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POTENTIALLY SEMI-STABLE DEFORMATION RINGS FOR DISCRETE SERIES EXTENDED TYPES

by Sandra Rozensztajn

Abstract. — We define deformation rings for potentially semi-stable deformations of fixed discrete series extended type in dimension 2. In the case of representations of the Galois group of $\mathbb{Q}_p$, we prove an analogue of the Breuil-Mézard conjecture for these rings. As an application, we give some results on the existence of congruences modulo p for newforms in $S_k(\Gamma_0(p))$.

Résumé (Anneaux de déformations potentiellement semi-stables pour les types étendus de la série discrète)

Nous définissons des anneaux de déformations pour les déformations potentiellement semi-stables ayant un type étendu de la série discrète fixé en dimension 2. Dans le cas des représentations du groupe de Galois de $\mathbb{Q}_p$, nous prouvons un analogue de la conjecture de Breuil-Mézard pour ces anneaux. Nous donnons comme application de ceci des résultats sur l'existence de congruences modulo p pour les formes nouvelles dans $S_k(\Gamma_0(p))$.

Contents

1. Introduction

Let $p > 2$ be a prime number, K a finite extension of $\mathbb{Q}_p$ with absolute Galois group G_K. Let $\overline{\rho}$ be a continuous representation of G_K of dimension 2 with coefficients in some finite field $\mathbb{F}$ of characteristic p. Let E be a finite extension of $\mathbb{Q}_p$ with residue field containing $\mathbb{F}$. There exists an $\mathscr{O}_E$-algebra $R^{\square}(\overline{\rho})$ parametrizing the framed deformations of $\overline{\rho}$ to $\mathscr{O}_E$-algebras. Kisin ([Kis08]) has shown that this ring has

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quotients $R^{\square,\psi}(w,\mathtt{t},\overline{\rho})$ that parametrize framed deformations ρ that are potentially semi-stable of given determinant (encoded by ψ), fixed Hodge-Tate weights (encoded by the Hodge-Tate type w) and fixed inertial type $\mathtt{t}$ (that is, the restriction to inertia of the Weil-Deligne representation $\mathrm{WD}(\rho)$ associated to ρ is isomorphic to a fixed smooth representation $\mathtt{t}$). We are interested in a variant of this situation: instead of considering deformations with a fixed inertial type $\mathtt{t}$, we consider deformations with a fixed extended type $\mathtt{t}'$, that is, such that the restriction to the Weil group of $\mathrm{WD}(\rho)$ is isomorphic to $\mathtt{t}'$, in the case when $\mathtt{t}'$ is a discrete series type (see the definition in Section 2.1). This problem was first considered in [BCDT01], in order to isolate some irreducible components of the deformation space parametrizing deformations with fixed inertial type. For a discrete series inertial type $\mathtt{t}$, we show that the ring $R^{\square,\psi}(w,\mathtt{t}',\overline{\rho})$ parametrizing deformations with fixed discrete series extended type $\mathtt{t}'$ extending $\mathtt{t}$ is the maximal reduced quotient of $R^{\square,\psi}(w,\mathtt{t},\overline{\rho})$ supported in some set of irreducible components of $\operatorname{Spec} R^{\square,\psi}(w,\mathtt{t},\overline{\rho})$. More precisely, depending on $\mathtt{t}$, adding the extra data of a $\mathtt{t}'$ either does not give any additional information, or divides the set of irreducible components in two parts.

Some important information about the geometry of the rings $R^{\square,\psi}(w,\mathtt{t},\overline{\rho})$ is given by the Breuil-Mézard conjecture ([BM02], proved for $K=\mathbb{Q}_p$ by Kisin [Kis09a] and Paškūnas [Paš15]) that relates the Hilbert-Samuel multiplicity of the special fiber of the ring to an automorphic multiplicity, computed in terms of smooth representations modulo p of $\mathrm{GL}_2(\mathscr{O}_K)$ attached to w and $\mathtt{t}$. Our main result is that when $K=\mathbb{Q}_p$ there is a similar formula for the Hilbert-Samuel multiplicity of the special fiber of $R^{\square,\psi}(w,\mathtt{t}',\overline{\rho})$ for a discrete series extended type $\mathtt{t}'$. More precisely, Gee and Geraghty have shown in [GG15] that for discrete series inertial types $\mathtt{t}$, the Breuil-Mézard conjecture can be reformulated using an automorphic multiplicity expressed in terms of representations not of $\mathrm{GL}_2(\mathscr{O}_K)$, but of $\mathscr{O}_D^\times$, where $\mathscr{O}_D$ is the ring of integers of the non-split quaternion algebra D over K. The formula we give for the multiplicity of the special fiber of $R^{\square,\psi}(w,\mathtt{t}',\overline{\rho})$ is in terms of representations of a quotient $\mathscr{G}$ of $D^\times$ containing $\mathscr{O}_D^\times$ as a subgroup of index 2. Using the local Langlands correspondence and the Jacquet-Langlands correspondence, we construct for each discrete series inertial type $\mathtt{t}$ a smooth representation $\sigma_{\mathscr{G}}(\mathtt{t})$ of $\mathscr{G}$ with coefficients in $\overline{\mathbb{Q}}_p$ (or a pair of such representations, depending on the inertial type $\mathtt{t}$). To a Hodge-Tate type w we attach a representation σ_w of $\mathscr{G}$ coming from an algebraic representation of GL_2 with highest weight given by w. The Hilbert-Samuel multiplicity is then given in terms of the multiplicity of the irreducible constituents of the reduction modulo p of $\sigma_{\mathscr{G}}(\mathtt{t})\otimes\sigma_w$, seen as representations of a finite group Γ through which all semi-simple representations modulo p of $\mathscr{G}$ factor. For $K=\mathbb{Q}_p$ we have the following theorem (see Theorem 3.5.1 for a more precise statement).

Theorem. — *Let $\overline{\rho}$ be a continuous representation of $G_{\mathbb{Q}_p}$ of dimension 2 with coefficients in $\overline{\mathbb{F}}_p$. There exists a positive linear form $\mu_{\overline{\rho}}$ on the Grothendieck ring of representations of Γ with values in $\mathbb{Z}$ satisfying the following property: for any discrete series inertial type $\mathtt{t}$, Hodge-Tate type w, character ψ lifting $\omega^{-1}\det\overline{\rho}$, and extended* *type* $\mathtt{t}'$ *compatible with* $(\mathtt{t},\psi)$*, there exists a choice of representation* $\sigma_{\mathscr{G}}(\mathtt{t})$ *of* $\mathscr{G}$ *such that we have:*

$$e(R^{\square,\psi}(w,\mathtt{t}',\overline{\rho})/\pi)=\mu_{\overline{\rho}}([\overline{\sigma_{\mathscr{G}}}(\mathtt{t})\otimes\overline{\sigma_w}]).$$

We deduce our result from the reformulation by [GG15] of the usual Breuil-Mézard conjecture, making use of modularity lifting theorems for modular forms on a quaternion algebra ramified at infinity and at primes dividing p.

One consequence of this formula is Corollary 3.5.9: except when $\overline{\rho}$ has some very specific form, then when the addition of the data of the extended types divides the deformation ring in two parts, these parts have the same multiplicity. This is to be expected when $\overline{\rho}$ is irreducible, as in this case it can easily be seen that the deformation rings corresponding to the two extended types are in fact isomorphic (see Remark 2.3.5). But this is much more surprising when $\overline{\rho}$ is reducible, as in this case there does not seem to be a natural way to relate the deformation rings corresponding to the two extended types.

We give a concrete application of our result to the existence of congruences modulo p for some modular forms. When $\mathtt{t}$ is trivial, the ring $R^{\square,\psi}(w,\mathtt{t},\overline{\rho})$ classifies semi-stable representations, and the extra data given by the extended type is the eigenvalues of the Frobenius of the associated filtered (ϕ,N)-module when the representation is not crystalline (there are only two possibilities for these eigenvalues if the determinant is fixed). If $f\in S_k(\Gamma_0(p))$ is a newform, this means that the extended type of $\rho_{f,p}|_{G_{\mathbb{Q}_p}}$ gives the value of the coefficient $a_p(f)=\pm p^{k/2-1}$. We give in Theorem 6.2.1 a criterion for the existence of a newform in $S_k(\Gamma_0(p))$ that is congruent to f modulo p but with the opposite value for a_p.

1.1. Plan of the article. — We define the deformation rings $R^{\square,\psi}(w,\mathtt{t}',\overline{\rho})$ for discrete series extended types in Section 2. In Section 3 we introduce the groups and representations that play a role in the automorphic side for the formula for the Hilbert-Samuel multiplicity of the special fiber of the rings $R^{\square,\psi}(w,\mathtt{t}',\overline{\rho})$ and state our main theorems. We give in Section 4 some results about modular forms for quaternion algebras ramified at infinity and at primes dividing p that we need in Section 5, where we prove the theorems. Section 6 is devoted to the application to modular forms.

1.2. Notation. — We fix a prime number $p>2$. We denote by K a finite extension of $\mathbb{Q}_p$, and by q the cardinality of its residue field. Let G_K be the absolute Galois group of K, I_K its inertia subgroup and W_K its Weil group. We denote by ε the p-adic cyclotomic character and ω its reduction modulo p. We normalize the Artin map of local class field theory $\mathrm{Art}_K:K^\times\to W_K^{ab}$ so that geometric Frobenius elements correspond to uniformizers. We denote by $\mathrm{unr}(a)$ the unramified character of W_K (or G_K) sending a geometric Frobenius to a, and also the unramified character of $K^\times$ sending a uniformizer to a. We denote by $\|\cdot\|$ the norm on W_K, that is, the character $\mathrm{unr}(q^{-1})$.

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2. Discrete series extended types and deformation rings

2.1. Extended types and Weil-Deligne representations. — An inertial type is a smooth representation $\mathtt{t}$ of I_K over $\overline{\mathbb{Q}}_p$ that extends to a representation of W_K. We define an extended type to be a smooth representation of W_K over $\overline{\mathbb{Q}}_p$.

We recall the following well-known classification for inertial types and extended types in dimension 2 when $p>2$ (see for example [Ima11, Lem. 2.1] for a proof).

Lemma 2.1.1. — *Let $\mathtt{t}$ be an extended type of degree 2. Then we are in exactly one of the following situations:*

(scal) *$\mathtt{t}|_{I_K}$ is scalar: there exist two smooth characters χ,χ' of W_K such that $\chi|_{I_K}=\chi'|_{I_K}$ and $\mathtt{t}=\chi\oplus\chi'$.*

(char) *There exist two smooth characters χ_1, χ_2 of W_K with distinct restrictions to I_K such that $\mathtt{t}=\chi_1\oplus\chi_2$.*

(red) *Let K' be the unramified quadratic extension of K. There exists a smooth character χ of $W_{K'}$ that does not extend to a character of W_K such that such that $\mathtt{t}=\operatorname{ind}_{W_{K'}}^{W_K}\chi$. In this case $\mathtt{t}|_{I_K}$ is reducible and is a sum of characters that do not extend to W_K.*

(irr) *There exist a ramified quadratic extension L of K and a smooth character χ of W_L that does not extend to a character of W_K such that $\mathtt{t}=\operatorname{ind}_{W_L}^{W_K}\chi$. In this case $\mathtt{t}|_{I_K}$ is irreducible.*

We call the inertial types corresponding to situation (scal), (red) or (irr) discrete series inertial types. We call the extended types corresponding to situation (red) or (irr), or to situation (scal) with $\chi'=\chi\otimes\|\cdot\|^{\pm1}$ discrete series extended types.

The following Proposition is an immediate consequence of the classification.

Proposition 2.1.2. — *Let $\mathtt{t}_1$ and $\mathtt{t}_2$ be two discrete series extended types with isomorphic restrictions to I_K. Then they differ by a twist by an unramified character.*

Let $\mathtt{t}$ be a discrete series extended type. If it is of the form (scal) *or* (irr) *then $\mathtt{t}$ is not isomorphic to $\mathtt{t}\otimes\operatorname{unr}(-1)$. If $\mathtt{t}$ is of the form* (red) *then $\mathtt{t}$ is isomorphic to $\mathtt{t}\otimes\operatorname{unr}(-1)$.*

Let $\mathtt{t}$ be a discrete series extended type. We call conjugate type of $\mathtt{t}$ the type $\mathtt{t}\otimes\operatorname{unr}(-1)$. Two types with isomorphic restriction to I_K are conjugate if and only if they have the same determinant. When $\mathtt{t}$ is of the form (scal) or (irr), two conjugate extended types are distinct, but they are isomorphic when $\mathtt{t}$ is of the form (red).

Let (r,N) be a Weil-Deligne representation of dimension 2, that is, a two-dimensional smooth representation r of the Weil group W_K and a nilpotent endomorphism N such that $Nr(x)=\|x\|^{-1}r(x)N$ for any $x\in W_K$. Let $\mathtt{t}$ be an inertial type; we say that (r,N) is of inertial type $\mathtt{t}$ if $r|_{I_K}$ is isomorphic to $\mathtt{t}$. Let $\mathtt{t}'$ be an extended type; we say that (r,N) is of extended type $\mathtt{t}'$ if r is isomorphic to $\mathtt{t}'$.

We say that (r,N) is a discrete series Weil-Deligne representation if either $r|_{I_K}$ is of the form (scal) and $N\neq 0$ or $r|_{I_K}$ is of the form (red) or (irr) (note that we can have $N\neq 0$ only when $r|_{I_K}$ is of the form (scal) and r is a twist of $1\oplus\|\cdot\|$). With this definition, discrete series inertial (resp. extended) types are exactly the restriction to I_K (resp. W_K) of discrete series Weil-Deligne representations. See Section 3.1 for a justification of this terminology.

2.2. Potentially semi-stable representations and discrete series extended types

2.2.1. *Filtered (ϕ,N)-modules with descent data*. — Let F be a finite extension of $\mathbb{Q}_p$, F_0 the maximal unramified extension of $\mathbb{Q}_p$ contained in F. Let E be a finite extension of $\mathbb{Q}_p$ (the coefficient field), that we suppose large enough.

A filtered (ϕ,N,F,E)-module is a free $F_0\otimes_{\mathbb{Q}_p}E$-module D of finite rank, endowed with a F_0-semi-linear, E-linear endomorphism ϕ and a $F_0\otimes E$ linear endomorphism N satisfying the commutation relation $N\phi=p\phi N$, with N nilpotent, ϕ an automorphism, and a decreasing filtration of $F\otimes_{F_0}D$ by $F\otimes_{\mathbb{Q}_p}E$-submodules $\mathrm{Fil}^i(F\otimes_{F_0}D)$ such that $\mathrm{Fil}^i(F\otimes_{F_0}D)=F\otimes_{F_0}D$ when i is small enough and $\mathrm{Fil}^i(F\otimes_{F_0}D)=0$ when i is large enough. We can define an admissibility condition for filtered (ϕ,N,F,E)-modules, we refer to [Fon94b] for the definition.

Let $\rho:G_F\to\mathrm{GL}(V)$ be a continuous representation, where V is a finite-dimensional E-vector space. If ρ is semi-stable, we can attach to it an admissible filtered (ϕ,N,F,E)-module by taking $D_{\mathrm{st}}(V)=(B_{\mathrm{st}}\otimes_{\mathbb{Q}_p}V)^{G_F}$. The functor $V\mapsto D_{\mathrm{st}}(V)$ gives an equivalence of categories between the category of semi-stable representations of G_F and the category of admissible filtered (ϕ,N,F,E)-modules which preserves dimension, and the Hodge-Tate weights of ρ are the indices i with $\mathrm{Fil}^{-i}(F\otimes_{F_0}D)\neq\mathrm{Fil}^{-i+1}(F\otimes_{F_0}D)$ (so that ε has its Hodge-Tate weights equal to 1).

Suppose now that we have $\rho:G_K\to\mathrm{GL}(V)$ a continuous representation such that ρ becomes semi-stable on a finite Galois extension F of K. Then we can attach to it an admissible filtered $(\phi,N,F/K,E)$-module, that is, an admissible filtered (ϕ,N,F,E)-module with descent data given by an action of $\mathrm{Gal}(F/K)$ which is F_0-semi-linear and E-linear and commutes with ϕ and N. The filtered (ϕ,N,F,E)-module is $D_{\mathrm{st}}^F(V)$, that is, $D_{\mathrm{st}}(V|_{G_F})$. This gives an equivalence of categories between the category of representations of G_K that become semi-stable over F and the category of admissible filtered $(\phi,N,F/K,E)$-modules.

2.2.2. *Weil-Deligne representation attached to a Galois representation*

Let $\rho:G_K\to\mathrm{GL}(V)$ be a continuous representation of G_K, where V is a finite-dimensional vector space over a finite extension E of $\mathbb{Q}_p$. If ρ is potentially semi-stable, we attach to its filtered $(\phi,N,F/K,E)$-module a Weil-Deligne representation $WD(\rho)$ as in [Fon94a] (see also [CDT99, App. B] for more detailed explanations of the construction and its properties). It does not depend on the field F over which K

becomes semi-stable, and moreover it does not depend on the filtration but only on ϕ, N and the action of $\mathrm{Gal}(F/K)$.

We say that ρ is of inertial type $\mathtt{t}$ if $\mathrm{WD}(\rho)$ is of type $\mathtt{t}$, and of extended type $\mathtt{t}'$ if $\mathrm{WD}(\rho)$ is. Note that $\mathrm{WD}(\rho)$ is of scalar inertial type if and only if ρ is semi-stable up to twist, and in this case $N \neq 0$ if and only if ρ is semi-stable but not crystalline up to twist.

2.3. Deformation rings. — In this section we fix a discrete series inertial type $\mathtt{t}$. Note that the notions of this Section will be interesting only when $\mathtt{t}$ is of the form (scal) or (irr) as we explain later.

