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Jean-Jérôme CASANOVA, Céline GRANDMONT, & Matthieu HILLAIRET

On an existence theory for a fluid-beam problem encompassing possible contacts

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ON AN EXISTENCE THEORY FOR A FLUID-BEAM PROBLEM ENCOMPASSING POSSIBLE CONTACTS

BY JEAN-JÉRÔME CASANOVA, CÉLINE GRANDMONT
& MATTHIEU HILLAIRET

ABSTRACT . — In this paper we consider a coupled system of pdes modeling the interaction between a two-dimensional incompressible viscous fluid and a one-dimensional elastic beam located on the upper part of the fluid domain boundary. We design a functional framework to define weak solutions in case of contact between the elastic beam and the bottom of the fluid cavity. We then prove that such solutions exist globally in time regardless a possible contact by approximating the beam equation by a damped beam and letting this additional viscosity vanish.

RÉSUMÉ (Existence de solution autorisant d'éventuels contacts pour un problème d'interaction fluide-structure)

Dans cet article, nous considérons un système couplé d'équations aux dérivées partielles modélisant l'interaction entre un fluide visqueux incompressible bi-dimensionnel et une poutre élastique mono-dimensionnelle située sur le bord supérieur du domaine fluide. Après avoir construit un cadre fonctionnel de solutions faibles autorisant les configurations où la poutre est en contact avec le fond de la cavité fluide, l'existence de solutions faibles, globale en temps, est démontrée, que des contacts se produisent ou non. La preuve repose sur l'analyse asymptotique d'un système couplé parabolique-parabolique pour lequel un terme de viscosité est ajouté à la structure, et dont on sait qu'il n'autorise pas les contacts. La limite de viscosité évanescence est alors solution de la formulation faible introduite et autorisant le contact.

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1. INTRODUCTION

In this paper we consider a fluid–structure system coupling a 2D homogeneous viscous incompressible fluid with a 1D elastic structure. When the elastic structure is at rest, the fluid domain is of rectangular type and the structure is located on the

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upper part of the fluid domain boundary. The fluid is described by the Navier–Stokes equations set in an unknown domain depending on the structure displacement that is assumed to be only transverse and that satisfies a beam equation. Since the fluid is viscous it sticks to the boundaries so that the fluid and the structure velocities are equal at the interface. Finally, the fluid applies a surface force on the structure. Such coupled nonlinear models can be viewed as toy models to describe the blood flow through large arteries.

The existence of a solution to the Cauchy problem associated with this kind of systems has been intensively studied in the last years. In [3, 11, 12] existence and uniqueness of a strong solution locally in time is proved in case additional viscosity is added to the structure equation (so that the structure displacement satisfies a damped Euler–Bernoulli equation). When no viscosity is added and in case the dynamics of the structure displacement is governed by a membrane equation, existence and uniqueness of a local strong solution is obtained in [8]. The beam case with no additional viscosity is investigated in [2], where existence of strong solution locally in time (or for small data) is proved but with a gap between the regularity of the initial conditions and the propagated regularity of the structure displacement. Existence of weak solutions is obtained in [4] for 3D-2D coupling where the structure behaviour is described by a viscous plate equation and in [6, 14] in the non-viscous case. Let us also mention weak existence results on fluid–shell models [10, 15]. Note that these results are obtained as long as the structure does not touch the bottom of the fluid cavity (or, in case of shells, as long as there is no self contact). More recently, in [7], the authors establish existence of a global-in-time strong solution in the 2D-1D case when the structure is governed by a damped Euler–Bernoulli equation. This global-in-time result is a consequence of a no contact one: it is proved therein that, for any $T > 0$, the structure does not touch the bottom of the cavity. The proof of this latter result relies strongly on the additional viscosity in the beam equation and on the control of the curvature of the structure.

The question we address here is: can we prove existence of a global weak solution regardless of a possible contact (for an undamped beam)? We aim to take advantage of the existence of global strong solution for a viscous structure and let the additional viscosity tend to zero. Our scheme is inspired by the one developed in [16] where the global existence of a weak solution is derived for a 2D fluid–solid coupled problem. However, in [16] the solids are viewed as inclusions whose viscosities are infinite. The fluid–solid problem is then approximated by a completely fluid problem with different viscosities in the inclusions and in the fluid. The viscosity of the inclusions is then sent to infinity. In contrast, in our case the parabolic–hyperbolic fluid–structure system is approximated by a parabolic–parabolic one by adding viscosity to the structure. We prove that, up to the extraction of a subsequence, the sequence of solutions of the damped system converges towards a weak solution (in a sense to be defined) of the undamped system. The main difficulties are to define functional and variational frameworks compatible with a possible contact and to prove the strong compactness of the velocities, also in case of a possible contact. Indeed the proof developed for

instance in [6], where the vanishing viscosity limit is also studied, strongly relies on the fact that the elastic structure does not touch the bottom of the fluid cavity.

1.1. THE FLUID-STRUCTURE MODEL. — We introduce now the damped coupled fluid–structure system. We refer to this system as $(FS)_\gamma$, where the subscript γ is used to track the dependency with respect to the “viscosity” of the structure. The configuration “at rest” of the fluid–structure system is assumed to be of the form $(0, L) \times (0, 1)$ where the elastic structure occupies the part of the boundary $(0, L) \times \{1\}$. The deformed fluid set is denoted by $F_{h_\gamma(t)}$. It depends on the structure vertical deformation $h_\gamma = 1 + \eta_\gamma$, where η_γ denotes the elastic vertical displacement. Thus, the deformed fluid configuration reads:

$$(1.1) \quad F_{h_\gamma(t)} = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < L, 0 < y < h_\gamma(t, x)\}.$$

The deformed elastic configuration is denoted by

$$\Gamma_{h_\gamma(t)} = \{(x, y) \in \mathbb{R}^2 \mid x \in (0, L), y = h_\gamma(t, x)\}.$$

The fluid velocity \mathbf{u}_γ and the fluid pressure p_γ satisfy the 2-D incompressible Navier–Stokes equations in the fluid domain:

$$(1.2) \quad \begin{aligned} \rho_f(\partial_t \mathbf{u}_\gamma + (\mathbf{u}_\gamma \cdot \nabla) \mathbf{u}_\gamma) - \operatorname{div} \sigma(\mathbf{u}_\gamma, p_\gamma) &= 0 \text{ in } F_{h_\gamma(t)}, \\ \operatorname{div} \mathbf{u}_\gamma &= 0 \text{ in } F_{h_\gamma(t)}, \end{aligned}$$

where $\sigma(\mathbf{u}_\gamma, p_\gamma)$ denotes the fluid stress tensor given by the Newton law:

$$\sigma(\mathbf{u}_\gamma, p_\gamma) = \mu(\nabla \mathbf{u}_\gamma + (\nabla \mathbf{u}_\gamma)^T) - p_\gamma I_2.$$

In the previous equations $\rho_f > 0$ and $\mu > 0$ are respectively the fluid density and viscosity. The structure displacement η_γ satisfies a damped Euler–Bernoulli beam equation:

$$(1.3) \quad \rho_s \partial_{tt} \eta_\gamma - \beta \partial_{xx} \eta_\gamma - \gamma \partial_{xx} \partial_t \eta_\gamma + \alpha \partial_x^4 \eta_\gamma = \phi(\mathbf{u}_\gamma, p_\gamma, \eta_\gamma) \text{ on } (0, L).$$

The constant $\rho_s > 0$ denotes the structure density and α, β, γ are non negative parameters. Through this paper we assume that $\alpha > 0$. This restriction guarantees sufficient regularity of the structure deformation in the compactness argument. The reader shall note for instance that we need $H^{1+\kappa} \cap W^{1,\infty}$ regularity in space for the deformation in Lemma 13.

The source term ϕ in the right-hand side of the beam equation arises from the action–reaction principle between the fluid and the structure. It represents the force applied by the fluid on the structure. It can be defined by the variational identity

$$(1.4) \quad \int_0^L \phi(\mathbf{u}_\gamma, p_\gamma, \eta_\gamma) \cdot \varphi(x, h_\gamma(t, x)) \mathbf{e}_2 \, dx = \int_{\Gamma_{h_\gamma(t)}} \sigma(\mathbf{u}_\gamma, p_\gamma) \mathbf{n}_\gamma \cdot \varphi|_{\Gamma_{h_\gamma(t)}} \mathbf{e}_2 \, d\Gamma_{h_\gamma(t)},$$

for any regular test function φ and where \mathbf{n}_γ denotes the unit exterior normal to the deformed interface:

$$\mathbf{n}_\gamma = \frac{1}{\sqrt{1 + (\partial_x \eta_\gamma)^2}} \begin{pmatrix} -\partial_x \eta_\gamma \\ 1 \end{pmatrix}.$$

Since the fluid is viscous the following kinematic condition holds true at the interface

$$(1.5) \quad \mathbf{u}_\gamma(t, x, h_\gamma(t, x)) = \partial_t \eta_\gamma(t, x) \mathbf{e}_2 \text{ on } (0, L) \times (0, T).$$

We complement the fluid and structure boundary conditions with

$$(1.6) \quad \begin{aligned} \mathbf{u}_\gamma &= 0 \text{ on } (0, L) \times \{0\}, \\ \eta_\gamma \text{ and } \mathbf{u}_\gamma &\text{ are } L\text{-periodic with respect to } x. \end{aligned}$$

Note that the kinematic condition (1.5) together with the incompressibility constraint of the fluid velocity imply that, by taking into account the boundary conditions (1.6),

$$(1.7) \quad \int_0^L \partial_t \eta_\gamma(t, x) \, dx = 0.$$

This condition states that the volume of the fluid cavity is preserved. It implies that the pressure p_γ is uniquely determined in contrast with classical fluid–solid interaction problems. Finally the fluid–structure system is completed with the following initial conditions

$$(1.8) \quad \begin{aligned} \eta_\gamma(0) &= \eta_\gamma^0 \text{ and } \partial_t \eta_\gamma(0) = \eta_\gamma^1 \text{ in } (0, L), \\ \mathbf{u}_\gamma(0) &= \mathbf{u}_\gamma^0 \text{ in } F_{h_\gamma^0} \text{ with } h_\gamma^0 = 1 + \eta_\gamma^0. \end{aligned}$$

REMARK 1. — As already noted in [4], due to the incompressibility constraint and the only transverse displacement of the beam

$$((\nabla \mathbf{u}_\gamma)^T \cdot \mathbf{n}_\gamma)_2 = 0, \text{ on } \Gamma_{h_\gamma(t)}.$$

It implies that the force applied by the fluid on the beam can be defined as follows

$$\int_0^L \phi(\mathbf{u}_\gamma, p_\gamma, \eta_\gamma) \cdot \varphi(x, h_\gamma(t, x)) \mathbf{e}_2 \, dx = \int_{\Gamma_{h_\gamma(t)}} (\nabla \mathbf{u}_\gamma - p_\gamma I_2) \mathbf{n}_\gamma \cdot \varphi|_{\Gamma_{h_\gamma(t)}} \mathbf{e}_2 \, d\Gamma_{h_\gamma(t)}.$$

For the same reason, a Korn equality also holds true

$$\int_{F_{h_\gamma(t)}} |\nabla \mathbf{u}_\gamma + (\nabla \mathbf{u}_\gamma)^T|^2 = 2 \int_{F_{h_\gamma(t)}} |\nabla \mathbf{u}_\gamma|^2.$$

The fluid–structure system (1.2)–(1.8) is denoted by $(\text{FS})_\gamma$ and $(\text{FS})_0$ corresponds to the system with $\gamma = 0$ for which we are going to prove the existence of a global weak solution. The case where $\gamma > 0$ is the one considered in [7]. It is proved therein that the structure does not touch the bottom of the fluid cavity, namely $\min_{x \in (0, L)} (1 + \eta_\gamma(t, x)) > 0$, for all t , implying the existence of a unique global strong solution. In the case $\gamma = 0$, it is proved in [6, 14] that a weak solution exists as long as the structure does not touch the bottom of the fluid cavity. In this paper, we investigate the vanishing viscosity limit (i.e., $\gamma \rightarrow 0$) and prove the convergence, up to the extraction of a subsequence, of the sequence of strong solutions $(\mathbf{u}_\gamma, \eta_\gamma)$ solutions of $(\text{FS})_\gamma$ defined on any time interval $(0, T)$ towards (\mathbf{u}, η) a weak solution (to be properly defined later on) of $(\text{FS})_0$. Note that at the limit we loose the no contact property and have only: $\min_{x \in (0, L)} (1 + \eta(t, x)) > 0$, for all t . One key issue is thus to define an appropriate framework in case of contact. Moreover, and as it is standard for this kind of fluid–structure coupled problem, another important difficulty comes from the obtention of strong compactness of the approximate velocities. Such a property is mandatory in order to pass to the limit in the convective terms.

To conclude this introductory part, we point out that we do not address here the uniqueness of solutions. One reason is the lack of contact dynamics that should be added in case of contact. In particular, for our definition of weak solutions, the structure velocity in contact points is not defined, moreover, the associated test-functions are required to have support far from the contact points (see Remark 22). Hence, it is likely that our definition of weak solution below allows various rebounds of the elastic structure in case of contact (and consequently various solutions), one (or several) of these solutions coming from the construction process we consider herein. This issue is now well-identified in the fluid–solid framework [17].

The rest of the paper splits into two sections. In the next section, we introduce and analyze a functional setting, we give the definition of weak solutions and state the main result. The last section is devoted to the proof of the existence result following standard steps: construction of a sequence of approximate solutions, derivation of compactness properties, passage to the limit. In the appendix, detailed proof of a technical lemma is given.

2. PROBLEM SETTING

In this section we first recall the energy estimates satisfied by any regular enough solution of the coupled problem. We then construct functional spaces and introduce a notion of weak solution relying on these energy estimates and that are compatible with a contact. Finally, we provide the rigorous statement of our main result and some technical lemma necessary to the following analysis.

2.1. ENERGY ESTIMATES. — Let $\gamma > 0$ and assume that $(\mathbf{u}_\gamma, \eta_\gamma)$ is a classical solution to $(\text{FS})_\gamma$. Let then multiply the first equation of Navier–Stokes system (1.2) by the fluid velocity \mathbf{u}_γ and integrate over $F_{h_\gamma(t)}$. Let also multiply the beam equation (1.3) with the structure velocity $\partial_t \eta_\gamma$ and integrate over $(0, L)$. By adding these two contributions, after integration by parts in space—and by taking into account the coupling conditions (definition (1.4) of ϕ and the kinematic condition (1.5)), the boundary conditions (1.6) together with the incompressibility constraint and Remark 1—we obtain

$$(2.1) \quad \frac{1}{2} \frac{d}{dt} \left(\rho_f \int_{F_{h_\gamma(t)}} |\mathbf{u}_\gamma(t, \mathbf{x})|^2 d\mathbf{x} + \rho_s \int_0^L |\partial_t \eta_\gamma(t, x)|^2 dx \right. \\ \left. + \beta \int_0^L |\partial_x \eta_\gamma(t, x)|^2 dx + \alpha \int_0^L |\partial_{xx} \eta_\gamma(t, x)|^2 dx \right) \\ + \gamma \int_0^L |\partial_{tx} \eta_\gamma(t, x)|^2 dx + \mu \int_{F_{h_\gamma(t)}} |\nabla \mathbf{u}_\gamma(t, \mathbf{x})|^2 d\mathbf{x} = 0.$$

Note that we have used here that the set $F_{h_\gamma(t)}$ moves with the velocity field \mathbf{u}_γ thanks to the equality of velocities at the interface (1.5), that implies

$$\int_{F_{h_\gamma(t)}} (\partial_t \mathbf{u}_\gamma(t, \mathbf{x}) + (\mathbf{u}_\gamma(t, \mathbf{x}) \cdot \nabla) \mathbf{u}_\gamma(t, \mathbf{x})) \cdot \mathbf{u}_\gamma(t, \mathbf{x}) d\mathbf{x} = \frac{1}{2} \frac{d}{dt} \int_{F_{h_\gamma(t)}} |\mathbf{u}_\gamma(t, \mathbf{x})|^2 d\mathbf{x}.$$

For $t > 0$, integrating (2.1) over $(0, t)$ leads to

$$\begin{aligned}
 (2.2) \quad & \frac{1}{2} \left(\rho_f \int_{F_{h_\gamma(t)}} |\mathbf{u}_\gamma(t, \mathbf{x})|^2 \, d\mathbf{x} + \rho_s \int_0^L |\partial_t \eta_\gamma(t, x)|^2 \, dx \right. \\
 & \quad + \beta \int_0^L |\partial_x \eta_\gamma(t, x)|^2 \, dx + \alpha \int_0^L |\partial_{xx} \eta_\gamma(t, x)|^2 \, dx \Big) \\
 & \quad + \gamma \int_0^t \int_0^L |\partial_{tx} \eta_\gamma(s, x)|^2 \, dx \, ds + \mu \int_0^t \int_{F_{h_\gamma(s)}} |\nabla \mathbf{u}_\gamma(s, \mathbf{x})|^2 \, d\mathbf{x} \, ds \\
 & = \frac{1}{2} \left(\rho_f \int_{F_{h_\gamma^0}} |\mathbf{u}_\gamma^0|^2 \, d\mathbf{x} + \rho_s \int_0^L |\eta_\gamma^1|^2 \, dx + \beta \int_0^L |\partial_x \eta_\gamma^0|^2 \, dx + \alpha \int_0^L |\partial_{xx} \eta_\gamma^0|^2 \, dx \right).
 \end{aligned}$$

As a consequence, we observe that, if $(\mathbf{u}_\gamma^0, \eta_\gamma^1, \eta_\gamma^0)$ are such that the right-hand side of (2.2) is uniformly bounded with respect to the viscosity parameter $\gamma > 0$, we have in particular

$$\eta_\gamma \text{ is uniformly bounded in } L^\infty(0, T; H_\#^2(0, L)) \cap W^{1,\infty}(0, T; L_\#^2(0, L)),$$

where the subscript $\#$ denotes spaces of periodic functions with respect to x . Thus the associated interface displacements $(\eta_\gamma)_{\gamma>0}$ are uniformly bounded at least in $C^0([0, T]; C_\#^1(0, L))$ thanks to the compact embedding

$$(2.3) \quad L^\infty(0, T; H_\#^2(0, L)) \cap W^{1,\infty}(0, T; L_\#^2(0, L)) \hookrightarrow C^{0,1-s}([0, T]; C_\#^{1,2s-3/2}(0, L)),$$

that holds for $3/4 < s < 1$. Then, there exists $M > 0$ depending on the initial data and independent of γ such that

$$(2.4) \quad 0 \triangleleft 1 + \eta_\gamma(t, x) \triangleleft M, \quad \forall (t, x) \in [0, T] \times [0, L], \quad \forall \gamma > 0.$$

Finally, to define our functional setting, we rely below on the assumption that the initial data $(\mathbf{u}^0, \eta^1, \eta^0)$ associated to $(\text{FS})_0$ do satisfy the assumption that the right-hand side of (2.2) is finite (for $\gamma = 0$). So that we have at-hand an upper bound $M > 0$ for the structure deformation $h = 1 + \eta$ for any physically reasonable solution. The above computations show also that, up to a good choice of regularized initial data the same functional framework can be used to describe the solutions to the damped system $(\text{FS})_\gamma$ (for $\gamma > 0$).

2.2. FUNCTIONAL SPACES. — We design now a functional framework compatible with possible contact between the structure and the bottom of the fluid cavity. The parameter $M > 0$ is fixed in the whole construction.

Given a non-negative function $h \in C_\#^1(0, L)$ such that $0 \triangleleft h \triangleleft M$ we recall that we denote:

$$F_h = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < L, 0 < y < h(x)\}.$$

In case h vanishes two crucial difficulties appear. First, the set F_h does not remain connected (see Figure 1, the domain below the graph splits into a connected component between the red dots and a connected component outside the red dots). In particular, if h is the deformation of a structure associated with a solution (\mathbf{u}, p, η)

to $(FS)_0$, we may expect that the condition (1.7) must be satisfied on each time-dependent connected component of the subset $\{x \in (0, L) \mid h(x) > 0\}$ and not only globally on $(0, L)$. Secondly, the boundaries of F_h contain at least one “cusp” so that it does not satisfy the cone property (see [1]). As a consequence, one must be careful in order to define a trace operator on $H^1(F_h)$.

