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**Pink's conjecture on unlikely intersections and families of semi-abelian varieties**

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## PINK'S CONJECTURE ON UNLIKELY INTERSECTIONS AND FAMILIES OF SEMI-ABELIAN VARIETIES

BY DANIEL BERTRAND & BAS EDIXHOVEN

**ABSTRACT.** — The Poincaré torsor of a Shimura family of abelian varieties can be viewed both as a family of semi-abelian varieties and as a mixed Shimura variety. We show that the special subvarieties of the latter cannot all be described in terms of the subgroup schemes of the former. This provides a counter-example to the relative Manin-Mumford conjecture, but also some evidence in favour of Pink's conjecture on unlikely intersections in mixed Shimura varieties. The main part of the article concerns mixed Hodge structures and the uniformisation of the Poincaré torsor, but other, more geometric, approaches are also discussed.

**RÉSUMÉ** (Sur la conjecture de Pink sur les intersections exceptionnelles et les familles de variétés semi-abéliennes)

Le toiseur de Poincaré d'une famille de Shimura de variétés abéliennes s'interprète à la fois comme une famille de variétés semi-abéliennes et comme une variété de Shimura mixte. Nous montrons que ses sous-variétés spéciales en ce deuxième sens ne peuvent pas toutes se décrire en termes de sous-schémas en groupes. Cela donne un contre-exemple à la conjecture de Manin-Mumford relative, mais témoigne aussi de la pertinence de la conjecture de Pink sur les intersections exceptionnelles dans les variétés de Shimura mixtes. L'essentiel de l'article porte sur les structures de Hodge mixtes, mais d'autres approches, de nature plus géométrique, sont aussi abordées.

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## 1. INTRODUCTION

In the unpublished preprint [25] Pink formulated a very influential conjecture (the equivalent Conjectures 1.1–1.3) on so-called “unlikely intersections” in mixed Shimura varieties. Here we merely recall the statement of his Conjecture 1.3:

if  $Y$  is a Hodge generic irreducible closed subvariety of a mixed Shimura variety  $S$ , then the union of the intersections of  $Y$  with the special subvarieties of  $S$  of codimension at least  $\dim(Y) + 1$  is not Zariski dense in  $Y$ .

We refer to [30] for more details on such intersections, and for their relations to the conjectures by Manin–Mumford, Mordell–Lang (which are now theorems), and André–Oort. See also [25], [24], and [18]. The André–Oort conjecture was recently proved for all  $A_g$  in [29].

In the last section of [25], Pink states a relative version of the Manin–Mumford conjecture for families of semi-abelian varieties, Conjecture 6.1:

if  $B \rightarrow X$  is a family of semi-abelian varieties over  $\mathbb{C}$  and  $Y$  is an irreducible closed subvariety in  $B$  that is not contained in any proper closed subgroup scheme of  $B \rightarrow X$ , then the union of the intersections of  $Y$  with algebraic subgroups of codimension at least  $\dim(Y) + 1$  of the fibres of  $B \rightarrow X$  is not Zariski dense in  $Y$ .

Furthermore, Theorem 6.3 of [25] claims that Conjecture 1.3 implies Conjecture 6.1. However, a counter-example to Conjecture 6.1 was given in the unpublished preprint [1], based on a relative version of a construction of Ribet ([16], [26]), leading to the notion of Ribet sections on certain semi-abelian schemes. But it was also shown in [1] that this counter-example was not in contradiction with Conjecture 1.3, and so, the error was in the proof of Theorem 6.3 (see Remark 5.4.4 at the end of Section 5 below). The conclusion is that the context of mixed Hodge structures *is* the right one for a relative Manin–Mumford conjecture for families of *semi*-abelian varieties: indeed, the image of a Ribet section is a special subvariety that can in general *not* be interpreted as a subgroup scheme (see Remark 5.4.2 below). However, for families of *abelian* varieties (that is, mixed Shimura varieties of Kuga type), Theorem 6.3 is correct, see [24, Prop. 4.6], [13, Prop. 3.4], and again Remark 5.4.4.

