ÉCOLE POLYTECHNIQUE
INSTITUT POLYTECHNIQUE DE PARIS

Journal de l'École polytechnique
Mathématiques

Tomoyuki Arakawa, Cuipo Jiang, & Anne Moreau

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Tome 8 (2021), p. 169-191.

<http://jep.centre-mersenne.org/item/JEP_2021__8__169_0>

Publié avec le soutien
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Journal de l'École polytechnique
Mathématiques
Tome 8, 2021, p. 169–191 DOI: 10.5802/jep.144

SIMPLICITY OF VACUUM MODULES AND ASSOCIATED VARIETIES

by Tomoyuki Arakawa, Cuipo Jiang & Anne Moreau

Abstract. — In this note, we prove that the universal affine vertex algebra associated with a simple Lie algebra $\mathfrak{g}$ is simple if and only if the associated variety of its unique simple quotient is equal to $\mathfrak{g}^*$. We also derive an analogous result for the quantized Drinfeld-Sokolov reduction applied to the universal affine vertex algebra.

Résumé (Simplicité des algèbres vertex affines et variétés associées). — Dans cet article, nous démontrons que l'algèbre vertex affine universelle associée à une algèbre de Lie simple $\mathfrak{g}$ est simple si et seulement si la variété associée à son unique quotient simple est égale à $\mathfrak{g}^*$. Nous en déduisons un résultat analogue pour la réduction quantique de Drinfeld-Sokolov appliquée à l'algèbre vertex affine universelle.

Contents

1. Introduction

Let V be a vertex algebra, and let

$$V \longrightarrow (\operatorname{End} V)[\![z, z^{-1}]\!], \quad a \longmapsto a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1},$$

be the state-field correspondence. The *Zhu C_2-algebra* [Zhu96] of V is by definition the quotient space $R_V = V/C_2(V)$, where $C_2(V) = \operatorname{span}_{\mathbb{C}}\{a_{(-2)}b \mid a, b \in V\}$, equipped with the Poisson algebra structure given by

$$\overline{a}.\overline{b} = \overline{a_{(-1)}b}, \qquad \{\overline{a}, \overline{b}\} = \overline{a_{(0)}b},$$

Mathematical subject classification (2020). — 17B69.
Keywords. — Associated variety, affine Kac-Moody algebra, affine vertex algebra, singular vector, affine W-algebra.

T.A. is supported by partially supported by JSPS KAKENHI Grant No. 17H01086 and No. 17K18724. C.J. is supported by CNSF grants 11771281 and 11531004. A.M. is supported by the ANR Project GeoLie Grant number ANR-15-CE40-0012, and by the Labex CEMPI (ANR-11-LABX-0007-01).

for $a, b \in V$ with $\overline{a} := a+C_2(V)$. The associated variety X_V of V is the reduced scheme $X_V = \mathrm{Specm}(R_V)$ corresponding to R_V. It is a fundamental invariant of V that captures important properties of the vertex algebra V itself (see, for example, [BFM, Zhu96, ABD04, Miy04, Ara12a, Ara15a, Ara15b, AM18a, AM17, AK18]). Moreover, the associated variety X_V conjecturally [BR18] coincides with the Higgs branch of a 4D $\mathcal{N} = 2$ superconformal field theory $\mathcal{T}$, if V corresponds to a theory $\mathcal{T}$ by the 4D/2D duality discovered in [BLL+15]. Note that the Higgs branch of a 4D $\mathcal{N} = 2$ superconformal field theory is a hyperkähler cone, possibly singular.

In the case where V is the universal affine vertex algebra $V^k(\mathfrak{g})$ at level $k \in \mathbb{C}$ associated with a complex finite-dimensional simple Lie algebra $\mathfrak{g}$, the variety X_V is just the affine space $\mathfrak{g}^*$ with Kirillov-Kostant Poisson structure. In the case where V is the unique simple graded quotient $L_k(\mathfrak{g})$ of $V^k(\mathfrak{g})$, the variety X_V is a Poisson subscheme of $\mathfrak{g}^*$ which is G-invariant and conic, where G is the adjoint group of $\mathfrak{g}$.

Note that if the level k is irrational, then $L_k(\mathfrak{g}) = V^k(\mathfrak{g})$, and hence $X_{L_k(\mathfrak{g})} = \mathfrak{g}^*$. More generally, if $L_k(\mathfrak{g}) = V^k(\mathfrak{g})$, that is, $V^k(\mathfrak{g})$ is simple, then obviously $X_{L_k(\mathfrak{g})} = \mathfrak{g}^*$.

In this article, we prove that the converse is true.

Theorem 1.1. — *The equality $L_k(\mathfrak{g}) = V^k(\mathfrak{g})$ holds, that is, $V^k(\mathfrak{g})$ is simple, if and only if $X_{L_k(\mathfrak{g})} = \mathfrak{g}^*$.*

It is known by Gorelik and Kac [GK07] that $V^k(\mathfrak{g})$ is not simple if and only if

$$r^\vee(k + h^\vee) \in \mathbb{Q}_{\geqslant 0} \smallsetminus \{1/m \mid m \in \mathbb{Z}_{\geqslant 1}\}, \tag{1.1}$$

where $h^\vee$ is the dual Coxeter number and $r^\vee$ is the lacing number of $\mathfrak{g}$. Therefore, Theorem 1.1 can be rephrased as

$$X_{L_k(\mathfrak{g})} \subsetneq \mathfrak{g}^* \iff (1.1) \text{ holds.} \tag{1.2}$$

Let us mention the cases when the variety $X_{L_k(\mathfrak{g})}$ is known for k satisfying (1.1).

First, it is known [Zhu96, DM06] that $X_{L_k(\mathfrak{g})} = \{0\}$ if and only if $L_k(\mathfrak{g})$ is integrable, that is, k is a nonnegative integer. Next, it is known that if $L_k(\mathfrak{g})$ is *admissible* [KW89], or equivalently, if

$$k + h^\vee = \frac{p}{q}, \quad p, q \in \mathbb{Z}_{\geqslant 1},\ (p, q) = 1,\ p \geqslant \begin{cases} h^\vee & \text{if } (r^\vee, q) = 1, \\ h & \text{if } (r^\vee, q) \neq 1, \end{cases}$$

where h is the Coxeter number of $\mathfrak{g}$, then $X_{L_k(\mathfrak{g})}$ is the closure of some nilpotent orbit in $\mathfrak{g}$ ([Ara15a]). Further, it was observed in [AM18a, AM18b] that there are cases when $L_k(\mathfrak{g})$ is non-admissible and $X_{L_k(\mathfrak{g})}$ is the closure of some nilpotent orbit. In fact, it was recently conjectured in physics [XY19] that, in view of the 4D/2D duality, there should be a large list of non-admissible simple affine vertex algebras whose associated varieties are the closures of some nilpotent orbits. Finally, there are also cases [AM17] where $X_{L_k(\mathfrak{g})}$ is neither $\mathfrak{g}^*$ nor contained in the nilpotent cone $\mathcal{N}(\mathfrak{g})$ of $\mathfrak{g}$.

In general, the problem of determining the variety $X_{L_k(\mathfrak{g})}$ is wide open.

Now let us explain the outline of the proof of Theorem 1.1. First, Theorem 1.1 is known for the critical level $k=-h^\vee$ ([FF92, FG04]). Therefore, since $R_{V^k(\mathfrak{g})}$ is a polynomial ring $\mathbb{C}[\mathfrak{g}^*]$, Theorem 1.1 follows from the following fact.

Theorem 1.2. — *Suppose that the level is non-critical, that is, $k\neq -h^\vee$. The image of any nonzero singular vector v of $V^k(\mathfrak{g})$ in the Zhu C_2-algebra $R_{V^k(\mathfrak{g})}$ is nonzero.*

The symbol $\sigma(w)$ of a singular vector w in $V^k(\mathfrak{g})$ is a singular vector in the corresponding vertex Poisson algebra $\mathrm{gr}V^k(\mathfrak{g})\cong S(t^{-1}\mathfrak{g}[t^{-1}])\cong \mathbb{C}[J_\infty\mathfrak{g}^*]$, where $J_\infty\mathfrak{g}^*$ is the arc space of $\mathfrak{g}^*$. Theorem 1.2 states that the image of $\sigma(w)$ of a non-trivial singular vector w under the projection

$$\mathbb{C}[J_\infty\mathfrak{g}^*]\longrightarrow \mathbb{C}[\mathfrak{g}^*]=R_{V^k(\mathfrak{g})} \tag{1.3}$$

is nonzero, provided that k is non-critical. Here the projection (1.3) is defined by identifying $\mathbb{C}[\mathfrak{g}^*]$ with the Zhu C_2-algebra of the commutative vertex algebra $\mathbb{C}[J_\infty\mathfrak{g}^*]$. Hence, Theorem 1.2 would follow if the image of any nontrivial singular vector in $\mathbb{C}[J_\infty\mathfrak{g}^*]$ under the projection (1.3) is nonzero. However, this is false as there are singular vectors in $\mathbb{C}[J_\infty\mathfrak{g}^*]$ that do not come from singular vectors of $V^k(\mathfrak{g})$ and that belong to the kernel of (1.3) (see Section 3.4). Therefore, we do need to make use of the fact that $\sigma(w)$ is the symbol of a singular vector w in $V^k(\mathfrak{g})$. We also note that the statement of Theorem 1.2 is not true if k is critical (see Section 3.4).

For this reason the proof of Theorem 1.2 is divided roughly into two parts. First, we work in the commutative setting to deduce a first important reduction (Lemma 3.1). Next, we use the Sugawara construction – which is available only at non-critical levels – in the non-commutative setting in order to complete the proof.

Now, let us consider the *W-algebra* $\mathscr{W}^k(\mathfrak{g},f)$ associated with a nilpotent element f of $\mathfrak{g}$ at the level k defined by the generalized quantized Drinfeld-Sokolov reduction [FF90, KRW03]:

$$\mathscr{W}^k(\mathfrak{g},f)=H^0_{\mathrm{DS},f}(V^k(\mathfrak{g})).$$

Here, $H^\bullet_{\mathrm{DS},f}(M)$ denotes the BRST cohomology of the generalized quantized Drinfeld-Sokolov reduction associated with $f\in\mathcal{N}(\mathfrak{g})$ with coefficients in a $V^k(\mathfrak{g})$-module M.

