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AN INTEGRAL MODEL OF THE PERFECTOID MODULAR CURVE

BY JUAN ESTEBAN RODRÍGUEZ CAMARGO

ABSTRACT. — We construct an integral model of the perfectoid modular curve. Studying this object, we prove some vanishing results for the coherent cohomology at perfectoid level. We use a local duality theorem at finite level to compute duals for the coherent cohomology of the integral perfectoid curve. Specializing to the structural sheaf, we can describe the dual of the completed cohomology as the inverse limit of the integral cusp forms of weight 2 and trace maps.

RÉSUMÉ (Un modèle entier de la courbe modulaire perfectoïde). — Nous construisons un modèle entier de la courbe modulaire perfectoïde. Avec cet objet nous montrons des résultats d'annulation de la cohomologie cohérente au niveau perfectoïde. Nous utilisons un théorème de dualité locale au niveau fini pour obtenir une dualité pour la cohomologie cohérente au niveau infini. Finalement, en considérant le faisceau structural, nous obtenons une description du dual de la cohomologie complétée en termes des formes modulaires cuspidales de poids 2 et des traces normalisées.

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INTRODUCTION

Throughout this article we fix a prime number p , C_p the p -adic completion of an algebraic closure of \mathbb{Q}_p , and $\{\zeta_m\}_{m \in \mathbb{N}} \subset C_p$ a compatible system of primitive roots of unity. Given a non-archimedean field K we let \mathcal{O}_K denote its valuation ring. We let $\overline{\mathbb{F}}_p$ be the residue field of \mathcal{O}_{C_p} and $\check{Z}_p = W(\overline{\mathbb{F}}_p) \subset C_p$ the ring of Witt vectors. Let Z_p^{cyc} and \check{Z}_p^{cyc} denote the p -adic completions of the p -adic cyclotomic extensions of Z_p and \check{Z}_p in C_p respectively.

MATHEMATICAL SUBJECT CLASSIFICATION (2020). — 14G35, 14F17, 14G45.

KEYWORDS. — Perfectoid modular curve, completed cohomology, coherent cohomology.

Let $M > 1$ be an integer and $\Gamma(M) \subset \mathrm{GL}_2(\mathbb{Z})$ the principal congruence subgroup of level M . We fix $N > 3$ an integer prime to p . For $n > 0$ we denote by $Y(Np^n)/\mathrm{Spec} Z_p$ the integral modular curve of level $\Gamma(Np^n)$ and $X(Np^n)$ its compactification, cf. [KM85]. We denote by $\mathfrak{X}(Np^n)$ the completion of $X(Np^n)$ along its special fiber, and by $\mathcal{X}(Np^n)$ its analytic generic fiber seen as an adic space over $\mathrm{Spa}(\mathbb{Q}_p, Z_p)$, cf. [Hub96].

In [Sch15], Scholze constructed the perfectoid modular curve of tame level $\Gamma(N)$. He proved that there exists a perfectoid space $\mathcal{X}(Np^\infty)$, unique up to a unique isomorphism, satisfying the tilde limit property

$$\mathcal{X}(Np^\infty) \sim \varprojlim_n \mathcal{X}(Np^n),$$

see [SW13, Def. 2.4.1] and [Hub96, Def. 2.4.2].

The first result of this paper is the existence of a Katz-Mazur integral model of the perfectoid modular curve. More precisely, we prove the following theorem, see Section 2 for the notion of a perfectoid formal scheme.

THEOREM 0.1. — *The inverse limit $\mathfrak{X}(Np^\infty) = \varprojlim_n \mathfrak{X}(Np^n)$ is a perfectoid formal scheme over $\mathrm{Spf} Z_p^{\mathrm{cyc}}$ whose analytic generic fiber is naturally isomorphic to the perfectoid modular curve $\mathcal{X}(Np^\infty)$.*

The integral perfectoid modular curve $\mathfrak{X}(Np^\infty)$ was previously constructed by Lurie in [Lur20], his method reduces the proof of perfectoidness to the ordinary locus via a mixed characteristic version of Kunz's theorem. The strategy in this paper is more elementary: we use faithfully flat descent to deduce perfectoidness of $\mathfrak{X}(Np^\infty)$ from the description of the stalks at the $\overline{\mathbb{F}}_p$ -points. Then, we deal with three different kind of points:

- The ordinary points where we use the Serre-Tate parameter to explicitly compute the deformation rings, cf. [Kat81, §2].
- The cusps where we have explicit descriptions provided by the Tate curve, cf. [KM85, §8–10].
- The supersingular points where even though we do not compute explicitly the stalk, one can prove that the Frobenius map is surjective modulo p .

It is worth mentioning that the study of the ordinary locus in Lurie's approach and the one presented in this document are very related, see [Lur20, Prop. 2.2] and Proposition 1.5 below.

As an application of the integral model we can prove vanishing results for the coherent cohomology of the perfectoid modular curve. Let $E^{\mathrm{sm}}/X(N)$ be the semi-abelian scheme extending the universal elliptic curve over $Y(N)$, cf. [DR73]. Let $e : X(N) \rightarrow E^{\mathrm{sm}}$ be the unit section and $\omega_E = e^* \Omega_{E^{\mathrm{sm}}/X(N)}^1$ the sheaf of invariant differentials. For $n > 0$ we denote by $\omega_{E,n}$ the pullback of ω_E to $X(Np^n)$, and $D_n \subset X(Np^n)$ the reduced cusp divisor. Let $k \in \mathbb{Z}$, we denote $\omega_{E,n}^k = \omega_{E,n}^{\otimes k}$ and $\omega_{E,n,\mathrm{cusp}}^k = \omega_{E,n}^k(-D_n)$.

Let $\omega_{E,\infty}^k$ be the pullback of ω_E^k to $\mathfrak{X}(Np^\infty)$, and $\omega_{E,\infty,\text{cusp}}^k$ the p -adic completion of the direct limit of the cuspidal modular sheaves $\omega_{E,n,\text{cusp}}^k$. In the following we consider almost mathematics with respect to the maximal ideal of Z_p^{cyc} .

THEOREM 0.2. — *Let $\mathcal{F} = \omega_{E,\infty}^k$ or $\omega_{E,\infty,\text{cusp}}^k$ and $\mathcal{F}_\mu^+ = \mathcal{F} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}(Np^\infty)}} \mathcal{O}_{\mathfrak{X}(Np^\infty)}^+$. There is an almost quasi-isomorphism of complexes*

$$R\Gamma_{\text{an}}(\mathfrak{X}(Np^\infty), \mathcal{F}_\mu^+) \simeq^{\text{ae}} R\Gamma(\mathfrak{X}(Np^\infty), \mathcal{F}).$$

Moreover, the following holds

- (1) *The cohomology complex $R\Gamma(\mathfrak{X}(Np^\infty), \mathcal{F})$ is concentrated in degree 0 if $k > 0$, degree $[0, 1]$ for $k = 0$, and degree 1 if $k < 0$.*
- (2) *For $k \in \mathbb{Z}$ and $i, s > 0$, we have $H^i(\mathfrak{X}(Np^\infty), \mathcal{F})/p^s = H^i(\mathfrak{X}(Np^\infty), \mathcal{F}/p^s)$ and $H^i(\mathfrak{X}(Np^\infty), \mathcal{F}) = \varprojlim_s H^i(\mathfrak{X}(Np^\infty), \mathcal{F}/p^s)$.*
- (3) *The cohomology groups $H^i(\mathfrak{X}(Np^\infty), \mathcal{F})$ are torsion-free.*

Next, we use Serre duality and Pontryagin duality to construct a local duality theorem for the modular curves at finite level. In the limit one obtains the following theorem

THEOREM 0.3. — *Let \mathfrak{X}_n be the connected component of $\mathfrak{X}(Np^n)_{\check{Z}_p}$ defined as the locus where the Weil pairing of the universal basis of $E[N]$ is equal to ζ_N . We denote $\mathfrak{X}_\infty = \varprojlim_n \mathfrak{X}_n$. Let $\mathcal{F} = \omega_{E,\infty}^k$ or $\omega_{E,\infty,\text{cusp}}^k$ and $\mathcal{F}_n = \omega_{E,n}^k$ or $\omega_{E,n,\text{cusp}}^k$ respectively. There is a natural $\text{GL}_2(\mathbb{Q}_p)$ -equivariant isomorphism*

$$\text{Hom}_{\check{Z}_p^{\text{cyc}}} (H^i(\mathfrak{X}_\infty, \mathcal{F}), \check{Z}_p^{\text{cyc}}) = \varprojlim_{n, \text{Tr}_n} H^{1-i}(X_n, \mathcal{F}_n^\vee \otimes \omega_{E,n,\text{cusp}}^2),$$

where the transition maps in the RHS are given by normalized traces, and \mathcal{F}_n^\vee is the dual sheaf of \mathcal{F}_n .

Finally, we specialize to the case $\mathcal{F} = \mathcal{O}_{\mathfrak{X}_\infty}$ where the completed cohomology appears. Let X_n be a connected component of $X(Np^n)_{\check{Z}_p}$ as in the previous theorem. Let $i > 0$ and let $\tilde{H}^i = \varprojlim_s \varinjlim_n H_{\text{ét}}^i(X_{n,C_p}, Z/p^s Z)$ be the completed i -th cohomology group, where $X_{n,C_p} = X_n \times_{\text{Spec } \check{Z}_p[\zeta_{p^n}]} \text{Spec } C_p$. Note that this is a slightly different version of Emerton’s completed cohomology [Eme06], where one considers the étale cohomology with compact supports of $Y_{n,C_p} \subset X_{n,C_p}$. Nevertheless, both cohomologies are related via the open and closed immersions $Y_n \subset X_n \supset D_n$. Following the same ideas of [Sch15, §4.2] one can show that $\tilde{H}^i \widehat{\otimes}_{Z_p} \mathcal{O}_{C_p}$ is almost equal to $H_{\text{an}}^i(\mathfrak{X}_{\infty,C_p}, \mathcal{O}_{\mathfrak{X}_\infty}^+)$, in particular it vanishes for $i > 2$ and $\tilde{H}^0 = Z_p$. Using the theorem above we obtain the following result.

THEOREM 0.4. — *There is a $\text{GL}_2(\mathbb{Q}_p)$ -equivariant almost isomorphism of almost \mathcal{O}_{C_p} -modules*

$$\text{Hom}_{\mathcal{O}_{C_p}} (\tilde{H}^1 \widehat{\otimes}_{Z_p} \mathcal{O}_{C_p}, \mathcal{O}_{C_p}) \simeq^{\text{ae}} \varprojlim_{n, \text{Tr}_n} H^0(X_{n,C_p}, \omega_{E,\infty,\text{cusp}}^2).$$

The outline of the paper is the following. In Section 1 we recall the construction of the integral modular curves at finite level; they are defined as the moduli space parametrizing elliptic curves endowed with a Drinfeld basis of the torsion subgroups, we will follow [KM85]. Then, we study the deformation rings of the modular curves at $\overline{\mathbb{F}}_p$ -points. For ordinary points we use the Serre-Tate parameter to describe the deformation ring at level $\Gamma(Np^n)$. We show that it represents the moduli problem parametrizing deformations of the p -divisible group $E[p^\infty]$, and a split of the connected-étale short exact sequence

$$0 \longrightarrow \widehat{E} \longrightarrow E[p^\infty] \longrightarrow E[p^\infty]^{\text{ét}} \longrightarrow 0.$$

For cusps we refer to the explicit computations of [KM85, §§8 & 10]. Finally, in the case of a supersingular point we prove that any element of the local deformation ring at level $\Gamma(Np^n)$ admits a p -th root modulo p at level $\Gamma(Np^{n+1})$.

In Section 2 we introduce the notion of a perfectoid formal scheme. We prove Theorem 0.1 reducing to the formal deformation rings at $\overline{\mathbb{F}}_p$ -points via faithfully flat descent. We will say some words regarding Lurie's construction of $\mathfrak{X}(Np^\infty)$. It is worth mentioning that the tame level $\Gamma(N)$ is taken only for a cleaner exposition, by a result of Kedlaya-Liu about quotients of perfectoid spaces by finite group actions [KL19, Th. 3.3.26], there are integral models of any tame level.

In Section 3, we use Serre and Pontryagin duality to define a local duality pairing for the coherent cohomology of vector bundles over an lci projective curve over a finite extension of Z_p .

In Section 4, we compute the dualizing complexes of the modular curves at finite level. We prove the cohomological vanishing of Theorem 0.2 and its comparison with the cohomology of the perfectoid modular curve. We prove the duality theorem at infinite level, Theorem 0.3, and specialize to $\mathcal{F} = \mathcal{O}_{\mathfrak{X}_\infty}$ to obtain Theorem 0.4.

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1. A BRIEF INTRODUCTION TO THE KATZ-MAZUR INTEGRAL MODULAR CURVES

Let $N > 3$ be an integer prime to p and $n \in \mathbb{N}$. Let $\Gamma(Np^n) \subset \text{GL}_2(Z)$ be the principal congruence subgroup of level Np^n .

DRINFELD BASES. — We recall the definition of a Drinfeld basis for the M -torsion of an elliptic curve

DEFINITION 1.1. — Let M be a positive integer, S a scheme and E an elliptic curve over S . A *Drinfeld basis* of $E[M]$ is a morphism of group schemes $\psi : (Z/MZ)^2 \rightarrow E[M]$ such that the following equality of effective divisors holds

$$(1.1) \quad E[M] = \sum_{(a,b) \in (Z/MZ)^2} \psi(a, b).$$

We also write $(P, Q) = (\psi(1, 0), \psi(0, 1))$ for the Drinfeld basis ψ .

REMARK 1.2. — The left-hand-side of (1.1) is an effective divisor of E/S being a finite flat group scheme over S . The right-hand-side is a sum of effective divisors given by the sections $\psi(a, b)$ of S to E . Furthermore, if M is invertible over S , a homomorphism $\psi : Z/MZ \rightarrow E[M]$ is a Drinfeld basis if and only if it is an isomorphism of group schemes, cf. [KM85, Lem. 1.5.3].

PROPOSITION 1.3. — Let E/S be an elliptic curve. Let (P, Q) be a Drinfeld basis of $E[M]$ and $e_M : E[M] \times E[M] \rightarrow \mu_M$ the Weil pairing. Then $e_M(P, Q) \in \mu_M^\times(S)$ is a primitive root of unity, i.e., a root of the M -th cyclotomic polynomial.

Proof. — [KM85, Th. 5.6.3].

Let $M > 3$. From [KM85, Th. 5.1.1 & Sch. 4.7.0], the moduli problem parametrizing elliptic curves E/S and Drinfeld bases (P, Q) of $E[M]$ is representable by an affine and regular curve over Z . We denote this curve by $Y(M)$ and call it the *(a ne) integral modular curve of level $\Gamma(M)$* . By an abuse of notation, we will write $Y(M)$ for its scalar extension to Z_p .

The j -invariant is a finite flat morphism of Z_p -schemes $j : Y(M) \rightarrow \mathbb{A}_{Z_p}^1$. The *compactified integral modular curve of level $\Gamma(M)$* , denoted by $X(M)$, is the normalization of $\mathbb{P}_{Z_p}^1$ in $Y(M)$ via the j -invariant. The *cusps* or the *boundary divisor D* is the closed reduced subscheme of $X(M)$ defined by $1/j = 0$. The curve $X(M)$ is projective over Z_p and a regular scheme. We refer to $X(M)$ and $Y(M)$ simply as the modular curves of level $\Gamma(M)$.

Let $E_{\text{univ}}/Y(M)$ be the universal elliptic curve and $(P_{\text{univ}, M}, Q_{\text{univ}, M})$ the universal Drinfeld basis of $E_{\text{univ}}[M]$. Let $\Phi_M(X)$ be the M -th cyclotomic polynomial, and let $Z_p[\mu_M^\times]$ denote the ring $Z_p[X]/(\Phi_M(X))$. The Weil pairing of $(P_{\text{univ}, M}, Q_{\text{univ}, M})$ induces a morphism of Z_p -schemes

$$e_M : Y(M) \longrightarrow \text{Spec } Z_p[\mu_M^\times].$$

The map e_M extends uniquely to a map $e_M : X(M) \rightarrow \text{Spec } Z_p[\mu_M^\times]$ by normalization. In addition, e_M is geometrically reduced, and has geometrically connected fibers.

Taking N as in the beginning of the section, and $n \in \mathbb{N}$ varying, we construct the commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & X(Np^{n+1}) & \longrightarrow & X(Np^n) & \longrightarrow & X(Np^{n-1}) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \text{Spec}(Z_p[\mu_{Np^{n+1}}^\times]) & \longrightarrow & \text{Spec}(Z_p[\mu_{Np^n}^\times]) & \longrightarrow & \text{Spec}(Z_p[\mu_{Np^{n-1}}^\times]) & \longrightarrow & \cdots \end{array}$$

the upper horizontal arrows being induced by the map

$$(P_{\text{univ},Np^{n+1}}, Q_{\text{univ},Np^{n+1}}) \longmapsto (pP_{\text{univ},Np^{n+1}}, pQ_{\text{univ},Np^{n+1}}) = (P_{\text{univ},Np^n}, Q_{\text{univ},Np^n}),$$

and the lower horizontal arrows by the natural inclusions. In fact, the commutativity of the diagram is a consequence of the compatibility of the Weil pairing with multiplication by p

$$e_{Np^{n+1}}(pP_{\text{univ},Np^{n+1}}, pQ_{\text{univ},Np^{n+1}}) = e_{Np^n}(P_{\text{univ},Np^n}, Q_{\text{univ},Np^n})^p,$$

cf. [KM85, Th. 5.5.7 & 5.6.3].

