)) \neq 0\}\Big) \leqslant 2\, A(a, b)\, C(a, b).$$

Proof. — Fix $z \in D^*_{a/b}$ and choose a lift $\widetilde{z} \in \widetilde{D}_{a,b}$. The set

$$O(\widetilde{z}) = \overline{R(\widetilde{\check{f}}_{2mq}{}^{-1}(\widetilde{\phi}_{\widetilde{z}})) \cap L(\widetilde{\phi}_{\widetilde{z}})}$$

has measure $C(a, b)$ (for the lifted measure $\widetilde{\mu}_\omega$) and projects onto a subset $O(z) \subset D_{a/b}$ satisfying $\mu_\omega(O(z)) \leqslant C(a, b)$. Moreover, we have

$$\widetilde{z}' \in \widetilde{O}(\widetilde{z}) \iff \widetilde{\check{I}}(\widetilde{z}') \cap \widetilde{\phi}_{\widetilde{z}} \neq \varnothing, \quad z' \in O(z) \iff \check{I}(z') \cap \phi_z \neq \varnothing.$$

Note that $\widetilde{\check{I}}(\widetilde{z})$ and $\widetilde{\check{I}}(\widetilde{z}')$ meet a common leaf if and only if $\widetilde{z}' \in \widetilde{O}(\widetilde{z})$ or $\widetilde{z} \in \widetilde{O}(\widetilde{z}')$ and that $\check{I}(z)$ and $\check{I}(z')$ meet a common leaf if and only if $z' \in O(z)$ or $z \in O(z')$. Obviously, $I(z)$ and $I(z')$ meet a common leaf if $\overline{\widehat{\tau}}(z, z', \mathcal{F}_1, f^{-b}(\mathcal{F}_1)) \neq 0$ and so, it holds that

$$\begin{aligned}\mu_\omega \times \mu_\omega\big(\{(z, z') \in W_{a/b} \mid \widehat{\tau}(z, z', \mathcal{F}_1, f^{-b}(\mathcal{F}_1)) \neq 0\}\big)&\\ \leqslant \int_{D^*_{a/b}} \mu_\omega(O(z))\, d\mu_\omega(z) + \int_{D^*_{a/b}} \mu_\omega(O(z'))\, d\mu_\omega(z')&\\ \leqslant 2A(a, b)\, C(a, b).&\end{aligned}$$

□

We can be more precise:

Lemma 4.2. — *The following bound holds*

$$\int_{W_{a/b}} \overline{\widehat{\tau}}(z, z', \mathcal{F}_1, f^{-b}(\mathcal{F}_1))\, d\mu_\omega(z)\, d\mu_\omega(z') \leqslant 8mb\, A(a, b)\, C(a, b).$$

Proof. — We keep the same notations as in Lemma 4.1. For every $(z,z')\in W_{a/b}$ define

$$\nu(z,z')=\#\{k\in\mathbb{Z}\mid \overline{\widehat{\tau}}(\widetilde{z},T^{-k}(\widetilde{z}'),\,\mathcal{F}_1,f^{-b}(\mathcal{F}_1))\neq 0\},$$

where $\widetilde{z},\widetilde{z}'$ are given lifts of z,z'. Now, for every $z\in D^*_{a/b}$, define a function $\gamma_z: D^*_{a/b}\to\mathbb{N}$ assigning to every point $z'\in D^*_{a/b}$ the number of its lifts that are in $O(\widetilde{z})$, where $\widetilde{z}$ is a given lift of z. Suppose that $\gamma_z(z')\neq 0$, or equivalently that $z'\in O(z)$. Then, for every $\ell\in\{1,\dots,\gamma_z(z')\}$ there exists a unique lift $\widetilde{z}'\in\widetilde{O}(\widetilde{z})$ of z' such that $\widetilde{I}(\widetilde{z}')$ meets the leaves $T^r(\widetilde{\phi}_{\widetilde{z}})$, $0\leqslant r<\ell$. Note also that

$$\int_{D^*_{a/b}}\gamma_z(z')\,d\mu_\omega(z')=\widetilde{\mu}_\omega(\widetilde{O}(\widetilde{z}))=C(a,b)$$

and that

$$\nu(z,z')\leqslant\gamma_z(z')+\gamma_{z'}(z).$$

To conclude, we refer to Corollary 3.13 that says that for every $(\widetilde{z},\widetilde{z}')\in W_{a/b}$, it holds that $|\widehat{\tau}(\widetilde{z},\widetilde{z}',\mathcal{F}_1,f^{-b}(\mathcal{F}_1))|\leqslant 4mb$. □