The aim of this article is to provide not only a published account of this story, sharpening the results of [1], but also a self-contained description of the involved mixed Hodge structures and the corresponding mixed Shimura varieties, made as accessible as possible.

The article is structured as follows. In Section 2 we present the (counter)example, in the case of complex elliptic curves with complex multiplications, and in Section 3 (which introduces a different viewpoint) for abelian schemes. In Sections 4 and 5 we give the description of the example in the context of mixed Shimura varieties whose pure part parametrises principally polarised abelian varieties. We show that it gives evidence in favour of Pink’s Conjecture 1.3. Finally, in Section 6 we give a description of the example, in the case of elliptic curves, in terms of generalised jacobians.

REMARK 1.1. — In each section, we construct Ribet sections under various denominations, namely  $t_\varphi$  in (2.1.1),  $r_f$  in Proposition 3.1,  $r_f^{\text{Sh}}$  in Theorem 5.2, and  $t_\varphi^J$  in (6.0.2). At each step, we prove their compatibility, as well as some of their properties. The main property, leading to the searched-for counterexample to Conjecture 6.1 of [25], is stated in Theorem 2.4 and asserts that the Ribet section  $t_\varphi$  maps torsion points of the base to torsion points of their fibres. The proof (with sharper additional properties) is given in terms of  $r_f$  in Proposition 3.3, of  $r_f^{\text{Sh}}$  in Proposition 5.3 and of  $t_\varphi^J$  in Theorem 6.1. So, these proofs have logically unnecessary overlaps, but their settings are sufficiently distinct to justify this presentation. We should mention that yet another construction of the Ribet sections was proposed in [1], based as in [16] on the theory of 1-motives. But as shown in [7], the latter is equivalent to the construction of  $t_\varphi$  in Section 2.

REMARK 1.2. — We will sometimes abbreviate “the image of a given section” by “the section”. On the other hand, the image of a Ribet section will be called a Ribet variety.

REMARK 1.3. — One may wonder if, in spite of the above mentioned error in Theorem 6.3 of [25], Pink's general Conjecture 1.3 can still be applied to the study of unlikely intersections in semi-abelian varieties. Bertrand, who could see this only under strong assumptions of simplicity (and only for Manin-Mumford), suggested that Edixhoven study the problem in full generality. And indeed, after this article was finished, Edixhoven found that everything in Sections 4 and 5 of [25] is correct, except the proof of the last statement, Theorem 5.7. That theorem states that Conjecture 1.3 implies Conjecture 5.1, the unlikely intersection variant of the Manin-Mumford conjecture for semi-abelian varieties. Moreover, he also showed that, with a small change, and a more detailed description of the special subvarieties of the mixed Shimura varieties involved, Pink's argument gives that Conjecture 1.3 implies Conjecture 5.2 (unlikely intersection generalisation of Mordell-Lang), and therefore, by Theorem 5.5 of [25], implies Conjecture 5.1. The details of this will appear in an article in preparation by Edixhoven.

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## 2. THE EXAMPLE WITH ELLIPTIC CURVES

The key player in the example in [1] is the Poincaré torsor  $P$  on a product  $E \times E^\vee$ , where  $E$  is a complex elliptic curve and where  $E^\vee$  is its dual.

To make  $P$  and  $E^\vee$  more explicit, we use the isomorphism  $\lambda: E \rightarrow E^\vee$  that sends a point  $P$  to the class of the invertible  $\mathcal{O}$ -module  $\mathcal{O}((-P) - 0)$ , isomorphic to  $\mathcal{O}(0 - P)$  (this is the unique principal polarisation of  $E$ ). In the notation of [22, §6],  $\lambda = \varphi_M$ , where  $M$  is the invertible  $\mathcal{O}$ -module  $\mathcal{O}(0)$  on  $E$ , and where  $\varphi_M$  sends  $P$  to the class of  $(\text{tr}_P^* M) \otimes_{\mathcal{O}} M^{-1}$ , with  $\text{tr}_P$  the translation by  $P$  map on  $E$ .