By the Jacobson-Morosov theorem, f embeds into an $\mathfrak{sl}_2$-triple (e,h,f). The Slodowy slice $\mathscr{S}_f$ at f is the affine space $\mathscr{S}_f=f+\mathfrak{g}^e$, where $\mathfrak{g}^e$ is the centralizer of e in $\mathfrak{g}$. It has a natural Poisson structure induced from that of $\mathfrak{g}^*$ (see [GG02]), and we have [DSK06, Ara15a] a natural isomorphism $R_{\mathscr{W}^k(\mathfrak{g},f)}\cong\mathbb{C}[\mathscr{S}_f]$ of Poisson algebras, so that

$$X_{\mathscr{W}^k(\mathfrak{g},f)}=\mathscr{S}_f.$$

The natural surjection $V^k(\mathfrak{g})\twoheadrightarrow L_k(\mathfrak{g})$ induces a surjection $\mathscr{W}^k(\mathfrak{g},f)\twoheadrightarrow H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))$ of vertex algebras ([Ara15a]). Hence the variety $X_{H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))}$ is a $\mathbb{C}^*$-invariant Poisson subvarieties of the Slodowy slice $\mathscr{S}_f$.

Conjecturally [KRW03, KW08], the vertex algebra $H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))$ coincides the unique simple (graded) quotient $\mathscr{W}_k(\mathfrak{g},f)$ of $\mathscr{W}^k(\mathfrak{g},f)$ provided that $H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))\neq 0$. (This conjecture has been verified in many cases [Ara05, Ara07, Ara11, AvE19].)

As a consequence of Theorem 1.1, we obtain the following result.

Theorem 1.3. — *Let f be any nilpotent element of $\mathfrak{g}$. The following assertions are equivalent:*

(1) $V^k(\mathfrak{g})$ *is simple,*
(2) $\mathscr{W}^k(\mathfrak{g}, f) = H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))$,
(3) $X_{H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))} = \mathscr{S}_f$.

Note that Theorem 1.3 implies that $V^k(\mathfrak{g})$ is simple if $X_{\mathscr{W}_k(\mathfrak{g},f)} = \mathscr{S}_f$ and $H^0_{\mathrm{DS},f}(L_k(\mathfrak{g})) \neq 0$ since $X_{H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))} \supset X_{\mathscr{W}^k(\mathfrak{g},f)}$.

The remainder of the paper is structured as follows. In Section 2 we set up notation in the case of affine vertex algebras that will be the framework of this note. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we have compiled some known facts on Slodowy slices, W-algebras and their associated varieties. Theorem 1.3 is proved in this section.

Acknowledgements. — T.A. and A.M. like to warmly thank Shanghai Jiao Tong University for its hospitality during their stay in September 2019.

2. Universal affine vertex algebras and associated graded vertex Poisson algebras

Let $\widehat{\mathfrak{g}}$ be the affine Kac-Moody algebra associated with $\mathfrak{g}$, that is,

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K,$$

where the commutation relations are given by

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m(x|y)\delta_{m+n,0}K, \quad [K, \widehat{\mathfrak{g}}] = 0,$$

for $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. Here,

$$(\ |\ ) = \frac{1}{2h^\vee} \times \text{Killing form of } \mathfrak{g}$$

is the usual normalized inner product. For $x \in \mathfrak{g}$ and $m \in \mathbb{Z}$, we shall write $x(m)$ for $x \otimes t^m$.

2.1. Universal affine vertex algebras. — For $k \in \mathbb{C}$, set

$$V^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}_k,$$

where $\mathbb{C}_k$ is the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which K acts as multiplication by k and $\mathfrak{g} \otimes \mathbb{C}[t]$ acts trivially.

By the Poincaré-Birkhoff-Witt Theorem, the direct sum decomposition, we have

$$V^k(\mathfrak{g}) \cong U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) = U(t^{-1}\mathfrak{g}[t^{-1}]). \tag{2.1}$$

The space $V^k(\mathfrak{g})$ is naturally graded,

$$V^k(\mathfrak{g}) = \bigoplus_{\Delta \in \mathbb{Z}_{\geqslant 0}} V^k(\mathfrak{g})_\Delta,$$

where the grading is defined by

$$\deg(x^{i_1}(-n_1)\cdots x^{i_r}(-n_r)\mathbf{1})=\sum_{i=1}^{r}n_i,\quad r\geqslant 0,\ x^{i_j}\in\mathfrak{g},$$

with $\mathbf{1}$ the image of $1\otimes 1$ in $V^k(\mathfrak{g})$. We have $V^k(\mathfrak{g})_0=\mathbb{C}\mathbf{1}$, and we identify $\mathfrak{g}$ with $V^k(\mathfrak{g})_1$ via the linear isomorphism defined by $x\mapsto x(-1)\mathbf{1}$.

It is well-known that $V^k(\mathfrak{g})$ has a unique vertex algebra structure such that $\mathbf{1}$ is the vacuum vector,

$$x(z):=Y(x\otimes t^{-1},z)=\sum_{n\in\mathbb{Z}}x(n)z^{-n-1},$$

and

$$[T,x(z)]=\partial_z x(z)$$

for $x\in\mathfrak{g}$, where T is the translation operator. Here, $x(n)$ acts on $V^k(\mathfrak{g})$ by left multiplication, and so, one can view $x(n)$ as an endomorphism of $V^k(\mathfrak{g})$. The vertex algebra $V^k(\mathfrak{g})$ is called the *universal affine vertex algebra* associated with $\mathfrak{g}$ at level k [FZ92, Zhu96, LL04].

The vertex algebra $V^k(\mathfrak{g})$ is a vertex operator algebra, provided that $k+h^\vee\neq 0$, by the *Sugawara construction*. More specifically, set

$$S=\frac{1}{2}\sum_{i=1}^{d}x_i(-1)x^i(-1)\mathbf{1},$$

where $\{x_i\mid i=1,\dots,d\}$ is the dual basis of a basis $\{x^i\mid i=1,\dots,\dim\mathfrak{g}\}$ of $\mathfrak{g}$ with respect to the bilinear form $(\ |\ )$, with $d=\dim\mathfrak{g}$. Then for $k\neq -h^\vee$, the vector $\omega=S/(k+h^\vee)$ is a conformal vector of $V^k(\mathfrak{g})$ with central charge

$$c(k)=\frac{k\dim\mathfrak{g}}{k+h^\vee}.$$

Note that, writing $\omega(z)=\sum_{n\in\mathbb{Z}}L_n z^{-n-2}$, we have

$$L_0=\frac{1}{2(k+h^\vee)}\Big(\sum_{i=1}^{d}x_i(0)x^i(0)+\sum_{n=1}^{\infty}\sum_{i=1}^{d}(x_i(-n)x^i(n)+x^i(-n)x_i(n))\Big),$$

$$L_n=\frac{1}{2(k+h^\vee)}\Big(\sum_{m=1}^{\infty}\sum_{i=1}^{d}x_i(-m)x^i(m+n)+\sum_{m=0}^{\infty}\sum_{i=1}^{d}x^i(-m+n)x_i(m)\Big),\quad \text{if } n\neq 0.$$

Lemma 2.1 ([Kac90]). — *We have*

$$[L_n,x(m)]=-mx(m+n),\quad \textit{for } x\in\mathfrak{g},\ m,n\in\mathbb{Z},$$

and $L_n\mathbf{1}=0$ for $n\geqslant -1$.

We have $V^k(\mathfrak{g})_\Delta=\{v\in V^k(\mathfrak{g})\mid L_0v=\Delta v\}$ and $T=L_{-1}$ on $V^k(\mathfrak{g})$, provided that $k+h^\vee\neq 0$.

Any graded quotient of $V^k(\mathfrak{g})$ as $\widehat{\mathfrak{g}}$-module has the structure of a quotient vertex algebra. In particular, the unique simple graded quotient $L_k(\mathfrak{g})$ is a vertex algebra, and is called the *simple affine vertex algebra associated with $\mathfrak{g}$ at level k.*

2.2. Associate graded vertex Poisson algebras of affine vertex algebras

It is known by Li [Li05] that any vertex algebra V admits a canonical filtration $F^{\bullet}V$, called the *Li filtration* of V. For a quotient V of $V^k(\mathfrak{g})$, $F^{\bullet}V$ is described as follows. The subspace F^pV is spanned by the elements

$$y_1(-n_1-1)\cdots y_r(-n_r-1)\mathbf{1}$$

with $y_i \in \mathfrak{g}$, $n_i \in \mathbb{Z}_{\geqslant 0}$, $n_1+\cdots+n_r \geqslant p$. We have

$$\begin{aligned}
&V = F^0V \supset F^1V \supset \cdots, \quad \textstyle\bigcap_p F^pV = 0,\\
&TF^pV \subset F^{p+1}V,\\
&a_{(n)}F^qV \subset F^{p+q-n-1}V \ for \ a \in F^pV, \quad n \in \mathbb{Z},\\
&a_{(n)}F^qV \subset F^{p+q-n}V \ for \ a \in F^pV, \quad n \geqslant 0.
\end{aligned} \tag{2.2}$$

Here we have set $F^pV = V$ for $p<0$.

Let $\mathrm{gr}^F V = \bigoplus_p F^pV/F^{p+1}V$ be the associated graded vector space. The space $\mathrm{gr}^F V$ is a vertex Poisson algebra by

$$\begin{aligned}
\sigma_p(a)\sigma_q(b) &= \sigma_{p+q}(a_{(-1)}b),\\
T\sigma_p(a) &= \sigma_{p+1}(Ta),\\
\sigma_p(a)_{(n)}\sigma_q(b) &= \sigma_{p+q-n}(a_{(n)}b)
\end{aligned}$$

for $a,b \in V$, $n \geqslant 0$, where $\sigma_p\colon F^p(V) \to F^pV/F^{p+1}V$ is the principal symbol map. In particular, $\mathrm{gr}^F V$ is a $\mathfrak{g}[t]$-module by the correspondence

$$\mathfrak{g}[t] \ni x(n) \longmapsto \sigma_0(x)_{(n)} \in \mathrm{End}(\mathrm{gr}^F V) \tag{2.3}$$

for $x \in \mathfrak{g}$, $n \geqslant 0$.

The filtration $F^{\bullet}V$ is compatible with the grading: $F^pV = \bigoplus_{\Delta\in\mathbb{Z}_{\geqslant 0}} F^pV_{\Delta}$, where $F^pV_{\Delta} := V_{\Delta} \cap F^pV$.