Let $\overline{\rho}$ be a continuous representation of G_K of dimension 2 over a finite field $\mathbb{F}$ of characteristic p. Let E be a finite extension of $\mathbb{Q}_p$ with residue field containing $\mathbb{F}$. We denote by $R^{\square}(\overline{\rho})$ the universal framed deformation $\mathscr{O}_E$-algebra of $\overline{\rho}$.

2.3.1. *Deformation rings of fixed inertial type*

Let $w = (n_\tau, m_\tau) \in (\mathbb{Z}_{\geqslant 0} \times \mathbb{Z})^{\mathrm{Hom}(K,\overline{\mathbb{Q}}_p)}$ be a Hodge-Tate type, $\mathtt{t}$ be an inertial type, and ψ a character of G_K. We are interested in lifts ρ of $\overline{\rho}$ that are potentially semi-stable, with determinant $\psi\varepsilon$, Hodge-Tate weights $(m_\tau, m_\tau + n_\tau + 1)_\tau$ (we then say that ρ has Hodge-Tate type w), and inertial type $\mathtt{t}$.

In [Kis08], Kisin shows that, after possibly enlarging E, there exists a quotient $R^{\square,\psi}(w,\mathtt{t},\overline{\rho})$ of $R^{\square}(\overline{\rho})$ that has the following properties.

Theorem 2.3.1

(1) *$R^{\square,\psi}(w,\mathtt{t},\overline{\rho})$ is p-torsion free, $R^{\square,\psi}(w,\mathtt{t},\overline{\rho})[1/p]$ is reduced and equidimensional.*

(2) *for any finite extension E'/E, a map $x : R^{\square}(\overline{\rho}) \to E'$ factors through $R^{\square,\psi}(w,\mathtt{t},\overline{\rho})$ if and only if the representation ρ_x is of determinant $\psi\varepsilon$, potentially semi-stable of Hodge-Tate type w, and of inertial type $\mathtt{t}$.*

Remark 2.3.2. — The ring $R^{\square,\psi}(w,\mathtt{t},\overline{\rho})$ can be non-zero only if w, ψ and $\mathtt{t}$ satisfy the equality: $\psi|_{I_K} = (\det \mathtt{t}) \prod_\tau \varepsilon_\tau^{n_\tau + 2m_\tau}$, where ε_τ is the Lubin-Tate character corresponding to τ, so that $\varepsilon = \prod_\tau \varepsilon_\tau$. In this case we say that ψ is compatible with $\mathtt{t}$ and w. Note that by [EG14, Lem. 4.3.1] the isomorphism class of $R^{\square,\psi}(w,\mathtt{t},\overline{\rho})$ does not depend on ψ as long as it is compatible with $\mathtt{t}$ and w.

2.3.2. *Irreducible components and extended types.* — Suppose that $\mathtt{t}$ is a discrete series inertial type, and let $\mathtt{t}'$ be an extended type such that $\mathtt{t}'|_{I_K}$ is isomorphic to $\mathtt{t}$. We define a subset of the set of irreducible components of $\operatorname{Spec} R^{\square,\psi}(w,\mathtt{t},\overline{\rho})$ by saying that an irreducible component is of type $\mathtt{t}'$ if:

(1) when $\mathtt{t}$ is of the form (red) or (irr), the irreducible component has an E'-point x with ρ_x of extended type $\mathtt{t}'$ for some finite extension E'/E;

(2) when $\mathtt{t}$ is of the form (scal), the irreducible component has an E'-point x with ρ_x of extended type $\mathtt{t}'$ that is not potentially crystalline.

There can exist a component of type $\mathtt{t}'$ only if $\det \mathtt{t}' = \mathrm{WD}(\psi\varepsilon)$, hence there are at most two such extended types and then they are conjugate if $\mathtt{t}$ is of the form (scal) or (irr), and at most one such extended type if $\mathtt{t}$ is of the form (red). We say that $\mathtt{t}'$ is compatible with $(\mathtt{t},\psi)$ if $\mathtt{t}'|_{I_K}$ is isomorphic to $\mathtt{t}$ and $\det \mathtt{t}' = \mathrm{WD}(\psi\varepsilon)$. If $\mathtt{t}$ is of the form (red) and $\mathtt{t}'$ is compatible to $(\mathtt{t},\psi)$ then all irreducible components of $\operatorname{Spec} R^{\square,\psi}(w,\mathtt{t},\overline{\rho})$ are of type $\mathtt{t}'$.

If a component is of type $\mathtt{t}'$, then for all closed points x over a finite extension E'/E, the representation ρ_x is of type $\mathtt{t}'$. In particular a component is of at most one extended type. This follows from the next proposition.

Proposition 2.3.3. — *Let $\mathscr{A}$ be an affinoid algebra which is a domain. Let $\rho : G_K \to \mathrm{GL}_2(\mathscr{A})$ be a continuous $\mathscr{A}$-linear representation such that for any closed point $x \in \mathrm{Max}(\mathscr{A})$, the representation ρ_x is potentially semi-stable, with the same discrete series inertial type $\mathtt{t}$, Hodge-Tate weights and determinant for all x. If $\mathtt{t}$ is scalar and at least one representation in the family is not potentially crystalline, or if $\mathtt{t}$ is not scalar, then the extended type is constant in the family.*

Proof. — Let F be a finite extension of K such that $\rho_x|_{G_F}$ is semi-stable for all x. Such an F exists and is determined by $\mathtt{t}$. Using the results of [BC08, §6.3] we can then construct a free $F_0\otimes\mathscr{A}$-module $D_{\mathrm{st}}(\rho)$ with a Frobenius ϕ, a monodromy operator N and an action of $\mathrm{Gal}(F/K)$ that are $\mathscr{A}$-linear and such that for all x, $\mathscr{A}/\mathfrak{m}_x\otimes_{\mathscr{A}} D_{\mathrm{st}}(\rho)$ is isomorphic to $D^F_{\mathrm{st}}(\rho_x)$. We can apply the method of the construction of the Weil-Deligne representation as given in [CDT99, App. B] to $D_{\mathrm{st}}(\rho)$. This gives a continuous representation $r : W_K \to \mathrm{GL}_2(\mathscr{A})$ and $N \in M_2(\mathscr{A})$ such that for all x, (r_x, N_x) is the Weil-Deligne representation attached to ρ_x. By assumption all representations r_x have the same restriction to inertia and the same determinant. If $\mathtt{t}$ is of the form (red), this implies that the isomorphism class of r_x is constant and hence the extended type is constant in the family.

Suppose now that $\mathtt{t}$ is of the form (irr), that is, $\mathtt{t} = (\mathrm{ind}_{I_L}^{I_K}\chi)|_{I_K}$ for some ramified quadratic extension L of K and some character χ of W_L that does not extend to W_K. Then $\mathtt{t}|_{I_L} = \chi\oplus\chi'$ for characters χ,χ' that do not extend to I_K. Fix also an element $\alpha \in I_K \smallsetminus I_L$. We can choose a basis (e_1,e_2) of $\mathscr{A}^2$ (after possibly replacing $\mathrm{Max}(\mathscr{A})$ by some admissible covering) such that for all x, we have $r_x(\beta)e_1 = \chi(\beta)e_1$ and $r_x(\beta)e_2 = \chi'(\beta)e_2$ for all $\beta \in I_L$, and $r_x(\alpha)e_1 = e_2$. Let Frob be any Frobenius element of W_K. Then the matrix in this basis of $r_x(\mathrm{Frob}^2)$ and of $r_x(\beta)$ for any $\beta \in I_K$ is constant, and the matrix of $r_x(\mathrm{Frob})$ can take only two possible values that determine the isomorphism class of r_x. As the matrix varies continuously with x and $\mathscr{A}$ is integral, it is constant, hence the isomorphism class of r_x is constant.

Suppose now that $\mathtt{t}$ is scalar. Let Frob be any Frobenius element of W_K. Then the isomorphism class of r_x is determined by the characteristic polynomial of $r_x(\mathrm{Frob})$. Let U be the Zariski open subset of $\mathrm{Max}(\mathscr{A})$ defined by the condition $N \neq 0$. Then on U the eigenvalues of $r_x(\mathrm{Frob})$ are of the form α_x and $q\alpha_x$ for some α_x. As the determinant of r_x is constant, α_x^2 is constant, hence the characteristic polynomial of $r_x(\mathrm{Frob})$ can take only two possible values on U. As U is Zariski dense in $\mathrm{Max}(\mathscr{A})$ (because $\mathscr{A}$ is a domain), it can only take two possible values on $\mathrm{Max}(\mathscr{A})$, and in fact only one by continuity. Hence the isomorphism class of r_x is constant. $\square$

2.3.3. *Deformation rings of fixed discrete series extended type*. — We define a quotient $R^{\square,\psi}(w,\mathtt{t}',\overline{\rho})$ of $R^{\square,\psi}(w,\mathtt{t},\overline{\rho})$ by taking the maximal reduced quotient supported on the set of irreducible components of $\operatorname{Spec} R^{\square,\psi}(w,\mathtt{t},\overline{\rho})$ that are of type $\mathtt{t}'$. We also define, following [GG15, §5], a ring $R^{\square,\psi}(w,\mathtt{t}^{\mathrm{ds}},\overline{\rho})$ corresponding to all the irreducible components of *some* extended type $\mathtt{t}'$.

If $\mathtt{t}$ is of the form (red), there is exactly one extended type $\mathtt{t}'$ that is compatible with $(\mathtt{t},\psi)$, and we have

$$R^{\square,\psi}(w,\mathtt{t}',\overline{\rho}) = R^{\square,\psi}(w,\mathtt{t},\overline{\rho}) = R^{\square,\psi}(w,\mathtt{t}^{\mathrm{ds}},\overline{\rho}).$$

If $\mathtt{t}$ is of the form (irr), $R^{\square,\psi}(w,\mathtt{t},\overline{\rho}) = R^{\square,\psi}(w,\mathtt{t}^{\mathrm{ds}},\overline{\rho})$, but there are two extended types that are compatible with $(\mathtt{t},\psi)$ so $R^{\square,\psi}(w,\mathtt{t}',\overline{\rho})$ can be different from $R^{\square,\psi}(w,\mathtt{t}^{\mathrm{ds}},\overline{\rho})$.

If $\mathtt{t}$ is scalar, $R^{\square,\psi}(w,\mathtt{t}^{\mathrm{ds}},\overline{\rho})$ is a quotient of $R^{\square,\psi}(w,\mathtt{t},\overline{\rho})$ supported only on the components containing points corresponding to representations that are not potentially crystalline and is generally different from $R^{\square,\psi}(w,\mathtt{t},\overline{\rho})$ (see also [GG15, Lem. 5.5] and the remarks preceding it). If $\mathtt{t}'$ is an extended type compatible with $(\mathtt{t},\psi)$, then $R^{\square,\psi}(w,\mathtt{t}',\overline{\rho})$ is a quotient of $R^{\square,\psi}(w,\mathtt{t}^{\mathrm{ds}},\overline{\rho})$, but it can be different from it as there are two possibilities for $\mathtt{t}'$.

Then we have the following properties.

Proposition 2.3.4

(1) $R^{\square,\psi}(w,\mathtt{t}',\overline{\rho})$ *is p-torsion free,* $R^{\square,\psi}(w,\mathtt{t}',\overline{\rho})[1/p]$ *is reduced and equidimensional.*

(2) *for all finite extensions E'/E, if a map $x : R^{\square}(\overline{\rho}) \to E'$ factors through $R^{\square,\psi}(w,\mathtt{t}',\overline{\rho})$ then the representation ρ_x is of determinant $\psi\varepsilon$, potentially semi-stable of Hodge-Tate type w and of extended type $\mathtt{t}'$.*

(3) *for all finite extensions E'/E, a map $x : R^{\square}(\overline{\rho}) \to E'$ such that the representation ρ_x is of determinant $\psi\varepsilon$, potentially semi-stable of Hodge-Tate type w and of extended type $\mathtt{t}'$ factors through $R^{\square,\psi}(w,\mathtt{t}',\overline{\rho})$.*

Proof. — Properties (1) and (2) follow from the analogous properties for the inertial type $\mathtt{t}$, and Proposition 2.3.3.

In the case not of the form (scal), property (3) follows from the fact that any ρ_x of inertial type $\mathtt{t}$ is of some discrete series extended type $\mathtt{t}'$.

Suppose now that $\mathtt{t}$ is scalar, we can suppose that $\mathtt{t}$ is trivial. Let x be as in (3): the map x factors through $R^{\square,\psi}(w,\mathtt{t},\overline{\rho})$ by Theorem 2.3.1 and it defines a representation $\rho : G_K \to \mathrm{GL}_2(\mathscr{O}_{E'})$ lifting $\overline{\rho}$ for some finite extension E' of E. If ρ is not crystalline, then by definition x factors through $R^{\square,\psi}(w,\mathtt{t}',\overline{\rho})$. Suppose now that ρ is crystalline. Then $\mathrm{WD}(\rho)$ is of the form $(r,0)$ with r isomorphic to $\mathtt{t}'$. As $\mathtt{t}'$ is a discrete series extended type with trivial restriction to inertia, this means that $r = \psi\otimes(1\oplus\|\cdot\|)$ for some unramified character ψ of W_K. In particular, we see that there exists a nonzero map $\mathrm{WD}(\rho) \to \mathrm{WD}(\rho) \otimes \|\cdot\|$. By [All14, Th. D], this means that the point x defining the representation ρ is a non-smooth point on $\operatorname{Spec} R^{\square,\psi}(w, \mathtt{t}, \overline{\rho})[1/p]$. We know from the proof of [Kis09a, Lem. A.3] that the union of the crystalline irreducible components of $\operatorname{Spec} R^{\square,\psi}(w, \mathtt{t}, \overline{\rho})[1/p]$ is smooth. So this means that x is a point of $\operatorname{Spec} R^{\square,\psi}(w, \mathtt{t}, \overline{\rho})[1/p]$ which is also on an irreducible component that contains non-crystalline points, and so is a point on $\operatorname{Spec} R^{\square,\psi}(w, \mathtt{t}', \overline{\rho})[1/p]$, and x factors through $R^{\square,\psi}(w, \mathtt{t}', \overline{\rho})$. $\square$

Remark 2.3.5. — Suppose that $\overline{\rho}$ is irreducible. Then as $\overline{\rho}$ is isomorphic to $\overline{\rho} \otimes \mathrm{unr}(-1)$, the map $\rho \mapsto \rho \otimes \mathrm{unr}(-1)$ induces an involution of $R^{\square,\psi}(w, \mathtt{t}^{\mathrm{ds}}, \overline{\rho})$ that exchanges $R^{\square,\psi}(w, \mathtt{t}', \overline{\rho})$ and $R^{\square,\psi}(w, \mathtt{t}'', \overline{\rho})$, where $\mathtt{t}'$ and $\mathtt{t}''$ are the conjugate extended types compatible with $(\mathtt{t}, \psi)$. In particular $R^{\square,\psi}(w, \mathtt{t}', \overline{\rho})$ and $R^{\square,\psi}(w, \mathtt{t}'', \overline{\rho})$ are isomorphic.

Remark 2.3.6. — For an inertial type of the form (char), the extended type is not constant on irreducible components. In fact it follows from [GM09, §3.2] that, in the case $K = \mathbb{Q}_p$, for an inertial type of this form there exists only a finite number of isomorphism classes of potentially semi-stable representations of given regular Hodge-Tate weights and extended type.

3. Multiplicities

Let D be the non-split quaternion algebra over K. In this section we consider all smooth representations as having coefficients in $\overline{\mathbb{Q}}_p$, unless otherwise specified.

3.1. Local Langlands and Jacquet-Langlands. — We denote by JL the local Jacquet-Langlands correspondence, that attaches to every irreducible smooth admissible representation of $D^\times$ a discrete series smooth representation of $\mathrm{GL}_2(K)$ (that is, supercuspidal or a twist of the Steinberg representation).

We denote by rec_p the local Langlands correspondence that attaches to each irreducible smooth admissible representation of $\mathrm{GL}_2(K)$ a Weil-Deligne representation of degree 2 with the normalization of [HT01, Introd.].

We set $\mathrm{LL}_D(\pi) = (\mathrm{rec}_p \circ \mathrm{JL}(\pi)) \otimes \|\cdot\|^{1/2}$, so that the image of LL_D is exactly the discrete series Weil-Deligne representations (r, N) (see Sections 3.5 and 4.5.1 for a justification of the normalization). We give some properties of LL_D: let ψ a character of $K^\times$, and denote by N_D the reduced norm $D \to K$. For $x \in \overline{\mathbb{Q}}_p^\times$, we denote by $\mathrm{unr}_D(x)$ the character of $D^\times$ given by $\mathrm{unr}(x) \circ \mathrm{N}_D$, and more generally for ψ a character of $K^\times$ we denote by ψ_D the character $\psi \circ \mathrm{N}_D$ of $D^\times$. Then $\mathrm{LL}_D(\psi_D) = (\psi \oplus \psi\|\cdot\|, N)$ with $N \neq 0$.

Let ϖ_D be a uniformizer of D. If $a \geqslant 1$, we set $U_D^a = 1 + \varpi_D^a \mathscr{O}_D$. It is an open compact subgroup of $D^\times$, which does not depend on the choice of ϖ_D. It follows from the explicit description of smooth representations of $D^\times$ (as can be found for example in [BH06, Chap. 13]) that any irreducible smooth representation of $D^\times$ that is not a character has one of the following forms:

(1) $\pi_D = \mathrm{ind}_{L^\times U_D^a}^{D^\times}\psi$ for some character ψ and some ramified quadratic extension L of K.

(2) $\pi_D = \mathrm{ind}_{L^\times U_D^a}^{D^\times}\rho$ for some irreducible representation ρ of $L^\times U_D^a$ of dimension 1 or q and L the unramified quadratic extension of K.

Proposition 3.1.1. — *Let r be a Weil-Deligne representation of dimension 2 that is of the form* (red) *or* (irr) *of Lemma* 2.1.1. *Then the following conditions are equivalent:*

(1) *the type of r is of the form* (red).

