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Mirror symmetry for extended affine Weyl groups

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# MIRROR SYMMETRY FOR EXTENDED AFFINE WEYL GROUPS

BY ANDREA BRINI & KAROLINE VAN GEMST

**ABSTRACT.** — We give a uniform, Lie-theoretic mirror symmetry construction for the Frobenius manifolds defined by Dubrovin–Zhang in [21] on the orbit spaces of extended affine Weyl groups, including exceptional Dynkin types. The B-model mirror is given by a one-dimensional Landau–Ginzburg superpotential constructed from a suitable degeneration of the family of spectral curves of the affine relativistic Toda chain for the corresponding affine Poisson–Lie group. As applications of our mirror theorem we give closed-form expressions for the flat coordinates of the Saito metric and the Frobenius prepotentials in all Dynkin types, compute the topological degree of the Lyashko–Looijenga mapping for certain higher genus Hurwitz space strata, and construct hydrodynamic bihamiltonian hierarchies (in both Lax–Sato and Hamiltonian form) that are root-theoretic generalisations of the long-wave limit of the extended Toda hierarchy.

**RÉSUMÉ** (Symétrie miroir pour les groupes de Weyl affines étendus). — Nous donnons une construction de symétrie miroir, de façon uniforme et par des méthodes de théorie de Lie, pour les variétés de Frobenius définies par Dubrovin–Zhang sur les orbites des groupes de Weyl affines étendus, y compris les types de Dynkin exceptionnels. Le modèle miroir est donné par un superpotentiel de Landau–Ginzburg construit à partir d'une dégénérescence convenable des courbes spectrales de la chaîne de Toda affine relativiste pour le groupe de Lie–Poisson affine correspondant. Nous fournissons également plusieurs applications de ce théorème miroir. Celles-ci incluent des expressions explicites pour les coordonnées plates pour la métrique de Saito et le prépotentiel de Frobenius en tout type de Dynkin ; le calcul du degré topologique de l'application de Lyashko–Looijenga pour certaines strates des espaces d'Hurwitz en genre supérieur ; et la construction de hiérarchies hydrodynamiques bi-hamiltoniennes (à la fois dans le formalisme de Lax–Sato et hamiltonien) qui donnent des généralisations de la limite de dispersion nulle de la hiérarchie de Toda étendue.

## CONTENTS

1. Introduction.....	908
2. Dubrovin–Zhang Frobenius manifolds.....	914
3. Landau–Ginzburg superpotentials from Lie theory.....	918
4. Mirror symmetry.....	936
5. Applications.....	948
Appendix. Character relations for $E_6$ and $E_7$ .....	953
References.....	955

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## 1. INTRODUCTION

Frobenius manifolds, introduced by B. Dubrovin in [16] as a coordinate free formulation of the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations of 2D topological field theory, have sat for a good quarter of a century at a key point of confluence of algebraic geometry, singularity theory, quantum field theory, and the theory of integrable systems. In algebraic geometry, they serve as a model for the quantum co-homology (genus zero Gromov–Witten theory) of smooth projective varieties; in singularity theory, they encode the existence of pencils of flat pairings on the base of the mini-versal deformations of hypersurface singularities; in physics, they codify the associativity of the chiral ring of topologically twisted  $N = (2, 2)$  supersymmetric field theories in two dimensions; and in the theory of integrable hierarchies, they provide a loop-space formulation of hydrodynamic bihamiltonian integrable hierarchies in  $1+1$  dimensions.

On top of the physics-inspired examples of Frobenius manifolds coming from Witten’s topological A- and B-twists of 2D-theories with four supercharges, an interesting source of Frobenius manifolds is well-known to arise in Lie theory. Let  $\mathfrak{g}_{\mathbb{R}}$  be a simple complex Lie algebra associated to an irreducible root system  $R$ , and write  $\mathfrak{h}_{\mathbb{R}}$  and  $\mathfrak{g}_{\mathbb{R}}^{(1)}$  for, respectively, its Cartan subalgebra and the associated untwisted affine Lie algebra. In a celebrated result [17], Dubrovin constructed a class of semi-simple polynomial Frobenius manifolds on the space of regular orbits of the reflection representation of  $\text{Weyl}(\mathfrak{g}_{\mathbb{R}})$  (and in fact, on the orbit spaces of the defining representation of any Coxeter group). A remarkable extension of this was provided by Dubrovin and Zhang [21], who defined a Frobenius manifold structure  $M_{\mathbb{R}}^{\text{DZ}}$  on quotients of  $\mathfrak{h}_{\mathbb{R}} \subset \mathbb{C}$  by a suitable semi-direct product  $\text{Weyl}(\mathfrak{g}_{\mathbb{R}}^{(1)}) \rtimes \mathbb{Z}$ . They furthermore provided a mirror symmetry construction for Dynkin type A,  $\mathfrak{g}_{A_{N-1}} = \mathfrak{sl}_N(\mathbb{C})$ , in terms of Laurent-polynomial one-dimensional Landau–Ginzburg models, which was later generalised to classical Lie algebras in [20]. A question raised by [21, 20] was whether a similar uniform mirror symmetry construction for all Dynkin types could be established, including exceptional Lie algebras.

This paper gives a constructive Lie-theoretic answer to this question, which is furthermore entirely explicit, and provides closed-form expressions for the flat coordinates of the analogue of the Saito–Sekiguchi–Yano metric and for the Frobenius prepotential. Our mirror theorem has simultaneous implications for singularity theory, integrable systems, the Gromov–Witten theory of Fano orbicurves, and Seiberg–Witten theory, some of which are explored here.

### 1.1. MAIN RESULTS

**1.1.1. Mirror symmetry for Dubrovin–Zhang Frobenius manifolds.** — Our main result is the following general mirror theorem for Dubrovin–Zhang Frobenius manifolds (see Theorem 3.6 for the complete statement, and Table 1.1 for details of the notation employed). Let  $H_{g,n}$  be the Hurwitz space of isomorphism classes  $[\lambda : C_g \rightarrow \mathbb{P}^1]$  of covers of the complex line by a genus  $g$  curve  $C_g$  with ramification profile at infinity

described by  $n \geq (Z_{>0})^{\ell(n)}$ ,  $\ell(n) > 1$ . Fixing a suitable meromorphic function  $\mu$  on  $H_{g,n}$  induces, as a particular case of a classical construction of Dubrovin [14, 16], a semi-simple Frobenius manifold structure  $H_{g,n}^{[\mu]}$  on  $H_{g,n}$ .

**THEOREM 1.1** (= Theorem 3.6). — *For any simple Dynkin type  $R$  there exists a highest weight  $\omega$  for the corresponding simple Lie algebra  $\mathfrak{g}$ , pairs of integers  $(g_\omega, n_\omega)$ , and an explicit embedding  $\iota_\omega : M_R^{\text{DZ}} \hookrightarrow H_{g_\omega, n_\omega}^{[\mu]}$ , such that  $\iota_\omega$  is a Frobenius manifold isomorphism onto its image  $M_\omega^{\text{LG}} := \iota_\omega(M_R^{\text{DZ}})$ .*

In other words,  $\iota_\omega$  identifies the Frobenius manifold  $M_R^{\text{DZ}}$  with a distinguished stratum  $M_\omega^{\text{LG}}$  of a Hurwitz space, which is an affine-linear subspace of the Frobenius manifold  $H_{g_\omega, n_\omega}^{[\mu]}$  in its natural set of flat coordinates. The datum of the covering map on  $H_{g_\omega, n_\omega}$  defines a one-dimensional B-model superpotential for  $M_R^{\text{DZ}}$  in terms of a family of (trigonometric) meromorphic functions  $M_\omega^{\text{LG}}$ , whose Landau–Ginzburg residue formulas determine the Dubrovin–Zhang flat pencil of metrics and the Frobenius product structure on  $TM_R^{\text{DZ}}$ .

Theorem 1.1 is proved in two main steps. Fixing  $\omega$  a dominant weight in a minimal non-trivial Weyl orbit, we first associate to  $M_R^{\text{DZ}}$  a family of spectral curves specified by the vanishing of the characteristic polynomial in the representation  $\rho_\omega$  for a pencil of group elements  $g(\lambda) \in G := \exp \mathfrak{g}$ . The construction of the family hinges on determining all character relations of the form  $\chi^{\wedge k \rho_\omega} = p_k^\omega(\chi_{\rho_1}, \dots, \chi_{\rho_{\ell_R}})$  in the Weyl character ring of  $G$ , where  $\rho_i$  is the  $i^{\text{th}}$  fundamental representation of  $G$ , and  $\ell_R$  is the rank of  $R$ . The resulting family sweeps a submanifold of a Hurwitz space  $H_{g_\omega, n_\omega}$ : the second step consists in establishing that this is a Frobenius submanifold of the Frobenius manifold  $H_{g_\omega, n_\omega}^{[\mu]}$  satisfying the defining properties of  $M_R^{\text{DZ}}$ .

Our construction is motivated by a conjectural relation of the almost-dual Frobenius manifold [19] for  $R$  of type ADE with the orbifold quantum co-homology of the associated simple surface singularity, as proposed in [8, 6], which is in turn described by a degeneration of a family of spectral curves for the relativistic Toda chain associated to (a co-extension of) the corresponding affine Poisson–Lie group of type ADE [24, 46]. The one-parameter family of group elements  $g(\lambda)$  in our construction is given by the Lax operator for the chain, where  $\lambda$  is the spectral parameter: the relation to the associated Dubrovin–Zhang Frobenius manifold of type ADE is suggested by analogous results for the simple Lie algebra case due to Lerche–Warner, Ito–Yang, and Dubrovin [35, 30, 19].

For the minimal choices, the target Hurwitz space  $H_{g_\omega, n_\omega}$  is a space of rational functions ( $g_\omega = 0$ ) for type  $R = A_{\ell_i}, B_{\ell_i}, C_{\ell_i}, D_{\ell_i}$  and  $G_2$ , and it is a space of meromorphic functions on higher genus curves in the other exceptional types. Note that different choices of  $\omega$  induce different families of spectral curves, and therefore different embeddings  $\iota_\omega : M_R^{\text{DZ}} \hookrightarrow H_{g_\omega, n_\omega}^{[\mu]}$  inside a parent Hurwitz space. Whilst we prove Theorem 1.1 for dominant weights  $\omega$  in a minimal non-trivial Weyl orbit, we also provide verifications that different non-minimal choices of  $\omega$  indeed give rise to isomorphic Frobenius manifolds.

1.1.2. *Application I: Frobenius prepotentials.* — The original Dubrovin–Zhang construction establishes the existence of a Frobenius manifold structure on  $M_{\mathbb{R}}^{\text{DZ}}$  by abstractly constructing a flat pencil of metrics  $\gamma + \lambda\eta$  on  $T^*M_{\mathbb{R}}^{\text{DZ}}$ , where  $\gamma$  arises from an extension of the Killing pairing to  $\mathfrak{h}_{\mathbb{R}} \subset \mathbb{C}$ , without reference to an actual system of flat coordinates for  $\eta$  (the analogue of the Saito–Sekiguchi–Yano metric for finite reflection groups). From Theorem 1.1, the metric  $\eta$  and Frobenius product on the base of the family of spectral curves can then be computed using Landau–Ginzburg residue formulas for the superpotential: the associativity of the Frobenius product reduces the analysis of the pole structure of the Landau–Ginzburg residues to the sole poles of the superpotential, giving closed-form expressions for the flat coordinates of  $\eta$  and its prepotential. We then obtain the following

**THEOREM 1.2** (= Lemma 4.1 and Examples in Section 4.1). — *For all  $\mathbb{R}$ , we provide flat coordinates for the Saito metric of the Dubrovin–Zhang pencil and closed-form prepotentials for  $M_{\mathbb{R}}^{\text{DZ}}$ .*

Our expressions recover results of [21, 20] for classical Lie algebras; the statements for exceptional Dynkin types are new. Theorem 1.1 is key to the determination of the prepotential: the Landau–Ginzburg calculation reduces the computation of flat coordinates for  $\eta$  and a distinguished subset of structure constants to straightforward residue calculations on the spectral curves, from which the entire product structure on the Frobenius manifold can be recovered using WDVV equations.

1.1.3. *Application II: Lyashko–Looijenga multiplicities of meromorphic functions*

The enumeration of isomorphism classes of covers of  $S^2$  with prescribed ramification over a point is a classical problem in topology and enumerative combinatorics, going back to Hurwitz’ formula for the case in which the covering surface is also a Riemann sphere. The result of the enumeration for a cover of arbitrary geometric genus  $g$  and branching profile  $\mathbf{n} = (n_0, \dots, n_m)$  is the Hurwitz number  $h_{g,\mathbf{n}}$ , whose significance straddles several domains in enumerative combinatorics [29, 28], representation theory of the symmetric group [27], moduli of curves [23], and mathematical physics [13, 5, 10]. It was first noticed by Arnold [2] that when the branching profile has maximal degeneration (i.e., for polynomial maps  $f : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ ) this problem is intimately related to considering the topology of the complement of the discriminant for the base of the type  $A_\ell$  mini-versal deformation, and in particular to the degree of the Lyashko–Looijenga mapping [37, 36]  $\text{LL} : \mathbb{C}[\mu] \dashrightarrow \mathbb{C}[\mu]$ , which assigns to a polynomial  $\lambda(\mu)$  the unordered set of its critical values  $\text{LL}(\lambda)(\mu) = \prod_{\lambda^o(\tilde{z})=0} (\mu - f(\tilde{z}))$ . This is a finite polynomial map [2, 36], inducing a stratification of  $\mathbb{C}[\mu]$  according to the degeneracy of the critical values of  $\lambda$ . The computation of the topological degree of this mapping on a given stratum, enumerating the number of polynomials sharing the same critical values counted with multiplicity, can usually be translated into a combinatorial problem enumerating some class of embedded graphs. This connection was used by Looijenga [36] to reprove Cayley’s formula for the enumeration of marked trees (corresponding to the co-dimension zero stratum), and by Arnold [2]

to encompass the case of Laurent polynomials (see also [33, 34] for generalisations to rational functions and discriminant strata). The extension of this combinatorial approach to arbitrary strata at higher genus, involving enumerations of suitable coloured oriented graphs ( $k$ -constellations), appears unwieldy [43]. However, when  $\lambda(\mu)$  is the Landau–Ginzburg superpotential of a semi-simple, conformal Frobenius manifold, the graded structure of the latter can be used to determine the Lyashko–Looijenga multiplicity of  $\lambda(\mu)$  by a direct application of the quasi-homogeneous Bézout theorem [3], with no combinatorics involved. In particular Theorem 1.1 has the following immediate consequence.

**THEOREM 1.3** (= Corollary 5.2). — *For all  $R$  we compute the Lyashko–Looijenga multiplicity of the stratum  $\iota_\omega(M_R^{\text{DZ}}) = M_\omega^{\text{LG}} \quad H_{g_\omega, n_\omega}^{[\mu]}$ .*

This includes, in particular, the higher genus Hurwitz spaces appearing for types  $R = E_n$  and  $F_4$  (see Table 5.1).

**1.1.4. Application III: the dispersionless extended type- $R$  Toda hierarchy.** — The datum of a semi-simple conformal Frobenius manifold is equivalent to the existence of a  $\tau$ -symmetric quasi-linear integrable hierarchy, which is bihamiltonian with respect to a Dubrovin–Novikov hydrodynamic Poisson pencil. Having a description of the Frobenius manifold in terms of a closed-form prepotential allows to give an explicit presentation of the hierarchy in terms of an infinite set of commuting 1+1 PDEs in normal coordinates. The loop-space version of Theorem 1.2 is then the following

**THEOREM 1.4** (= Proposition 5.3). — *For all  $R$ , we construct a bihamiltonian dispersionless hierarchy on the loop space  $LM_R^{\text{DZ}}$  in Hamiltonian form for the canonical Poisson pencil associated to  $M_R^{\text{DZ}}$ .*

For type  $A_n$  this integrable hierarchy is the zero-dispersion limit of Carlet’s extended bigraded Toda hierarchy [11], and for type  $D_n$  it is the long-wave limit of the Cheng–Milanov extended  $D$ -type hierarchy [12]. For simply-laced  $R$ , we expect that the principal hierarchies of Theorem 1.4 should coincide with the dispersionless limit of the Hirota integrable hierarchies constructed by Milanov–Shen–Tseng in [40]. The non-simply-laced cases are, to the best of our knowledge, new examples of hydrodynamic integrable hierarchies: our construction of the Landau–Ginzburg superpotential is highly suggestive that these should be obtained as symmetry reductions of the hierarchies in [40] by the usual folding procedure of the Dynkin diagram. Aside from laying the foundation for determining the prepotential of  $M_R^{\text{DZ}}$ , Theorem 1.1 also provides a dispersionless Lax formulation for the hierarchy as an explicit reduction of Krichever’s genus- $g_\omega$ ,  $\ell(n_\omega)$ -pointed universal Whitham hierarchy.

**1.2. FURTHER APPLICATIONS.** — We also highlight three further applications of Theorem 1.1, which are the subject of ongoing investigation and whose details will be provided in three separate publications.

1.2.1. *The orbifold Norbury–Scott conjecture.* — When  $R = A_1$ , the Frobenius manifold  $M_{\mathbb{R}}^{\text{DZ}}$  famously coincides with the quantum cohomology  $\text{QH}(\mathbb{P}^1, \mathbb{C})$  of the complex projective line. In [42], the authors propose a higher genus version of this statement and conjecture that the Chekhov–Eynard–Orantin topological recursion applied to the Landau–Ginzburg superpotential of  $\mathbb{P}^1$  computes the  $n$ -point, genus- $g$  Gromov–Witten invariants of  $\mathbb{P}^1$  with descendant insertions of the Kähler class (the “stationary” invariants) in terms of explicit residues on the associated spectral curve (see [22] for a proof). It was shown in [44] that for type  $R = A_\ell, D_\ell$  and  $E_\ell$  the Dubrovin–Zhang Frobenius manifolds  $M_{\mathbb{R}}^{\text{DZ}}$  are isomorphic to the orbifold quantum cohomology of the Fano orbicurves  $\mathcal{C}_{\mathbb{R}} = [\mathbb{C}^* \backslash n\mathbb{C}^2 / \mathbb{R}]$ , where  $\mathbb{R} < SU(2), j_{\mathbb{R}} j < 1$  is the McKay group of type  $R$ . In particular,

$$\mathcal{C}_{\mathbb{R}} \simeq \begin{cases} \mathbb{P}(1, \ell), & R = A_\ell, \\ \mathbb{P}_{2,2,\ell-2}, & R = D_\ell, \\ \mathbb{P}_{2,3,\ell-3} & R = E_\ell. \end{cases}$$

The construction of the LG superpotentials of Theorem 1.1 now associates a family of mirror spectral curves to the quantum cohomology of these orbifolds. As anticipated in [6], it is natural to conjecture that the Norbury–Scott theorem receives an orbifold generalisation through Theorem 1.1, whereby higher genus stationary Gromov–Witten invariants of  $\mathcal{C}_{\mathbb{R}}$  can be computed by residue calculus on the respective type  $R$  spectral curve mirrors. The investigation of the correct phrasing for the topological recursion is ongoing.

1.2.2. *Seiberg–Witten theory.* — For the case of polynomial Frobenius manifolds with  $R = A_\ell, D_\ell$  or  $E_\ell$ , it was noted by a number of authors [35, 30, 19] that the *odd periods* of the Frobenius manifold (in the language of [19]) give the quantum periods of the Seiberg–Witten family of curves dual to  $N = 2$  pure super Yang–Mills theory on  $\mathbb{R}^4$  with gauge group given by the compact real form of  $\exp(\mathfrak{g})$ . In [41], Nekrasov reformulated the Seiberg–Witten study of  $N = 2, d = 4$  gauge theories in the context of five-dimensional  $N = 1$  gauge theories compactified on a circle, by viewing the five-dimensional theory on  $\mathbb{R}^4 \times S^1$  with gauge group  $G$  as, effectively, a four-dimensional theory with gauge group the extended loop group  $\widehat{G}$ . In this context the classical Coulomb vacua are parametrised by orbits of the associated extended affine Weyl group. It is only natural to conjecture that the construction of odd periods and Picard–Fuchs system for four dimensional Seiberg–Witten theory from the polynomial Frobenius manifolds can be lifted to provide solutions of five-dimensional Seiberg–Witten theory using their Dubrovin–Zhang, extended affine counterpart; this can indeed be explicitly checked for simply-laced cases [9]. Considerations about folding in five dimensions also allow to treat non-simply laced Lie groups, which points to the existence of a new class of Frobenius manifolds having as monodromy group an extension of the *twisted* affine Weyl group.

1.2.3. *Saito determinants on discriminant strata.* — In [1], the authors consider semi-simple Frobenius manifolds embedded as discriminant strata on the Dubrovin–Hertling polynomial Frobenius structures on the orbits of the reflection representation of Coxeter groups. In particular, they use the Landau–Ginzburg mirror superpotentials to establish structural results on the determinant of the restriction of the Saito metrics to arbitrary strata. A specific question asked by [1] is how much of that story can be lifted to the study of the Dubrovin–Zhang Frobenius manifolds on extended affine Weyl group orbits. The Landau–Ginzburg presentation of Theorem 1.1 unlocks the power to employ the same successful methodology in the affine setting as well.

$R$	An irreducible root system
$\ell_R$	The rank of $R$
$\mathfrak{g}_R$	The complex simple Lie algebra with root system $R$
$G_R$	The simply connected complex simple Lie group $\exp(\mathfrak{g}_R)$
$\mathfrak{h}_R$ (resp. $T_R$ )	The Cartan subalgebra of $\mathfrak{g}_R$ (resp. Cartan torus of $G_R$ )
$g$ (resp. $h$ )	A regular element of $G_R$ (resp. $\mathfrak{h}_R$ )
$W_R/\widehat{W}_R/\widetilde{W}_R$	The Weyl/affine Weyl/extended affine Weyl group of Dynkin type $R$
$\{\alpha_1, \dots, \alpha_{\ell_R}\}$	The set of simple roots of $R$
$\{\omega_1, \dots, \omega_{\ell_R}\}$	The set of fundamental weights of $R$
$\Lambda_r(R)$ (resp. $\Lambda_r(R)^\pm$ )	The lattice of roots of $R$ (resp. the semi-group of positive/negative roots)
$\Lambda_w(R)$ (resp. $\Lambda_w(R)^\pm$ )	The lattice of all weights of $R$ (resp. the monoid of non-negative/non-positive weights)
$\rho_\omega$	The irreducible representation of $G_R$ with highest weight $\omega$
$\rho_i$	The $i^{\text{th}}$ fundamental representation of $G_R$ , $i = 1, \dots, \ell_R$
$\Gamma(\rho)$	The weight system of the representation $\rho$
$\text{Mult}_\rho(\omega)$	The dimension of the weight space of $\omega$ in the representation $\rho$
$\chi_\omega$ (resp. $\chi_i$ )	The formal character of $\rho_\omega$ (resp. $\rho_i$ )
$w$	The Weyl vector of $\mathfrak{g}_R$
$[i_1 \dots i_{\ell_R}]_R$	Components of a weight in the $\omega$ -basis of $R$
$(x_1, \dots, x_{\ell_R})$	Linear coordinates on $\mathfrak{h}_R$ w.r.t. the co-root basis $\{\alpha_1^\vee, \dots, \alpha_{\ell_R}^\vee\}$
$(Q_1, \dots, Q_{\ell_R})$	$(\exp(x_1), \dots, \exp(x_{\ell_R}))$
$C_R$ (resp. $K_R$ )	The Cartan (resp. symmetrised Cartan) matrix of $R$
$M_R^{\text{DZ}}/M_w^{\text{LG}}/H_{g,n}^{[\mu]}$ ( $M_R^{\text{DZ}}/M_w^{\text{LG}}/H_{g,n}$ )	Dubrovin–Zhang/Landau–Ginzburg/Hurwitz Frobenius manifolds (resp. their underlying complex manifolds)
$\mathfrak{n} \vdash d$	A padded partition of $d \in \mathbb{N}$ (i.e., parts are allowed to be zero)
$ \mathfrak{n} $ (resp. $\ell(\mathfrak{n})$ )	The length of a partition $\mathfrak{n}$ (resp. the number of parts in $\mathfrak{n}$ )

TABLE 1.1. Notation employed throughout the text.



1.3. ORGANISATION OF THE PAPER. — The paper is organised as follows. In Section 2 we recall the definition of the affine Lie theoretic Frobenius manifolds of Dubrovin–Zhang (DZ). In Section 3 we state how to construct Frobenius manifolds in terms of Landau–Ginzburg (LG) superpotentials defined on suitable strata of a Hurwitz space. We also recall the construction in [6], describing how to find LG-superpotentials for DZ-manifolds from the characteristic equation of a suitable degeneration of the Lax operator for the type R periodic relativistic Toda chain. This boils down to finding relations in the representation ring of  $G$ , which we determine for all Dynkin types giving explicit algebraic expressions for the corresponding superpotentials. In Section 4 we prove the mirror theorem, and determine in turn closed-form prepotentials for the corresponding Dubrovin–Zhang Frobenius manifolds, including the hitherto unknown exceptional cases in type  $E_6$ ,  $E_7$  and  $F_4$ . Finally, in Section 5, we discuss the applications to the extended R-type Toda hierarchies and the calculation of the multiplicities of the Lyashko–Looijenga map of  $M_\omega^{\text{LG}}$ . Our notation<sup>(1)</sup> is described in Table 1.1.

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## 2. DUBROVIN–ZHANG FROBENIUS MANIFOLDS

2.1. GENERALITIES ON FROBENIUS MANIFOLDS. — We start by recalling the basic definitions from the theory of Frobenius manifolds.

DEFINITION 2.1. — A (complex, holomorphic) Frobenius manifold is a 5-tuple  $M = (M, \cdot, \eta, e, E)$ , where  $M$  is a finite dimensional complex manifold such that at each point  $p \in M$ , the fibre  $T_p M$  of the holomorphic tangent bundle at  $p$  has the structure of a unital associative commutative algebra with multiplication  $\cdot$  and identity element  $e$ , varying holomorphically. Additionally,  $\eta$  is a flat, holomorphic, non-degenerate symmetric  $(0, 2)$ -tensor such that the Frobenius property holds:

$$\eta(X \cdot Y, Z) = \eta(X, Y \cdot Z), \quad \forall X, Y, Z \in (M, TM).$$

Moreover, the following properties are satisfied:

- (1) the unit vector field is horizontal,  $r e = 0$ , w.r.t. the Levi-Civita connection  $r$  associated to  $\eta$ ;
- (2) there exists a  $(0, 3)$ -tensor  $c \in (M, \text{Sym}^3 T^* M)$  such that  $r_w c(X, Y, Z)$  is totally symmetric  $\forall W, X, Y, Z \in (M, TM)$ ;

<sup>(1)</sup>To declutter the polynomial expressions of the prepotentials of  $M_{\mathbb{R}}^{\text{DZ}}$ , and in a slight departure from the conventions in the Frobenius manifolds literature, components of a chart of  $M_{\mathbb{R}}^{\text{DZ}}$  will consistently be written with lower indices: in particular the Einstein summation convention is *never* assumed.