The *Poincaré bundle*  $L$  on  $E \times E$  is then

$$(2.0.1) \quad L = \text{add}^* M \otimes_O \text{pr}_1^* M^{-1} \otimes_O \text{pr}_2^* M^{-1} \otimes_O 0^* M,$$

where  $\text{add}$ ,  $\text{pr}_1$ ,  $\text{pr}_2$ , and  $0$  are the addition map, the projections, and the constant map  $0$  from  $E \times E$  to  $E$ . It is isomorphic (with the isomorphism given by the choice of a non-zero element of the fibre  $M(0)$  of  $M$  at  $0$ , i.e., of a non-zero tangent vector of  $E$  at  $0$ ) to  $O(D)$ , with

$$(2.0.2) \quad D = \text{add}^{-1}0 - \text{pr}_1^{-1}0 - \text{pr}_2^{-1}0.$$

The fibre  $L(x, y)$  at a point  $(x, y)$  is given by:

$$(2.0.3) \quad L(x, y) = M(x+y) \otimes M(x)^{-1} \otimes M(y)^{-1} \otimes M(0).$$

In particular:  $L(x, 0) = M(x) \otimes M(x)^{-1} \otimes M(0)^{-1} \otimes M(0) = \mathbb{C}$ , and similarly for  $L(0, y)$ . Hence  $L$  is canonically trivial on the union of  $E \times \{0\}$  and  $\{0\} \times E$ . But let us remark that the pullback of  $L$  via  $\text{diag}: E \rightarrow E \times E$  has fibre at  $x$  equal to  $M(2x) \otimes M(x)^{-2} \otimes M(0)$ , hence is given by the divisor  $\sum_{P \in E[2]} P - 2 \cdot 0$  which is of degree 2 and linearly equivalent to  $2 \cdot 0$ .

The *Poincaré torsor*  $P$  is then the  $\mathbb{G}_m$ -torsor on  $E \times E$  (trivial locally for the Zariski topology) of isomorphisms from  $O$  to  $L$ :

$$(2.0.4) \quad P = \text{Isom}(O, L).$$

It is represented by a complex algebraic variety over  $E \times E$ , also denoted  $P$ . Its fibre  $P(x, y)$  over  $(x, y)$  is the  $\mathbb{C}^\times$ -torsor  $\text{Isom}(\mathbb{C}, L(x, y))$ .

The theorem of the cube ([22, §6]) says that any invertible  $O$ -module  $N$  on  $E^n$  with  $n \geq 3$ , whose restrictions to  $\ker(\text{pr}_i)$  are trivial for all  $i$  in  $\{1, \dots, n\}$ , is trivial. For every such  $N$ , for any non-zero element  $s_0$  of  $N(0, \dots, 0)$  there is a unique  $s$  in  $N(E^n)$  such that  $s(0) = s_0$  (the reason is that  $O(E^n) = \mathbb{C}$ ).

For example, the invertible  $O$ -module

$$\bigotimes_{I \subset \{1, 2, 3\}} \text{add}_I^* M^{(-1)^{\#I}} \quad \text{on } E \times E \times E,$$

where  $\text{add}_I: E^3 \rightarrow E$ ,  $(x_1, x_2, x_3) \mapsto \sum_{i \in I} x_i$ , is canonically trivial (canonically because its fibre at  $(0, 0, 0)$  is  $M(0)^{\otimes 4} \otimes M(0)^{\otimes -4} = \mathbb{C}$ ). Explicitly: for all points  $(x, y, z)$  of  $E^3$  we have

$$M(x+y+z) \otimes M(x+y)^{-1} \otimes M(x+z)^{-1} \otimes M(y+z)^{-1} \\ \otimes M(x) \otimes M(y) \otimes M(z) \otimes M(0)^{-1} = \mathbb{C}.$$

Similarly, the invertible  $O$ -modules on  $E^3$  with fibres

$$L(x, y+z) \otimes L(x, y)^{-1} \otimes L(x, z)^{-1} \quad \text{and} \quad L(x+y, z) \otimes L(x, z)^{-1} \otimes L(y, z)^{-1}$$

are canonically trivial. Therefore, for all points  $x, y$  and  $z$  of  $E$  we have:

$$(2.0.5) \quad L(x, y+z) = L(x, y) \otimes L(x, z), \quad L(x+y, z) = L(x, z) \otimes L(y, z).$$