(2) $\mathrm{LL}_D^{-1}(r) \simeq \mathrm{LL}_D^{-1}(r) \otimes \mathrm{unr}_D(-1)$.

(3) $\mathrm{LL}_D^{-1}(r) = \mathrm{ind}_{L^\times U_D^a}^{D^\times}\rho$ *for L the unramified quadratic extension of K, some a and some representation ρ of $L^\times U_D^a$.*

(4) *the restriction of* $\mathrm{LL}_D^{-1}(r)$ *to* $\mathscr{O}_D^\times$ *is the sum of two irreducible representations that differ by conjugation by ϖ_D.*

And the following conditions are equivalent:

(1) *the type of r of the form* (irr).

(2) $\mathrm{LL}_D^{-1}(r) \not\simeq \mathrm{LL}_D^{-1}(r) \otimes \mathrm{unr}_D(-1)$.

(3) $\mathrm{LL}_D^{-1}(r) = \mathrm{ind}_{L^\times U_D^a}^{D^\times}\psi$ *for some ramified quadratic extension L of K, some a and some character ψ of $L^\times U_D^a$.*

(4) *the restriction of* $\mathrm{LL}_D^{-1}(r)$ *to* $\mathscr{O}_D^\times$ *is irreducible.*

Proof. — Note first that LL_D is compatible with twists by characters, as this is the case for rec_p and JL.

(1) ⇔ (2) comes from Proposition 2.1.2.

(1) ⇔ (3) comes from the explicit descriptions of the local Langlands and Jacquet-Langlands correspondence (see [BH06]).

(3) ⇔ (4) is [GG15, Prop. 3.8] (see also [Gér78, §§5 & 6]). ☐

3.2. Representations attached to a discrete series inertial type

3.2.1. *Representations of $D^\times$ and $\mathscr{O}_D^\times$*. — Let $\mathtt{t}$ be some discrete series inertial type. Let (r, N) be some discrete series Weil-Deligne representation with $\mathtt{t} = r|_{I_K}$. Let $\pi_{\mathtt{t}} = \mathrm{LL}_D^{-1}(r, N)$, which depends on the choice of (r, N) only up to unramified twist. If (r, N) is of the form (scal) then $\pi_{\mathtt{t}}$ is a character of $D^\times$, so the restriction of $\pi_{\mathtt{t}}$ to $\mathscr{O}_D^\times$ is irreducible. As we have seen in Proposition 3.1.1, if (r, N) is of the form (irr) then $\pi_{\mathtt{t}}$ is still irreducible after restriction to $\mathscr{O}_D^\times$, and if (r, N) is of the form (red) then the restriction of $\pi_{\mathtt{t}}$ to $\mathscr{O}_D^\times$ is the sum of two irreducible constituents that differ by conjugation by ϖ_D. Let $\sigma_D(\mathtt{t})$ be one of the irreducible constituents of the restriction of $\pi_{\mathtt{t}}$ to $\mathscr{O}_D^\times$; it depends only on $\mathtt{t}$ and not on the choice of (r, N) (this is the same as the representation $\sigma_D(\mathtt{t})$ of [GG15, §3.1]). We can recover $\pi_{\mathtt{t}}$ from $\sigma_D(\mathtt{t})$ up to unramified twist. Hence we have the following property.

Proposition 3.2.1. — *Let π_D be a smooth irreducible representation of $D^\times$. Then* $\mathrm{Hom}_{\mathscr{O}_D^\times}(\sigma_D(\mathtt{t}), \pi_D) \neq 0$ *if and only if* $\mathrm{LL}_D(\pi_D)|_{I_K} \simeq \mathtt{t}$.

Remark 3.2.2. — As was already noted in [GG15], contrary to the case of GL_2, we see that the type for $D^\times$ is not unique, at least for representations of the form (red). On the other hand, as there are only one or two irreducible constituents for the restriction to $\mathscr{O}_D^\times$ of a smooth irreducible representation of $D^\times$, it is much easier to find a type.

3.2.2. *The group $\mathscr{G}_{\varpi_K}$.* — Let ϖ_K be a uniformizer of K and ϖ_D a uniformizer of D with $\varpi_D^2 = \varpi_K$. Let $\mathscr{G}_{\varpi_K} = D^\times/\varpi_K^{\mathbb{Z}}$. Then $\mathscr{G}_{\varpi_K}$ is isomorphic to the semi-direct product $\mathscr{O}_D^\times \rtimes_{\varpi_D} \{1,\iota\}$, where the action of ι on $\mathscr{O}_D^\times$ is by conjugation by ϖ_D. As a group, $\mathscr{G}_{\varpi_K}$ depends on ϖ_K, but not on the choice of ϖ_D such that $\varpi_D^2 = \varpi_K$. Let $\xi = \mathrm{unr}_D(-1)$, that is, the character of $\mathscr{G}_{\varpi_K}$ that is trivial on $\mathscr{O}_D^\times$ and sends ι to -1.

Let $\mathtt{t}$ be a discrete series inertial type. Using the representation $\pi_{\mathtt{t}}$ of Section 3.2.1, we can attach to $\mathtt{t}$ a smooth representation $\sigma_{\mathscr{G}}(\mathtt{t})$ of $\mathscr{G}_{\varpi_K}$, or equivalently a smooth representation of $D^\times$ that is trivial on ϖ_K: there is an unramified twist of $\pi_{\mathtt{t}}$ that is trivial on ϖ_K, as $\pi_{\mathtt{t}}(\varpi_K)$ is scalar.

The relation between $\sigma_{\mathscr{G}}(\mathtt{t})$ and $\sigma_D(\mathtt{t})$ is then given by Proposition 3.1.1. If $\mathtt{t}$ is of the form (scal) or (irr) then the representation $\sigma_{\mathscr{G}}(\mathtt{t})$ is defined only up to twist by ξ, and the restriction of $\sigma_{\mathscr{G}}(\mathtt{t})$ to $\mathscr{O}_D^\times$ is isomorphic to $\sigma_D(\mathtt{t})$, which is irreducible. If $\mathtt{t}$ is of the form (red), then there is only one possibility for $\sigma_{\mathscr{G}}(\mathtt{t})$ and the restriction of $\sigma_{\mathscr{G}}(\mathtt{t})$ to $\mathscr{O}_D^\times$ is isomorphic to the direct sum of $\sigma_D(\mathtt{t})$ and the representation $\sigma_D(\mathtt{t})^{\varpi_D}$ obtained from $\sigma_D(\mathtt{t})$ by conjugation by a uniformizer.

Remark 3.2.3. — The representation $\sigma_{\mathscr{G}}(\mathtt{t})$ of $\mathscr{G}_{\varpi_K}$ is the analogue in our situation of the representation $\sigma_{\tau'}$ of $\widetilde{U}_0(\ell)$ in [BCDT01, §1.2].

3.2.3. *Realizations of $\mathscr{G}_{\varpi_K}$.* — Let K' be the unramified quadratic extension of K. By fixing an embedding of K' into D and a basis of D as a K'-vector space, we can define an embedding $D^\times \to \mathrm{GL}_2(K')$, hence, after choosing $K' \to \overline{\mathbb{Q}}_p$, an embedding $u : D^\times \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$. All such embeddings are conjugate in $\mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ by the Skolem-Noether theorem.

Fix now a uniformizer ϖ_K of K, a square root ϖ_D of ϖ_K in D and a square root $\sqrt{\varpi_K}$ of ϖ_K in $\overline{\mathbb{Q}}_p$. With these choices we can define an embedding $\widetilde{u} : \mathscr{G}_{\varpi_K} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ by setting $\widetilde{u}|_{\mathscr{O}_D^\times} = u|_{\mathscr{O}_D^\times}$ and $\widetilde{u}(\iota) = \sqrt{\varpi_K}^{-1} u(\varpi_D)$.

Note that for each choice of ϖ_K and $\sqrt{\varpi_K}$, all the possible $\widetilde{u}$ corresponding to the various choices of u and ϖ_D are conjugate in $\mathrm{GL}_2(\overline{\mathbb{Q}}_p)$. Moreover, for varying choices of ϖ_K all the $\widetilde{u}|_{\mathscr{O}_D^\times}$ are conjugate.

3.3. Representations of Γ_K

3.3.1. *The group Γ_K.* — Let k be the residue field of K and ℓ its quadratic extension, so that $\mathscr{O}_D/\varpi_D \simeq \ell$. We define the group $\Gamma_K = \ell^\times \rtimes \{1,\iota\}$, where ι acts on $\ell^\times$ by the non-trivial k-automorphism of ℓ.

The quotient $\mathscr{G}_{\varpi_K}/(1+\varpi_D\mathscr{O}_D)$ is naturally isomorphic to the group Γ_K, and the map $\mathscr{G}_{\varpi_K} \to \Gamma_K$ extends the natural morphism $\mathscr{O}_D^\times \to \ell^\times$. As $1+\varpi_D\mathscr{O}_D$ is a pro-p-group, any semi-simple representation of $\mathscr{G}_{\varpi_K}$ in characteristic p factors through Γ_K.

3.3.2. *Irreducible representations of Γ_K in characteristic p.* — Let $\overline{\mathbb{F}}$ be an algebraic closure of k. Fix $\ell \to \overline{\mathbb{F}}$, and let q be the cardinality of k.

For an element a in $\mathbb{Z}/(q^2-1)\mathbb{Z}$, we denote by χ_a the character of $\ell^\times$ sending x to x^a.

For an element a in $\mathbb{Z}/(q-1)\mathbb{Z}$, we denote by δ_a the character of Γ_K such that δ_a coincides with $\chi_{a(q+1)} = \chi_a|_{k^\times} \circ N_{\ell/k}$ on $\ell^\times$ and $\delta_a(\iota) = 1$.

Let ξ be the character of Γ_K that is trivial on $\ell^\times$ and $\xi(\iota) = -1$.

Let $r_a = \operatorname{ind}_{\ell^\times}^{\Gamma_K} \chi_a$ for $a \in \mathbb{Z}/(q^2-1)\mathbb{Z}$. Then r_a is irreducible if and only if a is not divisible by $q+1$. If $a = (q+1)b$ then r_a is isomorphic to $\delta_b \oplus \xi\delta_b$.

Proposition 3.3.1. — *The irreducible representations of Γ_K with coefficients in $\overline{\mathbb{F}}$ are exactly the following:*

- *The characters δ_a for $a \in \mathbb{Z}/(q-1)\mathbb{Z}$.*
- *The characters $\xi\delta_a$ for $a \in \mathbb{Z}/(q-1)\mathbb{Z}$.*
- *The representations r_a for $a \in \mathbb{Z}/(q^2-1)\mathbb{Z}$ not divisible by $q+1$.*

Moreover, these representations are all distinct, except for the relation $r_a = r_{qa}$. Finally, ξr_a is isomorphic to r_a, and $r_a|_{\ell^\times} = \chi_a \oplus \chi_{qa}$.

The irreducible representations of Γ_K are the analogue in our situation of the Serre weights.

3.3.3. *Reduction modulo p of representations of $\mathscr{G}_{\varpi_K}$ attached to discrete series types*

Let $\sigma_{\mathscr{G}}(\mathtt{t})$ be a representation of $\mathscr{G}_{\varpi_K}$ attached to a discrete series inertial type $\mathtt{t}$ as in Section 3.2.2. As $\mathscr{G}_{\varpi_K}$ is compact, we can find an invariant lattice in $\sigma_{\mathscr{G}}(\mathtt{t})$, and consider the semi-simplification of the reduction modulo p of this representation. We denote by $\overline{\sigma_{\mathscr{G}}}(\mathtt{t})$ the representation of Γ_K that we obtain (it is semi-simple, independent of the choice of the invariant lattice and its restriction to $\ell^\times$ is independent of any choice).

Proposition 3.3.2. — *Each irreducible representation of Γ_K over $\overline{\mathbb{F}}$ has a lift in characteristic 0 that is of the form $\sigma_{\mathscr{G}}(\mathtt{t})$ for some discrete series inertial type $\mathtt{t}$.*

Proof. — Let δ be an irreducible representation of Γ_K of dimension 1. It is of the form $\chi \circ N_{\ell/k}$ or $\xi\chi \circ N_{\ell/k}$ for some character χ of $k^\times$. We define a scalar inertial type $\mathtt{t}_\delta$ by $\mathtt{t}_\delta = (\widetilde{\chi} \oplus \widetilde{\chi})|_{I_K}$, where $\widetilde{\chi}$ denotes the image by local class field theory of the Teichmüller lift of the character $\chi \circ (K^\times \to k^\times)$. Then we can choose $\sigma_{\mathscr{G}}(\mathtt{t}_\delta)$ so that $\overline{\sigma_{\mathscr{G}}}(\mathtt{t}_\delta)$ is isomorphic to δ (note that $\sigma_{\mathscr{G}}(\mathtt{t}_\delta)$ depends on δ and not only on $\mathtt{t}_\delta$).

Let r be an irreducible representation of Γ_K of dimension 2. There exists $a \in \mathbb{Z}/(q^2-1)\mathbb{Z}$ not divisible by $q+1$ such that $r|_{\ell^\times} = \chi_a \oplus \chi_{qa}$. Let K' be the unramified quadratic extension of K, and $\widetilde{\chi}_a : W_{K'} \to \overline{\mathbb{Q}}_p^\times$ the tame character given by the Teichmüller lift of $\chi_a \circ (K'^\times \to \ell^\times)$. We define an inertial type $\mathtt{t}_r$ of the form (red) by $\mathtt{t}_r = (\operatorname{ind}_{W_{K'}}^{W_K} \widetilde{\chi}_a)|_{I_K}$. Then $\overline{\sigma_{\mathscr{G}}}(\mathtt{t}_r)$ is isomorphic to r, as follows from the explicit constructions in [BH06, Chap. 13]. □

We denote by $\mathscr{R}(\Gamma_K)$ the Grothendieck ring of representations of Γ_K with coefficients in $\overline{\mathbb{F}}$. We denote by $[\sigma]$ the image in $\mathscr{R}(\Gamma_K)$ of a representation σ of Γ_K.

We now compute the reduction of some representations of $\mathscr{G}_{\varpi_K}$ attached to discrete series types. Let L be the ramified quadratic extension of K generated by the square root of ϖ_K, and fix an embedding of L into D. Any semi-simple representation of $L^\times U_D^a$ in characteristic p is trivial on $U_D^1 \cap L^\times U_D^a$ as this is a pro-p-group (recall that U_D^a was defined in Section 3.1). Any representation of $L^\times U_D^a$ that is trivial on the subgroup generated by ϖ_K factors through the image of $L^\times U_D^a$ inside $\mathscr{G}_{\varpi_K}$. Hence any semi-simple representation of $L^\times U_D^a$ in characteristic p that is trivial on ϖ_K factors through the subgroup $L^\times U_D^a/\langle U_D^1, \varpi_K^{\mathbb{Z}}\rangle$ of Γ_K. Let us call this subgroup Δ, it is equal to $k^\times \times \{1,\iota\} \subset \ell^\times \rtimes \{1,\iota\}$.

Proposition 3.3.3. — *Let L be as above and let $\theta = \operatorname{ind}_{L^\times U_D^a}^{D^\times} \psi$ for some smooth character ψ of $L^\times U_D^a$ with $\psi(\varpi_K)=1$. Then θ factors through $\mathscr{G}_{\varpi_K}$. Denote by $\overline{\theta}$ the semi-simple representation of Γ_K which is the reduction modulo p of θ. Let $n \in \mathbb{Z}/(q-1)\mathbb{Z}$ and $\alpha \in \{0,1\}$ be such that $\overline{\psi} = \xi^\alpha \chi_n|\Delta$ as a representation of Δ. We denote by $I(\overline{\psi})$ the set of irreducible representations of Γ_K with central character equal to $\overline{\psi}|_{k^\times}$. Then we have in $\mathscr{R}(\Gamma_K)$:*

(1) *if $\psi(-1)=-1$, that is, n is odd, then $I(\overline{\psi})$ consists of $(q+1)/2$ representations of dimension 2, and*

$$[\overline{\theta}] = q^{a-1}\Big(\sum_{r\in I(\overline{\psi})}[r]\Big),$$

(2) *if $\psi(1)=1$, that is, n is even, then $I(\overline{\psi})$ consists of $(q-1)/2$ representations of dimension 2 and 4 representations of dimension 1, and:*

$$\begin{aligned}[\overline{\theta}] = q^{a-1}\Big(\sum_{\substack{r\in I(\overline{\psi})\\ \dim(r)=2}}[r]\Big) &+ \frac{q^{a-1}+1}{2}[\xi^\alpha]([\delta_{n/2}]\\ &+[\delta_{(n+q-1)/2}]) + \frac{q^{a-1}-1}{2}[\xi^{\alpha+1}]([\delta_{n/2}]+[\delta_{(n+q-1)/2}]).\end{aligned}$$

Proof. — We proceed as in [BD14, §4]. We have $[L^\times U_D^1 : L^\times U_D^a] = q^{a-1}$ (note that the essential conductor of θ is $2a+1$). The reduction modulo p of $\operatorname{ind}_{L^\times U_D^a}^{L^\times U_D^1}\psi$ is the sum of $(q^{a-1}+1)/2$ copies of $\overline{\psi}$ and of $(q^{a-1}-1)/2$ copies of $\xi\overline{\psi}$. Let μ be a smooth character of $L^\times U_D^1$ in characteristic p with $\mu(\varpi_K)=1$, then $\operatorname{ind}_{L^\times U_D^1}^{\mathscr{G}_{\varpi_K}}\mu$ factors through Γ_K, and the representation of Γ_K that we obtain is $\operatorname{ind}_\Delta^{\Gamma_K}\mu$, which can be computed via Brauer characters. □

It follows from Proposition 3.1.1 that Proposition 3.3.3 gives $\overline{\sigma_{\mathscr{G}}}(\mathtt{t})$ when $\mathtt{t}$ is of type (irr) under some compatibility condition between $\mathtt{t}$ and ϖ_K. As we will see in Section 3.5 this compatibility condition is harmless. When $\mathtt{t}$ is scalar, $\overline{\sigma_{\mathscr{G}}}(\mathtt{t})$ is easy to compute as $\sigma_{\mathscr{G}}(\mathtt{t})$ is of dimension 1. The value of $\overline{\sigma_{\mathscr{G}}}(\mathtt{t})$ when $\mathtt{t}$ is of the form (red) could be immediately obtained from [BD14, Prop. 4.6]: as $\overline{\sigma_{\mathscr{G}}}(\mathtt{t}) = \xi\overline{\sigma_{\mathscr{G}}}(\mathtt{t})$, it is entirely determined by its restriction to $\ell^\times$. We do not give details as they are not really needed. Indeed, our goal in computing $\overline{\sigma_{\mathscr{G}}}(\mathtt{t})$ is to allow us to compute the multiplicity of some deformation rings as we shall see in Theorems 3.5.1 and 3.5.2, but for $\mathtt{t}$ of the form (red) this multiplicity can be computed by the formula coming from the Breuil-Mézard conjecture for GL_2. Note that complete results and computations can be found in [Tok15].