DEFORMATION RINGS AT $\overline{\mathbb{F}}_p$ -POINTS. — Let $k = \overline{\mathbb{F}}_p$ be an algebraic closure of \mathbb{F}_p . Let $\{\zeta_{Np^n}\}_{n \in \mathbb{N}}$ be a fixed sequence of compatible primitive Np^n -th roots of unity, set $\zeta_{p^n} = \zeta_{Np^n}^N$. Let $\check{Z}_p = W(k)$ denote the ring of integers of the p -adic completion of the maximal unramified extension of \mathbb{Q}_p . In the next paragraphs we will study the deformation rings of the modular curve at the closed points $X(Np^n)(k)$. We let $X(Np^n)_{\check{Z}_p}$ denote the compactified modular curve over \check{Z}_p of level $\Gamma(Np^n)$. Proposition 8.6.7 of [KM85] implies that $X(Np^n)_{\check{Z}_p} = X(Np^n) \times_{\text{Spec } Z_p} \text{Spec } \check{Z}_p$.

There is an isomorphism $\check{Z}_p[\mu_{Np^n}^\times] \cong \prod_{k \in (Z/NZ)^\times} \check{Z}_p[\zeta_{p^n}]$ given by fixing a primitive N -th root of unity in \check{Z}_p . Let $X(Np^n)_{\check{Z}_p}^\circ$ be the connected component of the modular curve which corresponds to the root ζ_N . In other words, $X(Np^n)_{\check{Z}_p}^\circ$ is the locus of $X(Np^n)_{\check{Z}_p}$ where $e_{Np^n}(P_{\text{univ},Np^n}, Q_{\text{univ},Np^n}) = \zeta_{Np^n}$. We denote $P_{\text{univ}}^{(n)} := NP_{\text{univ},Np^n}$ and $Q_{\text{univ}}^{(n)} := NQ_{\text{univ},Np^n}$.

Finally, given an elliptic curve E/S , we denote by \widehat{E} the completion of E along the identity section.

The ordinary points. — Let Art_k be the category of local artinian rings with residue field k , whose morphisms are the local ring homomorphisms compatible with the reduction to k . Any object in Art_k admits a unique algebra structure over \check{Z}_p . Let $\check{Z}_p[\zeta_{p^n}]\text{-Art}_k$ denote the subcategory of Art_k of objects endowed with an algebra structure of $\check{Z}_p[\zeta_{p^n}]$ compatible with the reduction to k . Following [Kat81], we use the Serre-Tate parameter to describe the deformation rings at ordinary k -points of $X(Np^n)_{\check{Z}_p}$.

Let E_0 be an ordinary elliptic curve over k and R an object in Art_k . A deformation of E_0 to R is a pair (E, ι) consisting of an elliptic curve E/R and an isomorphism $\iota : E \otimes_R k \rightarrow E_0$. We define the deformation functor $\text{Ell}_{E_0} : \text{Art}_k \rightarrow \text{Sets}$ by the rule

$$R \longmapsto \{(E, \iota) : \text{deformation of } E_0 \text{ to } R\} / \sim .$$

Then Ell_{E_0} sends an artinian ring R to the set of deformations of E_0 to R modulo isomorphism.

Let Q be a generator of the physical Tate module $T_p E_0(k) = T_p(E_0[p^\infty]^{\text{ét}})$. Let G_m be the multiplicative group over \check{Z}_p and \widehat{G}_m its formal completion along the identity. We have the following pro-representability theorem

THEOREM 1.4 ([Kat81, Th. 2.1])

(1) *The functor Ell_{E_0} is pro-representable by the formal scheme*

$$\text{Hom}_{Z_p}(T_p E_0(k) \otimes T_p E_0(k), \widehat{G}_m).$$

The isomorphism is given by the Serre-Tate parameter q , which sends a deformation E/R of E_0 to a bilinear form

$$q(E/R; \cdot, \cdot) : T_p E_0(k) \times T_p E_0(k) \longrightarrow \widehat{G}_m(R).$$

By evaluating at the fixed generator Q of $T_p E_0(k)$, we obtain the more explicit description

$$\text{Ell}_{E_0} = \text{Spf}(\check{Z}_p[[X]]),$$

where $X = q(E_{\text{univ}}/\text{Ell}_{E_0}; Q, Q) - 1$.

(2) *Let E_0 and E'_0 be ordinary elliptic curves over k , let $\pi_0 : E_0 \rightarrow E'_0$ be a homomorphism and $\pi_0^t : E'_0 \rightarrow E_0$ its dual. Let E and E' be liftings of E_0 and E'_0 to R respectively. A necessary and sufficient condition for π_0 to lift to a homomorphism $\pi : E \rightarrow E'$ is that*

$$q(E/R; \alpha, \pi^t(\beta)) = q(E'/R; \pi(\alpha), \beta)$$

for every $\alpha \in T_p E(k)$ and $\beta \in T_p E'(k)$.

We deduce the following proposition describing the ordinary deformation rings of finite level:

PROPOSITION 1.5. — *Let $x \in X(Np^n)_{\check{Z}_p}^\circ(k)$ be an ordinary point, say given by a triple (E_0, P_0, Q_0) , and write $(P_0^{(n)}, Q_0^{(n)}) = (NP_0, NQ_0)$. Let A_x denote the deformation ring of $X(Np^n)_{\check{Z}_p}^\circ$ at x . Then there is an isomorphism*

$$(1.2) \quad A_x \cong \check{Z}_p[\zeta_{p^n}][[X]][[T]] / ((1+T)^{p^n} - (1+X)) = \check{Z}_p[\zeta_{p^n}][[T]]$$

such that:

- (i) *the map (1.2) is $\check{Z}_p[\zeta_{p^n}]$ -linear.*
- (ii) *the variable $1+X$ is equal to the Serre-Tate parameter $q(E_{\text{univ}}/A_{\bar{x}}; Q, Q)$;*
- (iii) *the variable $1+T$ is equal to the Serre-Tate parameter*

$$q(E'_{\text{univ}}/A_{\bar{x}}, (\pi^t)^{-1}(Q), (\pi^t)^{-1}(Q))$$

of the universal deformation $\pi : E_{\text{univ}} \rightarrow E'_{\text{univ}}$ of the étale isogeny $\pi_0 : E_0 \rightarrow E_0/C_0$, with $C_0 = E_0[p^n]^{\text{ét}}$.

Proof. — The group scheme $E_0[N]$ is finite étale over k , which implies that a deformation of (E_0, P_0, Q_0) is equivalent to a deformation of $(E_0, P_0^{(n)}, Q_0^{(n)})$. The group $\mathrm{SL}_2(\mathbb{Z}/p^n\mathbb{Z})$ acts transitively on the set of Drinfeld bases of $E_0[p^n]$ with Weil pairing ζ_{p^n} . Without loss of generality, we can assume that $P_0^{(n)} = 0$ and that $Q_0^{(n)}$ generates $E_0[p^n](k)$, see [KM85, Th. 5.5.2]. Let E_{univ} denote the universal elliptic curve over A_x and $C \subset E_{\mathrm{univ}}[p^n]$ the subgroup generated by $Q_{\mathrm{univ}}^{(n)}$, it is an étale group lifting the étale group $C_0 = E_0[p^n]^{\mathrm{ét}}$. The base $(P_{\mathrm{univ}}^{(n)}, Q_{\mathrm{univ}}^{(n)})$ provides a splitting of the exact sequence

$$0 \longrightarrow \widehat{E}_{\mathrm{univ}} \longrightarrow E_{\mathrm{univ}}[p^n] \xrightarrow{Q_{\mathrm{univ}}^{(n)}} C_0 \longrightarrow 0.$$

Conversely, let R be an object in $\check{Z}_p[\zeta_{p^n}]\text{-Art}_k$ and E/R a deformation of E_0 . Let C be an étale subgroup of $E[p^n]$ of rank p^n . Then there exists a unique $Q^{(n)} \in C$ reducing to $Q_0^{(n)}$ modulo the maximal ideal. By Cartier duality, there is a unique $P^{(n)} \in \widehat{E}[p^n]$ such that $\epsilon(P^{(n)}, Q^{(n)}) = \zeta_{p^n}$. The pair $(P^{(n)}, Q^{(n)})$ is then a Drinfeld basis of $E[p^n]$ lifting $(P_0^{(n)}, Q_0^{(n)})$ (cf. [KM85, Prop. 1.11.2]). We have proved the equivalence of functors of $\check{Z}_p[\zeta_{p^n}]\text{-Art}_k$

$$\left\{ \begin{array}{l} \text{Deformations } E \text{ of } E_0 \text{ and} \\ \text{Drinfeld bases of } E[p^n] \\ \text{with Weil pairing } \zeta_{p^n} \end{array} \right\} / \sim \longleftrightarrow \left\{ \begin{array}{l} \text{Deformations } E \text{ of } E_0 \text{ and} \\ \text{étale subgroup } C \subset E[p^n] \text{ of rank } p^n \end{array} \right\} / \sim.$$

We also have a natural equivalence

$$\left\{ \begin{array}{l} \text{Deformations } E \text{ of } E_0 \text{ and} \\ \text{étale subgroup } C \subset E[p^n] \text{ of rank } p^n \end{array} \right\} / \sim \longleftrightarrow \left\{ \begin{array}{l} \text{Deformations of the étale isogeny} \\ \pi_0 : E_0 \rightarrow E_0/C_0 \end{array} \right\} / \sim.$$

Let $E'_{\mathrm{univ}}/\check{Z}_p[\zeta_{p^n}][[T]]$ denote the universal deformation of E_0/C_0 . The universal étale point $Q_{\mathrm{univ}}^{(n)}$ induces an étale isogeny of degree p^n over A_x

$$\pi : E_{\mathrm{univ}} \longrightarrow E'_{\mathrm{univ}}$$

lifting the quotient $\pi_0 : E_0 \rightarrow E_0/C_0$. Furthermore, the dual morphism $\pi^t : E'_{\mathrm{univ}} \rightarrow E_{\mathrm{univ}}$ induces an isomorphism of the physical Tate modules $\pi^t : T_p E'_{\mathrm{univ}}(k) \xrightarrow{\sim} T_p E_{\mathrm{univ}}(k)$. Let $Q \in T_p E_{\mathrm{univ}}(k)$ be the fixed generator, and $Q' \in T_p E'_{\mathrm{univ}}(k)$ its inverse under π^t . Theorem 1.4 implies

$$q(E_{\mathrm{univ}}; Q, Q) = q(E_{\mathrm{univ}}; Q, \pi^t(Q')) = q(E'_{\mathrm{univ}}; \pi(Q), Q') = q(E'_{\mathrm{univ}}; Q', Q')^{p^n}.$$

We obtain the isomorphism

$$A_x \cong \check{Z}_p[\zeta_{p^n}][[X, T]] / ((1 + T)^{p^n} - (1 + X)) = \check{Z}_p[\zeta_{p^n}][[T]],$$

where $X = q(E_{\mathrm{univ}}; Q, Q) - 1$ and $T = q(E'_{\mathrm{univ}}; Q', Q') - 1$.

REMARK 1.6. — Let $x \in X(Np^n)(k)$ be a closed ordinary point. The special fiber of the map $e_{Np^n} : X(Np^n) \rightarrow \text{Spec } Z_p[\mu_{Np^n}]$ is a union of Igusa curves with intersections at the supersingular points [KM85, Th. 13.10.3]. The Igusa curves are smooth over F_p [KM85, Th. 12.6.1], which implies that the deformation ring of $X(Np^n)$ at x is isomorphic to a power series ring $\check{Z}_p[[T_n]]$ (cf. discussion after [Wei13, Rem. 3.4.4]). The content of the previous proposition is the explicit relation between the variables T_n in the modular tower, see also [Lur20, Prop. 2.2].

The cusps. — Let $\text{Tate}(q)/Z_p((q))$ be the Tate curve, we recall from [KM85, Ch. 8.8] that it has j -invariant equal to

$$1/q + 744 + \dots .$$

We consider the ring $Z_p[[q]]$ as the completed stalk of $\mathbb{P}_{Z_p}^1$ at infinity. The Tate curve provides a description of the modular curve locally around the cusps, for that reason one can actually compute the formal deformation rings by means of this object, see [KM85] and [DR73]. In fact, let $\check{\text{Cusps}}[\Gamma(Np^n)]$ be the completion of the modular curve $X(Np^n)_{\check{Z}_p}$ along the cusps. From the theory developed in [KM85, Ch. 8 & 10], more precisely Theorems 8.11.10 and 10.9.1, we deduce the following proposition:

PROPOSITION 1.7. — We have an isomorphism of formal $\check{Z}_p[[q]]$ -schemes

$$\check{\text{Cusps}}([\Gamma(Np^n)]) \xrightarrow{\sim} \bigsqcup_{\Lambda \in \text{HomSurj}((Z/Np^n Z)^2, Z/Np^n Z)/\pm 1} \text{Spf}(\check{Z}_p[\zeta_{p^n}][[q^{1/Np^n}]]) .$$

The morphism $\check{\text{Cusps}}[\Gamma(Np^{n+1})] \rightarrow \check{\text{Cusps}}[\Gamma(Np^n)]$ is induced by the natural inclusion

$$\check{Z}_p[\zeta_{p^n}][[q^{1/Np^n}]] \longrightarrow \check{Z}_p[\zeta_{p^{n+1}}][[q^{1/Np^{n+1}}]]$$

on each respective connected component.

The supersingular points. — Let $(x_n \in X(Np^n)_{\check{Z}_p})_{n \in \mathbb{N}}$ be a sequence of compatible supersingular points and E_0 the elliptic curve defined over x_n . We denote by A_{x_n} the deformation ring of $X(Np^n)_{\check{Z}_p}$ at x_n . Let E_{univ}/A_{x_n} be the universal elliptic curve and $(P_{\text{univ}}^{(n)}, Q_{\text{univ}}^{(n)})$ the universal Drinfeld basis of $E_{\text{univ}}[p^n]$. We fix a formal parameter T of $\widehat{E}_{\text{univ}}$. Since x_n is supersingular, any p -power torsion point belongs to $\widehat{E}_{\text{univ}}$. We will use the following lemma as departure point:

LEMMA 1.8 ([KM85, Th. 5.3.2]). — The maximal ideal of the local ring A_{x_n} is generated by $T(P_{\text{univ}}^{(n)})$ and $T(Q_{\text{univ}}^{(n)})$.

By the Serre-Tate theorem [Kat81, Th. 1.2.1], and the general moduli theory of 1-dimensional formal groups over k [LT66], the deformation ring of $X(N)_{\check{Z}_p}$ at a supersingular point is isomorphic to $\check{Z}_p[[X]]$. Moreover, the p -multiplication modulo p can be written as $[p](T) \equiv V(T^p) \pmod p$, with $V \in k[[X]][[T]]$ the Verschiebung map $V : E_0^{(p)} \rightarrow E_0$. Without loss of generality we assume that V has the form

$$V(T) = XT + \dots + u(X)T^p + \dots ,$$

with $V(T) \equiv T^p \pmod{X}$. Using the Weierstrass preparation theorem we factorize $V(T)$ as

$$(1.3) \quad V(T) = T(X + \cdots + \tilde{u}(X)T^{p-1})(1 + XTR(X, T)),$$

where $\tilde{u}(0) = 1$ and $R \in k[[X, T]]$.

PROPOSITION 1.9. — *The parameter X is a p -power in A_{x_1}/p . Moreover, the generators $T(P_{\text{univ}}^{(n)})$ and $T(Q_{\text{univ}}^{(n)})$ of the maximal ideal of A_{x_n} are p -powers in $A_{x_{n+1}}/p$.*

Proof. — The second claim follows from the first and the equality $[p](T) \equiv V(T^p) \pmod{p}$. Consider $n = 1$ and write $P = P_{\text{univ}}^{(1)}$ and $Q = Q_{\text{univ}}^{(1)}$. Let $F : E_0 \rightarrow E_0^{(p)}$ and $V : E_0^{(p)} \rightarrow E_0$ denote the Frobenius and Verschiebung homomorphisms respectively. Using the action of $\text{GL}_2(Z/pZ)$, we can assume that P and $F(Q)$ are generators of $\ker F$ and $\ker V$ respectively (cf. [KM85, Th. 5.5.2]). We have the equality of divisors on $E_{\text{univ}}^{(p)}/(A_{x_n}/p)$

$$(1.4) \quad \ker V = \sum_{i=0}^{p-1} [i \cdot F(Q)].$$

The choice of the formal parameter T gives a formal parameter of $E_{\text{univ}}^{(p)}$ such that $T(F(Q)) = T(Q)^p$. Therefore, from (1.4) we see that the roots of $V(T)/T$ are $\{[i]^{(p)}(T(Q)^p)\}_{1 \leq i \leq p-1}$ where $[i]^{(p)}(T)$ is the i -multiplication of the formal group of $E_{\text{univ}}^{(p)}$. We obtain from (1.3)

$$\frac{X}{\tilde{u}(X)} = (-1)^{p-1} \prod_{i=1}^{p-1} ([i]^{(p)}(T(Q)^p)) = \left(\prod_{i=1}^{p-1} [i](T(Q)) \right)^p,$$

proving that $X/\tilde{u}(X)$ is a p -power in A_{x_1}/p . As $k[[X]] = k[[X/\tilde{u}(X)]]$ we are done.