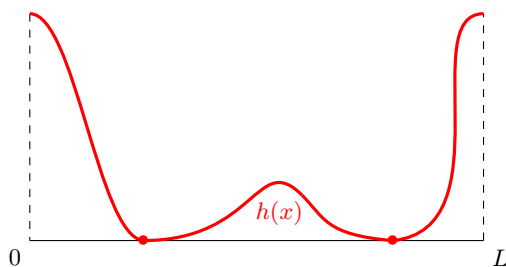


FIGURE 1. Example of a set with two “cusps”.

To overcome the second difficulty, we adapt the construction done in the context of fluid–solid problems in [16]. Namely, we extend the fluid velocity fields—by taking into account their trace on the structure—on a time–independent domain whose regularity does not suffer from possible contacts.

First, let us make precise some specific notations for the various domains used in the analysis. We introduce a virtual container $\Omega = (0, L) \times (-1, 2M)$. This set contains a part of the substrate $((x, y) \in (0, L) \times (-1, 0))$, the fluid film $((x, y) \in F_h)$ and a virtual medium containing an extension of the structure (what remains of Ω). Correspondingly, we also introduce three kinds of subsets of Ω . Given a continuous positive function h we define first a subgraph domain (containing the substrate and the fluid film)

$$F_h^- = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < L, -1 < y < h(x)\},$$

then the epigraph domain (corresponding to the virtual elastic medium)

$$S_h = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < L, h(x) < y < 2M\}.$$

Finally, for the analysis, we need also more general sets. Given $a, b : (0, L) \rightarrow \mathbb{R}$ such that $a \subset b$, we also define the set

$$\Omega_a^b = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < L, a(x) < y < b(x)\}.$$

We emphasize that there is some overlap between these notations. In particular, Ω, F_h, F_h^-, S_h can be seen as particular cases of sets of the form Ω_a^b .

For the study of non cylindrical time-dependent problems, we also need notations for space-time domains. We use the convention that notations for time-independent domains extend to the time-dependent case by adding a hat. More precisely,

we denote $\widehat{\Omega} = (0, T) \times \Omega$ and

$$\begin{aligned} \widehat{F}_h &= \bigcup_{t \in (0, T)} \{t\} \times F_{h(t)}, & \widehat{F}_h^- &= \bigcup_{t \in (0, T)} \{t\} \times F_{h(t)}^-, \\ \widehat{S}_h &= \bigcup_{t \in (0, T)} \{t\} \times S_{h(t)}, & \widehat{\Omega}_a^b &= \bigcup_{t \in (0, T)} \{t\} \times \Omega_{a(t)}^{b(t)}, \end{aligned}$$

where $h, a, b : (0, T) \times (0, L) \rightarrow \mathbb{R}$ are such that $h(t, x) > 0$ and $a(t, x) \leq b(t, x)$ for all $(t, x) \in (0, T) \times (0, L)$. We will also denote by \mathbf{n}_h the vector

$$\mathbf{n}_h = \frac{1}{\sqrt{1 + (\partial_x h)^2}} \begin{pmatrix} -\partial_x h \\ 1 \end{pmatrix}.$$

With these notations for the different sets, we introduce functional spaces to which our weak solutions will belong. The definition of these spaces is based on the following construction. Let us first introduce an extension operator:

DEFINITION 2. — Assume that $h \in C_{\sharp}^0(0, L)$ with $0 \leq h \leq M$. Let $\mathbf{v} \in \mathbf{L}_{\sharp}^2(F_h)$ and $d \in L_{\sharp}^2(0, L)$, we define the extension operator by

$$\bar{\mathbf{v}} = \begin{cases} d\mathbf{e}_2, & \text{in } S_h, \\ \mathbf{v}, & \text{in } F_h, \\ \mathbf{0}, & \text{in } \Omega_{-1}^0. \end{cases}$$

REMARK 3. — By construction, this extension operator defines a vector field $\bar{\mathbf{v}} \in \mathbf{L}_{\sharp}^2(\Omega)$. In the previous definition the used symbol $\bar{\mathbf{v}}$ involves only \mathbf{v} while the construction depends also on d . In what follows, this choice is justified as we consider functions \mathbf{v} and d satisfying the relation $\mathbf{v}|_{y=h} = d\mathbf{e}_2$, where $\mathbf{v}|_{y=h}$ denotes the function $x \mapsto \mathbf{v}(x, h(x))$ on $(0, L)$.

More precisely, when there is no contact this extension operator enjoys the following properties:

LEMMA 4. — Assume that $h \in W_{\sharp}^{1, \infty}(0, L)$ with $0 < h(x) \leq M$ for $x \in [0, L]$ and let $s \in (0, 1)$.

(1) If $s > 1/2$ and $\mathbf{v} \in \mathbf{H}_{\sharp}^s(F_h)$ is divergence free with $\mathbf{v}|_{y=0} = \mathbf{0}$, and $\mathbf{v}|_{y=h} = d\mathbf{e}_2$ with $d \in H_{\sharp}^s(0, L)$, we have that

$$\bar{\mathbf{v}} \in \mathbf{H}^s(\Omega), \quad \operatorname{div} \bar{\mathbf{v}} = 0 \text{ in } \Omega, \quad \bar{\mathbf{v}} \cdot \mathbf{e}_1 = 0 \text{ in } S_h.$$

(2) If $0 \leq s < 1/2$ and $\mathbf{v} \in \mathbf{H}_{\sharp}^s(F_h)$ is divergence free with $\mathbf{v} \cdot \mathbf{e}_2 = 0$ on $y = 0$, and $\mathbf{v}|_{y=h} \cdot \mathbf{n}_h = (0, d)^T \cdot \mathbf{n}_h$ on $(0, L)$ with $d \in H_{\sharp}^s(0, L)$, we have that:

$$\bar{\mathbf{v}} \in \mathbf{H}^s(\Omega), \quad \operatorname{div} \bar{\mathbf{v}} = 0 \text{ in } \Omega, \quad \bar{\mathbf{v}} \cdot \mathbf{e}_1 = 0 \text{ in } S_h.$$

(3) In both cases $0 \leq s < 1/2$ and $s > 1/2$ the extension operator is a bounded linear mapping of its arguments whose norm can be bounded with respect to M only:

$$\|\bar{\mathbf{v}}\|_{\mathbf{H}^s(\Omega)} \leq C(M) (\|d\|_{H_{\sharp}^s(0, L)} + \|\mathbf{v}\|_{\mathbf{H}^s(F_h)}).$$

Proof. — We note that in both cases $\bar{\mathbf{v}}$ is by construction piecewisely divergence free and belongs to \mathbf{H}^s (in the sets S_h, F_h, Ω_{-1}^0). Consequently, in case (2) the extension is straightforwardly in $\mathbf{H}^s(\Omega)$. Only the continuity of normal traces is required to yield a global divergence free vector field. In Case (1) we require continuity of the full trace to obtain an $\mathbf{H}^s(\Omega)$ vector field.

REMARK 5

(i) In the case $\min_{x \in [0, L]} h(x) > 0$, we may extend vector fields defined on F_h^- with a similar bar-operator. Then, similar results for this extension operator hold true.

(ii) In Lemma 4 and in what follows, in order to avoid to denote the trace by the classical symbol γ , which is reserved here to the added viscosity on the structure, we denote by $\mathbf{v}|_{y=h}$ the trace of \mathbf{v} defined as $\mathbf{v}|_{y=h}(x) = \mathbf{v}(x, h(x))$. We note that when $h(x) > 0$ for all $x \in [0, L]$, the associated linear trace operator is well-defined from $H_{\sharp}^1(F_h)$ into $H_{\sharp}^{1/2}(0, L)$. In the case where $h(x) > 0$ for all $x \in [0, L]$ it is well-defined from $H_{\sharp}^1(F_h^-)$ into $H_{\sharp}^{1/2}(0, L)$. It is easy to verify that

$$(2.5) \quad \|\mathbf{v}|_{y=h}\|_{H_{\sharp}^{1/2}(0, L)} \hookrightarrow C(\|h\|_{W_{\sharp}^{1, \infty}(0, L)})\|\mathbf{v}\|_{H_{\sharp}^1(F_h^-)}.$$

Consequently, for a $W_{\sharp}^{1, \infty}(0, L)$ -function h satisfying $0 \hookrightarrow h(x) \hookrightarrow M$, for $x \in [0, L]$ and for $s \in (0, 1)$, we set

$$(2.6) \quad K^s[h] = \{\mathbf{v} \in \mathbf{H}_{\sharp}^s(\Omega) \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} = 0 \text{ in } \Omega_{-1}^0, \mathbf{v} \cdot \mathbf{e}_1 = 0 \text{ in } S_h\},$$

$$(2.7) \quad X^s[h] = \{(\mathbf{w}, d) \in K^s[h] \times (H_{\sharp}^{2s}(0, L) \cap L_{\sharp, 0}^2(0, L)) \mid w_{2|(0, L) \times \{M\}} = d\},$$

where

$$L_{\sharp, 0}^2(0, L) = \{d \in L_{\sharp}^2(0, L) \mid \int_0^L d = 0\}.$$

When $s = 0$ we denote $K[h] = K^0[h]$ and $X[h] = X^0[h]$.

Under the assumptions of Lemma 4 we have that $\bar{\mathbf{v}} \in K^s[h]$ and $(\bar{\mathbf{v}}, d) \in X^s[h]$ in both cases $s \in (0, 1/2)$ and $s \in (1/2, 1)$. We emphasize that, for any $\mathbf{v} \in K[h]$, the divergence free condition implies that the trace on $(0, L) \times \{M\}$ of $v_2 = \mathbf{v} \cdot \mathbf{n}$ has a sense in $H_{\sharp}^{-1/2}(0, L)$. Similarly $\mathbf{v}|_{y=h} \cdot \mathbf{n}_h$ also makes sense in $H_{\sharp}^{-1/2}(0, L)$. Following the construction of the extension operator above, one expects this trace to represent the structure velocity.

Correspondingly, we introduce smooth variants of these functional spaces $\mathcal{K}[h]$ and $\mathcal{X}[h]$ defined by

$$(2.8) \quad \mathcal{K}[h] = \{\mathbf{w} \in C_{\sharp}^{\infty}(\Omega) \mid \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega, \mathbf{w} = 0 \text{ in } V(\Omega_{-1}^0), \\ \mathbf{w} \cdot \mathbf{e}_1 = 0 \text{ in } V(S_h)\},$$

$$(2.9) \quad \mathcal{X}[h] = \{(\mathbf{w}, d) \in \mathcal{K}[h] \times C_{\sharp}^{\infty}(0, L) \mid w_{2|(0, L) \times \{M\}} = d\}.$$

Here, we used “in $V(O)$ ” as a shortcut for the statement “in a neighbourhood of the open set O ”. Notations with calligraphic fonts will be involved in the definition of weak solutions as spaces for test-functions.

Before defining the weak solutions, we now verify that the previous coupled spaces encode the fluid–structure nature of the problem and behave correctly (from an analytical standpoint). Once again, h stands for a non-negative $W^{1,\infty}$ -function satisfying $0 \subset h \subset M$. The space $X[h]$ is endowed with the scalar product

$$(2.10) \quad \langle (\mathbf{u}, \dot{\eta}), (\mathbf{w}, d) \rangle_{X[h]} := \rho_f \int_{\Omega} \mathbf{u} \cdot \mathbf{w} + \rho_s \int_0^L \dot{\eta} d,$$

and we endow the spaces $X^s[h]$ with a Hilbert structure associated with the norms

$$\|(\mathbf{w}, d)\|_{X^s[h]} = \|\mathbf{w}\|_{H^s(\Omega)} + \|d\|_{H^{2s}(0,L)}.$$

For $s = 0$ this Hilbert-norm does not correspond to the scalar product as defined in (2.10) but the topologies are equivalent since ρ_f and ρ_s are both positive.

In order to prove the fluid–structure property, we show in the following lemma that, in the “virtual medium”, the velocity-fields in $X[h]$ coincide with a structure velocity.

LEMMA 6. — *Let $\mathbf{v} \in K[h]$. There exists $d \in L^2_{\sharp}(0, L)$ such that $\mathbf{v} = d\mathbf{e}_2$ in S_h .*

Proof. — By definition we have $\mathbf{v} = (0, v_2)^{\top}$ in S_h . Moreover, the divergence condition $\operatorname{div} \mathbf{v} = \partial_y v_2 = 0$ in S_h , implies that $v_2(x, y) = v_2(x)$ in S_h . Since $\mathbf{v} \in L^2_{\sharp}(\Omega)$ and $0 \subset h(x) \subset M, \forall x \in [0, L]$, we have $v_2 \in L^2_{\sharp}(0, L)$.

Given a divergence free $\mathbf{w} \in L^2_{\sharp}(\Omega)$ it is classical that we can construct a stream function $\Psi \in H^1_{\sharp}(\Omega)$ such that $\mathbf{w} = \nabla^{\perp} \Psi$. We show in the following lemma some additional properties satisfied by the stream function of an extended-field in $K[h]$:

LEMMA 7. — *Let $(\mathbf{w}, d) \in X[h]$ and set $I = \{x \in [0, L] \mid h(x) > 0\}$. There exists $\Psi \in H^1_{\sharp}(\Omega)$ such that $\mathbf{w} = \nabla^{\perp} \Psi$ which furthermore satisfies*

- $\Psi(x, y) = b(x)$ in S_h with $b \in H^1_{\sharp}(0, L) \cap H^1_0(I)$ and $d = \partial_x b$;
- $\Psi = 0$ in Ω^0_{-1} ;
- $\Psi = 0$ in $I^c \times (-1, 2M)$.

Proof. — We note that Ψ is defined up to an additive constant. However, in Ω^0_{-1} we have $\mathbf{w} = 0$, so that we fix this constant by choosing $\Psi = 0$ in Ω^0_{-1} . Then, due to the previous lemma $\mathbf{w}|_{(0,L) \times \{M\}} = d\mathbf{e}_2$ and the identity $\mathbf{w} = d\mathbf{e}_2$ holds in S_h . Thus $\partial_x \Psi = d$ in S_h and $\Psi(x, y) = b(x)$ in S_h , where $b \in H^1_{\sharp}(0, L)$ satisfies $\partial_x b = d$. Remark that the L -periodicity of b is ensured by $\int_0^L d(s) ds = 0$.

Concerning the last point of the lemma, we emphasize that, since $\Psi \in H^1_{\sharp}(\Omega)$, its trace is well-defined on vertical lines $x = cst$. Consequently, the value of Ψ on $I^c \times (-1, 2M)$ is well-defined, whatever the topological properties of I^c are. Now, given $a \in I^c$ (assuming I^c is non-empty), we have $h(a) = 0$ (by definition of I^c). The identity $\Psi(x, y) = b(x)$ in S_h , with $b \in H^1_{\sharp}(0, L)$, implies that $\Psi \in C^0_{\sharp}(S_h)$. In particular, Ψ is equal to a constant $b(a)$ on $\{a\} \times (-1, 2M)$. Moreover, the function Ψ is equal to 0 on Ω^0_{-1} . Finally, applying by a trace argument that $\Psi \in H^{1/2}(\{a\} \times (-1, 2M))$ and

using for example the following definition of the $H^{1/2}$ -norm

$$\|\Psi\|_{H^{1/2}(\{a\} \times (-1, 2M))}^2 = \int_{\{a\} \times (-1, 2M)} \Psi^2 + \int_{\{a\} \times (-1, 2M)} \int_{\{a\} \times (-1, 2M)} \frac{|\Psi(a, x) - \Psi(a, y)|^2}{|x - y|^2},$$

we get that the trace of Ψ cannot “jump” in $y = 0$. Therefore, we have $b(a) = 0$ and $\Psi = 0$ on $\{a\} \times (-1, 2M)$.

Reproducing the arguments of the proof with first-order derivatives, we note that, in case $(\mathbf{w}, d) \in X[h] \cap (K^1[h] \times H_{\#}^1(0, L))$, the stream function Ψ —that we construct in the above lemma—satisfies moreover:

- $b \in H_0^2(I)$
- $\Psi \in H_{\#}^2(\Omega)$ with $\nabla \Psi = 0$ on $I^c \times (-1, 2M)$.

We conclude this preliminary analysis of the space $X[h]$ by showing that we have density of smooth vector fields in $X[h]$. This is made precise in the following lemma:

LEMMA 8. — *The embedding $X[h] \cap (C_{\#}^{\infty}(\bar{\Omega}) \times C_{\#}^{\infty}(0, L)) \subset X[h]$ is dense.*

Proof. — The difficulty of this proof is to deal with the case where h has zeros. The main idea is to work with the stream function of the extended vector field. If we had $h(x) > 0$, for all $x \in [0, L]$, a contraction in y and a standard truncation and regularization argument on the stream function can be used. In the case where h has zeros, one first cuts off the zeros of h and then takes advantage of the better regularity of Ψ on the structure. A detailed proof is given in the appendix.

Correspondingly to the previous remark, we also have density of

$$X[h] \cap (C_{\#}^{\infty}(\bar{\Omega}) \times C_{\#}^{\infty}(0, L))$$

in $X[h] \cap (K^1[h] \times H_{\#}^1(0, L))$ endowed with the $K^1[h] \times H_{\#}^1(0, L)$ topology.

2.3. WEAK SOLUTIONS AND MAIN RESULT. — In this section we introduce first our weak formulation of $(\text{FS})_{\gamma}$.

We assume that the initial conditions $(\mathbf{u}^0, \eta^0, \eta^1)$ satisfy

$$(2.11) \quad \eta^0 \in H_{\#}^2(0, L) \text{ with } \min_{x \in [0, L]} (1 + \eta^0) > 0,$$

$$(2.12) \quad (\mathbf{u}^0, \eta^1) \in \mathbf{L}_{\#}^2(F_{h^0}) \times L_{\#,0}^2(0, L),$$

$$(2.13) \quad \text{div } \mathbf{u}^0 = 0 \text{ in } F_{h^0},$$

$$(2.14) \quad \mathbf{u}^0 \cdot \mathbf{n}^0 = 0 \text{ on } (0, L) \times \{0\} \text{ and } \mathbf{u}^0(\cdot, h_0(\cdot)) \cdot \mathbf{n}^0 = (0, \eta^1(\cdot))^T \cdot \mathbf{n}^0 \text{ on } (0, L).$$

We can then define $M > 0$ by (2.4) and construct the associated Ω . We have the following definition for a weak solution to $(\text{FS})_{\gamma}$:

DEFINITION 9. — Let $(\mathbf{u}^0, \eta^0, \eta^1)$ satisfying (2.11)–(2.13) and $\gamma > 0$. We say that a pair $(\bar{\mathbf{u}}_{\gamma}, \eta_{\gamma})$ is a weak solution to $(\text{FS})_{\gamma}$ if it satisfies the following items:

- (i) $(\bar{\mathbf{u}}_{\gamma}, \eta_{\gamma}) \in L^{\infty}(0, T; \mathbf{L}_{\#}^2(\Omega)) \times (L^{\infty}(0, T; H_{\#,0}^2(0, L)) \cap W^{1,\infty}(0, T; L_{\#}^2(0, L)))$ with $(\bar{\mathbf{u}}_{\gamma}(t), \partial_t \eta_{\gamma}(t)) \in X[h_{\gamma}(t)]$ for a.e. $t \in (0, T)$, $\nabla \bar{\mathbf{u}}_{\gamma} \in L^2(\widehat{F}_{h_{\gamma}}^-)$,

(ii) the kinematic condition

$$\mathbf{u}_\gamma(t, x, 1 + \eta_\gamma(t, x)) = \partial_t \eta_\gamma(t, x) \mathbf{e}_2 \quad \text{on } (0, T) \times (0, L),$$

(iii) For any $(\mathbf{w}_\gamma, d_\gamma) \in C_{\sharp}^\infty(\widehat{\Omega}) \times C_{\sharp}^\infty([0, T] \times [0, L])$ such that $(\mathbf{w}_\gamma(t), d_\gamma(t)) \in \times [h_\gamma(t)]$ for all $t \in [0, T]$ we have for a.e. $t \in (0, T)$

$$\begin{aligned} (2.15) \quad & \rho_f \int_{F_{h_\gamma(t)}} \mathbf{u}_\gamma(t) \cdot \mathbf{w}_\gamma(t) - \rho_f \int_0^t \int_{F_{h_\gamma(s)}} \mathbf{u}_\gamma \cdot \partial_t \mathbf{w}_\gamma + (\mathbf{u}_\gamma \cdot \nabla) \mathbf{w}_\gamma \cdot \mathbf{u}_\gamma \\ & + \rho_s \int_0^L \partial_t \eta_\gamma(t) d_\gamma(t) - \rho_s \int_0^t \int_0^L \partial_t \eta_\gamma \partial_t d_\gamma + \mu \int_0^t \int_{F_{h_\gamma(s)}} \nabla \mathbf{u}_\gamma : \nabla \mathbf{w}_\gamma \\ & + \beta \int_0^t \int_0^L \partial_x \eta_\gamma \partial_x d_\gamma + \alpha \int_0^t \int_0^L \partial_{xx} \eta_\gamma \partial_{xx} d_\gamma + \gamma \int_0^t \int_0^L \partial_{xt} \eta_\gamma \partial_x d_\gamma \\ & = \rho_f \int_{F_{h_0}} \mathbf{u}^0 \cdot \mathbf{w}_\gamma(0) + \rho_s \int_0^L \eta^1 d_\gamma(0), \end{aligned}$$

where $\mathbf{u}_\gamma = \overline{\mathbf{u}_\gamma}|_{\widehat{F}_{h_\gamma}}$.