3.4. A reformulation of a result of Gee and Geraghty. — Let $\Gamma = \Gamma_K$. We fix a uniformizer ϖ_K of K. We denote by w_0 the Hodge-Tate type $(0,0)_{\tau\in\mathrm{Hom}(K,\overline{\mathbb{Q}}_p)}$.

In the case $K=\mathbb{Q}_p$, let $w=(n,m)$ be a Hodge-Tate type. We set $|w| = n+2m$. We define a representation $\sigma_w = \mathrm{Sym}^n\otimes\det^m$ of $\mathrm{GL}_2(\overline{\mathbb{Q}}_p)$, hence of $\mathscr{G}=\mathscr{G}_{\varpi_{\mathbb{Q}_p}}$ via a realization of $\mathscr{G}$ as in Section 3.2.3. In particular σ_{w_0} is the trivial representation of $\mathscr{G}$. The isomorphism class of the restriction of σ_w to $\mathscr{O}_D^\times$ does not depend on the particular choice of a realization, as they are all conjugate in restriction to $\mathscr{O}_D^\times$. We can see the reduction modulo p of σ_w as a representation of Γ by restriction. We denote its image in $\mathscr{R}(\Gamma)$ by $\overline{\sigma_w}$, it does not depend on any choices made (including ϖ_K): indeed it is the restriction to Γ of the representation $\mathrm{Sym}^n\otimes\det^m$ of $\mathrm{GL}_2(\mathbb{F})$ via any embedding of Γ into $\mathrm{GL}_2(\mathbb{F})$, and all such embeddings are conjugate.

We denote by π a uniformizer of the field E of Section 2.3. For any noetherian local ring A, we denote by $e(A)$ the Hilbert-Samuel multiplicity $e(A,A)$ (see [Mat89] for the definition of the Hilbert-Samuel multiplicity, and also [Kis09a, §1.3] for properties relevant to our situation).

Let $\ell=\mathbb{F}_{q^2}$, and let $\mathscr{R}(\ell^\times)$ be the Grothendieck ring of representations of $\ell^\times$ with coefficients in $\overline{\mathbb{F}}$.

For $K=\mathbb{Q}_p$, we recall the following result ([GG15, Cor. 5.7]), which is the consequence of the main result of [GG15] and the usual formulation of the Breuil-Mézard conjecture proved in [Kis09a], [Paš15] and [HT15]. Here $\sigma_D(\mathtt{t})$ is, as in Section 3.2.1, a choice of irreducible sub-representation of the restriction of $\pi_{\mathtt{t}}$ to $\mathscr{O}_D^\times$.

Theorem 3.4.1. — *Let $\overline{\rho}: G_{\mathbb{Q}_p}\to\mathrm{GL}_2(\mathbb{F})$, and suppose that $p\geqslant 5$ if $\overline{\rho}$ is a twist of an extension of the trivial representation by the cyclotomic character. There exists a positive linear functional $i_{D,\overline{\rho}}:\mathscr{R}(\ell^\times)\to\mathbb{Z}$ such that for each discrete series inertial type $\mathtt{t}$, and each choice of $\sigma_D(\mathtt{t})$, we have*

$$e(R^{\square,\psi}(w,\mathtt{t}^{\mathrm{ds}},\overline{\rho})/\pi) = i_{D,\overline{\rho}}([\overline{\sigma_D}(\mathtt{t})\otimes\overline{\sigma_w}|_{\ell^\times}]).$$

We return to the case of a general K. We have the following well-known result (see for example [GS11, Lem. 3.5]).

Proposition 3.4.2. — *Let $\overline{\rho}$ be a continuous representation of G_K of dimension 2 with coefficients in $\mathbb{F}$. Suppose that $\overline{\rho}$ has a potentially semi-stable lift with scalar type $\mathtt{t}=\psi\oplus\psi$ and Hodge-Tate weights $(0,1)_{\tau\in\mathrm{Hom}(K,\overline{\mathbb{Q}}_p)}$ which is not potentially crystalline. Then $\overline{\rho}$ is an unramified twist of $\left(\begin{smallmatrix}\omega & *\\ 0 & 1\end{smallmatrix}\right)\otimes\overline{\psi}$.*

We deduce from this that when $\overline{\rho}$ is not a twist of an extension of the trivial character by the cyclotomic character, $R^{\square,\psi}(w_0, \mathtt{t}^{\mathrm{ds}}, \overline{\rho}) = R_{\mathrm{cr}}^{\square,\psi}(w_0, \mathtt{t}^{\mathrm{ds}}, \overline{\rho})$ for any discrete series type $\mathtt{t}$, where the second ring parametrizes only representations that are potentially crystalline. Hence we can deduce from the main result of [GG15] and [GK14, Th. A] the following.

Theorem 3.4.3. — *Let $\overline{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ a continuous representation that is not a twist of an extension of the trivial representation by the cyclotomic character. There exists a positive linear functional $i_{D,\overline{\rho}} : \mathscr{R}(\ell^\times) \to \mathbb{Z}$ such that for each discrete series inertial type $\mathtt{t}$, and each choice of $\sigma_D(\mathtt{t})$, we have*

$$e(R^{\square,\psi}(w_0, \mathtt{t}^{\mathrm{ds}}, \overline{\rho})/\pi) = i_{D,\overline{\rho}}([\overline{\sigma_D}(\mathtt{t})]).$$

Let $d_{\mathtt{t}} = 1$ if $\mathtt{t}$ has the form (scal) or (irr), and $d_{\mathtt{t}} = 2$ if $\mathtt{t}$ has the form (red), so that $d_{\mathtt{t}}$ is the number of irreducible components of $\sigma_{\mathscr{G}}(\mathtt{t})|_{\mathscr{O}_D^\times}$. Then we can give a reformulation of Theorems 3.4.1 and 3.4.3 in terms of representations of Γ.

Theorem 3.4.4. — *Let $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$, and suppose that $p \geqslant 5$ if $\overline{\rho}$ is a twist of an extension of the trivial representation by the cyclotomic character. There exists a positive linear functional $i_{\overline{\rho}} : \mathscr{R}(\Gamma) \to \mathbb{Z}$ such that for each discrete series inertial type $\mathtt{t}$ and each choice of $\sigma_{\mathscr{G}}(\mathtt{t})$ we have*

$$d_{\mathtt{t}} e(R^{\square,\psi}(w, \mathtt{t}^{\mathrm{ds}}, \overline{\rho})/\pi) = i_{\overline{\rho}}([\overline{\sigma_{\mathscr{G}}}(\mathtt{t}) \otimes \overline{\sigma_w}]).$$

Theorem 3.4.5. — *Let $\overline{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ a continuous representation that is not a twist of an extension of the trivial representation by the cyclotomic character. There exists a positive linear functional $i_{\overline{\rho}} : \mathscr{R}(\Gamma) \to \mathbb{Z}$ such that for each discrete series inertial type $\mathtt{t}$ and each choice of $\sigma_{\mathscr{G}}(\mathtt{t})$ we have*

$$d_{\mathtt{t}} e(R^{\square,\psi}(w_0, \mathtt{t}^{\mathrm{ds}}, \overline{\rho})/\pi) = i_{\overline{\rho}}([\overline{\sigma_{\mathscr{G}}}(\mathtt{t})]).$$

Proof of Theorems 3.4.4 *and* 3.4.5. — It follows from Theorems 3.4.1 (resp. 3.4.3) and the definition of $\sigma_{\mathscr{G}}(\mathtt{t})$ that we have the equality

$$d_{\mathtt{t}} e(R^{\square,\psi}(w, \mathtt{t}^{\mathrm{ds}}, \overline{\rho})/\pi) = i_{D,\overline{\rho}}([\overline{\sigma_{\mathscr{G}}}(\mathtt{t}) \otimes \overline{\sigma_w}|_{\ell^\times}]).$$

Set $i_{\overline{\rho}}([\gamma]) = i_{D,\overline{\rho}}([\gamma|_{\ell^\times}])$ for all irreducible representations γ of Γ to get the result. $\square$

In particular, we observe that

$$i_{\overline{\rho}}([\gamma]) = i_{\overline{\rho}}([\xi\gamma]) = (\dim \gamma) e(R^{\square,\psi}(w_0, \mathtt{t}_\gamma^{\mathrm{ds}}, \overline{\rho})/\pi)$$

for any irreducible representation γ of Γ, where $\mathtt{t}_\gamma$ is the inertial type defined in the proof of Proposition 3.3.2.

We denote by $W_\Gamma(\overline{\rho})$ the set of γ such that $i_{\overline{\rho}}([\gamma]) \neq 0$. This is the translation in the setting of representations of Γ of the (predicted) quaternionic Serre weights of [GS11]. Note in particular that, as in [GS11], the set $W_\Gamma(\overline{\rho})$ is determined by the existence of certain lifts of $\overline{\rho}$ that have all their Hodge-Tate weights equal to $(0, 1)$, which makes the situation with quaternion algebras simpler than the situation of

Serre weights for GL_2, since, for GL_2, one cannot in general lift a Serre weight as a type in characteristic 0.

3.5. Multiplicity formulas. — We now state our main theorems for the multiplicity of the special fiber of the discrete series extended type deformation rings, which we prove in Section 5.

Theorem 3.5.1. — *Let $\overline{\rho}$ be a continuous representation of $G_{\mathbb{Q}_p}$ of dimension 2 with coefficients in $\mathbb{F}$. Suppose that $p \geqslant 5$ if $\overline{\rho}$ is a twist of an extension of the trivial character by the cyclotomic character. Let $\overline{\psi} = \omega^{-1}\det\overline{\rho}$, $\varpi_{\mathbb{Q}_p}$ a uniformizer of $\mathbb{Q}_p$ and α a square root of $\overline{\psi}(\varpi_{\mathbb{Q}_p})^{-1}$.*

There exists a positive linear form $\mu_{\overline{\rho}}$ on $\mathscr{R}(\Gamma)$ with values in $\mathbb{Z}$ satisfying the following property: for any discrete series inertial type $\mathtt{t}$, Hodge-Tate type w, character ψ lifting $\overline{\psi}$ compatible with $\mathtt{t}$ and w, and $\mathtt{t}^+$ a discrete series extended type compatible with $(\mathtt{t}, \psi)$ we have:

$$e(R^{\square,\psi}(w, \mathtt{t}^+, \overline{\rho})/\pi) = \mu_{\overline{\rho}}([\overline{\sigma_{\mathscr{G}}}(\mathtt{t}) \otimes \overline{\sigma_w}])$$

for the choice of representation $\sigma_{\mathscr{G}}(\mathtt{t})$ of $\mathscr{G}_{\varpi_{\mathbb{Q}_p}}$ such that $\mathtt{t}^+ = \mathrm{LL}_D(\sigma_{\mathscr{G}}(\mathtt{t})) \otimes \mathrm{unr}(a\varpi_{\mathbb{Q}_p}^{|w|})^{-1}$ for some $a \in \overline{\mathbb{Z}}_p^\times$ lifting α.

Theorem 3.5.2. — *Let $\overline{\rho}$ be a continuous representation of G_K of dimension 2 with coefficients in $\mathbb{F}$ that is not a twist of an extension of the trivial character by the cyclotomic character. Let $\overline{\psi} = \omega^{-1}\det\overline{\rho}$, ϖ_K a uniformizer of K and α a square root of $\overline{\psi}(\varpi_K)^{-1}$.*

There exists a positive linear form $\mu_{\overline{\rho}}$ on $\mathscr{R}(\Gamma)$ with values in $\mathbb{Z}$ satisfying the following property: for any discrete series inertial type $\mathtt{t}$, character ψ lifting $\overline{\psi}$ compatible with $\mathtt{t}$ and w_0, and $\mathtt{t}^+$ a discrete series extended type compatible with $(\mathtt{t}, \psi)$ we have:

$$e(R^{\square,\psi}(w_0, \mathtt{t}^+, \overline{\rho})/\pi) = \mu_{\overline{\rho}}([\overline{\sigma_{\mathscr{G}}}(\mathtt{t})])$$

for the choice of representation $\sigma_{\mathscr{G}}(\mathtt{t})$ of $\mathscr{G}_{\varpi_K}$ such that $\mathtt{t}^+ = \mathrm{LL}_D(\sigma_{\mathscr{G}}(\mathtt{t})) \otimes \mathrm{unr}(a)^{-1}$ for some $a \in \overline{\mathbb{Z}}_p^\times$ lifting α.

Remark 3.5.3. — It follows from the definition of the compatibility of $\mathtt{t}^+$ with $(\mathtt{t}, \psi, w)$ that there exists indeed a choice of $\sigma_{\mathscr{G}}(\mathtt{t})$ satisfying the condition. If $\mathtt{t}^-$ is the extended type conjugate to $\mathtt{t}^+$, then the choices of $\sigma_{\mathscr{G}}(\mathtt{t})$ for $\mathtt{t}^+$ and $\mathtt{t}^-$ differ by multiplication by ξ.

In the case when $\mathtt{t}$ is of the form (red), recall that there is only one extended type $\mathtt{t}^+$ compatible with $(\mathtt{t}, \psi)$, and

$$R^{\square,\psi}(w, \mathtt{t}^{\mathrm{ds}}, \overline{\rho}) = R^{\square,\psi}(w, \mathtt{t}^+, \overline{\rho}).$$

There is no choice to be made for $\sigma_{\mathscr{G}}(\mathtt{t})$ as it is isomorphic to $\xi\sigma_{\mathscr{G}}(\mathtt{t})$.

We have the following proposition, which is a consequence of [Kis09a, Prop. 1.3.9].

Proposition 3.5.4. — *Let* $\mathtt{t}^+$, $\mathtt{t}^-$ *be the two distinct conjugate extended types compatible with* $(\mathtt{t},\psi)$ *with* $\mathtt{t}$ *of the form* (scal) *or* (irr). *Then*

$$e(R^{\square,\psi}(w,\mathtt{t}^+,\overline{\rho})/\pi)+e(R^{\square,\psi}(w,\mathtt{t}^-,\overline{\rho})/\pi)=e(R^{\square,\psi}(w,\mathtt{t}^{\mathrm{ds}},\overline{\rho})/\pi).$$

We have the following corollary ($\mathtt{t}_r$ and $\mathtt{t}_\delta$ are the inertial types defined in the proof of Proposition 3.3.2).

Corollary 3.5.5. — *We have*

$$\mu_{\overline{\rho}}([r])=e(R^{\square,\psi}(w_0,\mathtt{t}_r^{\mathrm{ds}},\overline{\rho})/\pi)$$

for any irreducible representation r *of* Γ *of dimension* 2*, and*

$$\mu_{\overline{\rho}}([\delta]+[\xi\delta])=e(R^{\square,\psi}(w_0,\mathtt{t}_\delta^{\mathrm{ds}},\overline{\rho})/\pi)$$

for any irreducible representation δ *of* Γ *of dimension* 1*. In particular, for any irreducible representation* γ *of* Γ*, we have*

$$\mu_{\overline{\rho}}([\gamma])+\mu_{\overline{\rho}}([\xi\gamma])=i_{\overline{\rho}}([\gamma]).$$

Proof. — Let r be an irreducible representation of Γ of dimension 2. Then $r=\overline{\sigma_{\mathscr{G}}}(\mathtt{t}_r)$ for some inertial type $\mathtt{t}_r$ of the form (red) by Proposition 3.3.2. Then as remarked in Section 2.3.3, if $\mathtt{t}_r^+$ is the extended type compatible with $(\mathtt{t}_r,\psi)$, then

$$R^{\square,\psi}(w_0,\mathtt{t}_r^+,\overline{\rho})=R^{\square,\psi}(w_0,\mathtt{t}_r^{\mathrm{ds}},\overline{\rho}),$$

hence the formula in this case. Let δ be an irreducible representation of Γ of dimension 1. Then

$$e(R^{\square,\psi}(w_0,\mathtt{t}_\delta^{\mathrm{ds}},\overline{\rho})/\pi)=e(R^{\square,\psi}(w_0,\mathtt{t}_\delta^+,\overline{\rho})/\pi)+e(R^{\square,\psi}(w_0,\mathtt{t}_\delta^-,\overline{\rho})/\pi),$$

where $\mathtt{t}_\delta^+$ and $\mathtt{t}_\delta^-$ are the two conjugate extended types compatible with $(\mathtt{t}_\delta,\psi)$. So we deduce the formula from Remark 3.5.3. The formula with $i_{\overline{\rho}}$ then follows from Theorems 3.4.4 and 3.4.5. $\square$

It follows from this corollary that $\mu_{\overline{\rho}}([\gamma])=0$ if $\gamma\notin W_\Gamma(\overline{\rho})$. We begin the definition of $\mu_{\overline{\rho}}$ by setting $\mu_{\overline{\rho}}([\gamma])=0$ for any irreducible γ not in $W_\Gamma(\overline{\rho})$. With this definition, the equalities of Theorem 3.5.1 and 3.5.2 hold for all $\mathtt{t},\psi,w$ (with $w=w_0$ if $K\neq\mathbb{Q}_p$) such that $R^{\square,\psi}(w,\mathtt{t}^{\mathrm{ds}},\overline{\rho})=0$.