This gives two composition laws on  $P$ : for  $\alpha: \mathbb{C} \rightarrow L(x, y)$  in  $P(x, y)$  and  $\beta: \mathbb{C} \rightarrow L(x, z)$  in  $P(x, z)$  we get  $\alpha \otimes \beta: \mathbb{C} \rightarrow L(x, y+z)$  in  $P(x, y+z)$ , and

similarly with the second variable fixed. With the first variable fixed,  $P$  is a commutative group-variety over  $E$ , via  $\text{pr}_1$ , whose fibres are extensions of  $E$  by  $\mathbb{G}_m$ , and similarly for  $\text{pr}_2$ ; for details, see Chapter I, Section 2.5 of [21] and the Proposition of Section 2.6 there. In particular,  $P$  is a *bi-extension* of  $E$  and  $E$  by  $\mathbb{G}_m$ : the two partial group laws commute with each other in the following sense. For  $x_1, x_2, y_1$  and  $y_2$  in  $E$ , and  $p_{i,j}$  in  $P(x_i, y_j)$ , the various ways of summing the  $p_{i,j}$  leads to the same result in  $P(x_1 + x_2, y_1 + y_2)$ . This is proved by considering the universal case  $T := E^4$ ,  $x_1 = \text{pr}_1$ ,  $x_2 = \text{pr}_2$ ,  $y_1 = \text{pr}_3$  ad  $y_2 = \text{pr}_4$ , and concluding that the trivialisations of

$$L(x_1 + x_2, y_1 + y_2) \otimes L(x_1, y_1)^{-1} \otimes L(x_1, y_2)^{-1} \otimes L(x_2, y_1)^{-1} \otimes L(x_2, y_2)^{-1}$$

corresponding to the various ways of summing are equal because they are so at  $(0, 0, 0, 0)$ : writing it out in terms of  $\mathcal{M}$  leads to the tensor product of as many  $\mathcal{M}(0)$ 's as  $\mathcal{M}(0)^{-1}$ 's.

With these preliminaries behind us, we can finally proceed to the construction of Ribet sections. Let  $\varphi$  be an endomorphism of  $E$  and let  $\bar{\varphi} := \lambda^{-1} \circ \varphi^\vee \circ \lambda$  be the conjugate of  $\varphi$ . Let

$$\gamma = (\text{id}, \varphi - \bar{\varphi}): E \longrightarrow E \times E, \quad P \longmapsto (P, (\varphi - \bar{\varphi})(P))$$

be the graph map attached to  $\varphi - \bar{\varphi}$ . The following fact was observed in [6]; see also [16] for a description in terms of 1-motives.

**PROPOSITION 2.1.** — *The invertible  $\mathcal{O}$ -module  $\gamma^*L$  on  $E$  is canonically trivial.*

*Proof.* — As this is the crucial ingredient of the example that we present in this article, we give two proofs: one for readers who prefer a computation using divisors, and one for those who prefer universal properties. But first we note that if  $\varphi = \bar{\varphi}$ , then  $\gamma = (\text{id}, 0)$  and  $\gamma^*L$  is canonically trivial because, as mentioned above,  $L$  is canonically trivial on  $E \times \{0\}$ . So in the first proof below we may and do assume that  $\varphi \neq \bar{\varphi}$ .