The regularity statements in the first item of the definition comes from the energy estimate (2.2) while the weak formulation (2.15) is obtained classically by multiplying the fluid equation (1.2) with \mathbf{w}_γ and the beam equation (1.3) with d_γ and performing formal integration by parts. As usual for this type of fluid–structure problem, the test functions depend on the solution and thus on the parameter γ , adding further nonlinearity to the system.

We recall that, from [4], [6], there exists a weak solution for $\gamma > 0$ as long as the beam does not touch the bottom of the fluid cavity. If $\gamma > 0$ again and the initial data are smooth enough, it is also proved in [7] that there exists a unique global in time strong solution such that $\min_{x \in [0, L]} h_\gamma(t, x) > 0$ for all $t > 0$. The main result of this paper is stated in the following theorem

THEOREM 10. — *Let $\rho_f > 0$, $\rho_s > 0$, $\mu > 0$, $\beta > 0$ and $\alpha > 0$. Suppose that $T > 0$ and that the initial conditions $(\mathbf{u}^0, \eta^0, \eta^1)$ satisfy (2.11)–(2.14).*

Then $(FS)_0$ has a weak solution (\mathbf{u}, η) on $(0, T)$. This solution satisfies furthermore for a.e. $t \in (0, T)$

$$\begin{aligned} (2.16) \quad & \frac{1}{2} \left(\rho_f \int_{F_{h(t)}} |\mathbf{u}(t, \mathbf{x})|^2 \, d\mathbf{x} + \rho_s \int_0^L |\partial_t \eta(t, x)|^2 \, dx \right. \\ & \left. + \beta \int_0^L |\partial_x \eta(t, x)|^2 \, dx + \alpha \int_0^L |\partial_{xx} \eta(t, x)|^2 \, dx \right) + \mu \int_0^t \int_{F_{h(s)}} |\nabla \mathbf{u}(s, \mathbf{x})|^2 \, d\mathbf{x} \, ds \\ & \leq \frac{1}{2} \left(\rho_f \int_{F_{h_0}} |\mathbf{u}^0|^2 \, d\mathbf{x} + \rho_s \int_0^L |\eta^1|^2 \, dx + \beta \int_0^L |\partial_x \eta^0|^2 \, dx + \alpha \int_0^L |\partial_{xx} \eta^0|^2 \, dx \right). \end{aligned}$$

Before detailing the proof of this result, we shall comment on the choice of test functions and the relations with a strong formulation of $(FS)_\gamma$, in particular in the case where contacts occur. Since we focus on the construction of weak solutions,

we stick here to a short description of formal arguments. Before contact (i.e., as long as $\min_{x \in [0, L]} h_\gamma(t, x) > \bar{\alpha}$ for some $\bar{\alpha} > 0$), we claim that our definition coincides with the definition of [4] and that the solutions constructed in [7] for smooth data match our definition also. In particular in this case we can choose test functions such that $(\mathbf{w}_\gamma(t), d_\gamma(t)) \in X^1[h_\gamma(t)]$. Before contact, we recover (1.2)–(1.3) from the weak formulation using a classical argument. First, we may take as test functions the vector fields $\mathbf{w}_\gamma \in C_c^\infty(\widehat{\Omega})$ with $d = 0$ and we recover (1.2) with a zero mean pressure p by adapting an argument of de Rham. Assuming that $(\mathbf{u}_\gamma, p_\gamma)$ is sufficiently smooth—we can also recover the beam equation (1.3) with the following construction that enables to extend structure test functions in the fluid domain

DEFINITION 11. — Let $\lambda > 0$ and $\zeta \in C^\infty(\mathbb{R})$ such that $\mathbb{1}_{[1, \infty)} \triangleleft \zeta \triangleleft \mathbb{1}_{[1/2, \infty)}$. Given $d \in L^2_{\#}{}^2(0, L)$, we define

$$R_\lambda(d)(x, y) = \nabla^\perp(b(x)\zeta(y/\lambda)) \quad \forall (x, y) \in \Omega,$$

where $b \in H^1_{\#}(0, L) \cap L^2_{\#}{}^2(0, L)$ satisfies $\partial_x b = d$.

We note that the above construction is well-defined since d is chosen to be mean free. We do not include the dependence on ζ in the name of our operator since it will be a given fixed function throughout the paper. The present lifting operator is a variant of the one introduced in [4]. It enjoys the following straightforward properties:

LEMMA 12. — Let $\lambda > 0$ and $h \in W^{1, \infty}_{\#}(0, L)$ satisfying $\lambda \triangleleft h(x) \triangleleft M, \forall x \in [0, L]$.

(1) R_λ is a linear continuous mapping from $H^s_{\#}(0, L) \cap L^2_{\#}{}^2(0, L)$ into $K^s[h]$ for arbitrary $s \in [0, 1]$.

(2) R_λ maps $C^\infty_{\#}(0, L) \cap L^2_{\#}{}^2(0, L)$ into $\mathcal{K}[h]$.

Consequently, before contact, for any arbitrary structure test function

$$d_\gamma \in C^\infty_{\#}(0, L) \cap L^2_{\#}{}^2(0, L)$$

we may consider

$$\mathbf{w}_\gamma := R_\lambda[d_\gamma] \in C^\infty_{\#}(\widehat{\Omega}),$$

so that $(\mathbf{w}_\gamma, d_\gamma)$ is an admissible test function in our weak formulation. Classical integration by parts argument then enables to recover the structure equation (1.3) multiplied by d_γ . We note that, in this way, we recover (1.3) up to a constant (indeed the test function d_γ is mean free), but this constant mode corresponds to the choice of the constant normalizing the pressure in order to match the global volume preserving constraint (1.7).

When contact occurs, we recover a similar set of equations, assuming, once again, that the solution is sufficiently regular. Let us consider for instance a simplified configuration such that, on some time interval (T_0, T_1) there exist C^1 -functions $(a_k^-, a_k^+) : (T_0, T_1) \rightarrow \mathbb{R}^2$ ($k \in \mathbb{N}$) such that

$$\{(t, x) \in (T_0, T_1) \times (0, L) \mid h_\gamma(t, x) > 0\} = \bigcup_{k \in \mathbb{N}} \bigcup_{t \in (T_0, T_1)} \{t\} \times (a_k^-(t), a_k^+(t)).$$

In that case, we can reproduce similar arguments developed in the no contact case to recover the Navier–Stokes equations and the structure equations in each connected component of the fluid domain. More precisely, we introduce

$$\begin{aligned}\widehat{F}_k &:= \{(t, x, y) \in (T_0, T_1) \times (0, L) \times (0, M) \mid a_k^-(t) < x < a_k^+(t), 0 < y < h_\gamma(t, x)\}, \\ \widehat{\Gamma}_k &:= \{(t, x) \in (T_0, T_1) \times (0, L) \mid a_k^-(t) < x < a_k^+(t)\}.\end{aligned}$$

First, by using that \mathbf{u}_γ is divergence free on \widehat{F}_k we obtain

$$\int_{a_k^-(t)}^{a_k^+(t)} \partial_t \eta_\gamma = 0.$$

Second, by taking as a fluid test function a velocity field \mathbf{w}_γ with compact support in \widehat{F}_k , we construct a pressure $p_{k,\gamma}$ on \widehat{F}_k so that (1.2) holds true. We recall that, at this point, $p_{k,\gamma}$ is defined up to a constant. The global pressure p_γ is then constructed by concatenating all the $(p_{k,\gamma})_k$ to yield a pressure on \widehat{F}_{h_γ} (that is defined up to a number of constants related to the number of parameters k). Third, we consider a mean free structure test function $d_\gamma \in C_c^\infty(\widehat{\Gamma}_k)$. Since d_γ has compact support in the open set where $h_\gamma > 0$, h_γ is bounded from below by some $\alpha_k > 0$ on $\widehat{\Gamma}_k$. So, instead of choosing the mean free anti-derivative b_γ of the structure test function d_γ in the definition of R_{α_k} , we choose the one that vanishes outside the support of d_γ . In that way, we construct a test function \mathbf{w}_γ such that $(\mathbf{w}_\gamma, d_\gamma)$ is adapted to our weak formulation. So, we obtain (1.3) on $\widehat{\Gamma}_k$ up to a constant which is afterward fixed by a suitable choice of the pressure on \widehat{F}_k . Note that the structure equation is recovered on each component $\widehat{\Gamma}_k$ and not on the whole interval $(0, L)$ and that the pressure is again uniquely defined but that there are more constants to fix than in the no contact case. To end up this remark, we emphasize that when $\min_{x \in [0, L]} h_\gamma(t, x) > 0, \forall t \in [0, T]$, the test functions can be chosen in $X[h_\gamma(t)]$. Moreover if $\min_{x \in [0, L]} h_\gamma(t, x) > 0$ the elastic test functions can be chosen independent of the solution and thus independent of the regularization parameter γ (see [4]). It is not the case when a contact occurs since, as we saw in the previous construction, we require $d_\gamma = 0$ in a neighbourhood of the contact points.

We end this part by giving a road map of our proof of Theorem 10. To obtain a solution of the variational problem for $\gamma = 0$ we consider the approximate fluid–structure system $(\text{FS})_\gamma$ with a viscosity $\gamma > 0$. From [8, Th. 1] this fluid–structure system $(\text{FS})_\gamma$, completed with regularized initial conditions $(\mathbf{u}_\gamma^0, \eta_\gamma^0, \eta_\gamma^1)$, admits a unique strong solution $(\mathbf{u}_\gamma, p_\gamma, \eta_\gamma)$ such that $\min_{x \in [0, L]} h_\gamma(t, x) > 0$. It ensures that the existence time interval does not depend on γ . Moreover, this solution satisfies the energy equality (2.2), and thus one can extract converging subsequences. We may then consider one cluster point of this sequence and show that this is a weak solution to $(\text{FS})_0$. One key point here is that strong compactness of the approximate velocity fields is needed to pass to the limit in the convective nonlinear terms. The classical Aubin–Lions lemma does not apply directly because of the time-dependency of fluid domains and of the divergence free constraint. Many different strategies may be used

to handle this difficulty [5, 6, 10, 13]. Here we follow the line of [16] where the existence of weak solutions for a fluid–solid problem beyond contact is proved. We first obtain compactness of a projection of the fluid and structure velocities. Roughly speaking the idea is to obtain compactness on fixed domains independent of time and of γ . So, we define an interface satisfying $0 \subset \underline{h} \subset h_\gamma$, which is regular enough and “close” to h_γ for all γ small enough and we prove compactness of the projections of the velocity fields on the coupled space associated to \underline{h} . We recover the compactness of the velocity fields by proving some continuity properties of the projection operators with respect to \underline{h} . In particular we prove that $X^s[\underline{h}]$ is a good approximation space of $X^s[h]$ in H^s for some $s > 0$ whenever \underline{h} is close to h . This part of the proof is purely related to the definition of the spaces $X[h]$. Consequently we detail the arguments as preliminaries in the next subsection. We emphasize again that, since one may loose the no contact property at the limit, this study on the compactness of approximate velocity fields requires specific constructions. Once compactness is obtained, we pass finally to the limit in the weak formulation. Again, as contact may occur in the limit problem, we cannot follow [4] to construct a dense family of test functions independent of γ to pass to the limit.

2.4. ON THE h -DEPENDENCIES OF THE SPACES $X^s[h]$. — In this section, we analyze the continuity properties of the sets $X^s[h]$ with respect to the parameter h . To start with, we remark that, given $h \in C_{\sharp}^0(0, L)$ satisfying $0 \subset h \subset M$ the space $X^s[h]$ is a closed subspace of

$$X^s := \mathbf{H}_{\sharp}^s(\Omega) \times H_{\sharp}^{2s}(0, L).$$

We can then construct the projector $\mathbb{P}^s[h] : X^s \rightarrow X^s[h]$. We analyze in this section the continuity properties of these projectors with respect to the function h . Our main result is the following lemma:

LEMMA 13. — Fix κ such that $0 < \kappa < 1/2$. Let h and \underline{h} belong to $H_{\sharp}^{1+\kappa}(0, L) \cap W_{\sharp}^{1,\infty}(0, L)$ with $0 \subset \underline{h} \subset h \subset M$ and set

$$(2.17) \quad \|h\|_{H_{\sharp}^{1+\kappa}(0,L)} + \|h\|_{W_{\sharp}^{1,\infty}(0,L)} + \|\underline{h}\|_{H_{\sharp}^{1+\kappa}(0,L)} + \|\underline{h}\|_{W_{\sharp}^{1,\infty}(0,L)} \subset A.$$

Let $s \in [0, \kappa/2)$ and $(\mathbf{w}, \dot{\eta}) \in X^s[h]$ enjoying the further property

$$(2.18) \quad \mathbf{w}|_{F_h^-} \in \mathbf{H}_{\sharp}^1(F_h^-).$$

Then, the following estimate holds true:

$$(2.19) \quad \|\mathbb{P}^s[\underline{h}](\mathbf{w}, \dot{\eta}) - (\mathbf{w}, \dot{\eta})\|_{X^s} \subset C_A(\|\underline{h} - h\|_{W_{\sharp}^{1,\infty}(0,L)}) \|\mathbf{w}\|_{\mathbf{H}_{\sharp}^1(F_h^-)},$$

where $C_A(x) \xrightarrow{x \rightarrow 0} 0$.

Proof. — The idea is to construct $(\mathbf{v}, b) \in X^s[\underline{h}]$ such that

$$(2.20) \quad \|(\mathbf{v}, b) - (\mathbf{w}, \dot{\eta})\|_{\mathbf{H}_{\sharp}^s(\Omega) \times H_{\sharp}^{2s}(0,L)} \subset C_A(\|\underline{h} - h\|_{W_{\sharp}^{1,\infty}(0,L)}) \|\mathbf{w}\|_{\mathbf{H}_{\sharp}^1(F_h^-)}.$$

The inequality (2.19) then follows from the minimality property of the projection. The proof is divided in five steps. The two first ones are devoted to the construction of (\mathbf{v}, d) . The latter ones concern the derivation of (2.20).

Step 1. Geometrical preliminaries. — Before going to the construction of a candidate (\mathbf{v}, d) we define and analyze a change of variables χ that maps $F_{\underline{h}}^-$ on F_h^- . For $(x, y) \in \Omega$, we set

$$\chi(x, y) = (x, m(x)(y + 1) - 1), \quad \text{where } m(x) = \frac{h(x) + 1}{\underline{h}(x) + 1}.$$

Clearly, χ realizes a one-to-one mapping between $F_{\underline{h}}^-$ and F_h^- . Thanks to the regularity assumptions on h and \underline{h} , we remark that $m \in W_{\sharp}^{1,\infty}(0, L) \cap H_{\sharp}^{1+\kappa}(0, L)$ —since both spaces are algebras—and that $\chi \in \mathbf{W}_{\sharp}^{1,\infty}(\Omega)$.

For the definition of \mathbf{v} , we shall transform \mathbf{w} into a vector field \mathbf{w}^x satisfying $\mathbf{w}^x \cdot \mathbf{e}_1 = 0$ on $S_{\underline{h}}$. To preserve simultaneously that \mathbf{w}^x is divergence free, one multiplies, in a standard way, the vector field by the cofactor of $\nabla\chi$. So, we now analyze the multiplier properties of $\text{Cof}(\nabla\chi)^{\top}$. First, we have

$$\text{Cof}(\nabla\chi)^{\top} = \begin{pmatrix} m(x) & 0 \\ -\partial_x m(x)(y + 1) & 1 \end{pmatrix} \in \mathbf{L}_{\sharp}^{\infty}(\Omega) \cap \mathbf{H}_{\sharp}^{\kappa}(\Omega).$$

Then, straightforward computations yield

$$(2.21) \quad \|m - 1\|_{W_{\sharp}^{1,\infty}(0,L)} \triangleleft C_A \|h - \underline{h}\|_{W_{\sharp}^{1,\infty}(0,L)},$$

so that,

$$(2.22) \quad \|\text{Cof}(\nabla\chi)^{\top} - \text{Id}\|_{\mathbf{L}_{\sharp}^{\infty}(\Omega)} \triangleleft C_A \|h - \underline{h}\|_{W_{\sharp}^{1,\infty}(0,L)},$$

with C_A a constant depending only on the upper A defined by (2.17). Finally we prove H^{σ} -estimates. To this end, we interpolate between L^2 and H^{κ} . Estimate (2.22) implies

$$\|\text{Cof}(\nabla\chi)^{\top} - \text{Id}\|_{\mathbf{L}_{\sharp}^2(\Omega)} \triangleleft C_A \|h - \underline{h}\|_{W_{\sharp}^{1,\infty}(0,L)}.$$

Then, we remark that

$$\|m - 1\|_{H_{\sharp}^{1+\kappa}(0,L)} \triangleleft C_A,$$

and that, for any given f , a and b regular functions defined on $(0, L)$, there holds

$$(2.23) \quad \|f\|_{H_{\sharp}^{\sigma}(\Omega_{\sharp}^b)} \triangleleft \|a - b\|_{L_{\sharp}^{\infty}(0,L)}^{1/2} \|f\|_{H_{\sharp}^{\sigma}(0,L)},$$

for $0 \triangleleft \sigma \triangleleft 1$. Consequently, we obtain also

$$\|\text{Cof}(\nabla\chi)^{\top} - \text{Id}\|_{\mathbf{H}_{\sharp}^{\kappa}(\Omega)} \triangleleft C_A \|m - 1\|_{H_{\sharp}^{1+\kappa}(0,L)}.$$

Using interpolation between the L^2 and the H^{κ} estimates finally leads to

$$(2.24) \quad \|\text{Cof}(\nabla\chi)^{\top} - \text{Id}\|_{\mathbf{H}_{\sharp}^{\sigma}(\Omega)} \triangleleft C_A \|h - \underline{h}\|_{W_{\sharp}^{1,\infty}(0,L)}^{(\kappa-\sigma)/\kappa},$$

for $0 \triangleleft \sigma \triangleleft \kappa$.