From Proposition 3.4.2 we deduce the next proposition.

Proposition 3.5.6. — *If* $\overline{\rho}$ *is a representation such that* $W_\Gamma(\overline{\rho})$ *contains a representation* δ *of dimension* 1 *then* $\overline{\rho}$ *is a twist of an extension of the trivial character by the cyclotomic character and there is at most one possible value for* δ *for which* $\mu_{\overline{\rho}}(\delta)\neq 0$.

Remark 3.5.7. — When $K=\mathbb{Q}_p$, $\overline{\rho}$ is a twist of an extension of the trivial character by the cyclotomic character if and only if $W_\Gamma(\overline{\rho})$ contains a representation δ of dimension 1, and then $i_{\overline{\rho}}(\delta)=1$. This follows from the explicit computations of deformation rings that can be found in [BM02, §5.2].

Proposition 3.5.8. — *If* $\overline{\rho}$ *is not a twist of an extension of the trivial character by the cyclotomic character then for any representation* γ *of* Γ *we have* $\mu_{\overline{\rho}}([\gamma])=\mu_{\overline{\rho}}([\xi\gamma])$.

Proof. — It suffices to prove this for representations γ that are irreducible. If $\dim\gamma = 2$ then $\xi\gamma = \gamma$ so the statement holds. If $\dim\gamma = 1$ then by Proposition 3.5.6 both sides of the equality are zero. $\square$

Corollary 3.5.9. — *Let $\overline{\rho}$ be a continuous representation of G_K of dimension 2 with coefficients in $\mathbb{F}$ which is not a twist of an extension of the trivial character by the cyclotomic character. Then for any discrete series inertial type $\mathtt{t}$, and any Hodge-Tate type w if $K = \mathbb{Q}_p$, or for $w = w_0$ if $K \neq \mathbb{Q}_p$, we have*

$$e(R^{\square,\psi}(w,\mathtt{t}^+,\overline{\rho})/\pi) = e(R^{\square,\psi}(w,\mathtt{t}^-,\overline{\rho})/\pi) = \frac{d_{\mathtt{t}}}{2} e(R^{\square,\psi}(w,\mathtt{t}^{\mathrm{ds}},\overline{\rho})/\pi).$$

Proof. — The first equality comes from Proposition 3.5.8 and Remark 3.5.3. The last equality follows from Proposition 3.5.4. $\square$

Remark 3.5.10. — If $\overline{\rho}$ is irreducible Corollary 3.5.9 holds even without Theorems 3.5.1 and 3.5.2, because of Remark 2.3.5.

Corollary 3.5.11. — *Let $K = \mathbb{Q}_p$. Suppose that there exists a representation δ of dimension 1 of Γ with $i_{\overline{\rho}}(\delta) \neq 0$ (that is, $\overline{\rho}$ is a twist of an extension of the trivial character by the cyclotomic character, and then $i_{\overline{\rho}}(\delta) = 1$). Then for any discrete series inertial type $\mathtt{t}$, Hodge-Tate type w, character ψ and pair of conjugate extended types $(\mathtt{t}^+,\mathtt{t}^-)$ compatible with (t,ψ), we have*

$$\begin{aligned} &\textit{either} \quad e(R^{\square,\psi}(w,\mathtt{t}^+,\overline{\rho})/\pi) = e(R^{\square,\psi}(w,\mathtt{t}^-,\overline{\rho})/\pi), \\ &\textit{or} \quad |e(R^{\square,\psi}(w,\mathtt{t}^+,\overline{\rho})/\pi) - e(R^{\square,\psi}(w,\mathtt{t}^-,\overline{\rho})/\pi)| = 1. \end{aligned}$$

The former takes place in particular when $R^{\square,\psi}(w,\mathtt{t}^{\mathrm{ds}},\overline{\rho}) = 0$, or $\mathtt{t}$ is of the form (red)*, or $\mathtt{t}$ is of the form* (irr) *with $\pi_{\mathtt{t}}$ of the form $\mathrm{ind}_{L^\times U_D^a}^{D^\times}\psi$ for some ramified quadratic extension L of K, some a and some character ψ of $L^\times U_D^a$ with $\psi(-1) = -1$ (see Proposition* 3.1.1 *for the notations).*

We see examples where we have $|e(R^{\square,\psi}(w,\mathtt{t}^+,\overline{\rho})/\pi) - e(R^{\square,\psi}(w,\mathtt{t}^-,\overline{\rho})/\pi)| = 1$ in Section 6.

Proof. — Note that we can choose $\varpi_{\mathbb{Q}_p}$ as we wish to compute the multiplicities. Let $\sigma_{\mathscr{G}}(\mathtt{t})$ be a choice of representation attached to $\mathtt{t}$ as in Section 3.2.2. We need to compute

$$[\overline{\sigma_{\mathscr{G}}(\mathtt{t})} \otimes \overline{\sigma_w} : \delta] - [\xi\overline{\sigma_{\mathscr{G}}(\mathtt{t})} \otimes \overline{\sigma_w} : \delta].$$

We do this using the results of Proposition 3.3.3 and the remarks that follow for $\overline{\sigma_{\mathscr{G}}(\mathtt{t})}$, and the Lemma below for $\overline{\sigma_w}$. $\square$

Lemma 3.5.12. — *In $\mathscr{R}(\Gamma_{\mathbb{Q}_p})$ we have that $[\det^m] = [\xi^m\delta_m]$ for all m and*

$$[\mathrm{Sym}^{2n}\mathbb{F}^2] = [\delta_n] + \sum_{i=1}^{n}[r_{n(p+1)+i(p-1)}] \quad \textit{and} \quad [\mathrm{Sym}^{2n+1}\mathbb{F}^2] = \sum_{i=0}^{n}[r_{n(p+1)+p+i(p-1)}]$$

for all $n \geqslant 0$. Moreover, $r_{n(p+1)+i(p-1)}$ is irreducible for

$$0 < i < (p+1)/2 \quad \textit{and} \quad (p+1)/2 < i < p+1,$$

and

$$r_{n(p+1)+(p-1)(p+1)/2} = \delta_{n+(p-1)/2} \oplus \xi\delta_{n+(p-1)/2} \quad \textit{and} \quad r_{n(p+1)} = \delta_n \oplus \xi\delta_n.$$

Our proof of Theorems 3.5.1 and 3.5.2 is by deducing them from the usual version of the Breuil-Mézard conjecture, in the cases where it is already known. We can hope that this method generalizes to the cases of the Breuil-Mézard conjecture that are not yet known, which leads us to the following.

Conjecture 3.5.13. — *Let $\overline{\rho}$ be a continuous representation of G_K of dimension 2 with coefficients in $\mathbb{F}$. There exists a positive linear form $\mu_{\overline{\rho}}$ on $\mathscr{R}(\Gamma)$ with values in $\mathbb{Z}$ satisfying the following property: for any discrete series inertial type $\mathtt{t}$, Hodge-Tate type w, character ψ lifting $\omega^{-1}\det\overline{\rho}$ compatible with $\mathtt{t}$ and w, and extended type $\mathtt{t}^+$ compatible with $(\mathtt{t},\psi)$, there exists a choice of representation $\sigma_{\mathscr{G}}(\mathtt{t})$ of $\mathscr{G}$ such that we have:*

$$e(R^{\square,\psi}(w,\mathtt{t}^+,\overline{\rho})/\pi) = \mu_{\overline{\rho}}([\overline{\sigma_{\mathscr{G}}}(\mathtt{t})\otimes\overline{\sigma_w}]).$$

When $\overline{\rho}$ is not a twist of an extension of the trivial character by the cyclotomic character, we have

$$e(R^{\square,\psi}(w,\mathtt{t}^+,\overline{\rho})/\pi) = e(R^{\square,\psi}(w,\mathtt{t}^-,\overline{\rho})/\pi),$$

where $\mathtt{t}^-$ is the extended type that is conjugate to $\mathtt{t}^+$.

4. Quaternionic modular forms

4.1. Global setting. — Let F be a totally real number field such that for all places $v\,|\,p$, F_v is isomorphic to K. We denote by Σ_p the set of places above p, and we assume that the number of infinite places of F has the same parity as the cardinality of Σ_p. Let B be the quaternion algebra with center F that is ramified exactly at the infinite places of F and at Σ_p, which exists thanks to the parity condition.

For all $v\in\Sigma_p$, we fix an isomorphism between B_v and the quaternion algebra D of Section 3. For any finite place v of F that is not in Σ_p, fix an isomorphism between B_v and $M_2(F_v)$ so that $\mathscr{O}_{B_v}^\times$ corresponds to $\mathrm{GL}_2(\mathscr{O}_{F_v})$. We fix $v_0\in\Sigma_p$ and denote $\Sigma_p\smallsetminus\{v_0\}$ by Σ'_p.

Let ϖ_K be a uniformizer of K. We denote by $\mathscr{G}$ the group $\mathscr{G}_{\varpi_K}$ of Section 3.2.2. We fix a uniformizer ϖ_D of D with $\varpi_D^2=\varpi_K$.

4.2. Modular forms. — We recall the theory of quaternionic modular forms (see for example [Tay06, §1], and also [Kha01, §4.1] and [GS11, §2] for the situation with a quaternion algebra ramified at p).

Denote by $\mathbb{A}_F^f\subset\mathbb{A}_F$ the ring of finite adeles of F. Let $U=\prod_v U_v$ be a compact open subgroup of $(B\otimes_F\mathbb{A}_F^f)^\times$ such that for all finite places v, $U_v\subset\mathscr{O}_{B_v}^\times$, and for all $v\in\Sigma_p$, $U_v=\mathscr{O}_{B_v}^\times$.

Let A be a topological $\mathbb{Z}_p$-algebra. For all $v\,|\,p$, let (σ_v,V_v) be a representation of U_v on a finite free A-module. We define a representation σ of U on $V=\otimes_{v\,|\,p}V_v$ by letting U_v act by σ_v for $v\,|\,p$ and letting U_v act trivially for $v\nmid p$. Let η be a continuous character $(\mathbb{A}_F^f)^\times/F^\times \to A^\times$ such that for all v, the restriction of σ and of η to $U_v \cap \mathscr{O}_{F_v}^\times$ coincide (such a character does not necessarily exist).

Let $S_{\sigma,\eta}(U,A)$ be the set of continuous functions $f : B^\times\backslash(B\otimes_F \mathbb{A}_F^f)^\times \to V$ such that:

- for all $g\in (B\otimes_F \mathbb{A}_F^f)^\times$ and $u\in U$, $f(gu)=\sigma(u)^{-1}f(g)$,
- for all $g\in (B\otimes_F \mathbb{A}_F^f)^\times$ and $z\in (\mathbb{A}_F^f)^\times$, $f(gz)=\eta(z)^{-1}f(g)$.

We can extend the action of U on (σ,V) to an action of $U(\mathbb{A}_F^f)^\times$: we let $(\mathbb{A}_F^f)^\times$ act via η. We say that U is small enough (see for example [Kis09a, §2.1.1]) if

$$\text{for all } t\in (B\otimes_F \mathbb{A}_F^f)^\times,\quad (U(\mathbb{A}_F^f)\cap t^{-1}D^\times t)/F^\times = 1.$$

In this case, the functor $(\sigma,V)\mapsto S_{\sigma,\eta}(U,A)$ is exact in (σ,V). In the following we will always assume that U is small enough.

Let now $(\widetilde{\sigma}_{v_0},V_{v_0})$ be a representation of $\mathscr{G}$ with coefficients in A, and for $v\in\Sigma_p'$, let (σ_v,V_v) be a representation of $U_v\simeq\mathscr{O}_D^\times$ as before. Let $\widetilde{\sigma}$ be the representation $\widetilde{\sigma}_{v_0}\otimes(\otimes_{v\in\Sigma_p'}\sigma_v)$ of $\mathscr{G}\times(\prod_{v\in\Sigma_p'}U_v)$ on $\otimes_{v\in\Sigma_p}V_v$. Let σ_{v_0} be the restriction of $\widetilde{\sigma}_{v_0}$ to $U_{v_0}=\mathscr{O}_D^\times$, we define as before σ a representation of U on $\otimes_{v\in\Sigma_p}V_v$, and we suppose that the character η exists. We define a space of modular forms $S_{\widetilde{\sigma},\eta}(U,A)$ by setting $S_{\widetilde{\sigma},\eta}(U,A)=S_{\sigma,\eta}(U,A)$. We will endow the space $S_{\widetilde{\sigma},\eta}(U,A)$ with an additional structure (a Hecke operator at v_0) in Section 4.4.1.

4.3. Hecke algebra. — The group $(B\otimes_F\mathbb{A}_F^f)^\times$ acts on the set of functions on $(B\otimes_F\mathbb{A}_F^f)^\times$ by $g\cdot f(z)=f(zg)$.

Let S be a finite set of places of F containing all places above p and all v such that U_v is not $\mathscr{O}_{B_v}^\times$, and $S'\subset S$ the set of places w such that U_w is not $\mathscr{O}_{B_w}^\times$. Let $T_S=\mathbb{Z}[T_v,S_v,U_{\varpi_w}]_{v\notin S, w\in S'}$ be a polynomial ring. We define an action of T_S on $S_{\sigma,\eta}(U,A)$ by:

- T_v is the action of $U_v\left(\begin{smallmatrix}\varpi_v & 0\\ 0 & 1\end{smallmatrix}\right)U_v$,
- S_v is the action of $U_v\left(\begin{smallmatrix}\varpi_v & 0\\ 0 & \varpi_v\end{smallmatrix}\right)U_v$,
- U_{ϖ_w} is the action of $U_w\left(\begin{smallmatrix}\varpi_w & 0\\ 0 & 1\end{smallmatrix}\right)U_w$,

where ϖ_v is a uniformizer of F_v. The actions of these operators commute, and the definition of the Hecke operators T_v and S_v does not depend on the choice of ϖ_v.

Let $\mathbb{T}_{\sigma,\eta}(U,A)$ be the A-algebra generated by the image of T_S in the ring of endomorphisms of $S_{\sigma,\eta}(U,A)$.

4.4. Hecke operators at places above p

4.4.1. *Hecke operators.* — We fix a representation $\widetilde{\sigma}_{v_0}$ of $\mathscr{G}$, and representations σ_v of U_v for $v\in\Sigma_p'$ as in Section 4.2. Consider the space of modular forms $S_{\widetilde{\sigma},\eta}(U,A)$ of Section 4.2.

We define an operator W_{v_0} acting on $S_{\widetilde{\sigma},\eta}(U,A)$ by $(W_{v_0}f)(g)=\widetilde{\sigma}_{v_0}(\iota)f(g\varpi_{D,v_0})$, where ϖ_{D,v_0} is the element of $(B\otimes_F\mathbb{A}_F^f)^\times$ that is equal to ϖ_D at v_0 and 1 everywhere else. One easily checks that $W_{v_0}f$ is indeed an element of $S_{\widetilde{\sigma},\eta}(U,A)$ if f is. Note that $W_{v_0}^2$ is multiplication by $\eta(\varpi_{K,v_0})^{-1}$, where $\varpi_{K,v_0}=\varpi_{D,v_0}^2$. It is clear from the definition that W_{v_0} commutes with the action of the Hecke algebra $\mathbb{T}_{\sigma,\eta}(U)$.

Suppose that A contains a square root α of $\eta(\varpi_{K,v_0})^{-1}$. Then we get a decomposition

$$S_{\widetilde{\sigma},\eta}(U)=S_{\widetilde{\sigma},\eta}(U,A)^{+}\oplus S_{\widetilde{\sigma},\eta}(U,A)^{-},$$

where $S_{\widetilde{\sigma},\eta}(U,A)^{\pm}$ denotes the subspace of $S_{\widetilde{\sigma},\eta}(U,A)$, where W_{v_0} acts by $\pm\alpha$. If we replace σ_{v_0} by $\xi\sigma_{v_0}$ without changing α, the space of modular forms $S_{\widetilde{\sigma},\eta}(U,A)$ is unchanged, W_{v_0} is replaced by $-W_{v_0}$ and the $+$ and $-$ subspaces are exchanged.

4.4.2. *The case of type* (red). — Consider the following special case: let (σ_{v_0},V_{v_0}) be a representation of $U_{v_0}\simeq\mathscr{O}_D^\times$ over A, and σ'_{v_0} the representation on V_{v_0} defined by $\sigma'_{v_0}(g)=\sigma_{v_0}(\varpi_D g\varpi_D^{-1})$. We can define a representation $(\widetilde{\sigma}_{v_0},\widetilde{V}_{v_0})$ of $\mathscr{G}$ by $\widetilde{V}_{v_0}=V_{v_0}\oplus V_{v_0}$, U_{v_0} acts by $(\sigma_{v_0},\sigma'_{v_0})$ and ι acts by $\widetilde{\sigma}_{v_0}(x,y)=(y,x)$. Fix representations at places $v\in\Sigma'_p$, a character η and representations σ and $\widetilde{\sigma}$ as in Section 4.2.

Let α be a square root of $\eta(\varpi_{K,v_0})^{-1}$. We have two embeddings

$$i_+,i_-:S_{\sigma,\eta}(U,A)\longrightarrow S_{\widetilde{\sigma},\eta}(U,A),$$

given by $i_\pm(f)(g)=(f(g),\pm\alpha^{-1}\widetilde{\sigma}_{v_0}(\iota)f(g\varpi_{D,v_0}))$. The image of $i_\pm$ is $S_{\widetilde{\sigma},\eta}(U,A)^{\pm}$ and i_++i_- is a isomorphism from $S_{\sigma,\eta}(U,A)^2$ to $S_{\widetilde{\sigma},\eta}(U,A)$.