*A proof by divisors.* — The fibre of  $\gamma^*L$  at 0 is  $L(0, 0) = \mathbb{C}$ , and  $L$  is isomorphic to  $\mathcal{O}(D)$  with

$$D = \text{add}^{-1}0 - \text{pr}_1^{-1}0 - \text{pr}_2^{-1}0$$

as in (2.0.2). So it suffices to show that  $\gamma^*D$  is linearly equivalent to the zero divisor on  $E$ . Let  $\alpha := \varphi - \bar{\varphi}$ . We note that

$$\text{add} \circ \gamma = \text{add} \circ (\text{id}, \alpha) = \text{id} + \alpha, \quad \text{pr}_1 \circ \gamma = \text{id}, \quad \text{and} \quad \text{pr}_2 \circ \gamma = \alpha.$$

Hence we have the following equalities of divisors on  $E$ :

$$(\text{id}, \alpha)^*D = (\text{id} + \alpha)^*0 - \text{id}^*0 - \alpha^*0 = \sum_{P \in \ker(\text{id} + \alpha)} P - 0 - \sum_{Q \in \ker(\alpha)} Q.$$

The degree of this divisor is zero because, in  $\text{End}(E)$ ,  $\alpha$  is imaginary, so we have

$$\text{deg}(\text{id} + \alpha) = (\text{id} + \alpha)(\text{id} + \bar{\alpha}) = \text{id} + \alpha\bar{\alpha} = 1 + \text{deg}(\alpha).$$

Any degree zero divisor on  $E$  is linearly equivalent to  $R - 0$ , with  $R$  the image of the divisor under the group morphism  $\text{Div}^0(E) \rightarrow E$  that sends each point to itself. So in our case  $R$  is the sum of the points in  $\ker(\text{id} + \alpha)$ , minus the sum of the points in  $\ker(\alpha)$ . These two kernels are finite commutative groups. For such a group, the sum of the elements is 0, except when its 2-primary part is cyclic and non-trivial, in which case it is the element of order 2. Let  $a := \varphi + \bar{\varphi}$  be the trace of  $\varphi$ ; it is in the subring  $\mathbb{Z}$  of  $\text{End}(E)$ . Then  $\alpha = -a + 2\varphi$ , and  $\text{id} + \alpha = (1 - a) + 2\varphi$ . So one of these has odd degree, and the other is divisible by 2 in  $\text{End}(E)$ , and so for none of them the 2-primary part of the kernel is cyclic and non-trivial.

*A proof by universal properties.* — We view  $E \times E$  as an  $E$ -scheme via  $\text{pr}_2$ . Then  $L$  is the universal invertible  $\mathcal{O}$ -module of degree 0 on  $E$  with given trivialisation at 0: for every complex algebraic variety  $S$  and every invertible  $\mathcal{O}$ -module  $N$  on  $E_S$ , fibrewise of degree 0, and with a given trivialisation  $\mathcal{O}_S \rightarrow \mathcal{O}^*N$ , there is a unique  $f: S \rightarrow E$  such that the pullback of  $L$  via  $\text{id} \times f: E_S \rightarrow E_E$  is isomorphic to  $N$ . Moreover, in this case there is a unique isomorphism  $g: N \rightarrow (\text{id} \times f)^*L$  that is compatible with the given trivialisations at 0. Of course, the analogous statements are true with  $\text{pr}_2$  replaced by  $\text{pr}_1$ .

Let us turn to  $\bar{\varphi}$ . It is defined as  $\lambda^{-1} \circ \varphi^\vee \circ \lambda$ . Hence, for  $y$  in  $E$ ,  $\bar{\varphi}(y)$  is obtained as follows:  $\lambda(y)$  is the isomorphism class of the invertible  $\mathcal{O}$ -module  $L|_{E \times \{y\}}$  on  $E$ , and then  $\lambda(\bar{\varphi}(y)) = (\lambda \circ \lambda^{-1} \circ \varphi^\vee \circ \lambda)y = \varphi^\vee(\lambda(y))$  corresponds (by the definition of  $\varphi^\vee$ ) to  $\varphi^*(L|_{E \times \{y\}})$ . By definition of  $\lambda$  and  $L$ ,  $\lambda(\bar{\varphi}(y))$  corresponds to  $L|_{E \times \{\bar{\varphi}(y)\}}$ . Hence the invertible  $\mathcal{O}$ -modules  $(\varphi \times \text{id})^*L$  and  $(\text{id} \times \bar{\varphi})^*L$  on  $E \times E$ , both trivialised on  $\{0\} \times E$ , are uniquely isomorphic on the fibres of the second projection. Hence we have a canonical isomorphism between  $(\text{id} \times \bar{\varphi})^*L$  and  $(\varphi \times \text{id})^*L$ .