Step 2. Construction of (\mathbf{v}, d) . — Let consider $(\mathbf{w}, \dot{\eta}) \in X^s[h]$ enjoying the further property

$$(2.25) \quad \mathbf{w}|_{F_h^-} \in \mathbf{H}_{\sharp}^1(F_h^-).$$

As mentioned previously, to define (\mathbf{v}, d) we first construct an intermediate vector field \mathbf{w}^χ obtained by the change of variables χ from \mathbf{w} . However we note that χ does not map Ω into Ω so that we must at first extend \mathbf{w} for $y > 2M$. Namely, we set

$$\tilde{\mathbf{w}}(x, y) = \begin{cases} \mathbf{w}(x, y), & \text{if } (x, y) \in \Omega, \\ \dot{\eta}(x)\mathbf{e}_2, & \text{if } y > 2M. \end{cases}$$

This extension preserves the divergence free constraint. We next define

$$\mathbf{w}^\chi = \text{Cof}(\nabla\chi)^\top \tilde{\mathbf{w}} \circ \chi.$$

The $\text{Cof}(\nabla\chi)^\top$ factor ensures that \mathbf{w}^χ is also divergence free. Next we define (\mathbf{v}, d) as

$$(2.26) \quad \mathbf{v} = \begin{cases} \mathbf{w}^\chi - w_2^\chi|_{y=0}\mathbf{e}_2, & \text{in } \Omega_0^{2M}, \\ 0, & \text{in } \Omega_{-1}^0, \end{cases} \quad d = \dot{\eta} - w_2^\chi|_{y=0}.$$

The first step is to verify that $(\mathbf{v}, d) \in X^s[h], \forall s < \kappa/2$. First, by taking into account assumption (2.25), since $\tilde{\mathbf{w}}$ coincides with \mathbf{w} in F_h^- , we have that $\tilde{\mathbf{w}} \in \mathbf{H}_{\sharp}^1(F_h^-)$. Thus since the change of variables χ maps $F_{\underline{h}}^-$ onto F_h^- , the above analysis of the regularity of χ and of $\text{Cof}(\nabla\chi)^\top$ implies that $\mathbf{w}^\chi \in \mathbf{H}_{\sharp}^s(F_h^-), \forall s < \kappa$ (see [9, Prop. B.1]). Moreover, by the change of variables, the boundary $y = 0$ is mapped to $y = m - 1$ which is lower than h and strictly greater than -1 . Hence, the trace of $\tilde{\mathbf{w}} \circ \chi$ on $y = 0$ is well-defined and belongs to $H_{\sharp}^{1/2}(0, L)$. But by definition we have

$$w_2^\chi|_{y=0} = -\partial_x m \tilde{w}_1 \circ \chi|_{y=0} + \tilde{w}_2 \circ \chi|_{y=0},$$

where $\partial_x m \in H_{\sharp}^s(0, L)$. Classical multiplier arguments thus imply that $w_2^\chi|_{y=0} \in H_{\sharp}^{2s}(0, L)$ for any $s < \kappa/2$ and that

$$(2.27) \quad \|w_2^\chi|_{y=0}\|_{H_{\sharp}^{2s}(0, L)} \leq C_A(\|h - \underline{h}\|_{W_{\sharp}^{1, \infty}(0, L)})\|\mathbf{w}\|_{H^1(F_h^-)},$$

where $C_A(x) \xrightarrow{x \rightarrow 0} 0$. Furthermore, we have by construction that

$$\mathbf{w}^\chi = \dot{\eta}\mathbf{e}_2 \text{ in } S_{\underline{h}}.$$

Consequently, thanks to the regularity of $\dot{\eta}$ and the one obtained on $w_2^\chi|_{y=0}$, we deduce that $\mathbf{v} \in \mathbf{H}^{2s}(S_{\underline{h}}) \subset \mathbf{H}^s(S_{\underline{h}})$. Finally $(\mathbf{v}, d) \in \mathbf{H}^s(\Omega) \times H^{2s}(0, L)$, for $s < \kappa/2$. Let us now check the divergence free constraint and the fluid–structure velocity matching. We have by construction that

$$\text{div } \mathbf{w}^\chi = 0 \text{ in } \Omega.$$

Thus \mathbf{v} satisfies

$$\text{div } \mathbf{v} = 0 \text{ in } \Omega_0^{2M} \text{ and } \text{div } \mathbf{v} = 0 \text{ in } \Omega_0^{-1}.$$

Since, by construction $(\mathbf{v} \cdot \mathbf{e}_2)|_{y=0} = 0$, we obtain $\operatorname{div} \mathbf{v} = 0$ in Ω . Moreover, $\operatorname{div} \mathbf{w}^\chi = 0$ on C_{-1}^0 with $\mathbf{w}^\chi = 0$ on $y = -1$. By integrating this divergence constraint we obtain the condition

$$\int_0^L w_2^\chi|_{y=0} = 0.$$

As a consequence, since $\dot{\eta} \in L^2_{\sharp,0}(0, L)$, we obtain $d \in L^2_{\sharp,0}(0, L)$.

We now check the remaining compatibility conditions of $X^s[h]$. For $y > \underline{h}$, we have $\mathbf{v} = (\dot{\eta} - w_2^\chi|_{y=0})\mathbf{e}_2$ so that \mathbf{v} satisfies

$$\mathbf{v} \cdot \mathbf{e}_1 = 0, \text{ on } S_{\underline{h}}, \quad v_2|_{y=M} = d.$$

This ends the proof that $(\mathbf{v}, d) \in X^s[\underline{h}]$.

Step 3. Splitting of $\|(\mathbf{w}, \dot{\eta}) - (\mathbf{v}, d)\|_{X^s}$. — Let first remark that

$$\mathbf{v} - \mathbf{w}^\chi = \begin{cases} -w_2^\chi|_{y=0}\mathbf{e}_2, & \text{in } \Omega_0^{2M}, \\ -\mathbf{w}^\chi, & \text{in } \Omega_{-1}^0, \end{cases} \quad \dot{\eta} - d = w_2^\chi|_{y=0}.$$

Consequently we have

$$\begin{aligned} \|(\mathbf{w}, \dot{\eta}) - (\mathbf{v}, d)\|_{X^s} &\leq \|(\mathbf{w}, \dot{\eta}) - (\mathbf{w}^\chi, d)\|_{X^s} + \|\mathbf{w}^\chi - \mathbf{v}\|_{\mathbf{H}^s_\sharp(\Omega)} \\ &\leq \|\mathbf{w} - \mathbf{w}^\chi\|_{\mathbf{H}^s_\sharp(\Omega)} + \|w_2^\chi|_{y=0}\|_{H^{2s}_\sharp(0,L)} \\ &\quad + \|w_2^\chi|_{y=0}\mathbf{e}_2\|_{\mathbf{H}^s_\sharp(\Omega_0^{2M})} + \|\mathbf{w}^\chi\|_{\mathbf{H}^s_\sharp(\Omega_{-1}^0)}. \end{aligned}$$

Recalling (2.23) we obtain the bound

$$\|w_2^\chi|_{y=0}\mathbf{e}_2\|_{\mathbf{H}^s_\sharp(\Omega_0^{2M})} \leq \sqrt{2M} \|w_2^\chi|_{y=0}\mathbf{e}_2\|_{\mathbf{H}^s_\sharp(0,L)} \leq \sqrt{2M} \|w_2^\chi|_{y=0}\|_{H^{2s}_\sharp(0,L)}.$$

Moreover, as $\mathbf{w} = 0$ in Ω_{-1}^0 , we remark that an estimate on $\|\mathbf{w}^\chi - \mathbf{w}\|_{\mathbf{H}^s(\Omega)}$ implies an estimate on $\|\mathbf{w}^\chi\|_{\mathbf{H}^s(\Omega_{-1}^0)}$. Finally (2.19) is implied by the following estimate

$$(2.28) \quad \|\mathbf{w} - \mathbf{w}^\chi\|_{\mathbf{H}^s_\sharp(\Omega)} + \|w_2^\chi|_{y=0}\|_{H^{2s}_\sharp(0,L)} \leq C_A(\|\underline{h} - h\|_{W^1_\sharp(0,L)}) \|\mathbf{w}\|_{\mathbf{H}^1_\sharp(F_h^-)}.$$

Thus we have to prove that $\mathbf{w} - \mathbf{w}^\chi = \mathbf{w} - \operatorname{Cof}(\nabla\chi)^\top \mathbf{w} \circ \chi$ and $w_2^\chi|_{y=0} = -\partial_x m w_1 \circ \chi|_{y=0} + w_2 \circ \chi|_{y=0}$ can be estimated with respect to the difference $h - \underline{h}$. This is the aim of the two next steps respectively.

Step 4. Estimating $\mathbf{w} - \mathbf{w}^\chi$. — We estimate the difference $\mathbf{w} - \mathbf{w}^\chi$ by considering successively each of the subdomains of Ω : $S_h, \Omega_{\underline{h}}^h, F_{\underline{h}}, \Omega_{-1}^0$.

Estimates in S_h . — In S_h , there holds $\mathbf{w} = \dot{\eta}\mathbf{e}_2$. By replacing in the definition, we have also $\mathbf{w}^\chi = \dot{\eta}\mathbf{e}_2$ in $S_{\underline{h}}$ and since $h > \underline{h}$ we infer $\mathbf{w} - \mathbf{w}^\chi = 0$ in S_h .

Estimates in $\Omega_{\underline{h}}^h$. — The identity $\mathbf{w}^\chi = \eta \mathbf{e}_2$ still holds in $\Omega_{\underline{h}}^h \subset \mathcal{S}_{\underline{h}}$ which leads to

$$\mathbf{w}(x, y) - \mathbf{w}^\chi(x, y) = \mathbf{w}(x, y) - \mathbf{w}(x, h(x)) = \int_y^{h(x)} \partial_y \mathbf{w}(x, z) \, dz.$$

We obtain then

$$\begin{aligned} \|\mathbf{w}^\chi - \mathbf{w}\|_{\mathbf{L}_{\sharp}^2(\Omega_{\underline{h}}^h)}^2 &= \int_0^L \int_{\underline{h}(x)}^{h(x)} \left| \int_y^{h(x)} \partial_y \mathbf{w}(x, z) \, dz \right|^2 \, dy \, dx \\ &\leq \int_0^L \int_{\underline{h}(x)}^{h(x)} \|h - \underline{h}\|_{L_{\sharp}^{\infty}(0, L)} \int_{\underline{h}(x)}^{h(x)} |\partial_y \mathbf{w}(x, z)|^2 \, dz \, dy \, dx \\ &\leq \|h - \underline{h}\|_{L_{\sharp}^{\infty}(0, L)}^2 \|\nabla \mathbf{w}\|_{\mathbf{L}_{\sharp}^2(F_h^-)}^2. \end{aligned}$$

Moreover,

$$\|\mathbf{w}^\chi - \mathbf{w}\|_{\mathbf{H}_{\sharp}^{1/2}(\Omega_{\underline{h}}^h)} \leq \|\mathbf{w}^\chi\|_{\mathbf{H}_{\sharp}^{1/2}(\Omega_{\underline{h}}^h)} + \|\mathbf{w}\|_{\mathbf{H}_{\sharp}^{1/2}(\Omega_{\underline{h}}^h)} \leq \|\mathbf{w}^\chi\|_{\mathbf{H}_{\sharp}^{1/2}(\Omega_{\underline{h}}^h)} + \|\mathbf{w}\|_{\mathbf{H}_{\sharp}^1(F_h^-)}.$$

We have $\mathbf{w}^\chi = \eta \mathbf{e}_2 = \mathbf{w}|_{y=h}$. Hence, recalling the trace continuity estimate (2.5), we obtain

$$\|\mathbf{w}^\chi\|_{\mathbf{H}_{\sharp}^{1/2}(0, L)} \leq C_A \|\mathbf{w}\|_{\mathbf{H}_{\sharp}^1(F_h^-)}.$$

We conclude that

$$\|\mathbf{w}^\chi - \mathbf{w}\|_{\mathbf{H}_{\sharp}^{1/2}(\Omega_{\underline{h}}^h)} \leq C_A \|\mathbf{w}\|_{\mathbf{H}_{\sharp}^1(F_h^-)},$$

and by interpolation with the previous L^2 -estimate, the following estimate in H^s holds true

$$(2.29) \quad \|\mathbf{w}^\chi - \mathbf{w}\|_{\mathbf{H}_{\sharp}^s(\Omega_{\underline{h}}^h)} \leq C_A \|h - \underline{h}\|_{W_{\sharp}^{1, \infty}(0, L)}^{1-2s} \|\mathbf{w}\|_{\mathbf{H}_{\sharp}^1(F_h^-)}.$$

Estimates in $F_{\underline{h}}$. — Consider the following splitting:

$$(2.30) \quad \mathbf{w}^\chi - \mathbf{w} = \text{Cof}(\nabla \chi)^\top (\mathbf{w} \circ \chi - \mathbf{w}) + (\text{Cof}(\nabla \chi)^\top - \text{I}_2) \mathbf{w}.$$

Thanks to [9, Prop. B.1] we obtain, for $s < s' \leq \kappa$ (see (2.24) for the estimate of the $H^{s'}$ -norm of the cofactor matrix):

$$\begin{aligned} \|(\text{Cof}(\nabla \chi)^\top - \text{I}_2) \mathbf{w}\|_{\mathbf{H}_{\sharp}^s(F_{\underline{h}})} &\leq C_A \|(\text{Cof}(\nabla \chi)^\top - \text{I}_2)\|_{\mathbf{H}_{\sharp}^{s'}(\Omega)} \|\mathbf{w}\|_{\mathbf{H}_{\sharp}^1(F_h^-)} \\ &\leq C_A \|h - \underline{h}\|_{W_{\sharp}^{1, \infty}(0, L)}^{(\kappa - s')/\kappa} \|\mathbf{w}\|_{\mathbf{H}_{\sharp}^1(F_h^-)}. \end{aligned}$$

Here we use the continuity of the multiplication $H^{s'}(F_{\underline{h}}^-) \times H^1(F_h^-) \rightarrow H^s(F_{\underline{h}}^-)$. The continuity constant of this mapping may depend on \underline{h} . But, by a standard change of variables argument, we see that it depends increasingly on $\|\underline{h}\|_{W_{\sharp}^{1, \infty}(0, L)}$ only. This constant thus depends on A only. We now take care of the first term of the right-hand

side of (2.30). Let first note that we can bound the L^2 -norm of $\mathbf{w} \circ \chi - \mathbf{w}$ as follows:

$$\begin{aligned}
& \int_0^L \int_0^{\underline{h}(x)} |\mathbf{w}(\chi(x, y)) - \mathbf{w}(x, y)|^2 dy dx \\
&= \int_0^L \int_0^{\underline{h}(x)} |\mathbf{w}(x, m(x)(y+1) - 1) - \mathbf{w}(x, y)|^2 dy dx \\
&\leq \int_0^L \int_0^{\underline{h}(x)} \left| \int_y^{m(x)(y+1)-1} \partial_y \mathbf{w}(x, z) dz \right|^2 dy dx \\
&\leq \int_0^L \int_0^{\underline{h}(x)} (m(x) - 1)(y+1) \int_y^{m(x)(y+1)-1} |\partial_y \mathbf{w}(x, z)|^2 dz dy dx \\
&\leq (M+1) \|m-1\|_{L^\infty(0,L)} \|\underline{h}\|_{L^\infty(0,L)} \|\nabla \mathbf{w}\|_{\mathbf{L}^2_\#(F_{\underline{h}}^-)}^2.
\end{aligned}$$

The previous estimate leads to

$$\begin{aligned}
\|\mathbf{w} \circ \chi - \mathbf{w}\|_{\mathbf{L}^2_\#(F_{\underline{h}})} &\leq \sqrt{(M+1)M} \|m-1\|_{L^\infty_\#(0,L)}^{1/2} \|\mathbf{w}\|_{\mathbf{H}^1_\#(F_{\underline{h}}^-)} \\
&\leq C_M \|h - \underline{h}\|_{W^{1,\infty}_\#(0,L)}^{1/2} \|\mathbf{w}\|_{\mathbf{H}^1_\#(F_{\underline{h}}^-)}.
\end{aligned}$$

Finally, since $\|\text{Cof}(\nabla \chi)^\top\|_{L^\infty(F_{\underline{h}})} \leq C_A$, we deduce

$$(2.31) \quad \|\text{Cof}(\nabla \chi)^\top(\mathbf{w} \circ \chi - \mathbf{w})\|_{\mathbf{L}^2_\#(F_{\underline{h}})} \leq C_A \|h - \underline{h}\|_{W^{1,\infty}_\#(0,L)}^{1/2} \|\mathbf{w}\|_{\mathbf{H}^1_\#(F_{\underline{h}}^-)}.$$

Next we remark that $\mathbf{w} \circ \chi - \mathbf{w}$ is bounded in $\mathbf{H}^1_\#(F_{\underline{h}}^-)$. Indeed $\mathbf{w} \in \mathbf{H}^1_\#(F_{\underline{h}}^-)$ and thus $\mathbf{w} \in \mathbf{H}^1_\#(F_{\underline{h}}^-)$. It implies also that $\mathbf{w} \circ \chi \in \mathbf{H}^1_\#(F_{\underline{h}})$ since χ belongs to $W^{1,\infty}_\#(\Omega)$ and maps $F_{\underline{h}}^-$ in $F_{\underline{h}}$. Consequently, we have

$$\|\mathbf{w} \circ \chi - \mathbf{w}\|_{\mathbf{H}^1_\#(F_{\underline{h}})} \leq \|\mathbf{w} \circ \chi\|_{\mathbf{H}^1_\#(F_{\underline{h}})} + \|\mathbf{w}\|_{\mathbf{H}^1_\#(F_{\underline{h}})} \leq C_A \|\mathbf{w}\|_{\mathbf{H}^1_\#(F_{\underline{h}}^-)}.$$

Next, thanks to the fact that $\|\text{Cof}(\nabla \chi)^\top\|_{\mathbf{H}^s_\#(F_{\underline{h}})} \leq C_A$ (see (2.24)), we have, since $0 \leq s < \kappa$

$$(2.32) \quad \begin{aligned} \|\text{Cof}(\nabla \chi)^\top(\mathbf{w} \circ \chi - \mathbf{w})\|_{\mathbf{H}^s_\#(F_{\underline{h}})} &\leq C_A \|\text{Cof}(\nabla \chi)^\top\|_{\mathbf{H}^s_\#(\Omega)} \|\mathbf{w}\|_{\mathbf{H}^1_\#(F_{\underline{h}}^-)} \\ &\leq C_A \|\mathbf{w}\|_{\mathbf{H}^1_\#(F_{\underline{h}}^-)}. \end{aligned}$$

By interpolating (2.31) and (2.32), we obtain

$$\|\text{Cof}(\nabla \chi)^\top(\mathbf{w} \circ \chi - \mathbf{w})\|_{\mathbf{H}^s_\#(F_{\underline{h}})} \leq C_A (\|h - \underline{h}\|_{W^{1,\infty}_\#(0,L)}) \|\mathbf{w}\|_{\mathbf{H}^1_\#(F_{\underline{h}}^-)}.$$

To summarize the estimates in $F_{\underline{h}}$ we have proved that

$$(2.33) \quad \|\mathbf{w}^\chi - \mathbf{w}\|_{\mathbf{H}^s_\#(F_{\underline{h}})} \leq C_A (\|h - \underline{h}\|_{W^{1,\infty}_\#(0,L)}) \|\mathbf{w}\|_{\mathbf{H}^1_\#(F_{\underline{h}}^-)}.$$