Note also that for every pair (a,b) defined as above, it holds that

$$\widetilde{\check{f}}_{2mb}=(\widetilde{f}_{|\widetilde{D}_{a/b}})^b-T^a$$

(or equivalently that $[\check{I}]=[I]^b_{|D_{a/b}}[T_{-a}]_{|D_{a/b}}$). Indeed both maps lift f^b and coincide on $S_{a/b}$. If ϕ is a leaf of $\mathcal{F}_1$, then $m_{\widetilde{\check{f}}_{2mb},\phi}$ is non positive and

$$\int_{D^*_{a/b}}m_{\widetilde{\check{f}}_{2mb},\phi}\,d\mu_\omega=-C(a,b).$$

We deduce that μ_ω has a rotation vector (which was obviously known) and that

$$\operatorname{rot}_{\widetilde{\check{f}}_{2mb}}(\mu_\omega)=\int_{D^*_{a/b}}m_{\widetilde{\check{f}}_{2mb},\phi}\,d\mu_\omega=-C(a,b).$$

We deduce from Lemma 4.2 that $\lambda_{f^b,\mathcal{F}_1}$ is integrable on $W_{a/b}$ for the measure $\mu_\omega\times\mu_\omega$ and that

$$\left|\int_{W_{a,b}}\lambda_{f^b,\mathcal{F}_1}\,d\mu_\omega(z)\,d\mu_\omega(z')\right|\leqslant 8mb\,A(a,b)\,C(a,b).$$

According to the results of Section 2, we deduce that μ_ω has a self linking number for $\widetilde{\check{f}}_{2mb}$ and that

$$\left|\operatorname{link}_{\widetilde{\check{f}}_{2mb}}(\mu_\omega)-\int_{D^*_{a/b}}\operatorname{rot}_{\widetilde{\check{f}}_{2mb}}(z)\,d\mu_\omega(z)\,d\mu_\omega(z')\right|\leqslant 8mb\,A(a,b)\,C(a,b).$$

But we know that $\widetilde{\check{f}}_{2mb}=\check{f}^b\circ T^{-a}$ and so we obtain

$$\left|b\operatorname{link}_{\widetilde{f}_{|\widetilde{D}_{a/b}}}(\mu_\omega)-aA(a,b)^2-A(a,b)\int_{D^*_{a/b}}\operatorname{rot}_{\widetilde{\check{f}}_{2mq}}(z)\,d\mu_\omega(z)\right|\leqslant 8mb\,A(a,b)\,C(a,b),$$

which can be written

$$\left|\operatorname{link}_{\widetilde{f}_{|\widetilde{D}_{a/b}}}(\mu_\omega)-\frac{a}{b}A(a,b)^2+\frac{A(a,b)\,C(a,b)}{b}\right|\leqslant 8mA(a,b)C(a,b).$$

One can find a sequence $(a_n,b_n)_{n\geqslant 0}$ such that $\lim_{n\to+\infty}b_n\alpha-a_n=0$. Writing

$$\left|\operatorname{link}_{\widetilde{f}_{|\widetilde{D}_{a_n/b_n}}}(\mu_\omega)-\frac{a_n}{b_n}A(a_n,b_n)^2+\frac{A(a_n,b_n)\,C(a_n,b_n)}{b}\right|\leqslant 8mA^2(a_n,b_n)\,C(a_n,b_n)$$

and letting n tend to $+\infty$, one obtains

$$\mathrm{link}_{\widetilde{f}_{|\widetilde{\mathbb{D}}}}(\mu_\omega) - \pi^2\alpha = 0.$$

But we know that $\mathrm{link}_{\widetilde{f}_{|\widetilde{\mathbb{D}}}}(\mu_\omega) = \widetilde{\mathrm{Cal}}(I)$, so we can conclude. □

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Manuscript received 15th July 2022
accepted 6th March 2023

Patrice Le Calvez, Sorbonne Université, Université Paris-Cité, CNRS, IMJ-PRG
F-75005, Paris, France
& Institut Universitaire de France
E-mail : patrice.le-calvez@imj-prg.fr
Url : https://webusers.imj-prg.fr/~patrice.le-calvez/