As  $L$  together with its trivialisations on  $E \times \{0\}$  and  $\{0\} \times E$  is symmetric (that is, invariant under the automorphism of  $E \times E$  that sends  $(x, y)$  to  $(y, x)$ ), we get a canonical isomorphism between  $(\text{id} \times \bar{\varphi})^*L$  and  $(\text{id} \times \varphi)^*L$ .

From (2.0.5), applied with  $x = \text{id}_E$ ,  $y = \varphi$  and  $z = -\bar{\varphi}$  we get a canonical isomorphism, on  $E$ , from  $\gamma^*L$  to  $(\text{id}, \varphi)^*L \otimes (\text{id}, -\bar{\varphi})^*L$ . Applying it again, but now with  $x = \text{id}_E$ ,  $y = \bar{\varphi}$  and  $z = -\bar{\varphi}$ , we get a canonical isomorphism from  $\mathcal{O}$  to  $(\text{id}, -\bar{\varphi})^*L \otimes (\text{id}, \bar{\varphi})^*L$ , giving us a canonical isomorphism from  $(\text{id}, -\bar{\varphi})^*L$  to  $(\text{id}, \bar{\varphi})^*L^{-1}$ . Combining, we see that

$$\begin{aligned} \gamma^*L &= (\text{id}, \varphi)^*L \otimes (\text{id}, -\bar{\varphi})^*L = (\text{id}, \varphi)^*L \otimes (\text{id}, \bar{\varphi})^*L^{-1} \\ &= (\text{id}, \varphi)^*L \otimes (\text{id}, \varphi)^*L^{-1} = \mathcal{O}. \end{aligned} \quad \square$$

Now we view  $P$  as a group variety over  $E$  via  $\text{pr}_1: E \times E \rightarrow E$ . The canonical trivialisation

$$(2.1.1) \quad t_\varphi: \mathcal{O} \longrightarrow \gamma^*L = (\text{id}, \alpha)^*L$$

on  $E$  gives, for every  $x$  in  $E$ , an element  $t_\varphi(x)$  in  $\text{Isom}(\mathbb{C}, L(x, \alpha(x)))$ , hence an element in  $P(x, \alpha(x))$ . As such,  $t_\varphi$  is a section of the group variety  $P$  over  $E$ , which we call the *Ribet section attached to  $\varphi$* .

Following [1], we will now show that if  $\bar{\varphi} \neq \varphi$ , then  $t_\varphi$  gives a counterexample to Conjecture 6.1 of [25].

LEMMA 2.2. — *Let  $\mathbb{G}_m \hookrightarrow G \twoheadrightarrow E$  be an extension whose class in the group  $\text{Ext}(E, \mathbb{G}_m)$  is not torsion. Then the only connected algebraic subgroups of  $G$  are  $\{0\}$ ,  $\mathbb{G}_m$  and  $G$ .*

*Proof.* — Let  $H$  be a connected algebraic subgroup of  $G$ . Then  $\dim(H)$  is 0, 1 or 2. If it is 0 then  $H = \{0\}$ , and if it is 2 then  $H = G$ , so we assume it is 1, and that  $H$  is not equal to  $\mathbb{G}_m$ . Then  $H \rightarrow E$  is surjective, and since  $\mathbb{G}_m \cap H$  is a finite group,  $H \rightarrow E$  is an unramified cover. As  $H$  is connected, it is itself an elliptic curve, and there is an  $n \in \mathbb{Z}_{>0}$  and a factorisation  $n \cdot : E \rightarrow H \rightarrow E$ . This means that the extension  $\mathbb{G}_m \hookrightarrow G \twoheadrightarrow E$  is split after pullback via  $n \cdot : E \rightarrow E$ , hence its class is torsion.  $\square$

LEMMA 2.3. — *If  $\varphi \neq \bar{\varphi}$ , then the union over all  $n \in \mathbb{Z}$  of the images  $(n \cdot t_\varphi)(E)$  of the sections  $n \cdot t_\varphi$  is Zariski dense in  $P$ .*