Estimates in Ω_{-1}^0 . — The function \mathbf{w} is equal to zero and we have to estimate only \mathbf{w}^χ . As previously we obtain a first bound in L^2 involving $C_A(\|h - \underline{h}\|_{W^{1,\infty}_\#(0,L)})$ and then we prove that \mathbf{w}^χ is bounded in some H^s and we conclude using interpolation. For the L^2 -norm we have

$$\|\mathbf{w}^\chi\|_{\mathbf{L}^2_\#(\Omega_{-1}^0)} \leq \|(\text{Cof} \nabla \chi)^\top\|_{\mathbf{L}^\infty_\#(\Omega_{-1}^0)} \|\mathbf{w} \circ \chi\|_{\mathbf{L}^2_\#(\Omega_{-1}^0)},$$

and

$$\begin{aligned} \|\mathbf{w} \circ \chi\|_{\mathbf{L}^2_{\sharp}(\Omega^0_{-1})} &= \int_0^L \int_{-1}^0 |\mathbf{w}(\chi(x, y))|^2 dy dx \\ &= \int_0^L \int_0^{m(x)-1} |\mathbf{w}(x, y)|^2 \frac{dy}{m(x)} dx \\ &\leq (1 + M) \int_0^L \int_0^{m(x)-1} \left| \int_0^y \partial_y \mathbf{w}(x, z) dz \right|^2 dy dx \\ &\leq (1 + M) \int_0^L \int_0^{m(x)-1} y \int_0^y |\partial_z \mathbf{w}(x, z)|^2 dz dy dx \\ &\leq \frac{(1 + M)}{2} \|m - 1\|_{L^\infty_{\sharp}(0, L)}^2 \|\mathbf{w}\|_{\mathbf{H}^1_{\sharp}(F_h^-)}^2. \end{aligned}$$

Hence, using the estimates above with (2.21)–(2.22), we conclude

$$\|\mathbf{w}^X\|_{\mathbf{L}^2_{\sharp}(\Omega^0_{-1})} \leq C_A \|h - \underline{h}\|_{W^{1, \infty}_{\sharp}(0, L)} \|\mathbf{w}\|_{\mathbf{H}^1_{\sharp}(F_h^-)}.$$

Moreover, for any $0 < \sigma < \kappa$, we have

$$\begin{aligned} \|\mathbf{w}^X\|_{\mathbf{H}^{\sigma}_{\sharp}(\Omega^0_{-1})} &\leq C_1 \|(\text{Cof} \nabla \chi)^{\top}\|_{\mathbf{H}^{\sigma}_{\sharp}(\Omega^0_{-1})} \|\mathbf{w} \circ \chi\|_{\mathbf{H}^1_{\sharp}(\Omega^0_{-1})} \\ &\leq C_A \|\mathbf{w}\|_{\mathbf{H}^1_{\sharp}(F_h^-)}, \end{aligned}$$

and using interpolation up to choose $\sigma \in (s, \kappa)$

$$\|\mathbf{w}^X\|_{\mathbf{H}^s_{\sharp}(\Omega^0_{-1})} \leq C_A (\|h - \underline{h}\|_{W^{1, \infty}_{\sharp}(0, L)}) \|\mathbf{w}\|_{\mathbf{H}^1_{\sharp}(F_h^-)}.$$

To summarize we have proved that

$$(2.34) \quad \|\mathbf{w} - \mathbf{w}^X\|_{\mathbf{H}^s_{\sharp}(\Omega^0_{-1})} \leq C_A (\|h - \underline{h}\|_{W^{1, \infty}_{\sharp}(0, L)}) \|\mathbf{w}\|_{\mathbf{H}^1_{\sharp}(F_h^-)}.$$

Finally, combining (2.29)–(2.33)–(2.34), we obtain the expected estimate

$$(2.35) \quad \|\mathbf{w} - \mathbf{w}^X\|_{\mathbf{H}^s_{\sharp}(\Omega)} \leq C_A (\|h - \underline{h}\|_{W^{1, \infty}_{\sharp}(0, L)}) \|\mathbf{w}\|_{\mathbf{H}^1_{\sharp}(F_h^-)}.$$

Step 5. Estimating $w_2^X|_{y=0}$. — First, we recall that

$$w_2^X|_{y=0} = -\partial_x m(\cdot) w_1(\cdot, m(\cdot) - 1) + w_2(\cdot, m(\cdot) - 1).$$

This term is first estimated in L^2 , then in $H^{2\sigma}$ for $0 < \sigma < \kappa/2$, and the final estimate is obtained by interpolation. First, let us estimate the L^2 -norm

$$\begin{aligned} \|-\partial_x m(\cdot) w_1(\cdot, m(\cdot) - 1)\|_{L^2(0, L)}^2 &\leq \|\partial_x m\|_{L^\infty_{\sharp}(0, L)}^2 \int_0^L |w_1(x, m(x) - 1)|^2 dx \\ &\leq C_A \int_0^L (m(x) - 1) \int_0^{m(x)-1} |\partial_y w_1(x, y)|^2 dy dx \\ &\leq C_A \|m - 1\|_{L^\infty_{\sharp}(0, L)} \|\mathbf{w}\|_{\mathbf{H}^1_{\sharp}(F_h^-)} \\ &\leq C_A \|h - \underline{h}\|_{L^\infty_{\sharp}(0, L)} \|\mathbf{w}\|_{\mathbf{H}^1_{\sharp}(F_h^-)}. \end{aligned}$$

A similar estimate can be computed for $\|w_2(\cdot, m(\cdot) - 1)\|_{L^2_{\sharp}(0, L)}$ so that, we obtain

$$\|w_2^X|_{y=0}\|_{L^2_{\sharp}(0, L)} \leq C_A \|h - \underline{h}\|_{W^{1, \infty}_{\sharp}(0, L)} \|\mathbf{w}\|_{\mathbf{H}^1_{\sharp}(F_h^-)}.$$

For $0 < \sigma < \kappa/2$ we obtain, similarly to (2.27)

$$\|w_2^X|_{y=0}\|_{H_\#^{2\sigma}(0,L)} \leq C_A(\|h - \underline{h}\|_{W_\#^{1,\infty}(0,L)}) \|w\|_{\mathbf{H}_\#^1(F_{\underline{h}}^-)}.$$

Using interpolation we finally obtain (up to choose $s \leq \sigma < \kappa/2$) that

$$\|w_2^X|_{y=0}\|_{H_\#^{2s}(0,L)} \leq C_A(\|h - \underline{h}\|_{W_\#^{1,\infty}(0,L)}) \|w\|_{\mathbf{H}_\#^1(F_{\underline{h}}^-)},$$

which concludes the proof of (2.28) and the proof of the lemma is completed.

3. PROOF OF THEOREM 10

This section is devoted to the proof of existence of weak solutions of $(\text{FS})_0$. So, we fix $T > 0$ and initial data $(\mathbf{u}^0, \eta^0, \eta^1)$ satisfying (2.11)–(2.14). We recall that the strategy is to approximate this problem by a sequence of viscous problems $(\text{FS})_\gamma$, $\gamma > 0$, for which existence results are available. The proof is divided into three steps. First, we analyze the Cauchy theory of $(\text{FS})_\gamma$ when $\gamma > 0$ and prove that the sequence of solutions converges, up to a subsequence, when $\gamma \rightarrow 0$. We show in particular that possible weak limits are candidates to be weak solutions up to obtaining strong compactness of approximate velocities in L^2 . As explained in the introduction, this strong compactness property is the cornerstone of the analysis. Our proof builds on the projection/approximation argument provided by [16] in the fluid–solid case. In our fluid–elastic setting, it requires to build a uniform bound by below \underline{h} of the sequence of approximate structure deformations (in order to construct a fluid domain independent of γ on which the Navier–Stokes equations are satisfied by the sequence of approximate solutions to be able to apply Aubin–Lions Lemma for projections of the velocities). The second step of the proof is devoted to the construction of \underline{h} and the analysis of its properties. We then complete the proof of the L^2 -strong compactness. This last step relies in particular on the continuity result obtained in subsection 2.4.

3.1. STEP 1. CONSTRUCTION OF A CANDIDATE WEAK-SOLUTION. — Let us recall the strong existence result on $(0, T)$ stated in [7, Th. 1]. Given $\gamma > 0$ and initial data $(\mathbf{u}_\gamma^0, \eta_\gamma^0, \eta_\gamma^1)$ satisfying

$$(3.1) \quad (\eta_\gamma^0, \eta_\gamma^1) \in H_\#^3(0, L) \times H_\#^1(0, L),$$

$$(3.2) \quad \mathbf{u}_\gamma^0 \in \mathbf{H}_\#^1(F_{h_\gamma^0}), \quad \text{div } \mathbf{u}_\gamma^0 = 0 \text{ in } F_{h^0},$$

$$(3.3) \quad \mathbf{u}_\gamma^0(x, 0) = 0, \text{ and } \mathbf{u}_\gamma^0(x, h_\gamma^0(x)) = \eta_\gamma^1(x) \mathbf{e}_2, \quad \forall x \in [0, L],$$

$$(3.4) \quad \min_{x \in [0, L]} h_\gamma^0(x) > 0 \text{ and } \int_0^L \eta_\gamma^1(x) \, dx = 0,$$

the system $(\text{FS})_\gamma$ admits a unique strong solution defined on $(0, T)$. This solution satisfies moreover $\min_{x \in [0, L]} h_\gamma(t, x) > 0$ for all $t \in [0, T]$.

In order to apply this result we now explain the construction of a sequence of regular initial data $(\mathbf{u}_\gamma^0, \eta_\gamma^0, \eta_\gamma^1)_{\gamma > 0}$ approximating $(\mathbf{u}^0, \eta^0, \eta^1)$. First we construct

$\eta_\gamma^0 \in H_\#^3(0, L)$ by a standard convolution of η^0 with a regularizing kernel. Since η^0 satisfies (2.11), this sequence is uniformly bounded in $H_\#^2(0, L)$ and satisfies

$$\begin{aligned} \eta_\gamma^0 &\longrightarrow \eta^0, \text{ in } H_\#^2(0, L), \\ \|\eta_\gamma^0 - \eta^0\|_{C_\#^0([0, L])} &\leq \gamma \|\eta^0\|_{H_\#^2(0, L)} \leq C\gamma. \end{aligned}$$

Since $\min_{x \in [0, L]} h^0(x) > 0$, there exists $\lambda > 0$ verifying $\min_{x \in [0, L]} h_\gamma^0(x) > \lambda > 0$ for γ small enough. We next construct $\eta_\gamma^1 \in H_\#^1(0, L) \cap L_{\#,0}^2(0, L)$. Since η^1 satisfies (2.12), this second sequence enjoys the following properties:

$$\begin{aligned} \eta_\gamma^1 &\longrightarrow \eta^1, \text{ in } L_\#^2(0, L), \\ \|\eta_\gamma^1\|_{L_{\#,0}^2(0, L)} &\leq \|\eta^1\|_{L_\#^2(0, L)} \leq C. \end{aligned}$$

We now build the approximate initial velocity fields \mathbf{u}_γ^0 . A key difficulty here is to match the continuity of velocity field at the structure interface together with preserving the divergence free condition, taking into account that the approximation is defined on an approximate domain depending on γ . To handle this difficulty, we first define the extension of η^1 to the whole domain using the operator R_λ as defined in Definition 11. Next we consider $\overline{\mathbf{u}^0} - R_\lambda(\eta^1)$ which is in $K[h^0]$ and satisfies moreover $\overline{\mathbf{u}^0} - R_\lambda(\eta^1) = \mathbf{0}$ in $S_{h^0} \cup \Omega_{-1}^0$. Then we introduce the vertical contraction operator denoted by

$$\mathbf{v} \longmapsto \mathbf{v}_\sigma(x, y) = (\sigma v_1(x, \sigma y), v_2(x, \sigma y)) \quad \forall \sigma > 0.$$

We emphasize that this contraction operator preserves the divergence free constraint. By choosing $\sigma_\gamma = 1 + 2C\gamma/\lambda$ (with the constant C above), we have $(\overline{\mathbf{u}^0} - R_\lambda(\eta^1))_{\sigma_\gamma} = \mathbf{0}$ in $S_{h_\gamma^0} \cup \Omega_{-1}^0$, and $(\overline{\mathbf{u}^0} - R_\lambda(\eta^1))_{\sigma_\gamma}$ converges to $\overline{\mathbf{u}^0} - R_\lambda(\eta^1)$ in $L_\#^2(\Omega)$ when $\gamma \rightarrow 0$. Moreover, $(\overline{\mathbf{u}^0} - R_\lambda(\eta^1))_{\sigma_\gamma}$ belongs to $L_\#^2(F_{h_\gamma^0})$, is divergence free and satisfies $(\overline{\mathbf{u}^0} - R_\lambda(\eta^1))_{\sigma_\gamma} \cdot \mathbf{n} = 0$ on $\Gamma_{h_\gamma^0}$ and $(0, L) \times \{0\}$. Thus we approximate thanks to standard arguments (by truncation and regularization of the stream function for instance) this function by a divergence free function $(\overline{\mathbf{u}^0} - R_\lambda(\eta^1))_\gamma$ in $\mathbf{H}_\#^1(F_{h_\gamma^0})$ vanishing in a neighbourhood of $\Gamma_{h_\gamma^0}$ and $(0, L) \times \{0\}$. We may then set

$$\mathbf{u}_\gamma^0 = \left((\overline{\mathbf{u}^0} - R_\lambda(\eta^1))_\gamma + R_\lambda(\eta_\gamma^1) \right)_{|F_{h_\gamma^0}}.$$

Straightforward computations show that \mathbf{u}_γ^0 satisfies (3.2)–(3.3). Moreover, remarking that the operator R_λ is continuous from $L_{\#,0}^2(0, L)$ into $L_\#^2(\Omega)$ we have as γ goes to zero

$$\begin{aligned} \overline{\mathbf{u}_\gamma^0} &\longrightarrow \overline{\mathbf{u}^0}, \text{ in } L_\#^2(\Omega), \\ \|\mathbf{u}_\gamma^0\|_{L^2(F_{h_\gamma^0})} &\leq C(\|\eta^1\|_{L_\#^2(0, L)} + \|\mathbf{u}^0\|_{L^2(F_h^0)}) \leq C, \end{aligned}$$

where C does not depend on γ .

We now apply the result on existence of a strong solution for the viscous problem (FS) $_{\gamma>0}$. For fixed $\gamma > 0$ the unique solution $(\mathbf{u}_\gamma, \eta_\gamma)$ is global in time so that it exists

on any time interval $(0, T)$. The first step is to verify that $\bar{\mathbf{u}}_\gamma$ as defined by

$$\bar{\mathbf{u}}_\gamma = \begin{cases} \partial_t \eta_\gamma \mathbf{e}_2, & \text{in } S_{h_\gamma}, \\ \mathbf{u}_\gamma, & \text{in } F_{h_\gamma}, \\ \mathbf{0}, & \text{in } \Omega_{-1}^0, \end{cases}$$

together with η_γ is a pair of weak solution to $(\text{FS})_\gamma$ in the sense of Definition 9. First, we note that $(\mathbf{u}_\gamma, \eta_\gamma)$ satisfies estimate (2.2) so that we can define a constant M involved in the definition of our weak solution framework. By construction we have that

$$\eta_\gamma \in L^\infty(0, T; H_\#^2(0, L)) \cap W^{1,\infty}(0, T; L_{\#,0}^2(0, L)).$$

Moreover,

$$\bar{\mathbf{u}}_\gamma \in L^\infty(0, T; L_\#^2(\Omega)), \quad \nabla \bar{\mathbf{u}}_\gamma|_{\widehat{F}_{h_\gamma}^-} = \mathbb{1}_{\widehat{F}_{h_\gamma}^-} \nabla \mathbf{u}_\gamma \in L^2(\widehat{F}_{h_\gamma}^-),$$

and Lemma 4, implies that

$$(\bar{\mathbf{u}}_\gamma(t), \eta_\gamma(t)) \in X[h_\gamma(t)] \quad \text{for a.e. } t \in (0, T).$$

Thus the regularity statement (i) of Definition 9 is satisfied. Moreover the solution satisfies the kinematic condition $\mathbf{u}_\gamma(t, x, 1 + \eta_\gamma(t, x)) = \partial_t \eta_\gamma(t, x) \mathbf{e}_2$ is satisfied on $(0, L)$ which implies that $\partial_t \eta_\gamma \in L^2(0, T; H_\#^{1/2}(0, L))$ as the trace of $\bar{\mathbf{u}}_\gamma|_{\widehat{F}_{h_\gamma}^-}$. Remember here that $\min_{x \in [0, L]} h_\gamma(t, x) > 0$ for all $t \in [0, T]$ so that F_{h_γ} is a Lipschitz domain and consequently $\bar{\mathbf{u}}_\gamma|_{y=h_\gamma}$ is well defined. Thus the second item (ii) of Definition 9 holds true. Then, we note that, thanks to the regularity of solutions constructed in [7, Th. 1], the system (1.2)–(1.3) is satisfied pointwise so that we can multiply the system with test functions $(\mathbf{w}_\gamma, d_\gamma)$ for which the requirements in item (iii) of Definition 9 are satisfied and obtain (2.15) after integration by parts.

Moreover, we note that the solution $(\mathbf{u}_\gamma, \eta_\gamma)$ satisfies the energy estimate (2.2) with a right hand side that converges to

$$\frac{1}{2} \left(\rho_f \int_{F_{h_0}} |\mathbf{u}^0|^2 \, d\mathbf{x} + \rho_s \int_0^L |\eta^1|^2 \, dx + \beta \int_0^L |\partial_x \eta^0|^2 \, dx + \alpha \int_0^L |\partial_{xx} \eta^0|^2 \, dx \right).$$

when $\gamma \rightarrow 0$. Consequently, the sequence $(\bar{\mathbf{u}}_\gamma, \eta_\gamma)_{\gamma > 0}$ satisfies the following bounds:

(3.5) $\bar{\mathbf{u}}_\gamma$ is uniformly bounded in γ in $L^\infty(0, T; \mathbf{L}_\#^2(F_{h_\gamma(t)}))$,

(3.6) $\|\nabla \bar{\mathbf{u}}_\gamma\|_{L^2(\widehat{F}_{h_\gamma}^-)}$ is uniformly bounded in γ ,

(3.7) η_γ is uniformly bounded in γ in $L^\infty(0, T; H_\#^2(0, L)) \cap W^{1,\infty}(0, T; L_\#^2(0, L))$.

Furthermore, the structure velocity $\partial_t \eta_\gamma$ is bounded uniformly with respect to γ is $L^2(0, T; H_\#^{1/2}(0, L))$ as the trace $\bar{\mathbf{u}}_\gamma|_{y=h_\gamma}$.

Finally the sequence $(\bar{\mathbf{u}}_\gamma)_{\gamma > 0}$ satisfies additional uniform estimates that are summarized in the following lemma:

LEMMA 14. — *The sequence $(\bar{\mathbf{u}}_\gamma)_{\gamma > 0}$ is bounded in $L^4(\widehat{\Omega})$ and in $L^2(0, T; \mathbf{H}_\#^s(\Omega))$ for arbitrary $s < 1/2$.*

Proof. — The bound in $L^2(0, T; \mathbf{H}_\#^s(\Omega))$ comes again from Lemma 4. We next take care of the $L^4(\widehat{\Omega})$ uniform bound. To prove it, it is sufficient to obtain independent uniform bounds for the restrictions of $\bar{\mathbf{u}}_\gamma$ to \widehat{S}_{h_γ} and to $\widehat{F}_{h_\gamma}^-$ for the L^4 -norm.

In \widehat{S}_{h_γ} we already know that $\bar{\mathbf{u}}_\gamma = \partial_t \eta_\gamma \mathbf{e}_2$, where

$$\partial_t \eta_\gamma \text{ is bounded in } L^2(0, T; H_\#^{1/2}(0, L)) \cap L^\infty(0, T; L_\#^2(0, L)).$$

By interpolation, we obtain that $\partial_t \eta_\gamma$ —and consequently resp. $\bar{\mathbf{u}}_\gamma$ —is uniformly bounded in $L^4(0, T; L_\#^4(0, L))$, resp. $L^4(\widehat{S}_{h_\gamma})$.