COROLLARY 1.10. — *The Frobenius $\varphi : \varinjlim_n A_{x_n}/p \rightarrow \varinjlim_n A_{x_n}/p$ is surjective.*

Proof. — By induction on the graded pieces of the filtration defined by the ideal $(T(P^n), T(Q^n))$, one shows that A_{x_n}/p is in the image of the Frobenius restricted to $A_{x_{n+1}}/p$.

REMARK 1.11. — The completed local ring at a geometric supersingular point \bar{x} of $X(Np^n)$ is difficult to describe. For example, its reduction modulo p is the quotient of the power series ring $k[[X, Y]]$ by some explicit principal ideal which is written in terms of the formal group law of E at \bar{x} [KM85, Th. 13.8.4]. Weinstein gives in [Wei16] an explicit description of the deformation ring at a supersingular point of the modular curve at level $\Gamma(Np^\infty)$. In fact, Weinstein finds an explicit description of the deformation ring at infinite level of the Lubin-Tate space parametrizing 1-dimensional formal \mathcal{O}_K -modules of arbitrary height. In particular, he proves that the \mathfrak{m}_{x_0} -adic completion of the direct limit $\varinjlim_n A_{x_n}$ is a perfectoid ring. Corollary 1.10 says that the p -adic completion of $\varinjlim_n A_{x_n}$ is perfectoid, which is a slightly stronger result.

2. CONSTRUCTION OF THE PERFECTOID INTEGRAL MODEL

PERFECTOID FORMAL SPACES. — In this section we introduce a notion of perfectoid formal scheme which is already considered in [BMS18, Lem. 3.10], though not explicitly defined. We start with the affine pieces

DEFINITION 2.1. — An *integral perfectoid ring* is a topological ring R containing a non zero divisor π such that $p \in \pi^p R$, satisfying the following conditions:

- (i) the ring R is endowed with the π -adic topology. Moreover, it is separated and complete;
- (ii) the Frobenius morphism $\varphi : R/\pi R \rightarrow R/\pi^p R$ is an isomorphism.

We call π satisfying the previous conditions a *pseudo-uniformizer* of R .

REMARK 2.2. — The previous definition of integral perfectoid rings is well suited for p -adic completions of formal schemes. We do not consider the case where the underlying topology is not generated by a non-zero divisor, for example, the ring $W(\overline{\mathbb{F}}_p)[[X^{1/p^\infty}, Y^{1/p^\infty}]]$ which is the (p, X, Y) -adic completion of the ring $W(\overline{\mathbb{F}}_p)[X^{1/p^\infty}, Y^{1/p^\infty}]$. As is pointed out in [BMS18, Rem. 3.8], the notion of being integral perfectoid does not depend on the underlying topology, however to construct a formal scheme it is necessary to fix one.

Let R be an integral perfectoid ring with pseudo-uniformizer π , we attach to R the formal scheme $\mathrm{Spf} R$ defined as the π -adic completion of $\mathrm{Spec} R$. We say that $\mathrm{Spf} R$ is a perfectoid formal affine scheme. The following lemma says that the standard open subschemes of $\mathrm{Spf} R$ are perfectoid

LEMMA 2.3. — Let $f \in R$. Then $R\langle f^{-1} \rangle = \varprojlim_n R/\pi^n[f^{-1}]$ is an integral perfectoid ring.

Proof. — Let $n, k > 0$, as π is not a zero divisor we have a short exact sequence

$$0 \longrightarrow R/\pi^n \xrightarrow{\pi^k} R/\pi^{n+k} \longrightarrow R/\pi^k \longrightarrow 0.$$

Localizing at f and taking inverse limits on n we obtain

$$0 \longrightarrow R\langle f^{-1} \rangle \xrightarrow{\pi^k} R\langle f^{-1} \rangle \longrightarrow R/\pi^k[f^{-1}] \longrightarrow 0.$$

Then $R\langle f^{-1} \rangle$ is π -adically complete and π is not a zero-divisor. On the other hand, localizing at f the Frobenius map $\varphi : R/\pi \xrightarrow{\sim} R/\pi^p$ one gets

$$\varphi : R/\pi[f^{-1}] \xrightarrow{\sim} R/\pi^p[f^{-p}] = R/\pi^p[f^{-1}],$$

which proves that $R\langle f^{-1} \rangle$ is an integral perfectoid ring.

DEFINITION 2.4. — A *perfectoid formal scheme* \mathfrak{X} is a formal scheme which admits an affine cover $\mathfrak{X} = \bigcup_i U_i$ by perfectoid formal affine schemes.

Let F be equal to \mathbb{Q}_p or $\mathbb{F}_p((t))$, \mathcal{O}_F denote the ring of integers of F and ϖ be a uniformizer of \mathcal{O}_F . Let $\mathfrak{Int}\text{-}\mathfrak{Perf}_{\mathcal{O}_F}$ be the category of perfectoid formal schemes over \mathcal{O}_F whose structural morphism is adic, i.e., the category of perfectoid formal schemes $\mathfrak{X}/\mathrm{Spf} \mathcal{O}_F$ such that $\varpi\mathcal{O}_{\mathfrak{X}}$ is an ideal of definition of $\mathcal{O}_{\mathfrak{X}}$. Let $\mathcal{P}\mathrm{erf}_F$ be the category of perfectoid spaces over $\mathrm{Spa}(F, \mathcal{O}_F)$.

PROPOSITION 2.5. — *Let R be an integral perfectoid ring and π a pseudo-uniformizer. The ring $R[1/\pi]$ is a perfectoid ring in the sense of Fontaine [Fon13]. Furthermore, there is a unique “generic fiber” functor*

$$(-)_{\eta} : \mathfrak{Int}\text{-}\mathfrak{Perf}_{\mathcal{O}_F} \longrightarrow \mathcal{P}\mathrm{erf}_F$$

extending $\mathrm{Spf} R \rightarrow \mathrm{Spa}(R[1/\varpi], R^+)$, where R^+ is the integral closure of R in $R[1/\varpi]$. Moreover, given \mathfrak{X} a perfectoid formal scheme over \mathcal{O}_F , its generic fiber is universal for morphisms from perfectoid spaces to \mathfrak{X} . Namely, if \mathcal{Y} is a perfectoid space and $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}^+) \rightarrow (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is a morphism of locally and topologically ringed spaces, then there is a unique map $\mathcal{Y} \rightarrow \mathfrak{X}_{\mu}$ making the following diagram commutative

$$\begin{array}{ccc} (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}^+) & \longrightarrow & (\mathfrak{X}_{\mu}, \mathcal{O}_{\mathfrak{X}_{\mu}}^+) \\ & \searrow \quad \swarrow & \\ & (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) & \end{array}$$

REMARK 2.6. — The universal property of the functor $(-)_{\mu}$ is Huber’s characterization of the generic fiber of formal schemes in the case of perfectoid spaces, see [Hub94, Prop. 4.1].

Proof. — The first statement is [BMS18, Lem. 3.21]. For the construction of the functor, let \mathfrak{X} be a perfectoid formal scheme over \mathcal{O}_F . One can define \mathfrak{X}_{η} to be the gluing of the affinoid spaces $\mathrm{Spa}(R[1/\varpi], R^+)$ for $\mathrm{Spf} R \subset \mathfrak{X}$ an open perfectoid formal affine subscheme, this is well-defined after Lemma 2.3.

We prove the universal property of the generic fiber functor. Let $\mathcal{Y} \in \mathcal{P}\mathrm{erf}_K$ and let $f : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}^+) \rightarrow (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ be a morphism of locally and topologically ringed spaces. First, if $\mathcal{Y} = \mathrm{Spa}(S, S^+)$ is affinoid perfectoid and $\mathfrak{X} = \mathrm{Spf} R$ is perfectoid formal affine, f is determined by the global sections map $f^* : R \rightarrow S^+$. Then, there exists a unique map of affinoid perfectoid rings $f_{\eta}^* : (R[1/\varpi], R^+) \rightarrow (S, S^+)$ extending f^* . By gluing morphisms from affinoid open subsets for a general \mathcal{Y} , one gets that $\mathfrak{X}_{\eta} := \mathrm{Spa}(R[1/\varpi], R^+)$ satisfies the universal property. For an arbitrary \mathfrak{X} , one can glue the generic fibers of the open perfectoid formal affine subschemes of \mathfrak{X} .

We end this subsection with a theorem which reduces the proof of the perfectoidness of the integral modular curve at any tame level to the level $\Gamma(Np^{\infty})$.

THEOREM 2.7 (Kedlaya-Liu). — *Let A be a perfectoid ring on which a finite group G acts by continuous ring homomorphisms. Then the invariant subring A^G is a perfectoid ring. Moreover, if $R \subset A$ is an open integral perfectoid subring of A then R^G is an open integral perfectoid subring of A^G .*

Proof. — The first statement is [KL19, Th. 3.3.26]. The second statement follows from the description of open perfectoid subrings of A as p -power closed subrings of A° , i.e., open subrings of A° such that $x^p \in R$ implies $x \in R$, see [Mor17, Cor. 2.2].

THE MAIN CONSTRUCTION. — Let $\mathfrak{X}(Np^\infty)$ denote Scholze’s perfectoid modular curve [Sch15]. Let Z_p^{cyc} be the p -adic completion of the p -adic cyclotomic integers $\varinjlim_n Z_p[\mu_{p^n}]$. Let $\mathfrak{X}(Np^n)$ be the completion of $X(Np^n)$ along its special fiber. We have the following theorem.

THEOREM 2.8. — *The inverse limit $\mathfrak{X}(Np^\infty) := \varprojlim_n \mathfrak{X}(Np^n)$ is a p -adic perfectoid formal scheme, it admits a structural map to $\text{Spf } Z_p^{\text{cyc}}[\mu_N]$, and its generic fiber is naturally isomorphic to the perfectoid modular curve $\mathfrak{X}(Np^\infty)$. Furthermore, let $n > 0$, let $\text{Spec } R \subset X(Np^n)$ be an affine open subscheme, $\text{Spf } \widehat{R}$ its p -adic completion and $\text{Spf } \widehat{R}_\infty$ the inverse image in $\mathfrak{X}(Np^\infty)$. Then $\widehat{R}_\infty = (\widehat{R}_\infty[1/p])^\circ$ and*

$$(\text{Spf } \widehat{R}_\infty)_\eta = \text{Spa}(\widehat{R}_\infty[1/p], \widehat{R}_\infty).$$

REMARK 2.9. — The previous result gives a different proof of Scholze’s theorem that the generic fiber $\mathfrak{X}(Np^\infty)$ is a perfectoid space by more elementary means.

Proof. — The maps between the (formal) modular curves are finite and flat. Then $\mathfrak{X}(Np^\infty) := \varprojlim_n \mathfrak{X}(Np^n)$ is a flat p -adic formal scheme over Z_p . Fix $n_0 > 0$, let $\text{Spec } R \subset X(Np^{n_0})$ and $\text{Spf } \widehat{R} \subset \mathfrak{X}(Np^{n_0})$ be as in the theorem. For $n > n_0$, let $\text{Spec } R_n$ (resp. $\text{Spf } \widehat{R}_n$) denote the inverse image of $\text{Spec } R$ (resp. $\text{Spf } \widehat{R}$) in $\text{Spec } X(Np^n)$ (resp. $\mathfrak{X}(Np^n)$). Let $R_\infty := \varinjlim_n R_n$ and let \widehat{R}_∞ be its p -adic completion.

CLAIM. — \widehat{R}_∞ is an integral perfectoid ring, equal to $(\widehat{R}_\infty[1/p])^\circ$.

Suppose that the claim holds, it is left to show that $\mathfrak{X}(Np^\infty)_\eta$ is the perfectoid modular curve $\mathfrak{X}(Np^\infty)$. There are natural maps of locally and topologically ringed spaces

$$(\mathfrak{X}(Np^n), \mathcal{O}_{\mathfrak{X}(Np^n)}^+) \longrightarrow (\mathfrak{X}(Np^n), \mathcal{O}_{\mathfrak{X}(Np^n)}).$$

We have $\mathfrak{X}(Np^\infty) \sim \varprojlim_n \mathfrak{X}(Np^n)$, where we use the notion of tilde limit [SW13, Def. 2.4.1]. Then, by p -adically completing the inverse limit of the tower, we obtain a map of locally and topologically ringed spaces

$$(\mathfrak{X}(Np^\infty), \mathcal{O}_{\mathfrak{X}(Np^\infty)}^+) \longrightarrow (\mathfrak{X}(Np^\infty), \mathcal{O}_{\mathfrak{X}(Np^\infty)}).$$

This provides a map $f : \mathfrak{X}(Np^\infty) \rightarrow (\mathfrak{X}(Np^\infty))_\eta$. Since

$$(\text{Spf } \widehat{R}_\infty)_\eta = \text{Spa}(\widehat{R}_\infty[1/p], \widehat{R}_\infty) \sim \varprojlim_n \text{Spa}(\widehat{R}_n[1/p], \widehat{R}_n),$$

and the tilde limit is unique in the category of perfectoid spaces [SW13, Prop. 2.4.5], the map f is actually an isomorphism.

Proof of the claim. — First, by [Heu19, Lem. A.2.2.3] the ring $(\widehat{R}_n[1/p])^\circ$ is the integral closure of \widehat{R}_n in its generic fiber. By [Bha17, Lem. 5.1.2] and the fact that R_n is a regular ring one gets that $\widehat{R}_n = (\widehat{R}_n[1/p])^\circ$ for all $n > n_0$. As R_∞ is faithfully flat over R_n for all n , one easily checks that $\widehat{R}_\infty \cap \widehat{R}_n[1/p] = \widehat{R}_n$. Moreover, R_∞ is integrally closed in its generic fiber, and by Lemma 5.1.2 of loc. cit. again one obtains that \widehat{R}_∞ is integrally closed in $\widehat{R}_\infty[1/p]$. Let $x \in \widehat{R}_n[1/p]$ be power bounded in $\widehat{R}_\infty[1/p]$, then $px^s \in \widehat{R}_\infty$ for all $s \in \mathbb{N}$, in particular $\{px^s\}_{s \in \mathbb{N}} \subset \widehat{R}_n$ which implies that $x \in \widehat{R}_n$. This shows that $\varinjlim_n \widehat{R}_n$ is dense in $(\widehat{R}_\infty[1/p])^\circ$, taking p -adic completions one gets $\widehat{R}_\infty = (\widehat{R}_\infty[1/p])^\circ$.

The Weil pairings evaluated at the universal Drinfeld basis $(P_{\text{univ}, Np^n}, Q_{\text{univ}, Np^n})$ of $E[Np^n]$ induce compatible morphisms $\mathfrak{X}(Np^n) \rightarrow \text{Spf } Z_p[\mu_{Np^n}]$. Taking inverse limits one gets the structural map $\mathfrak{X}(Np^\infty) \rightarrow \text{Spf } Z_p^{\text{cyc}}[\mu_N]$. In particular, there exists $\pi \in \widehat{R}_\infty$ such that $\pi^p = pa$ with $a \in Z_p^{\text{cyc}, \times}$. To prove that \widehat{R}_∞ is integral perfectoid we need to show that the absolute Frobenius map

$$\varphi : R_\infty/\pi \longrightarrow R_\infty/p$$

is an isomorphism. The strategy is to prove this fact for the completed local rings of the stalks of $\text{Spec } R_\infty/p$ and use faithfully flat descent.

Injectivity is easy, it follows from the fact that R_∞ is integrally closed in $R_\infty[1/p]$. To show that φ is surjective, it is enough to prove that the absolute Frobenius is surjective after a profinite étale base change. Indeed, the relative Frobenius is an isomorphism for profinite étale base changes. Let $S = R \otimes_{Z_p} \check{Z}_p$ and let $\widehat{S} = \widehat{R} \widehat{\otimes}_{Z_p} \check{Z}_p$ be the p -adic completion of S . We use similar notation for $S_n = R_n \otimes_{Z_p} \check{Z}_p$, \widehat{S}_n , S_∞ and \widehat{S}_∞ . We have to show that the absolute Frobenius

$$\varphi : S_\infty/\pi \longrightarrow S_\infty/p$$

is surjective.