*Proof.* — Let  $Z$  be the Zariski closure of the union of the  $(n \cdot t_\varphi)(E)$ . Let  $x$  in  $E$  be of infinite order. Then  $y := \alpha(x)$  is of infinite order as well. The point  $t_\varphi(x)$  of the extension  $P_x$  of  $E$  by  $\mathbb{G}_m$  has image  $y$  in  $E$ . The Zariski closure in  $P_x$  of  $\{n \cdot t_\varphi(x) : n \in \mathbb{Z}\}$  is a closed subgroup  $H$  of  $P_x$ . The image of  $H$  in  $E$  is closed ( $H \rightarrow E$  is a morphism of algebraic groups), and contains  $y$ , hence is equal to  $E$ . Hence  $\dim(H)$  is 1 or 2. Assume that  $\dim(H) = 1$ . By Lemma 2.2 the extension class of  $P_x$  is torsion, but that contradicts that this class, being  $\lambda(x)$ , is not torsion. We conclude that  $\dim(H) = 2$ , and  $H = P_x$ . Hence  $Z$  contains all  $P_x$  with  $x$  not torsion. Then  $Z = P$ .  $\square$

THEOREM 2.4. — *For every torsion point  $x$  in  $E$ ,  $t_\varphi(x)$  is torsion in  $P_x$ .*

*Proof.* — We will give three proofs: one in the context of abelian schemes and biextensions (Proposition 3.3), one, more elementary, using generalised jacobians of elliptic curves with a double point in Section 6, and a third proof, using the description of  $t_\varphi(E)$  as a special subvariety of a mixed Shimura variety (Proposition 5.3). We refer to [1, §1], for the initial proof of Theorem 2.4, based on the theory of 1-motives.  $\square$

We now explain why the closed subvariety  $Y := t_\varphi(E)$  in the family of semi-abelian varieties  $B := P$  over  $X := E$  is a counter-example to [25, Conj. 6.1] when  $\varphi - \bar{\varphi} \neq 0$ . First of all,  $Y$  is not contained in a proper subvariety of  $B$  that is a subgroup scheme of  $B$  over  $X$  because of Lemma 2.3.

Secondly,  $d := \dim(Y) = 1$ , hence according to the conjecture, the intersection of  $Y$  with the set  $B^{[>1]}$  that is the union, over all  $x$  in  $X$ , of all subgroups of  $B_x$  of codimension  $> 1$ , should not be Zariski dense in  $Y$ . However,  $B^{[>1]}$  is the set of points that are torsion in their fibre, and Theorem 2.4 says that the intersection is infinite.



### 3. THE EXAMPLE WITH ABELIAN SCHEMES

In this section we consider abelian schemes, but even in the case of elliptic curves, this section provides a new point of view on Ribet sections and their properties. We recommend Chapter I of [21] and references therein for further details about biextensions, duality and pairings.

Let  $S$  be a scheme,  $A$  an abelian scheme over  $S$ , and  $A^\vee$  its dual ([11, §I.1]). Let  $L$  be the universal line bundle on  $A \times_S A^\vee$ , rigidified, compatibly, at  $\{0\} \times A^\vee$  and  $A \times \{0\}$ ; it identifies  $A$  with the dual of  $A^\vee$ . Then  $P = \text{Isom}_{A \times_S A^\vee}(O, L)$  is the Poincaré  $\mathbb{G}_m$ -torsor on  $A \times_S A^\vee$ , and as described in the previous section in the case of elliptic curves, it is a biextension of  $A$  and  $A^\vee$  by  $\mathbb{G}_m$ . In particular, over  $A^\vee$ ,  $P$  is the universal extension of  $A$  by  $\mathbb{G}_m$ , and over  $A$ ,  $P$  is the universal extension of  $A^\vee$  by  $\mathbb{G}_m$ . Proposition 2.1 extends to the present situation as follows (see [6], [7], [19, §8.3]).