We would like to apply a similar interpolation argument on the domain $F_{h_\gamma}^-$ using the uniform bounds on the restrictions of $\bar{\mathbf{u}}_\gamma$ and its gradient in $F_{h_\gamma}^-$. To track the dependencies with respect to h_γ to be able to ensure that the interpolation argument leads to uniform bounds, we use a change of variables. We denote by

$$\tilde{\mathbf{u}}_\gamma(t, x, z) = \bar{\mathbf{u}}_\gamma(t, x, (h_\gamma(t, x) + 1)z - 1) \quad \forall (t, x, z) \in (0, T) \times \Omega_0^1.$$

Since $0 \triangleleft h_\gamma \triangleleft M$, straightforward computations show that, for a.e. $t \in (0, T)$ we have

$$\|\bar{\mathbf{u}}_\gamma(t)\|_{L_\#^4(F_{h_\gamma(t)})} \triangleleft (\|h_\gamma(t)\|_{L_\#^\infty(0, L)} + 1)^{1/4} \|\tilde{\mathbf{u}}_\gamma(t)\|_{L_\#^4(\Omega_0^1)},$$

and

$$\begin{aligned} \|\tilde{\mathbf{u}}_\gamma(t)\|_{L_\#^2(\Omega_0^1)} &\triangleleft \|\bar{\mathbf{u}}_\gamma(t)\|_{L_\#^2(F_{h_\gamma(t)})}, \\ \|\nabla \tilde{\mathbf{u}}_\gamma(t)\|_{L_\#^2(\Omega_0^1)} &\triangleleft (1 + \|h_\gamma(t)\|_{W_\#^{1, \infty}(0, L)}) \|\nabla \bar{\mathbf{u}}_\gamma\|_{L_\#^2(F_{h_\gamma(t)}^-)}. \end{aligned}$$

However, the following interpolation inequality holds true

$$\|\tilde{\mathbf{u}}_\gamma(t)\|_{L_\#^4(\Omega_0^1)} \triangleleft C \|\tilde{\mathbf{u}}_\gamma(t)\|_{L_\#^2(\Omega_0^1)}^{1/2} \|\nabla \tilde{\mathbf{u}}_\gamma(t)\|_{L_\#^2(\Omega_0^1)}^{1/2},$$

and thus

$$\|\bar{\mathbf{u}}_\gamma(t)\|_{L_\#^4(F_{h_\gamma(t)}^-)} \triangleleft C(1 + \|h_\gamma(t)\|_{W_\#^{1, \infty}(0, L)}) \|\mathbf{u}_\gamma(t)\|_{L_\#^2(F_{h_\gamma(t)}^-)}^{1/2} \|\nabla \tilde{\mathbf{u}}_\gamma(t)\|_{L_\#^2(\Omega_0^1)}^{1/2}.$$

Applying (3.5) and (3.6) together with the bound for h_γ in $L^\infty(0, T; W_\#^{1, \infty}(0, L))$ coming from (3.7), and thus uniform in γ , we obtain the desired bound on $\bar{\mathbf{u}}_\gamma$ which is uniformly bounded in $L^4(F_{h_\gamma}^-)$.

We now prove the existence of cluster points of the sequence $(\bar{\mathbf{u}}_\gamma, \eta_\gamma)_{\gamma > 0}$. First, thanks to (3.7), and to the compact embedding (2.3)

$$\begin{aligned} \eta_\gamma &\rightarrow \eta \text{ uniformly in } C^0([0, T]; C_\#^1(0, L)), \\ \eta_\gamma &\rightharpoonup \eta \text{ weakly-}\star \text{ in } W^{1, \infty}(0, T; L_\#^2(0, L)). \end{aligned}$$

Next, using the energy estimates (2.2) and Lemma 14, we may construct a divergence free function $\bar{\mathbf{u}} \in L^\infty(0, T; \mathbf{L}_\#^2(\Omega)) \cap L^4(\widehat{\Omega})$, such that, up to a subsequence, the following convergences hold:

$$\begin{aligned} \bar{\mathbf{u}}_\gamma &\rightharpoonup \bar{\mathbf{u}}, \text{ weakly-}\star \text{ in } L^\infty(0, T; \mathbf{L}_\#^2(\Omega)), \\ \bar{\mathbf{u}}_\gamma &\rightharpoonup \bar{\mathbf{u}}, \text{ weakly in } L^4(\widehat{\Omega}). \end{aligned}$$

We now verify that any cluster point $(\bar{\mathbf{u}}, \eta)$ of the sequence $(\mathbf{u}_\gamma, \eta_\gamma)_{\gamma>0}$ enjoys the properties of Definition 9, which defines the weak solutions of the limit coupled system $(\text{FS})_0$. For simplicity, we do not relabel the sequence converging to $(\bar{\mathbf{u}}, \eta)$. We first note that, for fixed $\gamma > 0$, we have

$$\int_0^L \partial_t \eta_\gamma = 0.$$

This property is conserved in the weak limit so that $\partial_t \eta$ is mean free globally in time. Next, we verify that $(\bar{\mathbf{u}}(t), \partial_t \eta) \in X[h(t)]$ for a.e. $t \in (0, T)$. The divergence free condition is verified at the limit. Moreover, we note that, the property $\bar{\mathbf{u}}_\gamma = 0$ on $\widehat{\Omega}_{-1}^0$ is preserved in the weak limit. Furthermore, since η_γ uniformly converges towards η , we easily obtain $\bar{\mathbf{u}} = \partial_t \eta \mathbf{e}_2$, in \widehat{S}_h , by testing the weak convergence of $\bar{\mathbf{u}}_\gamma$ and $\partial_t \eta_\gamma \mathbf{e}_2$, which are equal in \widehat{S}_{h_γ} , against functions $\varphi \in C_c^\infty(\widehat{S}_h)$.

Consequently, $(\bar{\mathbf{u}}(t), \partial_t \eta) \in X[h(t)]$. We now prove that $\bar{\mathbf{u}}$ has better regularity in \widehat{F}_h^- as stated in the following lemma. In what follows, we set $\rho_\gamma = \mathbb{1}_{F_\gamma}$, $\rho_\gamma^- = \mathbb{1}_{F_{h_\gamma}^-}$ and $\rho^- = \mathbb{1}_{F_h^-}$.

LEMMA 15. — We have $\nabla \bar{\mathbf{u}} \in L^2(\widehat{F}_h^-)$ and the sequence $\rho_\gamma^- \nabla \bar{\mathbf{u}}_\gamma$ converges to $\rho^- \nabla \bar{\mathbf{u}}$ weakly in $L^2(\widehat{\Omega})$.

REMARK 16. — In the previous statement, we use the convention that if $O \subset \widehat{\Omega}$ and $f \in L^2(O)$ then $\mathbb{1}_O f \in L^2(\widehat{\Omega})$ is the extension by 0 of this $L^2(O)$ -function.

Proof. — We remind that $\nabla \bar{\mathbf{u}}_\gamma \in L^2(\widehat{F}_{h_\gamma}^-)$ so that $\rho_\gamma^- \nabla \bar{\mathbf{u}}_\gamma$ corresponds to the extension by $\mathbf{0}$ of this vector field. Because of (3.6), $\rho_\gamma^- \nabla \bar{\mathbf{u}}_\gamma$ is uniformly bounded in $L^2(\widehat{\Omega})$. Thus $\rho_\gamma^- \nabla \bar{\mathbf{u}}_\gamma$ converges weakly to some \mathbf{z} in $L^2(0, T; L^2_\#(\Omega))$. Thanks to the uniform convergence of h_γ to h , we may then compute \mathbf{z} by testing the weak convergence of $\rho_\gamma^- \nabla \bar{\mathbf{u}}_\gamma$ against functions $\varphi \in C_c^\infty(\widehat{S}_h)$ and $\varphi \in C_c^\infty(\widehat{F}_h^-)$ respectively. This implies that $\mathbf{z}|_{\widehat{S}_h} = \mathbf{0}$ and $\mathbf{z}|_{\widehat{F}_h^-} = (\nabla \bar{\mathbf{u}})|_{\widehat{F}_h^-}$, which ends the proof.

REMARK 17. — The previous lemma gives the H^1 space regularity of $\bar{\mathbf{u}}|_{F_{h(t)}^-}$ (for a.e. $t \in (0, T)$). Since $F_{h(t)}^-$ a Lipschitz domain, it enables us to define an $H^{1/2}$ trace of $\bar{\mathbf{u}}$ on $\partial F_{h(t)}^-$.

This concludes the proof that $(\bar{\mathbf{u}}, \eta)$ satisfies item (i) of Definition 9. We also prove at first that the weak cluster point satisfies the expected energy estimate. Indeed, for any arbitrary small $\varepsilon > 0$, thanks to the strong convergence of η_γ to η , we have that $h_\gamma > h - \varepsilon$ for γ sufficiently small. We may apply then classical weak limit arguments to pass to the limit in the energy estimate satisfied by the $(\mathbf{u}_\gamma, \eta_\gamma)$. Consequently for

almost every t we have

$$\begin{aligned} & \frac{1}{2} \left(\rho_f \int_{F_{h(t)-\varepsilon}^-} |\mathbf{u}(t, \mathbf{x})|^2 \, d\mathbf{x} + \rho_s \int_0^L |\partial_t \eta(t, x)|^2 \, dx \right. \\ & \quad \left. + \beta \int_0^L |\partial_x \eta(t, x)|^2 \, dx + \alpha \int_0^L |\partial_{xx} \eta(t, x)|^2 \, dx \right) + \mu \int_0^t \int_{F_{h(s)-\varepsilon}^-} |\nabla \mathbf{u}(s, \mathbf{x})|^2 \, d\mathbf{x} \, ds \\ \hookrightarrow & \liminf_{\gamma \rightarrow 0} \left[\frac{1}{2} \left(\rho_f \int_{F_{h(t)-\varepsilon}^-} |\mathbf{u}_\gamma(t, \mathbf{x})|^2 \, d\mathbf{x} + \rho_s \int_0^L |\partial_t \eta_\gamma(t, x)|^2 \, dx \right. \right. \\ & \quad \left. \left. + \beta \int_0^L |\partial_x \eta_\gamma(t, x)|^2 \, dx + \alpha \int_0^L |\partial_{xx} \eta_\gamma(t, x)|^2 \, dx \right) \right. \\ & \quad \left. + \mu \int_0^t \int_{F_{h(s)-\varepsilon}^-} |\nabla \mathbf{u}_\gamma(s, \mathbf{x})|^2 \, d\mathbf{x} \, ds \right] \\ \hookrightarrow & \lim_{\gamma \rightarrow 0} \frac{1}{2} \left(\rho_f \int_{F_{h_0}^0} |\mathbf{u}_\gamma^0|^2 \, d\mathbf{x} + \rho_s \int_0^L |\eta_\gamma^1|^2 \, dx + \beta \int_0^L |\partial_x \eta_\gamma^0|^2 \, dx + \alpha \int_0^L |\partial_{xx} \eta_\gamma^0|^2 \, dx \right) \\ = & \frac{1}{2} \left(\rho_f \int_{F_{h_0}^0} |\mathbf{u}^0|^2 \, d\mathbf{x} + \rho_s \int_0^L |\eta^1|^2 \, dx + \beta \int_0^L |\partial_x \eta^0|^2 \, dx + \alpha \int_0^L |\partial_{xx} \eta^0|^2 \, dx \right). \end{aligned}$$

Since ε is arbitrary, we obtain the expected energy inequality.

We now show that any limit $(\bar{\mathbf{u}}, \eta)$ also satisfies items (ii) and (iii) of Definition 9 of weak solutions under the further assumption that the following lemma holds true:

LEMMA 18. — *Up to the extraction of a subsequence that we do not relabel, we have $(\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma) \xrightarrow{\gamma \rightarrow 0} (\rho \bar{\mathbf{u}}, \partial_t \eta)$ strongly in $L^2(\widehat{\Omega}) \times L^2((0, T) \times (0, L))$.*

So, fix $(\mathbf{w}, d) \in C^\infty(\widehat{\Omega}) \times C^\infty(0, L)$ such that $(\mathbf{w}(t), d(t)) \in \times [h(t)]$ for all $t \in (0, T)$. Due to the uniform convergence of h_γ and the special structure of $\times [h(t)]$ for which we require that $\mathbf{w} = 0$ in the neighbourhood of Ω_{-1}^0 and $\mathbf{w} \cdot \mathbf{e}_1 = 0$ in the neighbourhood of $S_h(t)$, there exists $\gamma_0 > 0$ such that, for all $0 < \gamma < \gamma_0$, $(\mathbf{w}(t), d(t)) \in X[h_\gamma(t)]$ for all $t \in (0, T)$. Hence, (\mathbf{w}, d) is a test function for any γ small enough and for a.e. $t \in (0, T)$, $(\bar{\mathbf{u}}_\gamma, \eta_\gamma)$ satisfies

$$\begin{aligned} (3.8) \quad & \rho_f \int_\Omega \rho_\gamma \bar{\mathbf{u}}_\gamma(t) \cdot \mathbf{w}(t) - \rho_f \int_0^t \int_\Omega \rho_\gamma \bar{\mathbf{u}}_\gamma \cdot \partial_t \mathbf{w} + (\rho_\gamma \bar{\mathbf{u}}_\gamma \cdot \nabla) \mathbf{w} \cdot \rho_\gamma \bar{\mathbf{u}}_\gamma \\ & \quad + \rho_s \int_0^L \partial_t \eta_\gamma(t) d(t) - \rho_s \int_0^t \int_0^L \partial_t \eta_\gamma \partial_t d + \mu \int_0^t \int_\Omega \rho_\gamma \nabla \mathbf{u}_\gamma : \nabla \mathbf{w} \\ & \quad + \int_0^t \int_0^L \beta \partial_x \eta_\gamma \partial_x d + \alpha \partial_{xx} \eta_\gamma \partial_{xx} d + \gamma \int_0^t \int_0^L \partial_{tx} \eta_\gamma \partial_x d \\ & \quad = \rho_f \int_\Omega \rho_\gamma \bar{\mathbf{u}}_\gamma^0 \cdot \mathbf{w}(0) + \rho_s \int_0^L \eta_\gamma^1 d(0). \end{aligned}$$

Let us recall that, thanks to Lemma 18, $\rho_\gamma \bar{\mathbf{u}}_\gamma$ strongly converges in $L^2(\widehat{\Omega})$. The convergence of $\rho_\gamma \nabla \mathbf{u}_\gamma$ is proved in Lemma 15 and we can pass to the limit all the

terms of (3.8). The pair $(\bar{\mathbf{u}}, \eta)$ satisfies, for a.e. $t \in (0, T)$,

$$\begin{aligned} & \rho_f \int_{\Omega} \rho \bar{\mathbf{u}}(t) \cdot \mathbf{w}(t) - \rho_f \int_0^t \int_{\Omega} \rho \bar{\mathbf{u}} \cdot \partial_t \mathbf{w} + (\rho \bar{\mathbf{u}} \cdot \nabla) \mathbf{w} \cdot \rho \bar{\mathbf{u}} + \rho_s \int_0^L \partial_t \eta(t) d(t) \\ & - \rho_s \int_0^t \int_0^L \partial_t \eta \partial_t d + \mu \int_0^t \int_{\Omega} \rho \mathbf{u} : \nabla \mathbf{w} + \int_0^t \int_0^L \beta \partial_x \eta \partial_x d + \alpha \partial_{xx} \eta \partial_{xx} d \\ & = \rho_f \int_{\Omega} \rho \bar{\mathbf{u}}^0 \cdot \mathbf{w}(0) + \rho_s \int_0^L \eta^1 d(0), \end{aligned}$$

which is a rewriting of the variational formulation (2.15). Thus the item (iii) of Definition 9 is satisfied. The last point to verify is the kinematic condition at the fluid–structure interface. We know that

$$\mathbf{u}_{\gamma}(t, x, h_{\gamma}(t, x)) = \partial_t \eta_{\gamma}(t, x) \mathbf{e}_2.$$

From Lemma 18 the right hand side converges strongly in $L^2(0, T; L^2_{\sharp}(0, L))$ towards $\partial_t \eta$. It converges also weakly in $L^2(0, T; H^{1/2}_{\sharp}(0, L))$. The left hand side is the trace of the function $(x, z) \mapsto \mathbf{u}_{\gamma}(t, x, (1 + h_{\gamma}(t, x))z + h_{\gamma}(t, x))$ on $z = 0$. Thanks to the previous convergences this function converges strongly in $L^2(\widehat{\Omega}_0^1)$ and weakly in $L^2(0, T; H^1_{\sharp}(\Omega_0^1))$ towards $\mathbf{u}(t, x, (1 + h(t, x))z + h(t, x))$. Hence by continuity of the trace we obtain

$$\mathbf{u}(t, x, h(t, x)) = \partial_t \eta(t, x) \mathbf{e}_2,$$

so that item (ii) of Definition 9, which completes the proof of Theorem 10.

It thus remains to prove Lemma 18. As is usual for fluid–structure problems, the sequence of domains is unknown and depends on time and here on the viscosity parameter, so that standard Aubin–Lions lemma cannot be applied directly to obtain compactness of the velocities. One key point is to build a piecewise in time, regular enough in space, interface, dealing with possible contact, close to the sequence of interfaces but always lower. The construction of this artificial interface is the aim of the next subsection. We then conclude the proof in the final subsection. Thanks to the variational formulation and to this well chosen interface “from below” we obtain bounds on the time derivative of an L^2 projection of the velocities, for which we are able to apply an adapted version of the Aubin–Lions lemma. It implies that the sequence of velocities is nearly compact. Next the key idea is to use that the velocities can be approximated, in H^s for some $s > 0$, by velocities associated to the interface “from below” so that we can “fill” the gap. This relies on the continuity properties of the H^s -projector operator obtained in Lemma 13.

3.2. STEP 2. CONSTRUCTION OF AN INTERFACE FROM BELOW. — Before the construction of the interface from below, we analyze a simple method to approximate a given stationary deformation from below. Namely, given $h \in H^2_{\sharp}(0, L)$ satisfying $h > 0$ and $\mu > 0$, we denote

$$\underline{h}_{\mu} := [h - \mu]_{+},$$

where the subscript + denotes here the positive part of functions. The properties of this approximation process are summarized in the following lemma:

LEMMA 19. — Let $h \in H_{\#}^2(0, L)$ with $h > 0$ and $\mu > 0$. Then $\underline{h}_{\mu} \in C_{\#}^0(0, L)$ and, given $\kappa \in (0, 1/2)$, there exists a constant C independent of μ and h for which:

$$(3.9) \quad \|\underline{h}_{\mu}\|_{H_{\#}^{1+\kappa}(0, L)} + \|\underline{h}_{\mu}\|_{W_{\#}^{1, \infty}(0, L)} \leq C \|h\|_{H_{\#}^2(0, L)},$$

$$(3.10) \quad \|\underline{h}_{\mu} - h\|_{W_{\#}^{1, \infty}(0, L)} \leq \mu + \sup_{\{x \in [0, L] \mid h(x) \leq \mu\}} |h'(x)|.$$

Proof. — Let $\mu > 0$ and $h \in H_{\#}^2(0, L)$ non-negative. The first statement $\underline{h}_{\mu} \in C_{\#}^0(0, L)$ is standard. We prove the two inequalities (3.9), (3.10) successively.

Step 1: proof of inequality (3.9). — Since $h \in C_{\#}^0(0, L)$, the subset $\{x \in (0, L) \mid h(x) > \mu\}$ is open. We may then construct at most denumerable sets $\{c_i, i \in \mathbb{N}\}$ and $\{d_i, i \in \mathbb{N}\}$ such that $\{x \in (0, L) \mid h(x) > \mu\} = \bigcup_{i \in \mathbb{N}} (c_i, d_i)$. The successive derivatives of $[h - \mu]_+$ then read

$$[h - \mu]_+' = h' \mathbf{1}_{h > \mu}, \quad [h - \mu]_+'' = h'' \mathbf{1}_{h > \mu} + \sum_{i \in \mathbb{N}} \delta_{c_i} h'(c_i) - \delta_{d_i} h'(d_i),$$

To show that $\underline{h}_{\mu} \in H_{\#}^{1+\kappa}(0, L)$ and have a bound on its norm, we now prove that $[h - \mu]_+'' \in H_{\#}^{\kappa-1}(0, L)$.