Let $x = (x_{n_0}, x_{n_0+1}, \dots, x_n, \dots)$ be a $\overline{\mathbb{F}}_p$ -point of $\text{Spf } \widehat{S}_\infty$ which is an inverse limit of $\overline{\mathbb{F}}_p$ -points of $\text{Spf } \widehat{S}_n$. Write x_{n_0} simply by x_0 . Then, it is enough to show that φ is surjective after taking the stalk at x . Let S_{n,x_n} be the localization of S_n at the prime x_n and $S_{\infty,x} = \varinjlim_n S_{n,x_n}$. Let \mathfrak{S}_{n,x_n} be the completion of S_{n,x_n} along its maximal ideal. Recall that the ring S_n is finite flat over S , this implies that $\mathfrak{S}_{n,x_n} = S_{n,x_n} \otimes_{S_{x_0}} \widehat{S}_{x_0}$.

The scheme $X(Np^n)$ is of finite type over Z_p , in particular every point has a closed point as specialization. Thus, by faithfully flat descent, we are reduced to prove that for every $\overline{\mathbb{F}}_p$ -point $x \in \text{Spf } S_\infty/p = \varprojlim \text{Spf } S_n/p$, the \widehat{S}_{x_0} -base change of

$$\varphi : S_{\infty,x}/\pi \longrightarrow S_{\infty,x}/p$$

is surjective (even an isomorphism). We have the following commutative diagram

$$\begin{CD} S_{\infty,x}/\pi \otimes_{S_{x_0}} \widehat{S}_{x_0} @>\varphi \otimes \text{id}>> S_{\infty,x}/p \otimes_{\varphi, S_{x_0}} \widehat{S}_{x_0} \\ @| @| \\ \varinjlim \mathfrak{S}_{n,x_n}/\pi @>>> \varinjlim (\mathfrak{S}_{n,x_n}/p \otimes_{\varphi, \widehat{S}_{x_0}} \widehat{S}_{x_0}). \end{CD}$$

The ring R_n is of finite type over Z_p , so that the absolute Frobenius $\varphi : R_n/\pi \rightarrow R_n/p$ is finite. This implies that $\mathfrak{S}_{n,x_n}/p$ is a finite $\widehat{S_{x_0}}$ -module via the module structure induced by the Frobenius. Then, the following composition is an isomorphism

$$\mathfrak{S}_{n,x_n}/p \otimes_{\varphi, \widehat{S_{x_0}}} \widehat{S_{x_0}} \longrightarrow \varprojlim_m \mathfrak{S}_{n,x_n}/(p, \mathfrak{m}_S^{mp}) \cong \mathfrak{S}_{n,x_n}/p,$$

where \mathfrak{m}_S is the maximal ideal of S_{x_0} . Thus, we are reduced to prove that the absolute Frobenius $\varphi : \varinjlim_n \mathfrak{S}_{n,x_n}/\pi \rightarrow \varinjlim_n \mathfrak{S}_{n,x_n}/p$ is surjective. Finally, we deal with the cusps, the supersingular and the ordinary points separately; we use the descriptions of Section 1:

- *In the ordinary case*, the local ring \mathfrak{S}_{n,x_n} is isomorphic to $\check{Z}_p[\zeta_{p^n}][[X_n]]$. From the proof of Proposition 1.5, one checks that the inclusion $\mathfrak{S}_{n,x_n} \rightarrow S_{n+1,x_{n+1}}$ is given by $X_n = (1 + X_{n+1})^p - 1$. Then, one obtains the surjectivity of Frobenius when reducing modulo p .

- *The supersingular case* is Corollary 1.10.

- Finally, if we are dealing with a *cusp* x , the ring \mathfrak{S}_{n,x_n} is isomorphic to $\check{Z}_p[\zeta_{p^n}][[q^{1/Np^n}]]$ and $\mathfrak{S}_{n,x_n} \rightarrow S_{n+1,x_{n+1}}$ is the natural inclusion by Proposition 1.7. The surjectivity of φ is clear.

RELATION WITH LURIE’S STACK. — In this subsection we make more explicitly the relation between Lurie’s construction of $\mathfrak{X}(Np^\infty)$ and the one presented in this article. The key result is the following theorem.

THEOREM 2.10 ([Lur20, Th. 1.9]). — *Let $\pi \in Z_p[\mu_{p^2}]$ be a pseudo-uniformizer such that $\pi^p = a\pi$ where a is a unit. For $n > 3$ there exists a unique morphism $\theta : \mathfrak{X}(Np^n)/\pi \rightarrow \mathfrak{X}(Np^{n-1})/p$ making the following diagram commutative⁽¹⁾*

$$\begin{array}{ccc} \mathfrak{X}(Np^n)/p & \xrightarrow{\varphi} & \mathfrak{X}(Np^n)/\pi \\ \downarrow & \swarrow \theta & \downarrow \\ \mathfrak{X}(Np^{n-1})/p & \xrightarrow{\varphi} & \mathfrak{X}(Np^{n-1})/\pi, \end{array}$$

where φ is the absolute Frobenius.

This theorem can be deduced from the local computations made in Section 1. Indeed, let $x_n \in \mathfrak{X}(Np^n)(\overline{\mathbb{F}}_p)$ be a $\overline{\mathbb{F}}_p$ -point and $x_{n-1} \in \mathfrak{X}(Np^{n-1})(\overline{\mathbb{F}}_p)$ its image. We have proved that there exists a unique map of the deformation rings at the points x_{n-1} and x_n

$$\theta^* : \widehat{O}_{\mathfrak{X}(Np^{n-1}),x_{n-1}}/p \longrightarrow \widehat{O}_{\mathfrak{X}(Np^n),x_n}/\pi$$

⁽¹⁾The assumption $n > 3$ is only to guarantee that $O(\mathfrak{X}(Np^{n-1}))$ contains π .

making the following diagram commutative

$$\begin{array}{ccc}
 \widehat{\mathcal{O}}_{\mathfrak{X}(Np^n), x_n}/p & \xleftarrow{\varphi^*} & \widehat{\mathcal{O}}_{\mathfrak{X}(Np^n), x_n}/\pi \\
 \uparrow & \nearrow \theta^* & \uparrow \\
 \widehat{\mathcal{O}}_{\mathfrak{X}(Np^{n-1}), x_{n-1}}/p & \xleftarrow{\varphi^*} & \widehat{\mathcal{O}}_{\mathfrak{X}(Np^{n-1}), x_{n-1}}/\pi.
 \end{array}$$

This corresponds to Propositions 1.5, 1.7 and 1.9 for x_n ordinary, a cusp and a supersingular point respectively. Then, one constructs θ using faithfully flat descent from the completed local rings to the localized local rings at x_n , and gluing using the uniqueness of θ^* .

3. COHOMOLOGY AND LOCAL DUALITY FOR CURVES OVER \mathcal{O}_K

Let K be a finite extension of \mathbb{Q}_p and \mathcal{O}_K its valuation ring. In this section we recall the Grothendieck-Serre duality theorem for local complete intersection (lci) projective curves over \mathcal{O}_K , we will follow [Har66]. Then, we use Pontryagin duality to define a local duality pairing of coherent cohomologies.

Let X be a locally noetherian scheme and $D(X)$ the derived category of \mathcal{O}_X -modules. We use subscripts c, qc on $D(X)$ for the derived category of \mathcal{O}_X -modules with coherent and quasi-coherent cohomology, the subscript fTd refers to the subcategory of complexes with finite Tor dimension. We use superscripts +, -, b for the derived category of bounded below, bounded above and bounded complexes respectively. For instance, $D_c^b(X)_{\text{fTd}}$ is the derived category of bounded complexes of \mathcal{O}_X -modules of finite Tor dimension and coherent cohomology. If $X = \text{Spec } A$ is affine, we set $D(A) := D_{\text{qc}}(X)$, the derived category of A -modules.

DEFINITION 3.1. — Let $f : X \rightarrow Y$ be a morphism of schemes.

(1) The map f is *embeddable* if it factors as $X \xrightarrow{\iota} S \rightarrow Y$ where ι is a finite morphism and S is smooth over Y .

(2) The map f is *projectively embeddable* if it factors as composition $X \xrightarrow{\iota} \mathbb{P}_Y^n \rightarrow Y$ for some $n > 0$, where ι is a finite morphism.

(3) The map f is a *local complete intersection* if locally on Y and X it factors as $X \xrightarrow{\iota} S \rightarrow Y$, where S is a smooth Y -scheme, and ι is a closed immersion defined by a regular sequence of S . The length of the regular sequence is called the codimension of X in S .

THEOREM 3.2 (Hartshorne). — *Let $f : X \rightarrow Y$ be a projectively embeddable morphism of noetherian schemes of finite Krull dimension. Then there exist an exceptional inverse image functor $f^! : D(Y) \rightarrow D(X)$, a trace map $\text{Tr} : Rf_* f^! \rightarrow 1$ in $D_{\text{qc}}^+(Y)$, and an adjunction*

$$\theta : Rf_* R\text{Hom}_X(\mathcal{F}, f^! \mathcal{G}) \xrightarrow{\sim} R\text{Hom}_Y(Rf_* \mathcal{F}, \mathcal{G})$$

for $F \in D_{\text{qc}}^-(X)$ and $G \in D_{\text{qc}}^+(Y)$.

Moreover, the formation of the exceptional inverse image is functorial. More precisely, given a composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ with f, g and gf projectively embeddable, there is a natural isomorphism $(gf)^! \cong f^!g^!$. This functor commutes with flat base change. Namely, let $u : Y' \rightarrow Y$ be a flat morphism, $f' : X' \rightarrow Y'$ the base change of f to Y' and $v : X' \rightarrow X$ the projection. Then there is a natural isomorphism of functors $v^*f^! = f'^!u^*$.

Proof. — We refer to [Har66, Th. III.8.7] for the existence of $f^!$, its functoriality and compatibility with flat base change. See Theorems III.10.5 and III.11.1 of loc. cit. for the existence of Tr and the adjunction θ respectively.

EXAMPLE 3.3. — Let $f : X \rightarrow Y$ be a morphism of finite type of noetherian schemes of finite Krull dimension.

(1) We can define the functor $f^!$ for finite morphisms as

$$f^!\mathcal{F} = f^{-1} \mathbf{R} \text{Hom}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{F}) \quad \text{for } \mathcal{F} \in \mathbf{D}(Y).$$

The duality theorem in this case is equivalent to the (derived) \otimes -Hom adjunction, see [Har66, §III.6].

(2) Let f be smooth of relative dimension n , then one has $f^!\mathcal{F} = \mathcal{F} \otimes \omega_{X/Y}^\circ[n]$ where $\omega_{X/Y}^\circ = \Lambda^n \Omega_{X/Y}^1$, see [Har66, §III.2].

LEMMA 3.4. — Let $f : X \rightarrow Y$ be an lci morphism of relative dimension n between locally noetherian schemes of finite Krull dimension. Then $f^!\mathcal{O}_Y = \omega_{X/Y}^\circ[n]$ with $\omega_{X/Y}^\circ$ an invertible \mathcal{O}_X -module.

Proof. — Working locally on Y and X , we may assume that f factors as $X \xrightarrow{\iota} S \xrightarrow{g} Y$, where g is a smooth morphism of relative dimension m , and ι is a regular closed immersion of codimension $m - n$ defined by an ideal $I = (f_1, \dots, f_{m-n})$. Let $\omega_{S/Y}^\circ = \Lambda^m \Omega_{S/Y}^1$ be the sheaf of m -differentials of S over Y , then

$$\begin{aligned} f^!\mathcal{O}_Y &= \iota^!g^!\mathcal{O}_Y \\ &= \iota^{-1} \mathbf{R} \text{Hom}_{\mathcal{O}_S}(\iota_*\mathcal{O}_X, \omega_{S/Y}^\circ[m]) \\ &= \iota^{-1} \mathbf{R} \text{Hom}_{\mathcal{O}_S}(\mathcal{O}_S/I, \omega_{S/Y}^\circ)[m]. \end{aligned}$$

Let $K(\underline{f})$ be the Koszul complex of the regular sequence $\underline{f} = (f_1, \dots, f_{m-n})$. Then $K(\underline{f})$ is a flat resolution of \mathcal{O}_S/I , its dual $K(\underline{f})^\vee = \text{Hom}_{\mathcal{O}_S}(K(\underline{f}), \mathcal{O}_S)$ is a flat resolution of $(\mathcal{O}_S/I)[- (m - n)]$. Therefore

$$\begin{aligned} f^!\mathcal{O}_Y &\simeq \iota^{-1} \text{Hom}_{\mathcal{O}_S}(K(\underline{f}), \omega_{S/Y}^\circ)[m] \\ &= \iota^{-1} K(\underline{f})^\vee \otimes \omega_{S/Y}^\circ[m] \\ &\simeq \iota^{-1}(\mathcal{O}_S/I \otimes^L \omega_{S/Y}^\circ[n]) \\ &= \iota^{-1}((\omega_{S/Y}^\circ/I)[n]) = \iota^*\omega_{S/Y}^\circ[n], \end{aligned}$$

which is an invertible sheaf of \mathcal{O}_X -modules as required.

REMARK 3.5. — Let $f : X \rightarrow Y$ be a regular closed immersion of codimension n defined by the ideal I . From the proof of Lemma 3.4 one can deduce that $f^! \mathcal{O}_Y = \Lambda^n f^*(I/I^2)^\vee[-n]$ is the normal sheaf concentrated in degree n .

The compatibility of $f^!$ with tensor products allows us to compute $f^!F$ in terms of $f^*\mathcal{F}$ and $f^! \mathcal{O}_Y$:

PROPOSITION 3.6 ([Har66, Prop. III.8.8]). — *Let $f : X \rightarrow Y$ be an embeddable morphism of locally noetherian schemes of finite Krull dimension. Then there are functorial isomorphisms*

- (1) $f^! \mathcal{F} \otimes^L Lf^* \mathcal{G} \rightarrow f^!(\mathcal{F} \otimes^L \mathcal{G})$ for $\mathcal{F} \in D_{qc}^+(Y)$ and $\mathcal{G} \in D_{qc}^b(Y)_{\text{fTd}}$.
- (2) $R Hom_X(Lf^* \mathcal{F}, f^! \mathcal{G}) \rightarrow f^!(R Hom_Y(\mathcal{F}, \mathcal{G}))$ for $\mathcal{F} \in D_c^-(Y)$ and $\mathcal{G} \in D_{qc}^+(Y)$.

Moreover, if f is an lci morphism, then $f^! \mathcal{O}_Y$ is invertible and we have $f^! \mathcal{G} \cong f^! \mathcal{O}_Y \otimes Lf^* \mathcal{G}$ for $\mathcal{G} \in D_{qc}^b(Y)_{\text{fTd}}$. We call $f^! \mathcal{O}_Y$ the dualizing sheaf of f .

We now prove the local duality theorem for vector bundles over lci projective curves:

PROPOSITION 3.7. — *Let $f : X \rightarrow \text{Spec } \mathcal{O}_K$ be an lci projective curve, and let $\omega_{X/\mathcal{O}_K}^\circ$ be the dualizing sheaf of f , i.e., the invertible sheaf such that $\omega_{X/\mathcal{O}_K}^\circ[1] = f^! \mathcal{O}_K$. Let \mathcal{F} be a locally free \mathcal{O}_X -module of finite rank, then:*

- (1) $R f_* \mathcal{F}$ is representable by a perfect complex of length $[0, 1]$;
- (2) we have a perfect pairing

$$H^0(X, \mathcal{F} \otimes K/\mathcal{O}_K) \times H^1(X, \mathcal{F}^\vee \otimes \omega_{X/\mathcal{O}_K}^\circ) \longrightarrow K/\mathcal{O}_K$$

given by the composition of the cup product and the trace $\text{Tr} : Rf_* \omega_{X/\mathcal{O}_K}^\circ \rightarrow \mathcal{O}_K$.

Proof. — As \mathcal{F} is a vector bundle and f is projective of relative dimension 1, the cohomology groups $R^i f_* \mathcal{F}$ are finitely generated over \mathcal{O}_K and concentrated in degrees 0 and 1. Then, $R f_* \mathcal{F}$ is quasi-isomorphic to a complex

$$0 \longrightarrow M_0 \xrightarrow{d} M_1 \longrightarrow 0,$$

with M_1 and M_2 finite free \mathcal{O}_K -modules. Moreover, the complex

$$0 \longrightarrow M_0 \otimes K/\mathcal{O}_K \xrightarrow{d \otimes 1} M_1 \otimes K/\mathcal{O}_K \longrightarrow 0$$

is quasi-isomorphic to $R f_*(\mathcal{F} \otimes K/\mathcal{O}_K)$ in $D(\mathcal{O}_K)$, see [Mum08, Th. 5.2].