We will make use of this remark in the following situation: $\widetilde{\sigma}_{v_0}$ is of the form $\sigma_{\mathscr{G}}(\mathtt{t})\otimes\sigma_{\mathrm{alg}}$ for σ_{alg} the restriction to $\mathscr{G}$ of some algebraic representation of GL_2 (by an embedding as in Section 3.2.3), $\mathtt{t}$ is an inertial type of the form (red) and $\sigma_{\mathscr{G}}(\mathtt{t})$ is the $\mathscr{G}$-representation attached to $\mathtt{t}$ in Section 3.2.2.

4.5. Galois representations attached to quaternionic modular forms

4.5.1. *General results*. — Suppose now that A is a p-adic field E containing the unramified quadratic extension K' of K and a square root $\sqrt{\varpi_K}$ of ϖ_K. Then there is an embedding $\widetilde{u}$ of $\mathscr{G}$ into $\mathrm{GL}_2(E)$ as in Section 3.2.3. Suppose that for all $v\,|\,p$, the representation σ_v of Section 4.2 is of the form $\sigma_{v,\mathrm{alg}}\otimes\sigma_{v,\mathrm{sm}}$, where $\sigma_{v,\mathrm{sm}}$ is a smooth representation of U_v, and $\sigma_{v,\mathrm{alg}}$ is the restriction to U_v of an algebraic representation of GL_2 via $\widetilde{u}|_{\mathscr{O}_D^\times}$. We always assume that either $K=\mathbb{Q}_p$ or $\sigma_{v,\mathrm{alg}}$ is trivial for all v. If $K=\mathbb{Q}_p$, $\sigma_{v,\mathrm{alg}}$ is the restriction of a representation of the form $\mathrm{Sym}^{n_v}E^2\otimes\det^{m_v}$ and $k=n_v+2m_v+1$ does not depend on v.

We recall the construction and properties of Galois representations associated to eigenforms in $S_{\sigma,\eta}(U,E)$. See for example [Kis09b, §3.1.14] for the link between these spaces of modular forms and the classical spaces of automorphic representations, from which we deduce the properties of the Galois representations attached to them. Choose embeddings i_p, i_∞ of E into $\overline{\mathbb{Q}}_p$ and $\mathbb{C}$ respectively.

Let $\sigma_{p,\mathrm{alg}}=\otimes_v\sigma_{v,\mathrm{alg}}$ and $\sigma_{p,\mathrm{sm}}=\otimes_v\sigma_{v,\mathrm{sm}}$. Let $\sigma_{\mathbb{C},\mathrm{alg}}=\sigma_{\mathrm{alg}}\otimes_E\mathbb{C}$, $\sigma_{\mathbb{C},\mathrm{sm}}=\sigma_{\mathrm{sm}}\otimes_E\mathbb{C}$ and $\sigma_{\mathbb{C}}=\sigma_{\mathbb{C},\mathrm{alg}}\otimes\sigma_{\mathbb{C},\mathrm{sm}}$, acting on the space $W_{\mathbb{C}}$. Then $\sigma_{\mathbb{C},\mathrm{alg}}$ can be viewed as a representation of $B_\infty^\times=(B\otimes_{\mathbb{Q}}\mathbb{R})^\times$, and $\sigma_{\mathbb{C},\mathrm{sm}}$ is a smooth representation of $U_p=\otimes_vU_v$. Let U'_p be a compact open subgroup of U_p contained in $\ker\sigma_{p,\mathrm{sm}}$, and U'

the compact subgroup of $(B\otimes_F \mathbb{A}_F)^\times$ which is the same as U but with U_p replaced by U'_p.

Let $C^\infty(B^\times\backslash(B\otimes_F\mathbb{A}_F)^\times/U')$ the space of smooth functions with values in $\mathbb{C}$. It is endowed with a right action of $B_\infty^\times$.

We denote by $(\sigma^\vee, W^\vee)$ the dual of a representation (σ, W). Let

$$\phi : S_{\sigma,\eta}(U,E) \longrightarrow \mathrm{Hom}_{B_\infty^\times}(W_\mathbb{C}^\vee, C^\infty(B^\times\backslash(B\otimes_F\mathbb{A}_F)^\times/U'))$$

be the map defined by

$$\phi(f) = w \longmapsto \big(x\mapsto w(\sigma_{\mathbb{C},\mathrm{alg}}(x_\infty)^{-1}\sigma_{p,\mathrm{alg}}(x_p)f(x^\infty))\big),$$

where $x=(x^\infty,x_\infty)\in(B\otimes_F\mathbb{A}_F^f)^\times\times B_\infty^\times$. We denote by

$$\phi_w(f)\in C^\infty(B^\times\backslash(B\otimes_F\mathbb{A}_F)^\times/U')$$

the element $\phi(f)(w)$ for $w\in W_\mathbb{C}^\vee$.

Let $\pi=\otimes_v\pi_v$ be the irreducible automorphic representation of $B^\times$ generated by some non-zero $\phi_w(f)$ for $w\in W_\mathbb{C}^\vee$ and $f\in S_{\sigma,\eta}(U,E)$ that is an eigenform for T_S. Then π_∞ is isomorphic to $W_{alg,\mathbb{C}}^\vee$, and has central character

$$\eta_\mathbb{C}(z) = N_{F/\mathbb{Q}}(z_\infty)^{1-k}N_{F/\mathbb{Q}}(z_p)^{k-1}\eta(z_p)^{-1}.$$

Let $\rho_f : G_F\to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be the Galois representation attached to π, so that for all v not in S, the characteristic polynomial $\rho_f(\mathrm{Frob}_v)$ is $X^2-t_vX+N(v)s_v$, where Frob_v is an arithmetic Frobenius at v, and t_v and s_v are the eigenvalues of the Hecke operators T_v and S_v acting on f. Then ρ_f has determinant $\varepsilon\eta$ and for all $v\,|\,p$, $\rho_f|_{G_{F_v}}$ is potentially semi-stable with Hodge-Tate weights (m_v, m_v+n_v+1) if $K=\mathbb{Q}_p$ and $(0,1)_{\tau\in\mathrm{Hom}(K,\overline{\mathbb{Q}}_p)}$ otherwise, and $\mathrm{WD}(\rho_f|_{G_{F_v}})^{F-ss}$ is isomorphic to $\mathrm{LL}_D(\pi_v^\vee)$.

4.5.2. *Structure at p.* — Let v be in Σ_p. Let $\varphi:\sigma_\mathbb{C}^\vee\to\pi$ be given by $w\mapsto\phi_w(f)$, and $\varphi_v:\sigma_\mathbb{C}^\vee\to\pi_v$ be the projection to π_v. It is a non-zero U_v-equivariant morphism (where U_v acts on $\sigma_\mathbb{C}^\vee$ via its action by $\sigma_{v,\mathrm{sm}}$), hence $\pi_v|_{U_v}$ contains some irreducible constituent of $\sigma_{v,\mathrm{sm}}^\vee$. In particular if $\sigma_{v,\mathrm{sm}}$ is a copy of representations $\sigma_D(\mathtt{t}_v)$ attached to some discrete series inertial type $\mathtt{t}_v$ as in Section 3.2.1, $\rho_f|_{G_{F_v}}$ is of type $\mathtt{t}_v$ by Proposition 3.2.1.

Fix $v_0\in\Sigma_p$ and suppose that $\sigma_{v_0,\mathrm{sm}}$ is in fact a representation of $\mathscr{G}$. By the embedding $\widetilde{u}$, the representation $\sigma_{v_0,\mathrm{alg}}$ also extends to a representation of $\mathscr{G}$. Suppose that $W_{v_0}f=\alpha f$, where W_{v_0} is the Hecke operator defined in Section 4.4.1. Let μ be the central character of $\sigma_{v_0,\mathrm{alg}}$. We extend the representation $\sigma_{v_0,\mathrm{sm}}^\vee$ of $\mathscr{G}$ to a representation of $B_{v_0}^\times = D^\times$ by $\sigma_{v_0,\mathrm{sm}}^\vee\otimes\mathrm{unr}(\alpha\mu(\sqrt{\varpi_K}))$ (here $\sigma_{v_0,\mathrm{sm}}$ is seen as a representation of $D^\times$ via the canonical map $D^\times\to\mathscr{G}$). Then φ_{v_0} is equivariant for the action of the group $D^\times$ so that π_{v_0} is isomorphic to $\sigma_{v_0,\mathrm{sm}}^\vee\otimes\mathrm{unr}(\alpha\mu(\sqrt{\varpi_K}))$ if $\sigma_{v_0,\mathrm{sm}}$ is irreducible. This gives the following.

Lemma 4.5.1. — *If $\sigma_{v_0,\mathrm{sm}}=\sigma_\mathscr{G}(\mathtt{t})$ for some discrete series inertial type $\mathtt{t}$, $\sigma_{v_0,\mathrm{alg}}$ has central character μ and if $W_{v_0}f=\alpha f$ then $\rho_f|_{G_{F_{v_0}}}$ is of extended type $\mathrm{LL}_D(\sigma_\mathscr{G}(\mathtt{t}))\otimes\mathrm{unr}(\alpha\mu(\sqrt{\varpi_K}))^{-1}$.*

5. Proof of the main theorems

5.1. Notation. — In this section we fix K and a continuous representation $\overline{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$.

When $K = \mathbb{Q}_p$, we assume that $p \geqslant 5$ when $\overline{\rho}$ is a twist of an extension of the trivial character by the cyclotomic character (we need this condition to apply the results of [Paš15] and [HT15]).

When $K \neq \mathbb{Q}_p$, we assume that $\overline{\rho}$ is not a twist of an extension of the trivial character by the cyclotomic character, and whenever a Hodge-Tate type w appears we always mean $w = w_0$.

We fix a uniformizer ϖ_K of K.

Let $\overline{\psi}$ be the character $\omega^{-1}\det\overline{\rho}$ of G_K, which we see also as a character $K^\times$ via local class field theory. We fix $\alpha \in \overline{\mathbb{F}}_p$ such that $\alpha^2 = \overline{\psi}(\varpi_K)^{-1}$.

For any irreducible representation γ of $\Gamma = \Gamma_K$, we fix an inertial type $\mathtt{t}_\gamma$ and a representation $\sigma_{\mathscr{G}}(\mathtt{t}_\gamma)$ as in the proof of Proposition 3.3.2, a lift ψ_γ of $\overline{\psi}$ that is compatible with $\mathtt{t}_\gamma$ and w_0, and an extended type $\mathtt{t}_\gamma^+$ such that $\mathtt{t}_\gamma^+$ is compatible with $(\mathtt{t}_\gamma, \psi_\gamma)$ and $\mathtt{t}_\gamma^+ = \mathrm{LL}_D(\sigma_{\mathscr{G}}(\mathtt{t}_\gamma)) \otimes \mathrm{unr}(a_\gamma)^{-1}$ for an a_γ lifting α.

5.2. Definition of $\mu_{\overline{\rho}}$. — We are now able to define the linear form $\mu_{\overline{\rho}}$: we define it to be the linear form on $\mathscr{R}(\Gamma)$ such that $\mu_{\overline{\rho}}([\gamma]) = e(R^{\square,\psi_\gamma}(w_0, \mathtt{t}_\gamma^+, \overline{\rho})/\pi)$ for any irreducible representation γ of Γ. It is clear that $\mu_{\overline{\rho}}(\gamma) = 0$ if $i_{\overline{\rho}}(\gamma) = 0$.

We must now prove that $\mu_{\overline{\rho}}$ satisfies the properties claimed in Theorems 3.5.1 and 3.5.2.

Let $\mathtt{t}$ be a discrete series inertial type, w a Hodge-Tate type (with $w = w_0$ if $K \neq \mathbb{Q}_p$), ψ a lift of $\overline{\psi}$ that is compatible with $\mathtt{t}$ and w, and $\mathtt{t}^+$ an extended type compatible with $(\mathtt{t}, \psi)$.

If $R^{\square,\psi}(w, \mathtt{t}^{\mathrm{ds}}, \overline{\rho}) = 0$ then we have $\mu_{\overline{\rho}}([\overline{\sigma_{\mathscr{G}}}(\mathtt{t}) \otimes \overline{\sigma}_w]) = 0$ by the results of Section 3.4. So we need only prove the equalities of Theorems 3.5.1 and 3.5.2 when $R^{\square,\psi}(w, \mathtt{t}^{\mathrm{ds}}, \overline{\rho}) \neq 0$. This is the object of the rest of this section.

5.3. Global realization in characteristic p. — We start by realizing $\overline{\rho}$ in some global Galois representation.

Proposition 5.3.1. — *There exist a totally real field F and a continuous irreducible representation $\overline{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}})$, such that:*

(1) *the number of places of F above p has the same parity as the number of infinite places of F,*

(2) *for all $v \mid p$, F_v is isomorphic to K,*

(3) *for all $v \mid p$, $\overline{r}|_{G_{F_v}} \simeq \overline{\rho}$,*

(4) *$\overline{r}$ is unramified outside p,*

(5) *$\overline{r}$ is totally odd,*

(6) *$\overline{r}$ is modular,*

(7) *the restriction of $\overline{r}$ to $G_{F(\zeta_p)}$ is absolutely irreducible, and if $p = 5$, $\overline{r}$ does not have projective image isomorphic to $PGL_2(\mathbb{F}_5)$.*

Proof. — All conditions except the first follow from [GK14, Cor. A.3]. We can ensure that the first condition is satisfied by taking an F such that the number of places of F above p and the number of infinite places of F are both even. Indeed, note that the proof of [GK14, Cor. A.3] starts (in [GK14, Prop. A.1]) by considering an auxiliary number field E which is totally real and such that for all $v \mid p$, E_v is isomorphic to K, and the F we get is a finite extension of E. If we impose to E the additional condition that $2[K : \mathbb{Q}_p]$ divides $[E : \mathbb{Q}]$, then the parity condition will be satisfied. $\square$

5.4. Global realizations in characteristic 0

5.4.1. *Global data*. — From now on we fix a field F and a representation $\overline{r}$ satisfying the conditions of Proposition 5.3.1.

Let Σ_p be the set of places of F above p. We fix a $v_0 \in \Sigma_p$ and denote $\Sigma_p \smallsetminus \{v_0\}$ by Σ'_p as before.

Let B be the quaternion algebra with center F that is ramified exactly at the infinite places of F and at all places in Σ_p. Such a B exists thanks to condition (1) of Proposition 5.3.1. Let D be the non-split quaternion algebra over K. Let $\mathscr{O}_B$ be a maximal order in B. For all v not dividing p, we fix an isomorphism $\mathscr{O}_{B_v}^\times \simeq \mathrm{GL}_2(\mathscr{O}_{F_v})$, and for all $v \in \Sigma_p$, we fix an isomorphism $\mathscr{O}_{B_v}^\times \simeq \mathscr{O}_D^\times$.

Let ϖ_D be a uniformizer of $D = B_{v_0}$ such that $\varpi_D^2 = \varpi_K$, where ϖ_K is our fixed uniformizer of K. We set $\mathscr{G} = \mathscr{G}_{\varpi_K}$.

We choose an auxiliary place $v_1 \nmid p$ such that $Nv_1 \neq 1 \pmod p$, the ratio of the eigenvalues of $\overline{r}(\mathrm{Frob}_{v_1})$ is not $Nv_1^{\pm 1}$, and the characteristic of v_1 is large enough so that for any quadratic extension F' of F and any ζ a root of unity in F', $v_1 \nmid \zeta + \zeta^{-1} - 2$. The existence of such a place v_1 follows from [DDT97, Lem. 4.11] and [Kis09a, Lem. 2.2.1].

We let U be the compact open subgroup of $(B \otimes_F \mathbb{A}_F^f)^\times$ such that $U_v = \mathscr{O}_D^\times$ for $v \in \Sigma_p$, $U_v = \mathrm{GL}_2(\mathscr{O}_{F_v})$ for $v \notin \Sigma_p$ and $v \neq v_1$, and finally U_{v_1} is the set of elements of $\mathrm{GL}_2(\mathscr{O}_{F_{v_1}})$ that are upper triangular unipotent modulo v_1. The last condition we imposed on v_1 ensures that U is small enough in the sense of Section 4.2 (see [Kis09a, §2.1.1].).

5.4.2. *Modular lift*. — We want know to show that the representation $\overline{r}$ can be lifted to an appropriate modular Galois representation.

Lemma 5.4.1. — *For all $v \in \Sigma_p$, let $\mathtt{t}_v$ be an inertial type such that $\gamma_v = \sigma_D(\mathtt{t}_v)$ is an irreducible representation of $\ell^\times$, and ψ_v be a character of G_{F_v}. Suppose that the ring*

$$R_p = \widehat{\otimes}_{v \in \Sigma_p} R^{\square, \psi_v}(w_0, \mathtt{t}_v^{\mathrm{ds}}, \overline{\rho})$$

is not zero. Let $\sigma = \otimes_{v \in \Sigma_p} \sigma_D(\mathtt{t}_v)$. Then there exists η satisfying the compatibility conditions with σ of Section 4.2, and which restricts to ψ_v on $F_v^\times$ for all $v \in \Sigma_p$, and the space of modular forms $S_{\sigma,\eta}(U, \mathscr{O})$ contains an eigenform f whose associated Galois representation has its reduction modulo p isomorphic to $\overline{r}$.

Proof. — For the existence and construction of η, see [GK14, §5.4.1].

We now prove the existence of the eigenform f.