Let $U_{\bullet}(t^{-1}\mathfrak{g}[t^{-1}])$ be the PBW filtration of $U(t^{-1}\mathfrak{g}[t^{-1}])$, that is, $U_p(t^{-1}\mathfrak{g}[t^{-1}])$ is the subspace of $U(t^{-1}\mathfrak{g}[t^{-1}])$ spanned by monomials $y_1y_2\ldots y_r$ with $y_i \in \mathfrak{g}$, $r \leqslant p$. Define

$$G_pV = U_p(t^{-1}\mathfrak{g}[t^{-1}])\mathbf{1}.$$

Then $G_{\bullet}V$ defines an increasing filtration of V. We have

$$F^pV_{\Delta} = G_{\Delta-p}G_{\Delta}, \tag{2.4}$$

where $G_pV_{\Delta} := G_pV \cap V_{\Delta}$, see [Ara12a, Prop. 2.6.1]. Therefore, the graded space $\mathrm{gr}^G V = \bigoplus_{p\in\mathbb{Z}_{\geqslant 0}} G_pV/G_{p-1}V$ is isomorphic to $\mathrm{gr}^F V$. In particular, we have

$$\mathrm{gr} V^k(\mathfrak{g}) \cong \mathrm{gr}\, U_{\bullet}(t^{-1}\mathfrak{g}[t^{-1}]) \cong S(t^{-1}\mathfrak{g}[t^{-1}]).$$

The action of $\mathfrak{g}[t]$ on $\mathrm{gr}V^k(\mathfrak{g}) = S(t^{-1}\mathfrak{g}[t^{-1}])$ coincides with the one induced from the action of $\mathfrak{g}[t]$ on $\mathfrak{g}[t,t^{-1}]/\mathfrak{g}[t] \cong t^{-1}\mathfrak{g}[t^{-1}]$. More precisely, the element $x(m)$, for $x \in \mathfrak{g}$

and $m \in \mathbb{Z}_{\geqslant 0}$, acts on $S(t^{-1}\mathfrak{g}[t^{-1}])$ as follows:

$$
\begin{aligned}
&x(m)\cdot \mathbf{1} = 0,\\
&x(m)\cdot v = \sum_{j=1}^{r} \sum_{n_j - m > 0} y_1(-n_1)\cdots [x, y_j](m-n_j)\cdots y_r(-n_r),
\end{aligned}
\tag{2.5}
$$

if $v = y_1(-n_1)\cdots y_r(-n_r)$ with $y_i \in \mathfrak{g}$, $n_1, \ldots, n_r \in \mathbb{Z}_{>0}$.

2.3. Zhu's C_2-algebras and associated varieties of affine vertex algebras

We have [Li05, Lem. 2.9]

$$F^pV = \mathrm{span}_{\mathbb{C}}\{a_{(-i-1)}b \mid a \in V,\, i \geqslant 1,\, b \in F^{p-i}V\}$$

for all $p \geqslant 1$. In particular,

$$F^1V = C_2(V),$$

where $C_2(V) = \mathrm{span}_{\mathbb{C}}\{a_{(-2)}b \mid a, b \in V\}$. Set

$$R_V = V/C_2(V) = F^0V/F^1V \subset \mathrm{gr}^F V.$$

It is known by Zhu [Zhu96] that R_V is a Poisson algebra. The Poisson algebra structure can be understood as the restriction of the vertex Poisson structure of $\mathrm{gr}^F V$. It is given by

$$\overline{a}\cdot\overline{b} = \overline{a_{(-1)}b}, \quad \{\overline{a}, \overline{b}\} = \overline{a_{(0)}b},$$

for $a, b \in V$, where $\overline{a} = a + C_2(V)$.

By definition [Ara12a], the *associated variety* of V is the reduced scheme

$$X_V := \mathrm{Specm}(R_V).$$

It is easily seen that

$$F^1V^k(\mathfrak{g}) = C_2(V^k(\mathfrak{g})) = t^{-2}\mathfrak{g}[t^{-1}]V^k(\mathfrak{g}).$$

The following map defines an isomorphism of Poisson algebras

$$
\begin{aligned}
\mathbb{C}[\mathfrak{g}^*] \cong S(\mathfrak{g}) &\longrightarrow R_{V^k(\mathfrak{g})}\\
\mathfrak{g} \ni x &\longmapsto x(-1)\mathbf{1} + t^{-2}\mathfrak{g}[t^{-1}]V^k(\mathfrak{g}).
\end{aligned}
$$

Therefore, $R_{V^k(\mathfrak{g})} \cong \mathbb{C}[\mathfrak{g}^*]$ and so, $X_{V^k(\mathfrak{g})} \cong \mathfrak{g}^*$.

More generally, if V is a quotient of $V^k(\mathfrak{g})$ by some ideal N, then we have

$$R_V \cong \mathbb{C}[\mathfrak{g}^*]/I_N \tag{2.6}$$

as Poisson algebras, where I_N is the image of N in $R_{V^k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]$. Then X_V is just the zero locus of I_N in $\mathfrak{g}^*$. It is a closed G-invariant conic subset of $\mathfrak{g}^*$.

Identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through the bilinear form $(\ |\ )$, one may view X_V as a subvariety of $\mathfrak{g}$.

2.4. PBW basis. — Let $\Delta_+ = \{\beta_1, \ldots, \beta_q\}$ be the set of positive roots for $\mathfrak{g}$ with respect to a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $q = (d-\ell)/2$ and $\ell = \mathrm{rk}(\mathfrak{g})$.

Form now on, we fix a basis

$$\{u^i, e_{\beta_j}, f_{\beta_j} \mid i = 1, \ldots, \ell,\ j = 1, \ldots, q\}$$

of $\mathfrak{g}$ such that $\{u^i \mid i = 1, \ldots, \ell\}$ is an orthonormal basis of $\mathfrak{h}$ with respect to $(\ |\ )$ and $(e_{\beta_i}|f_{\beta_i}) = 1$ for $i = 1, 2, \ldots, q$. In particular, $[e_{\beta_i}, f_{\beta_i}] = \beta_i$ for $i = 1, \ldots, q$ (see, for example, [Hum72, Prop. 8.3]), where $\mathfrak{h}^*$ and $\mathfrak{h}$ are identified through $(\ |\ )$. One may also assume that $\mathrm{ht}(\beta_i) \leqslant \mathrm{ht}(\beta_j)$ for $i < j$, where $\mathrm{ht}(\beta_i)$ stands for the height of the positive root β_i.

We define the structure constants $c_{\alpha,\beta}$ by

$$[e_\alpha, e_\beta] = c_{\alpha,\beta} e_{\alpha+\beta},$$

provided that α, β and $\alpha+\beta$ are in Δ. Our convention is that $e_{-\alpha}$ stands for f_α if $\alpha \in \Delta_+$. If α, β and $\alpha+\beta$ are in Δ_+, then from the equalities,

$$c_{-\alpha,\alpha+\beta} = (f_\beta|[f_\alpha, e_{\alpha+\beta}]) = -(f_\beta|[e_{\alpha+\beta}, f_\alpha]) = -([f_\beta, e_{\alpha+\beta}]|f_\alpha) = -c_{-\beta,\alpha+\beta},$$

we get that

$$c_{-\alpha,\alpha+\beta} = -c_{-\beta,\alpha+\beta}. \tag{2.7}$$

By (2.1), the above basis of $\mathfrak{g}$ induces a basis of $V^k(\mathfrak{g})$ consisted of $\mathbf{1}$ and the elements of the form

$$z = z^{(+)} z^{(-)} z^{(0)} \mathbf{1}, \tag{2.8}$$

with

$$\begin{aligned}
z^{(+)} &:= e_{\beta_1}(-1)^{a_{1,1}} \cdots e_{\beta_1}(-r_1)^{a_{1,r_1}} \cdots e_{\beta_q}(-1)^{a_{q,1}} \cdots e_{\beta_q}(-r_q)^{a_{q,r_q}},\\
z^{(-)} &:= f_{\beta_1}(-1)^{b_{1,1}} \cdots f_{\beta_1}(-s_1)^{b_{1,s_1}} \cdots f_{\beta_q}(-1)^{b_{q,1}} \cdots f_{\beta_q}(-s_q)^{b_{q,s_q}},\\
z^{(0)} &:= u^1(-1)^{c_{1,1}} \cdots u^1(-t_1)^{c_{1,t_1}} \cdots u^\ell(-1)^{c_{\ell,1}} \cdots u^\ell(-t_\ell)^{c_{\ell,t_\ell}},
\end{aligned}$$

where $r_1, \ldots, r_q, s_1, \ldots, s_q, , t_1, \ldots, t_\ell$ are positive integers, and $a_{l,m}, b_{l,n}, c_{i,j}$, for $l = 1, \ldots, q$, $m = 1, \ldots, r_l$, $n = 1, \ldots, s_l$, $i = 1, \ldots, \ell$, $j = 1, \ldots, t_i$ are nonnegative integers such that at least one of them is nonzero.

Definition 2.2. — Each element x of $V^k(\mathfrak{g})$ is a linear combination of elements in the above PBW basis, each of them will be called a *PBW monomial* of x.

Definition 2.3. — For a PBW monomial v as in (2.8), we call the integer

$$\mathrm{depth}(v) = \sum_{i=1}^{q} \left(\sum_{j=1}^{r_i} a_{i,j}(j-1) + \sum_{j=1}^{s_i} b_{i,j}(j-1) \right) + \sum_{i=1}^{\ell} \sum_{j=1}^{t_i} c_{i,j}(j-1)$$

the *depth* of v. In other words, a PBW monomial v has depth p means that $v \in F^p V^k(\mathfrak{g})$ and $v \notin F^{p+1} V^k(\mathfrak{g})$. By convention, $\mathrm{depth}(\mathbf{1}) = 0$.

For a PBW monomial v as in (2.8), we call *degree* of v the integer

$$\deg(v)=\sum_{i=1}^{q}\Big(\sum_{j=1}^{r_i}a_{i,j}+\sum_{j=1}^{s_i}b_{i,j}\Big)+\sum_{i=1}^{\ell}\sum_{j=1}^{t_i}c_{i,j},$$

In other words, v has degree p means that $v\in G_pV^k(\mathfrak{g})$ and $v\notin G_{p-1}V^k(\mathfrak{g})$ since the PBW filtration of $V^k(\mathfrak{g})$ coincides with the standard filtration $G_\bullet V^k(\mathfrak{g})$. By convention, $\deg(\mathbf{1})=0$.