The duality theorem 3.2 gives a quasi-isomorphism

$$R f_*(\mathcal{F}^\vee \otimes \omega_{X/\mathcal{O}_K}^\circ)[1] = R f_* R Hom_X(\mathcal{F}, f^! \mathcal{O}_K) \simeq R Hom_{\mathcal{O}_K}(R f_* \mathcal{F}, \mathcal{O}_K).$$

This implies that $R f_*(\mathcal{F}^\vee \otimes \omega_{X/\mathcal{O}_K}^\circ)$ is quasi-isomorphic to $0 \rightarrow M_1^\vee \xrightarrow{d^\vee} M_0^\vee \rightarrow 0$. Finally, Pontryagin duality for \mathcal{O}_K implies $\text{Hom}_{\mathcal{O}_K}(\ker(d \otimes 1), K/\mathcal{O}_K) = \text{coker } d^\vee$, which translates in the desired statement.

REMARK 3.8. — The previous proposition relates two notions of duality. Namely, Serre and Pontryagin duality. We can deduce the following facts:

(1) The \mathcal{O}_K -module $H^0(X, \mathcal{F} \otimes K/\mathcal{O}_K)$ is co-free of rank r , that is isomorphic to $(K/\mathcal{O}_K)^r$, if and only if $H^1(X, \mathcal{F}^\vee \otimes \omega_{X/\mathcal{O}_K}^\circ)$ is free of rank r . In that case, the module $H^0(X, \mathcal{F})$ is free and $H^0(X, \mathcal{F})/p^n \rightarrow H^0(X, \mathcal{F}/p^n)$ is an isomorphism for all $n \in \mathbb{N}$. Furthermore, Serre duality provides a perfect pairing

$$H^0(X, \mathcal{F}) \times H^1(X, \mathcal{F}^\vee \otimes \omega_{X/\mathcal{O}_K}^\circ) \longrightarrow \mathcal{O}_K.$$

(2) The \mathcal{O}_K -module $H^0(X, \mathcal{F})$ (resp. $H^1(X, \mathcal{F} \otimes K/\mathcal{O}_K)$) is free (resp. co-free) for any finite locally free \mathcal{O}_X -module.

(3) In the notation of the previous proof, Pontryagin duality implies

$$\mathrm{Hom}_{\mathcal{O}_K}(\mathrm{coker}(d \otimes 1), K/\mathcal{O}_K) = \ker d^\vee,$$

which is equivalent to a perfect pairing

$$H^1(X, \mathcal{F} \otimes K/\mathcal{O}_K) \times H^0(X, \mathcal{F}^\vee \otimes \omega_{X/\mathcal{O}_K}^\circ) \longrightarrow K/\mathcal{O}_K.$$

4. COHOMOLOGY OF MODULAR SHEAVES

Let $N > 3$ be an integer prime to p . Let $X(Np^n)$ be the modular curve over \mathbb{Z}_p of level $\Gamma(Np^n)$. Let $\check{Z}_p = W(\overline{\mathbb{F}}_p)$ and let $X(Np^n)_{\check{Z}_p}$ be the extension of scalars of $X(Np^n)$ to \check{Z}_p . We denote by $X_n := X(Np^n)_{\check{Z}_p}^\circ$ the connected component of $X(Np^n)_{\check{Z}_p}$ given by fixing the Weil pairing $e_N(P_N, Q_N) = \zeta_N$, where (P_N, Q_N) is the universal basis of $E[N]$ and $\zeta_N \in \check{Z}_p$ a primitive N -th root of unity. We also write $X = X_0$. Let $\mathcal{O}_n = \check{Z}_p[\mu_{p^n}]$ be the n -th cyclotomic extension of \check{Z}_p , $\mathcal{O}^{\mathrm{cyc}}$ the p -adic completion of $\varinjlim_n \mathcal{O}_n$, K_n and K^{cyc} the field of fractions of \mathcal{O}_n and $\mathcal{O}^{\mathrm{cyc}}$ respectively. We set $\mathcal{O} = \check{Z}_p$ and $K = \mathcal{O}[1/p]$. Let $\pi_n : X_n \rightarrow \mathrm{Spec} \mathcal{O}_n$ denote the structural map defined by the Weyl pairing of the universal basis of $E[p^n]$. We also denote $p_n : X_n \rightarrow X_{n-1}$ the natural morphism induced by p -multiplication of Drinfeld bases.

Let E^{sm}/X be the semi-abelian scheme over X extending the universal elliptic curve to the cusps, cf [DR73]. Let $e : X \rightarrow E^{\mathrm{sm}}$ be the unit section and $\omega_E := e^* \Omega_{E^{\mathrm{sm}}/X}^1$ the modular sheaf, i.e., the sheaf of invariant differentials of E^{sm} over X . For $k \in \mathbb{Z}$ we define $\omega_E^k = \omega_E^{\otimes k}$ the sheaf of modular forms of weight k , we denote by $\omega_{E,n}^k$ the pullback of ω_E^k to X_n . Let $D_n \subset X_n$ be the (reduced) cusp divisor and $\omega_{E,n,\mathrm{cusp}}^k := \omega_{E,n}^k(-D_n)$ the sheaf of cusp forms of weight k over X_n . By an abuse of notation we will also write D_n for the pullback $p_{n+1}^* D_n$ to X_{n+1} , by Proposition 1.7 we have that $D_n = pD_{n+1}$.

Finally, we let \mathfrak{X}_n be the completion of X_n along its special fiber and $\mathfrak{X}_\infty = \varprojlim_n \mathfrak{X}_n$ the integral perfectoid modular curve, see Theorem 2.8. Let \mathcal{X}_n be the analytic generic fiber of X_n and $\mathcal{X}_\infty \sim \varinjlim_n \mathcal{X}_n$ the Scholze's perfectoid modular curve.

DUALIZING SHEAVES OF MODULAR CURVES. — Consider the tower of modular curves

$$\begin{array}{ccccccc} \cdots & \longrightarrow & X_{n+1} & \xrightarrow{p_{n+1}} & X_n & \xrightarrow{p_n} & X_{n-1} & \longrightarrow & \cdots \\ & & \downarrow \pi_{n+1} & & \downarrow \pi_n & & \downarrow \pi_{n-1} & & \\ \cdots & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{n+1}) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_n) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{n-1}) & \longrightarrow & \cdots \end{array}$$

Since X_n is regular of finite type over \mathcal{O}_n , it is a local complete intersection. This implies that the sheaf $\omega_n^\circ := \pi_n^! \mathcal{O}_n$ is invertible. The modular curve X/\mathcal{O} is smooth of relative dimension 1, then we have that $\omega_0^\circ = \Omega_{X_0/\mathcal{O}}^1$, cf. Example 3.3(2). On the other hand, the Kodaira-Spencer map provides an isomorphism $KS : \omega_{E,\text{cusp}}^2 \cong \Omega_{X/\mathcal{O}}^1$.

Let $X'_{n-1} = X_{n-1} \times_{\text{Spec } \mathcal{O}_{n-1}} \text{Spec } \mathcal{O}_n$, and by an abuse of notation $p_n : X_n \rightarrow X'_{n-1}$ the induced map. Let $\pi'_{n-1} : X'_{n-1} \rightarrow \mathcal{O}_n$ be the structural map and $\text{pr}_1 : X'_{n-1} \rightarrow X_{n-1}$ the first projection. We also write $\omega_{E,n-1}^k$ for the pullback of $\omega_{E,n-1}^k$ to X'_{n-1} . Note that the compatibility of the exceptional inverse image functor with flat base change (Theorem 3.2) implies that $\pi_{n-1}^! \mathcal{O}_n \cong \text{pr}_1^* \omega_{n-1}^\circ = \omega_{n-1}^\circ \otimes_{\mathcal{O}_{n-1}} \mathcal{O}_n$.

PROPOSITION 4.1. — *There exists a natural isomorphism*

$$\xi_n : p_n^*(\omega_{n-1}^\circ)(D_{n-1} - D_n) \xrightarrow{\sim} \omega_n^\circ$$

induced by the normalized trace $\frac{1}{p} \text{Tr}_n : \mathcal{O}_{X_n} \rightarrow p_n^! \mathcal{O}_{X'_{n-1}}$. Moreover, the composition of $\xi_n \circ \dots \circ \xi_1$ with the Kodaira-Spencer map gives an isomorphism $\omega_{E,n,\text{cusp}}^2 \cong \omega_n^\circ$.

Proof. — By Proposition 3.6 we have an isomorphism

$$(4.1) \quad \xi'_n : p_n^! \mathcal{O}_{X'_{n-1}} \otimes p_n^* \omega_{n-1}^\circ \xrightarrow{\sim} p_n^! \omega_{n-1}^\circ = \omega_n^\circ.$$

The map p_n is finite flat, then $p_n^! \mathcal{O}_{X'_{n-1}} = p_n^{-1} \text{Hom}_{\mathcal{O}_{X'_{n-1}}} (p_{n,*} \mathcal{O}_{X_n}, \mathcal{O}_{X'_{n-1}})$ by Example 3.3(1). By Lemma 3.4, the sheaf $p_n^! \mathcal{O}_{X'_{n-1}}$ is invertible as X'_{n-1} is an lci projective curve. We claim that the trace $\text{Tr}_n : \mathcal{O}_{X_n} \rightarrow p_n^! \mathcal{O}_{X'_{n-1}}$ induces an isomorphism $\frac{1}{p} \text{Tr}_n : \mathcal{O}_{X_n}(D_{n-1} - D_n) \cong p_n^! \mathcal{O}_{X'_{n-1}}$. It suffices to consider the ordinary points and the cusps, indeed, the supersingular points are of codimension 2 in X_n .

Let $x \in X'_{n-1}(\overline{\mathbb{F}}_p)$ be an ordinary point. We have a cartesian square

$$(4.2) \quad \begin{array}{ccc} \bigsqcup_{x_n \mapsto x} \text{Spf } \widehat{\mathcal{O}}_{X_n, x_n} & \longrightarrow & X_n \\ \downarrow & & \downarrow p_n \\ \text{Spf } \widehat{\mathcal{O}}_{X'_{n-1}, x} & \longrightarrow & X'_{n-1}. \end{array}$$

By Proposition 1.5 we have isomorphisms

$$\widehat{\mathcal{O}}_{X'_{n-1}, x} \cong W(\overline{\mathbb{F}}_p)[\zeta_{p^n}][[T_{n-1}]], \quad \widehat{\mathcal{O}}_{X_n, x_n} \cong W(\overline{\mathbb{F}}_p)[\zeta_{p^n}][[T_n]]$$

with relations $(1+T_n)^p = 1+T_{n-1}$. Taking the different ideal of the finite flat extension $\widehat{\mathcal{O}}_{X_n, x_n} / \widehat{\mathcal{O}}_{X'_{n-1}, x}$, one finds

$$\text{Hom}_{\widehat{\mathcal{O}}_{X'_{n-1}, x}} (\widehat{\mathcal{O}}_{X_n, x_n}, \widehat{\mathcal{O}}_{X'_{n-1}, x}) \cong \frac{1}{p} \widehat{\mathcal{O}}_{X_n, x_n} \cdot \text{Tr}_n.$$

On the other hand, let $x \in X'_{n-1}(\overline{\mathbb{F}}_p)$ be a cusp. We have a cartesian square (4.2) and by Proposition 1.7 isomorphisms

$$\widehat{\mathcal{O}}_{X'_{n-1}, x} \cong W(\overline{\mathbb{F}}_p)[\zeta_{p^n}][[q^{1/p^{n-1}}]], \quad \widehat{\mathcal{O}}_{X_n, x_n} \cong W(\overline{\mathbb{F}}_p)[\zeta_{p^n}][[q^{1/p^n}]].$$

Taking the different ideal we obtain the equality

$$Hom_{\widehat{\mathcal{O}}_{X'_{n-1},x}}(\widehat{\mathcal{O}}_{X_n,x_n}, \widehat{\mathcal{O}}_{X'_{n-1},x}) \cong \frac{1}{p} q^{1/p^n - 1/p^{n-1}} \widehat{\mathcal{O}}_{X_n,x_n} \cdot \text{Tr}_n.$$

The previous computations show that the trace of $\mathcal{O}_{X_n}/\mathcal{O}_{X'_{n-1}}$ induces an isomorphism of invertible sheaves

$$\frac{1}{p} \text{Tr}_n : \mathcal{O}_{X_n}(D_{n-1} - D_n) \xrightarrow{\sim} p_n^! \mathcal{O}_{X'_{n-1}}.$$

Then, from (4.1) we have an isomorphism

$$\xi_n : \mathcal{O}_{X_n}(D_{n-1} - D_n) \otimes p_n^* \omega_{n-1}^\circ \longrightarrow \omega_n^\circ,$$

with $\xi_n = \xi'_n \circ (\frac{1}{p} \text{Tr}_n \otimes 1)$.

The isomorphism $\omega_{E,n,\text{cusps}}^2 \cong \omega_n^\circ$ follows by a straightforward induction on the composition $\xi_n \circ \dots \circ \xi_1$, and the Kodaira-Spencer map $KS : \omega_{E,\text{cusp}}^2 \cong \Omega_{X/\mathcal{O}}^1$.

LEMMA 4.2. — *Let $x \in X'_{n-1}(\overline{\mathbb{F}}_p)$ be an ordinary point or a cusp. Let $\widetilde{\text{Tr}}_n : p_{n,*} \mathcal{O}_{X_n}(D_{n-1} - D_n) \rightarrow \mathcal{O}_{X'_{n-1}}$ be the normalized trace map $\frac{1}{p} \text{Tr}_n$. Then the completed localization of $\widetilde{\text{Tr}}_n$ at x is surjective. Moreover, if \mathcal{F} is a quasi-coherent sheaf over X'_{n-1} , the composition $\mathcal{F} \rightarrow p_{n,*} p_n^* \mathcal{F} \xrightarrow{\widetilde{\text{Tr}}_n} \mathcal{F}$ is multiplication by p .*

Proof. — Localizing at x we find

$$\widetilde{\text{Tr}}_n = \bigoplus_{\substack{x_n \mapsto x}} (\frac{1}{p} \text{Tr}_n) : \bigoplus_{x_n \mapsto x} \widehat{\mathcal{O}}_{X_n,x_n} \otimes (q^{1/p^n - 1/p^{n-1}}) \longrightarrow \widehat{\mathcal{O}}_{X'_{n-1},x},$$

where $q^{1/p^n - 1/p^{n-1}}$ is invertible if x is ordinary, or a generator of D_{n-1} if it is a cusp. The explicit descriptions found in the previous proposition show that $\widetilde{\text{Tr}}_n$ is surjective on each direct summand. Finally, looking at an ordinary point x , it is clear that there are p different points x_n in the fiber of x , this implies $\widetilde{\text{Tr}}_n(1) = p$.

VANISHING OF COHERENT COHOMOLOGY. — In order to prove vanishing theorems for the coherent cohomology over the perfectoid modular curve, we first need some vanishing results at finite integral level. We have the following proposition

PROPOSITION 4.3. — *For all $n \in \mathbb{N}$ the following holds:*

- (1) $H^0(X_n, \omega_{E,n}^k \otimes_{\mathcal{O}_n} K_n/\mathcal{O}_n) = H^0(X_n, \omega_{E,n,\text{cusp}}^k \otimes_{\mathcal{O}_n} K_n/\mathcal{O}_n) = 0$ for $k < 0$.
- (2) $H^1(X_n, \omega_{E,n}^k) = H^1(X_n, \omega_{E,n,\text{cusp}}^k) = 0$ for $k > 2$.
- (3) $H^0(X_n, \mathcal{O}_{X_n}(-D_n) \otimes_{\mathcal{O}_n} K_n/\mathcal{O}_n) = H^1(X_n, \omega_{E,n}^2) = 0$ and $H^0(X_n, \mathcal{O}_{X_n} \otimes_{\mathcal{O}_n} K_n/\mathcal{O}_n) = H^1(X_n, \omega_{E,n,\text{cusp}}^2) \otimes_{\mathcal{O}_n} (K_n/\mathcal{O}_n) = K_n/\mathcal{O}_n$.

Proof. — By Propositions 3.7 and 4.1, (1) and (2) are equivalent. Similarly, by Remark 3.8(1), and Proposition 4.1, it is enough to show (3) for \mathcal{O}_{X_n} and $\mathcal{O}_{X_n}(-D_n)$.

Let ν_n be the closed point of $\text{Spec } \mathcal{O}_n$ and $\varpi \in \mathcal{O}_n$ a uniformizer, we write $\nu = \nu_0$ for the closed point of $\text{Spec } \mathcal{O}$. It suffices to prove $H^0(X_n, \omega_{E,n}^k/\varpi) = H^0(X_{n,\nu_n}, \omega_{E,n}^k) = 0$ for $k < 0$. Indeed, for $s > 1$, the short exact sequence

$$0 \longrightarrow \omega_{E,n}^k/\varpi^s \xrightarrow{\varpi} \omega_{E,n}^k/\varpi^{s+1} \longrightarrow \omega_{E,n}^k/\varpi \longrightarrow 0$$

induces a left exact sequence in global sections

$$0 \longrightarrow H^0(X_n, \omega_{E,n}^k/\varpi^s) \longrightarrow H^0(X_n, \omega_{E,n}^k/\varpi^{s+1}) \longrightarrow H^0(X_n, \omega_{E,n}^k/\varpi).$$

An inductive argument on s shows $H^0(X_n, \omega_{E,n}^k/\varpi^s) = 0$ for all $s > 1$.