(3) there exists  $E \in (M, TM)$  such that  $r E$  is covariantly constant, and the corresponding 1-parameter group of diffeomorphisms acts by conformal transformations of the metric and the product tensor.

A complex Frobenius manifold is *semisimple* if the set

$$\text{Discr}(M) := \{v \in T_p M \mid v \cdot v = 0\}$$

has positive complex codimension. Whenever  $E$  is in the group of units of  $(T_p M, \cdot)$ , one may define a second flat metric,  $\gamma \in (M, \text{Sym}^2 T^* M)$ , by

$$\gamma(E \cdot X, Y) = \eta(X, Y).$$

A key consequence of Definition 2.1 is the existence of a one-parameter affine family of flat metrics on  $T^* M$ . The non-degenerate pairings  $\eta$  and  $\gamma$  on  $(M, TM)$  define dual cotangent metrics  $\eta^*$  and  $\gamma^*$ , the Gram matrices of which are inverses of those of  $\eta$  and  $\gamma$ . Then Definition 2.1 implies that  $\gamma^*$  and  $\eta^*$  form a flat pencil of metrics: i.e.,  $\gamma^* + \lambda \eta^*$  is a flat metric for any  $\lambda \in \mathbb{C}$ , and its Christoffel symbols satisfy  $(\gamma^* + \lambda \eta^*)_{,i} = (\gamma^*)_{,i} + \lambda (\eta^*)_{,i}$ .

Since the metric  $\eta$  is flat, a Frobenius manifold carries a canonical affine equivalence class of charts given by flat frames for  $\eta$ . Spelling out the Frobenius manifold axioms in one such chart  $(t_1, \dots, t_{\dim M})$  on some sufficiently small open  $U \subset M$  amounts to the local existence of a holomorphic function  $F$  (the *prepotential*) with the following properties (we denote  $\partial_i$  as a shorthand for  $\partial/\partial t_i$ ):

- (i) the Gram matrix  $\eta_{ij} = \eta(\partial_i, \partial_j) = \partial_{ijk}^3 F$ , is constant and non-degenerate;
- (ii)  $e = \partial_{t_1}$ ;
- (iii)  $c_{ijk} = c(\partial_i, \partial_j, \partial_k) = \eta(\partial_i, \partial_j, \partial_k) = \partial^3 F / \partial t_i \partial t_j \partial t_k$ ;
- (iv)  $E = \sum_i d_i t_i \partial_i + \sum_i r_i \partial_i$ ;
- (v)  $\gamma_{ij} = \sum_k E^k c_{kij}$ ;
- (vi)  $\partial_i \cdot \partial_j = \sum_k c_{ij}^k \partial_k$ , where  $c_{ij}^k := \sum_m \eta^{km} c_{mij}$ , and  $\eta^{ij} := (\eta)_{ij}^{-1}$ ;
- (vii)  $F$  satisfies the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations,

$$\sum_{k,\ell} \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k} \eta^{k\ell} \frac{\partial^3 F}{\partial t_\ell \partial t_m \partial t_n} = j \cdot m.$$

**2.2. FROBENIUS MANIFOLDS FROM EXTENDED AFFINE WEYL GROUPS.** — We will here give a condensed description of the Dubrovin–Zhang construction of semi-simple Frobenius manifold structures on the space of regular orbits of extended affine Weyl groups, which follows closely the account given in [6].

Let  $\mathfrak{g}_R$  be a rank- $\ell_R$  complex simple Lie algebra associated to a root system  $R$ ,  $\mathfrak{h}_R$  the associated Cartan subalgebra,  $\dim \mathfrak{h}_R = \ell_R$ , and  $W_R$  the Weyl group. The construction of Dubrovin–Zhang Frobenius manifolds depends on a canonical choice of a marked node in the Dynkin diagram of  $R$ , which will be labelled  $\bar{k} \in \{1, \dots, \ell_R\}$ , and we let  $\alpha_{\bar{k}}$  and  $\omega_{\bar{k}}$  denote the corresponding simple root and fundamental weight, respectively. This node is an “attaching” vertex for the external nodes in the diagram, that is, the one which if removed splits the Dynkin diagram into disconnected A-type

pieces: we depict this choice of marking on the Dynkin diagrams with one vertex added, corresponding to the  $\alpha^-$  root, in Figure 2.1. Except for  $R = A_{\ell}$ , where any node can be chosen,  $\bar{k}$  marks the fundamental representation  $\rho_{\bar{k}}$  of  $\mathfrak{g}_R$  of highest dimension.

The action of  $W_R$  on  $\mathfrak{h}_R$  may be lifted to an action of the affine Weyl group  $\widehat{W}_R = W_R \ltimes \frac{1}{\tau}(\mathbb{R})$ , with  $\frac{1}{\tau}(\mathbb{R})$  being the lattice of co-roots:

$$\begin{aligned} \widehat{W}_R \curvearrowright \mathfrak{h}_R & \curvearrowright \mathfrak{h}_R, \\ ((w, \alpha^-), h) & \curvearrowright w(h) + \alpha^-. \end{aligned}$$

Then the *extended* affine Weyl group  $\widetilde{W}_R$  is defined as the semi-direct product  $\widetilde{W}_R := \widehat{W}_R \ltimes \mathbb{C}$  acting on  $\mathfrak{h}_R \times \mathbb{C}$  by

$$(2.1) \quad \begin{aligned} \widetilde{W}_R \curvearrowright \mathfrak{h}_R \times \mathbb{C} & \curvearrowright \mathfrak{h}_R \times \mathbb{C}, \\ ((w, \alpha^-, \ell), (h, v)) & \curvearrowright (w(h) + \alpha^- + \ell\omega_{\bar{k}}, v - \ell). \end{aligned}$$

Let  $\mathcal{H}_R$  denote the hyperplane arrangement associated to the root system  $R$ , and  $\mathfrak{h}_R^{\text{reg}} := \mathfrak{h}_R \setminus \mathcal{H}_R$  be the set of regular elements in  $\mathfrak{h}_R$ . The restriction of (2.1) to  $\mathfrak{h}_R^{\text{reg}} \times \mathbb{C}$  is then a free affine action, whose quotient defines the regular orbit space of the extended affine Weyl group of  $R$  with marked node  $\bar{k}$  as

$$(2.2) \quad M_R^{\text{DZ}} := (\mathfrak{h}_R^{\text{reg}} \times \mathbb{C}) / \widetilde{W}_R = T_R^{\text{reg}} / W_R \times \mathbb{C},$$

where  $T_R^{\text{reg}} = \exp(\mathfrak{h}_R^{\text{reg}})$  is the image of the set of regular elements of  $\mathfrak{h}_R^{\text{reg}}$  under the exponential map to the maximal torus  $T_R$ .

**REMARK 2.2.** — Let  $(x_1, \dots, x_{\ell_R})$  be linear coordinates on  $\mathfrak{h}_R$  w.r.t. the co-root basis  $f_{\alpha_1}, \dots, f_{\alpha_{\ell_R}}$ , and extend these to linear coordinates  $(x_1, \dots, x_{\ell_R}; x_{\ell_R+1})$  on  $\mathfrak{h}_R \times \mathbb{C}$ . Writing  $Q_i = e^{x_i}$ , we denote  $I_R := \mathbb{C}[Q_1, \dots, Q_{\ell_R+1}]$  the ring of regular functions on the algebraic torus  $T_R \times \mathbb{C}^* \simeq (\mathbb{C}^*)^{\ell_R+1}$ . By its definition in (2.2),  $M_R^{\text{DZ}}$  is a smooth complex manifold homeomorphic to a Zariski open subset of the affine GIT quotient  $\text{Spec } I_R / \widetilde{W}_R$ . Note that, in [21, §1, Def. & Main Lem.], the authors consider a partial compactification of the latter to an affine scheme  $\text{Spec } A_R$ , where  $A_R$  is a polynomial subring, satisfying suitable boundedness conditions at infinity, of the Laurent polynomial ring  $I_R$ . Then  $M_R^{\text{DZ}}$  also sits in the underlying affine variety as an open submanifold.

The linear coordinates  $(x_1, \dots, x_{\ell_R}; x_{\ell_R+1})$  on  $\mathfrak{h}_R^{\text{reg}} \times \mathbb{C}$  can serve as local coordinates on the regular orbit space. By orthogonal extension of minus the Cartan–Killing form on  $\mathfrak{h}_R$ , we define a non-degenerate pairing  $\xi$  on  $\mathfrak{h}_R \times \mathbb{C}$  by

$$\xi(\partial_{x_i}, \partial_{x_j}) := \begin{cases} (K_R)_{ij} & \text{if } i, j < \ell_R + 1, \\ d_{\bar{k}} & \text{if } i = j = \ell_R + 1, \\ 0 & \text{otherwise,} \end{cases}$$

with  $x_{\ell_R+1}$  parametrising linearly the right summand in  $\mathfrak{h}_R \times \mathbb{C}$ . Here,  $d_i := \hbar\omega_i, \omega_{\bar{k}}^j$ , with  $\hbar\alpha, \beta^i$  being the pairing on  $\mathfrak{h}_R$  induced by the restriction of the Killing form on

R	$\bar{k}$	Canonically marked affine Dynkin diagram
$A_\ell$	$1, \dots, \ell$	
$B_\ell$	$\ell - 1$	
$C_\ell$	$\ell$	
$D_\ell$	$\ell - 2$	
$E_6$	3	
$E_7$	3	
$E_8$	3	
$F_4$	2	
$G_2$	2	

FIGURE 2.1. Affine Dynkin diagrams with canonical markings, as in [21, Table 1]. The node corresponding to the affine root is marked in black, and the canonical marked node is indicated with a  $\times$ .

the Cartan subalgebra. The quotient map  $@ : T_{\mathbb{R}}^{\text{reg}} \subset \mathbb{C} \rightarrow M_{\mathbb{R}}^{\text{DZ}}$  from (2.2) defines a principal  $W_{\mathbb{R}}$ -bundle on  $M_{\mathbb{R}}^{\text{DZ}}$ : a section  $\tilde{\sigma}_i$ , lifts a (sufficiently small) open  $U \subset M_{\mathbb{R}}^{\text{DZ}}$  to the  $i^{\text{th}}$  sheet of the cover  $V_i \cong \tilde{\sigma}_i^{-1}(U) \cong V_1 \times_{W_{\mathbb{R}}} U$ . The following reconstruction theorem holds [21, Th. 2.1]:

**THEOREM 2.3.** — *There exists a unique (up to isomorphism) semisimple Frobenius structure  $M_{\mathbb{R}}^{\text{DZ}} = (M_{\mathbb{R}}^{\text{DZ}}, e, E, \eta, \cdot)$  satisfying the following properties in flat coordinates  $(t_1, \dots, t_{\ell_{\mathbb{R}}+1})$  for  $\eta$ :*

$$(DZ-I) \quad e = \partial_{t_{\bar{k}}};$$

$$(DZ-II) \quad E = \frac{1}{d_{\bar{k}}} \partial_{x_{\ell_{\mathbb{R}}+1}} = \sum_{j=1}^{\ell_{\mathbb{R}}} \frac{d_j}{d_{\bar{k}}} t_j \partial_{t_j} + \frac{1}{d_{\bar{k}}} \partial_{t_{\ell_{\mathbb{R}}+1}};$$

$$(DZ-III) \quad \text{the intersection form is } \gamma = \tilde{\sigma}_i \xi;$$

$$(DZ-IV) \quad \text{the prepotential is polynomial in } t_1, \dots, t_{\ell_{\mathbb{R}}+1}, e^{t_{\ell_{\mathbb{R}}+1}}.$$

Such Frobenius manifolds will always be of charge one, or equivalently, the prepotential will be a degree 2 quasi-homogeneous function of its arguments.

### 3. LANDAU–GINZBURG SUPERPOTENTIALS FROM LIE THEORY

**3.1. ONE-DIMENSIONAL LG MIRROR SYMMETRY.** — Hurwitz spaces are moduli spaces parametrising ramified covers of the Riemann sphere. A point in such a space is an equivalence class  $[\lambda : C_g \twoheadrightarrow \mathbb{P}^1]$ , where  $C_g$  is a genus- $g$  smooth complex projective curve (compact Riemann surface) and  $\lambda$  is a morphism to the complex projective line realising  $C_g$  as a branched cover of  $\mathbb{P}^1$ ; the equivalence relation is here given by automorphisms of the cover.

We consider Hurwitz spaces with fixed ramification over infinity. Let the preimage of  $\infty$  consist of  $m+1$  distinct points, denoted by  $\infty_i \in C_g$  for  $i = 0, \dots, m$ , with  $\lambda$  having degree  $n_i + 1$  near  $\infty_i$ . The corresponding Hurwitz space will be denoted  $H_{g,n}$ , where  $n := (n_0, \dots, n_m)$ . This is a connected complex manifold (and in fact, an irreducible quasi-projective complex algebraic variety [25]) of dimension  $d_{g,n} := \dim(H_{g,n}) = 2g + 2m + \sum_{i=0}^m n_i$ . We will write  $\pi$ ,  $\lambda$  and  $\infty_i$  for, respectively, the universal family, the universal map, and the sections marking the  $\infty_i$ , as per the following commutative diagram:

$$\begin{array}{ccccc} C_g & \xrightarrow{\quad} & C_{g,n} & \xrightarrow{\quad \lambda \quad} & \mathbb{P}^1 \\ P_i \uparrow & & \downarrow & \uparrow & \downarrow \pi \\ & & \text{pt} & & \\ & & [\lambda] \xrightarrow{\quad} & H_{g,n} & \end{array}$$

We furthermore denote by  $d = d_{\pi}$  the relative differential with respect to the universal family and  $p_i^{\text{cr}} \in C_g = \pi^{-1}([\lambda])$  the critical points  $d\lambda = 0$  of the universal map. By the Riemann existence theorem, the critical values of  $\lambda$ ,  $\infty_i$ ,  $\infty_j$  serve as local coordinates away from the closed subsets in  $H_{g,n}$  in which  $u_i = u_j$  for  $i \neq j$ , whose union is called the discriminant. Additionally, there is an action on a Hurwitz space given by the affine subgroup of the  $\text{PGL}_2(\mathbb{C})$ -action on the target,

$$(3.1) \quad (C, \lambda) \mapsto (C, a\lambda + b), \quad u_i \mapsto au_i + b,$$

for  $a, b \in \mathbb{C}$ , and  $\delta_i = 1, \dots, d_{g,n}$ .

On the complement of the discriminant, we can associate a family of semi-simple, commutative, unital  $\mathbb{C}$ -algebra structures on the tangent fibres at  $(u_1, \dots, u_{d_{g,n}})$

by positing that the coordinate vector fields in the  $u$ -chart are the idempotents of the algebra,

$$(3.2) \quad \partial_{u_i} \partial_{u_j} = \delta_{ij} \partial_{u_i}.$$

The unit and Euler vector field,

$$(3.3) \quad e = \sum_{i=1}^{d_{g;n}} \partial_{u_i}, \quad E = \sum_{i=1}^{d_{g;n}} u_i \partial_{u_i},$$

arise here as the generators of the affine action in (3.1) by, respectively, translations and rescalings.

What remains to be constructed to define a full-fledged Frobenius manifold structure on  $H_{g,n}$  is a flat non-degenerate symmetric pairing playing the role of  $\eta$ , such that the vector fields  $e$  and  $E$  are, respectively, horizontal and linear under its Levi-Civita connection. This will depend on additional data [16, Lect. 5], as we now explain in the generality we will require.

**DEFINITION 3.1.** — A meromorphic function  $\mu : C_{g,n} \dashrightarrow \mathbb{P}^1$  on the universal family is  $\lambda$ -admissible if it satisfies the following properties:

- (i)  $\mu$  does not factor through  $\lambda$ , i.e.,  $\exists g : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$  s.t.  $\mu = g \circ \lambda$ ;
- (ii)  $0 \notin d\mu \in H^1(C_{g,n}/H_{g,n})$ ;
- (iii)  $\text{div}(\mu) = \sum_{i=0}^{\ell} a_i \tau_i(H_{g,n})$  for  $a_i \in \mathbb{Z}$ .

The datum of a  $\lambda$ -admissible projection  $\mu$  allows to define extra structure on the universal curve, as follows. We can first of all canonically associate to it a relative one-form given by  $d \log \mu \in H^1(C_{g,n}/H_{g,n})(1_0 + \dots + 1_m)$ : this is an exact third-kind differential on the fibres of the universal curve, which has simple poles at  $\tau_i$  with residues  $\text{Res}_{\tau_i} d \log \mu = a_i$ . The second ingredient that the  $\mu$ -projection provides is a notion of a meromorphic Ehresmann connection on  $TC_{g,n}$ , defined in terms of its singular foliation by level sets of  $\mu$ . Let  $p \in C_{g,n}$  with  $d\mu(p) \neq 0$ ,  $m := \mu(p)$ , so that the leaf  $C_{g,n}^{(m)} := \overline{p^\theta} \subset C_{g,n} \mid \mu(p^\theta) = m$  is locally transverse to the fibres of the universal curve. Let  $U$  be a small neighbourhood of  $\pi(p) \subset H_{g,n}$ ; by transversality, there is a canonical local holomorphic section  $\sigma_m : U \rightarrow \pi^{-1}(U)$  of  $C_{g,n}$ , lifting  $U$  to  $\pi^{-1}(U) \setminus C_{g,n}^{(m)}$ . Accordingly, holomorphic vector fields  $X \in (U, TH_{g,n})$  are lifted to local holomorphic sections  $(\sigma_m)_* X$  of  $TC_{g,n}$  which are tangent to the leaves of the foliation. This defines locally around  $p$  a holomorphic derivation as

$$\delta_X^{(\mu)} f := L_{(\sigma_m)_* X} f.$$

Globally, however, the leaves of the  $\mu$ -foliation will fail to be transverse to the fibres of the universal curve at the critical locus of  $\mu$ : the derivation  $\delta_X^{(\mu)}$  will therefore take values in the ring of meromorphic functions on  $C_{g,n}$ , with poles on the ramification divisor of  $\mu$ :

$$(3.4) \quad \delta_X^{(\mu)} : H^0(C_{g,n}, \mathcal{O}_{C_{g,n}}) \dashrightarrow H^0(C_{g,n}, \mathcal{K}_{C_{g,n}}),$$

$$f \mapsto (\delta_X^{(\mu)} f)(p) := (L_{(\sigma_{\mu(p)})_* X} f)(p).$$

In more low-brow terms, and in local coordinates  $p = (u_1, \dots, u_{d_{g,n}}; \mu)$  on the complement of  $d\mu = 0$ , the derivation  $\delta_{\partial_{u_i}}^{(\mu)} f$  is simply the partial derivative taken with respect to  $u_i$  whilst keeping  $\mu$  constant. The meromorphicity of the derivation near an order- $r$  ramification point  $q^{cr} \in C_{g,n}$  with  $\mu(q^{cr}) = m(u_1, \dots, u_{d_{g,n}})$ ,  $d\mu(q^{cr}) = 0$ , is then just expressing that

$$\delta_{\partial_{u_i}}^{(\mu)} (\mu(p) - m)^{1/r} = (r-1)(\mu(p) - m)^{(1-r)/r} \partial_{u_i} m$$

has a pole of order  $r-1$  at  $p = q^{cr}$  as soon as  $\partial_{u_i} m(u) \neq 0$ .

With these definitions at hand, we can define a Frobenius manifold structure  $H_{g,n}^{[\mu]} := (H_{g,n}, \eta, e, E)$  on the Hurwitz space  $H_{g,n}$ . For later convenience, we will introduce an additional parametric dependence of the Frobenius structure on a normalisation factor  $N \in \mathbb{C}^*$ : this will be immaterial *per se* in the comparison with the Frobenius manifolds of Theorem 2.3, as such factor can be scaled away by a Frobenius manifold isomorphism given by a time- $1/(2N)$  flow along the Euler vector field,<sup>(2)</sup> but it will be helpful in simplifying the notation of the proof of Theorem 3.6. In terms of the datum of  $(\lambda, \mu, N)$ , the metric  $\eta$  is defined by the residue formula

$$(3.5) \quad \eta(X, Y) := N \sum_i \operatorname{Res}_{p_i^{cr}} \frac{\delta_X^{(\mu)} \lambda \delta_Y^{(\mu)} \lambda}{d\lambda} \left( \frac{d\mu}{\mu} \right)^2,$$

for  $X, Y \in (H_{g,n}, TH_{g,n})$ . Furthermore, combining (3.2) and (3.5) the 3-tensor  $c(X, Y, Z)$  is defined by the LG formula

$$(3.6) \quad c(X, Y, Z) := \eta(X, Y \cdot Z) := N \sum_i \operatorname{Res}_{p_i^{cr}} \frac{\delta_X^{(\mu)} \lambda \delta_Y^{(\mu)} \lambda \delta_Z^{(\mu)} \lambda}{d\lambda} \left( \frac{d\mu}{\mu} \right)^2,$$

which clearly satisfies the Frobenius property. Moreover, the second flat pairing is obtained upon replacing  $\lambda$  by  $\log \lambda$  in (3.5),

$$(3.7) \quad \gamma(X, Y) := N \sum_i \operatorname{Res}_{p_i^{cr}} \frac{\delta_X^{(\mu)} \log \lambda \delta_Y^{(\mu)} \log \lambda}{d \log \lambda} \left( \frac{d\mu}{\mu} \right)^2.$$

**PROPOSITION 3.2.** — *The residue formulas (3.5)–(3.7) define a Frobenius manifold structure  $H_{g,n}^{[\mu]} := (H_{g,n}, \eta, e, E)$ , which is semi-simple outside the discriminant of  $H_{g,n}$ .*

The statement of the Proposition is a direct specialisation of the Main Lemma and the proof of Th. 5.1 in [16, Lect. 5], where Frobenius structures are constructed on (suitable covers) of Hurwitz spaces  $H_{g,n}$ . These depend on a choice of a *primary differential*  $\phi$ : a meromorphic relative 1-form on  $\pi : C_{g,n} \rightarrow H_{g,n}$  satisfying suitable admissibility conditions, classified in five Types I–V in [16, Lect. 5]. If  $\mu$  is  $\lambda$ -admissible, the case  $\phi := \frac{1}{\mu} d \log \mu$  considered here (an exact third kind differential having at most simple poles at the poles of  $\lambda$ ) is readily seen to satisfy the criteria of Type III given in [16, Lect. 5], and therefore leads to an honest Frobenius manifold. We will

<sup>(2)</sup>The Frobenius manifolds of Theorem 2.3 have charge  $d = 1$ , hence their prepotentials are quasi-homogeneous of degree  $3 - d = 2$ . A time- $s$  Euler flow hence scales them by  $2s$ .

call the marked meromorphic function  $\lambda$  a *Landau-Ginzburg (LG) superpotential* for the Frobenius manifold, with primary differential  $\overline{N}(d\mu)/\mu$ .

**3.2. SUPERPOTENTIALS FOR EXTENDED AFFINE WEYL GROUPS.** — We give here a general method for the construction of spectral curves associated to a finite relativistic Toda chains, as anticipated in [6], for arbitrary Dynkin types.

Let  $\omega \in \mathfrak{h}_w^+(\mathbb{R})$  be the highest weight of a non-trivial irreducible representation  $\rho_\omega \in \text{Rep}(G_{\mathbb{R}})$  of minimal dimension. In particular,  $\rho_\omega$  is quasi-minuscule, i.e., all non-zero weights in the weight system  $(\rho_\omega)$  are in the same irreducible orbit under the action of the Weyl group, and it is minuscule (quasi-minuscule with no zero weights) for all  $\mathbb{R} \notin B_\ell, E_8, F_4$  and  $G_2$ . Consider the characteristic polynomial of  $g \in G$  in the representation  $\rho_\omega$ ,

$$(3.8) \quad Q_\omega(\chi_1, \dots, \chi_{\ell_{\mathbb{R}}}; \mu) = \det_{\rho_\omega}(g - \mu \mathbf{1}) = \sum_{k=0}^{\dim \rho_\omega} (\mu)^{(\dim \rho_\omega - k)} \chi^{\wedge k} \rho_\omega(g),$$

where the second equality is the co-factor expansion of the determinant. Recall that the representation ring of a simple Lie group is an integral polynomial ring generated by the fundamental representations,

$$(3.9) \quad \chi^{\wedge k} \rho_\omega(g) = p_k^\omega(\chi_1, \dots, \chi_{\ell_{\mathbb{R}}}) \in \mathbb{Z}[\chi_1, \dots, \chi_{\ell_{\mathbb{R}}}],$$

where  $\chi_i(g) := \text{Tr}_{\rho_i}(g)$  is the  $i^{\text{th}}$  fundamental character. Since  $\rho_\omega$  is quasi-minuscule,  $Q_\omega$  factorises as

$$(3.10) \quad Q_\omega = (1 - \mu)^{z_0} Q_\omega^{\text{red}} = (1 - \mu)^{z_0} \prod_{0 \neq \omega^0 \in (\rho_\omega)} (e^{\omega^0 h} - \mu),$$

where  $z_0$  is the dimension of the zero weight space of  $\rho_\omega$ , and  $e^h$  with  $[e^h] = [g]$  is a choice of Cartan torus element conjugate to  $g$ : in particular,  $z_0 = 0$  and  $Q_\omega^{\text{red}} = Q_\omega$  for  $\mathbb{R} \notin B_\ell, E_8, F_4$ , or  $G_2$ .

Define now

$$(3.11) \quad P_\omega(w_0, \dots, w_{\ell_{\mathbb{R}}+1}; \lambda, \mu) := Q_\omega^{\text{red}}(\chi_i = w_i - \delta_{i\bar{k}} \lambda / w_0; \mu),$$

and consider, as  $w := (w_0; w_1, \dots, w_{\ell_{\mathbb{R}}}) \in \mathbb{C} \times \mathbb{C}^{\ell_{\mathbb{R}}}$  varies, the family of plane algebraic curves in  $\text{Spec} \mathbb{C}[\lambda, \mu]$  with fibre at  $w$  given by  $C_w^{(\omega)} := \mathbb{V}(P_\omega)$ . We compactify and desingularise the fibres over  $w$  by taking the normalisation  $\overline{C_w^{(\omega)}}$  of their closure in  $\mathbb{P}^2$ . Marking the  $\lambda$ -projection  $(\lambda, \mu) \xrightarrow{\mathcal{P}} \lambda \in \mathbb{P}^1$  and varying  $w$  defines a subvariety  $M_\omega^{\text{LG}}$  of the Hurwitz space  $H_{g_\omega, n_\omega}$ , where  $g_\omega = h^{1,0}(\overline{C_w^{(\omega)}})$ , and  $n_\omega$  records the ramification profile at infinity of the  $\lambda$ -projection.

**REMARK 3.3.** — It is not obvious that the above defines an immersion of  $M_\omega^{\text{LG}}$  into  $H_{g_\omega, n_\omega}$ . This will follow from the “rectification statement” in Lemma 4.1, according to which  $M_\omega^{\text{LG}}$  embeds as a dimension- $(\ell_{\mathbb{R}} + 1)$  affine hyperplane into  $H_{g_\omega, n_\omega}$  in the respective flat coordinate systems.