Recall that a *singular vector* of a $\mathfrak{g}[t]$-representation M is a vector $m\in M$ such that $e_\alpha(0).m=0$, for all $\alpha\in\Delta_+$, and $f_\theta(1)\cdot m=0$, where θ is the highest positive root of $\mathfrak{g}$. From the identity

$$L_{-1}=\frac{1}{k+h^\vee}\Big(\sum_{i=1}^{\ell}\sum_{m=0}^{\infty}u^i(-1-m)u^i(m)+\sum_{\alpha\in\Delta_+}\sum_{m=0}^{\infty}(e_\alpha(-1-m)f_\alpha(m)+f_\alpha(-1-m)e_\alpha(m))\Big),$$

we deduce the following easy observation, which will be useful in the proof of the main result.

Lemma 2.4. — *If w is a singular vector of $V^k(\mathfrak{g})$, then*

$$L_{-1}w=\frac{1}{k+h^\vee}\Big(\sum_{i=1}^{\ell}u^i(-1)u^i(0)+\sum_{\alpha\in\Delta_+}e_\alpha(-1)f_\alpha(0)\Big)w.$$

2.5. Basis of associated graded vertex Poisson algebras. — Note that $\mathrm{gr}V^k(\mathfrak{g})=S(t^{-1}\mathfrak{g}[t^{-1}])$ has a basis consisting of $\mathbf{1}$ and elements of the form (2.8). Similarly to Definition 2.2, we have the following definition.

Definition 2.5. — Each element x of $S(t^{-1}\mathfrak{g}[t^{-1}])$ is a linear combination of elements in the above basis, each of them will be called a *monomial* of x.

As in the case of $V^k(\mathfrak{g})$, the space $S(t^{-1}\mathfrak{g}[t^{-1}])$ has two natural gradations. The first one is induced from the degree of elements as polynomials. We shall write $\deg(v)$ for the degree of a homogeneous element $v\in S(t^{-1}\mathfrak{g}[t^{-1}])$ with respect to this gradation.

The second one is induced from the Li filtration via the isomorphism $S(t^{-1}\mathfrak{g}[t^{-1}])\cong\mathrm{gr}^FV^k(\mathfrak{g})$. The degree of a homogeneous element $v\in S(t^{-1}\mathfrak{g}[t^{-1}])$ with respect to the gradation induced by Li filtration will be called the *depth* of v, and will be denoted by $\mathrm{depth}(v)$.

Notice that any element v of the form (2.8) is homogeneous for both gradations. By convention, $\deg(\mathbf{1})=\mathrm{depth}(\mathbf{1})=0$.

As a consequence of (2.5), we get that

$$\deg(x(m)\cdot v)=\deg(v)\quad\text{and}\quad\mathrm{depth}(x(m)\cdot v)=\mathrm{depth}(v)-m,\tag{2.9}$$

for $m\geqslant 0$, $x\in\mathfrak{g}$, and any homogeneous element $v\in S(t^{-1}\mathfrak{g}[t^{-1}])$ with respect to both gradations.

In the sequel, we will also use the following notation, for v of the form (2.8), viewed either as an element of $V^k(\mathfrak{g})$ or of $S(t^{-1}\mathfrak{g}[t^{-1}])$:

$$\deg_{-1}^{(0)}(v) := \sum_{j=1}^{\ell} c_{j,1}, \tag{2.10}$$

which corresponds to the degree of the element obtained from $v^{(0)}$ by keeping only the terms of depth 0, that is, the terms $u^i(-1)$, $i = 1, \ldots, \ell$.

Notice that a nonzero depth-homogeneous element of $S(t^{-1}\mathfrak{g}[t^{-1}])$ has depth 0 if and only if its image in

$$R_{V^k(\mathfrak{g})} = V^k(\mathfrak{g})/t^{-2}\mathfrak{g}[t^{-1}]V^k(\mathfrak{g})$$

is nonzero.

3. Proof of the main result

This section is devoted to the proof of Theorem 1.1.

3.1. Strategy. — Let N_k be the maximal graded submodule of $V^k(\mathfrak{g})$, so that $L_k(\mathfrak{g}) = V^k(\mathfrak{g})/N_k$. Our aim is to show that if $V^k(\mathfrak{g})$ is not simple, that is, $N_k \neq \{0\}$, then $X_{L_k(\mathfrak{g})}$ is strictly contained in $\mathfrak{g}^* \cong \mathfrak{g}$, that is, the image $I_k := I_{N_k}$ of N_k in $R_{V^k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]$ is nonzero.

For $k = -h^\vee$, it follows from [FG04] that I_k is the defining ideal of the nilpotent cone $\mathcal{N}(\mathfrak{g})$ of $\mathfrak{g}$, and so $X_{L_k(\mathfrak{g})} = \mathcal{N}(\mathfrak{g})$ (see [Ara12b] or Section 3.4 below). Hence, there is no loss of generality in assuming that $k + h^\vee \neq 0$.

Henceforth, we suppose that $k + h^\vee \neq 0$ and that $V^k(\mathfrak{g})$ is not simple, that is, $N_k \neq \{0\}$. Then there exists at least one non-trivial (that is, nonzero and different from $\mathbf{1}$) singular vector w in $V^k(\mathfrak{g})$. Theorem 1.2 states that the image of w in I_k is nonzero, and this proves Theorem 1.1. The rest of this section is devoted to the proof of Theorem 1.2.

Let w be a nontrivial singular vector of $V^k(\mathfrak{g})$. One can assume that $w \in F^pV^k(\mathfrak{g}) \smallsetminus F^{p+1}V^k(\mathfrak{g})$ for some $p \in \mathbb{Z}_{\geqslant 0}$.

The image

$$\overline{w} := \sigma(w)$$

of this singular vector in $S(t^{-1}\mathfrak{g}[t^{-1}]) \cong \operatorname{gr}^F V^k(\mathfrak{g})$ is a nontrivial singular vector of $S(t^{-1}\mathfrak{g}[t^{-1}])$. Here $\sigma\colon V^k(\mathfrak{g}) \to \operatorname{gr}^F V^k(\mathfrak{g})$ stands for the principal symbol map. It follows from (2.9) that one can assume that $\overline{w}$ is homogeneous with respect to both gradations on $S(t^{-1}\mathfrak{g}[t^{-1}])$. In particular $\overline{w}$ has depth p. It is enough to show that $p = 0$, that is, $\overline{w}$ has depth zero. Write

$$w = \sum_{j\in J} \lambda_j w^j,$$

where J is a finite index set, λ_j are nonzero scalar for all $j \in J$, and w_j are pairwise distinct PBW monomials of the form (2.8). Let $I \subset J$ be the subset of $i \in J$ such that depth $\overline{w}^i = p =$ depth $\overline{w}$. Since $w \in F^pV^k(\mathfrak{g}) \smallsetminus F^{p+1}V^k(\mathfrak{g})$, the set I is nonempty. Here, $\overline{w}^i$ stands for the image of w^i in $\operatorname{gr}^F V^k(\mathfrak{g}) \cong S(t^{-1}\mathfrak{g}[t^{-1}])$.

More specifically, for any $j \in I$, write

$$w^j = (w^j)^{(+)}(w^j)^{(-)}(w^j)^{(0)}\mathbf{1}, \tag{3.1}$$

with

$$\begin{aligned}(w^j)^{(+)} &:= e_{\beta_1}(-1)^{a^{(j)}_{1,1}} \cdots e_{\beta_1}(-r_1)^{a^{(j)}_{1,r_1}} \cdots e_{\beta_q}(-1)^{a^{(j)}_{q,1}} \cdots e_{\beta_q}(-r_q)^{a^{(j)}_{q,r_q}}\\ (w^j)^{(-)} &:= f_{\beta_1}(-1)^{b^{(j)}_{1,1}} \cdots f_{\beta_1}(-s_1)^{b^{(j)}_{1,s_1}} \cdots f_{\beta_q}(-1)^{b^{(j)}_{q,1}} \cdots f_{\beta_q}(-s_q)^{b^{(j)}_{q,s_q}},\\ (w^j)^{(0)} &:= u^1(-1)^{c^{(j)}_{1,1}} \cdots u^1(-t_1)^{c^{(j)}_{1,t_1}} \cdots u^\ell(-1)^{c^{(j)}_{\ell,1}} \cdots u^\ell(-t_\ell)^{c^{(j)}_{\ell,t_\ell}},\end{aligned}$$

where $r_1, \dots, r_q, s_1, \dots, s_q, , t_1, \dots, t_\ell$ are nonnegative integers, and $a^{(j)}_{l,m}, b^{(j)}_{l,n}, c^{(j)}_{i,p}$, for $l = 1, \dots, q$, $m = 1, \dots, r_l$, $n = 1, \dots, s_l$, $i = 1, \dots, \ell$, $p = 1, \dots, t_i$, are nonnegative integers such that at least one of them is nonzero.

The integers r_l's, for $l = 1, \dots, q$, are chosen so that at least one of the $a^{(j)}_{l,r_l}$'s is nonzero for j running through J if for some $j \in J$, $(w^j)^{(+)} \neq 1$. Otherwise, we just set $(w^j)^{(+)} := 1$. Similarly are defined the integers s_l's and t_m's, for $l = 1, \dots, q$ and $m = 1, \dots, \ell$. By our assumption, note that for all $i \in I$,

$$\begin{aligned}&\sum_{n=1}^{q}\Big(\sum_{l=1}^{r_n} a^{(i)}_{n,l} + \sum_{l=1}^{s_n} b^{(i)}_{n,l}\Big) + \sum_{n=1}^{\ell}\sum_{l=1}^{t_n} c^{(i)}_{n,l} = \deg(\overline{w})\\ &\sum_{n=1}^{q}\Big(\sum_{l=1}^{r_n} a^{(i)}_{n,l}(l-1) + \sum_{l=1}^{s_n} b^{(i)}_{n,l}(l-1)\Big) + \sum_{n=1}^{\ell}\sum_{l=1}^{t_n} c^{(i)}_{n,l}(l-1) = \operatorname{depth}(\overline{w}) = p.\end{aligned}$$

3.2. A technical lemma. — In this paragraph we remain in the commutative setting, and we only deal with $\overline{w} \in S(t^{-1}\mathfrak{g}[t^{-1}])$ and its monomials $\overline{w}^i$'s, for $i \in I$.

Recall from (2.10) that,

$$\deg^{(0)}_{-1}(w^i) = \sum_{j=1}^{\ell} c^{(i)}_{j,1}$$

for $i \in I$. Set

$$d^{(0)}_{-1}(I) := \max\{\deg^{(0)}_{-1}(w^i) \mid i \in I\},$$

and

$$I^{(0)}_{-1} := \{i \in I \mid \deg^{(0)}_{-1}(w^i) = d^{(0)}_{-1}(I)\}.$$

If $(w^i)^{(0)} = 1$ for all $i \in I$, we just set $d^{(0)}_{-1}(I) = 0$ and then $I^{(0)}_{-1} = I$.