Let $\lambda \in H^0(X_{n,\nu_n}, \omega_{E,n}^k)$ be non-zero. Applying the action of $SL_2(Z/p^nZ)$, we can assume that λ is non-zero in an open dense subscheme of X_{n,ν_n} . In fact, this holds for some linear combination $\sum_{\gamma \in SL_2(Z/p^nZ_p)} a_n \gamma^* \lambda$ with $a_n \in \overline{\mathbb{F}}_p$. The norm $N_{X_{n,\nu_n}/X_\nu}(\omega_{E,n}^k)$ of $\omega_{E,n}^k$ to X_ν is $\omega_{E,n}^{kd}$, where $d = \deg(X_{n,\nu_n}/X_\nu)$. Hence if $k < 0$, the sheaf $N_{X_{n,\nu_n}/X_\nu}(\omega_{E,n}^k)$ has negative degree in the smooth curve X_ν . This implies that

$$H^0(X_\nu, N_{X_{n,\nu_n}/X_\nu}(\omega_{E,n}^k)) = 0 \quad \text{and} \quad N_{X_{n,\nu_n}/X_\nu}(\lambda) = 0,$$

a contradiction. Therefore $H^0(X_{n,\nu_n}, \omega_{E,n}^k) = 0$ for $k < 0$. Since $\omega_{E,n,\text{cusp}}^k = \omega_{E,n}^k(-D_n)$, we trivially deduce $H^0(X_{n,\nu_n}, \omega_{E,n,\text{cusp}}^k) = 0$.

The results for \mathcal{O}_{X_n} and $\mathcal{O}_{X_n}(-D_n)$ are clear as X_n/\mathcal{O}_n is proper, flat, geometrically connected and has geometrically reduced fibers.

REMARK 4.4. — Strictly speaking, we can apply Proposition 3.7 only for projective curves over a finite extension of Z_p . However, as the formation of coherent cohomology is compatible with affine flat base change of the base, the conclusion of loc. cit. holds in the situation of the previous proposition.

COROLLARY 4.5. — *Let $\mathcal{F} = \omega_{E,n}^k$ or $\omega_{E,n,\text{cusp}}^k$ for $k \neq 1$, the following holds.*

(1) *The cohomology groups $H^0(X_n, \mathcal{F} \otimes K/\mathcal{O})$ and $H^1(X_n, \mathcal{F})$ are cofree and free \mathcal{O}_n -modules respectively.*

(2) *We have a perfect duality pairing*

$$H^0(X_n, \mathcal{F} \otimes K/\mathcal{O}) \times H^1(X_n, \mathcal{F}^\vee \otimes \omega_{n,\text{cusp}}^2) \longrightarrow K_n/\mathcal{O}_n.$$

Proof. — Part (2) is Proposition 3.7. Part (1) follows from Remark 3.8(1) and the previous proposition. Indeed, if $k < 0$, the vanishing of $H^0(X_n, \mathcal{F} \otimes K_n/\mathcal{O}_n)$ implies that $H^1(X_n, \mathcal{F})$ is torsion-free. As the cohomology group is of finite type over \mathcal{O}_n , it is a finite free \mathcal{O}_n -module. The other cases are proved in a similar way.

Next, we will prove some cohomological vanishing results for the modular sheaves ω_E^k and $\omega_{E,\text{cusp}}^k$ at infinite level. Particularly, we will show that the cohomology of ω_E^k over \mathfrak{X}_∞ is concentrated in degree 0 if $k > 0$. The case $k > 2$ will follow from Proposition 4.3, one can also argue directly for $k = 2$. What is remarkable is the vanishing for $k = 1$, in which case we use the perfectoid nature of \mathfrak{X}_∞ .

Let $\omega_{E,\infty}^k$ be the pullback of ω_E^k to \mathfrak{X}_∞ . Let $m > n$, note that we have an inequality of divisors $D_m \subset D_n$. Then, $\mathcal{O}_{X_m}(-D_n) \subset \mathcal{O}_{X_m}(-D_m)$, and the pullback of $\omega_{E,n,\text{cusp}}^k$ injects into $\omega_{E,m,\text{cusp}}^k$. We define $\omega_{E,\infty,\text{cusp}}^k$ as the p -adic completion of the direct limit $\varinjlim_n \omega_{E,n,\text{cusp}}^k$, if $k = 0$ we simply write $\mathcal{O}_{\mathfrak{X}_\infty}(-D_\infty)$ for $\omega_{E,\infty,\text{cusp}}^0$. The sheaf $\omega_{E,\infty,\text{cusp}}^k$ is no longer a coherent sheaf over \mathfrak{X}_∞ ; its reduction modulo p is a direct limit of line bundles which is not stationary at the cusps. One way to think about

an element in $\omega_{E,\infty,\text{cusp}}^k$ is via q -expansions: the completed localization of $\omega_{E,\infty}^k$ at a cusp $x = (x_0, x_1, \dots) \in \mathfrak{X}_\infty$ is isomorphic to

$$\mathcal{O}^{\text{cyc}}[[q^{1/p^\infty}]] := \varprojlim_s \left(\varinjlim_n \mathcal{O}^{\text{cyc}}[[q^{1/p^n}]] \right) / (p, q)^s.$$

Then, an element $f \in \omega_{E,\infty,x}^k$ can be written as a power series

$$f = \sum_{m \in \mathbb{Z}[1/p]_{\geq 0}} a_m q^m$$

satisfying certain convergence conditions. The element f belongs to the localization at x of $\omega_{E,\infty,\text{cusp}}^k$ if and only if $a_0 = 0$. For a detailed treatment of the cusps at perfectoid level we refer to [Heu20], particularly Theorem 3.17.

THEOREM 4.6. — *The following holds*

- (1) *The cohomology complexes $R\Gamma(\mathfrak{X}_\infty, \omega_{E,\infty}^k)$ and $R\Gamma(\mathfrak{X}_\infty, \omega_{E,\infty,\text{cusp}}^k)$ are concentrated in degree $[0, 1]$ for all $k \in \mathbb{Z}$.*
- (2) *For all $m, i > 0$ and $k \in \mathbb{Z}$, we have $H^i(\mathfrak{X}_\infty, \omega_{E,\infty}^k/p^m) = \varinjlim_n H^i(X_n, \omega_{E,n}^k/p^m)$ and $H^i(\mathfrak{X}_\infty, \omega_{E,\infty,\text{cusp}}^k/p^m) = \varinjlim_n H^i(X_n, \omega_{E,n,\text{cusp}}^k/p^m)$.*
- (3) *The sheaves $\omega_{E,\infty}^k$ and $\omega_{E,\infty,\text{cusp}}^k$ have cohomology concentrated in degree 0 for $k > 0$. Similarly, the sheaves $\omega_{E,\infty}^k$ and $\omega_{E,\infty,\text{cusp}}^k$ have cohomology concentrated in degree 1 for $k < 0$.*
- (4) $H^0(\mathfrak{X}_\infty, \mathcal{O}_{\mathfrak{X}_\infty}(-D_\infty)) = 0$ and $H^0(\mathfrak{X}_\infty, \mathcal{O}_{\mathfrak{X}_\infty}) = \mathcal{O}^{\text{cyc}}$.

Proof. — Let $\mathcal{F} = \omega_{E,\infty}^k$ or $\omega_{E,\infty,\text{cusp}}^k$ and $\mathcal{F}_n = \omega_{E,n}^k$ or $\omega_{E,n,\text{cusp}}^k$ respectively. We show (1) assuming part (2). By evaluating \mathcal{F} at formal affine perfectoids of \mathfrak{X}_∞ arising from finite level, one can use [Sch13, Lem. 3.18] to deduce that $\mathcal{F} = R\varprojlim_s \mathcal{F}/p^s$: the case $\mathcal{F} = \omega_{E,\infty}^k$ is clear as it is a line bundle. Otherwise, we know that

$$\mathcal{F}/p^s = \varinjlim_n \mathcal{F}_n/p^s = \varinjlim_n (\mathcal{F}_n/p^s \otimes_{X_n} \mathcal{O}_{\mathfrak{X}_\infty})$$

is a direct limit of $\mathcal{O}_{\mathfrak{X}_\infty}/p^s$ -line bundles, so that it is a quasi-coherent sheaf over \mathfrak{X}_∞ , and the system $\{\mathcal{F}/p^s\}_{s \in \mathbb{N}}$ satisfies the Mittag-Leffler condition on formal affine perfectoids. One obtains the quasi-isomorphism

$$R\Gamma(\mathfrak{X}_\infty, \mathcal{F}) = R\varprojlim_s R\Gamma(\mathfrak{X}_\infty, \mathcal{F}/p^s)$$

whose cohomology translates into short exact sequences

$$(4.3) \quad 0 \longrightarrow R^1\varprojlim_s H^{i-1}(\mathfrak{X}_\infty, \mathcal{F}/p^s) \longrightarrow H^i(\mathfrak{X}_\infty, \mathcal{F}) \longrightarrow \varprojlim_s H^i(\mathfrak{X}_\infty, \mathcal{F}/p^s) \longrightarrow 0.$$

But part (2) implies that $H^i(\mathfrak{X}_\infty, \mathcal{F}/p^s) = \varinjlim_n H^i(X_n, \mathcal{F}_n/p^s)$ for all $s \in \mathbb{N}$. As X_n is a curve over \mathcal{O}_n and \mathcal{F}_n/p^s is supported in its special fiber, we know that $H^i(X_n, \mathcal{F}_n/p^s) = 0$ for $i > 2$ and that the inverse system $\{H^1(X_n, \mathcal{F}_n/p^s)\}_{s \in \mathbb{N}}$ satisfies the ML condition. This implies that $H^i(\mathfrak{X}_\infty, \mathcal{F}/p^s) = 0$ for $i > 2$ and that the ML condition holds for $\{H^1(\mathfrak{X}_\infty, \mathcal{F}/p^s)\}_{s \in \mathbb{N}}$. From (4.3) one obtains that $H^i(\mathfrak{X}_\infty, \mathcal{F}) = 0$ for $i > 2$.

We prove part (2). Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be a finite affine cover of X , let \mathfrak{U}_n (resp. \mathfrak{U}_∞) be its pullback to X_n (resp. \mathfrak{X}_∞). As $\mathcal{F}/p^s = \varinjlim_n \mathcal{F}_n/p^s$ is a quasi-coherent $\mathcal{O}_{\mathfrak{X}_\infty}/p^s$ -module, and the (formal) schemes \mathfrak{X}_∞ and X_n are separated, we can use the Čech complex of \mathfrak{U}_n (resp. \mathfrak{U}_∞) to compute the cohomology groups. By definition we have

$$C^\bullet(\mathfrak{U}_\infty, \mathcal{F}/p^s) = \varinjlim_n C^\bullet(\mathfrak{U}_n, \mathcal{F}_n/p^s),$$

then (2) follows as filtered direct limits are exact.

The vanishing results of Proposition 4.3 imply (3) for $k < 0$ and $k > 2$. Let $k = 1, 2$ and $p^{1/p} \in \mathcal{O}^{\text{cyc}}$ be such that $|p^{1/p}| = |p|^{1/p}$. As \mathfrak{X}_∞ is integral perfectoid, the Frobenius $F : \mathfrak{X}_\infty/p \rightarrow \mathfrak{X}_\infty/p^{1/p}$ is an isomorphism. Moreover,

$$F^*(\omega_{E,\infty}^k/p^{1/p}) = \omega_{E,\infty}^{pk}/p \quad \text{and} \quad F^*(\omega_{E,\infty,\text{cusp}}^k/p^{1/p}) = \omega_{E,\infty,\text{cusp}}^{pk}/p$$

(notice that $F^*(D_n) = pD_n = D_{n-1}$). Then, Proposition 4.3(2) implies

$$(4.4) \quad H^1(\mathfrak{X}_\infty, \omega_{E,\infty}^k/p^{1/p}) \cong H^1(\mathfrak{X}_\infty, \omega_{E,\infty}^{pk}/p) = 0,$$

similarly for $\omega_{E,\infty,\text{cusp}}^k$. By induction on s , one shows that $H^1(\mathfrak{X}_\infty, \omega_{E,\infty}^k/p^s) = 0$ and that $H^0(\mathfrak{X}_\infty, \omega_{E,\infty}^k/p^{s+1}) \rightarrow H^0(\mathfrak{X}_\infty, \omega_{E,\infty}^k/p^s)$ is surjective for all $s \in \mathbb{N}$ (resp. for $\omega_{E,\infty,\text{cusp}}^k$). Taking derived inverse limits one gets

$$H^1(\mathfrak{X}_\infty, \omega_{E,\infty}^k) = H^1(\mathfrak{X}_\infty, \omega_{E,\infty,\text{cusps}}^k) = 0$$

and

$$H^0(\mathfrak{X}_\infty, \omega_{E,\infty}^k) = \varprojlim_s H^0(\mathfrak{X}_\infty, \omega_{E,\infty}^k/p^s)$$

(resp. for $\omega_{E,\infty,\text{cusp}}^k$). This proves (3) for $k = 1, 2$.

Finally, part (4) follows from part (2), Proposition 4.3(3), and the fact that

$$H^0(\mathfrak{X}_\infty, \mathcal{O}_{\mathfrak{X}_\infty}) = \varprojlim_s H^0(\mathfrak{X}_\infty, \mathcal{O}_{\mathfrak{X}_\infty}/p^s)$$

by (4.3) (resp. for $\mathcal{O}_{\mathfrak{X}_\infty}(-D_\infty)$).

COROLLARY 4.7. — *Let $\mathcal{F} = \omega_{E,\infty}^k$ or $\omega_{E,\infty,\text{cusp}}^k$ for $k \in \mathbb{Z}$. Then $H^i(\mathfrak{X}_\infty, \mathcal{F})/p^s = H^i(\mathfrak{X}_\infty, \mathcal{F}/p^s)$ and $H^i(\mathfrak{X}_\infty, \mathcal{F}) = \varprojlim_s H^i(\mathfrak{X}_\infty, \mathcal{F}/p^s)$ for all $i, s > 0$. In particular, the cohomology groups $H^i(\mathfrak{X}_\infty, \mathcal{F})$ are p -adically complete and separated. Moreover, they are all torsion-free.*

Proof. — The case $k \neq 0$ follows since the cohomology complexes $R\Gamma(\mathfrak{X}_\infty, \mathcal{F}/p^s)$ are concentrated in only one degree, and $R\Gamma(\mathfrak{X}_\infty, \mathcal{F}) = R\varprojlim_s R\Gamma(\mathfrak{X}_\infty, \mathcal{F}/p^s)$. The case $k = 0$ follows by part (4) of the previous theorem. Namely,

$$H^0(\mathfrak{X}_\infty, \mathcal{O}_{\mathfrak{X}_\infty}(-D_\infty)/p^s) = 0 \quad \text{and} \quad H^0(\mathfrak{X}_\infty, \mathcal{O}_{\mathfrak{X}_\infty}/p^s) = \mathcal{O}^{\text{cyc}}/p^s$$

for all $s > 0$. Hence, the inverse system of H^0 -cohomology groups satisfy the Mittag-Leffler condition, and the $R^1\varprojlim$ appearing in the derived inverse limit disappears for the H^1 -cohomology.

As an application of the previous vanishing theorem, we obtain vanishing results for the coherent cohomology of the perfectoid modular curve. Let $(\mathcal{X}_\infty, \mathcal{O}_{\mathcal{X}_\infty}^+) \rightarrow (\mathfrak{X}_\infty, \mathcal{O}_{\mathfrak{X}_\infty})$ be the natural map of locally and topologically ringed spaces provided by the generic fiber functor, see Proposition 2.5 and Theorem 2.8. We define

$$\omega_{E,\eta}^{k,+} := \omega_{E,\infty}^k \otimes_{\mathcal{O}_{\mathfrak{X}_\infty}} \mathcal{O}_{\mathcal{X}_\infty}^+ \quad \text{and} \quad \omega_{E,\text{cusp},\eta}^{k,+} := \omega_{E,\infty,\text{cusp}}^k \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}_\infty}} \mathcal{O}_{\mathcal{X}_\infty}^+,$$

where the completed tensor product is with respect to the p -adic topology. As usual, we denote $\mathcal{O}_{\mathcal{X}_\infty}^+(-D_\infty) = \omega_{E,\text{cusp}}^{0,+}$. In the following we consider almost mathematics with respect to the maximal ideal of \mathcal{O}^{cyc} .