In the next two Sections we will calculate the character relations (3.9), and therefore determine explicitly the polynomials  $P_\omega$ . The following Proposition is an anticipated consequence of this direct calculation.



**PROPOSITION 3.4.** — Let  $\tilde{P}_\omega^{(j)}(w_0, \dots, w_{\ell_R}; \mu) := [\lambda^j]P_\omega$  be the  $j^{\text{th}}$  coefficient of  $P_\omega$  in the variable  $\lambda$ , and let  $j_\omega^{\max} := \deg_\lambda P_\omega$ . Then  $\tilde{P}_\omega^{(j_\omega^{\max})}$  is a product of a monomial  $\mu^{a_\omega}$  and cyclotomic polynomials  $k_i^\omega(\mu)$

$$(3.12) \quad \tilde{P}_\omega^{(j_\omega^{\max})}(w_0, \dots, w_{\ell_R}; \mu) = \mu^{a_\omega} \prod_{i=1}^{b_\omega} k_i^\omega(\mu),$$

with  $a_\omega \in \mathbb{Z}_{>0}$  and  $b_\omega, k_i^\omega \in \mathbb{Z}_{>0}$ . Moreover,

$$(3.13) \quad \tilde{P}_\omega^{(j)}(w_0, \dots, w_{\ell_R}; 0) = \delta_{j0},$$

$$(3.14) \quad \lim_{\mu \rightarrow 0} (\mu^\theta)^{\dim \rho_\omega} \tilde{P}_\omega^{(j)}(w_0, \dots, w_{\ell_R}; 1/\mu^\theta) = \delta_{j0}.$$

**REMARK 3.5.** — The projection  $\lambda : \overline{C_w^{(\omega)}} \dashrightarrow \mathbb{P}^1$  can only possibly have poles at  $\mu = 1$  or at the zeroes of  $\tilde{P}_\omega^{(j_\omega^{\max})}$ . The first equality in the Proposition, (3.12), entails then that  $\delta_{\partial_{w_i}}^{(\mu)} \tilde{P}_\omega^{(j_\omega^{\max})} = 0$  since the r.h.s. is constant in  $w$  at fixed  $\mu$ . In particular, the ramification profile  $n_\omega$  is independent of  $w$ .

The second part of the Proposition implies that the zeroes of  $\mu$  must occur at points that are poles of  $\lambda$ , since by (3.13) the equation  $P_\omega(\lambda, \mu)|_{\mu=0} = 0$  has no solutions for finite  $\lambda$ . Likewise, replacing  $\mu$  by  $\mu = 1/\mu^\theta$  reveals that poles of  $\mu$  are also poles of  $\lambda$  by (3.14). Therefore,  $d \log \mu$  has at most simple poles at the poles of  $\lambda$ , showing in particular that  $\mu$  satisfies Property (iii) in Definition 3.1, with Properties (i) and (ii) being obvious, and is therefore  $\lambda$ -admissible.

Following the discussion of Section 3.1, and fixing  $N_\omega \in \mathbb{C}^*$ , we can then define a family of semi-simple, commutative, unital Frobenius algebras on  $TM_\omega^{\text{LG}}$  via (3.5)–(3.6):

$$(3.15) \quad \eta(\partial_{w_i}, \partial_{w_j}) = N_\omega \sum_{\ell} \text{Res}_{p_\ell^{\text{cr}}} \frac{\delta_{\partial_{w_i}}^{(\mu)} \lambda \delta_{\partial_{w_j}}^{(\mu)} \lambda}{d\lambda} \left( \frac{d\mu}{\mu} \right)^2,$$

$$(3.16) \quad \eta(\partial_{w_i}, \partial_{w_j}, \partial_{w_k}) = N_\omega \sum_{\ell} \text{Res}_{p_\ell^{\text{cr}}} \frac{\delta_{\partial_{w_i}}^{(\mu)} \lambda \delta_{\partial_{w_j}}^{(\mu)} \lambda \delta_{\partial_{w_k}}^{(\mu)} \lambda}{d\lambda} \left( \frac{d\mu}{\mu} \right)^2,$$

$$(3.17) \quad \gamma(\partial_{w_i}, \partial_{w_j}) = N_\omega \sum_{\ell} \text{Res}_{p_\ell^{\text{cr}}} \frac{\delta_{\partial_{w_i}}^{(\mu)} \log \lambda \delta_{\partial_{w_j}}^{(\mu)} \log \lambda}{d \log \lambda} \left( \frac{d\mu}{\mu} \right)^2,$$

where  $p_\ell^{\text{cr}} \in g_\ell$  are the ramification points of  $\lambda : \overline{C_w^{(\omega)}} \dashrightarrow \mathbb{P}^1$ . This doesn't yet give a Frobenius manifold, or indeed a Frobenius submanifold of  $H_{g_\omega, n_\omega}^{[\mu]}$ , as  $\eta$  is not guaranteed to be non-degenerate or flat at this stage. The following statement establishes that this is the case.

**THEOREM 3.6** (Mirror symmetry for DZ Frobenius manifolds). — *The Landau–Ginzburg formulas (3.15)–(3.17) define a semi-simple conformal Frobenius submanifold  $\iota_\omega : M_\omega^{\text{LG}} = (M_\omega^{\text{LG}}, \eta, e, E, \gamma) \hookrightarrow H_{g_\omega, n_\omega}^{[\mu]}$ . In particular, (3.15) and (3.17) give flat,*

non-degenerate metrics on  $TM_\omega^{\text{LG}}$ , and the identity and Euler vector fields read

$$e = w_0^{-1} \partial_{w_{\bar{k}}}, \quad E = w_0 \partial_{w_0}.$$

Furthermore,

$$M_\omega^{\text{LG}} \simeq M_{\mathbb{R}}^{\text{DZ}}.$$

The explicit embedding  $\iota_\omega : M_\omega^{\text{LG}} \hookrightarrow H_{g_\omega, n_\omega}$  is described by the relations (3.10) in the character ring of  $G$ , setting the coefficients associated to the interior of the Newton polytope of  $P_\omega^{\text{red}}(\lambda, \mu)$  to be the polynomials  $p_k^\omega(w_1, \dots, w_r)$ .

The proof of Theorem 3.6 requires two key steps:

- (1) computing the exterior relations (3.10) in the Weyl character ring of  $G$ : this was solved for simply-laced cases in [4, 7], and we complete the solution here in full generality;
- (2) proving that the LG formulas (3.15)–(3.17), combined with the reconstruction theorem Theorem 2.3, establish the mirror statement of Theorem 3.6.

In the remainder of this Section we perform the first step and construct explicitly the family of LG mirror duals to type-R Dubrovin–Zhang Frobenius manifolds. We will then devote Section 4 to show how the second step leads to a proof of Theorem 3.6.

**3.3. SUPERPOTENTIALS FOR CLASSICAL LIE GROUPS.** — In the following we present the construction of the spectral curve for the classical root systems  $R = A_\ell, B_\ell, C_\ell, D_\ell$  independently, and show how our construction for a weight  $\omega$  corresponding to a minimal-dimensional representation  $\rho_\omega$  recovers the mirror results of [20] for these cases. We will use the shorthand notation  $\varepsilon_i := \chi^{\wedge^i \rho_\omega}$  for the exterior characters of  $\rho_\omega$ .

**3.3.1.  $R = A_\ell$ .** — The Dynkin diagram for a finite  $A_\ell$  is shown in Figure 2.1. In this case we can choose any (non-affine) node to be the marked one, since the removal of any node from the corresponding finite Dynkin diagram results in two disconnected  $A$ -type pieces, with ranks adding up to  $\ell - 1$ .

A choice of minimal, nontrivial, irreducible representation  $\rho_\omega := \rho_1 = (\ell + 1)$  for  $\text{SL}_C(\ell + 1)$  is the defining  $(\ell + 1)$ -dimensional representation, the other choice corresponding to its dual representation,  $\rho_\ell = \wedge^\ell$ . We then have that  $\varepsilon_i = \chi_i$  for  $i = 1, \dots, \ell$ , and  $\varepsilon_0 = \varepsilon_{\ell+1} = 1$ . Thus (3.8) becomes

$$(3.18) \quad P_{[10\dots 0]_{A_\ell}} = \frac{(\ell - 1)^{\bar{k}} \lambda \mu^{\bar{k}}}{w_0} + 1 + (\ell - 1)^{\ell+1} \mu^{\ell+1} + \sum_{i=1}^{\ell} (\ell - 1)^i w_i \mu^i,$$

which defines a family of genus 0 curves. Setting (3.18) equal to zero and solving for  $\lambda$  gives

$$(3.19) \quad \lambda = \frac{(\ell - 1)^{\bar{k}} w_0 (1 + (\ell - 1)^{\ell+1} \mu^{\ell+1} + \sum_{i=1}^{\ell} (\ell - 1)^i w_i \mu^i)}{\mu^{\bar{k}}},$$

which is, for every point in the moduli space, a meromorphic function of  $\mu$  with poles at 0 and  $1$  of orders  $\bar{k}, \ell + 1 - \bar{k}$ , respectively. We see that we have  $\ell + 1$

parameters  $w_0, \dots, w_\ell$ , and so the resulting Frobenius manifold is  $\ell + 1$  dimensional.<sup>(3)</sup> In particular, it is an  $\ell + 1$  dimensional submanifold of the Hurwitz space  $H_{0, n_\omega}$ , with ramification profile  $n_\omega = (\bar{k} - 1, \ell - \bar{k})$ . This Hurwitz space, however, is of dimension  $2 + \bar{k} - 1 + \ell - \bar{k} = \ell + 1$ , and so the DZ-Frobenius manifold associated to  $A_\ell$  is isomorphic to (a full-dimensional ball inside) its associated Hurwitz space. This, as we will see, will not be the case for the other Dynkin types.

3.3.2.  $R = B_\ell$ . — For  $\ell > 2$ , the minimal, nontrivial, irreducible representation of  $\text{Spin}(2\ell + 1)$  is the defining representation  $\rho_1 = (\mathbf{2}\ell + \mathbf{1})$  of the special orthogonal group in  $(2\ell + 1)$ -dimensions,<sup>(4)</sup> which is the irreducible representation with highest weight  $\omega_1$ . In this case the marked node is  $\bar{k} = \ell - 1$ , as depicted in Figure 2.1.

For  $i < \ell$ , the  $i^{\text{th}}$  fundamental representation  $\rho_i$  of  $B_\ell$  is the  $i^{\text{th}}$  exterior power of  $(\mathbf{2}\ell + \mathbf{1})$ . For  $i = \ell$ , the decomposition of the tensor square of  $\rho_\ell$  leads to

$$p_i^{[10\dots 0]_{B_\ell}} = \begin{cases} \chi_i & \text{if } i < \ell, \\ \chi_\ell^2 - \sum_{j=0}^{\ell-1} \chi_j & \text{if } i = \ell. \end{cases}$$

Together with the self-duality relation  $p_i^{[10\dots 0]_{B_\ell}} = p_{2\ell+1-i}^{[10\dots 0]_{B_\ell}}$ , we get that the curve is the zero locus of

$$(3.20) \quad P_{[10\dots 0]_{B_\ell}} = \frac{(\ell - 1)^\ell (\mu - 1)(\mu + 1)^2 \mu^{\ell-1} \lambda}{w_0} + \sum_{i=0}^{\ell} (\ell - 1)^i \mu^i (1 - \mu^{2(\ell-i)+1}) \varepsilon_i,$$

with

$$\varepsilon_i = \begin{cases} 1 & \text{if } i = 0, \\ w_i & \text{if } 1 < i < \ell, \\ w_\ell^2 - \sum_{j=0}^{\ell-1} w_j & \text{if } i = \ell. \end{cases}$$

Note that (3.20) has a factor of  $(\mu - 1)$ , since  $(\mathbf{2}\ell + \mathbf{1})$  has a one-dimensional zero weight space. Setting to zero the reduced characteristic polynomial  $P_{[10\dots 0]_{B_\ell}}^{\text{red}} = P_{[10\dots 0]_{B_\ell}} / (\mu - 1)$  gives

$$(3.21) \quad \lambda = \frac{(\ell - 1)^\ell w_0}{\mu^{\ell-1} (\mu + 1)^2} \sum_{j=0}^{2\ell} \mu^j \left( \sum_{i=0}^{\min(j, 2\ell-j)} (\ell - 1)^i \varepsilon_i \right).$$

For each point in the moduli space, this is a rational function in  $\mu$  with three poles at 0,  $-1$ , and  $1$  of orders  $\ell - 1, 2, \ell - 1$ , respectively. Hence,  $M_{[10\dots 0]_{B_\ell}}^{\text{LG}}$  is a sublocus in the  $(2\ell + 1)$ -dimensional Hurwitz space  $H_{0, n_\omega}$ , with  $n_\omega = (\ell - 2, 1, \ell - 2)$ . The latter carries an involution given by sending  $\mu \mapsto 1/\mu$ , and  $M_{[10\dots 0]_{B_\ell}}^{\text{LG}}$  is characterised as the  $(\ell + 1)$ -dimensional stratum that is fixed by the involution.

<sup>(3)</sup>This is, in fact, the case for all DZ-manifolds associated to a simple Lie algebra of rank  $\ell$ .

<sup>(4)</sup>For  $\ell = 2$ , the 4-dimensional spin representation  $\rho_2$  is both minimal and minuscule. It is also isomorphic to the vector representation  $\rho_1$  of  $C_2$ , which is included in the discussion of the next section.

3.3.3.  $R = C_\ell$ . — The minimal, nontrivial, irreducible representation for  $Sp(2\ell)$  is the defining representation  $\rho_1 = (\mathbf{2}\ell)$  of the rank  $2\ell$  symplectic group. Again, this representation corresponds to the one in which  $\omega_1$  is highest weight. The canonical node is the  $\ell^{\text{th}}$  node, as shown in Figure 2.1.

The exterior powers  $\wedge^i \rho_1$  are reducible, with only fundamental representations appearing as direct summands in their decomposition, giving the character relations

$$p_i^{[10\dots 0]_{C_\ell}} = \begin{cases} \sum_{j=0}^{i/2} \chi_{2j} & \text{for } i \text{ even,} \\ \sum_{j=0}^{(i-1)/2} \chi_{2j+1} & \text{for } i \text{ odd.} \end{cases}$$

From this, and the fact that  $\bar{k} = \ell$  for  $C_\ell$ , we see that the characteristic polynomial (3.8) is

$$P_{[10\dots 0]_{C_\ell}} = \frac{(-1)^\ell \mu^\ell \lambda}{w_0} + \sum_{i=0}^{\ell-1} (-1)^i \varepsilon_i \mu^i (1 + \mu^{2(\ell-i)}) + (-1)^\ell \varepsilon_\ell \mu^\ell,$$

with  $\varepsilon_{2i} = \sum_{j=0}^i \chi_{2j}$ ,  $\varepsilon_{2i+1} = \sum_{j=0}^i \chi_{2j+1}$  (and  $\varepsilon_0 = 1$  as usual). Setting equal to zero and solving for  $\lambda$  gives

$$\lambda = \frac{(-1)^{\ell-1} w_0 \left( \sum_{i=0}^{\ell-1} (-1)^i \varepsilon_i \mu^i (1 + \mu^{2(\ell-i)}) + (-1)^\ell \varepsilon_\ell \mu^\ell \right)}{\mu^\ell},$$

which is a rational function in  $\mu$  with two poles at 0 and  $-1$  both of order  $\ell$ . Hence, the associated covering Hurwitz space is  $H_{0, n_\omega}$ , with  $n_\omega = (\ell-1, \ell-1)$ , which has dimension  $2\ell$ . As before, there is an involution on this Hurwitz space sending  $\mu \mapsto -1/\mu$ , with  $M_{[10\dots 0]_{C_\ell}}^{\text{LG}}$  being the  $(\ell+1)$ -dimensional stratum that is fixed by it.

3.3.4.  $R = D_\ell$ . — For  $\ell > 4$ , the minimal, nontrivial, irreducible representation of  $Spin(2\ell)$  is the defining vector<sup>(5)</sup> representation  $\rho_1 = (\mathbf{2}\ell)_v$  of  $SO(2\ell)$ , which corresponds to the irreducible representation with highest weight  $\omega_1$ . The canonical node is the one with label  $\ell-2$ , as shown in Figure 2.1.

The character relations for  $D_\ell$  were found in [4] to be

$$\begin{aligned} p_i^{[10\dots 0]_{D_\ell}} &= \chi_i, \quad i < \ell-1, \\ p_{\ell-1}^{[10\dots 0]_{D_\ell}} &= \chi_{\ell-1} \chi_\ell \begin{cases} \sum_{j=0}^{(\ell/2)-2} \chi_{2j+1} & \text{if } \ell \text{ is even,} \\ \sum_{j=0}^{(\ell-3)/2} \chi_{2j} & \text{if } \ell \text{ is odd,} \end{cases} \\ p_\ell^{[10\dots 0]_{D_\ell}} &= \chi_{\ell-1}^2 + \chi_\ell^2 \begin{cases} \sum_{j=0}^{(\ell/2)-1} \chi_{2j} & \text{if } \ell \text{ is even,} \\ \sum_{j=0}^{(\ell-3)/2} \chi_{2j+1} & \text{if } \ell \text{ is odd,} \end{cases} \end{aligned}$$

and  $p_i^{[10\dots 0]_{D_\ell}} = p_{2\ell-i}^{[10\dots 0]_{D_\ell}}$ , so that

$$(3.22) \quad P_{[10\dots 0]_{D_\ell}} = \frac{(-1)^\ell \mu^{\ell-2} (\mu^2 - 1)^2 \lambda}{w_0} + \sum_{i=0}^{\ell-1} (-1)^i \varepsilon_i \mu^i (1 + \mu^{2(\ell-i)}) + (-1)^\ell \mu^\ell \varepsilon_\ell,$$

<sup>(5)</sup>For  $\ell = 4$ , this can be any of the irreducible 8-dimensional representations, related by triality.

where as before we denote  $\varepsilon_i(w_1, \dots, w_\ell) = p_i^{[10\dots 0]_{D_\ell}}(\chi_j = w_j)$ , with  $\varepsilon_0 = 1$ . Setting (3.22) equal to zero and solving for  $\lambda$  gives

$$(3.23) \quad \lambda = (-1)^{\ell-1} \frac{w_0 \left( \sum_{i=0}^{\ell-1} (-1)^i \varepsilon_i \mu^i (1 + \mu^{2(\ell-i)}) + (-1)^\ell \mu^\ell \varepsilon_\ell \right)}{\mu^{\ell-2} (\mu^2 - 1)^2},$$

which, for every point  $w$ , is a rational function in  $\mu$  with four poles at  $0, 1, 1, -1$  of orders  $\ell-2, \ell-2, 2, 2$ , respectively. Hence, the parent Hurwitz space is  $H_{0; n_\omega}$ , where  $n_\omega = (\ell-3, \ell-3, 1, 1)$ , which has dimension  $2\ell+2$ . Once more this hosts an involution obtained by sending  $\mu \mapsto 1/\mu$ , identifying  $M_{[10\dots 0]_{D_\ell}}^{LG}$  as its fixed locus.

### 3.4. COMPARISON WITH THE DUBROVIN–STRACHAN–ZHANG–ZUO CONSTRUCTION

For the case of  $R = A_\ell$ , an LG-superpotential was already found in the original paper [16], with<sup>(6)</sup>

$$(3.24) \quad \lambda = \sum_{j=0}^{k+m} b_j \mu^{m-j},$$

where  $b_j \in \mathbb{C}$  and  $b_0 b_{k+m} \neq 0$ . Moreover, in [20], the authors construct a three-integer parameter family of superpotentials of the form<sup>(7)</sup>

$$(3.25) \quad \lambda^{DSZZ}(\ell, k, m) = \frac{4^m \mu^m \sum_{j=0}^{\ell} a_j 2^{-2(j+k+m)} (\mu + 1/\mu)^{2(j+k+m)}}{(\mu - 1)^m}.$$

The key result of [20] is an identification of (3.25) with a superpotential for a Dubrovin–Zhang Frobenius manifold of type  $B_\ell, C_\ell, D_\ell$ , possibly with a non-canonical choice of marked node in the Dynkin diagram, for suitable choices of  $(\ell, k, m)$ . In particular, the mirror theorem for the canonical label  $\bar{k}$  is obtained by setting  $(\ell, k, m)$  equal to  $(\ell, \ell-1, 1)$ ,  $(\ell, \ell, 0)$ , and  $(\ell, \ell-2, 1)$ , respectively. We shall now show that the results of [16] and [20] coincide with our construction in the previous section.

3.4.1.  $R = A_\ell$ . — By using the fact that  $k + m = \ell + 1$ , (3.24) becomes

$$\lambda = \frac{\sum_{j=0}^{\ell+1} b_j \mu^{\ell+1-j}}{\mu^k},$$

which is the same as (3.19) by

$$b_i = (-1)^k w_0 \begin{cases} 1 & \text{if } i = 0, \\ (-1)^{\ell+1} & \text{if } i = \ell + 1, \\ w_i & \text{otherwise.} \end{cases}$$

<sup>(6)</sup>To relate this to the expression in [16] let  $\mu = e^{i\phi}$ .

<sup>(7)</sup>To relate this to the expression in [20] let  $\mu = e^{2i\phi}$ .

3.4.2.  $R = B_\ell$ . — In the case of  $B_\ell$  we consider (3.25) with  $k = \ell - 1$ ,  $m = 1$ , which is

$$(3.26) \quad \lambda^{\text{DSZZ}}(\ell, \ell - 1, 1) = \frac{4\mu \sum_{j=0}^{\ell} a_j 2^{-2(\ell-j)} \left(\frac{\rho}{\mu} + 1/\frac{\rho}{\mu}\right)^{2(\ell-j)}}{(\mu - 1)^2}.$$

Simplifying (3.26) gives:

$$\frac{w_0}{(\mu - 1)^2} \sum_{j=0}^{\ell} \frac{a_j 2^{-2(\ell-j-1)} (\mu + 1)^{2(\ell-j)}}{w_0 \mu^{\ell-j-1}} = \frac{(-1)^\ell w_0}{(\mu + 1)^2 \mu^{\ell-1}} \sum_{\beta=0}^{2\ell} (-1)^\beta C_\beta \mu^\beta,$$

where we have used the binomial theorem and let  $\mu \nabla \mu$ , with

$$C_\beta = \frac{(-1)^\ell}{w_0} \sum_{j, \alpha; j+\alpha=\beta} a_j 2^{-2(\ell-j-1)} \binom{2(\ell-j)}{\alpha} = \frac{(-1)^\ell}{w_0} \sum_{j=0}^{\beta} a_j 2^{-2(\ell-j-1)} \binom{2(\ell-j)}{\beta-j}.$$

On the other hand, the superpotential constructed from the spectral curve, (3.21), is given by

$$\lambda_{B_\ell} = \frac{(-1)^\ell w_0}{\mu^{\ell-1} (\mu + 1)^2} \sum_{j=0}^{2\ell} \mu^j \left( \sum_{i=0}^{\min(j, 2\ell-j)} (-1)^i \varepsilon_i \right),$$

which we can write as

$$(3.27) \quad \lambda_{B_\ell} = \frac{(-1)^\ell w_0}{\mu^{\ell-1} (\mu + 1)^2} \sum_{j=0}^{2\ell} b_j \mu^j,$$

with  $b_j = \sum_{i=0}^{\min(j, 2\ell-j)} (-1)^i \varepsilon_i$ ; note that  $b_j = b_{2\ell-j}$ . This means that we want to match up  $b_i = (-1)^i C_i$ , hence

$$(3.28) \quad \sum_{j=0}^{\min(i, 2\ell-i)} (-1)^j \varepsilon_j = \frac{(-1)^{\ell+i}}{w_0} \sum_{j=0}^i a_j 2^{-2(\ell-j-1)} \binom{2(\ell-j)}{i-j}.$$

We claim that

$$(3.29) \quad \varepsilon_i = \frac{(-1)^\ell}{w_0} \sum_{j=0}^{\ell} a_j 2^{-2(\ell-i-1)} \binom{2\ell-2j+1}{i-j}.$$

*Proof.* — The  $i = 0$  case is clear giving  $\varepsilon_0 = \frac{(-1)^\ell}{w_0} 2^{-2(\ell-1)} a_0$  obtained by taking  $j = 0$ .

So suppose  $0 < i < \ell$ . Then from (3.28) we get

$$\begin{aligned} \varepsilon_i &= \frac{(-1)^\ell}{w_0} \sum_{j=0}^i a_j 2^{-2(\ell-j-1)} \binom{2(\ell-j)}{i-j} + \frac{(-1)^\ell}{w_0} \sum_{j=0}^{i-1} a_j 2^{-2(\ell-j-1)} \binom{2(\ell-j)}{i-1-j} \\ &= \frac{(-1)^\ell}{w_0} \left( a_i 2^{-2(\ell-i-1)} + \sum_{j=0}^{i-1} a_j 2^{-2(\ell-j-1)} \left( \binom{2(\ell-j)}{i-j} + \binom{2(\ell-j)}{i-1-j} \right) \right). \end{aligned}$$

Furthermore,

$$\binom{2(\ell - j)}{i - j} + \binom{2(\ell - j)}{i - 1 - j} = \frac{(2(\ell - j))!}{(i - j)!(2\ell - i - j)!} + \frac{(2(\ell - j))!}{(i - j - 1)!(2\ell - i - j + 1)!} = \binom{2\ell - 2j + 1}{i - j},$$

which gives the result for  $i \leq \ell$ . Hence, since  $\varepsilon_i = \varepsilon_{2\ell+1-i}$ , we have (3.29) for all  $i$ .