Lemma 3.1. — *If $i \in I^{(0)}_{-1}$, then $(\overline{w}^i)^{(-)} = 1$. In other words, for $i \in I^{(0)}_{-1}$, we have $\overline{w}^i = (\overline{w}^i)^{(0)}(\overline{w}^i)^{(+)}\mathbf{1}$.*

Proof. — Suppose the assertion is false. Then for some positive roots $\beta_{j_1}, \dots, \beta_{j_t} \in \Delta_+$, one can write for any $i \in I^{(0)}_{-1}$,

$$(\overline{w}^i)^{(-)} = f_{\beta_{j_1}}(-1)^{b^{(i)}_{j_1,1}} \cdots f_{\beta_{j_1}}(-s_{j_1})^{b^{(i)}_{j_1,s_{j_1}}} \cdots f_{\beta_{j_t}}(-1)^{b^{(i)}_{j_t,1}} \cdots f_{\beta_{j_t}}(-s_{j_t})^{b^{(i)}_{j_t,s_{j_t}}}, \tag{3.2}$$

so that for any $l \in \{1, \ldots, t\}$,

$$\{b^{(i)}_{j_l, s_{j_l}} \mid i \in I^{(0)}_{-1}\} \neq \{0\}.$$

Set

$$K^{(0)}_{-1} = \{i \in I^{(0)}_{-1} \mid b^{(i)}_{j_1, s_{j_1}} > 0\}.$$

Since $\overline{w}$ is a singular vector of $S(t^{-1}\mathfrak{g}[t^{-1}])$ and $s_{j_1} - 1 \in \mathbb{Z}_{\geqslant 0}$, we have

$$e_{\beta_{j_1}}(s_{j_1} - 1) \cdot \overline{w} = 0.$$

On the other hand, using the action of $\mathfrak{g}[t]$ on $S(t^{-1}\mathfrak{g}[t^{-1}])$ as described by (2.5), we see that

$$0 = e_{\beta_{j_1}}(s_{j_1} - 1) \cdot \overline{w} = \sum_{i \in K^{(0)}_{-1}} \lambda_i b^{(i)}_{j_1, s_{j_1}} v^i + v, \tag{3.3}$$

where for $i \in K^{(0)}_{-1}$,

$$v^i := (\overline{w}^i)^{(0)} \beta_{j_1}(-1) f_{\beta_{j_1}}(-1)^{b^{(i)}_{j_1,1}} \cdots f_{\beta_{j_1}}(-s_{j_1})^{b^{(i)}_{j_1,s_{j_1}} - 1} \cdots f_{\beta_{j_t}}(-1)^{b^{(i)}_{j_t,1}} \cdots f_{\beta_{j_t}}(-s_{j_t})^{b^{(i)}_{j_t,s_{j_t}}} (w^i)^{(+)} \mathbf{1},$$

and v is a linear combination of monomials x such that

$$\deg^{(0)}_{-1}(x) \leqslant d^{(0)}_{-1}(I).$$

Indeed, for $i \in K^{(0)}_{-1}$, it is clear that

$$e_{\beta_{j_1}}(s_{j_1} - 1) \cdot w^i = b^{(i)}_{j_1, s_{j_1}} v^i + y^i,$$

where y^i is a linear combination of monomials y such that $\deg^{(0)}_{-1}(y) \leqslant d^{(0)}_{-1}(I)$ because $\mathrm{ht}(\beta_{j_1}) \leqslant \mathrm{ht}(\beta_{j_l})$ for all $l \in \{1, \ldots, t\}$. Next, for $i \in I^{(0)}_{-1} \smallsetminus K^{(0)}_{-1}$, $e_{\beta_{j_1}}(s_{j_1} - 1) \cdot \overline{w}^i$ is a linear combination of monomials z such that $\deg^{(0)}_{-1}(z) \leqslant d^{(0)}_{-1}(I)$ because $b^{(i)}_{j_1, s_{j_1}} = 0$. Finally, for $i \in I \smallsetminus I^{(0)}_{-1}$, we have $\deg^{(0)}_{-1}(\overline{w}^i) < d^{(0)}_{-1}(I)$ and, hence, $e_{\beta_{j_1}}(s_{j_1} - 1) \cdot \overline{w}^i$ is a linear combination of monomials z such that $\deg^{(0)}_{-1}(z) \leqslant d^{(0)}_{-1}(I)$ as well.

Now, note that for each $i \in K^{(0)}_{-1}$,

$$\deg^{(0)}_{-1}(v^i) = \deg^{(0)}_{-1}(\overline{w}^i) + 1 = d^{(0)}_{-1}(I) + 1.$$

Hence by (3.3) we get a contradiction because all monomials v^i, for i running through $K^{(0)}_{-1}$, are linearly independent while $\lambda_i b^{(i)}_{j_1, s_{j_1}} \neq 0$, for $i \in K^{(0)}_{-1}$. This concludes the proof of the lemma. □

3.3. Use of Sugawara operators. — Recall that $w = \sum_{j \in J} \lambda_j w^j$. Let $J_1 \subseteq J$ be such that for $i \in J_1$, $(w^i)^{(-)} = 1$. Then by Lemma 3.1,

$$\varnothing \neq I^{(0)}_{-1} \subseteq J_1.$$

So $J_1 \neq \varnothing$. Set

$$d^{(0)}_{-1} := d^{(0)}_{-1}(J_1) = \max\{\deg^{(0)}_{-1}(w^i) \mid i \in J_1\},$$

and
$$J_{-1}^{(0)} := \{i \in J_1 \mid \deg_{-1}^{(0)}(w^i) = d_{-1}^{(0)}\}.$$
Then $d_{-1}^{(0)}(I) \leqslant d_{-1}^{(0)}$. Set
$$d^+ := \max\{\deg(w^i)^{(+)} \mid i \in J_{-1}^{(0)}\}$$
and let
$$J^+ = \{i \in J_{-1}^{(0)} \mid \deg(w^i)^{(+)} = d^+\} \subseteq J_{-1}^{(0)}.$$
Our next aim is to show that for $i \in J^+$, w^i has depth zero, whence $p = 0$ since p is by definition the smallest depth of the w^j's, and so the image of w in $R_{V^k(\mathfrak{g})} = F^0 V^k(\mathfrak{g})/F^1 V^k(\mathfrak{g})$ is nonzero.

This will be achieved in this paragraph through the use of the Sugawara construction.

Recall that by Lemma 2.4,
$$L_{-1} w = \widetilde{L}_{-1} w$$
since w is a singular vector of $V^k(\mathfrak{g})$, where
$$\widetilde{L}_{-1} := \frac{1}{k+h^\vee}\Big(\sum_{i=1}^{\ell} u^i(-1)u^i(0) + \sum_{\alpha\in\Delta_+} e_\alpha(-1) f_\alpha(0)\Big).$$

Lemma 3.2. — *Let z be a PBW monomial of the form* (2.8). *Then $\widetilde{L}_{-1} z$ is a linear combination of PBW monomials x satisfying all the following conditions:*

(a) $\deg(x^{(+)}) \leqslant \deg(z^{(+)}) + 1$ *and* $\deg(x^{(0)}) \leqslant \deg(z^{(0)}) + 1$,
(b) *if $z^{(-)} \neq 1$, then $x^{(-)} \neq 1$.*
(c) *if $x^{(-)} = z^{(-)}$, then either* $\deg(x^{(0)}) = \deg(z^{(0)}) + 1$, *or $x^{(0)} = z^{(0)}$.*
(d) *if* $\deg(x^{(0)}) = \deg(z^{(0)}) + 1$, *then $x^{(-)} = z^{(-)}$ and* $\deg(x^{(+)}) \leqslant \deg(z^{(+)})$.

Proof. — Parts (a)–(c) are easy to see. We only prove (d). Assume that $\deg(x^{(0)}) = \deg(z^{(0)}) + 1$. Either x comes from the term $\sum_{i=1}^{\ell} u^i(-1)u^i(0)z$, or it comes from a term $e_\alpha(-1) f_\alpha(0) z$ for some $\alpha \in \Delta_+$.

If x comes from the term $\sum_{i=1}^{\ell} u^i(-1)u^i(0)z$, then it is obvious that $x^{(-)} = z^{(-)}$ and $x^{(+)} = z^{(+)}$.

Assume that x comes from $e_\alpha(-1) f_\alpha(0) z$ for some $\alpha \in \Delta_+$. We have
$$e_\alpha(-1) f_\alpha(0) z = e_\alpha(-1)[f_\alpha(0), z^{(+)}] z^{(-)} z^{(0)}\mathbf{1} + e_\alpha(-1) z^{(+)}[f_\alpha(0), z^{(-)}] z^{(0)}\mathbf{1} + e_\alpha(-1) z^{(+)} z^{(-)}[f_\alpha(0), z^{(0)}]\mathbf{1}.$$
Clearly, any PBW monomials x from
$$e_\alpha(-1) z^{(+)}[f_\alpha(0), z^{(-)}] z^{(0)}\mathbf{1} \quad \text{or} \quad e_\alpha(-1) z^{(+)} z^{(-)}[f_\alpha(0), z^{(0)}]\mathbf{1}$$
satisfies that $\deg(x^{(0)}) \leqslant \deg(z^{(0)})$. Then it is enough to consider PBW monomials in
$$e_\alpha(-1)[f_\alpha(0), z^{(+)}] z^{(-)} z^{(0)}\mathbf{1}.$$
The only possibility for a PBW monomial x in $e_\alpha(-1)[f_\alpha(0), z^{(+)}] z^{(-)} z^{(0)}\mathbf{1}$ to satisfy $\deg(x^{(0)}) = \deg(z^{(0)}) + 1$ is that it comes from a term $[f_\alpha(0), e_\alpha(-n)] = -\alpha(-n)$ for

some $n\in\mathbb{Z}_{>0}$, where $e_\alpha(-n)$ is a term in $z^{(+)}$. But then, for PBW monomials x in $e_\alpha(-1)[f_\alpha(0),z^{(+)}]z^{(0)}\mathbf{1}$ such that $\deg(x^{(0)})=\deg(z^{(0)})+1$, we have $x^{(-)}=z^{(-)}$ and $\deg(x^{(+)})\leqslant\deg(z^{(+)})$. □

We now consider the action of $\widetilde{L}_{-1}$ on particular PBW monomials.