Suppose first that none of the $\mathtt{t}_v$ is scalar. Then each $R^{\square,\psi_v}(w_0,\mathtt{t}_v^{\mathrm{ds}},\overline{\rho})$ parametrizes only potentially crystalline representations. Corollary 3.1.7 of [Gee11] applied to our situation gives that for each irreducible component of $\operatorname{Spec} R_p[1/p]$, there exists a lift r of $\overline{r}$ that is modular, unramified outside p, with determinant $\eta\varepsilon$, potentially crystalline with Hodge-Tate weights $(0,1)$ at each $v\in\Sigma_p$ and for each $F_v\to\overline{\mathbb{Q}}_p$ and defining a point on the given irreducible component. The representation r comes from some automorphic form on a quaternion algebra over F. Thanks to the local conditions on r, we can suppose that r comes from a modular form f on B, and that $f\in S_{\sigma,\eta}(U,\mathscr{O})$. This proves the claim in this case, and in particular whenever $\overline{\rho}$ is not isomorphic to a twist of $\left(\begin{smallmatrix}\omega & * \\ 0 & 1\end{smallmatrix}\right)$ by Proposition 3.5.6.

When $\overline{\rho}$ is isomorphic to a twist of $\left(\begin{smallmatrix}\omega & * \\ 0 & 1\end{smallmatrix}\right)$ (in the case $K=\mathbb{Q}_p$), Corollary 3.1.7 of [Gee11] is not enough: for $\dim\gamma_v=1$, we need points on $\operatorname{Spec} R^{\square,\psi_v}(w_0,\mathtt{t}_v^{\mathrm{ds}},\overline{\rho})[1/p]$ corresponding to potentially semi-stable representations that are not potentially crystalline. We make use of [BD14, Th. 3.2.2] instead: it allows us to choose r such that each $r|_{G_{F_v}}$ is not potentially crystalline when $\dim\gamma_v=1$. Note that in this situation, each $R^{\square,\psi_v}(w_0,\mathtt{t}_{\gamma_v}^{\mathrm{ds}},\overline{\rho})$ is irreducible. $\square$

Let $\mathtt{t}$ be a discrete series inertial type and w be a Hodge-Tate type, and let ψ be the character defined in Section 5.1.

Proposition 5.4.2. — *Suppose that $R^{\square,\psi}(w,\mathtt{t}^{\mathrm{ds}},\overline{\rho})\neq 0$. There exists a character η of $(\mathbb{A}_F^f)^\times/F^\times$ which restricts to ψ on $F_v^\times$ for all $v\in\Sigma_p$ such that, for $\sigma=\otimes_{v\in\Sigma_p}(\sigma_D(\mathtt{t})\otimes\sigma_w)$, the space of modular forms $S_{\sigma,\eta}(U,\mathscr{O})$ contains an eigenform f whose associated Galois representation has its reduction modulo p isomorphic to $\overline{r}$.*

Proof. — As before, the existence of η satisfying the compatibility conditions with σ of Section 4.2 and whose restriction to $F_v^\times$ coincides ψ for all $v\in\Sigma_p$, comes from [GK14, 5.4.1].

If $R^{\square,\psi}(w,\mathtt{t}^{\mathrm{ds}},\overline{\rho})\neq 0$, by Theorems 3.4.1 and 3.4.3 there exist an irreducible representation γ of $\prod_{v\in\Sigma_p}\ell^\times$, $\gamma=\otimes_{v\in\Sigma_p}\gamma_v$ appearing as an irreducible constituent of $\otimes_{v\in\Sigma_p}(\overline{\sigma_D}(\mathtt{t})\otimes\overline{\sigma_w})$ and characters ψ_v that are equal to a finite order character times the cyclotomic character such that $\widehat{\otimes}_{v\in\Sigma_p}R^{\square,\psi_v}(w_0,\mathtt{t}_v^{\mathrm{ds}},\overline{\rho})\neq 0$, where $\mathtt{t}_v$ is the inertial type attached to the representation γ_v as in the proof of Proposition 3.3.2.

We can apply Lemma 5.4.1 to the family of types $(\mathtt{t}_v)$, as by construction $\sigma_D(\mathtt{t}_v)=\gamma_v$ is an irreducible representation of $\ell^\times$. Then the result follows from [GS11, Lem. 2.1]. $\square$

In particular in the conditions of Proposition 5.4.2, $r_f|_{G_{F_{v_0}}}$ has determinant $\varepsilon\psi$, inertial type $\mathtt{t}$ and Hodge-Tate type w, where r_f is the Galois representation attached to f as in Section 4.5.1.

5.5. Patching. — Let $(\mathtt{t}_v, w_v)_{v\in\Sigma_p}$ be a family of discrete series inertial types and Hodge-Tate types, with $\mathtt{t}_{v_0} = \mathtt{t}$ and $w_{v_0} = w$. Let also $(\psi_v)_{v\in\Sigma_p}$ be a family of characters of $G_K = G_{F_v}$.

We suppose in this paragraph that there exists a character η of $(\mathbb{A}_F^f)^\times/F^\times$ that restricts to ψ_v on $F_v^\times$ for all $v \in \Sigma_p$ and such that for $\sigma = \otimes_{v\in\Sigma_p}(\sigma_D(\mathtt{t}_v)\otimes\sigma_{w_v})$, the space of modular forms $S_{\sigma,\eta}(U,\mathscr{O})$ contains an eigenform f whose associated Galois representation has its reduction modulo p isomorphic to $\overline{r}$.

Consider now

$$\widetilde{\sigma} = (\sigma_{\mathscr{G}}(\mathtt{t}))\otimes\sigma_w)\otimes(\otimes_{v\in\Sigma_p'}(\sigma_D(\mathtt{t}_v)\otimes\sigma_{w_v}).$$

It is a representation of $\mathscr{G}\times\prod_{v\in\Sigma_p'} U_v$, where $U_v = \mathscr{O}_D^\times$. The space of modular forms $S_{\widetilde{\sigma},\eta}(U,\mathscr{O})$ is either the same as $S_{\sigma,\eta}(U,\mathscr{O})$ as an $\mathscr{O}$-module and a T_S-algebra (if $\mathtt{t}$ is of the form (scal) or (irr), as in this case $\sigma_{\mathscr{G}}(\mathtt{t})$ acts on the same space as $\sigma_D(\mathtt{t})$) or is two copies of $S_{\sigma,\eta}(U,\mathscr{O})$ (if $\mathtt{t}$ is of the form (red), as in this case $\sigma_{\mathscr{G}}(\mathtt{t})$ acts on a space which is two copies $\sigma_D(\mathtt{t})$, see Section 4.4.2). In any case, it contains a copy of the form f given by Proposition 5.4.2. Let $\mathfrak{m}$ be a maximal ideal of T_S containing p such that f is in $S_{\widetilde{\sigma},\eta}(U,\mathscr{O})_{\mathfrak{m}}$.

Let $\mathbb{T}_{\widetilde{\sigma},\eta}(U,\mathscr{O})_{\mathfrak{m}}$ be the image of $T_{S,\mathfrak{m}}$ in the endomorphism ring of $S_{\widetilde{\sigma},\eta}(U,\mathscr{O})_{\mathfrak{m}}$. Any eigenform in $S_{\widetilde{\sigma},\eta}(U,\mathscr{O})_{\mathfrak{m}}$ has an associated Galois representation with residual representation isomorphic to $\overline{r}$, which is absolutely irreducible. By the main result of [Tay89] and the Jacquet-Langlands correspondence (see [Tay06, Lem. 1.3]) we deduce that we have a Galois representation

$$r_{\mathfrak{m}} : G_F \longrightarrow \mathrm{GL}_2(\mathbb{T}_{\widetilde{\sigma},\eta}(U,\mathscr{O})_{\mathfrak{m}})$$

coming from all the eigenforms in $S_{\widetilde{\sigma},\eta}(U,\mathscr{O})_{\mathfrak{m}}$. In particular, $r_{\mathfrak{m}}$ is unramified outside p.

Let $\mathscr{R}$ be the universal deformation ring for deformations of $\overline{r}$ that are unramified outside Σ_p and $\mathscr{R}^\square$ the framed analogue. Then we have a map $\mathscr{R}\to\mathbb{T}_{\widetilde{\sigma},\eta}(U,\mathscr{O})_{\mathfrak{m}}$ coming from $r_{\mathfrak{m}}$, so $\mathscr{R}$ acts on $S_{\widetilde{\sigma},\eta}(U,\mathscr{O})_{\mathfrak{m}}$ via the Hecke operators.

Let $R^\square(\overline{\rho})$ the ring classifying lifts of $\overline{\rho}$, and $\mathscr{R}_p^\square = \widehat{\otimes}_{v\in\Sigma_p} R^\square(\overline{\rho})$. Let R_p be

$$\widehat{\otimes}_{v\in\Sigma_p} R^{\square,\psi_v}(w_v, \mathtt{t}_v^{\mathrm{ds}}, \overline{\rho})$$

seen as a quotient of $\mathscr{R}_p^\square$, and $\mathscr{R}' = \mathscr{R}^\square\otimes_{\mathscr{R}_p^\square} R_p$. The ring $\mathscr{R}'$ is the universal ring for lifts of $\overline{r}$ that are unramified outside p and potentially semi-stable of inertial type $\mathtt{t}_v$, of some discrete series extended type, of Hodge-Tate type w_v and determinant $\varepsilon\psi$ at each place $v\in\Sigma_p$. Let $M_0 = \mathscr{R}^\square\otimes_{\mathscr{R}} S_{\widetilde{\sigma},\eta}(U,\mathscr{O})_{\mathfrak{m}}$. Then the action of $\mathscr{R}^\square$ on M_0 factors through $\mathscr{R}'$ by the results recalled in Section 4.5.2.

We decompose the reduction $\overline{\sigma}$ of $\widetilde{\sigma}$ modulo p as a direct sum $\overline{\sigma} = \oplus_\gamma\gamma^{\oplus n_\gamma}$, with $\gamma = \gamma_{v_0}\otimes(\otimes_{v\in\Sigma_p'}\gamma_v)$, of representations of $\Gamma\times\prod_{v\in\Sigma_p'}\ell^\times$. This gives a decomposition $\overline{\sigma} = \oplus_\gamma\gamma^{n_\gamma}$ as a representation of $\mathscr{G}\times\prod_{v\in\Sigma_p'} U_v$.

Using the techniques of [Kis09a, §(2.2)] we construct the following objects:

(1) a ring R_∞ which is a power series ring on R_p (this is $\overline{R}_\infty$ of [Kis09a]),

(2) a ring S_∞ which is a power series ring on $\mathscr{O}$ (this is $\mathscr{O}[[\Delta_\infty]]$ of [Kis09a]),

(3) an (S_∞, R_∞)-module M_∞ that is free as an S_∞-module, and such that M_0 is a quotient of M_∞,

(4) a (S_∞, R_∞)-linear operator W_{v_0} on M_∞ compatible with the Hecke operator W_{v_0} on M_0 defined in Section 4.4.1, with $W_{v_0}^2 = \psi_{v_0}(\varpi_K)^{-1}$,

(5) a decomposition $M_\infty \otimes \mathbb{F} = \oplus_\gamma \overline{M}_{\infty,\gamma}^{\oplus n_\gamma}$ as (S_∞, R_∞)-module, such that each $\overline{M}_{\infty,\gamma}$ is a finite free $S_\infty \otimes \mathbb{F}$-module, and such that moreover $\overline{M}_{\infty,\gamma}$ does not depend on $(\mathtt{t}_{v_0}, w_{v_0})$ or the $(\mathtt{t}_v, w_v)$, in a sense that is explained below.

The only part which is not a copy of the arguments of [Kis09a] is (4). The module M_∞ is built by patching modules $M_n = \mathscr{R}^\square \otimes_{\mathscr{R}} S_{\widetilde{\sigma},\eta}(U_n, \mathscr{O})_{\mathfrak{m}_n}$ for some choice of compact open subgroup U_n which is maximal at v for all $v \mid p$, and for some choice of maximal ideal $\mathfrak{m}_n$ (with $U_0 = U$ and $\mathfrak{m}_0 = \mathfrak{m}$). In particular, for each n there is a Hecke operator W_{v_0} on M_n as in Section 4.4.1 with $W_{v_0}^2 = \eta(\varpi_{K,v_0})^{-1} = \psi_{v_0}(\varpi_K)^{-1}$ which is compatible with the surjective map $M_n \to M_0$. Moreover, the action of W_{v_0} commutes with the right action of the subgroup of $(B \otimes_F \mathbb{A}_F^f)^\times$ of elements that are trivial at v_0, hence W_{v_0} is R_∞- and S_∞-linear. Once a square root of $\eta(\varpi_{K,v_0})^{-1}$ is fixed, then for each n we have a decomposition $M_n = M_n^+ \oplus M_n^-$ in sub-(S_∞, R_∞)-modules according to the eigenvalues of W_{v_0}, and this decomposition is compatible to the decomposition $M_0^+ \oplus M_0^-$. We apply the patching argument not only to M_n, but to the decomposition $M_n^+ \oplus M_n^-$, which gives a decomposition $M_\infty = M_\infty^+ \oplus M_\infty^-$ and an operator W_{v_0} on M_∞ with the required properties. Note that M_∞^+ and M_∞^- are also free as S_∞-modules.

Let γ be an irreducible smooth representation of $\mathscr{G} \times \prod_{v \in \Sigma'_p} U_v$ in characteristic p, and $\widetilde{\gamma}$ a smooth lift of γ (as in Proposition 3.3.2 for the part which is a representation of $\mathscr{G}$ and by Teichmüller lift for the parts which are representations of $U_v = \mathscr{O}_D^\times$). By Lemma 5.4.1, there exists a character η_γ of $(\mathbb{A}_F^f)^\times / F^\times$ lifting $\overline{\eta}$ and characters $(\psi_{\gamma,v})_{v \in \Sigma_p}$ satisfying the conditions at the beginning of this Paragraph, so we can make the constructions with the space of modular forms $S_{\widetilde{\gamma},\eta_\gamma}(U, \mathscr{O})_{\mathfrak{m}}$. We denote by $M_{\infty,\gamma}$ the patched module we obtain (although it depends not only on γ but also on other data). Then (5) means that $\overline{M}_{\infty,\gamma}$ is isomorphic to the reduction modulo p of $M_{\infty,\gamma}$. We also have a Hecke operator W_{v_0} on $M_{\infty,\gamma}$ and a decomposition $M_{\infty,\gamma} = M_{\infty,\gamma}^+ \oplus M_{\infty,\gamma}^-$ that reduces to $\overline{M}_{\infty,\gamma} = \overline{M}_{\infty,\gamma}^+ \oplus \overline{M}_{\infty,\gamma}^-$.

5.6. Equality of multiplicities. — Recall that $d_{\mathtt{t}} = 2$ if $\mathtt{t}$ is of the form (red) and $d_{\mathtt{t}} = 1$ otherwise. As in [Kis09a, Lem. (2.2.11)], M_∞ has rank 0 or $d_{\mathtt{t}}$ at each generic point of R_∞ and

$$e(M_\infty/\pi M_\infty, R_\infty/\pi R_\infty) \leqslant d_{\mathtt{t}} e(R_\infty/\pi R_\infty),$$

with equality if and only if the support of M_∞ is all of $\operatorname{Spec} R_\infty$ (we already know that it is a union of irreducible components of $\operatorname{Spec} R_\infty$).

Our main ingredient is the following Proposition, which we prove using the results of Section 3.4 and the methods of [Kis09a, §(2.2)].

Proposition 5.6.1. — *We have* $e(M_\infty/\pi M_\infty, R_\infty/\pi R_\infty) = d_{\mathtt{t}} e(R_\infty/\pi R_\infty)$.

Proof. — Suppose first that $w_v = w_0$ for all $v \in \Sigma_p$. By reasoning as in the proof of Proposition 5.4.2, we see that the support of the module M_0 meets each irreducible component of $\operatorname{Spec} R_\infty[1/p]$. The irreducible components of $\operatorname{Spec} R_\infty[1/p]$ are connected components, hence we can apply the criterion of [GK14, Lem. 4.3.8]: the equality of Proposition 5.6.1 holds if and only if the support of M_0 meets every irreducible component of $\operatorname{Spec} R_\infty[1/p]$.

Let us now return to the case without conditions on the w_v (in the case $K = \mathbb{Q}_p$). It follows from the decomposition $M_\infty \otimes \mathbb{F} = \oplus_\gamma \overline{M}_{\infty,\gamma}^{\oplus n_\gamma}$ that

$$e(M_\infty/\pi, R_\infty/\pi) = \sum_\gamma n_\gamma e(\overline{M}_{\infty,\gamma}, R_\infty/\pi).$$

Moreover,

$$e(R_\infty/\pi R_\infty) = \prod_{v\in\Sigma_p} e(R^{\square,\psi_v}(w_v, \mathtt{t}^{\mathrm{ds}}_{\gamma_v}, \overline{\rho})/\pi).$$

As $K = \mathbb{Q}_p$, it follows from the theorems of Section 3.4 that we have

$$d_{\mathtt{t}} e(R_\infty/\pi R_\infty) = \sum_\gamma n_\gamma (\dim \gamma_{v_0}) \prod_{v\in\Sigma_p} e(R^{\square,\psi}(w_0, \mathtt{t}^{\mathrm{ds}}_{\gamma_v}, \overline{\rho})/\pi).$$

We denote by $R_{\infty,\gamma}$ the ring that is the analogue of R_∞ but with $(w_0, \mathtt{t}_{\gamma_v})$ instead of $(w_v, \mathtt{t}_v)$ for all $v \in \Sigma_p$, and the characters $(\psi_{\gamma,v})_{v\in\Sigma_p}$ defined at the end of Section 5.5 instead of (ψ_v). Let also $R_{p,\gamma}$ be the analogue of R_p.