3.4.3.  $R = C_\ell$ . — In the case of  $C_\ell$ , we consider (3.25) with  $k = \ell, m = 0$  which is

$$(3.30) \quad \lambda^{\text{DSZZ}}(\ell, \ell, 0) = \sum_{j=0}^{\ell} a_j 2^{-2(\ell-j)} \left( \rho_{\mu} + \frac{1}{\mu} \right)^{2(\ell-j)}.$$

Simplifying (3.30) gives:

$$\frac{(1)^\ell w_0}{\mu^\ell} \sum_{j=0}^{\ell} \frac{(1)^\ell a_j 2^{-2(\ell-j)} (\mu + 1)^{2(\ell-j)}}{w_0 \mu^{-j}} = \frac{(1)^\ell w_0}{\mu^\ell} \sum_{\beta=0}^{2\ell} C_\beta \mu^\beta,$$

where we again have used the binomial theorem, and with

$$C_\beta = \frac{(1)^\ell w_0}{w_0} \sum_{j=0}^{\beta} a_j 2^{-2(\ell-j)} \binom{2(\ell-j)}{\beta-j}.$$

Thus, the equivalence is obtained in the case of  $C_\ell$  by letting

$$\varepsilon_i = \frac{(1)^{\ell+i} w_0}{w_0} \sum_{j=0}^i a_j 2^{-2(\ell-j)} \binom{2(\ell-j)}{i-j}.$$

3.4.4.  $R = D_\ell$ . — For  $D_\ell$ , we want to consider (3.25) with  $k = \ell - 2, m = 1$ , which is of the form

$$\lambda^{\text{DSZZ}}(\ell, \ell - 2, 1) = \frac{4\mu \sum_{j=0}^{\ell} a_j 2^{-2(\ell-j-1)} \left( \rho_{\mu} + 1/\rho_{\mu} \right)^{2(\ell-j-1)}}{(\mu - 1)^2}.$$

This is equivalent to

$$\frac{(1)^\ell w_0}{(\mu - 1)^2 \mu^{\ell-2} (\mu + 1)^2} \sum_{j=0}^{\ell} \frac{(1)^\ell a_j (\mu + 1)^{2(\ell-j)}}{2^{2(\ell-j-2)} w_0 \mu^{-j}} = \frac{(1)^\ell w_0}{\mu^{\ell-2} (\mu^2 - 1)^2} \sum_{\beta=0}^{2\ell} C_\beta \mu^\beta,$$

with

$$C_\beta = \frac{(1)^\ell w_0}{w_0} \sum_{j=0}^{\beta} a_j 2^{-2(\ell-j-2)} \binom{2(\ell-j)}{\beta-j},$$

where, again, the binomial theorem has been used. Hence, the map

$$\varepsilon_i = \frac{(1)^{\ell+i} w_0}{w_0} \sum_{j=0}^i a_j 2^{-2(\ell-j-2)} \binom{2(\ell-j)}{\beta-j}$$

gives the equivalence.

3.5. SUPERPOTENTIALS FOR EXCEPTIONAL LIE GROUPS. — We present here the construction of the spectral curve for the exceptional types  $E_6$ ,  $E_7$ ,  $F_4$ , and  $G_2$ . The  $E_8$  case was treated extensively in [6] and [7], and we only give a very brief presentation here.

As for the classical cases, the construction of the superpotential hinges on determining the character relations (3.9) for all  $k$ . Explicitly, for all dominant weights  $\varpi = \sum_i \varpi_i \omega_i \in \Lambda^+(\mathbb{R})$  we should determine  $N_{\varpi}^{(\omega, k)} \in \mathbb{Z}$  such that

$$(3.31) \quad p_k^\omega = \sum_{\varpi \in \Lambda^+(\mathbb{R})} N_{\varpi}^{(\omega, k)} \prod_{i=1}^{\ell_{\mathbb{R}}} \chi_i^{\varpi_i}.$$

DEFINITION 3.7. — A set of dominant weights  $\omega \in \Lambda^+(\rho_\omega)$  is called *pivotal* for  $\omega$  if

$$\varpi \in \Lambda^+(\rho_\omega), \varpi \leq \omega, \varpi^0 \leq \varpi.$$

Here  $\varpi^0 \leq \varpi$  denotes the canonical partial ordering of weights, i.e.,

$$\varpi^0 \leq \varpi \iff \varpi - \varpi^0 = \sum_{i=1}^{\ell_{\mathbb{R}}} n_i \alpha_i \quad \text{with } n_i > 0.$$

Definition 3.7 then states that a set  $\omega$  of dominant weights is pivotal for a representation  $\rho_\omega$  if it is contained in the weight system of the exterior algebra of  $\rho_\omega$ , and all the other weights in  $\Lambda^+(\rho_\omega)$  are lower, in the partial order, than some element of  $\omega$ .

It will be useful, in the following, to consider pivotal sets that are as small as possible. As for the classical cases, we take  $\omega$  to sit in a minimal non-trivial orbit of  $W_{\mathbb{R}}$ , as described in Table 3.1.

EXAMPLE 3.1. — Consider the decomposition of the exterior algebra of  $\rho_\omega$  into irreducible representations,

$$(3.32) \quad \Lambda^+(\rho_\omega) = \bigoplus_{\omega^0} \text{Mult}_{\Lambda^+(\rho_\omega)}(\omega^0) \rho_{\omega^0},$$

and let  $\omega$  denote the finite set of dominant weights appearing on the r.h.s. with non-zero multiplicity. Then  $\omega$  is pivotal for  $\omega$ , although not necessarily of minimal cardinality. Suppose e.g.  $\omega = \alpha_0$  is the highest root, so that  $\rho_\omega = \mathfrak{g}$  is the adjoint representation. Then the dominant weight given by twice the Weyl vector,  $2w = 2 \sum_i \omega_i = \sum_{\alpha > 0} \alpha$ , appears with multiplicity  $\text{Mult}_{\Lambda^+(\mathfrak{g})}(w) = 1$  in (3.32), and it is higher than any other highest weight in the decomposition of  $\Lambda^+(\rho_\omega)$  into irreducibles. In this case,  $\alpha_0 = \bar{r}wg$  is pivotal and of minimal cardinality.



R	$\omega$	$\rho_\omega$	$j \omega $
$E_6$	[100000] (resp. [000010])	$27_{E_6}$ (resp. $\overline{27}_{E_6}$ )	111
$E_7$	[0000010]	$56_{E_7}$	907
$E_8$	[00000010]	$248_{E_8}$	950077
$F_4$	[0001]	$26_{F_4}$	74
$G_2$	[10]	$7_{G_2}$	5

TABLE 3.1. Highest weights of minimal representations for exceptional root systems. The last column indicates the cardinality of their sets of admissible exponents (Definition 3.9).

LEMMA 3.8. — *The sets of dominant weights*

$$(3.33) \quad \omega := \begin{cases} \{[010120], [120010], [200200], [110110], [001030], [030000], [020020], [000041], [000050]\}, & \omega = [100000]_{E_6}, \\ \{[0002022], [0001113], [0100132], [0101041], [1001051], [1001061], [0011031], [0010070], [0000204], [0110050], [0010070], [1100070]\}, & \omega = [0000010]_{E_7}, \\ [1000090], [0003011], [0020040], [0004000], [0000105], [00000100], [0000006]\}, & \omega = [00000010]_{E_8}, \\ f[2222222]g, & \omega = [0001]_{F_4}, \\ f[0022]g, & \omega = [10]_{G_2}, \\ f[20]g, & \end{cases}$$

are pivotal and of minimal cardinality for  $\omega$ .

For  $R = E_8, F_4$  and  $G_2$  the weight system of  $\rho$  is the set of short roots of  $R$ , and the single element of its minimal pivotal set is then the sum of the positive short roots. For  $R = E_6, E_7$  the pivotal sets of minimal cardinality in (3.33) can be constructed by direct inspection of the weight system.

DEFINITION 3.9. — Let  $\omega$  be as in Lemma 3.8. We call the finite set

$$I_\omega := \left\{ \iota \in (Z_{>0})^{\ell_R} \mid \exists \varpi = \sum_k \varpi_k \omega_k \in \omega \text{ s.t. } \sum_k (C_R)_{jk}^{-1} (\iota_k - \varpi_k) \in Z_{\leq 0} \right\}$$

the set of *admissible exponents* of the exterior algebra  $\wedge^\iota \rho_\omega$ .

In other words,  $\iota$  is admissible if and only if the corresponding dominant weight  $\varpi_\iota := \sum_i \iota_i \omega_i \in \omega$  for some weight  $\varpi^\theta$  in the minimal pivotal sets of Lemma 3.8. We will use the short-hand notation  $\varpi_\iota = \varpi$  when this happens. The terminology is justified by the following

LEMMA 3.10. — *Let  $\iota \in (Z_{>0})^{\ell_R}$ ,  $\iota \notin I_\omega$ . Then  $N_{\varpi_\iota}^{(\omega, k)} = 0$  for all  $k$ .*

*Proof.* — Consider the representation space version of (3.31),

$$(3.34) \quad \wedge^k \rho_\omega = \bigoplus_l N_{\varpi_l}^{(\omega, k)} \bigotimes_{i=1}^{\ell_{\mathbb{R}}} \rho_i^{l_i}.$$

By Definition 3.7, we have  $\varpi = \omega$  for all  $\varpi \in (\wedge^k \rho_\omega)$ , and furthermore, by Definition 3.9, the  $\ell_{\mathbb{R}}$ -tuple of its coefficients in the basis of fundamental weights is admissible. Equivalently, if  $l^\theta \in (Z_{>0})^{\ell_{\mathbb{R}}} \cap I_\omega$  is not admissible, then the corresponding dominant weight  $\varpi_{l^\theta} := \sum_i l_i^\theta \omega_i \notin (\wedge^k \rho_\omega)$  is not in the exterior algebra: taking multiplicities of (3.34) at  $\varpi_{l^\theta}$  then gives

$$(3.35) \quad 0 = \text{Mult}_{\wedge^k \rho_\omega}(\varpi_{l^\theta}) = \sum_l N_{\varpi_l}^{(\omega, k)} \text{Mult}_{\rho_i^{l_i}}(\varpi_{l^\theta}).$$

Further notice that

$$(3.36) \quad \text{Mult}_{\rho_i^{l_i}}(\varpi_{l^\theta}) = \begin{cases} 1, & l^\theta = l, \\ 0, & l^\theta \notin l \text{ and } l^\theta - l \notin (Z_{>0})^{\ell_{\mathbb{R}}}. \end{cases}$$

Both equalities in (3.36) are consequences of the (elementary) fact that the weights of a tensor product representation are given by the sums of the weights of its factors,  $(V \otimes W) = \sum_{\nu \in V} \sum_{\omega \in W} \nu + \omega$ . In particular, the first equality states that the weight space of  $\varpi_l$  in  $\rho_i^{l_i}$  is 1-dimensional, corresponding to the unique decomposition of  $\varpi_l = \sum_i l_i \omega_i$  as a sum of highest weights for each factor; and the second is the assertion that  $\varpi_{l^\theta} \notin (\rho_i^{l_i})$  when  $l_j^\theta > l_j$  for some  $j$ . As a consequence, (3.35) is a linear homogeneous system in the unknowns  $N_{\varpi_l}^{(\omega, k)}$  with trivial kernel: the integral matrix  $M$  with coefficients  $(M)_{l^\theta, l} := \text{Mult}_{\rho_i^{l_i}}(\varpi_{l^\theta})$ , in a choice of basis where the column  $(M)_{l^\theta}$  is to the left of the column  $(M)_l$  whenever  $l^\theta - l \in (Z_{>0})^{\ell_{\mathbb{R}}}$ , is upper-triangular and with ones on the diagonal. The claim then follows.

By Lemma 3.10, the sum in the polynomial character decomposition (3.31) localises on the set of admissible exponents, whose cardinality  $|J|_{\omega, j}$  is shown in Table 3.1.

**COROLLARY 3.11.** — *Fix a bijection  $\sigma : j_1, \dots, j_l \omega_j \in I_\omega$  inducing a total order on the set of admissible exponents, and let  $N_\omega \in \text{Mat}_{|J|_{\omega, j} \times \dim \rho_\omega}(\mathbb{Z})$  with  $(N_\omega)_{\ell k} := N_{\varpi_{\sigma(\ell)}}^{(\omega, k)}$ . Then, there exist explicit rational matrices  $A_\omega \in \text{GL}_{|J|_{\omega, j}}(\mathbb{Q})$ ,  $B_\omega \in \text{Mat}_{|J|_{\omega, j} \times \dim \rho_\omega}(\mathbb{Q})$  such that  $N_\omega = A_\omega B_\omega$ . In particular,  $N_\omega$  is explicitly computable for all  $\omega$ .*

*Proof.* — Fix  $Q := \{(Q_1^{(\kappa)}, \dots, Q_{\ell_{\mathbb{R}}}^{(\kappa)}) \in \mathbb{Q}^{\ell_{\mathbb{R}}}\}_{\kappa=1}^{|J|_{\omega, j}}$  rational points in  $T_{\mathbb{R}}$  in general position, and define

$$(3.37) \quad (D_\omega)_{l, \kappa} := \prod_{i=1}^{\ell_{\mathbb{R}}} \chi_i^{l_i} (Q_1^{(\kappa)}, \dots, Q_{\ell_{\mathbb{R}}}^{(\kappa)}),$$

$$(3.38) \quad (B_\omega)_{\kappa, \ell} := \chi^{\wedge \ell \rho_\omega} (Q_1^{(\kappa)}, \dots, Q_{\ell_{\mathbb{R}}}^{(\kappa)}).$$

Evaluating (3.31) at  $Q$  amounts to saying that

$$(3.39) \quad D_\omega N_\omega = B_\omega.$$

The identification of the Weyl character ring as an integral polynomial ring generated by the fundamental characters, (3.9), assures that  $\det D_\omega \neq 0$  for general  $Q$ , and (3.39) is a rank- $|J|_{\omega, j}$  linear problem over the rationals. We then have  $A_\omega = D_\omega^{-1}$ ,

and furthermore, the integrality of the coefficients in (3.9) assures that  $N_\omega \geq 2$ . Furthermore, given any such  $Q$ , the rational matrices  $A_\omega$ ,  $B_\omega$ , and  $N_\omega$  are computable in an entirely explicit manner, as follows: the fundamental characters on the r.h.s. of (3.37) are given as

$$\chi_i(Q_1^{(\kappa)}, \dots, Q_{\ell_R}^{(\kappa)}) = \sum_{\omega^0 \geq 2} \prod_{(\rho_i)_{j=1}^{\ell_R}} (Q_j^{(\kappa)})^{\omega_j^0},$$

and their value can be computed from the known expressions of the elements of the fundamental weight systems  $(\rho_i)$ ,  $i = 1, \dots, \ell_R$ . The exterior characters in (3.38) can similarly be computed from the knowledge of  $(\rho_\omega)$  alone to evaluate the power sum virtual characters of  $\rho_\omega$ ,

$$\chi_{\rho_\omega}((Q_1^{(\kappa)})^n, \dots, (Q_{\ell_R}^{(\kappa)})^n) = \sum_{\omega^0 \geq 2} \prod_{(\rho_\omega)_{j=1}^{\ell_R}} (Q_j^{(\kappa)})^{n\omega_j^0},$$

from which the exterior characters  $\chi^{\wedge k} \rho_\omega$  can be recovered using the Girard–Newton identities,

$$k \chi^{\wedge k} \rho_\omega = \sum_{n=1}^k (-1)^{n-1} \chi^{\wedge k-n} \rho_\omega \chi_{\rho_\omega}.$$

The integral linear system (3.39) can then be efficiently solved explicitly for  $N_\omega$  using Dixon's  $p$ -adic lifting algorithm.

Using (3.8)–(3.11), (3.31) and the complete calculation of the coefficients  $N_{\overline{\omega}_i}^{(\omega, k)}$  from Lemma 3.10 and Corollary 3.11 we can construct Landau–Ginzburg superpotentials for all  $R$ , as we now describe for the remaining exceptional cases.

3.5.1.  $R = E_6$ . — The Dynkin diagram for the affine  $E_6$  root system is given in Figure 2.1, for which the canonical label is  $\overline{k} = 3$ . In this case there are two nontrivial minimal irreducible representations, the 27-dimensional fundamental representation  $\rho_1 = (\mathbf{27})$  with highest weight  $\omega_1$  and its dual representation  $\rho_5 = (\overline{\mathbf{27}})$  with highest weight  $\omega_5$ , related by complex conjugation.

The character relations (3.31) are given explicitly for  $\omega = \omega_1$  in the appendix. The resulting family of spectral curves has fibres which are hyperelliptic curves of genus 5, with Newton polygon as shown in Figure 3.1, and ramification profile over  $\overline{1}$

$$\left( \overbrace{(3, 6)}^{\mu=0}, \overbrace{(6, 3)}^{\mu=1}, \overbrace{(3)}^{\mu=\varepsilon_3^j} \right),$$

where  $\varepsilon_3$  is a primitive third root of unity. This realises  $M_{[100000]E_6}^{\text{LG}}$  as a 7-dimensional subvariety of the 42-dimensional Hurwitz space  $H_{g_\omega, n_\omega}$  with  $g_\omega = 5$  and  $n_\omega = (5, 5, 2, 2, 2, 2, 2)$ , and explicit embedding described by (A.1).

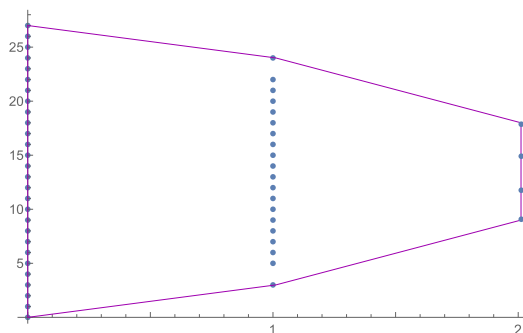


FIGURE 3.1. Newton polygon for the  $E_6$ -spectral curve.

3.5.2.  $R = E_7$ . — The Dynkin diagram for the affine  $E_7$  root system is given in Figure 2.1, in which we see that the canonical label is  $\bar{k} = 3$ . For this case there is a unique choice of minimal representation, corresponding to the 56-dimensional fundamental representation having highest weight  $\omega_6$ . Choosing this representation gives character relations which we include in the appendix for  $k = 1, \dots, 11$ . The resulting family of spectral curves has fibres of genus 33, with a degree 3 morphism to  $\mathbb{P}^1$  inducing a  $3 : 1$  branched cover of the Riemann sphere with ramification profile over  $1$

$$\left( \overbrace{12, 6, 4}^{\mu=0}, \overbrace{12, 6, 4}^{\mu=1}, \overbrace{2}^{\mu=1}, \overbrace{4}^{\mu=i} \right).$$

Hence,  $M_\omega^{\text{LG}}$  is an 8-dimensional submanifold in the 130-dimensional Hurwitz space  $H_{g_\omega, n_\omega}$ , with  $g_\omega = 33$  and  $n_\omega = (11, 5, 3, 11, 5, 3, 1, 1, 3, 3)$ . The associated Newton polygon is shown in Figure 3.2.

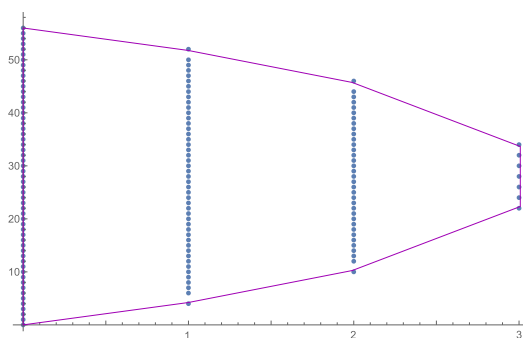


FIGURE 3.2. Newton polygon for the  $E_7$ -spectral curve.

3.5.3.  $R = E_8$ . — As mentioned, this case was thoroughly treated in [6], [7], and we will provide only a brief presentation here. The Dynkin diagram for the affine  $E_8$  root system is given in Figure 2.1, and the canonical label is, as for all the  $E$ -types,  $\bar{k} = 3$ .

In this case, the minimal, nontrivial, irreducible representation is the 248-dimensional adjoint representation. Explicit character relations were given in [7], where their derivation is explained in detail. It is shown in [6] that the resulting curve is of genus 128 and induces a cover of the Riemann sphere with ramification over 1 at  $\mu = 0, 1$ , in addition to second, third and fifth roots of unity, with ramification profile given in [6, Eq. (5.34)]. Explicit flat coordinates and prepotential can also be found in [6]. The resulting parent Hurwitz space is of dimension 518.

3.5.4.  $R = F_4$ . — The Dynkin diagram for the affine root system of type  $F_4$  is shown in Figure 2.1. In this case the canonical node is the one corresponding to the fundamental weight  $\omega_2$ ,  $\omega = [0001]_{F_4}$ , and  $\rho_\omega$  will be the 26-dimensional fundamental representation, i.e., the irreducible representation of highest weight  $\omega_4$ . The character relations (3.9) are then given as follows:

$$\begin{aligned}
 p_1^{[0001]_{F_4}} &= \chi_4, \\
 p_2^{[0001]_{F_4}} &= \chi_1 + \chi_3, \\
 p_3^{[0001]_{F_4}} &= \chi_2 + \chi_1\chi_4 - \chi_4, \\
 p_4^{[0001]_{F_4}} &= \chi_1^2 + \chi_3\chi_1 - \chi_4^2 - \chi_2, \\
 p_5^{[0001]_{F_4}} &= \chi_1^2\chi_4 - \chi_4^3 - \chi_1\chi_4 - 2\chi_2\chi_4 + \chi_3\chi_4 + \chi_4 + \chi_3^2 - \chi_2 + \chi_3, \\
 p_6^{[0001]_{F_4}} &= \chi_1^3 - \chi_1^2 - \chi_4^2\chi_1 - 3\chi_2\chi_1 + \chi_3\chi_4\chi_1 + \chi_4\chi_1 - \chi_1 - \chi_4^3 + \chi_4^2 - \chi_2 \\
 &\quad - \chi_2\chi_4 + \chi_3\chi_4 + \chi_4, \\
 p_7^{[0001]_{F_4}} &= \chi_1\chi_4^3 - \chi_4^3 - \chi_1\chi_4^2 + \chi_2\chi_4^2 - 2\chi_1\chi_4 - 2\chi_2\chi_4 - 3\chi_1\chi_3\chi_4 + 2\chi_3\chi_4 \\
 &\quad + \chi_4 + 2\chi_4^2 + \chi_1^2 - \chi_1 + \chi_1\chi_2 - \chi_2 + \chi_1^2\chi_3 - \chi_1\chi_3 - 2\chi_2\chi_3, \\
 p_8^{[0001]_{F_4}} &= \chi_4^2\chi_1^2 - \chi_1^3 - 2\chi_3\chi_1^2 + \chi_2\chi_1 - \chi_3\chi_1 + \chi_2\chi_4\chi_1 - \chi_3\chi_4\chi_1 - \chi_4\chi_1 \\
 &\quad + \chi_4^3 + \chi_3\chi_4^3 - 2\chi_3^2 - \chi_4^2 - \chi_2 + 3\chi_2\chi_3 + \chi_3 - 3\chi_3^2\chi_4 + 2\chi_2\chi_4 \\
 &\quad - 2\chi_3\chi_4 + \chi_4, \\
 p_9^{[0001]_{F_4}} &= \chi_4^5 - \chi_1\chi_4^3 - 4\chi_3\chi_4^3 - 2\chi_4^3 + 2\chi_1\chi_4^2 + 4\chi_2\chi_4^2 + \chi_1\chi_3\chi_4^2 \\
 &\quad + 2\chi_3^2\chi_4 - \chi_2 + \chi_1\chi_2\chi_4 + \chi_2^2 + 2\chi_1\chi_3\chi_4 + 3\chi_3\chi_4 - 2\chi_1^2 + \chi_2\chi_4 \\
 &\quad - 2\chi_1\chi_3^2 - 2\chi_1\chi_2 + 2\chi_1\chi_4 - 2\chi_1^2\chi_3 - 2\chi_1\chi_3 - \chi_2\chi_3, \\
 p_{10}^{[0001]_{F_4}} &= \chi_1\chi_4^4 - 5\chi_3\chi_4^3 - 2\chi_4^3 - 2\chi_1^2\chi_4^2 + 3\chi_2\chi_4^2 - 3\chi_1\chi_3\chi_4^2 - \chi_3\chi_4^2 \\
 &\quad + 5\chi_3^2\chi_4 + \chi_4^5 - \chi_1\chi_4^3 + 3\chi_1\chi_4 + \chi_1\chi_2\chi_4 + \chi_2\chi_4 + 3\chi_1\chi_3\chi_4 \\
 &\quad + \chi_2\chi_3\chi_4 + 4\chi_3\chi_4 - \chi_4 + \chi_1^3 - \chi_2^2 + \chi_1\chi_3^2 + 3\chi_3^2 - \chi_1\chi_2 \\
 &\quad + \chi_2 + 2\chi_1^2\chi_3 + 3\chi_1\chi_3 - 3\chi_2\chi_3, \\
 p_{11}^{[0001]_{F_4}} &= \chi_1\chi_4^4 + \chi_2\chi_4^3 - \chi_3\chi_4^3 + \chi_4^3 - \chi_1^2\chi_4^2 - 2\chi_1\chi_4^2 - 2\chi_2\chi_4^2 \\
 &\quad - 5\chi_1\chi_3\chi_4^2 - 2\chi_4^2 + 2\chi_3^2\chi_4 - \chi_1\chi_4\chi_2\chi_4 + \chi_1\chi_3\chi_4 - 3\chi_2\chi_3\chi_4 \\
 &\quad + \chi_3^3 + 3\chi_1^2 + \chi_2^2 + 4\chi_1\chi_3^2 + 2\chi_3^2 + \chi_1 + 3\chi_1\chi_2 + 3\chi_1^2\chi_3 \\
 &\quad + 5\chi_1\chi_3 + 2\chi_2\chi_3 + \chi_3,
 \end{aligned}
 \tag{3.40}$$

$$\begin{aligned}
 p_{12}^{[0001]_{F_4}} &= \chi_4^4 \chi_4^5 \chi_1 \chi_4^4 + 3\chi_1 \chi_4^3 + 2\chi_3 \chi_4^3 + 2\chi_1^2 \chi_4^2 + \chi_3^2 \chi_4^2 \chi_1 \chi_4^2 \\
 &\quad \chi_2 \chi_4^2 + \chi_3 \chi_4^2 \quad 5\chi_1 \chi_4 \quad 4\chi_2 \chi_4 \quad 5\chi_1 \chi_3 \chi_4 \quad 3\chi_2 \chi_3 \chi_4 + \chi_3 \chi_4 \\
 &\quad + \chi_4 \chi_1^3 + 2\chi_2^2 \chi_1 \chi_3^2 + 3\chi_1 \chi_2 \quad 3\chi_1 \chi_3 \quad 2\chi_2 \chi_3 + \chi_1 \chi_3 \chi_4^2, \\
 (3.40) \quad p_{13}^{[0001]_{F_4}} &= 2\chi_4^4 \quad 2\chi_4^5 \quad 2\chi_1 \chi_4^4 + 4\chi_1 \chi_4^3 \quad 2\chi_2 \chi_4^3 + 6\chi_3 \chi_4^3 + 4\chi_1^2 \chi_4^2 + 2\chi_3^2 \chi_4^2 \\
 &\quad 2\chi_1 \chi_4^2 \quad 2\chi_2 \chi_4^2 + 4\chi_1 \chi_3 \chi_4^2 \quad 2\chi_4^2 \quad 2\chi_1^2 \chi_4 \quad 4\chi_3^2 \chi_4 \quad 2\chi_1 \chi_4 \\
 &\quad + 2\chi_1 \chi_2 \chi_4 \quad 4\chi_1 \chi_3 \chi_4 + 2\chi_2 \chi_3 \chi_4 \quad 4\chi_3 \chi_4 \quad 2\chi_3^3 \quad 4\chi_1^2 + 2\chi_4^3 \\
 &\quad 2\chi_2^2 \quad 4\chi_1 \chi_3^2 \quad 4\chi_3^2 \quad 4\chi_1 \chi_2 \quad 4\chi_1^2 \chi_3 \quad 4\chi_1 \chi_3 \quad 2\chi_2 \chi_3 + 2,
 \end{aligned}$$

with  $p_{26}^{[0001]_{F_4}} = p_i^{[0001]_{F_4}}$ . Note that the above relations in the character ring follow from those for  $R = E_6$  and  $\rho = (\mathbf{27})$  or  $\rho = (\overline{\mathbf{27}})$  by folding; in particular  $M_{F_4, [0001]_{F_4}}^{LG}$  sits inside  $M_{[100000]_{E_6}}^{LG}$  as the fixed locus of the involution  $w_1 \ \$ w_5, w_2 \ \$ w_4$ . The generic fibre  $\overline{C_w^{\omega_4}}$  is a genus 4 hyperelliptic curve with ramification over 7

$$\left( \overbrace{3, 6}^{\mu=0}, \overbrace{3, 6}^{\mu=1}, \overbrace{3}^{\mu=\varepsilon_3}, \overbrace{3}^{\mu=\varepsilon_3^2} \right),$$

where  $\varepsilon_3$  is a primitive third root of unity, and the associated Newton polygon is shown in Figure 3.3. The extended affine  $F_4$ -Frobenius manifold is thus realised as a 5-dimensional submanifold of the 36-dimensional Hurwitz space  $H_{g_\omega, n_\omega}$ , with  $g_\omega = 4$  and  $n_\omega = (5, 5, 2, 2, 2, 2)$ .