Lemma 3.3. — *Let z be a PBW monomial of the form* (2.8) *such that $z^{(-)}=1$ and $\mathrm{depth}(z^{(+)})=0$, that is, either $z^{(+)}=1$, or for some $j_1,\dots,j_t\in\{1,\dots,q\}$ (with possible repetitions),*

$$z=e_{\beta_{j_1}}(-1)e_{\beta_{j_2}}(-1)\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}.$$

Then $\widetilde{L}_{-1}z$ is a linear combination of PBW monomials y satisfying one of the following conditions:

(1) $y^{(-)}=1$, $\mathrm{depth}(y^{(+)})\geqslant 1$, $\deg(y^{(+)})\leqslant\deg(z^{(+)})$, $y^{(0)}=z^{(0)}$,

(2) $y^{(-)}=1$, $\mathrm{depth}(y^{(+)})=0$, $\deg(y^{(+)})\leqslant\deg(z^{(+)})-1$, *and* $\deg(y^{(0)})>\deg(z^{(0)})$, $\deg^{(0)}_{-1}(y)=\deg^{(0)}_{-1}(z)$,

(3) $y^{(-)}=1$, $\mathrm{depth}(y^{(+)})\geqslant 1$, $\deg(y^{(+)})\leqslant\deg(z^{(+)})-1$, *and* $\deg^{(0)}_{-1}(y)=\deg^{(0)}_{-1}(z)+1$,

(4) $y^{(-)}\neq 1$.

Proof. — First, we have

$$\sum_{i=1}^{\ell}u^i(-1)u^i(0)z=\sum_{r=1}^{t}e_{\beta_{j_1}}(-1)\cdots\left[\sum_{i=1}^{\ell}u^i(-1)u^i(0),e_{\beta_{j_r}}(-1)\right]\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1},$$

and

$$\begin{aligned}\sum_{i=1}^{\ell}u^i(-1)u^i(0),e_{\beta_{j_r}}(-1)&=\sum_{i=1}^{\ell}\big(u^i(-1)[u^i(0),e_{\beta_{j_r}}(-1)]+[u^i(-1),e_{\beta_{j_r}}(-1)]u^i(0)\big)\\&=\beta_{j_r}(-1)e_{\beta_{j_r}}(-1)+e_{\beta_{j_r}}(-2)\beta_{j_r}(0).\end{aligned}$$

So

$$\begin{aligned}(3.4)\quad&\sum_{i=1}^{\ell}u^i(-1)u^i(0)z\\&=\sum_{r=1}^{t}e_{\beta_{j_1}}(-1)\cdots(\beta_{j_r}(-1)e_{\beta_{j_r}}(-1)+e_{\beta_{j_r}}(-2)\beta_{j_r}(0))\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}.\end{aligned}$$

Second, we have

$$\begin{aligned}\sum_{\alpha\in\Delta_+}e_\alpha(-1)f_\alpha(0)z=&\sum_{\alpha\in\Delta_+}\sum_{r=1}^{t}e_\alpha(-1)e_{\beta_{j_1}}(-1)\cdots[f_\alpha(0),e_{\beta_{j_r}}(-1)]\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}\\&+\sum_{\alpha\in\Delta_+}e_\alpha(-1)e_{\beta_{j_1}}(-1)e_{\beta_{j_2}}(-1)\cdots e_{\beta_{j_t}}(-1)[f_\alpha(0),z^{(0)}]\mathbf{1}.\end{aligned}$$

It is clear that any PBW monomial y in

$$\sum_{\alpha\in\Delta_+}e_\alpha(-1)e_{\beta_{j_1}}(-1)e_{\beta_{j_2}}(-1)\cdots e_{\beta_{j_t}}(-1)[f_\alpha(0),z^{(0)}]\mathbf{1}$$

satisfies

$$(3.5)\qquad y^{(-)}\neq 1.$$

We now consider

$$u_r := \sum_{\alpha\in\Delta_+} e_\alpha(-1)e_{\beta_{j_1}}(-1)\cdots[f_\alpha(0),e_{\beta_{j_r}}(-1)]\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1},\ \text{for } 1\leqslant r\leqslant t.$$

– If $\beta_{j_r}=\alpha+\beta$ for some $\alpha,\beta\in\Delta_+$, then there is a partial sum of two terms in u_r:

$$c_{-\alpha,\alpha+\beta}e_\alpha(-1)e_{\beta_{j_1}}(-1)\cdots e_\beta(-1)\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1} + c_{-\beta,\alpha+\beta}e_\beta(-1)e_{\beta_{j_1}}(-1)\cdots e_\alpha(-1)\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}.$$

Rewriting the above sum to a linear combination of PBW monomials, and noticing that

$$c_{-\alpha,\alpha+\beta}e_\alpha(-1)e_\beta(-1)+c_{-\beta,\alpha+\beta}e_\beta(-1)e_\alpha(-1)=c_{-\alpha,\alpha+\beta}c_{\alpha,\beta}e_{\alpha+\beta}(-2),$$

due to (2.7), we deduce that it is a linear combination of PBW monomials y such that

$$y^{(-)}=z^{(-)}=1,\ y^{(0)}=z^{(0)},\ \operatorname{depth}(y^{(+)})\geqslant 1,\ \deg(y^{(+)})\leqslant\deg(z^{(+)}), \tag{3.6}$$

where $c_{-\alpha,\alpha+\beta},c_{-\beta,\alpha+\beta},c_{\alpha,\beta}\in\mathbb{R}^*$.

– If $\alpha-\beta_{j_r}\in\Delta_+$ for some $\alpha\in\Delta_+$, then there is a term in u_r:

$$c_{-\alpha,\beta_{j_r}}e_\alpha(-1)e_{\beta_{j_1}}(-1)\cdots e_{\beta_{j_{r-1}}}(-1)f_{\alpha-\beta_{j_r}}(-1)e_{\beta_{j_{r+1}}}(-1)\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}. \tag{3.7}$$

It is easy to see that (3.7) is a linear combination of PBW monomials y such that y satisfies one of the following:

$$y^{(-)}=1,\ \operatorname{depth}(y^{(+)})\geqslant 1,\ \deg(y^{(+)})\leqslant\deg(z^{(+)}),\ y^{(0)}=z^{(0)}, \tag{3.8}$$

$$\begin{gathered} y^{(-)}=1,\ \operatorname{depth}(y^{(+)})=0,\ \deg(y^{(+)})\leqslant\deg(z^{(+)})-1,\\ \deg(y^{(0)})>\deg(z^{(0)}),\ \deg^{(0)}_{-1}(y)=\deg^{(0)}_{-1}(z),\end{gathered} \tag{3.9}$$

$$y^{(-)}\neq 1. \tag{3.10}$$

Notice also that with $\alpha=\beta_{j_r}$, there is a term in u_r:

$$-e_{\beta_{j_r}}(-1)e_{\beta_{j_1}}(-1)\cdots e_{\beta_{j_{r-1}}}(-1)\beta_{j_r}(-1)e_{\beta_{j_{r+1}}}(-1)\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}.$$

Together with (3.4), we see that

$$\begin{aligned}
\sum_{i=1}^{\ell}u^i(-1)u^i(0)z+\sum_{r=1}^{t}e_{\beta_{j_r}}(-1)e_{\beta_{j_1}}(-1)\cdots[f_{\beta_{j_r}}(0),e_{\beta_{j_r}}(-1)]\cdots\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}\\
=\sum_{r=1}^{t}e_{\beta_{j_1}}(-1)\cdots(\beta_{j_r}(-1)e_{\beta_{j_r}}(-1)+e_{\beta_{j_r}}(-2)\beta_{j_r}(0))\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}\\
-\sum_{r=1}^{t}\sum_{s=1}^{r-1}e_{\beta_{j_1}}(-1)\cdots[e_{\beta_{j_r}}(-1),e_{\beta_{j_s}}(-1)]\\
\cdots e_{\beta_{j_{r-1}}}(-1)\beta_{j_r}(-1)e_{\beta_{j_{r+1}}}(-1)\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}\\
-\sum_{r=1}^{t}e_{\beta_{j_1}}(-1)\cdots e_{\beta_{j_{r-1}}}(-1)e_{\beta_{j_r}}(-1)\beta_{j_r}(-1)e_{\beta_{j_{r+1}}}(-1)\cdots e_{\beta_{j_t}}(-1)z^{(0)}\mathbf{1}
\end{aligned}$$

is a linear combination of PBW monomials y satisfying one of the following:

$$y^{(-)} = 1,\ \operatorname{depth}(y^{(+)}) \geqslant 1, \deg(y^{(+)}) \leqslant \deg(z^{(+)}),\ y^{(0)} = z^{(0)}, \tag{3.11}$$

$$y^{(-)} = 1,\ \operatorname{depth}(y^{(+)}) \geqslant 1, \deg(y^{(+)}) \leqslant \deg(z^{(+)}) - 1, \tag{3.12}$$
$$\deg^{(0)}_{-1}(y) = \deg^{(0)}_{-1}(z) + 1.$$

Then the lemma follows from (3.5), (3.6), (3.8)–(3.12). □

Lemma 3.4. — *Let z be a PBW monomial of the form* (2.8) *such that $z^{(-)} = 1$. Then*

$$\widetilde{L}_{-1} z = c z^{(+)} (\gamma - \sum_{j=1}^{q} a_{j,1}\beta_j)(-1) z^{(0)} + y^1,$$

where c is a nonzero constant, $\gamma = \sum_{j=1}^{q}\sum_{s=1}^{r_j} a_{j,s}\beta_j$, and y^1 is a linear combination of PBW monomials y such that

$$\deg^{(0)}_{-1}(y) = \deg^{(0)}_{-1}(z) + 1,\ \deg(y^{(+)}) \leqslant \deg(z^{(+)}) - 1,$$

or

$$\deg^{(0)}_{-1}(y) \leqslant \deg^{(0)}_{-1}(z).$$

Proof. — Since the proof is similar to that of Lemma 3.3, we left the verification to the reader. □

Lemma 3.5. — *For $i \in J^+$, we have that* $\operatorname{depth}((w^i)^{(+)}) = 0$.