Then the image of R_∞ and $R_{\infty,\gamma}$ in the endomorphisms of $\overline{M}_{\infty,\gamma}$ is the same, as follows from (4) of the properties of patching, hence

$$e(\overline{M}_{\infty,\gamma}, R_\infty/\pi) = e(\overline{M}_{\infty,\gamma}, R_{\infty,\gamma}/\pi).$$

Moreover,

$$e(R_{\infty,\gamma}/\pi) = \prod_{v\in\Sigma_p} e(R^{\square,\psi_{\gamma,v}}(w_0, \mathtt{t}^{\mathrm{ds}}_{\gamma_v}, \overline{\rho})/\pi),$$

and we have $e(\overline{M}_{\infty,\gamma}, R_{\infty,\gamma}/\pi) = (\dim \gamma_{v_0}) e(R_{\infty,\gamma}/\pi)$ by applying the part of Proposition 5.6.1 that we have already proved to $\overline{M}_{\infty,\gamma}$ and $R_{\infty,\gamma}$, which concludes the proof of Proposition 5.6.1 in the general case. □

5.7. Action of the Hecke operator at p. — We apply the results of the preceding paragraphs in the situation coming from Proposition 5.4.2, so in particular $w_v = w$ and $\mathtt{t}_v = \mathtt{t}$ and $\psi_v = \psi$ for all $v \in \Sigma_p$. Recall that we have chosen a square root α of $\overline{\eta}(\varpi_{K,v_0})^{-1} = \overline{\psi}(\varpi_K)^{-1}$.

Let $(\mathtt{t}^+, \mathtt{t}^-) = (\mathtt{t}^+_{v_0}, \mathtt{t}^-_{v_0})$ be a pair of conjugate extended types compatible to $(\mathtt{t}, \psi)$. We set

$$\begin{aligned} R^+_\infty &= R^{\square,\psi}(w_{v_0}, \mathtt{t}^+_{v_0}, \overline{\rho}) \otimes_{R^{\square,\psi}(w_{v_0}, \mathtt{t}^{\mathrm{ds}}_{v_0}, \overline{\rho})} R_\infty, \\ R^-_\infty &= R^{\square,\psi}(w_{v_0}, \mathtt{t}^-_{v_0}, \overline{\rho}) \otimes_{R^{\square,\psi}(w_{v_0}, \mathtt{t}^{\mathrm{ds}}_{v_0}, \overline{\rho})} R_\infty \end{aligned}$$

(so that $R_\infty = R^+_\infty = R^-_\infty$ when $\mathtt{t}$ is of the form (red), and these rings differ only when $\mathtt{t}$ is of the form (scal) or (irr)).

We make a choice for $\sigma_{\mathscr{G}}(\mathtt{t})$ so that

$$\mathtt{t}^+ = \mathrm{LL}_D(\sigma_{\mathscr{G}}(\mathtt{t})) \otimes \mathrm{unr}(a\sqrt{\varpi_K}^{|w|})^{-1}$$

for a lift a of α with $a^2 = \psi(\varpi_K)^{-1}$ (recall that $|w| = n+2m$ if $K = \mathbb{Q}_p$ and $w=(n,m)$, and set $|w_0| = 0$ for any K).

We can consider the Hecke operator W_{v_0} acting on all the spaces of modular forms that we have defined. This gives a decomposition $M_\infty = M_\infty^+ \oplus M_\infty^-$ as in Section 5.5, where M_∞^+ is the submodule on which W_{v_0} acts by a lift of α, and decompositions $M_n = M_n^+ \oplus M_n^-$ for the modules M_n.

The action of R_∞ on M_∞^+ factors through R_∞^+, and similarly for R_∞^-. Indeed this is true for each $M_n = M_n^+ \oplus M_n^-$ by Lemma 4.5.1.

We can do the same thing for each irreducible representation γ of Γ: recall that for each γ we made a choice in Section 5.1 of an inertial type $\mathtt{t}_\gamma$ and a representation $\sigma_{\mathscr{G}}(\mathtt{t}_\gamma)$ of $\mathscr{G}$ such that $\overline{\sigma_{\mathscr{G}}}(\mathtt{t}_\gamma)$ is isomorphic to γ and an extended type

$$\mathtt{t}_\gamma^+ = \mathrm{LL}_D(\sigma_{\mathscr{G}}(\mathtt{t}_\gamma)) \otimes \mathrm{unr}(a_\gamma)^{-1}$$

for an a_γ lifting α with $W_{v_0}^2 = a_\gamma^2$ on $M_{\infty,\gamma}$. Let

$$\begin{aligned}\mathtt{t}_\gamma^- &= \mathrm{LL}_D(\xi\sigma_{\mathscr{G}}(\mathtt{t}_\gamma)) \otimes \mathrm{unr}(a_\gamma)^{-1}\\ &= \mathrm{LL}_D(\sigma_{\mathscr{G}}(\mathtt{t}_{\xi\gamma})) \otimes \mathrm{unr}(a_\gamma)^{-1}\\ &= \mathrm{LL}_D(\sigma_{\mathscr{G}}(\mathtt{t}_\gamma)) \otimes \mathrm{unr}(-a_\gamma)^{-1}.\end{aligned}$$

For γ of dimension 2 we set $R_{\infty,\gamma}^+ = R_{\infty,\gamma}^- = R_{\infty,\gamma}$ and for γ of dimension 1 we set

$$\begin{aligned}R_{\infty,\gamma}^+ &= R^{\square,\psi}(w_0,\mathtt{t}_\gamma^+,\overline{\rho}) \otimes_{R^{\square,\psi}(w_0,\mathtt{t}_\gamma^{\mathrm{ds}},\overline{\rho})} R_{\infty,\gamma},\\ R_{\infty,\gamma}^- &= R^{\square,\psi}(w_0,\mathtt{t}_\gamma^-,\overline{\rho}) \otimes_{R^{\square,\psi}(w_0,\mathtt{t}_\gamma^{\mathrm{ds}},\overline{\rho})} R_{\infty,\gamma}.\end{aligned}$$

Then for all γ the action of $R_{\infty,\gamma}$ on $\overline{M}_{\infty,\gamma}^\pm$ factors through $R_{\infty,\gamma}^\pm$ as before. Note that $\mathtt{t}_\gamma^- = \mathtt{t}_{\xi\gamma}^+$ and $R_{\infty,\gamma}^- = R_{\infty,\xi\gamma}^+$.

Note that the decompositions $M_\infty = M_\infty^+ \oplus M_\infty^-$ and $\overline{M}_{\infty,\gamma} = \overline{M}_{\infty,\gamma}^+ \oplus \overline{M}_{\infty,\gamma}^-$ for all γ are compatible, that is,

$$\overline{M}_\infty^\pm \otimes \mathbb{F} = \oplus_\gamma (\overline{M}_{\infty,\gamma}^\pm)^{n_\gamma}.$$

In particular, we have $e(M_\infty^\pm/\pi, R_\infty^\pm/\pi) = e(M_\infty^\pm/\pi, R_\infty/\pi)$, hence

$$e(M_\infty^+/\pi, R_\infty^+/\pi) + e(M_\infty^-/\pi, R_\infty^-/\pi) = e(M_\infty/\pi, R_\infty/\pi).$$

We also have that $e(M_\infty^\pm/\pi, R_\infty^\pm/\pi) \leqslant e(R_\infty^\pm/\pi)$ by the same argument as in [Kis09a, Lem. (2.2.11)]. Moreover,

$$d_{\mathtt{t}} e(R_\infty/\pi) = e(R_\infty^+/\pi) + e(R_\infty^-/\pi)$$

(see Proposition 3.5.4). Hence we deduce from Proposition 5.6.1 that

$$e(M_\infty^\pm/\pi, R_\infty^\pm/\pi) = e(R_\infty^\pm/\pi) \quad\text{and}\quad e(\overline{M}_{\infty,\gamma}^\pm, R_{\infty,\gamma}^\pm/\pi) = e(R_{\infty,\gamma}^\pm/\pi) \text{ for all } \gamma.$$

Finally we get that

$$e(R^{\square,\psi}(w,\mathtt{t}^+,\overline{\rho})/\pi)=\sum_\gamma[\overline{\sigma_{\mathscr{G}}}(\mathtt{t})\otimes\overline{\sigma_w}:\gamma]e(R^{\square,\psi}(w_0,\mathtt{t}_\gamma^+,\overline{\rho})/\pi),$$

as $e(R_\infty/\pi)\neq 0$. As the right-hand side is $\mu_{\overline{\rho}}([\overline{\sigma_{\mathscr{G}}}(\mathtt{t})\otimes\overline{\sigma_w}])$ by the definition of $\mu_{\overline{\rho}}$ in Section 5.2, we get that

$$e(R^{\square,\psi}(w,\mathtt{t}^+,\overline{\rho})/\pi)=\mu_{\overline{\rho}}([\overline{\sigma_{\mathscr{G}}}(\mathtt{t})\otimes\overline{\sigma_w}]),$$

which finishes the proof of Theorems 3.5.1 and 3.5.2.

6. Application

6.1. Computation of weights. — In this section we suppose $p\geqslant 5$.

Let $\overline{\rho}$ be a continuous representation $G_{\mathbb{Q}_p}\to\mathrm{GL}_2(\mathbb{F})$ such that $\overline{\rho}|_{I_p}=\left(\begin{smallmatrix}\omega & *\\ 0 & 1\end{smallmatrix}\right)\otimes\omega^n$. We compute $\mu_{\overline{\rho}}$ in this case for the choice $\varpi_{\mathbb{Q}_p}=p$.

For a representation of the form

$$\overline{\rho}=\begin{pmatrix}\omega & *\\ 0 & 1\end{pmatrix}\otimes\omega^n\otimes\mathrm{unr}(x)$$

then $\overline{\psi}(p)=x^2$. In order to apply Theorem 3.5.1 we need to make a choice of a square root of $\overline{\psi}(p)^{-1}$, and we take this square root to be $\alpha=x^{-1}$.

Lemma 6.1.1. — *Let $\overline{\rho}|_{I_p}=\left(\begin{smallmatrix}\omega & *\\ 0 & 1\end{smallmatrix}\right)\otimes\omega^n$ for some n with $*$ "très ramifié" (and non-zero), then $\mu_{\overline{\rho}}(\xi^n\delta_n)=1$ and all other $\mu_{\overline{\rho}}(\gamma)$ are 0.*

Let $\overline{\rho}|_{I_p}=\left(\begin{smallmatrix}\omega & *\\ 0 & 1\end{smallmatrix}\right)\otimes\omega^n$ for some n with $*$ "peu ramifié" but non-zero, then $\mu_{\overline{\rho}}(\xi^n\delta_n)=1$ and $\mu_{\overline{\rho}}(r_{n(p+1)+p-1})=2$ and all other $\mu_{\overline{\rho}}(\gamma)$ are 0.

Let $\overline{\rho}|_{I_p}=\left(\begin{smallmatrix}\omega & 0\\ 0 & 1\end{smallmatrix}\right)\otimes\omega^n$ for some n, then $\mu_{\overline{\rho}}(\xi^n\delta_n)=1$ and $\mu_{\overline{\rho}}(r_{n(p+1)+p-1})=4$ and all other $\mu_{\overline{\rho}}(\gamma)$ are 0.

Proof. — We can compute $e(R^{\square,\psi}(w,\mathrm{triv}^{\mathrm{ds}},\overline{\rho})/\pi)$ for any Hodge-Tate type w using the formula coming from the Breuil-Mézard conjecture for GL_2 and the list of the Serre weights with their multiplicities given in [BM02, §2.1.2]. Then we compare this to the formula for this multiplicity given by Theorem 3.5.1, using also the formula given by Lemma 3.5.12.

We compute first $e(R^{\square,\psi}(w,\mathrm{triv}^{\mathrm{ds}},\overline{\rho})/\pi)$ for Hodge-Tate types of the form $w=(0,m)$. We get that $\mu_{\overline{\rho}}(\xi^\alpha\delta_m)=0$ for $\alpha=0,1$ if m is not equal to n modulo $p-1$, and $\mu_{\overline{\rho}}(\delta_n)+\mu_{\overline{\rho}}(\xi\delta_n)=1$. Using the computations in [BM02, §5.2.1] we see that in fact $\mu_{\overline{\rho}}(\xi^n\delta_n)=1$.

By computing $e(R^{\square,\psi}(w,\mathrm{triv}^{\mathrm{ds}},\overline{\rho})/\pi)$ for Hodge-Tate types w of the form (a,b) for $a>0$ we can find the value of $\mu_{\overline{\rho}}(r)$ for representations r of dimension 2. $\square$

6.2. An application to congruences modulo p in $S_k(\Gamma_0(p))$. — Let f be a newform in $S_k(\Gamma_0(p))$. Then $a_p(f)=\pm p^{k/2-1}$. We denote by ρ_f the p-adic Galois representation associated to f, r_f its reduction modulo p, and $r_{f,p}$ its restriction to a decomposition group $G_{\mathbb{Q}_p}$ at p.

Theorem 6.2.1. — *Let $k > 2$ be an even integer, f a newform in $S_k(\Gamma_0(p))$.*

(1) *Suppose that $r_{f,p}$ is of the form*

$$\begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix} \otimes \omega^{k/2-1} \otimes \mathrm{unr}(x)$$

for some x and $*$ "très ramifié" (and non-zero) and $k \leqslant 2p+2$. Then

$$x^{-1} = (-1)^{k/2-1}\overline{(a_p(f)/p^{k/2-1})}.$$

In particular, there does not exist a newform g in $S_k(\Gamma_0(p))$ congruent to f modulo p such that $a_p(g) = -a_p(f)$.

(2) *Suppose that either $r_{f,p}|_{I_p}$ is not of the form $\left(\begin{smallmatrix} \omega & * \\ 0 & 1 \end{smallmatrix}\right) \otimes \omega^{k/2-1}$ with $*$ "très ramifié" or that $k > 2p+2$. If $r_f|_{G_{\mathbb{Q}(\zeta_p)}}$ is absolutely irreducible then there exists a newform $g \in S_k(\Gamma_0(p))$ congruent to f modulo p such that $a_p(g) = -a_p(f)$.*

Proof. — Let $u_p(f) = a_p(f)p^{1-k/2}$. By the results of [Sai09], ρ_f is a semi-stable, non-crystalline representation with extended type

$$\mathtt{t}_f = (\|\cdot\|^{k/2-1} \oplus \|\cdot\|^{k/2}) \otimes \mathrm{unr}(u_p(f)^{-1}) = (1 \oplus \|\cdot\|) \otimes \mathrm{unr}(u_p(f)^{-1}p^{1-k/2}).$$

Let w_k be the Hodge-Tate type $(k-2, 0)$ and $\psi_k = \varepsilon^{k-2}$.

Suppose first that $r_{f,p}$ is of the form

$$\begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix} \otimes \omega^{k/2-1} \otimes \mathrm{unr}(x),$$

with $*$ "très ramifié". By the existence of f, $R^{\square,\psi_k}(w_k, \mathtt{t}_f, r_{f,p})$ is non-zero. With the normalization $\varpi_{\mathbb{Q}_p} = p$ as before, there is a choice of $i \in \mathbb{Z}/2\mathbb{Z}$ with $\sigma_{\mathscr{G}}(\mathrm{triv}) = \xi^i$ such that

$$\mathtt{t}_f = (1 \oplus \|\cdot\|) \otimes \mathrm{unr}((-1)^i) \otimes \mathrm{unr}(y^{-1}p^{1-k/2})$$

for some y lifting x^{-1}, and then

$$e(R^{\square,\psi_k}(w_k, \mathtt{t}_f, r_{f,p})/\pi) = \mu_{r_{f,p}}([\xi^i \sigma_{w_k}]).$$

As $k \leqslant 2p+2$, by Lemma 3.5.12 and Lemma 6.1.1 this can be non-zero only if $i = k/2-1$, that is, $y = (-1)^{k/2-1}u_p(f)$, which gives the result (note that we could apply the same method for $f \in S_k(\Gamma_0(Np))$ new at p for any N such that $p \nmid N$).

Suppose now that either $r_{f,p}|_{I_p}$ is not of the form $\left(\begin{smallmatrix} \omega & * \\ 0 & 1 \end{smallmatrix}\right) \otimes \omega^{k/2-1}$ with $*$ "très ramifié" or that $k > 2p+2$. By the existence of f, $R^{\square,\psi_k}(w_k, \mathtt{t}_f, r_{f,p})$ is non-zero and then Corollary 3.5.9 or the computations of Lemma 6.1.1 show that both $R^{\square,\psi_k}(w_k, \mathtt{t}_f, r_{f,p})$ and $R^{\square,\psi_k}(w_k, \mathtt{t}'_f, r_{f,p})$ are non-zero when $k > 2$, where $\mathtt{t}'_f$ is the extended type conjugate to $\mathtt{t}_f$.

Suppose now that moreover $r_f|_{G_{\mathbb{Q}(\zeta_p)}}$ is absolutely irreducible. Let B be the quaternion algebra over $\mathbb{Q}$ that is ramified exactly at p and ∞. There exists a modular form f' on B such that the automorphic representations attached to f and f' correspond to each other via Jacquet-Langlands, and more precisely we can take for f' an eigenform in $S_{\sigma,\eta}(U, \mathscr{O})$ for $\sigma = \sigma_{\mathrm{alg}} = \mathrm{Sym}^{k-2}\mathscr{O}^2$ and some character η that restricts to ψ_k at p, and U as in Section 5.3. Then we are in the situation of Section 5.5, from which we retain the notations. In particular Proposition 5.6.1 holds, hence the module $M_0[1/p]$ meets each irreducible component of $\operatorname{Spec} R_p[1/p]$. As $\operatorname{Spec} R_p[1/p]$ has irreducible components of both possible extended types, there exists an eigenform g' in $S_{\sigma,\eta}(U, \mathscr{O})$ such that the Galois representation attached to g' has an extended type at p which is conjugate to that of r_f. Let $g \in S_k(\Gamma_0(p))$ be an eigenform such that the automorphic representations attached to g and g' correspond via Jacquet-Langlands, then g is the form we were looking for. $\square$

The first part of Theorem 6.2.1 can be seen as a generalization of Conjecture 4 of [CS04] which was proved in [AB07] (see also [BP11]).

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