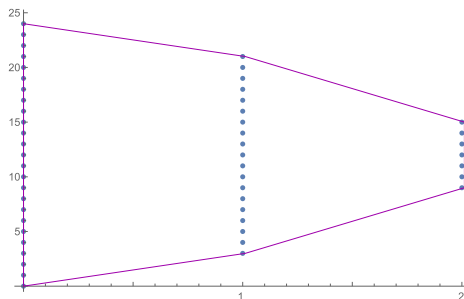


FIGURE 3.3. Newton polygon for the  $F_4$ -spectral curve.

3.5.5.  $R = G_2$ . — The Dynkin diagram for the affine  $G_2$  root system is given in Figure 2.1, and the canonical label is  $\bar{k} = 2$ . In this case,  $\rho_{\omega_1} = (\mathbf{7})$  is the 7-dimensional fundamental representation which is the irreducible representation with highest weight  $\omega_1$ . We obtain the character relations

$$(3.41) \quad p_1^{[10]_{G_2}} = \chi_1, \quad p_2^{[10]_{G_2}} = \chi_1 + \chi_2, \quad p_3^{[10]_{G_2}} = \chi_1^2 - \chi_2,$$

and  $p_7^{[10]_{G_2}} = p_i^{[10]_{G_2}}$ , hence

$$P_{[10]_{G_2}}(\lambda, \mu; w_0, w_1, w_2) = \sum_{i=0}^3 (-1)^i p_i^{[10]_{G_2}}(w_1, w_2 - \lambda/w_0) \mu^i (1 - \mu^7)^{2i},$$

and solving for  $\lambda$

$$(3.42) \quad \lambda = \frac{w_0}{\mu^2(\mu + 1)^2} \left( \mu^6 + (1 - w_1)\mu^5 + (1 + w_2)\mu^4 + (1 - w_1^2 + 2w_2)\mu^3 + (1 + w_2)\mu^2 + (1 - w_1)\mu + 1 \right).$$

As was the case for  $R = F_4$ , the same superpotential could be obtained from the LG model of  $\omega = [1000]_{D_4}$  by the order three folding of the  $D_4$  Dynkin diagram, and  $M_{[10]_{G_2}}^{\text{LG}}$  sits inside  $M_{[1000]_{D_4}}^{\text{LG}}$  as the fixed locus of the triality action sending  $(w_1, w_3, w_4) \mapsto (w_{\varepsilon(1)}, w_{\varepsilon(3)}, w_{\varepsilon(4)})$  with  $\varepsilon \in S_3$ . The outcome is a family of rational functions in  $\mu$ , with three poles at  $\mu = 0, -1, 1$ , all of order two. This means that the resulting Frobenius manifold is a 3-dimensional sublocus in the 7-dimensional Hurwitz space  $H_{0, n_\omega}$ , with  $n_\omega = (1, 1, 1)$ .

#### 4. MIRROR SYMMETRY

Having constructed  $M_\omega^{\text{LG}}$  for all Dynkin types, we now move on to proving Theorem 3.6.

As in [16, Lect. 5], and with notation as in Section 3.1, define local coordinates  $\kappa_i$  in a neighbourhood of  $\gamma_i$  by Lagrange inversion of  $\lambda(\kappa) = \kappa_i^{n_i+1} + O(1)$ . Consider then the following functions on  $H_{g_\omega, n_\omega}$ :

$$(4.1a) \quad \tau_{i;\alpha} := \text{Res}_{\gamma_i} \kappa_i^{-\alpha} \log \mu \, d\lambda, \quad \alpha = 1, \dots, n_i,$$

$$(4.1b) \quad \tau_j^{\text{ext}} := \text{p.v.} \int_{\gamma_0}^{\gamma_j} d \log \mu, \quad j = 1, \dots, \ell(n_\omega),$$

$$(4.1c) \quad \tau_k^{\text{res}} := \text{Res}_{\gamma_k} \lambda \, d \log \mu, \quad k = 0, \dots, \ell(n_\omega),$$

where the principal value, p.v., indicates subtraction of the divergent part in  $\kappa_i$ .

LEMMA 4.1. — *There exist complex numbers  $f_{i,\alpha}^{(r)}, q_j, r_k \in \mathbb{C}$  and holomorphic functions  $f_{t_i} : M_\omega^{\text{LG}} \rightarrow \mathbb{C}, g_{i=1, \dots, \ell_R+1}$  such that*

$$(4.2) \quad \tau_j^{\text{ext}}|_{M_\omega^{\text{LG}}} = q_j t_{\ell_R+1}, \quad \tau_{i;\alpha}^j|_{M_\omega^{\text{LG}}} = \sum_{r=1}^{\ell_R} f_{i,\alpha}^{(r)} t_r, \quad \tau_k^{\text{res}}|_{M_\omega^{\text{LG}}} = r_k t_{\bar{k}}.$$

Moreover,  $(t_1, \dots, t_{\ell_R+1})$  are flat coordinates for the metric (4.8) on  $M_\omega^{\text{LG}}$ .

Proof. — Suppose first that  $R$  is any of the root systems  $A_\ell, B_\ell, C_\ell, D_\ell$  or  $G_2$ . Then, from the discussion of Sections 3.4 and 3.5.5, we have  $g_\omega = 0$ , and  $M_\omega^{\text{LG}} \subset H_{0, n_\omega}$  is a space of rational functions. The statement of the Lemma is then a specialisation of [16, Th. 5.1], which in particular asserts that  $\tau_{i;\alpha}, \tau_j^{\text{ext}}$  and  $\tau_k^{\text{res}}$  are a complete set of flat coordinates for the metric (3.5) on a genus zero Hurwitz space; and since  $M_\omega^{\text{LG}}$  is a fixed-locus of an involution acting linearly in these coordinates, it is specified by a linear condition of the form (4.2). For the remaining four exceptional cases, the linear relations in (4.2) and the constancy of the Gram matrix of the metric (3.15) in the chart  $(t_1, \dots, t_{\ell_R+1})$  follow from a direct residue calculation from (4.1a)–(4.1c),

and from making use of the explicit form of the superpotential from (3.11) for each Dynkin type; we omit the details here.<sup>(8)</sup>

LEMMA 4.2. — *Let  $(t_1, \dots, t_{\ell_R+1})$  be flat coordinates for (3.15) as in Lemma 4.1. Then  $t_{\ell_R+1} = \log w_0/d_{\bar{k}}$  and, for all  $i = 1, \dots, \ell_R$ ,*

$$t_i(w_0, \dots, w_{\ell_R}) \geq w_0^{d_i/d_{\bar{k}}} Z[w_1, \dots, w_{\ell_R}].$$

Moreover, the change of variables  $w \mathcal{V} t(w)$  has a polynomial inverse

$$w_i(t_1, \dots, t_{\ell_R+1}) \geq Q[t_1, \dots, t_{\ell_R+1}, e^{t_{\ell_R+1}}].$$

Proof. — A direct calculation from (4.1a)–(4.1c), using as above the explicit form of (3.11), shows that the flat coordinates  $(t_1, \dots, t_{\ell_R+1})$  are related (up to normalisation) to  $(w_0, \dots, w_{\ell_R})$  as

$$t_i = w_0^{d_i/d_{\bar{k}}} t_i(w), \quad t_{\ell_R+1} = \frac{\log w_0}{d_{\bar{k}}},$$

where  $t_i(w)$  are explicit integral polynomials in  $(w_1, \dots, w_{\ell_R})$ . Moreover, it can be verified directly that

$$\begin{aligned} \partial_{w_j} t_i &= 0 && \text{if } d_j > d_i, \\ \deg_{w_j} t_i &= 1 && \text{if } d_j = d_i, \end{aligned}$$

implying that the inverse function  $t \mathcal{V} w(t)$  is a polynomial in  $(t_1, \dots, t_{\ell_R}, e^{t_{\ell_R+1}})$  with rational coefficients.

The previous two Lemmas put us in a position to prove Theorem 3.6.

Proof of Theorem 3.6. — Consider the change-of-variables

$$w_i(x_1, \dots, x_{\ell_R}) = \chi_i(e^h) = \sum_{\omega^0 \geq (\rho_i)} \prod_{j=1}^{\ell_R} e^{\omega_j^0 x_j}, \quad i = 1, \dots, \ell_R.$$

Further identifying

$$(4.3) \quad w_0 = e^{c_\omega x_{\ell_R+1}},$$

for  $c_\omega \geq \mathbb{C}^*$ , extends this to a local analytic isomorphism sending

$$M_{\mathbb{R}}^{\text{DZ}} \ni (x_1, \dots, x_{\ell_R}; x_{\ell_R+1}) \xrightarrow{\sim} (w_0; w_1, \dots, w_{\ell_R}) \geq M_{\omega}^{\text{LG}}.$$

We shall now prove that this is, in fact, an isomorphism of Frobenius structures upon checking the four defining properties, (DZ-I) through (DZ-IV), of the reconstruction Theorem 2.3.

(DZ-I) and (DZ-II). Define holomorphic vector fields  $e, E \geq (M_{\omega}^{\text{LG}}, TM_{\omega}^{\text{LG}})$  as

$$(4.4) \quad e := \frac{\partial_{w_{\bar{k}}}}{w_0}, \quad E := w_0 \partial_{w_0}.$$

<sup>(8)</sup>All calculations can be made available to the interested reader upon request.



Then, the one-parameter group of isomorphisms generated by the horizontal lift  $\delta_e^{(\mu)}$  (resp.  $\delta_E^{(\mu)}$ ) to the universal curve acts on the superpotential by translation (resp. conformal transformations) on the superpotential. To see this, note that, by (3.11),

$$(4.5) \quad \begin{aligned} w_0 &\mapsto aw_0 & \lambda &\mapsto a\lambda, \\ w_{\bar{k}} &\mapsto w_{\bar{k}} + b/w_0 & \lambda &\mapsto \lambda + b. \end{aligned}$$

Since the unit and Euler vector field (3.3) of a Hurwitz–Frobenius manifold are characterised as the generators of the affine action (3.1) on the superpotential, (4.5) identifies  $e$  with the unit and  $E$  with the Euler vector field of  $M_{\omega}^{LG}$ . To verify (DZ-I) and (DZ-II), it then remains to check that the expressions in (4.4) coincide with those for the respective vector fields of  $M_{\mathbb{R}}^{DZ}$ . A simple calculation from Lemma 4.2 shows that

$$\begin{aligned} e &= \frac{1}{w_0} \sum_{i=1}^{\ell_{\mathbb{R}+1}} \frac{\partial t_i}{\partial w_{\bar{k}}} \partial_{t_i} = \partial_{t_{\bar{k}}}, \\ E = w_0 \partial_{w_0} &= w_0 \sum_{i=1}^{\ell_{\mathbb{R}+1}} \frac{\partial t_i}{\partial w_0} \partial_{t_i} = \frac{\partial_{t_{\ell_{\mathbb{R}+1}}}}{d_{\bar{k}}} + \sum_{i=1}^{\ell_{\mathbb{R}}} \frac{d_i t_i}{d_{\bar{k}}} \partial_{t_i}, \end{aligned}$$

thereby matching the expression of the unit and the Euler vector fields in Theorem 2.3.

(DZ-III). Let us consider now the Gram matrix of the intersection pairing on  $M_{\omega}^{LG}$  in the  $x$ -chart. Consider first the argument of the residues in (3.17),

$$(4.6) \quad \gamma_{ij}(p) := \frac{\delta_{\partial_{x_i}}^{(\mu)} \lambda \delta_{\partial_{x_j}}^{(\mu)} \lambda}{\lambda \mu^2 \partial_{\mu} \lambda} d\mu(p),$$

so that

$$\gamma(\partial_{x_i}, \partial_{x_j}) = \sum_{\ell} \operatorname{Res}_{p=p_{\ell}^{cr}} \gamma_{ij}(p).$$

From (4.6), we deduce that the pole structure of  $\gamma_{ij}(p)$  is as follows:

- (1) it has simple poles at the critical points  $\bar{r} p_{\ell}^{cr} g$ , for which  $d\lambda(p_{\ell}^{cr}) = 0$ ;
- (2) it has at most simple poles at  $\lambda(p) = 0$ , and only when both  $i, j \notin \ell_{\mathbb{R}+1}$ : indeed, from (4.3) we have that  $\delta_{\partial_{x_{\ell_{\mathbb{R}+1}}}^{(\mu)}} \lambda = c_{\omega} \lambda$ , thereby cancelling the zeroes at the denominator when either  $i$  or  $j = \ell_{\mathbb{R}+1}$ ;
- (3) it has at most simple poles at  $\mu(p) = 0$  (for which  $\lambda(p) = 1$ : see Remark 3.5) and only when  $i = j = \ell_{\mathbb{R}+1}$ . To see why, notice that locally at a point  $p^{\flat}$  near  $\mu(p) = 0$  we have

$$\lambda(p^{\flat}) = e^{c_{\omega} x_{\ell_{\mathbb{R}+1}}} \mu(p^{\flat})^{-q_p} (r_p + O(\mu)),$$

where  $q(p) \in \mathbb{Z}_{>0}$  and  $r(p) \in \mathbb{C}$ . Then, the denominator in (4.6) has a leading Puiseux asymptotics in  $\mu$  of the form  $\mu^{1-2q_p}$ , resulting from the combination of the order  $q_p$  divergence of  $\lambda(\mu)$ , the order  $q_p + 1$  divergence of  $\partial_{\mu} \lambda(\mu)$ , and the double zero of  $\mu^2$ . For the numerator, we have  $\delta_{\partial_{x_i}}^{(\mu)} \lambda = \mu^{\delta_{i, \ell_{\mathbb{R}+1}} - q_p}$ , since  $r(p)$  is  $w$ -independent by Proposition 3.4, so that

$$\operatorname{ord}_{\mu(p)=0} \gamma_{ij} = 1 - \delta_{i, \ell_{\mathbb{R}+1}} - \delta_{j, \ell_{\mathbb{R}+1}};$$

(4) superficially, there may be further poles to be expected at the critical points  $\bar{r}q_m^{cr}g$  of the  $\mu$ -projection,  $d\mu(q_m^{cr}) = 0$ , that is, where the Ehresmann connection (3.4) induced by the  $\mu$ -foliation is singular and  $\delta_{\partial x_i}^{(\mu)}\lambda$  possibly develops a pole. However these singularities are offset by a vanishing of the same order of  $d\mu/\partial\mu\lambda$ , so that

$$(4.7) \quad \text{ord}_{q_m^{cr}} \gamma_{ij} = 0.$$

Based on the above, we turn the contour around and equate the sum of residues at the critical points in (3.17) to a much more manageable sum of residues at poles and zeroes of  $\mu$  and  $\lambda$ . When  $i = j = \ell_R + 1$ , we only have poles at  $\mu = 0$ , and

$$\begin{aligned} \gamma(\partial_{x_{\ell_R+1}}, \partial_{x_{\ell_R+1}}) &= N_\omega \sum_{\mu(p)=0} \text{Res}_{p^\theta=p} \gamma_{\ell_R+1, \ell_R+1}(p^\theta) \\ &= N_\omega c_\omega^2 \sum_{\mu(p)=0} \text{Res}_{p^\theta=p} \frac{\lambda}{\mu^2 \partial_\mu \lambda} d\mu(p^\theta) \\ &= N_\omega c_\omega^2 \sum_{\mu(p)=0} \frac{1}{\text{ord}_p \lambda} = \sum_{i: \mu(\gamma_i)=0} \frac{N_\omega c_\omega^2}{(n_\omega)_i}. \end{aligned}$$

Setting  $c_\omega = \sqrt{d_{\bar{k}}} (N_\omega \sum_{i: \mu(\gamma_i)=0} 1/(n_\omega)_i)^{1/2}$  gives  $\gamma(\partial_{x_{\ell_R+1}}, \partial_{x_{\ell_R+1}}) = d_{\bar{k}}$ .

Suppose now that  $j = \ell_R + 1, i < \ell_R + 1$ . Our analysis above shows that  $\gamma_{ij}$  is regular outside the critical locus of  $\lambda$ , and therefore

$$\gamma(\partial_{x_i}, \partial_{x_{\ell_R+1}}) = \gamma(\partial_{x_{\ell_R+1}}, \partial_{x_i}) = \delta_{i, \ell_R+1} d_{\bar{k}}.$$

Finally, let's look at  $i, j < \ell_R + 1$ . In this case, outside  $\bar{r}p_\ell^{cr}g$ ,  $\gamma_{ij}$  has only simple poles at the zeroes of  $\lambda$ , i.e., when

$$0 = P_\omega(0, \mu) = Q_\omega^{\text{red}}(\mu) = \prod_{0 \notin \omega^\theta \geq (\rho_\omega)} (\mu - e^{\omega^\theta(x)}) \quad (\mu = e^{\omega^\theta(x)}, \omega^\theta \geq (\rho_\omega),$$

where we used the shorthand notation  $\omega^\theta(x) := \sum_{n=1}^{\ell_R} \omega_n^\theta x_n$ . The evaluation of the residue at  $(\lambda, \mu) = (0, e^{\omega^\theta(x)})$  gives

$$\begin{aligned} \text{Res}_{\mu=e^{\omega^\theta(x)}} \gamma_{ij} &= \frac{\delta_{\partial x_i}^{(\mu)} \lambda \delta_{\partial x_j}^{(\mu)} \lambda}{\mu^2 (\partial_\mu \lambda)^2} \Big|_{\mu=e^{\omega^\theta(x)}} = \frac{\partial_{x_i} Q_\omega^{\text{red}} \partial_{x_j} Q_\omega^{\text{red}}}{\mu^2 (\partial_\mu Q_\omega^{\text{red}})^2} \Big|_{\mu=e^{\omega^\theta(x)}} \\ (4.8) \quad &= \frac{e^{2\omega^\theta(x)} \omega_i^\theta \omega_j^\theta \prod_{\omega^{00}, \omega^{000} \notin \omega^\theta} (e^{\omega^\theta(x)} - e^{\omega^{00}(x)}) (e^{\omega^\theta(x)} - e^{\omega^{000}(x)})}{e^{2\omega^\theta(x)} \prod_{\omega^{00}, \omega^{000} \notin \omega^\theta} (e^{\omega^\theta(x)} - e^{\omega^{00}(x)}) (e^{\omega^\theta(x)} - e^{\omega^{000}(x)})} \\ &= \omega_i^\theta \omega_j^\theta, \end{aligned}$$

where the second equality makes use of the implicit function theorem. Summing over  $\omega^\theta$  gives

$$\begin{aligned}
 \gamma(\partial_{x_i}, \partial_{x_j}) &= N_\omega \sum_{\omega^\theta \in \mathcal{Z}(\rho_\omega)} \omega_i^\theta \omega_j^\theta = N_\omega \operatorname{Tr}(\rho_\omega(h_i)\rho_\omega(h_j)) \\
 (4.9) \qquad \qquad &= N_\omega C_2(\rho_\omega)(K_{\mathbb{R}})_{ij} \\
 &= N_\omega \frac{(\omega, \omega + 2w) \dim_{\mathbb{C}} \rho_\omega}{\dim_{\mathbb{C}} \mathfrak{g}} (K_{\mathbb{R}})_{ij}.
 \end{aligned}$$

In (4.9) the first equality is immediate; the second reflects the fact that the resulting trace gives a non-degenerate Ad-invariant bilinear form on  $\mathfrak{h}_{\mathbb{R}}^*$ , hence proportional to the Killing pairing since  $\mathfrak{g}$  is simple; and the final step identifies the proportionality factor with the corresponding quadratic Casimir eigenvalue for the representation  $\rho_\omega$  in the appropriate normalisation (see e.g. [26, Lect. 25.1]). Picking  $N_\omega := \dim_{\mathbb{C}} \mathfrak{g} / (\omega, \omega + 2w) \dim_{\mathbb{C}} \rho_\omega$  concludes the identification of (3.17) with the intersection pairing in Theorem 2.3.

(DZ-IV). The final missing piece to invoke the reconstruction theorem, Theorem 2.3, is to prove the quasi-polynomiality of the prepotential of  $M_\omega^{\text{LG}}$ . Consider the change-of-variables<sup>(9)</sup>

$$(4.10) \qquad v_i := \begin{cases} \log w_0 & \text{if } i = 0, \\ w_i & \text{otherwise,} \end{cases}$$

and define

$${}_{ijk}(p) := N_\omega \frac{\delta_{\partial_{v_i}}^{(\mu)} \lambda \delta_{\partial_{v_j}}^{(\mu)} \lambda \delta_{\partial_{v_k}}^{(\mu)} \lambda}{\mu^2 \partial_\mu \lambda} d\mu(p).$$

Then the symmetric (0,3) tensor (3.16) in the  $v$ -chart reads

$$(4.11) \qquad c(\partial_{v_i}, \partial_{v_j}, \partial_{v_k}) = \sum_{\ell} \operatorname{Res}_{p=p_\ell^{\text{cr}}} {}_{ijk}(p).$$

The same analysis as the one we carried out to verify **[DZ-III]** reveals that  ${}_{ijk}$  has poles at the critical points (i) and the poles (iii) of  $\lambda$ . Additionally, it has poles at the critical points (iv) of the  $\mu$ -projection, due to the singularities of the derivation  $\delta_X^{(\mu)}$  in (3.4). As noted in (4.7), in the case of the intersection pairing, the poles of  $\delta_{\partial_{x_i}}^{(\mu)} \lambda$  and  $\partial_\mu \lambda$  in (4.6) were cancelling out between the numerator and denominator in the Puiseux expansion of  ${}_{ijk}(p)$  near  $q_m^{\text{cr}}$ . On the other hand, the additional factor containing  $\delta_{\partial_{w_k}}^{(\mu)} \lambda$  in the numerator of (3.16) may give rise to a pole with non-vanishing residue for  ${}_{ijk}(p)$  at  $q_m^{\text{cr}}$ .

<sup>(9)</sup>This might look like an unnecessary piece of trivial additional notation at this stage: it just amounts to replacing  $w_0$ -derivatives of the superpotential with logarithmic  $w_0$ -derivatives. The reader will forgive us as this will help making the discussion of polynomiality completely manifest at the end of the argument: indeed, while components  $c(\partial_{w_i}, \partial_{w_j}, \partial_{w_k})$  of the  $c$ -tensor in the  $w$ -chart are Laurent monomials (with coefficients in  $\mathbb{Q}[w_1, \dots, w_{\ell_{\mathbb{R}}}]$  in  $w_0$  with exponent  $2 - \delta_{i0} - \delta_{j0} - \delta_{k0}$ , and in particular have negative power for  $i = j = k = 0$ , the components  $c(\partial_{v_i}, \partial_{v_j}, \partial_{v_k})$  of the  $c$ -tensor in the  $v$ -chart will always be elements of  $e^{2v_0} \mathbb{Q}[v_1, \dots, v_{\ell_{\mathbb{R}}}]$  (see (4.14)).

To compute (4.11) we determine individually the residues of  $i_{jk}(p)$  using the known expression of  $P_\omega(\lambda, \mu)$  from (3.11). By Proposition 3.4, the  $\mu$ -coordinate of the poles of  $\lambda$  have simple  $v$ -independent expressions, which are either  $\mu = 0, 1$ , or a root of unity (see (3.14)). The calculation of the corresponding residues is straightforward, and we find

$$(4.12) \quad \text{Res}_{p=1} \ i_{jk}(p) \in e^{2v_0} \mathbb{Q}[v_1, \dots, v_{\ell_{\mathbb{R}}}], \quad r = 1, \dots, \ell(n_\omega).$$

On the other hand, the  $\mu$ -coordinates of the critical points  $\tilde{f}_{p_\ell}^{cr} g$  of  $\lambda$  (resp.  $\tilde{f}_{q_m}^{cr} g$  of  $\mu$ ) are given by the roots of  $\text{Discr}_\mu(P_\omega)(\mu)$  (resp.  $\text{Discr}_\lambda(P_\omega)(\mu)$ ). When  $\deg_\lambda P_\omega > 1$ , these are both high degree polynomials in  $\mu$  with  $v$ -dependent coefficients, and therefore  $\mu(p_\ell^{cr})$  and  $\mu(q_m^{cr})$  are given by complicated (algebraic) hypergeometric functions of  $(e^{v_0}, v_1, \dots, v_{\ell_{\mathbb{R}}+1})$ . Since  $i_{jk}$  has in general non-vanishing residues at  $p = q_m^{cr}$ , turning the contour around in the sum in (4.11) will pick up some intricate hypergeometric contributions from these points, making it difficult to provide a manifest proof of the polynomiality of their sum. One exception however is when  $i = 0$ : since  $\delta_{\partial_{v_0}}^{(\mu)} \lambda = \lambda$ , the same count of the order of divergence as in (4.7) shows that

$$(4.13) \quad \text{ord}_{q_m^{cr}} \ o_{jk} = 0 \Rightarrow \text{Res}_{p=q_m^{cr}} \ o_{jk} = 0.$$

In this case, the residue theorem implies that, closing the contour around the complement of  $\tilde{f}_{p_\ell}^{cr} g$ , (4.11) equates to a sum of residues coming only from the poles of  $\lambda$ ,

$$(4.14) \quad c(\partial_{v_0}, \partial_{v_j}, \partial_{v_k}) = \sum_r \text{Res}_{p=1} \ o_{jk}(p) \in e^{2v_0} \mathbb{Q}[v_1, \dots, v_{\ell_{\mathbb{R}}}],$$

where we used (4.12). In the same vein, we also obtain

$$(4.15) \quad \eta(\partial_{v_j}, \partial_{v_k}) = \sum_r \text{Res}_{p=1} \ \frac{1}{\lambda} \ o_{jk}(p) \in e^{v_0} \mathbb{Q}[v_1, \dots, v_{\ell_{\mathbb{R}}}].$$

To compute the remaining components  $\tilde{c}_{ijk}(v) := c(\partial_{v_i}, \partial_{v_j}, \partial_{v_k})$ , we use the associativity of the Frobenius product in the  $v$ -chart:

$$(4.16) \quad \sum_{k,\ell} (\tilde{c}_{ijk} \tilde{\eta}^{k\ell} \tilde{c}_{0\ell m} - \tilde{c}_{imk} \tilde{\eta}^{k\ell} \tilde{c}_{0\ell j}) = 0,$$

where  $\tilde{\eta}^{k\ell} \in e^{-v_0} \mathbb{Q}(w_1, \dots, w_{\ell_{\mathbb{R}}})$  is the inverse of the Gram matrix of (4.15). The set of equations (4.16) give an a priori overconstrained inhomogeneous linear system for the unknowns  $\tilde{c}_{ijk}(v)$  with all  $i, j, k \neq 0$ . It is not at all obvious, in principle, that a unique solution of (4.16) exists; and even so, that such a solution is a polynomial (as opposed to rational) function of  $(e^{v_0}, v_1, \dots, v_{\ell_{\mathbb{R}}})$ . That this is the case, however, can be shown from a direct calculation from (4.14)–(4.15), which gives

$$c(\partial_{v_i}, \partial_{v_j}, \partial_{v_k}) \in e^{2v_0} \mathbb{Q}[v_1, \dots, v_{\ell_{\mathbb{R}}}]$$

Using Lemma 4.2 together with (4.10), we finally deduce that

$$(4.17) \quad c(\partial_{t_i}, \partial_{t_j}, \partial_{t_k}) = \frac{\partial F_{\mathbb{R}}}{\partial t_i \partial t_j \partial t_k} \in \mathbb{Q}[t_1, \dots, t_{\ell_{\mathbb{R}}+1}, e^{t_{\ell_{\mathbb{R}}+1}}].$$

The claim now follows upon invoking Theorem 2.3.