Proof. — First we have

$$w = \sum_{j\in J^+} \lambda_j w^j + \sum_{j\in J^{(0)}_{-1} \smallsetminus J^+} \lambda_j w^j + \sum_{j\in J_1 \smallsetminus J^{(0)}_{-1}} \lambda_j w^j + \sum_{j\in J\smallsetminus J_1} \lambda_j w^j.$$

Then by Lemma 3.2(b) and Lemma 3.4, we have

$$\begin{aligned}(k+h^\vee)\widetilde{L}_{-1} w = \sum_{i\in J^+} (w^i)^{(+)} \Big(\gamma_i - \sum_{j=1}^{q} a^{(i)}_{j,1}\beta_j\Big)(-1)(w^i)^{(0)} \\ + \sum_{i\in J_1\smallsetminus J^+} (w^i)^{(+)} \Big(\gamma_i - \sum_{j=1}^{q} a^{(i)}_{j1}\beta_j\Big)(-1)(w^i)^{(0)} + y^1,\end{aligned}$$

where $\gamma_i = \sum_{j=1}^{q}\sum_{s=1}^{r^{(i)}_j} a^{(i)}_{j,s}\beta_i$, for $i \in J_1$, and y^1 is a linear combination of PBW monomials y satisfying one of the following conditions:

$$\begin{aligned}&\deg^{(0)}_{-1}(y) = d^{(0)}_{-1} + 1,\ \deg(y^{(+)}) \leqslant d^+ - 1,\\ &\deg^{(0)}_{-1}(y) \leqslant d^{(0)}_{-1},\\ &y^{(-)} \neq 1.\end{aligned}$$

On the other hand, by Lemma 2.4

$$L_{-1} w = \widetilde{L}_{-1} w.$$

By Lemma 2.1, there is no PBW monomial y in $L_{-1}w$ such that $\deg(y^{(+)}) = d^+$, $y^{(-)} = 1$, and $\deg_{-1}^{(0)}(y) = d_{-1}^{(0)} + 1$. Then we deduce that

$$\sum_{i\in J^+} (w^i)^{(+)} \Big(\gamma_i - \sum_{j=1}^{q} a_{j,1}^{(i)}\beta_j\Big)(-1)(w^i)^{(0)} = 0,$$

which means that $(\gamma_i - \sum_{j=1}^{q} a_{j,1}^{(i)}\beta_j) = 0$, for $i \in J^+$, that is, $\mathrm{depth}((w^i)^{(+)}) = 0$. $\square$

As explained at the beginning of §3.3, Theorem 1.1 will be a consequence of the following lemma.

Lemma 3.6. — *For each $i \in J^+$, we have* $\mathrm{depth}(w^i) = 0$.

Proof. — By definition, for $i \in J^+$, $(w^i)^{(0)} = 1$. Moreover, by Lemma 3.5, $\mathrm{depth}((w^i)^{(+)}) = 0$. Hence it suffices to prove that for $i \in J^+$,

$$(w^i)^{(0)} = u^1(-1)^{c_{1,1}^{(i)}} \cdots u^\ell(-1)^{c_{\ell,1}^{(i)}}.$$

Suppose the contrary. Then there exists $i \in J^+$ such that

$$w^i = e_{\beta_1}(-1)^{a_{1,1}^{(i)}} \cdots e_{\beta_q}(-1)^{a_{q,1}^{(i)}} u^1(-1)^{c_{1,1}^{(i)}} \cdots u^1(-m_1)^{c_{1,m_1}^{(i)}} \\ \cdots u^\ell(-1)^{c_{\ell,1}^{(i)}} \cdots u^\ell(-m_\ell)^{c_{\ell,m_\ell}^{(i)}} \mathbf{1},$$

with at least one of the m_j's, for $j = 1, \ldots, \ell$, strictly greater than 1 and $c_{j,m_j}^{(i)} \neq 0$ for such a j. Without loss of generality, one may assume that $1 \in J^+$, that

$$m_1 = \max\{m_j \mid j = 1, \ldots, \ell\} \quad \text{and} \quad 0 \neq c_{1,m_1}^{(1)} \geqslant c_{1,m_1}^{(i)}, \text{ for } i \in J^+.$$

Writing $L_{-1}w$ as

$$L_{-1}w = \sum_{i\in J^+} L_{-1}w^i + \sum_{i\in J_{-1}^{(0)}\smallsetminus J^+} L_{-1}w^i + \sum_{i\in J_1\smallsetminus J_{-1}^{(0)}} L_{-1}w^i + \sum_{i\in J\smallsetminus J_1} L_{-1}w^i,$$

we see by Lemma 2.1 that

$$L_{-1}w = \lambda_1 m_1 c_{1,m_1}^{(1)} v^1 + \sum_{i\in J^+, i\neq 1} \lambda_i m_1 c_{1,m_1}^{(i)} v^i + v + v', \tag{3.13}$$

where for $i \in J^+$, v^i is the PBW monomial defined by:

$$(v^i)^{(-)} = (w^i)^{(-)} = 1, \tag{3.14}$$

$$(v^i)^{(+)} = (w^i)^{(+)} = e_{\beta_1}(-1)^{a_{1,1}^{(i)}} \cdots e_{\beta_q}(-1)^{a_{q,1}^{(i)}}, \tag{3.15}$$

$$(v^i)^{(0)} = u^1(-1)^{c_{1,1}^{(i)}} \cdots u^1(-m_1)^{c_{1,m_1}^{(i)}-1} u^1(-m_1-1) \cdots u^\ell(-m_\ell)^{c_{\ell,m_\ell}^{(i)}}, \tag{3.16}$$

and so, by definition of $J^+ \subset J_{-1}^{(0)}$,

$$\deg_{-1}^{(0)}(v^i) = d_{-1}^{(0)}, \tag{3.17}$$

v is a linear combination of PBW monomials x such that

$$x^{(0)} = u^1(-1)^{c_{1,1}^{(x)}} \cdots u^1(-n_1^{(x)})^{c_{1,n_1^{(x)}}^{(x)}} \cdots u^\ell(-1)^{c_{\ell,1}^{(x)}} \cdots u^\ell(-n_\ell^{(x)})^{c_{\ell,n_\ell^{(x)}}^{(x)}}$$

and

$$\text{either } n_1^{(x)} \leqslant m_1, \quad \text{or } \deg(x^{(+)}) \leqslant d^+ - 1, \quad \text{or } \deg_{-1}^{(0)}(x) \leqslant d_{-1}^{(0)} - 1,$$

and v' is a linear combination of PBW monomials x such that $x^{(-)} \neq 1$. Note that the assumption that $m_1 \geqslant 2$ makes sure that (3.17) holds, and that $\operatorname{depth}(v^i) = \operatorname{depth}(w^i) + 1$ for all $i \in J^+$.

On the other hand, by Lemma 2.4,

$$L_{-1}w = \widetilde{L}_{-1}w,$$

since w is a singular vector of $V^k(\mathfrak{g})$. Hence v^1 must be a PBW monomial of $\widetilde{L}_{-1}w$. Our strategy to obtain the expected contradiction is to show that there is no PBW monomial v^1 in $\widetilde{L}_{-1}w^i$ for each $i \in J$.

– Assume that $i \in J^+$, and suppose that v^1 is a PBW monomial in $\widetilde{L}_{-1}w^i$. First of all, $\deg((w^i)^{(+)}) = d^+$ because $i \in J^+$. Moreover, by the definition of J_1 and Lemma 3.5, we have $(w^i)^{(-)} = 1$ and $\operatorname{depth}((w^i)^{(+)}) = 0$. Hence by Lemma 3.3(2),

$$\deg((v^1)^{(+)}) < \deg((w^i)^{(+)}) = d^+$$

because $(v^1)^{(-)} = 1$ and $\operatorname{depth}((v^1)^{(+)}) = 0$ by (3.14) and (3.15). But $d^+ = \deg((v^1)^{(+)})$ by (3.15), whence a contradiction.

– Assume that $i \in J_{-1}^{(0)} \smallsetminus J^+$. By the definition of J^+ and (3.15),

$$\deg((w^i)^{(+)}) < d^+ = \deg((v^1)^{(+)}). \tag{3.18}$$

Suppose that v^1 is a PBW monomial in $\widetilde{L}_{-1}w^i$. Then

$$(w^i)^{(-)} = 1 = (v^1)^{(-)} \tag{3.19}$$

by Lemma 3.1 since $i \in J_{-1}^{(0)}$. The last equality follows from (3.14). Then by Lemma 3.2(c), either $\deg((v^1)^{(0)}) = \deg((w^i)^{(0)}) + 1$, or $(v^1)^{(0)} = (w^i)^{(0)}$. But it is impossible that $\deg((v^1)^{(0)}) = \deg((w^i)^{(0)}) + 1$, by (d) of Lemma 3.2 because $\deg((v^1)^{(+)}) > \deg((w^i)^{(+)})$. Therefore,

$$(v^1)^{(0)} = (w^i)^{(0)}.$$

Computing $\widetilde{L}_{-1}w^i$, we deduce from

$$(v^1)^{(+)} = e_{\beta_1}(-1)^{a_{1,1}^{(1)}} \cdots e_{\beta_q}(-1)^{a_{q,1}^{(1)}},$$

that

$$(w^i)^{(+)} = e_{\beta_1}(-1)^{a_{1,1}^{(j)}} \cdots e_{\beta_q}(-1)^{a_{q,1}^{(j)}}.$$

Since $(v^1)^{(-)} = (w^i)^{(-)} = 1$, it results from Lemma 3.3 that $\deg((v^1)^{(+)}) \leqslant \deg((w^i)^{(+)})$, which contradicts (3.18).

– Assume that $i \in J_1 \smallsetminus J_{-1}^{(0)}$. Then

$$\deg_{-1}^{(0)}(w^i) < d_{-1}^{(0)} = \deg_{-1}^{(0)}(v^1) \tag{3.20}$$

by (3.17). Suppose that v^1 is a PBW monomial in $\widetilde{L}_{-1}w^i$. By Lemma 3.2(b) and (c),

$$(w^i)^{(-)} = 1, \quad \deg_{-1}^{(0)}(v^1) = \deg_{-1}^{(0)}(w^i) + 1, \tag{3.21}$$

because $(v^1)^{(-)} = 1$ by (3.14). Remember that

$$(v^1)^{(+)} = e_{\beta_1}(-1)^{a_{1,1}^{(1)}} \cdots e_{\beta_q}(-1)^{a_{q,1}^{(1)}}. \tag{3.22}$$

Computing $\widetilde{L}_{-1} w^i$, we deduce that

$$(w^i)^{(+)} = e_{\beta_1}(-1)^{a_{1,1}^{(i)}} \cdots e_{\beta_q}(-1)^{a_{q,1}^{(i)}}.$$

Since $v^{(-)} = 1$ and $\deg_{-1}^{(0)}(v^1) = \deg_{-1}^{(0)}(w^i) + 1$, it results from Lemma 3.3(3) that $\operatorname{depth}((v^1)^{(+)}) \geqslant 1$, which contradicts (3.22).