Using the H^2 -regularity of h we obtain that

$$h'' \mathbf{1}_{h > \mu} \in L_{\#}^2(0, L) \subset H_{\#}^{\kappa-1}(0, L).$$

Moreover, for any test function $\varphi \in D(0, L)$, we have:

$$|\langle \delta_{c_i}, \varphi \rangle| = |\varphi(c_i)| \leq \|\varphi\|_{C_{\#}(0, L)} \leq C \|\varphi\|_{H_{\#}^{1-\kappa}(0, L)},$$

where $C > 0$ stands for the constant associated with the Sobolev embedding

$$H_{\#}^{1-\kappa}(0, L) \subset C_{\#}^0(0, L).$$

Hence $\delta_{c_i} \in H^{\kappa-1}(0, L)$, with a norm independent of c_i . To show that $[h - \mu]_+'' \in H_{\#}^{\kappa-1}(0, L)$ it then remains to prove that the sums

$$\sum_{i \in \mathbb{N}} \delta_{c_i} h'(c_i), \quad \sum_{i \in \mathbb{N}} \delta_{d_i} h'(d_i),$$

do converge normally in the Banach space $H_{\#}^{\kappa-1}(0, L)$.

Let $i \in \mathbb{N}$, since $h(c_i) = h(d_i) = \mu$, there exists $b_i \in (c_i, d_i)$ such that $h'(b_i) = 0$. This implies

$$|h'(c_i)| \leq |b_i - c_i|^{1/2} \|h\|_{H^2(c_i, d_i)} \leq |d_i - c_i|^{1/2} \|h\|_{H^2(c_i, d_i)},$$

and thus

$$\begin{aligned}
 \sum_{i \in I_\mu} \|h'(c_i)\delta_{c_i}\|_{H_\#^{\kappa-1}(0,L)} &\leq C \sum_{i \in I_\mu} |d_i - c_i|^{1/2} \|h\|_{H^2(c_i,d_i)} \\
 (3.11) \qquad \qquad \qquad &\leq C \left(\sum_{i \in I_\mu} |d_i - c_i| \right)^{1/2} \left(\sum_{i \in I_\mu} \|h\|_{H^2(c_i,d_i)}^2 \right)^{1/2} \\
 &\leq CL^{1/2} \|h\|_{H_\#^2(0,L)}.
 \end{aligned}$$

Consequently, we have

$$\sum_{i \in I_\mu} \delta_{c_i} h'(c_i) \in H_\#^{\kappa-1}(0, L) \quad \text{and} \quad \left\| \sum_{i \in I_\mu} \delta_{c_i} h'(c_i) \right\|_{H_\#^{\kappa-1}(0,L)} \leq CL^{1/2} \|h\|_{H_\#^2(0,L)}.$$

The argument is similar for $\sum_{i \in I_\mu} \delta_{d_i} h'(d_i)$. This completes the proof of estimate (3.9).

Step 2: proof of estimate (3.10). — Since $h \in H_\#^2(0, L)$, thanks to the continuous embedding of $H_\#^1(0, L)$ in $L_\#^\infty(0, L)$, we deduce that $[h - \mu]_+ \in W_\#^{1,\infty}(0, L)$. Furthermore, it is clear that, for any $x \in [0, L]$,

$$|[h - \mu]_+(x) - h(x)| \leq \mu, \quad [h - \mu]'_+(x) - h'(x) = -h'(x)\mathbf{1}_{h \leq \mu}(x).$$

This implies that (3.10) is satisfied.

The next lemma ensures that the right-hand side of estimate (3.10) goes to zero when μ goes to zero:

LEMMA 20. — *Let $h \in C_\#^1(0, L)$ with $h > 0$. The following limit holds*

$$\sup_{\{x \in [0, L] \mid h(x) \leq \mu\}} |h'(x)| \xrightarrow{\mu \rightarrow 0} 0.$$

Proof. — Since $h \in C_\#^1(0, L)$ we have that $\{x \in [0, L] \mid h(x) \leq \mu\}$ is a compact subset of $[0, L]$ and that there exists $x_\mu \in [0, L]$ such that:

$$\sup_{\{x \in [0, L] \mid h(x) \leq \mu\}} |h'(x)| = |h'(x_\mu)|.$$

Note that, by construction, we have $h(x_\mu) \leq \mu$.

Using the compactness of $[0, L]$ we have $x_\mu \rightarrow \bar{x} \in [0, L]$ as μ goes to zero (up to a subsequence). Using the continuity of h and passing to the limit in the inequality $h(x_\mu) \leq \mu$ we obtain $h(\bar{x}) = 0$. Moreover, using that $h(x) > 0$ we deduce that \bar{x} is a local minimum of h and thus that $h'(\bar{x}) = 0$. Finally the continuity of h' ensures that $|h'(x_\mu)| \xrightarrow{\mu \rightarrow 0} |h'(\bar{x})| = 0$.

We recall that we consider a sequence $(h_\gamma)_{\gamma > 0}$ that converges to h strongly in the sense of $C^0([0, T]; C_\#^1(0, L))$ with $h \in L^\infty(0, T; H_\#^2(0, L)) \cap W^{1,\infty}(0, T; L_\#^2(0, L))$. We now are in a position to build a family of approximating interfaces “from below”

of any h_γ for γ small enough. Namely, given $\delta > 0$ we construct a piecewise-constant (in time) function \underline{h}_δ such that there exists $\gamma_0 > 0$ for which

$$(3.12) \quad \|\underline{h}_\delta\|_{L^\infty(0,T;W_\#^{1,\infty}(0,L))} \leq C,$$

$$(3.13) \quad \underline{h}_\delta \leq h_\gamma, \quad \forall \gamma \leq \gamma_0,$$

$$(3.14) \quad \|\underline{h}_\delta - h_\gamma\|_{L^\infty(0,T;W_\#^{1,\infty}(0,L))} \leq \delta, \quad \forall \gamma \leq \gamma_0,$$

with C depending only on initial data.

So, let us fix $\delta > 0$ and introduce parameters $\varepsilon > 0$, $N \in \mathbb{N}$ to be made precise later on. We construct our piecewise approximation as follows. We consider the subdivision of the time interval $[0, T] = \bigcup_{0 \leq k \leq N} I_k$ with $I_k = [k\Delta t, (k+1)\Delta t)$, $\Delta t = T/(N+1)$ and we fix $t_k \in I_k$ such that $\|h(t_k)\|_{H_\#^2(0,L)} \leq \|h\|_{L^\infty(0,T;H_\#^2(0,L))}$. On each time interval I_k we then define

$$(3.15) \quad \underline{h}_\delta(t, x) = [h - 2\varepsilon]_+(t_k, x), \quad t \in I_k.$$

Estimate (3.12) follows directly from Lemma 19 and estimate (3.7). Now, up to a good choice for the parameters ε and N , \underline{h}_δ satisfies the two properties (3.13) and (3.14) for γ small enough. First, let us prove that $\underline{h}_\delta \leq h_\gamma$ for all γ sufficiently small and N sufficiently large (depending on ε). We recall that by interpolation, we have $h \in C^{0,\theta}([0, T]; C_\#^1(0, L))$ for $\theta \in (0, 1/4)$ (see embedding (2.3)). Consequently, for any $k \leq N$, we have

$$\|h(t, x) - h(t_k, x)\|_{L_\#^\infty(0,L)} \leq C\Delta t^\theta, \quad \forall t \in I_k.$$

Similarly, since h_γ converges to h in $C^0([0, T]; C_\#^1(0, L))$ we can find $\gamma_0 > 0$ such that, for $\gamma \leq \gamma_0$,

$$\|h(t, x) - h_\gamma(t, x)\|_{W_\#^{1,\infty}(0,L)} \leq \varepsilon, \quad \forall t \in (0, T).$$

Assuming that N is chosen such that (with the above constant C)

$$C\Delta t^\theta = C\left(\frac{T}{N+1}\right)^\theta < \varepsilon,$$

we have, for any $k \leq N$,

$$h_\gamma(t, x) > h(t, x) - \varepsilon > h(t_k, x) - 2\varepsilon, \quad \forall (t, x) \in I_k \times (0, L), \forall \gamma \leq \gamma_0.$$

Taking the positive part in the previous inequality (recall that $h_\gamma > 0$) we obtain (3.13).

We now estimate the difference between \underline{h}_δ and h_γ for $\gamma \leq \gamma_0$. Given $k \leq N$ we have, for all $\gamma \leq \gamma_0$,

$$\begin{aligned} \|\underline{h}_\delta - h_\gamma\|_{L^\infty(I_k; W_\#^{1,\infty}(0,L))} & \leq \|\underline{h}_\delta - h\|_{L^\infty(I_k; W_\#^{1,\infty}(0,L))} + \|h - h_\gamma\|_{L^\infty(I_k; W_\#^{1,\infty}(0,L))}, \\ & \leq \|[h - 2\varepsilon]_+(t_k) - h(t_k)\|_{W_\#^{1,\infty}(0,L)} + \|h(t_k) - h\|_{L^\infty(I_k; W_\#^{1,\infty}(0,L))} + \varepsilon, \\ & \leq \|[h - 2\varepsilon]_+(t_k) - h(t_k)\|_{W_\#^{1,\infty}(0,L)} + 2\varepsilon. \end{aligned}$$

Applying Lemma 19, this entails

$$\|[h - 2\varepsilon]_+(t_k) - h(t_k)\|_{W_\#^{1,\infty}(0,L)} \leq 2\varepsilon + \sup_{\{x \in [0,L], h(t_k, x) \leq 2\varepsilon\}} |\partial_x h(t_k, x)|.$$

Finally we obtain

$$\|\underline{h}_\delta - h_\gamma\|_{L^\infty((0,T);W_{\sharp}^{1,\infty}(0,L))} \leq 4\varepsilon + \sup_{\{x \in [0,L], h(t_k,x) \leq 2\varepsilon\}} |\partial_x h(t_k, x)|.$$

Consequently, applying Lemma 20 and choosing $\varepsilon > 0$ sufficiently small with the corresponding $N \in \mathbb{N}$ and $\gamma_0 > 0$ we obtain that the interface \underline{h}_δ satisfies (3.14).

3.3. STEP 3. L^2 -COMPACTNESS OF THE VELOCITIES. — In this section we study the L^2 -convergence of the pair $(\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma)$ stated in Lemma 18. We know that, up to a subsequence that we do not relabel, $\rho_\gamma \bar{\mathbf{u}}_\gamma \rightharpoonup \rho \bar{\mathbf{u}}$ weakly in $L^2(0, T; \mathbf{L}_{\sharp}^2(\Omega))$, and $\partial_t \eta_\gamma \rightharpoonup \partial_t \eta$ in $L^2(0, T; L_{\sharp}^2(0, L))$. To prove the strong convergence of the sequence $(\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma)$ to $(\rho \bar{\mathbf{u}}, \partial_t \eta)$ it remains to show that the following convergence holds true:

$$(3.16) \quad \rho_f \int_0^T \int_{\Omega} |\rho_\gamma \bar{\mathbf{u}}_\gamma|^2 + \rho_s \int_0^T \int_0^L |\partial_t \eta_\gamma|^2 \xrightarrow{\gamma \rightarrow 0} \rho_f \int_0^T \int_{\Omega} |\rho \bar{\mathbf{u}}|^2 + \rho_s \int_0^T \int_0^L |\partial_t \eta|^2.$$

We recall that we endow $X^0 := \mathbf{L}^2(\Omega) \times L_{\sharp}^2(0, L)$ with the scalar product:

$$((\bar{\mathbf{v}}, \dot{\eta}), (\bar{\mathbf{w}}, d))_{X^0} = \rho_f \int_{\Omega} \bar{\mathbf{v}} \cdot \bar{\mathbf{w}} + \rho_s \int_0^L \dot{\eta} d.$$

In particular, with these notations, the right hand side of (3.16) also reads:

$$\int_0^T ((\rho_\gamma \bar{\mathbf{u}}, \partial_t \eta_\gamma), (\bar{\mathbf{u}}, \partial_t \eta_\gamma))_{X^0}.$$

By restriction, this bilinear form enables to consider any element of X^0 as an element of $(X^s)'$ via the formula

$$(3.17) \quad \langle (\bar{\mathbf{v}}, \dot{\eta}), (\bar{\mathbf{w}}, d) \rangle_{(X^s)', X^s} = ((\bar{\mathbf{v}}, \dot{\eta}), (\bar{\mathbf{w}}, d))_{X^0} \quad \forall ((\bar{\mathbf{v}}, \dot{\eta}), (\bar{\mathbf{w}}, d)) \in X^0 \times X^s.$$

In what follows we use this identification without mentioning it.

To obtain (3.16), we show actually that, up to extract again a denumerable times subsequences, we can prove that the error terms

$$\text{Err}_\gamma := \rho_f \int_0^T \int_{\Omega} |\rho_\gamma \bar{\mathbf{u}}_\gamma|^2 + \rho_s \int_0^T \int_0^L |\partial_t \eta_\gamma|^2 - \left(\rho_f \int_0^T \int_{\Omega} |\rho \bar{\mathbf{u}}|^2 + \rho_s \int_0^T \int_0^L |\partial_t \eta|^2 \right)$$

satisfy $\limsup_{\gamma \rightarrow 0} |\text{Err}_\gamma| \leq \tilde{\varepsilon}$ for any arbitrary small $\tilde{\varepsilon}$. We shall compute ε with respect to the parameter $\delta > 0$ fixing the interface from below \underline{h}_δ satisfying (3.13)–(3.14) as in the previous subsection. So let us fix such a $\delta > 0$. We recall that the related interface \underline{h}_δ is constant on a family of intervals $(I_k)_{0 \leq k \leq N}$ covering $[0, T]$. Below, we denote $\underline{h}_{\delta,k}$ the value of \underline{h}_δ on I_k . We then split the time integral and

introduce the projector $\mathbb{P}^s[\underline{h}_{\delta,k}]$ for a given $s < 1/2$. This yields:

$$\begin{aligned} \text{Err}_\gamma &= \sum_{k=0}^N \int_{I_k} ((\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma), (\bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma))_{X^0} - ((\rho \bar{\mathbf{u}}, \partial_t \eta), (\bar{\mathbf{u}}, \partial_t \eta))_{X^0} \\ &= \sum_{k=0}^N \int_{I_k} ((\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma), (\bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma) - \mathbb{P}^s[\underline{h}_{\delta,k}](\bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma))_{X^0} \\ &\quad + \int_{I_k} (\mathbb{P}^s[\underline{h}_{\delta,k}](\bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma), (\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma))_{X^0} \\ &\quad - \int_{I_k} ((\rho \bar{\mathbf{u}}, \partial_t \eta), (\bar{\mathbf{u}}, \partial_t \eta) - \mathbb{P}^s[\underline{h}_{\delta,k}](\bar{\mathbf{u}}, \partial_t \eta))_{X^0} \\ &\quad - \int_{I_k} (\mathbb{P}^s[\underline{h}_{\delta,k}](\bar{\mathbf{u}}, \partial_t \eta), (\rho \bar{\mathbf{u}}, \partial_t \eta))_{X^0}. \end{aligned}$$

Then, since $\mathbb{P}^s[\underline{h}_{\delta,k}](\bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma) \in X^s[\underline{h}_{\delta,k}] \subset X[\underline{h}_{\delta,k}]$ and thanks the identification (3.17), we write

$$\begin{aligned} (\mathbb{P}^s[\underline{h}_{\delta,k}](\bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma), (\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma))_{X^0} &= (\mathbb{P}^s[\underline{h}_{\delta,k}](\bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma), \mathbb{P}[\underline{h}_{\delta,k}](\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma))_{X^0} \\ &= \langle \mathbb{P}[\underline{h}_{\delta,k}](\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma), \mathbb{P}^s[\underline{h}_{\delta,k}](\bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma) \rangle_{(X^s)', X^s}. \end{aligned}$$

Proceeding similarly with the limit term, we obtain the following splitting

$$\text{Err}_\gamma = \sum_{k=0}^N \text{Err}_{\gamma,k}^{\text{app}} + \text{Err}_{\gamma,k}^{\text{conv}} - \text{Err}_k^{\text{app}},$$

where, for arbitrary $k \in N$, we denote

$$\begin{aligned} \text{Err}_{\gamma,k}^{\text{app}} &= \int_{I_k} \left((\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma), (\bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma) - \mathbb{P}^s[\underline{h}_{\delta,k}](\bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma) \right)_{X^0} \\ \text{Err}_k^{\text{app}} &= \int_{I_k} \left((\rho \bar{\mathbf{u}}, \partial_t \eta), (\bar{\mathbf{u}}, \partial_t \eta) - \mathbb{P}^s[\underline{h}_{\delta,k}](\bar{\mathbf{u}}, \partial_t \eta) \right)_{X^0} \\ \text{Err}_{\gamma,k}^{\text{conv}} &= \int_{I_k} \langle \mathbb{P}[\underline{h}_{\delta,k}](\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma), \mathbb{P}^s[\underline{h}_{\delta,k}](\bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma) \rangle_{(X^s)', X^s} \\ &\quad - \langle \mathbb{P}[\underline{h}_{\delta,k}](\rho \bar{\mathbf{u}}, \partial_t \eta), \mathbb{P}^s[\underline{h}_{\delta,k}](\bar{\mathbf{u}}, \partial_t \eta) \rangle_{(X^s)', X^s}. \end{aligned}$$

For the two first type of terms we use the fact that the projection on $X^s[\underline{h}_{\delta,k}]$ has good approximation properties. So, to estimate the error terms $\text{Err}_{\gamma,k}^{\text{app}}$, we use Lemma 13 for $\kappa = 1/4$. Indeed, from the bound (3.6) and Lemma (14), we know that $(\bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma)$ satisfies, for a.e. $t \in I_k$, $(\bar{\mathbf{u}}_\gamma(t), \partial_t \eta_\gamma(t)) \in X^s[h_\gamma]$, for $s < 1/2$, with $\bar{\mathbf{u}}_\gamma \in H_{\sharp}^1(F_{h_\gamma}^-)$. Moreover, we remark that both interfaces h_γ and h_δ belong to $H_{\sharp}^{1+\kappa}(0, L) \cap W_{\sharp}^{1,\infty}(0, L)$ and that, thanks to the definition of h_δ , there exists $A > 0$ independent of γ and δ , such that

$$\begin{aligned} \|h_\gamma\|_{L^\infty(0,T;H_{\sharp}^{1+\kappa}(0,L))} + \|h_\gamma\|_{L^\infty(0,T;W_{\sharp}^{1,\infty}(0,L))} \\ + \|\underline{h}_\delta\|_{L^\infty(0,T;H_{\sharp}^{1+\kappa}(0,L))} + \|\underline{h}_\delta\|_{L^\infty(0,T;W_{\sharp}^{1,\infty}(0,L))} \in A. \end{aligned}$$

Finally by construction h_δ and h_γ are close in $W_\#^{1,\infty}(0, L)$ and (3.14) is satisfied. Hence, Lemma 13 implies, for $s < 1/8$

$$\begin{aligned} & \|P^s[\underline{h}_{\delta,k}](\bar{\mathbf{u}}_\gamma(t, \cdot), \partial_t \eta_\gamma(t, \cdot)) - (\bar{\mathbf{u}}_\gamma(t, \cdot), \partial_t \eta_\gamma(t, \cdot))\|_{X^0} \\ & \quad \hookrightarrow \|P^s[\underline{h}_{\delta,k}](\bar{\mathbf{u}}_\gamma(t, \cdot), \partial_t \eta_\gamma(t, \cdot)) - (\bar{\mathbf{u}}_\gamma(t, \cdot), \partial_t \eta_\gamma(t, \cdot))\|_{X^s} \\ & \quad \hookrightarrow C_A(\delta) \|\nabla \bar{\mathbf{u}}_\gamma(t, \cdot)\|_{L^2_\#(F_{h_\gamma(t)})}, \end{aligned}$$

with $\lim_{x \rightarrow 0} C_A(x) = 0$. Using Cauchy–Schwartz inequality, we deduce from the previous estimate that

$$\begin{aligned} & \sum_{k=0}^N |\text{Err}_{\gamma,k}^{\text{app}}| \\ & \quad \hookrightarrow \sum_{k=0}^N \int_{I_k} \|P^s[\underline{h}_{\delta,k}](\bar{\mathbf{u}}_\gamma(t, \cdot), \partial_t \eta_\gamma(t, \cdot)) - (\bar{\mathbf{u}}_\gamma(t, \cdot), \partial_t \eta_\gamma(t, \cdot))\|_{X^0} \|(\bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma)\|_{X^0} \\ & \quad \hookrightarrow C_A(\delta) \|(\bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma)\|_{L^\infty(0,T;X^0)} \int_0^T \|\nabla \bar{\mathbf{u}}_\gamma\|_{L^2(F_{h_\gamma(t)})}. \end{aligned}$$