REMARK 4.3. — The explicit form of (3.11) was crucially used in the computation of (4.14)–(4.15), as well as in determining the flat coordinates in Lemma 4.2. Despite the disadvantage of having to carry out a separate analysis of each of the seven Dynkin series, an immediate spin-off of this method is that closed-form polynomial expressions for the flat coordinates and, from (4.17), the prepotential are rather powerfully produced from straightforward residue computations. We will illustrate this in detail in the next Section.

4.1. EXAMPLES. — The construction of closed-form flat frames for  $\eta$  and prepotentials in all Dynkin types does not follow directly from the original Dubrovin–Zhang construction of the flat pencil.<sup>(10)</sup> One of the advantages of the mirror formulation in Theorem 3.6 is that these can now be easily computed using the Landau–Ginzburg formalism.

#### 4.1.1. Classical root systems

EXAMPLE 4.1 ( $R = B_3$ ). — From (3.21) we have the superpotential

$$\lambda_{B_3} = \frac{w_0}{\mu^2(\mu+1)^2} \left( \mu^6 + 1 + (\mu^5 + \mu)(w_1 - 1) + (\mu^4 + \mu^2)(-w_1 + w_2 + 1) + \mu^3(-w_3^2 + 2w_2 + 2) \right),$$

which gives flat coordinates

$$t_4 = \frac{\log w_0}{2}, \quad t_1 = w_0^{1/2}(w_1 + 1), \quad t_3 = w_0^{1/2}w_3, \quad t_2 = w_0(w_1 + w_2 + 2).$$

We obtain the prepotential for  $B_3$  to be

$$F_{B_3} = t_4 t_2^2 + \frac{1}{2} t_1^2 t_2 + t_3^2 t_2 - \frac{1}{48} t_1^4 - \frac{1}{24} t_3^4 + 2t_1 t_3^2 e^{t_4} + t_1^2 e^{2t_4} + 2t_3^2 e^{2t_4} + \frac{1}{2} e^{4t_4},$$

which is seen to be equivalent to the free energy in [21, Ex. 2.7] by  $F \mapsto F/2$ .

EXAMPLE 4.2 ( $R = B_4$ ). — For  $B_4$ , (3.21) is

$$\lambda_{B_4} = \frac{w_0}{\mu^3(\mu+1)^2} \left( \mu^8 + 1 + (\mu^7 + \mu)(w_1 - 1) + (\mu^6 + \mu^2)(-w_1 + w_2 + 1) + (\mu^5 + \mu^3)(w_1 - w_2 + w_3 - 1) + \mu^4(w_4^2 - 2w_1 - 2w_3) \right),$$

which gives flat coordinates

$$t_5 = \frac{\log w_0}{3}, \quad t_1 = w_0^{1/3}(w_1 + 1), \quad t_2 = w_0^{1/2}w_4, \quad t_4 = w_0^{2/3}(4w_1 - w_1^2 + 6w_2 + 11), \\ t_3 = w_0(2w_1 + w_2 + w_3 + 2).$$

<sup>(10)</sup>A priori one could try to construct a quasi-homogeneous polynomial ansatz for the prepotential, impose the associativity of the Frobenius product, and solve for the coefficients. This typically results in a large system of non-linear equations, owing to the non-linearity of the WDVV equations, whose solution is already unviable e.g. for  $R = E_6$ . The proof of Theorem 3.6 shows that the Landau–Ginzburg formulas reduce this to an explicit calculation of residues and a relatively small-rank, *linear* problem.

In this case, the resulting prepotential is given by

$$F_{B_4} = \frac{1}{1944}t_1^4t_4 + \frac{1}{9720}t_1^6 + \frac{1}{648}t_1^2t_4^2 + \frac{1}{9}t_1t_4t_3 + \frac{1}{24}t_2^4 + t_2^2t_3 + \frac{1}{1944}t_3^3 \\ + t_5t_3^2 + \frac{1}{3}t_1^2t_2^2e^{t_5} + \frac{1}{3}t_2^2t_4e^{t_5} + \frac{1}{36}t_1^4e^{2t_5} + \frac{1}{18}t_1^2t_4e^{2t_5} + 2t_1t_2^2e^{2t_5} \\ + \frac{1}{36}t_4^2e^{2t_5} + 2t_2^2e^{3t_5} + \frac{1}{2}t_1^2e^{4t_5} + \frac{1}{3}e^{6t_5}.$$

EXAMPLE 4.3 ( $R = C_3$ ). — Here, the superpotential is given by

$$\lambda_{C_3} = \frac{w_0(\mu^6 + 1) + (\mu^5 + \mu)w_1 + (\mu^4 + \mu^2)(w_2 + 1) + \mu^3(w_1 + w_3)}{\mu^3},$$

which leads to the following flat coordinates:

$$t_4 = \frac{\log w_0}{3}, \quad t_1 = w_0^{1/3}w_1, \quad t_2 = w_0^{2/3}(w_1^2 + 6(w_2 + 1)), \quad t_3 = w_0(w_1 + w_3).$$

In this case, the prepotential is

$$F_{C_3} = t_4t_3^2 + \frac{1}{9}t_1t_2t_3 + \frac{1}{9720}t_1^6 + \frac{1}{1944}t_1^4t_2 + \frac{1}{648}t_1^2t_2^2 + \frac{1}{1944}t_2^3 \\ + \frac{1}{36}t_1^4e^{2t_4} + \frac{1}{18}t_1^2t_2e^{2t_4} + \frac{1}{36}t_2^2e^{2t_4} + \frac{1}{2}t_1^2e^{4t_4} + \frac{1}{3}e^{6t_4},$$

which is the same as the one found in Example 2.8 in [21] after letting  $F \mapsto F/2$ , and  $t_2 \mapsto 6t_2$ .

EXAMPLE 4.4 ( $R = D_4$ ). — For  $\ell = 4$ , (3.23) becomes

$$\lambda_{D_4} = \frac{w_0(\mu^8 + w_1(\mu^7 + \mu) + w_2(\mu^6 + \mu^2) + (w_1 + w_3w_4)(\mu^5 + \mu^3) + 1)}{\mu^2(\mu^2 + 1)^2},$$

which has poles at  $\mu = 0, 1, -1, 1$ , all of order 2. The resulting flat coordinates are

$$t_5 = \frac{\log w_0}{2}, \quad t_1 = w_0^{1/2}w_1, \quad t_3 = w_0^{1/2}(w_3 + w_4), \\ t_4 = w_0^{1/2}(w_3 - w_4), \quad t_2 = w_0(w_2 + 2),$$

which leads to the prepotential

$$F_{D_4} = t_5t_2^2 + \frac{1}{4}t_3^2t_2 + \frac{1}{4}t_4^2t_2 + \frac{1}{48}t_1^4 + \frac{1}{2}t_1^2t_2 + \frac{1}{384}t_3^4 + \frac{1}{64}t_3^2t_4^2 + \frac{1}{384}t_4^4 \\ + \frac{1}{2}t_1t_3^2e^{t_5} + \frac{1}{2}t_1t_4^2e^{t_5} + t_1^2e^{2t_5} + \frac{1}{2}t_3^2e^{2t_5} + \frac{1}{2}t_4^2e^{2t_5} + \frac{1}{2}e^{4t_5}.$$

EXAMPLE 4.5 ( $R = D_5$ ). — For  $\ell = 5$ , (3.23) is given by

$$\lambda_{D_5} = \frac{w_0}{\mu^3(\mu^2 + 1)^2} \left( \mu^{10} + w_1(\mu^9 + \mu) + w_2(\mu^8 + \mu^2) + w_3(\mu^7 + \mu^3) \right. \\ \left. + (w_2 + w_4w_5 + 1)(\mu^6 + \mu^4) + (w_4^2 + w_5^2 + 2w_1 + 2w_3)\mu^5 + 1 \right),$$

which has poles at  $\mu = 0, 1, 1,$  and  $1,$  of orders 3, 3, 2, and 2, respectively. We can compute the flat coordinates

$$t_6 = \frac{\log w_0}{3}, \quad t_1 = w_0^{1/3} w_1, \quad t_4 = w_0^{1/2} (w_4 - w_5), \quad t_5 = w_0^{1/2} (w_4 + w_5), \\ t_2 = w_0^{2/3} (w_1^2 - 6(w_2 + 2)), \quad t_3 = w_0(2w_1 + w_3),$$

which give the prepotential

$$F_{D_5} = \frac{t_1^6}{9720} + \frac{1}{36} e^{2t_6} t_1^4 + \frac{t_2 t_1^4}{1944} + \frac{1}{2} e^{4t_6} t_1^2 + \frac{1}{648} t_2^2 t_1^2 + \frac{1}{12} e^{t_6} t_4^2 t_1^2 \\ + \frac{1}{2} e^{2t_6} t_4^2 t_1 + \frac{1}{2} e^{2t_6} t_5^2 t_1 + \frac{1}{9} t_2 t_3 t_1 + \frac{e^{6t_6}}{3} + \frac{t_4^4}{384} + \frac{t_5^4}{384} + \frac{t_2^3}{1944} \\ + \frac{1}{2} e^{3t_6} t_4^2 + \frac{1}{12} e^{t_6} t_2 t_4^2 + \frac{1}{4} t_3 t_4^2 + \frac{1}{2} e^{3t_6} t_5^2 + \frac{1}{64} t_4^2 t_5^2 + \frac{1}{12} e^{t_6} t_2 t_5^2 \\ + \frac{1}{12} e^{t_6} t_5^2 t_1^2 + \frac{1}{18} e^{2t_6} t_2 t_1^2 + \frac{1}{4} t_3 t_5^2 + \frac{1}{36} e^{2t_6} t_2^2 + t_3^2 t_6.$$

#### 4.1.2. Exceptional root systems

EXAMPLE 4.6 ( $R = E_6$ ). — For the exceptional cases, we find expressions for  $\lambda$  near any ramification point by Puiseux expansions. By doing so, we obtain the following flat coordinates for  $R = E_6$ ,

$$t_1 = w_0^{1/3} w_1, t_2 = w_0^{2/3} (w_1^2 - 6w_2 - 12w_5), t_3 = w_0(2w_1 w_5 + w_3 + 3w_6 + 3), \\ t_4 = w_0^{2/3} (w_5^2 + 12w_1 + 6w_4), t_5 = w_0^{1/3} w_5, t_6 = w_0^{1/2} (w_6 + 2), t_7 = \frac{\log(w_0)}{6},$$

and the corresponding prepotential is given by

$$F_{E_6} = \frac{1}{19440} t_1^6 + \frac{1}{72} e^{2t_7} t_5 t_1^4 + \frac{1}{3888} t_2 t_1^4 + \frac{1}{6} e^{6t_7} t_1^3 + \frac{1}{6} e^{3t_7} t_6 t_1^3 + \frac{5}{18} e^{4t_7} t_5^2 t_1^2 \\ + \frac{1}{36} e^{t_7} t_5^2 t_6 t_1^2 + \frac{1}{36} e^{4t_7} t_4 t_1^2 + \frac{1}{36} e^{t_7} t_6 t_4 t_1^2 + \frac{1}{36} e^{2t_7} t_5 t_2 t_1^2 \\ + \frac{1}{72} e^{2t_7} t_5^4 t_1 + \frac{1}{72} e^{2t_7} t_4^2 t_1 + \frac{1}{2} e^{8t_7} t_5 t_1 + e^{5t_7} t_5 t_6 t_1 + \frac{1}{36} e^{2t_7} t_5^2 t_4 t_1 \\ + \frac{1}{6} e^{3t_7} t_6 t_2 t_1 + \frac{1}{19440} t_5^6 + \frac{1}{96} t_6^4 + \frac{1}{6} e^{6t_7} t_5^3 + \frac{1}{3} e^{3t_7} t_6^3 + \frac{1}{3888} t_4^3 \\ + \frac{1}{3888} t_2^3 + \frac{1}{2} e^{6t_7} t_6^2 + \frac{1}{72} e^{2t_7} t_5 t_2^2 + \frac{1}{2} t_7 t_3^2 + \frac{1}{6} e^{3t_7} t_5^3 t_6 + \frac{1}{3888} t_5^4 t_4 \\ + \frac{1}{6} e^{3t_7} t_5 t_6 t_4 + \frac{1}{36} e^{4t_7} t_5^2 t_2 + \frac{1}{36} e^{t_7} t_5^2 t_6 t_2 + \frac{1}{36} e^{4t_7} t_4 t_2 + \frac{1}{36} e^{t_7} t_6 t_4 t_2 \\ + \frac{1}{4} t_6^2 t_3 + \frac{1}{18} t_5 t_4 t_3 + \frac{1}{2} e^{2t_7} t_5 t_6^2 t_1 + \frac{1}{12} e^{12t_7} + \frac{1}{1296} t_2^2 t_1^2 \\ + \frac{1}{18} t_2 t_3 t_1 + \frac{1}{1296} t_5^2 t_4^2.$$

EXAMPLE 4.7 ( $R = E_7$ ). — Similarly to the  $E_6$ -case, by taking Puiseux expansions of the spectral curve, we obtain the following flat coordinates

$$\begin{aligned} t_8 &= \frac{\log(w_0)}{12}, \quad t_6 = w_0^{1/4} w_6, \quad t_1 = w_0^{1/3} (w_1 + 2), \quad t_7 = w_0^{1/2} (2w_6 + w_7), \\ t_5 &= w_0^{1/2} (w_6^2 + 8w_1 + 4w_5 + 12), \quad t_2 = w_0^{2/3} (w_1^2 + 26w_1 + 6w_2 + 12w_5 + 26), \\ t_4 &= w_0^{3/4} (5w_6^3 + 24(6w_1 - w_5 + 21)w_6 + 96w_4 + 288w_7), \\ t_3 &= w_0(3w_1^2 + 2w_1(w_5 + 8) + 3w_6^2 + 3w_6w_7 + 2w_2 + w_3 + 7w_5 + 14). \end{aligned}$$

The prepotential for  $E_7$  takes the form

$$\begin{aligned} F_{E_7} &= \frac{1}{4128768} t_6^8 + \frac{1}{18432} 49e^{6t_8} t_6^6 + \frac{1}{18432} e^{2t_8} t_1 t_6^6 + \frac{1}{294912} t_5 t_6^6 + \frac{1}{384} e^{3t_8} t_7 t_6^5 \\ &+ \frac{19}{192} e^{12t_8} t_6^4 + \frac{5}{288} e^{4t_8} t_1^2 t_6^4 + \frac{1}{49152} t_5^2 t_6^4 + \frac{1}{8} e^{8t_8} t_1 t_6^4 + \frac{1}{1536} 13e^{6t_8} t_5 t_6^4 \\ &+ \frac{1}{576} e^{4t_8} t_2 t_6^4 + \frac{1}{4} e^{9t_8} t_7 t_6^3 + \frac{1}{576} e^{t_8} t_1^2 t_7 t_6^3 + \frac{25}{96} e^{5t_8} t_1 t_7 t_6^3 + \frac{7}{384} e^{3t_8} t_7 t_5 t_6^3 \\ &+ \frac{1}{9216} e^{6t_8} t_4 t_6^3 + \frac{1}{9216} e^{2t_8} t_1 t_4 t_6^3 + \frac{1}{147456} t_5 t_4 t_6^3 + \frac{1}{6} e^{18t_8} t_6^2 + \frac{1}{288} e^{2t_8} t_1^4 t_6^2 \\ &+ \frac{1}{24576} t_5^3 t_6^2 + \frac{5}{8} e^{10t_8} t_1^2 t_6^2 + \frac{5}{8} e^{6t_8} t_7^2 t_6^2 + \frac{1}{8} e^{2t_8} t_1 t_7^2 t_6^2 + \frac{5}{512} e^{6t_8} t_5^2 t_6^2 \\ &+ \frac{1}{288} e^{2t_8} t_2^2 t_6^2 + \frac{1}{589824} t_4^2 t_6^2 + \frac{1}{2} e^{14t_8} t_1 t_6^2 + \frac{1}{32} e^{12t_8} t_5 t_6^2 + \frac{1}{24} e^{4t_8} t_1^2 t_5 t_6^2 \\ &+ \frac{1}{144} e^{2t_8} t_1^2 t_2 t_6^2 + \frac{1}{24} e^{6t_8} t_1 t_2 t_6^2 + \frac{1}{96} e^{4t_8} t_5 t_2 t_6^2 + \frac{1}{384} e^{3t_8} t_7 t_4 t_6^2 + \frac{1}{3} e^{3t_8} t_7^3 t_6 \\ &+ \frac{1}{6} e^{3t_8} t_1^3 t_7 t_6 + \frac{7}{6} e^{7t_8} t_1^2 t_7 t_6 + e^{11t_8} t_1 t_7 t_6 + \frac{1}{4} e^{9t_8} t_7 t_5 t_6 + \frac{1}{96} e^{t_8} t_1^2 t_7 t_5 t_6 \\ &+ \frac{1}{6} e^{7t_8} t_7 t_2 t_6 + \frac{1}{6} e^{3t_8} t_1 t_7 t_2 t_6 + \frac{1}{96} e^{t_8} t_7 t_5 t_2 t_6 + \frac{1}{576} e^{4t_8} t_1^2 t_4 t_6 + \frac{1}{49152} t_5^2 t_4 t_6 \\ &+ \frac{1}{1536} e^{2t_8} t_1 t_5 t_4 t_6 + \frac{1}{576} e^{4t_8} t_2 t_4 t_6 + \frac{1}{384} t_4 t_3 t_6 + \frac{1}{24} e^{24t_8} t_1^6 \\ &+ \frac{1}{96} t_7^4 + \frac{1}{49152} t_5^4 + \frac{1}{6} e^{12t_8} t_1^3 + \frac{1}{384} e^{6t_8} t_5^3 + \frac{1}{3888} t_2^3 + \frac{1}{4} e^{16t_8} t_1^2 + \frac{1}{2} e^{12t_8} t_7^2 \\ &+ e^{8t_8} t_1 t_7^2 + \frac{1}{64} e^{12t_8} t_5^2 + \frac{1}{64} e^{4t_8} t_1^2 t_5^2 + \frac{1}{72} e^{8t_8} t_2^2 + \frac{1}{1296} t_1^2 t_2^2 + \frac{1}{72} e^{4t_8} t_1 t_2^2 \\ &+ \frac{1}{18432} e^{6t_8} t_4^2 + \frac{1}{18432} e^{2t_8} t_1 t_4^2 + \frac{1}{294912} t_5 t_4^2 + \frac{1}{2} t_8 t_3^2 + \frac{1}{72} e^{4t_8} t_1^5 \\ &+ \frac{1}{8} e^{10t_8} t_1^2 t_5 + \frac{1}{8} e^{6t_8} t_7^2 t_5 + \frac{1}{8} e^{2t_8} t_1 t_7^2 t_5 + \frac{1}{3888} t_1^4 t_2 + \frac{1}{36} e^{4t_8} t_1^3 t_2 + \frac{1}{36} e^{8t_8} t_1^2 t_2 \\ &+ \frac{1}{144} e^{2t_8} t_1^2 t_5 t_2 + \frac{1}{24} e^{6t_8} t_1 t_5 t_2 + \frac{1}{576} e^{t_8} t_1^2 t_7 t_4 + \frac{1}{96} e^{5t_8} t_1 t_7 t_4 + \frac{1}{384} e^{3t_8} t_7 t_5 t_4 \\ &+ \frac{1}{4} t_7^2 t_3 + \frac{1}{128} t_5^2 t_3 + \frac{1}{18} t_1 t_2 t_3 + \frac{1}{2949120} t_4 t_5^2 + \frac{1}{1536} e^{2t_8} t_1 t_5 t_6^4 + \frac{1}{576} e^{t_8} t_7 t_2 t_6^3 \\ &+ \frac{5}{24} e^{6t_8} t_1^3 t_6^2 + \frac{1}{576} e^{t_8} t_7 t_2 t_4 + \frac{1}{6} e^{4t_8} t_7^2 t_2 + \frac{1}{24} e^{6t_8} t_1^3 t_5 + \frac{1}{288} e^{2t_8} t_1^4 t_5 \\ &+ \frac{1}{288} e^{2t_8} t_5 t_2^2 + \frac{2}{3} e^{4t_8} t_1^2 t_7^2 + \frac{5}{36} e^{8t_8} t_1^4 + \frac{1}{1536} e^{6t_8} t_5 t_4 t_6 + \frac{5}{16} e^{5t_8} t_1 t_7 t_5 t_6 \\ &+ \frac{1}{64} e^{3t_8} t_7 t_5^2 t_6 + \frac{1}{512} e^{2t_8} t_1 t_5^2 t_6^2 + \frac{1}{8} e^{8t_8} t_1 t_5 t_6^2. \end{aligned}$$



EXAMPLE 4.8 ( $R = F_4$ ). — In this case we obtain flat coordinates

$$t_5 = \frac{1}{6} \log(w_0), \quad t_4 = w_0^{1/3}(w_4 + 1), \quad t_3 = w_0^{2/3}(w_4^2 + 16w_4 + 6w_3 + 6w_1 + 11),$$

$$t_2 = w_0(2w_4^2 + 6w_4 + w_4w_1 + w_3 + w_2 + 4w_1 + 5), \quad t_1 = w_0^{1/2}(w_4 + w_1 + 2).$$

The resulting prepotential is

$$F_{F_4} = \frac{1}{9720}t_4^6 + \frac{1}{36}e^{2t_5}t_4^5 + \frac{5}{18}e^{4t_5}t_4^4 + \frac{1}{36}e^{t_5}t_1t_4^4 + \frac{1}{432}t_3t_4^4 + \frac{1}{3}e^{6t_5}t_4^3 + \frac{1}{3}e^{3t_5}t_1t_4^3$$

$$+ \frac{1}{4}e^{2t_5}t_3t_4^3 + \frac{1}{2}e^{8t_5}t_4^2 + \frac{1}{2}e^{2t_5}t_1^2t_4^2 + \frac{1}{32}t_3^2t_4^2 + e^{5t_5}t_1t_4^2 + \frac{1}{4}e^{4t_5}t_3t_4^2$$

$$+ \frac{9}{16}e^{2t_5}t_3^2t_4 + \frac{3}{2}e^{3t_5}t_1t_3t_4 + \frac{1}{2}t_3t_2t_4 + \frac{1}{12}e^{12t_5}t_1^4 + \frac{1}{3}e^{3t_5}t_1^3 + \frac{3t_3^3}{64}$$

$$+ \frac{9}{16}e^{4t_5}t_3^2 + \frac{9}{16}e^{t_5}t_1t_3^2 + \frac{1}{2}t_5t_2^2 + \frac{1}{4}t_1^2t_2 + \frac{1}{2}e^{6t_5}t_1^2 + \frac{1}{4}e^{t_5}t_1t_3t_4^2.$$

EXAMPLE 4.9 ( $R = G_2$ ). — For  $G_2$ , we obtain flat coordinates

$$(4.18) \quad t_3 = \frac{\log(w_0)}{6}, \quad t_1 = w_0^{1/2}(w_1 + 1), \quad t_2 = w_0(2w_1 + w_2 + 2).$$

In this case, the prepotential takes the form

$$(4.19) \quad F_{G_2} = \frac{1}{2}t_2^2t_3 + \frac{1}{4}t_2t_1^2 + \frac{1}{96}t_1^4 + \frac{1}{3}t_1^3e^{3t_3} + \frac{1}{2}t_1^2e^{6t_3} + \frac{1}{12}e^{12t_3},$$

which matches exactly the expression found in [21, Ex. 2.4].

4.2. NON-MINIMAL IRREDUCIBLE REPRESENTATIONS. — It is argued in [6], based on the isomorphism of Toda flows on Prym–Tyurin varieties associated to different representations, [38, 39], that the Frobenius manifold obtained from the construction of Section 3.2 is independent of the choice of highest weight  $\omega$ .