COROLLARY 4.8. — *The following holds.*

- (1) *The cohomology complexes $R\Gamma_{\text{an}}(\mathcal{X}_\infty, \omega_{E,\eta}^{k,+})$ and $R\Gamma_{\text{an}}(\mathcal{X}_\infty, \omega_{E,\text{cusp},\eta}^{k,+})$ of almost \mathcal{O}^{cyc} -modules are concentrated in degrees $[0, 1]$ for all $k \in \mathbb{Z}$.*
- (2) *The sheaves $\omega_{E,\eta}^{k,+}$ and $\omega_{E,\text{cusp},\eta}^{k,+}$ have cohomology almost concentrated in degree 0 for $k > 0$. Similarly, the sheaves $\omega_{E,\eta}^{k,+}$ and $\omega_{E,\text{cusp},\eta}^{k,+}$ have cohomology almost concentrated in degree 1 for $k < 0$.*
- (3) *$H_{\text{an}}^0(\mathcal{X}_\infty, \mathcal{O}_{\mathcal{X}_\infty}^+(-D_\infty)) = 0$ and $H_{\text{an}}^0(\mathcal{X}_\infty, \mathcal{O}_{\mathcal{X}_\infty}^+) = \mathcal{O}^{\text{cyc}}$.*

Proof. — We first prove the corollary for $\mathcal{F} = \omega_{E,\infty}^k$. Let \mathcal{F}_η^+ denote the pullback of \mathcal{F} to $(\mathcal{X}_\infty, \mathcal{O}_{\mathcal{X}_\infty}^+)$. Let $\mathfrak{U} = \{\mathcal{U}_i\}_{i \in I}$ be an open cover of \mathfrak{X}_∞ given by formal affine perfectoids arising from finite level such that $\omega_{E,\infty}|_{\mathcal{U}_i}$ is trivial. By Theorem 2.8, the generic fiber $\mathcal{U}_{i,\eta}$ of \mathcal{U}_i is an open affinoid perfectoid subspace of \mathcal{X}_∞ . Let $\mathfrak{U}_\eta := \{\mathcal{U}_{i,\eta}\}_{i \in I}$, note that \mathfrak{U}_η is a covering of \mathcal{X}_∞ and that the restriction of \mathcal{F}_η^+ to $\mathcal{U}_{i,\eta}$ is trivial. By Scholze’s Almost Acyclicity Theorem for affinoid perfectoids, $\mathcal{F}_\eta^+|_{\mathcal{U}_{i,\eta}}$ is almost acyclic for all $i \in I$. The Čech-to-derived functor spectral sequence gives us an almost quasi-isomorphism

$$C^\bullet(\mathfrak{U}_\eta, \mathcal{F}_\eta^+) \simeq R\Gamma_{\text{an}}(\mathcal{X}_\infty, \mathcal{F}_\eta^+).$$

On the other hand, by the proof of Theorem 4.6 there is a quasi-isomorphism

$$C^\bullet(\mathfrak{U}, \mathcal{F}) \simeq R\Gamma(\mathfrak{X}_\infty, \mathcal{F}).$$

But by definition of \mathcal{F}_η^+ , and the fact that $\mathcal{O}_{\mathcal{X}_\infty}^+(\mathcal{U}_{i,\eta}) = \mathcal{O}_{\mathfrak{X}_\infty}(\mathcal{U}_i)$ by Theorem 2.8, we actually have an almost equality $C^\bullet(\mathfrak{U}_\eta, \mathcal{F}_\eta^+) \stackrel{\text{ae}}{=} C^\bullet(\mathfrak{U}, \mathcal{F})$. In other words, there is an almost quasi-isomorphism $R\Gamma_{\text{an}}(\mathcal{X}_\infty, \mathcal{F}_\eta^+) \stackrel{\text{ae}}{\simeq} R\Gamma(\mathfrak{X}_\infty, \mathcal{F})$.

Let $\mathcal{F}_{\text{cusp}} = \omega_{E,\infty,\text{cusp}}^k$ and $\mathcal{F}_{\text{cusp},\eta}^+$ its pullback to $(\mathcal{X}_\infty, \mathcal{O}_{\mathcal{X}_\infty}^+)$. To prove that

$$R\Gamma_{\text{an}}(\mathcal{X}_\infty, \mathcal{F}_{\text{cusp},\eta}^+) \stackrel{\text{ae}}{\simeq} R\Gamma(\mathfrak{X}_\infty, \mathcal{F}_{\text{cusp}}),$$

we argue as follows: note that we can write $\mathcal{F}_{\text{cusp}} = \mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}_\infty}} \mathcal{O}_{\mathfrak{X}_\infty}(-D_\infty)$. To apply the same argument as before we only need to show that $\mathcal{O}_{\mathcal{X}_\infty}^+(-D_\infty)$ is almost acyclic over affinoid perfectoids of \mathcal{X}_∞ . Let $V(D_\infty) \subset \mathcal{X}_\infty$ be the perfectoid closed subspace defined by the cusps. Note that $\mathcal{O}_{\mathcal{X}_\infty}(-D_\infty)$ is the ideal sheaf of $V(D_\infty)$, see the proof of [Sch15, Th. IV.2.1] or the explicit description of the completed stalks at the

cusps of the integral perfectoid modular curve. Then, we have an almost short exact sequence for all $s \in \mathbb{N}$

$$(4.5) \quad 0 \longrightarrow O_{\mathcal{X}_\infty}^+(-D_\infty)/p^s \longrightarrow O_{\mathcal{X}_\infty}^+/p^s \longrightarrow O_{V(D_\infty)}^+/p^s \longrightarrow 0.$$

As the intersection of an affinoid perfectoid of \mathcal{X}_∞ with $V(D_\infty)$ is affinoid perfectoid, and the second map of (4.5) is surjective when evaluating at affinoid perfectoids of \mathcal{X}_∞ , Scholze’s almost acyclicity implies that $O_{\mathcal{X}_\infty}^+(-D_\infty)/p^s$ is almost acyclic in affinoid perfectoids. Taking inverse limits and noticing that $\{O_{\mathcal{X}_\infty}^+(-D_\infty)/p^s\}_{s \in \mathbb{N}}$ satisfies the ML condition in affinoid perfectoids, we get that $O_{\mathcal{X}_\infty}^+(-D_\infty)$ is almost acyclic in affinoid perfectoids of \mathcal{X}_∞ . The corollary follows from the vanishing results at the level of formal schemes.

REMARK 4.9. — As it was mentioned to me by Vincent Pilloni, the cohomological vanishing of the modular sheaves at infinite level provides many different exact sequences involving modular forms and the completed cohomology of the modular tower (to be defined in the next subsection). Namely, the primitive comparison theorem permits to compute the C_p -scalar extension of the completed cohomology as $H_{\text{an}}^1(\mathcal{X}_{\infty, C_p}, O_{\mathcal{X}_\infty})$. On the other hand, the Hodge-Tate exact sequence

$$0 \longrightarrow \omega_E^{-1} \otimes_{O_X} \widehat{O}_X \longrightarrow T_p E \otimes_{\widehat{Z}_p} \widehat{O}_X \longrightarrow \omega_E \otimes_{O_X} \widehat{O}_X \longrightarrow 0$$

gives a short exact sequence over \mathcal{X}_∞

$$(4.6) \quad 0 \longrightarrow \omega_{E, \eta}^{-1} \longrightarrow O_{\mathcal{X}_{\infty, C_p}}^{\oplus 2} \longrightarrow \omega_{E, \eta} \longrightarrow 0$$

via the universal trivialization of $T_p E$. Then, taking the cohomology of (4.6) one obtains an exact sequence

$$0 \rightarrow C_p^{\oplus 2} \rightarrow H_{\text{an}}^0(\mathcal{X}_{\infty, C_p}, \omega_{E, \eta}) \longrightarrow H_{\text{an}}^1(\mathcal{X}_{\infty, C_p}, \omega_{E, \eta}^{-1}) \longrightarrow H_{\text{an}}^1(\mathcal{X}_{\infty, C_p}, O_{\mathcal{X}_\infty})^{\oplus 2} \rightarrow 0.$$

Another example is given by tensoring (4.6) with ω_E and taking cohomology. One finds

$$0 \rightarrow C_p \rightarrow H_{\text{an}}^0(\mathcal{X}_{\infty, C_p}, \omega_{E, \eta})^{\oplus 2} \longrightarrow H_{\text{an}}^0(\mathcal{X}_{\infty, C_p}, \omega_{E, \eta}^2) \longrightarrow H_{\text{an}}^1(\mathcal{X}_{\infty, C_p}, O_{\mathcal{X}_\infty}) \rightarrow 0.$$

It may be interesting a more careful study of these exact sequences.

DUALITY AT INFINITE LEVEL. — Let $\mathcal{F} = \omega_{E, \infty}^k$ or $\omega_{E, \infty, \text{cusp}}^k$ for $k \in \mathbb{Z}$, let $\mathcal{F}_n = \omega_{E, n}^k$ or $\omega_{E, n, \text{cusp}}^k$ respectively. Let C be a non archimedean field extension of K^{cyc} and \mathcal{O}_C its valuation ring. Let $\mathfrak{X}_{\infty, C}$ be the extension of scalars of the integral modular curve to \mathcal{O}_C . Corollary 4.7 says that the cohomology groups $H^i(\mathfrak{X}_\infty, \mathcal{F})$ are torsion-free, p -adically complete and separated. In particular, we can endow $H^i(\mathfrak{X}_{\infty, C}, \mathcal{F})[1/p]$ with an structure of C -Banach space with unit ball $H^i(\mathfrak{X}_{\infty, C}, \mathcal{F})$. The local duality theorem extends to infinite level in the following way.

THEOREM 4.10. — *Let \mathcal{F} and \mathcal{F}_n be as above, and let $\mathcal{F}_n^\vee = \text{Hom}_{\mathcal{O}_{X_n}}(\mathcal{F}_n, \mathcal{O}_{X_n})$ be the dual sheaf of \mathcal{F}_n . There is a $\text{GL}_2(\mathbb{Q}_p)$ -equivariant isomorphism of topological \mathcal{O}_C -modules*

$$(4.7) \quad \text{Hom}_{\mathcal{O}_C}(\mathbb{H}^i(\mathfrak{X}_{\infty, C}, \mathcal{F}), \mathcal{O}_C) \cong \varprojlim_{n, \widetilde{\text{Tr}}_n} \mathbb{H}^{1-i}(X_{n, \mathcal{O}_C}, \mathcal{F}_n^\vee \otimes \omega_{E, n, \text{cusp}}^2).$$

The LHS is endowed with the weak topology, the RHS is endowed with the inverse limit topology, $\widetilde{\text{Tr}}_n$ are the normalized traces of Proposition 4.1, and the extension of scalars is given by $X_{n, \mathcal{O}_C} = X_n \times_{\text{Spec } \mathcal{O}_n} \text{Spec } \mathcal{O}_C$.

REMARK 4.11

(1) We could restate the previous theorem using $\omega_n^\circ = \pi_n^! \mathcal{O}_n$ instead of $\omega_{E, n, \text{cusp}}^2$, the trace $\widetilde{\text{Tr}}_n$ would be replaced by the Serre duality trace relative to the morphism $X_{n+1, \mathcal{O}_C} \rightarrow X_{n, \mathcal{O}_C}$. Note that even though the ring \mathcal{O}_C is not noetherian, all the objects involved are defined as pullbacks of objects which live over a finite extension of \mathbb{Z}_p , see Remark 4.4.

(2) Let $\mathcal{F}_\eta^+ = \mathcal{F} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}_\infty}} \mathcal{O}_{\mathfrak{X}_\infty}^+$ be the pullback of \mathcal{F} to \mathcal{X}_∞ , denote $\mathcal{F}_\eta = \mathcal{F}_\eta^+[1/p]$. By Corollary 4.8 we know that

$$\mathbb{H}^i(\mathcal{X}_\infty, \mathcal{F}_\eta) = \mathbb{H}^i(\mathfrak{X}_\infty, \mathcal{F})[1/p].$$

Thus, $\mathbb{H}^i(\mathcal{X}_{\infty, C}, \mathcal{F}_\eta)$ can be endowed with an structure of C -Banach space. Its dual is given by

$$\mathbb{H}^i(\mathcal{X}_{\infty, C}, \mathcal{F}_\eta)^* = \left(\varprojlim_{m, \widetilde{\text{Tr}}_m} \mathbb{H}^{1-i}(X_{m, \mathcal{O}_C}, \mathcal{F}_m^\vee \otimes \omega_{E, m, \text{cusp}}^2) \right)[1/p].$$

(3) Let $R_n : \mathbb{Z}_p[\zeta_N]^{\text{cyc}} \rightarrow \mathbb{Z}_p[\zeta_{Np^n}]$ denote the n -th normalized Tate trace, and let X'_n be the connected component of $X(N, p^n)_{\mathbb{Z}_p[\zeta_N]}$ corresponding to ζ_N . There is a natural injective map

$$\varprojlim_{m, \widetilde{\text{Tr}}_m} \mathbb{H}^{1-i}(X'_{m, \mathbb{Z}_p[\zeta_N]^{\text{cyc}}}, \mathcal{F}_m^\vee \otimes \omega_{E, m, \text{cusp}}^2) \longrightarrow \varprojlim_{n, R_n \circ \widetilde{\text{Tr}}_n} \mathbb{H}^{1-i}(X'_n, \mathcal{F}_n^\vee \otimes \omega_{E, n, \text{cusp}}^2).$$

However, this map is not surjective in general; the RHS is profinite while the LHS is not compact.

Before proving Theorem 4.10 let us say some words about the inverse limit of (4.7), it can be described as the kernel of the map

$$\prod_n \mathbb{H}^{1-i}(X_{n, \mathcal{O}_C}, \mathcal{F}_n^\vee \otimes \omega_{E, n, \text{cusp}}^2) \xrightarrow{1 - \widetilde{\text{Tr}}_n} \prod_n \mathbb{H}^{1-i}(X_{n, \mathcal{O}_C}, \mathcal{F}_n^\vee \otimes \omega_{E, n, \text{cusp}}^2).$$

Moreover, Corollary 4.7 says that the factors in the products are p -adically complete, separated and torsion-free. The following lemma implies that the inverse limit is always p -adically complete and separated

LEMMA 4.12. — *Let N, M be torsion-free, p -adically complete and separated \mathbb{Z}_p -modules, and $f : N \rightarrow M$ a \mathbb{Z}_p -linear map. Then $\ker f$ is torsion-free, p -adically complete and separated.*

Proof. — It is clear that $\ker f$ is torsion-free. The map f is continuous for the p -adic topology, in particular $\ker f \subset N$ is a closed sub-module. Since M is torsion-free, one has that $\ker f \cap p^s N = p^s \ker f$ for all $s > 1$. Then,

$$\ker f = \varprojlim_s (\ker f / (\ker f \cap p^s N)) = \varprojlim_s \ker f / p^s \ker f,$$

proving the lemma.

Next, we recall the $\mathrm{GL}_2(\mathbb{Q}_p)$ -action in both sides of (4.7). Without loss of generality we take $C = K^{\mathrm{cyc}}$. Let $\chi : \mathrm{Gal}(\mathcal{O}^{\mathrm{cyc}}/\mathcal{O}) \xrightarrow{\sim} \mathbb{Z}_p^\times$ be the cyclotomic character. We define $\psi : \mathrm{GL}_2(\mathbb{Q}_p) \rightarrow \mathrm{Gal}(\mathcal{O}^{\mathrm{cyc}}/\mathcal{O})$ to be $\psi(g) = \chi^{-1}(p^{-v_p(\det g)} \det g)$, where $v_p : \mathbb{Q}_p^* \rightarrow \mathbb{Z}$ denotes the p -adic valuation. Fix $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ and $n > 0$. Let $m > 1$ be such that $\Gamma(p^m) \subset \Gamma(p^n) \cap g\Gamma(p^n)g^{-1}$, write $c_g : \mathrm{GL}_2(\mathbb{Q}_p) \rightarrow \mathrm{GL}_2(\mathbb{Q}_p)$ for the conjugation $x \mapsto gxg^{-1}$. We denote by $X(Np^n)_{c(g)}$ be the modular curve of level $\Gamma(N) \cap \Gamma(p^n) \cap g\Gamma(p^n)g^{-1}$, let $X_{n,c(g)}$ be the locus where the Weil pairing of the universal basis of $E[N]$ is equal to $\zeta_N \in \check{\mathbb{Z}}_p$. We let ω° be the dualizing sheaf of $X_{n,c(g)}$, i.e., the exceptional inverse image of $\mathcal{O}_{n,c(g)} := H^0(X_{n,c(g)}, \mathcal{O}_{X_{n,c(g)}})$ over $X_{n,c(g)}$.

The maps

$$\Gamma(p^m) \hookrightarrow \Gamma(p^n) \cap g\Gamma(p^n)g^{-1} \xrightarrow{c_{g^{-1}}} g^{-1}\Gamma(p^n)g \cap \Gamma(p^n) \hookrightarrow \Gamma(p^n)$$

induce maps of modular curves

$$X_m \xrightarrow{q_1} X_{n,c(g)} \xrightarrow{g} X_{n,c(g^{-1})} \xrightarrow{q_2} X_n,$$

with g an isomorphism. Notice that the modular sheaves ω_E^k are preserved by the pullbacks of q_1, q_2 and g . Let \mathcal{F} and \mathcal{F}_n be as in Theorem 4.10, we have induced maps of cohomology

$$R\Gamma(X_n, \mathcal{F}_n/p^s) \xrightarrow{q_1^* \circ g^* \circ q_2^*} R\Gamma(X_m, \mathcal{F}_m/p^s).$$

Taking direct limits we obtain a map

$$R\Gamma(\mathfrak{X}_\infty, \mathcal{F}/p^s) \xrightarrow{g^*} R\Gamma(\mathfrak{X}_\infty, \mathcal{F}/p^s).$$

Finally, taking derived inverse limits one gets the action of $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ on the cohomology $R\Gamma(\mathfrak{X}_\infty, \mathcal{F})$.