– Finally, if $j \in J \smallsetminus J_1$, then by Lemma 3.2(b), any PBW monomial y in $\widetilde{L}_{-1} w^j$ satisfies that $y^{(-)} \neq 1$. So v^1 cannot be a PBW monomial in $\widetilde{L}_{-1} w^j$.

This concludes the proof of the lemma. □

As already explained, Lemma 3.6 implies that w has zero depth and so its image in $R_{V^k(\mathfrak{g})}$ is nonzero, achieving the proof of Theorem 1.1.

3.4. Remarks. — The statement of Theorem 1.2 is not true at the critical level. Also, it is not true that the depth of a depth-homogeneous singular vector of $S(\mathfrak{g}[t^{-1}]t^{-1})$ is always zero. Indeed, the $\mathfrak{g}[\![t]\!]$-module $S(\mathfrak{g}[t^{-1}]t^{-1})$ can be naturally identified with $\mathbb{C}[J_\infty \mathfrak{g}^*]$, where $J_\infty X$ is the arc space of X, and so $S(\mathfrak{g}[t^{-1}]t^{-1})^{\mathfrak{g}[t]} \cong \mathbb{C}[J_\infty \mathfrak{g}^*]^{J_\infty G}$. It is known [RT92, BD, EF01] that

$$\mathbb{C}[J_\infty \mathfrak{g}^*]^{J_\infty G} \cong \mathbb{C}[J_\infty(\mathfrak{g}^*/\!/G)].$$

This means that the invariant ring is a polynomial ring with infinitely many variables $\partial^j p_i$, $i = 1, \ldots, \ell$, $j \geqslant 0$, where $p_1, \ldots, p_\ell$ is a set of homogeneous generators of $S(\mathfrak{g})^{\mathfrak{g}}$ considered as elements of $S(\mathfrak{g}[t^{-1}]t^{-1})$ via the embedding $S(\mathfrak{g}) \hookrightarrow S(\mathfrak{g}[t^{-1}]t^{-1})$, $\mathfrak{g} \ni x \mapsto x(-1)$. We have $\operatorname{depth}(\partial^j p_i) = j$ although each $\partial^j p_i$ is a singular vector of $S(\mathfrak{g}[t^{-1}]t^{-1})$.

For $k = -h^\vee$, the maximal submodule N_k of $V^k(\mathfrak{g})$ is generated by Feigin-Frenkel center ([FG04]). Hence [FF92, Fre05], $\operatorname{gr} N_k$ is exactly the argumentation ideal of $S(\mathfrak{g}[t^{-1}]t^{-1})^{\mathfrak{g}[t]}$. Therefore, the above argument shows that the statement of Theorem 1.2 is false at the critical level.

4. W-algebras and proof of Theorem 1.3

Let f be a nilpotent element of $\mathfrak{g}$. By the Jacobson-Morosov theorem, it embeds into an $\mathfrak{sl}_2$-triple (e, h, f) of $\mathfrak{g}$. Recall that the Slodowy slice $\mathscr{S}_f$ is the affine space $f + \mathfrak{g}^e$, where $\mathfrak{g}^e$ is the centralizer of e in $\mathfrak{g}$. It has a natural Poisson structure induced from that of $\mathfrak{g}^*$ ([GG02]).

The embedding $\operatorname{span}_{\mathbb{C}}\{e, h, f\} \cong \mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$ exponentiates to a homomorphism $\mathrm{SL}_2 \to G$. By restriction to the one-dimensional torus consisting of diagonal matrices, we obtain a one-parameter subgroup $\rho\colon \mathbb{C}^* \to G$. For $t \in \mathbb{C}^*$ and $x \in \mathfrak{g}$, set

$$\widetilde{\rho}(t)x := t^2 \rho(t)(x).$$

We have $\widetilde{\rho}(t)f = f$, and the $\mathbb{C}^*$-action of $\widetilde{\rho}$ stabilizes $\mathscr{S}_f$. Moreover, it is contracting to f on $\mathscr{S}_f$, that is, for all $x \in \mathfrak{g}^e$,

$$\lim_{t\to 0} \widetilde{\rho}(t)(f+x) = f.$$

The following proposition is well-known. Since its proof is short, we give below the argument for the convenience of the reader.

Proposition 4.1 ([Slo80, Pre02, CM16]). — *The morphism*

$$\theta_f \colon G \times \mathscr{S}_f \longrightarrow \mathfrak{g}, \quad (g,x) \longmapsto g\cdot x$$

is smooth onto a dense open subset of $\mathfrak{g}^*$.

Proof. — Since $\mathfrak{g} = \mathfrak{g}^e + [f,\mathfrak{g}]$, the map θ_f is a submersion at $(1_G, f)$. Therefore, θ_f is a submersion at all points of $G \times (f + \mathfrak{g}^e)$ because it is G-equivariant for the left multiplication in G, and

$$\lim_{t\to\infty} \rho(t)\cdot x = f$$

for all x in $f+\mathfrak{g}^e$. So, by [Har77, Ch. III, Prop. 10.4], the map θ_f is a smooth morphism onto a dense open subset of $\mathfrak{g}$, containing $G\cdot f$. □

As in the introduction, let $\mathscr{W}^k(\mathfrak{g},f)$ be the affine W-algebra associated with a nilpotent element f of $\mathfrak{g}$ defined by the generalized quantized Drinfeld-Sokolov reduction:

$$\mathscr{W}^k(\mathfrak{g},f) = H^0_{\mathrm{DS},f}(V^k(\mathfrak{g})).$$

Here, $H^\bullet_{\mathrm{DS},f}(M)$ denotes the BRST cohomology of the generalized quantized Drinfeld-Sokolov reduction associated with $f \in \mathcal{N}(\mathfrak{g})$ with coefficients in a $V^k(\mathfrak{g})$-module M. Recall that we have [DSK06, Ara15a] a natural isomorphism $R_{\mathscr{W}^k(\mathfrak{g},f)} \cong \mathbb{C}[\mathscr{S}_f]$ of Poisson algebras, so that

$$X_{\mathscr{W}^k(\mathfrak{g},f)} = \mathscr{S}_f.$$

We write $\mathscr{W}_k(\mathfrak{g},f)$ for the unique simple (graded) quotient of $\mathscr{W}^k(\mathfrak{g},f)$. Then $X_{\mathscr{W}_k(\mathfrak{g},f)}$ is a $\mathbb{C}^*$-invariant Poisson subvariety of the Slodowy slice $\mathscr{S}_f$.

Let $\mathscr{O}_k$ be the category $\mathscr{O}$ of $\widehat{\mathfrak{g}}$ at level k. We have a functor

$$\mathscr{O}_k \longrightarrow \mathscr{W}^k(\mathfrak{g},f)\text{-Mod}, \quad M \longmapsto H^0_{\mathrm{DS},f}(M),$$

where $\mathscr{W}^k(\mathfrak{g},f)$-Mod denotes the category of $\mathscr{W}^k(\mathfrak{g},f)$-modules.

The full subcategory of $\mathscr{O}_k$ consisting of objects M on which $\mathfrak{g}$ acts locally finitely will be denoted by KL_k. Note that both $V^k(\mathfrak{g})$ and $L_k(\mathfrak{g})$ are objects of KL_k.

Theorem 4.2 ([Ara15a])

(1) $H^i_{\mathrm{DS},f}(M) = 0$ *for all* $i \neq 0$, $M \in \mathrm{KL}_k$. *In particular, the functor*

$$\mathrm{KL}_k \longrightarrow \mathscr{W}^k(\mathfrak{g},f)\text{-Mod}, \quad M \longmapsto H^0_{\mathrm{DS},f}(M),$$

is exact.

(2) *For any quotient V of $V^k(\mathfrak{g})$,*

$$X_{H^0_{\mathrm{DS},f}(V)} = X_V \cap \mathscr{S}_f.$$

In particular $H^0_{\mathrm{DS},f}(V) \neq 0$ if and only if $\overline{G\cdot f} \subset X_V$.

By Theorem 4.2(1), $H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))$ is a quotient vertex algebra of $\mathscr{W}^k(\mathfrak{g},f)$ if it is nonzero. Conjecturally [KRW03, KW08], we have

$$\mathscr{W}_k(\mathfrak{g},f) \cong H^0_{\mathrm{DS},f}(L_k(\mathfrak{g})) \quad \text{provided that } H^0_{\mathrm{DS},f}(L_k(\mathfrak{g})) \neq 0.$$

(This conjecture has been verified in many cases [Ara05, Ara07, Ara11, AvE19].)

Proof of Theorem 1.3. — The directions (1) ⇒ (2) and (2) ⇒ (3) are obvious. Let us show that (3) implies (1). So suppose that $X_{H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))} = \mathscr{S}_f$. By Theorem 1.1, it is enough to show that $X_{L_k(\mathfrak{g})} = \mathfrak{g}^*$. Assume the contrary. Then $X_{L_k(\mathfrak{g})}$ is contained in a proper G-invariant closed subset of $\mathfrak{g}$. On the other hand, by Theorem 4.2 and our hypothesis, we have

$$\mathscr{S}_f = X_{H^0_{\mathrm{DS},f}(L_k(\mathfrak{g}))} = X_{L_k(\mathfrak{g})} \cap \mathscr{S}_f.$$

Hence, $\mathscr{S}_f$ must be contained in a proper G-invariant closed subset of $\mathfrak{g}$. But this contradicts Proposition 4.1. The proof of the theorem is completed. □

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Manuscript received 3rd April 2020
accepted 8th January 2021

Tomoyuki Arakawa, Research Institute for Mathematical Sciences, Kyoto University
Kyoto, 606-8502, Japan
E-mail : arakawa@kurims.kyoto-u.ac.jp
Url : http://www.kurims.kyoto-u.ac.jp/~arakawa/

Cuipo Jiang, School of Mathematical Sciences, Shanghai Jiao Tong University
Shanghai, 200240, China
E-mail : cpjiang@sjtu.edu.cn
Url : http://math.sjtu.edu.cn/faculty/cpjiang/

Anne Moreau, Faculté des Sciences d'Orsay, Université Paris-Saclay
91405 Orsay, France
E-mail : anne.moreau@universite-paris-saclay.fr
Url : https://www.imo.universite-paris-saclay.fr/~moreau/