Let us verify this explicitly for  $R = G_2$ . In this case picking  $\omega = \omega_2$  gives the second smallest-dimensional non-trivial irreducible representation of  $G_2$ , which is the 14-dimensional adjoint representation  $\rho_{\omega_2} = \mathfrak{g}_2$ . By the same method of the previous section we obtain

$$\begin{aligned} p_0^{[01]G_2} &= 1, \\ p_1^{[01]G_2} &= \chi_2, \\ p_2^{[01]G_2} &= \chi_1^3 - \chi_1^2 - \chi_1(2\chi_2 + 1), \\ p_3^{[01]G_2} &= \chi_1^4 - \chi_1^3 - \chi_1^2(3\chi_2 + 1) + \chi_1 + 2\chi_2^2 + \chi_2, \\ p_4^{[01]G_2} &= \chi_1^3(\chi_2 - 1) - \chi_1^2(\chi_2 - 1) - \chi_1(2\chi_2^2 - \chi_2 - 1) - \chi_2^2 + \chi_2, \\ p_5^{[01]G_2} &= \chi_1^5 - 2\chi_1^4 - 5\chi_1^3\chi_2 + \chi_1^2(3\chi_2 + 2) + \chi_1(6\chi_2^2 + 5\chi_2 - 1) + \chi_2^3 + 2\chi_2^2, \\ p_6^{[01]G_2} &= \chi_1^4 - 3\chi_1^3\chi_2 + \chi_1^2(\chi_2^2 - \chi_2 - 2) + \chi_1\chi_2(4\chi_2 + 3) + 2\chi_2^2 + \chi_2, \\ p_7^{[01]G_2} &= +4\chi_1^4 + 2\chi_1^3(3\chi_2 + 1) + 2\chi_1^2(\chi_2^2 - 2\chi_2 - 3) \\ &\quad - 2\chi_1\chi_2(4\chi_2 + 3) - 2\chi_2^3 - 6\chi_2^2 + 2 - 2\chi_1^5, \end{aligned}$$

and  $p_i^{[01]G_2} = p_{14-i}^{[01]G_2}$  by reality of the adjoint representation. The characteristic polynomial (3.8) then factorises as

$$P_{[01]G_2}^{\text{red}} = (\mu - 1)^2 P_{G_2, \text{short}} P_{G_2, \text{long}},$$

with the three factors corresponding to the three irreducible Weyl orbits of the adjoint representation associated to the zero, short, and long roots of  $G_2$ :

$$\begin{aligned} P_{G_2, \text{short}} &= P_{[10]G_2}^{\text{red}}, \\ P_{G_2, \text{long}} &= \mu^6 + \mu^5 (w_1 - w_2 + 1) + \mu^4 (w_1^3 - 3w_2w_1 - w_1 - 2w_2 + 1) \\ &\quad + \mu^3 (2w_1^3 - w_1^2 - 4w_2w_1 - 2w_1 - w_2^2 - 4w_2 + 1) \\ &\quad + \mu^2 (w_1^3 - 3w_2w_1 - w_1 - 2w_2 + 1) + \mu (w_1 - w_2 + 1) + 1. \end{aligned}$$

For any  $w$ , the curve  $\overline{C_w^{(1)}} = \overline{\mu P_{G_2, \text{long}} = 0}g$  is a copy of  $P^1$ , and the  $\lambda$ -projection has ramification profile

$$\left( \overbrace{1, 2}^{\mu=0}, \overbrace{1, 2}^{\mu=1} \right).$$

The embedding  $\iota_{[01]G_2} : M_{[01]G_2}^{\text{LG}} \dashrightarrow H_{0, (0, 1, 0, 1)}$  gives the same flat coordinates (4.18) as for the case  $\omega = [10]_{G_2}$ , and up to scaling the prepotential coincides with the prepotential (4.19), as expected.

**4.3. NON-CANONICAL DYNKIN MARKING.** — In [6], it was proposed that the family of Frobenius algebras obtained in (3.11) through the shift of  $w_j \mapsto w_j + \delta_{ij} \lambda/w_0$  for any  $i$  should give the Frobenius structure corresponding to Dynkin node  $\alpha_i$ . Theorem 3.6 shows that this is the case for  $i = \bar{k}$ , and this proposal is consistent with the analysis of the generalised type-A mirrors of [21]. However we now show that the conjecture is false in this form at the stated level of generality. Considering the case  $R = G_2$ , we see that shifting  $w_1$  instead of  $w_2$  in (3.41) yields

$$(4.20) \quad P_{[10]G_2, i=1} = \left(\frac{\lambda}{w_0}\right)^2 \mu^3 + \frac{\lambda}{w_0} (\mu^5 - 2w_1\mu^3 - \mu - 2) + \mu^6 + (\mu^5 + \mu)(1 - w_1) + (\mu^4 + \mu^2)(w_2 + 1) - \mu^3(w_1^2 - 2w_2 - 1) + 1.$$

By computing the metric  $\eta = \sum_{i,j} \tilde{\eta}_{ij} dw^i dw^j$  we get

$$(4.21) \quad \tilde{\eta}_{ij} = \begin{pmatrix} \frac{8w_1 + 1}{4w_0} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is clearly a singular matrix. Hence, (4.20) cannot define a Frobenius manifold structure, and the conjecture fails in this generality.

**REMARK 4.4.** — Opening for allowed changes in the primary differential, i.e., scalings and translations of  $d \log \mu$  which are type I (Legendre) transformations [16], under which the intersection form is invariant, does not resolve the nondegeneracy, and neither does attempting to change the pole structure manually akin to that of (3.25). Thus, the problem of finding Frobenius manifolds associated to non-canonical nodes

for exceptional groups is still open, and indeed even the existence of non-canonical Frobenius manifold structures for DZ-manifolds of exceptional types is, at present, unknown.

### 5. APPLICATIONS

5.1. **TOPOLOGICAL DEGREES OF LYASHKO–LOOIJENGA MAPS.** — The semi-simple locus of a generically semi-simple,  $n$ -dimensional Frobenius manifold  $M$  is topologically a covering of finite multiplicity over a quotient by  $S_n$  of the complement of their discriminant, with covering map

$$(5.1) \quad \text{LL} : M \rightarrow (C^n \setminus \text{Discr}_M) / S_n \\ t \mapsto (e_1(u(t)), \dots, e_n(u(t)))g$$

assigning to  $t \in M$  the unordered set of its canonical coordinates in the form of their elementary symmetric polynomials  $e_i(u_1, \dots, u_n)$ . When  $M$  has a Landau–Ginzburg description as a holomorphic family of meromorphic functions, the application LL is the classical Lyashko–Looijenga (LL) mapping, sending a meromorphic function to the unordered set of its critical values.

As anticipated in Section 1.1.3, a direct corollary of Theorem 3.6 is the computation of the degree of the LL map of  $M_\omega^{\text{LG}} \rightarrow M_{\mathbb{R}}^{\text{DZ}}$ . The calculation of the LL-degree can be tackled combinatorially through the enumeration of certain embedded graphs [32, §1.3.2], which unfortunately proves to be intractable for a general stratum of a Hurwitz space of arbitrary genus and ramification profile. In the case of  $\text{deg LL}(M_\omega^{\text{LG}})$ , however, its realisation as a conformal Frobenius manifold allows to bypass the problem altogether by the use of the quasi-homogeneous Bézout theorem. To this aim, note that

$$(5.2) \quad \det(z - E(t)) = \prod_{i=1}^{\ell_{\mathbb{R}}+1} (z - u_i(t)) = \sum_{j=0}^{\ell_{\mathbb{R}}+1} (-1)^j e_j(u_1(t), \dots, u_{\ell_{\mathbb{R}}+1}(t)) z^{\ell_{\mathbb{R}}+1-j}.$$

Setting  $T := e^{t_{\ell_{\mathbb{R}}+1}}$ , it follows from Theorems 2.3 and 3.6 that the LL-map (5.1) is polynomial in  $(t_1, \dots, t_{\ell_{\mathbb{R}}+1}, T)$  since both the product and the Euler vector field are in (5.2). Moreover, it follows from the definition (3.2) of the quantum product in canonical coordinates that the canonical idempotents  $\partial_{u_i}$  have weight  $-1$  under the Euler scaling, meaning that  $e_i(u)$  is quasi-homogeneous of degree  $i$ . We can then avail ourselves of the graded generalisation of Bézout’s theorem (see e.g. [32, Th. 3.3]).

**THEOREM 5.1.** — *Let  $F : A^n \rightarrow A^n$  be a finite morphism induced by a quasi-homogeneous map  $F : C[y_1, \dots, y_n] \rightarrow C[x_1, \dots, x_n]$  with degrees  $p_i$  (resp.  $q_i$ ) for  $y_i$  (resp.  $x_i$ ) for  $i = 1, \dots, n$ . Then*

$$\text{deg } F = \prod_{i=1}^n \frac{p_i}{q_i}.$$

An immediate consequence of Theorems 2.3, 3.6 and 5.1 is the following

**COROLLARY 5.2.** — *The degree of the LL-map of the Hurwitz stratum  $M_\omega^{\text{LG}}$  is*

$$\frac{(\ell_{\mathbb{R}} + 1)! (\omega_{\bar{k}}, \omega_{\bar{k}})^{\ell_{\mathbb{R}}}}{\prod_{j=1}^{\ell_{\mathbb{R}}} (\omega_j, \omega_{\bar{k}})}.$$

We collect in Table 5.1 the calculation of the degrees for the minimal choices of weight  $\omega$ . Our expectation that  $M_\omega^{\text{LG}} \simeq M_{\mathbb{R}}^{\text{DZ}}$  for any of the infinitely many choices of dominant weight  $\omega \in \Lambda^+(\mathbb{R})$  implies that the same formula (5.2) would hold for the Hurwitz strata associated to those non-minimal choices by the construction of Section 3.2. For type  $A_\ell$ , Corollary 5.2 recovers Arnold’s formula for the LL-degree of the space of complex trigonometric polynomials, as was already shown in [21]. The fifth column indicates how  $\iota_\omega(M_{\mathbb{R}}^{\text{DZ}})$  sits as a stratum inside the parent Hurwitz space  $H_{g_\omega, n_\omega}$ , either through explicit character relations in  $\text{Rep}(G_{\mathbb{R}})$  or as a fixed locus of an automorphism of the Hurwitz space induced by the folding of the Dynkin diagram.

$\mathcal{R}$	$g_\omega$	$n_\omega$	$d_{g_\omega, n_\omega}$	$\iota_\omega(M_{\mathbb{R}}^{\text{DZ}})$	$\text{deg}(\text{LL})$
$A_\ell$	0	$(\bar{k}-1, \ell-\bar{k})$	$\ell+1$	$H_{g_\omega, n_\omega}^{[\mu]}$	$\frac{(\ell+1-\bar{k})^{\ell+1} \bar{k}(\bar{k}-1)^\ell}{(\bar{k}-1)!(\ell-\bar{k})!}$
$B_\ell$	0	$(\ell-2, \ell-2, 1)$	$2\ell+1$	$(H_{g_\omega, n_\omega}^{[\mu]})^{\mu_2}$	$2(\ell+1)\ell(\ell-1)^{\ell-1}$
$C_\ell$	0	$(\ell-1, \ell-1)$	$2\ell$	$(H_{g_\omega, n_\omega}^{[\mu]})^{\mu_2}$	$(\ell+1)\ell^\ell$
$D_\ell$	0	$(\ell-3, \ell-3, 1, 1)$	$2\ell+2$	$(H_{g_\omega, n_\omega}^{[\mu]})^{\mu_2}$	$4\ell(\ell^2-1)(\ell-2)^{\ell-2}$
$E_6$	5	$(5, 5, 2, 2, 2, 2)$	42	(A.1)	$2^3 \cdot 3^6 \cdot 5 \cdot 7$
$E_7$	33	$(11, 5, 3, 11, 5, 3, 1, 1, 3, 3)$	130	(A.2)	$2^{12} \cdot 3^3 \cdot 5 \cdot 7$
$E_8$	128	$(29, 29, 14, 14, 14, 14, 14, 14, 9, 9, 9, 9, 5, 5, 4, 4, 4, 4, 4, 2, 2, 0, 0)$	518	[6, 7]	$2^4 \cdot 3^5 \cdot 5^5 \cdot 7$
$F_4$	4	$(5, 5, 2, 2, 2, 2)$	36	(3.40); $(M_{[100000]E_6}^{\text{LG}})^{\mu_2}$	$2^3 \cdot 3^3 \cdot 5$
$G_2$	0	$(1, 1, 1)$	7	(3.41); $(M_{[1000]D_4}^{\text{LG}})^{S_3}$	12

TABLE 5.1. Lyashko–Looijenga degrees for all Dynkin types.

5.2. THE TYPE-R EXTENDED TODA HIERARCHY. — Another notable consequence of the determination of the prepotential and the superpotential of  $M_{\mathbb{R}}^{\text{DZ}}$  in Theorem 3.6 is the construction of a dispersionless bihamiltonian hierarchy of integrable PDEs on the loop space  $\text{LM}_{\mathbb{R}}^{\text{DZ}}$ . This is amenable to a presentation both in normal and dispersionless Lax forms, and generalises the dispersionless limit of the extended bi-graded Toda hierarchy of [11, 21] (corresponding to  $\mathbb{R} = A_\ell$ ) to general Dynkin types.

We first recall the general theory of principal hierarchies associated to Frobenius manifolds as formulated by Dubrovin [15]. Let  $M$  be an  $n$ -dimensional semi-simple

complex Frobenius manifold,  $\tilde{f}t_\alpha : M \rightarrow \mathbb{C}g_{\alpha=1}^n$  a coordinate chart for  $M$  which is flat for the metric  $\eta$ , and  $LM = \tilde{f}u : S^1 \rightarrow Mg$  the formal loop space of  $M$  – an element  $u \in LM$  being an  $n$ -tuple  $u = (u^1, \dots, u^n)$  with  $u_i \in \mathbb{C}[[X, X^{-1}]]$  a formal Laurent series in a periodic coordinate  $X \in S^1$ . Let  $(\eta^{\alpha\beta})$  and  $(\gamma^{\alpha\beta})$ ,  $\alpha, \beta = 1, \dots, n$ , denote the Gram matrices of the cotangent metrics  $\eta$  and  $\gamma$  on  $M$ . The loop space  $LM$  can be endowed with a bihamiltonian pencil of hydrodynamic Poisson brackets

$$(5.3) \quad \tilde{f}u^\alpha(X), u^\beta(Y)g_{[\lambda]} = (\gamma^{\alpha\beta} + \lambda\eta^{\alpha\beta})\delta^\theta(X - Y) + \sum_{\delta} \frac{\alpha\beta}{\delta}(u)\partial_X u^\delta \delta(X - Y),$$

where  $\frac{\alpha\beta}{\delta}$  denotes the Christoffel symbol of the Levi-Civita connection of  $\gamma$  in the flat coordinate chart  $t$  for  $\eta$ . Let  $r$  denote the affine, torsion free connection on  $T(M \rightarrow \mathbb{C}^*)$  defined by

$$(5.4) \quad \begin{aligned} r_V W &= i_V d_M W + zV \lrcorner W, \\ r_{\partial_z} W &= i_{\partial_z} d_{\mathbb{C}^*} W - E \lrcorner W - z^{-1}VW, \end{aligned}$$

where  $z \in \mathbb{C}^*$ ,  $V(z)$ ,  $W(z) \in (TM)$ , and  $V \in (\text{End}(TM))$  is the grading operator defined in flat coordinates as  $V_\beta^\alpha = (1 - n/2)\delta_\beta^\alpha + \partial_\beta E^\alpha$ . The Frobenius manifold axioms imply that  $r$  is flat [16, Lect. 6], and a basis of horizontal sections  $\sum_\alpha W_\beta^\alpha \partial_\alpha$  can be taken to have the form  $W_\beta^\alpha = \sum_\nu \eta^{\alpha\nu} \partial_\beta h_\nu(u, z) z^V z^R$  for some constant matrix  $R$  (determined by the monodromy data of the Frobenius manifold; see [18, Lect. 2]) and  $h(u, z) \in (O_M)[[z]]$ . Furthermore, the Taylor coefficients  $h_{\alpha,m}(u) := [z^{m+1}]h_\alpha$  define Hamiltonian densities for which the corresponding local Hamiltonians,

$$H_{\alpha,m}[u] := \int_{S^1} h_{\alpha,m}(u(X)) dX,$$

are in involution with respect to (5.3) for all  $\lambda$ ,

$$(5.5) \quad \tilde{f}H_{\alpha,m}, H_{\beta,n}g_{[\lambda]} = 0.$$

The corresponding involutive Hamiltonian flows

$$(5.6) \quad \begin{aligned} \partial_{T_{\alpha,m}} u^\beta &:= \tilde{f}u^\beta, H_{\alpha,m}g_{[\lambda]} \\ &= \sum_{\delta\epsilon} \left[ (\gamma^{\beta\delta} + \lambda\eta^{\beta\delta}) \partial_{u^\delta u^\epsilon}^2 h_{\alpha,m}(u) u_X^\epsilon + \frac{\beta\delta}{\epsilon} \partial_{u^\delta} h_{\alpha,m}(u) \partial_X u^\epsilon \right] \end{aligned}$$

for  $\alpha = 1, \dots, n$  and  $m = 0, \dots, 1$  define an integrable hierarchy of quasi-linear PDEs on  $LM$ , called the *principal hierarchy* of  $M$ ; the dependent variables  $u^\alpha = u^\alpha(X, T)$  are called the normal coordinates of the hierarchy. This hierarchy moreover satisfies the  $\tau$ -symmetry condition

$$\partial_{T_{\mu,m}} h_{\nu,n}((u(X, T))) = \partial_{T_{\nu,n}} h_{\mu,m}((u(X, T))) = \frac{\partial^3 \log \tau(X, T)}{\partial X \partial T_{\mu,m} \partial T_{\nu,n}},$$

for some function  $\tau(X; (T_{\mu,m})_{\mu,m})$ . In particular,  $u^\alpha(X, T) = \partial_{X, T_{\alpha,0}}^2 \log \tau$ .

**5.2.1. The dispersionless extended R-type Toda hierarchy: Hamiltonian and Lax–Sato form.** — In the case of the Dubrovin–Zhang Frobenius manifolds of type R, Theorem 3.6 can be used effectively to solve the problem of finding the fundamental solutions of (5.4) and obtain the presentation of the principal hierarchy in normal coordinates (5.6). An immediate adaptation of [16, Prop. 6.3] gives the following

**PROPOSITION 5.3.** — *With conventions as in Lemma 4.1, let  $\tilde{h}_{i,\alpha}, \tilde{h}_j^{\text{ext}}, \tilde{h}_k^{\text{res}}$  be the flat coordinates for the deformed connection (5.4) on  $M_{\mathbb{R}}^{\text{DZ}} \subset \mathbb{C}^*$  normalised such that  $\tilde{h}_{i,\alpha} = \tau_{i,\alpha} + O(z), \tilde{h}_j^{\text{ext}} = \tau_j^{\text{ext}} + O(z), \tilde{h}_k^{\text{res}} = \tau_k^{\text{res}} + O(z)$ . Then,*

$$(5.7) \quad \tilde{h}_{i,\alpha}(\tau, z) = \frac{n_i + 1}{\alpha} \text{Res}_{\gamma_i} \kappa_i^\alpha {}_1F_1\left(1, 1 + \frac{\alpha}{n_i + 1}; z\lambda(\mu)\right) \frac{d\mu}{\mu},$$

$$(5.8) \quad \tilde{h}_j^{\text{ext}}(\tau, z) = \text{p.v.} \int_{\gamma_0}^{\gamma_j} e^{z\lambda} \frac{d\mu}{\mu}, \quad \tilde{h}_k^{\text{res}}(\tau, z) = \text{Res}_{\gamma_i} \frac{e^{z\lambda}}{z} \frac{1}{\mu} d\mu,$$

where  ${}_1F_1(a, b; x) = \sum_{n=0}^{\infty} (a)_n x^n / (b)_n n!$  is Kummer’s confluent hypergeometric function, and  $(a)_n := (a + n)! / (a)!$ .

We call the bihamiltonian integrable hierarchy defined by (5.6) and (5.7)–(5.8) the *dispersionless extended R-type Toda hierarchy*. For  $R = A_n$ , this coincides with the dispersionless limit of the bi-graded Toda hierarchy of [11]. The adjective “extended” refers to the Hamiltonian flows generated by  $H_j^{\text{ext}}[u]$ , which are higher order versions of the space translation: when  $R$  is simply-laced these encode the non-stationary Gromov–Witten invariants of the type- $R$   $\mathbb{P}^1$  orbifold given by insertions of descendants of the identity. We refer to the Hamiltonian flows generated by  $H_{i,\alpha}[u]$  and  $H_k^{\text{res}}[u]$  as the *stationary flows* of the hierarchy.

Theorem 3.6 also provides a dispersionless Lax–Sato description of (5.6) and (5.7)–(5.8) as a specific reduction of the universal genus- $g_\omega$  Whitham hierarchy with  $\ell(n_\omega)$  punctures [31, 15]. Let  $\tilde{C}_w^{(\omega)}$  be the universal covering of  $\overline{C}_w^\omega \subset \mathbb{P}^1 \times \mathbb{G}_i$ , the fibre at  $w$  of the Landau–Ginzburg family of Theorem 3.6, viewed as an analytic variety. Following [16], we consider second and third kind differentials  $\tilde{\omega}_{i,\alpha}, \tilde{\omega}_j^{\text{ext}}, \tilde{\omega}_k^{\text{res}}$  defined on  $\tilde{C}_w^{(\omega)}$  such that

$$\begin{aligned} \tilde{\omega}_{i,\alpha;m} &= \frac{1}{n_i + 1} \left[ \left( \frac{\alpha}{n_i + 1} \right)_{m+1} \right]^{-1} d\lambda^{\alpha/(n_i+1)+m} + \text{regular}, \\ \tilde{\omega}_{j;m}^{\text{ext}} &= \begin{cases} \frac{d\psi_m(\lambda)}{n_j + 1} + \text{regular near } \gamma_i, \\ \frac{d\psi_m(\lambda)}{n_0 + 1} + \text{regular near } \gamma_0, \end{cases} \\ \tilde{\omega}_{i;m}^{\text{res}} &= d\left( \frac{\lambda^{m+1}}{(m+1)!} \right) + \text{regular}, \end{aligned}$$

where  $\psi_m(\lambda) := \lambda^m / m! (\log \lambda - H_m)$ , and  $H_m$  is the  $m^{\text{th}}$  harmonic number. Then, (5.6) and (5.7)–(5.8) are equivalent to the dispersionless Lax system

$$\partial_{T_{(i,\alpha);m}} \lambda = f\lambda, q_{i,\alpha;m} g_{\text{LS}}, \quad \partial_{T_{i;m}^{\text{ext}}} \lambda = f\lambda, q_{j;m}^{\text{ext}} g_{\text{LS}}, \quad \partial_{T_{i;m}^{\text{res}}} \lambda = f\lambda, q_{i;m}^{\text{res}} g_{\text{LS}},$$

where  $q_{i,\alpha;m}(\mu) := \int^\mu \tilde{\omega}_{i,\alpha;m}, q_{j;m}^{\text{ext}}(\mu) := \int^\mu \tilde{\omega}_{j;m}^{\text{ext}}$  and  $q_{i;m}^{\text{res}}(\mu) := \int^\mu \tilde{\omega}_{i;m}^{\text{res}}$ , and

$$f(\mu, X), g(\mu, X) g_{\text{LS}} := \mu(\partial_\mu f \partial_X g - \partial_X f \partial_\mu g).$$

Having a closed-form superpotential for  $M_{\mathbb{R}}^{\text{DZ}}$  from Theorem 3.6 in particular provides explicit expressions for the Lax–Sato and Hamiltonian densities.

REMARK 5.4. — In [45], a deformation scheme for the genus zero universal Whitham hierarchy is introduced in terms of a Moyal-type quantisation of the dispersionless Lax–Sato formalism. It would be intriguing to apply these ideas to the cases when  $M_\omega^{LG}$  embeds into a genus  $g_\omega = 0$  Hurwitz space, and verify that the resulting dispersionful deformation of the Principal Hierarchy is compatible with the hierarchy obtained by the quantisation of the underlying semi-simple CohFT.

EXAMPLE 5.1 ( $R = G_2$ ). — Let us consider for example  $R = G_2$ . We construct explicitly the whole tower of Hamiltonian densities for the stationary flows of the type- $G_2$  dispersionless Toda hierarchy. In principle, these can be computed (up to a triangular linear transformation in the flow variables  $T^{\alpha,m}$ ) by imposing the recursion relation, coming from the first line of (5.4),

$$(5.9) \quad \partial_{t_\alpha t_\beta}^2 h_{\gamma,m} = \sum_\delta c_{\alpha\beta}^\delta \partial_{t_\delta} h_{\gamma,m-1}$$

with  $\gamma = 1, 2, 3$ ,  $m > 0$ ,  $h_{\gamma,0} = \sum_\delta \eta_{\gamma,\delta} t_\delta$ , and  $c_{\alpha\beta}^\gamma$  are the structure constants of the quantum product determined by the prepotential (4.19). While the recursion (5.9) is ostensibly very hard to solve directly, the combination of Theorem 3.6 and Proposition 5.3 allows to give closed forms for the stationary Hamiltonians  $H_{\gamma,m}$ ,  $\gamma = 1, 3$ , parametrically in  $m$ . This is most easily achieved in flat coordinates  $(x_0, x_1, x_2)$  for the second metric  $\gamma$ , and in Hamiltonian form for the corresponding Poisson bracket. From (3.42) we have

$$\lambda(\mu) = \frac{w_0}{\mu^2(\mu+1)^2} \prod_{i=1}^6 (\mu - a_i(x)),$$

with

$$a_1 = e^{x_1}, \quad a_2 = e^{-x_1+x_2}, \quad a_3 = e^{2x_1-x_2}, \quad a_4 = e^{-x_1}, \quad a_5 = e^{x_1-x_2}, \quad a_6 = e^{x_2-2x_1}.$$

Labelling the punctures at  $\mu = 0, -1$  and  $1$  as  $1_0, 1_1$  and  $1_2$  respectively we have:

$$\begin{aligned} \tilde{h}_{0;m}^{\text{res}} &= \tilde{h}_{2;m}^{\text{res}} = h_{3,m}, & \tilde{h}_{1;m}^{\text{res}} &= 0, \\ \tilde{h}_{0,1/2;m} &= \tilde{h}_{2,1/2;m} = h_{1,m}, & \tilde{h}_{1,1/2;m} &= 2\tilde{h}_{0,1/2;m}. \end{aligned}$$

From (5.7) and (5.8) we then get that

$$\begin{aligned} h_{1,m} &= \sum_{j=0}^{2m+1} \sum_{\substack{k_1, \dots, k_6=0, \dots, 2m+1 \\ \sum_i k_i=2m+1-j}} \frac{(2m+1)_j e^{(m+1/2)x_0}}{j!(3/2)_m} \prod_{i=1}^6 \frac{(m-k_i+3/2)_{k_i}}{a_i(x)^{k_i} (m-1/2)_{k_i}}, \\ h_{3,m} &= \sum_{j=0}^{2m} \sum_{\substack{k_1, \dots, k_6=0, \dots, m \\ \sum_i k_i=2m-j}} \frac{(2m)_j e^{mx_0}}{j!m!} \prod_{i=1}^6 \frac{a_i(x)^{m-k_i} (m-k_i+1)_{k_i}}{k_i!}, \end{aligned}$$

where  $(a)_m = (a+m)/ (m)$  is the usual notation for the Pochhammer symbol. In these coordinates, the Gram matrix of the second metric  $\gamma$  and its inverse read

$$(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 12 & 6 \\ 0 & 6 & 4 \end{pmatrix}, \quad (\gamma^{-1}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 1/2 \\ 0 & 1/2 & 1 \end{pmatrix}.$$

Let  $x_i = f_i(t_1, t_2, t_3)$ ,  $i = 0, 1, 2$  be the change-of-variables expressing the  $x$ -coordinates in the flat coordinate chart  $(t_1, t_2, t_3)$  for the first metric  $\eta$ , and define accordingly  $w^i = f_i(u^1, u^2, u^3)$  for the corresponding dependent variables for the principal hierarchy. In these coordinates, the second Poisson bracket takes the form (recall that  $\gamma^{ij} := (\gamma^{-1})_{ij}$ )

$$\bar{w}^i(X), w^j(Y)g_0 = \gamma^{ij} \delta^0(X - Y)$$

and the stationary flows are given by

$$\frac{\partial w^i}{\partial T_{j,m}} = \bar{w}^i, H_{j,m}g_0 = \sum_{k=0,1,2} \gamma^{ik} \partial_{w^k} h_{j,m}(w) \partial_X w^k, \quad j = 1, 3.$$