Then we use the uniform estimates (3.5), (3.6) to obtain that there exists a constant C_1 (depending only on initial data and T) such that, for $\gamma \hookrightarrow \gamma_0$,

$$(3.18) \quad \sum_{k=0}^N |\text{Err}_{\gamma,k}^{\text{app}}| \hookrightarrow C_1 C_A(\delta).$$

Similarly we have

$$(3.19) \quad \sum_{k=0}^N |\text{Err}_k^{\text{app}}| \hookrightarrow C_1 C_A(\delta).$$

To complete the proof, the following term remains to be estimated

$$\limsup_{\gamma \rightarrow 0} \sum_{k=0}^N |\text{Err}_{\gamma,k}^{\text{conv}}|.$$

At first, we prove that, for a fixed $k \hookrightarrow N$ and up to a subsequence, $P[\underline{h}_{\delta,k}](\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma)$ converges strongly to $P[\underline{h}_{\delta,k}](\rho \bar{\mathbf{u}}, \partial_t \eta)$ in $L^2(I_k; (X^s[\underline{h}_{\delta,k}])')$. Note that, since we have:

$$(\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma) \rightharpoonup (\rho \bar{\mathbf{u}}, \partial_t \eta) \text{ in } L^2(I_k; X^s[\underline{h}_{\delta,k}]) - w,$$

the only difficulty relies on showing that the sequence $P[\underline{h}_{\delta,k}](\rho \bar{\mathbf{u}}, \partial_t \eta)$ is relatively compact in $L^2(I_k; (X^s[\underline{h}_{\delta,k}])')$. To do so we apply an adapted version of Aubin–Lions lemma that can be found in [5, §4.3] that reads

LEMMA 21. — *Let us consider three Hilbert spaces M_i , $i = 1, 2, 3$ and two operators $T : M_0 \mapsto M_1$ and $S : M_0 \mapsto M_2$ satisfying*

- *T and S are two linear compact operators,*
- *$Su = 0$ implies $Tu = 0$.*

If (u_n) is bounded in $L^2(0, T; M_0)$ and $(\partial_t S u_n)$ is bounded in $L^2(0, T; M_2)$, then $T u_n$ is a compact set of $L^2(0, T; M_1)$.

We are going to use this version of Aubin–Lions lemma with the triplet

$$(X^0, (X^s[h_{\delta,k}])', (X^1[h_{\delta,k}])')$$

and with $S = i_1 \circ P[h_{\delta,k}]$, $T = i_s \circ P[h_{\delta,k}]$, where i_ℓ denotes the injection of $X[h_{\delta,k}]$ into $(X^\ell[h_{\delta,k}])'$. The mappings i_ℓ are indeed injective functions since Lemma 8 implies that the continuous embedding $X^\ell[h_{\delta,k}] \subset X[h_{\delta,k}]$ is dense, for $\ell > 0$ and these densities imply that $X[h_{\delta,k}]$ is continuously embedded in $(X^\ell[h_{\delta,k}])'$, for $\ell > 0$. Moreover, thanks to Rellich–Kondrachov theorem the embeddings $X^\ell[h_{\delta,k}] \subset X[h_{\delta,k}]$ are compact. The dual of a compact operator being compact, $X[h_{\delta,k}]$ is compactly embedded in $(X^\ell[h_{\delta,k}])'$ for $\ell > 0$. Consequently $i_1 \circ P[h_{\delta,k}]$ and $i_s \circ P[h_{\delta,k}]$ are compact linear operators. Moreover $i_1 \circ P[h_{\delta,k}](\mathbf{w}, b) = 0$ implies $P[h_{\delta,k}](\mathbf{w}, b) = 0$ so that the second point is clearly satisfied.

Next applying (3.5), we have that $(\bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma)$ is uniformly bounded in $L^2(0, T; X^0)$ (with respect to γ). Thus the sequence $(\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma)$ is bounded in γ in $L^2(I_k; X^0)$. We must now obtain a uniform bound for $\partial_t P[h_{\delta,k}](\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma)$ in $L^p(I_k; (X^1[h_{\delta,k}])')$. Precisely, we look for an estimate of the type

$$\left| - \int_{I_k} (P[h_{\delta,k}](\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma), \partial_t(\mathbf{w}, b))_{X^0} \right| \leq C \int_{I_k} \|(\mathbf{w}(t), d(t))\|_{X^1[h_{\delta,k}]}^2 dt,$$

for any $(\mathbf{w}, d) \in L^2(I_k; X^1[h_{\delta,k}])$. To obtain such an estimate we use the variational formulation (2.15) satisfied by $(\bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma)$. We consider $(\mathbf{w}, d) \in C_c^\infty(I_k; X^1[h_{\delta,k}])$. This is an admissible test function since $h_\delta \leq h_\gamma$ and since, in the case where $\gamma > 0$ for which $\min_{x \in [0, L]} h_\gamma(t, x) > 0$, $\forall t \in [0, T]$, we can consider test functions in $C^\infty([0, T]; X^1[h_\gamma(t)])$. We obtain

$$\begin{aligned} \int_{I_k} (P[h_{\delta,k}](\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma), \partial_t(\mathbf{w}, b))_{X^0} &= -\rho_f \int_{I_k} \rho_\gamma \bar{\mathbf{u}}_\gamma \cdot \partial_t \mathbf{w} - \rho_s \int_{I_k} \partial_t \eta_\gamma \partial_t b \\ &= \rho_f \int_{I_k} \int_{F_{h_\gamma(t)}} (\mathbf{u}_\gamma \cdot \nabla) \mathbf{w} \cdot \mathbf{u}_\gamma - \mu \int_{I_k} \int_{F_{h_\gamma(t)}} \nabla \mathbf{u}_\gamma : \nabla \mathbf{w} \\ &\quad - \beta \int_{I_k} \int_0^L \partial_x \eta_\gamma \partial_x b + \alpha \int_{I_k} \int_0^L \partial_{xx} \eta_\gamma \partial_{xx} b + \gamma \int_{I_k} \int_0^L \partial_{tx} \eta_\gamma \partial_x b. \end{aligned}$$

The nonlinear convection term is estimated using the L^4 -regularity of $\bar{\mathbf{u}}_\gamma$ stated in Lemma 14,

$$\begin{aligned} \left| \rho_f \int_{I_k} \int_\Omega (\rho_\gamma \bar{\mathbf{u}}_\gamma \cdot \nabla) \mathbf{w} \cdot \rho_\gamma \bar{\mathbf{u}}_\gamma \right| &\leq \rho_f \int_{I_k} \|\bar{\mathbf{u}}_\gamma(t)\|_{L^4(\Omega)}^2 \|\nabla \mathbf{w}(t)\|_{L^2(\Omega)} dt \\ &\leq \rho_f \|\bar{\mathbf{u}}_\gamma\|_{L^4(\hat{\Omega})}^2 \|\nabla \mathbf{w}\|_{L^2(I_k; L^2(\Omega))}. \end{aligned}$$

The other terms are estimated directly and we obtain

$$\left| \int_{I_k} (P[h_{\delta,k}](\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma), \partial_t(\mathbf{w}, b))_{X^0} \right| \leq C \|(\mathbf{w}, b)\|_{L^2(I_k; X^1[h_{\delta,k}])},$$

where C depends only on the initial data. The previous inequality implies that $\partial_t P[h_{\delta,k}](\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma)$ is bounded in γ in $L^2(I_k; (X^1[h_{\delta,k}])')$. It then follows from

the adapted version of Aubin–Lions lemma that $\mathbb{P}[\underline{h}_{\delta,k}](\rho_\gamma \bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma)$ is compact in $L^2(I_k; (X^s[\underline{h}_{\delta,k}])')$. Moreover, since there holds:

$$(\bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma) \rightharpoonup (\bar{\mathbf{u}}, \partial_t \eta) \text{ in } L^2(I_k; X^s) - w, \text{ for } s < 1/2,$$

we also have that $\mathbb{P}^s[\underline{h}_{\delta,k}](\bar{\mathbf{u}}_\gamma, \partial_t \eta_\gamma)$ converges weakly to $\mathbb{P}^s[\underline{h}_{\delta,k}](\bar{\mathbf{u}}, \partial_t \eta)$ in $L^2(I_k; X^s)$. Combining a strong and a weak convergence result leads to

$$(3.20) \quad \lim_{\gamma \rightarrow 0} \text{Err}_{\gamma,k}^{\text{conv}} = 0 \quad \forall k \subset N.$$

Finally, combining (3.18)–(3.20), we conclude that

$$\limsup_{\gamma \rightarrow 0} |\text{Err}_\gamma| \subset C_1 C_A(\delta)$$

for arbitrary $\delta > 0$. We conclude the proof by remarking that $C_A(\delta) \rightarrow 0$ when $\delta \rightarrow 0$. For completeness, we remark that in the computations of bounds for Err_γ we only extract subsequences when we apply the Aubin–Lions lemma. Since we perform extraction a finite number of times for any value of the parameter δ that we can choose in a denumerable sets (i.e., a sequence converging to 0), our proof induces indeed denumerable extractions of subsequences.

REMARK 22. — In the final weak formulation we consider fluid test functions that vanishes in the neighbourhood of the bottom of the fluid cavity and that are only transverse in the neighbourhood of the interface. Note that we could have also considered fluid test functions that vanish in the neighbourhood of any contact point or, at least, at the contact point. It imposes in particular the structure test functions to be zero near each contact point so that they depend implicitly on the solution.

APPENDIX. PROOF OF LEMMA 8

This appendix is devoted to a density lemma in the space $X[h]$. We first recall the statement of the lemma to be proved and proceed to the proof.

LEMMA 23. — *Let $h \in C_{\sharp}^0(0, L)$ satisfy $0 \subset h(x) \subset M, \forall x \in [0, L]$. The embedding $X[h] \cap (C^\infty(\bar{\Omega}) \times C_{\sharp}^\infty(0, L)) \subset X[h]$ is dense.*

Proof. — First notice that the main difficulty here comes from the potential contact i.e., the points where h is equal to zero. If there is no contact we may construct explicitly a smooth approximating sequence of any pair in $X[h]$ by adapting the arguments of [4], see also the construction of approximate initial data in Section 3.1.

When h vanishes, we propose an alternative proof: in this case we obtain that $(X[h] \cap C^\infty(\bar{\Omega}) \times C_{\sharp}^\infty(0, L))^\perp = \{(\mathbf{0}, 0)\}$. So, let $(\mathbf{u}, \dot{\eta}) \in X[h]$ and assume it satisfies:

$$(A.1) \quad \rho_f \int_{\Omega} \mathbf{u} \cdot \mathbf{w} + \rho_s \int_0^L \dot{\eta} d = 0, \quad \forall (\mathbf{w}, d) \in X[h] \cap C^\infty(\bar{\Omega}) \times C_{\sharp}^\infty(0, L).$$

Using Lemma 7 there exists $\Psi \in H_{\sharp}^1(\Omega)$ and $b \in H_{\sharp}^1(0, L)$ such that $\mathbf{u} = \nabla^\perp \Psi$ with $\Psi = b(x)$ in S_h and $\Psi = 0$ in $\Omega_{-1}^0 \cup I^c \times (-1, 2M)$, where $I = \{x \in [0, L] \mid h(x) > 0\}$.

To complete the proof, we obtain that \mathbf{u} vanishes in $I \times (-1, 2M)$. Since I is an open subset of $(0, L)$ we may construct an at most denumerable I' such that

$$I = \bigsqcup_{i \in I'} (a_i, b_i),$$

where the (a_i, b_i) are the connected components of I . It is now sufficient to prove that, for arbitrary $i \in I'$ there holds $\mathbf{u} = 0$ in $\Omega_i = (a_i, b_i) \times (-1, M)$. This is obtained by a suitable choice of functions (\mathbf{w}, d) in (A.1).

Let fix $i \in I'$ and $\varepsilon > 0$ small enough. Consider $\chi_\varepsilon \in C^\infty((a_i, b_i))$ such that

$$\begin{aligned} \chi_\varepsilon &= 1 \text{ on } [a_i + \varepsilon, b_i - \varepsilon], \quad \text{supp}(\chi_\varepsilon) \subset [a_i + \varepsilon/2, b_i - \varepsilon/2], \\ \|\chi'_\varepsilon\|_{L^\infty(\mathbb{R})} &< \frac{1}{\varepsilon} \quad \|\chi''_\varepsilon\|_{L^\infty(\mathbb{R})} < \frac{1}{\varepsilon^2}. \end{aligned}$$

Existence of such a truncation function is classical. We now introduce

$$\mathbf{w}_\varepsilon = \nabla^\perp(\chi_\varepsilon \Psi), \quad d_\varepsilon = \partial_x(\chi_\varepsilon b).$$

It is straightforward that $(\mathbf{w}_\varepsilon, d_\varepsilon) \in \mathbf{L}^2((a_i, b_i) \times (-1, M)) \times L^2_0((a_i, b_i))$ and has support in $(a_i + \varepsilon, b_i - \varepsilon)$. On the other hand, there exists $\delta_\varepsilon > 0$ such that $h(x) > \delta_\varepsilon$ on $(a_i + \varepsilon/2, b_i - \varepsilon/2)$. Setting $h_\varepsilon = \max(h, \delta_\varepsilon)$ we have then that $h_\varepsilon \in C^0_\#(0, L)$ does not vanish and $(\mathbf{w}_\varepsilon, d_\varepsilon) \in X[h_\varepsilon]$. Consequently, we may reproduce the arguments in the case of a non vanishing deformation to approximate $(\mathbf{w}_\varepsilon, d_\varepsilon)$ by a sequence of pairs in $X[h_\varepsilon] \cap (C^\infty(\bar{\Omega}) \times C^\infty_\#(0, L))$. Moreover, we emphasize that, by construction, this sequence has support in $(a_i + \varepsilon/2, b_i - \varepsilon/2) \times (-1, 2M)$ also so that it is actually a sequence of $X[h] \cap (C^\infty(\bar{\Omega}) \times C^\infty_\#(0, L))$ that approximates $(\mathbf{w}_\varepsilon, d_\varepsilon)$ in $X[h]$ also.

Consequently the identity (A.1) holds true for $(\mathbf{w}_\varepsilon, d_\varepsilon)$ also and we have

$$\int_{\Omega_i} \mathbf{u} \cdot \nabla^\perp(\chi_\varepsilon \Psi) + \int_{a_i}^{b_i} \dot{\eta} \partial_x(\chi_\varepsilon b) = 0.$$

But, recalling that $\nabla^\perp \Psi = \mathbf{u}$ and $\partial_x b = \dot{\eta}$ we may expand the differential operators to yield that:

$$0 = \int_{\Omega_i} \mathbf{u} \cdot \nabla^\perp(\chi_\varepsilon \Psi) + \int_{a_i}^{b_i} \dot{\eta} z'_\varepsilon = \int_{\Omega_i} |\mathbf{u}|^2 \chi_\varepsilon + \int_{a_i}^{b_i} |\dot{\eta}|^2 \chi_\varepsilon + \int_{\Omega_i} \mathbf{u} \cdot \nabla^\perp(\chi_\varepsilon) \Psi + \int_{a_i}^{b_i} \dot{\eta} b \chi'_\varepsilon.$$

Since χ_ε depends on the x -variable only and χ'_ε vanishes on $\{a_i, a_i + \varepsilon, b_i - \varepsilon, b_i\}$, we have, by integrating by parts:

$$(A.2) \quad - \int_{\Omega_i} \mathbf{u} \cdot \nabla^\perp(\chi_\varepsilon) \Psi = \int_{a_i}^{a_i + \varepsilon} \int_{-1}^{2M} \frac{\Psi^2}{2} \chi''_\varepsilon + \int_{b_i - \varepsilon}^{b_i} \int_{-1}^{2M} \frac{\Psi^2}{2} \chi''_\varepsilon.$$

Similarly we prove the following equality:

$$(A.3) \quad - \int_{a_i}^{b_i} \dot{\eta} b(x) \chi'_\varepsilon = \int_{a_i}^{a_i + \varepsilon} \frac{b(x)^2}{2} \chi''_\varepsilon + \int_{b_i - \varepsilon}^{b_i} \frac{b(x)^2}{2} \chi''_\varepsilon.$$

Since $\Psi = 0$ on $\{a_i\} \times (-1, 2M)$ a standard Poincaré inequalities entails that:

$$\begin{aligned} \int_{a_i}^{a_i+\varepsilon} \int_{-1}^{2M} \frac{\Psi^2}{2} &\leq \frac{\varepsilon^2}{4} \int_{a_i}^{a_i+\varepsilon} \int_{-1}^{2M} |\nabla \Psi|^2 = \frac{\varepsilon^2}{4} \int_{a_i}^{a_i+\varepsilon} |\mathbf{u}|^2, \\ \int_{a_i}^{a_i+\varepsilon} \frac{b(x)^2}{2} &\leq \frac{\varepsilon^2}{4} \int_{a_i}^{a_i+\varepsilon} |\partial_x b(x)|^2 = \frac{\varepsilon^2}{4} \int_{a_i}^{a_i+\varepsilon} |\dot{\eta}|^2. \end{aligned}$$

We have a similar identity for integrals involving $(b_i - \varepsilon, b_i)$ by using that $\Psi = 0$ on $\{b_i\} \times \{-1, 2M\}$. Using finally that L^∞ -estimate on χ_ε'' in (A.2)–(A.3) we conclude

$$(A.4) \quad \int_{a_i+\varepsilon}^{b_i-\varepsilon} \int_{-1}^{2M} |\mathbf{u}|^2 + \int_{a_i+\varepsilon}^{b_i-\varepsilon} |\dot{\eta}|^2 \\ \leq \frac{1}{4} \left(\int_{a_i}^{a_i+\varepsilon} \int_{-1}^{2M} |\mathbf{u}|^2 + \int_{a_i}^{a_i+\varepsilon} |\dot{\eta}|^2 + \int_{b_i-\varepsilon}^{b_i} \int_{-1}^{2M} |\mathbf{u}|^2 + \int_{b_i-\varepsilon}^{b_i} |\dot{\eta}|^2 \right).$$

Since $(\mathbf{u}, \dot{\eta})$ are both L^2 -functions, the right-hand side of this identity vanishes when $\varepsilon \rightarrow 0$. So, letting $\varepsilon \rightarrow 0$ we obtain $(\mathbf{u}, \dot{\eta}) = (\mathbf{0}, 0)$ in Ω_i . This ends the proof.

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JEAN-JÉRÔME CASANOVA, CEREMADE, UMR CNRS 7534, Université Paris-Dauphine, PSL Research University
Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France
E-mail : casanova@ceremade.dauphine.fr
Url : <https://www.ceremade.dauphine.fr/~casanova/>

CÉLINE GRANDMONT, Inria Paris
75012 Paris, France
& Sorbonne Université, UMR 7598 LJLL
75005 Paris, France
E-mail : celine.grandmont@inria.fr
Url : <https://team.inria.fr/commedia/grandmont/>

MATHIEU HILLAIRET, IMAG, Univ Montpellier, CNRS
Montpellier, France
E-mail : matthieu.hillairet@umontpellier.fr
Url : <http://hillairet.perso.math.cnrs.fr/>