The action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on cohomology is not $\mathcal{O}^{\mathrm{cyc}}$ -linear. In fact, it is ψ -semi-linear; this can be shown by considering the Cartan decomposition

$$\mathrm{GL}_2(\mathbb{Q}_p) = \bigsqcup_{n_1 > n_2} \mathrm{GL}_2(\mathbb{Z}_p) \begin{pmatrix} p^{n_1} & 0 \\ 0 & p^{n_2} \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_p)$$

and using the compatibility of the Weil pairing with the determinant.

The action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\varprojlim_{n, \Gamma_n} H^{1-i}(X_{n, \mathcal{O}^{\mathrm{cyc}}}, \mathcal{F}_n^\vee \otimes \omega_{E,n, \mathrm{cusp}}^2)$ is defined in such a way that the isomorphism (4.7) is equivariant. Namely, there is a commutative

diagram of local duality pairings provided by the functoriality of Serre duality (4.8)

$$\begin{array}{ccc}
 H^{1-i}(X_{m,\mathcal{O}^{\text{cyc}}}, \mathcal{F}_m^\vee \otimes \omega_{E,m,\text{cusp}}^2) \times H^i(X_{m,\mathcal{O}^{\text{cyc}}}, \mathcal{F}_m \otimes K/\mathcal{O}) & \longrightarrow & K^{\text{cyc}}/\mathcal{O}^{\text{cyc}} \\
 \downarrow \widetilde{\text{Tr}}_{q_1} & & q_1^* \uparrow \\
 H^{1-i}(X_{n,c(g),\mathcal{O}^{\text{cyc}}}, (g \circ q_2)^* \mathcal{F}_n^\vee \otimes \omega^\circ) \times H^i(X_{n,c(g),\mathcal{O}^{\text{cyc}}}, (g \circ q_2)^* \mathcal{F}_n \otimes K/\mathcal{O}) & \longrightarrow & K^{\text{cyc}}/\mathcal{O}^{\text{cyc}} \\
 \downarrow g^{-1*} & & g^* \uparrow \\
 H^{1-i}(X_{n,c(g^{-1}),\mathcal{O}^{\text{cyc}}}, q_2^* \mathcal{F}_n^\vee \otimes \omega^\circ) \times H^i(X_{n,c(g^{-1}),\mathcal{O}^{\text{cyc}}}, q_2^* \mathcal{F} \otimes K/\mathcal{O}) & \longrightarrow & K^{\text{cyc}}/\mathcal{O}^{\text{cyc}} \\
 \downarrow \widetilde{\text{Tr}}_{q_2} & & q_2^* \uparrow \\
 H^{1-i}(X_{n,\mathcal{O}^{\text{cyc}}}, \mathcal{F}_n^\vee \otimes \omega_{E,n,\text{cusp}}^2) \times H^i(X_{n,\mathcal{O}^{\text{cyc}}}, \mathcal{F}_n \otimes K/\mathcal{O}) & \longrightarrow & K^{\text{cyc}}/\mathcal{O}^{\text{cyc}}.
 \end{array}$$

The maps $\widetilde{\text{Tr}}_{q_1}$ and $\widetilde{\text{Tr}}_{q_2}$ are induced by the Serre duality traces of q_1 and q_2 respectively, cf. Remark 4.11(1). Thus, the right action of $g \in \text{GL}_2(\mathbb{Q}_p)$ on a tuple $\underline{f} = (f_n) \in \varprojlim_{n, \widetilde{\text{Tr}}_n} H^{1-i}(X_{n,\mathcal{O}^{\text{cyc}}}, \mathcal{F}_n^\vee \otimes \omega_{E,n,\text{cusp}}^2)$ is given by $\underline{f}|_g = ((f|_g)_n)_{n \in \mathbb{N}}$, where

$$(f|_g)_n = \widetilde{\text{Tr}}_{q_2} \circ g^{-1*} \circ \widetilde{\text{Tr}}_{q_1}(f_m)$$

for m big enough, and q_1, q_2 as in (4.8).

Proof of Theorem 4.10. — Without loss of generality we take $C = K^{\text{cyc}}$. Let $\mathcal{F} = \omega_{E,\infty}^k$ or $\omega_{E,\infty,\text{cusp}}^k$. By Corollary 4.7 we have

$$H^i(\mathfrak{X}_\infty, \mathcal{F}) \otimes (K/\mathcal{O}) = H^i(\mathfrak{X}_\infty, \mathcal{F} \otimes K/\mathcal{O}).$$

Therefore

$$\begin{aligned}
 \text{Hom}_{\mathcal{O}^{\text{cyc}}} (H^i(\mathfrak{X}_\infty, \mathcal{F}), \mathcal{O}^{\text{cyc}}) &= \text{Hom}_{\mathcal{O}^{\text{cyc}}} (H^i(\mathfrak{X}_\infty, \mathcal{F}) \otimes K/\mathcal{O}, K^{\text{cyc}}/\mathcal{O}^{\text{cyc}}) \\
 &= \text{Hom}_{\mathcal{O}^{\text{cyc}}} (H^i(\mathfrak{X}_\infty, \mathcal{F} \otimes K/\mathcal{O}), K^{\text{cyc}}/\mathcal{O}^{\text{cyc}}).
 \end{aligned}$$

On the other hand, we have

$$H^i(\mathfrak{X}_\infty, \mathcal{F} \otimes K/\mathcal{O}) = \varprojlim_{n, p_n^*} H^i(X_{n,\mathcal{O}^{\text{cyc}}}, \mathcal{F}_n \otimes K/\mathcal{O}),$$

where the transition maps are given by pullbacks. By local duality, Proposition 3.7, we have a natural isomorphism

$$\begin{aligned}
 \text{Hom}_{\mathcal{O}^{\text{cyc}}} (H^i(\mathfrak{X}_\infty, \mathcal{F}), \mathcal{O}^{\text{cyc}}) &= \varprojlim_{n, p_n^*} \text{Hom}_{\mathcal{O}^{\text{cyc}}} (H^i(X_{n,\mathcal{O}^{\text{cyc}}}, \mathcal{F}_n \otimes K/\mathcal{O}), K^{\text{cyc}}/\mathcal{O}^{\text{cyc}}) \\
 &= \varprojlim_{n, \widetilde{\text{Tr}}_n} H^{1-i}(X_{n,\mathcal{O}^{\text{cyc}}}, \mathcal{F}_n^\vee \otimes \omega_{E,n,\text{cusp}}^2).
 \end{aligned}$$

The isomorphism is $\text{GL}_2(\mathbb{Q}_p)$ -equivariant by the diagram (4.8).

We end this section with an application of the local duality theorem at infinite level to the completed cohomology. We let $\mathcal{X}_{n,\text{pro-ét}}$ be the pro-étale site of the finite level modular curve as in [Sch13, §3], and $\mathcal{X}_{\infty,\text{pro-ét}}$ the pro-étale site of the perfectoid modular curve as in [SW20, Lect. 8].

DEFINITION 4.13. — Let $i > 0$. The i -th completed cohomology group of the modular tower $\{\mathcal{X}_n\}_{n>0}$ is defined as

$$\tilde{H}^i := \varprojlim_s \varinjlim_n H_{\text{ét}}^i(\mathcal{X}_{n, C_p}, Z/p^s Z).$$

REMARK 4.14. — The previous definition of completed cohomology is slightly different from the one of [Eme06]. Indeed, Emerton consider the étale cohomology with compact support of the affine modular curve Y_n . Let $j : Y_n \rightarrow X_n$ be the inclusion and $\iota : D_n \rightarrow X_n$ be the cusp divisor, both constructions are related by taking the cohomology of the short exact sequence

$$0 \rightarrow j_!(Z/p^s Z) \rightarrow Z/p^s Z \rightarrow \iota_* \iota^* Z/p^s Z \rightarrow 0.$$

Moreover, the cohomology at the cusps can be explicitly computed, and many interesting cohomology classes already appear in \tilde{H}^1 .

We recall some important completed sheaves in the pro-étale site. Let \mathcal{W} denote \mathcal{X}_n or \mathcal{X}_∞

– We denote $\widehat{Z}_p = \varprojlim_s Z/p^s Z$, the p -adic completion over $\mathcal{W}_{\text{pro-ét}}$ of the locally constant sheaf Z .

– Let $\widehat{O}_{\mathcal{W}}^+ = \varprojlim_s O_{\mathcal{W}}^+/p^s$ be the p -adic completion of the structural sheaf of bounded functions over $\mathcal{W}_{\text{pro-ét}}$.

By [Sch13, Lem. 3.18] the sheaf $\widehat{O}_{\mathcal{W}}^+$ is the derived inverse limit of the projective system $\{O_{\mathcal{W}}^+/p^s\}_s$. On the other hand, the repleteness of the pro-étale site and [BS14, Prop. 3.1.10] implies that \widehat{Z}_p is also the derived inverse limit of $\{Z/p^s Z\}_s$. We have the following proposition

PROPOSITION 4.15. — Let $i > 0$, there is a short exact sequence

$$0 \rightarrow R^1 \varprojlim_s H_{\text{ét}}^{i-1}(\mathcal{X}_{\infty, C_p}, Z/p^s Z) \rightarrow H_{\text{pro-ét}}^i(\mathcal{X}_{\infty, C_p}, \widehat{Z}_p) \rightarrow \tilde{H}^i \rightarrow 0.$$

Proof. — As $\widehat{Z}_p = R \varprojlim_s Z/p^s Z$, the Grothendieck spectral sequence for derived limits gives short exact sequences for $i > 0$

$$\begin{aligned} 0 \rightarrow R^1 \varprojlim_s H_{\text{pro-ét}}^{i-1}(\mathcal{X}_{\infty, C_p}, Z/p^s Z) &\rightarrow H_{\text{pro-ét}}^i(\mathcal{X}_{\infty, C_p}, \widehat{Z}_p) \\ &\rightarrow \varprojlim_s H_{\text{pro-ét}}^i(\mathcal{X}_{\infty, C_p}, Z/p^s Z) \rightarrow 0. \end{aligned}$$

Lemma 3.16 of [Sch13] implies that $H_{\text{pro-ét}}^i(\mathcal{X}_{\infty, C_p}, Z/p^s Z) = H_{\text{ét}}^i(\mathcal{X}_{\infty, C_p}, Z/p^s Z)$. On the other hand, [Sch12, Cor. 7.18] gives an isomorphism

$$H_{\text{ét}}^i(\mathcal{X}_{\infty, C_p}, Z/p^s Z) = \varinjlim_n H_{\text{ét}}^i(\mathcal{X}_{n, C_p}, Z/p^s Z),$$

the proposition follows.

Next, we relate the completed cohomologies \tilde{H}^i with the coherent cohomology of \mathcal{X}_∞ via the Primitive Comparison Theorem. This strategy is the same as the one presented by Scholze in [Sch15, Ch. IV] for Emerton’s completed cohomology. In the following we work with the almost-setting with respect to the maximal ideal of \mathcal{O}_{C_p} .

PROPOSITION 4.16 ([Sch15, Th. IV.2.1]). — *There are natural almost isomorphisms*

$$(4.9) \quad \tilde{H}^i \widehat{\otimes}_{Z_p} \mathcal{O}_{C_p} = {}^{\text{ae}} H^i_{\text{pro-ét}}(\mathcal{X}_{\infty, C_p}, \widehat{\mathcal{O}}_{\mathcal{X}_{\infty}}^+) = {}^{\text{ae}} H^i(\mathfrak{X}_{\infty}, \mathcal{O}_{\mathfrak{X}_{\infty}}) \widehat{\otimes}_{\mathcal{O}_{\text{cyc}}} \mathcal{O}_{C_p}.$$

In particular, $\tilde{H}^i = 0$ for $i > 2$, the $\mathbb{R}^1 \varprojlim_s$ of Proposition 4.15 vanishes, and the \tilde{H}^i are torsion-free, p -adically complete and separated.

Proof. — By the Primitive Comparison Theorem [Sch13, Th. 5.1], there are almost quasi-isomorphisms for all $n, s, i \in \mathbb{N}$

$$H^i_{\text{ét}}(\mathcal{X}_{n, C_p}, Z/p^s Z) \otimes_{Z_p} \mathcal{O}_{C_p} = {}^{\text{ae}} H^i_{\text{ét}}(\mathcal{X}_{n, C_p}, \mathcal{O}_{\mathcal{X}_n}^+ / p^s).$$

Taking direct limits on n , and using [Sch12, Cor., 7.18] one gets

$$(4.10) \quad H^i_{\text{ét}}(\mathcal{X}_{\infty, C_p}, Z/p^s Z) \otimes_{Z_p} \mathcal{O}_{C_p} = {}^{\text{ae}} H^i_{\text{ét}}(\mathcal{X}_{\infty, C_p}, \mathcal{O}_{\mathcal{X}_{\infty}}^+ / p^s).$$

Namely, we have $\mathcal{O}_{\mathcal{X}_{\infty}}^+ / p^s = \varinjlim_n \mathcal{O}_{\mathcal{X}_n}^+ / p^s$ as sheaves in the étale site of \mathcal{X}_{∞} . In fact, let U_{∞} be an affinoid perfectoid in the étale site of \mathcal{X}_{∞} which factors as a composition of rational localizations and finite étale maps. By [Sch12, Lem. 7.5] there exists $n_0 > 0$ and an affinoid space $U_{n_0} \in \mathcal{X}_{n_0, \text{ét}}$ such that $U_{\infty} = \mathcal{X}_{\infty} \times_{\mathcal{X}_{n_0}} U_{n_0}$. For $n > n_0$ denote the pullback of U_{n_0} to $\mathcal{X}_{n, \text{ét}}$ by U_n , then $U_{\infty} \sim \varprojlim_{n > n_0} U_n$ and $\mathcal{O}^+(U_{\infty}) / p^s = \varinjlim_{n > n_0} \mathcal{O}^+(U_n) / p^s$.

The sheaf $\mathcal{O}_{\mathcal{X}_{\infty}}^+ / p^s$ is almost acyclic on affinoid perfectoids, this implies that the RHS of (4.10) is equal to $H^i_{\text{an}}(\mathcal{X}_{\infty, C_p}, \mathcal{O}_{\mathcal{X}_{\infty}}^+ / p^s)$. Then, the proof of Corollary 4.8 allows us to compute the above complex using the formal model \mathfrak{X}_{∞}

$$(4.11) \quad H^i_{\text{an}}(\mathcal{X}_{\infty, C_p}, \mathcal{O}_{\mathcal{X}_{\infty}}^+ / p^s) = {}^{\text{ae}} H^i(\mathfrak{X}_{\infty}, \mathcal{O}_{\mathfrak{X}_{\infty}} / p^s) \otimes_{\mathcal{O}_{\text{cyc}}} \mathcal{O}_{C_p}.$$

Corollary 4.7 shows that the inverse system $\{H^i(\mathfrak{X}_{\infty}, \mathcal{O}_{\mathfrak{X}_{\infty}} / p^s)\}_s$ satisfy the Mittag-Leffler condition. As \mathcal{O}_{C_p} / p^s is a faithfully flat $Z/p^s Z$ -algebra, the inverse system $\{H^i_{\text{ét}}(\mathcal{X}_{\infty, C_p}, Z/p^s Z)\}_s$ also satisfies the Mittag-Leffler condition. One deduces from Proposition 4.15 that

$$(4.12) \quad H^i_{\text{pro-ét}}(\mathcal{X}_{\infty, C_p}, \widehat{Z}_p) = \tilde{H}^i.$$

We also obtain that $\tilde{H}^i / p^s = H^i_{\text{ét}}(\mathcal{X}_{\infty, C_p}, Z/p^s Z)$ for all $i \in \mathbb{N}$. Taking inverse limits in (4.10), and using (4.11) and (4.12) one obtains the corollary.

We obtain a description of the dual of the completed cohomology in terms of cuspidal modular forms of weight 2:

THEOREM 4.17. — *There is a $\text{GL}_2(\mathbb{Q}_p)$ -equivariant isomorphism of almost \mathcal{O}_{C_p} -modules*

$$\text{Hom}_{\mathcal{O}_{C_p}}(\tilde{H}^1 \widehat{\otimes}_{Z_p} \mathcal{O}_{C_p}, \mathcal{O}_{C_p}) = {}^{\text{ae}} \varprojlim_{n, \text{Tr}_n} H^0(X_{n, \mathcal{O}_{C_p}}, \omega_{E, n, \text{cusp}}^2).$$

Proof. — This is a consequence of Proposition 4.16 and the particular case of Theorem 4.10 when $\mathcal{F} = \mathcal{O}_{\mathfrak{X}_{\infty}}$ and $C = C_p$.

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