#### APPENDIX. CHARACTER RELATIONS FOR $E_6$ AND $E_7$

A.1.  $R = E_6$ . — The character relations for  $\rho = \rho_{\omega_1}$  and  $k = 1, \dots, 13$  are given below. The remaining character relations for  $k > 13$  are obtained from the complex conjugation relation

$$p_k^{[100000]_{E_6}}(\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6) = p_{27-k}^{[000010]_{E_6}}(\chi_5, \chi_4, \chi_3, \chi_2, \chi_1, \chi_6).$$

From Lemma 3.10 and Corollary 3.11 we find:

$$\begin{aligned} p_0^{[100000]_{E_6}} &= 1, \\ p_1^{[100000]_{E_6}} &= \chi_1, \\ p_2^{[100000]_{E_6}} &= \chi_2, \\ p_3^{[100000]_{E_6}} &= \chi_3, \\ p_4^{[100000]_{E_6}} &= \chi_1 \chi_5^2 - \chi_2 \chi_5 + \chi_4 + \chi_4 \chi_6, \\ p_5^{[100000]_{E_6}} &= \chi_1^2 - 2\chi_5^2 \chi_1 + 2\chi_4 \chi_1 + \chi_4^2 + \chi_5 \chi_6^2 + \chi_2 - 2\chi_3 \chi_5 + \chi_5 - \chi_2 \chi_6 - \chi_5 \chi_6, \\ p_6^{[100000]_{E_6}} &= \chi_1 \chi_5 - \chi_5^3 - \chi_2 \chi_5^2 + 2\chi_4 \chi_5 - 2\chi_1 \chi_6 \chi_5 + \chi_4 \chi_6 \chi_5 + \chi_6^3 \\ &\quad + 2\chi_1 \chi_2 - 2\chi_3 + \chi_2 \chi_4 - 3\chi_3 \chi_6, \\ p_7^{[100000]_{E_6}} &= 2\chi_2^2 + \chi_5 \chi_2 - 2\chi_5 \chi_6 \chi_2 + \chi_3 \chi_5^2 + \chi_1 \chi_6^2 + \chi_4 \chi_6^2 - 3\chi_1 \chi_3 \\ &\quad - 2\chi_3 \chi_4 + \chi_4 - \chi_5^2 \chi_6 - \chi_1 \chi_6 + \chi_4 \chi_6, \\ p_8^{[100000]_{E_6}} &= \chi_2 \chi_5^3 - \chi_1 \chi_6 \chi_5^2 + \chi_6^2 \chi_5 + \chi_1 \chi_2 \chi_5 - 2\chi_3 \chi_5 - 3\chi_2 \chi_4 \chi_5 + \chi_3 \chi_6 \chi_5 \\ &\quad - 2\chi_6 \chi_5 + \chi_5 - \chi_1^2 - \chi_2 \chi_6^2 + \chi_2 \chi_3 + \chi_1 \chi_4 + \chi_1^2 \chi_6 - \chi_2 \chi_6 + 2\chi_1 \chi_4 \chi_6, \\ p_9^{[100000]_{E_6}} &= \chi_1 \chi_5^4 - \chi_6 \chi_5^3 + \chi_2 \chi_5^2 - 4\chi_1 \chi_4 \chi_5^2 + \chi_2 \chi_6 \chi_5^2 - \chi_2^2 \chi_5 - \chi_1 \chi_6^2 \chi_5 \\ &\quad + \chi_4 \chi_5 + 3\chi_4 \chi_6 \chi_5 + \chi_6^3 + \chi_3^2 + 2\chi_1 \chi_4^2 + \chi_1 \chi_2 - 6\chi_3 + 4\chi_1^2 \chi_4 - 4\chi_2 \chi_4 \\ &\quad + 4\chi_1 \chi_3 \chi_5 - 2\chi_2 \chi_4 \chi_6 - 3\chi_3 \chi_6 - 4\chi_1 \chi_5 + 3, \end{aligned}$$



$$\begin{aligned}
p_{10}^{[100000]E_6} &= \chi_5^5 - 5\chi_4\chi_5^3 + \chi_1\chi_6\chi_5^3 - \chi_6^2\chi_5^2 - \chi_1\chi_2\chi_5^2 + 5\chi_3\chi_5^2 - \chi_5^2 - 2\chi_1^2\chi_5 \\
&\quad + \chi_2\chi_3\chi_5 + 4\chi_1\chi_4\chi_5 + \chi_1^2\chi_6\chi_5 - 2\chi_2\chi_6\chi_5 - 3\chi_1\chi_4\chi_6\chi_5 + \chi_1\chi_6^2 \\
&\quad + \chi_1^2\chi_2 - 5\chi_1\chi_3 + 2\chi_1\chi_2\chi_4 - 5\chi_3\chi_4 - \chi_2^2\chi_6 - 2\chi_1\chi_6 + \chi_1\chi_3\chi_6 \\
&\quad + \chi_2\chi_5 + 5\chi_4^2\chi_5 + 2\chi_4\chi_6^2 + 2\chi_1 - \chi_4\chi_6, \\
p_{11}^{[100000]E_6} &= \chi_6\chi_5^4 - \chi_2\chi_5^3 + \chi_1\chi_3\chi_5^2 - \chi_4\chi_5^2 + 2\chi_1\chi_6\chi_5^2 - 4\chi_4\chi_6\chi_5^2 - 2\chi_6^2\chi_5 \\
&\quad + 3\chi_3\chi_5 + 3\chi_2\chi_4\chi_5 - 2\chi_1\chi_2\chi_6\chi_5 + 3\chi_3\chi_6\chi_5 - \chi_6\chi_5 + \chi_1\chi_2^2 + 2\chi_4^2 \\
&\quad + 2\chi_1^2\chi_6^2 - 2\chi_2\chi_6^2 + 2\chi_2 - 3\chi_1^2\chi_3 + \chi_2\chi_3 + \chi_2^2\chi_4 + \chi_1\chi_4 - 2\chi_1\chi_3\chi_4 \\
&\quad + \chi_2\chi_6 + 2\chi_4^2\chi_6 - 2\chi_1^2\chi_6 - \chi_1\chi_2\chi_5, \\
p_{12}^{[100000]E_6} &= 2\chi_6\chi_1^3 + \chi_5^2\chi_1^2 - \chi_4\chi_1^2 - 2\chi_2\chi_5\chi_1^2 - \chi_5^2\chi_6\chi_1^2 + \chi_4\chi_6\chi_1^2 + 3\chi_5\chi_6^2\chi_1 \\
&\quad - 3\chi_2\chi_3\chi_1 + \chi_2\chi_4\chi_5\chi_1 + \chi_5\chi_1 - 5\chi_2\chi_6\chi_1 - 2\chi_5\chi_6\chi_1 + \chi_2^3 + \chi_3\chi_5^3 \\
&\quad - \chi_5^3 - 2\chi_6^3 + 3\chi_3^2 - \chi_3 + \chi_2\chi_4 + 3\chi_2^2\chi_5 - 3\chi_3\chi_4\chi_5 + 2\chi_4\chi_5 + \chi_5^3\chi_6, \\
&\quad + 2\chi_2\chi_1 + 6\chi_3\chi_6 - 3\chi_4\chi_5\chi_6 - \chi_1^3 - \chi_2\chi_5^2\chi_6 \\
p_{13}^{[100000]E_6} &= \chi_1^4 - 2\chi_5^2\chi_1^3 + 2\chi_4\chi_1^3 + \chi_4^2\chi_1^2 - 3\chi_2\chi_1^2 - 2\chi_3\chi_5\chi_1^2 - \chi_5\chi_1^2 - \chi_2\chi_6\chi_1^2 \\
&\quad + 2\chi_5^3\chi_1 + 2\chi_2\chi_5^2\chi_1 - 2\chi_6^2\chi_1 + \chi_3\chi_1 - 4\chi_2\chi_4\chi_1 + \chi_2^2\chi_5\chi_1 - 4\chi_4\chi_5\chi_1 \\
&\quad + 3\chi_3\chi_6\chi_1 + \chi_4\chi_5\chi_6\chi_1 - \chi_6\chi_1 + 2\chi_1 + \chi_2^2 - 2\chi_2\chi_4^2 + \chi_3\chi_5^2 + \chi_2\chi_4\chi_5^2 \\
&\quad + \chi_5^2\chi_6^2 - 3\chi_3\chi_4 + 2\chi_4 + \chi_2\chi_5 - \chi_2\chi_3\chi_5 + \chi_2^2\chi_6 - 2\chi_5^2\chi_6 - 2\chi_2\chi_5\chi_6 \\
&\quad + 4\chi_5\chi_6\chi_1^2 - \chi_5^3\chi_6\chi_1.
\end{aligned}$$

A.2.  $R = E_7$ . — We include here the character relations for  $\rho = \rho_{\omega_6}$  and  $k = 1, \dots, 11$ ; note that by reality, we have  $p_{56}^{\omega_6, k} = p_k^{\omega_6}$ . The full set of character relations for  $k$  up to 28 is available upon request.

$$\begin{aligned}
p_0^{[0000010]E_7} &= 1, \\
p_1^{[0000010]E_7} &= \chi_6, \\
p_2^{[0000010]E_7} &= \chi_5 + 1, \\
p_3^{[0000010]E_7} &= \chi_4 + \chi_6, \\
p_4^{[0000010]E_7} &= \chi_3 + \chi_5 + 1, \\
p_5^{[0000010]E_7} &= (1 - \chi_1)\chi_4 + (\chi_1^2 + \chi_1 + \chi_2 + \chi_5 + 1)\chi_6 + \chi_2\chi_7, \\
p_6^{[0000010]E_7} &= (1 - 2\chi_5)\chi_1^2 - 2\chi_1^3 + (\chi_6^2 - \chi_7\chi_6 + \chi_7^2 + 4\chi_2 - 2\chi_3 + 2\chi_5 + 2)\chi_1 \\
&\quad - \chi_3 + 2\chi_5 + 2\chi_2(\chi_5 + 1) + \chi_4\chi_6 - \chi_4\chi_7 - \chi_6\chi_7 + \chi_5^2 + \chi_2^2 + 1, \\
p_7^{[0000010]E_7} &= \chi_4(2 - \chi_1^2 + \chi_1 + \chi_2 + 2\chi_5) + (1 - \chi_1^3 + (2\chi_2 + \chi_5 + 3)\chi_6\chi_1 + 2\chi_2 \\
&\quad - 2\chi_3 + \chi_5 + \chi_7)(2\chi_1^2 + (\chi_2 - 2\chi_5 + 1)\chi_1 + \chi_7^2 + 3\chi_2 - 3\chi_3),
\end{aligned}$$

$$\begin{aligned}
p_8^{[0000010]E_7} &= (2\chi_6^2 + \chi_4\chi_6 \quad 2\chi_7\chi_6 + \chi_7^2 + 4\chi_2 \quad 2\chi_1^3 \quad 4\chi_3 + 2\chi_5 \quad 2\chi_4\chi_7)\chi_1 \\
&\quad + \chi_2^2 + 2\chi_4^2 + \chi_6^2 + \chi_5\chi_7^2 + \chi_7^2 \quad 2\chi_3 \quad 3\chi_3\chi_5 + 3\chi_4\chi_6 + \chi_4\chi_7 \\
&\quad + \chi_2(\chi_6^2 + \chi_7\chi_6 + \chi_7^2 \quad 2\chi_3 + 2\chi_5) + (\chi_3 \quad 2\chi_5 \quad \chi_6\chi_7 + 2)\chi_1^2 \\
&\quad \chi_5\chi_6\chi_7 + \chi_6\chi_7, \\
p_9^{[0000010]E_7} &= (\chi_1 + 2)\chi_6^3 \quad 2\chi_1\chi_7\chi_6^2 + (\chi_1^3 + \chi_1^2 + (\chi_7^2 + 2\chi_2 \quad 2\chi_3 \quad 2\chi_5)\chi_1 \\
&\quad 3\chi_3 \quad 2\chi_5 + \chi_2(\chi_5 + 1) \quad \chi_5^2 \quad 1)\chi_6 + ((\chi_2 + \chi_3 + \chi_5 + 2)\chi_1 \\
&\quad (\chi_5 + 2)\chi_1^2 + \chi_5^2 \quad \chi_3 + 3\chi_5 + 2\chi_2(\chi_5 + 1) + 2)\chi_7 \\
&\quad + \chi_4(\chi_1^3 \quad \chi_1^2 + (3\chi_2 + \chi_5 + 2)\chi_1 \\
&\quad \chi_7^2 \quad 2\chi_2 + \chi_3 + 3\chi_5 \quad \chi_6\chi_7 + 4), \\
p_{10}^{[0000010]E_7} &= (\chi_6^2 \quad 2\chi_7\chi_6 + \chi_7^2 \quad (4\chi_2 + 1)\chi_5 + \chi_4(\chi_6 + \chi_7))\chi_1^2 \\
&\quad (\chi_4^2 + 3\chi_7\chi_4 \quad 4\chi_5^2 \quad \chi_2\chi_6^2 \quad 2\chi_6^2 + \chi_7^2 \quad 3\chi_2\chi_6\chi_7 \quad 3\chi_6\chi_7)\chi_1 \\
&\quad + \chi_6\chi_7^3 + \chi_3^2 + 4\chi_2\chi_5^2 + 2\chi_5^2 + \chi_2\chi_6^2 \quad 5\chi_5\chi_6^2 \quad 5\chi_6^2 \quad \chi_2\chi_7^2 \quad \chi_5\chi_7^2 \\
&\quad + 5\chi_5 \quad 4\chi_2\chi_4\chi_6 + 3\chi_4\chi_6 + \chi_4\chi_5\chi_6 \quad 2\chi_2\chi_4\chi_7 + \chi_4\chi_7 + 2\chi_2\chi_6\chi_7 \\
&\quad + 2\chi_2\chi_5 + \chi_5\chi_1^4 \quad \chi_6\chi_7\chi_1^3 + (4\chi_6^2 + \chi_7^2 \quad 2)\chi_5\chi_1\chi_1 + 2\chi_5\chi_6\chi_7 + 3\chi_6^4 \\
&\quad + \chi_6\chi_7 + \chi_3 (6\chi_6^2 \quad 3\chi_7\chi_6 + (4\chi_1 + 5)\chi_5 + 4) \quad \chi_7^2 + 2\chi_2^2\chi_5 + 3, \\
p_{11}^{[0000010]E_7} &= \chi_6\chi_1^5 \quad \chi_7\chi_1^4 + 2\chi_4\chi_1^3 \quad 5\chi_2\chi_6\chi_1^3 \quad 5\chi_6\chi_1^3 + \chi_5\chi_7\chi_1^3 \quad \chi_6^3\chi_1^2 \\
&\quad \chi_6\chi_7^2\chi_1^2 + \chi_4\chi_1^2 \quad \chi_4\chi_5\chi_1^2 + \chi_2\chi_6\chi_1^2 + 5\chi_3\chi_6\chi_1^2 \quad 4\chi_5\chi_6\chi_1^2 + 3\chi_6\chi_1^2 \\
&\quad + 4\chi_2\chi_7\chi_1^2 \quad 2\chi_5\chi_7\chi_1^2 + 3\chi_7\chi_1^2 + 4\chi_6^3\chi_1 + \chi_7^3\chi_1 + \chi_4\chi_6^2\chi_1 \quad 6\chi_2\chi_4\chi_1 \\
&\quad + \chi_3\chi_4\chi_1 \quad 3\chi_4\chi_1 + 4\chi_4\chi_5\chi_1 + 5\chi_2^2\chi_6\chi_1 \quad 2\chi_5^2\chi_6\chi_1 + 8\chi_2\chi_6\chi_1 \\
&\quad 8\chi_3\chi_6\chi_1 + 4\chi_2\chi_5\chi_6\chi_1 \quad \chi_6\chi_1 + \chi_5^2\chi_7\chi_1 + \chi_6^2\chi_7\chi_1 \quad \chi_2\chi_7\chi_1 \\
&\quad 4\chi_3\chi_7\chi_1 \quad 3\chi_2\chi_5\chi_7\chi_1 + \chi_5\chi_7\chi_1 \quad 2\chi_4\chi_6\chi_7\chi_1 \quad 3\chi_7\chi_1 \\
&\quad + 2\chi_5\chi_6^3 + \chi_4\chi_5^2 \quad \chi_4\chi_6^2 \quad \chi_4\chi_7^2 + 2\chi_2\chi_6\chi_7^2 + \chi_5\chi_6\chi_7^2 + \chi_6\chi_7^2 \quad 7\chi_2\chi_4 \\
&\quad + 5\chi_3\chi_4 + 3\chi_4 + 2\chi_2\chi_4\chi_5 + 3\chi_4\chi_5 + 3\chi_2^2\chi_6 \quad \chi_5^2\chi_6 \\
&\quad + 2\chi_2\chi_6 \quad 5\chi_2\chi_3\chi_6 \quad 2\chi_3\chi_6 + 7\chi_2\chi_5\chi_6 \quad 5\chi_3\chi_5\chi_6 \quad \chi_5\chi_6 \\
&\quad + \chi_6 \quad 2\chi_2^2\chi_7 \quad \chi_4^2\chi_7 + \chi_5^2\chi_7 \quad \chi_2\chi_6^2\chi_7 \quad 2\chi_5\chi_6^2\chi_7 \quad 2\chi_6^2\chi_7 \quad 2\chi_2\chi_7 \\
&\quad + 3\chi_3\chi_7 \quad \chi_2\chi_5\chi_7 + \chi_3\chi_5\chi_7 + 3\chi_5\chi_7 + 2\chi_4\chi_6\chi_7 + 2\chi_7.
\end{aligned}$$

## REFERENCES

- [1] G. ANTONIOU, M. FEIGIN & I. A. B. STRACHAN – “The Saito determinant for Coxeter discriminant strata”, 2020, arXiv: 2008.10133.
- [2] V. I. ARNOLD – “Topological classification of complex trigonometric polynomials and the combinatorics of graphs with an identical number of vertices and edges”, *Funktsional. Anal. i Prilozhen.* **30** (1996), no. 1, p. 1–17, 96.
- [3] V. I. ARNOLD, S. M. GUSEIN-ZADE & A. N. VARCHENKO – *Singularities of differentiable maps. Vol. I*, Monographs in Math., vol. 82, Birkhäuser Boston, Inc., Boston, MA, 1985.
- [4] G. BOROT & A. BRINI – “Chern–Simons theory on spherical Seifert manifolds, topological strings and integrable systems”, *Adv. Theo. Math. Phys.* **22** (2018), no. 2, p. 305–394.

- [5] V. BOUCHARD & M. MARINO – “Hurwitz numbers, matrix models and enumerative geometry”, in *From Hodge theory to integrability and TQFT tt\*-geometry*, Proc. Sympos. Pure Math., vol. 78, American Mathematical Society, Providence, RI, 2008, p. 263–283.
- [6] A. BRINI – “ $E_8$  spectral curves”, *Proc. London Math. Soc. (3)* **121** (2020), no. 4, p. 954–1032.
- [7] ———, “Exterior powers of the adjoint representation and the Weyl ring of  $E_8$ ”, *J. Algebra* **551** (2020), p. 301–341.
- [8] A. BRINI, R. CAVALIERI & D. ROSS – “Crepant resolutions and open strings”, *J. reine angew. Math.* **755** (2019), p. 191–245.
- [9] A. BRINI & K. OSUGA – “Five-dimensional gauge theories and the local B-model”, *Lett. Math. Phys.* **112** (2022), article no. 44 (48 pages).
- [10] N. CAPORASO, L. GRIGUOLO, M. MARINO, S. PASQUETTI & D. SEMINARA – “Phase transitions, double-scaling limit, and topological strings”, *Phys. Rev. D* **75** (2007), no. 4, article no. 046004 (24 pages).
- [11] G. CARLET – “The extended bigraded Toda hierarchy”, *J. Phys. A* **39** (2006), no. 30, p. 9411–9435.
- [12] J. CHENG & T. MILANOV – “The extended D-Toda hierarchy”, *Selecta Math. (N.S.)* **27** (2021), no. 2, article no. 24 (85 pages).
- [13] M. J. CRESCIMANNO & W. TAYLOR – “Large  $N$  phases of chiral QCD in two-dimensions”, *Nuclear Phys. B* **437** (1995), p. 3–24.
- [14] B. DUBROVIN – “Hamiltonian formalism of Whitham type hierarchies and topological Landau-Ginzburg models”, *Comm. Math. Phys.* **145** (1992), p. 195–207.
- [15] ———, “Integrable systems in topological field theory”, *Nuclear Phys. B* **379** (1992), no. 3, p. 627–689.
- [16] ———, “Geometry of 2D topological field theory”, in *Integrable systems and quantum groups* (M. Francaviglia & S. Greco, eds.), Lect. Notes in Math., vol. 1620, Springer-Verlag, Berlin, 1996, p. 120–348.
- [17] ———, “Differential geometry of the space of orbits of a Coxeter group”, in *Integrable systems*, Surv. Differ. Geom., vol. 4, Int. Press, Boston, MA, 1998, p. 181–211.
- [18] ———, “Painlevé transcendents and two-dimensional topological field theory”, in *The Painlevé property*, CRM Ser. Math. Phys., Springer, New York, 1999, p. 287–412.
- [19] ———, “On almost duality for Frobenius manifolds”, in *Geometry, topology, and mathematical physics*, Amer. Math. Soc. Transl. Ser. 2, vol. 212, American Mathematical Society, Providence, RI, 2004, p. 75–132.
- [20] B. DUBROVIN, I. STRACHAN, Y. ZHANG & D. ZUO – “Extended affine Weyl groups of BCD-type: Their Frobenius manifolds and Landau–Ginzburg superpotentials”, *Adv. Math.* **351** (2019), p. 897–946.
- [21] B. DUBROVIN & Y. ZHANG – “Extended affine Weyl groups and Frobenius manifolds”, *Compositio Math.* **111** (1998), no. 2, p. 167–219.
- [22] P. DUNIN-BARKOWSKI, N. ORANTIN, S. SHADRIN & L. SPITZ – “Identification of the Givental formula with the spectral curve topological recursion procedure”, *Comm. Math. Phys.* **328** (2014), p. 669–700.
- [23] T. EKEDAH, S. LANDO, M. SHAPIRO & A. VAINSHTEIN – “Hurwitz numbers and intersections on moduli spaces of curves”, *Invent. Math.* **146** (2001), p. 297–327.
- [24] V. V. FOCK & A. MARSHAKOV – “Loop groups, clusters, dimers and integrable systems”, in *Geometry and quantization of moduli spaces*, Adv. Courses Math. CRM Barcelona, Birkhäuser/Springer, Cham, 2016, p. 1–66.
- [25] W. FULTON – “Hurwitz schemes and irreducibility of moduli of algebraic curves”, *Ann. of Math. (2)* **90** (1969), p. 542–575.
- [26] W. FULTON & J. HARRIS – *Representation theory*, Graduate Texts in Math., vol. 129, Springer-Verlag, New York, 1991.
- [27] I. P. GOULDEN & D. M. JACKSON – “The combinatorial relationship between trees, cacti and certain connection coefficients for the symmetric group”, *European J. Combin.* **13** (1992), no. 5, p. 357–365.
- [28] I. P. GOULDEN, D. M. JACKSON & A. VAINSHTEIN – “The number of ramified coverings of the sphere by the torus and surfaces of higher genera”, *Ann. Comb.* **4** (2000), no. 1, p. 27–46.

- [29] A. GOUPIL & G. SCHAEFFER – “Factoring  $n$ -cycles and counting maps of given genus”, *European J. Combin.* **19** (1998), no. 7, p. 819–834.
- [30] K. ITO & S.-K. YANG – “Flat coordinates, topological Landau-Ginzburg models and the Seiberg-Witten period integrals”, *Phys. Lett. B* **415** (1997), p. 45–53.
- [31] I. M. KRICHEVER – “The tau function of the universal Whitham hierarchy, matrix models and topological field theories”, *Comm. Pure Appl. Math.* **47** (1994), p. 437.
- [32] S. K. LANDO – “Ramified coverings of the two-dimensional sphere and intersection theory in spaces of meromorphic functions on algebraic curves”, *Uspehi Mat. Nauk* **57** (2002), no. 3(345), p. 29–98.
- [33] S. K. LANDO & D. ZVONKINE – “On multiplicities of the Lyashko–Looijenga mapping on strata of the discriminant”, *Funct. Anal. Appl.* **33** (1999), no. 3, p. 178–188.
- [34] ———, “Counting ramified coverings and intersection theory on spaces of rational functions. I. Cohomology of Hurwitz spaces”, *Moscow Math. J.* **7** (2007), no. 1, p. 85–107.
- [35] W. LERCHE & N. P. WARNER – “Exceptional SW geometry from ALE fibrations”, *Phys. Lett. B* **423** (1998), no. 1-2, p. 79–86.
- [36] E. LOOIJENGA – “The complement of the bifurcation variety of a simple singularity”, *Invent. Math.* **23** (1974), p. 105–116.
- [37] O. V. LYASHKO – “The geometry of bifurcation diagrams”, *Uspehi Mat. Nauk* **34** (1979), no. 3(207), p. 205–206.
- [38] A. MCDANIEL & L. SMOLINSKY – “A Lie-theoretic Galois theory for the spectral curves of an integrable system. I”, *Comm. Math. Phys.* **149** (1992), no. 1, p. 127–148.
- [39] ———, “A Lie-theoretic Galois theory for the spectral curves of an integrable system. II”, *Trans. Amer. Math. Soc.* **349** (1997), no. 2, p. 713–746.
- [40] T. MILANOV, Y. SHEN & H.-H. TSENG – “Gromov–Witten theory of Fano orbifold curves and ADE-Toda Hierarchies”, *Geom. Topol.* **20** (2016), no. 4, p. 2135–2218.
- [41] N. NEKRASOV – “Five dimensional gauge theories and relativistic integrable systems”, *Nuclear Phys. B* **531** (1998), no. 1-3, p. 323–344.
- [42] P. NORBURY & N. SCOTT – “Gromov–Witten invariants of  $P^1$  and Eynard–Orantin invariants”, *Geom. Topol.* **18** (2014), no. 4, p. 1865–1910.
- [43] D. PANOV & D. ZVONKINE – “Counting meromorphic functions with critical points of large multiplicities”, *J. Math. Sci.* **126** (2005), no. 2, p. 1095–1110.
- [44] P. ROSSI – “Gromov–Witten theory of orbicurves, the space of tri-polynomials and symplectic field theory of Seifert fibrations”, *Math. Ann.* **348** (2010), no. 2, p. 265–287.
- [45] B. M. SZABLIKOWSKI & M. BŁASZAK – “Dispersionful analog of the Whitham hierarchy”, *J. Math. Phys.* **49** (2008), no. 8, p. 082701, 20.
- [46] H. WILLIAMS – “Double Bruhat cells in Kac–Moody groups and integrable systems”, *Lett. Math. Phys.* **103** (2013), p. 389–419.

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