**PROPOSITION 3.1.** — *Let  $S$  be a scheme,  $A$  an abelian scheme over  $S$ ,  $P$  the Poincaré torsor on  $A \times_S A^\vee$ ,  $f: A^\vee \rightarrow A$  a morphism of group schemes,  $f^\vee: A^\vee \rightarrow (A^\vee)^\vee = A$  its dual, and*

$$\alpha := f - f^\vee: A^\vee \longrightarrow A.$$

*The restriction of  $P$  to the graph of  $\alpha$  has a unique section  $r_f$*

$$\begin{array}{ccccc} \mathbb{G}_{m,A^\vee} & \hookrightarrow & P & \twoheadrightarrow & A_{A^\vee} = A \times_S A^\vee \\ & \searrow & \downarrow r_f & \nearrow & \uparrow \\ & & A^\vee & \xrightarrow{(\alpha, \text{id})} & \end{array}$$

*with value 1 at the origin.*

*Proof.* — We start in a more general situation: let  $A_1$  and  $A_2$  be abelian schemes over  $S$ ,  $P_1$  and  $P_2$  their Poincaré torsors, and  $f: A_1 \rightarrow A_2$ . Then the dual  $f^\vee: A_2^\vee \rightarrow A_1^\vee$  is defined by the condition that the pullback of the universal extension

$$\mathbb{G}_{m,A_2^\vee} \hookrightarrow P_2 \twoheadrightarrow (A_2)_{A_2^\vee} = A_2 \times_S A_2^\vee$$

by  $f \times \text{id}: A_1 \times_S A_2^\vee \rightarrow A_2 \times_S A_2^\vee$  is isomorphic to the pullback of the universal extension

$$\mathbb{G}_{m,A_1^\vee} \hookrightarrow P_1 \twoheadrightarrow (A_1)_{A_1^\vee} = A_1 \times_S A_1^\vee$$

by  $\text{id} \times f^\vee: A_1 \times_S A_2^\vee \rightarrow A_1 \times_S A_1^\vee$ . Such an isomorphism is unique, hence

$$\text{for all } T \rightarrow S, x \in A_1(T), y \in A_2^\vee(T): P_1(x, f^\vee y) = P_2(fx, y).$$

Now we specialise to the case where  $A_1 = A_2^\vee$ . Then  $A_1 \times_S A_1^\vee = A_2^\vee \times_S A_2$ , with Poincaré torsors  $P_1$  and  $\sigma^*P_2$ , where  $\sigma: A_2^\vee \times_S A_2 \rightarrow A_2 \times_S A_2^\vee$  is the coordinate switch. Then we have, for  $T \rightarrow S$ ,  $x \in A_1(T) = A_2^\vee(T)$  and  $y \in A_2^\vee(T)$ :

$$(3.1.1) \quad P_2(fx, y) = P_1(x, f^\vee y) = P_2(f^\vee y, x).$$

Now we restrict to the case  $y = x$ , where we have  $P_2(fx, x) = P_2(f^\vee x, x)$ . Then additivity in the first factor gives that

$$\begin{aligned}
 P_2(\alpha x, x) &= P_2((f - f^\vee)x, x) = P_2(fx - f^\vee x, x) \\
 (3.1.2) \qquad &= P_2(fx, x) \otimes P_2(f^\vee x, x)^{-1} \\
 &= \text{Hom}(P_2(fx, x), P_2(f^\vee x, x)) = \mathbb{G}_{mT}.
 \end{aligned}$$

Now we take  $A_2 = A$ , and define  $r_f: A^\vee \rightarrow P$  by letting it send  $x$  to the  $T$ -point of  $P(\alpha x, x)$  corresponding to the unit section of  $\mathbb{G}_{mT}$  via the isomorphism in (3.1.2).

By construction,  $r_f(0) = 1$ . This condition makes it unique, as two such sections differ by a factor in  $O(A^\vee)^\times = O(S)^\times$ , with value 1 at  $0 \in A^\vee(S)$ .  $\square$

**REMARK 3.2.** — When  $A \rightarrow S$  is a complex elliptic curve  $E$ , and  $\lambda: E \rightarrow E^\vee$  is as in Section 2, and  $\varphi$  is in  $\text{End}(E)$ , and  $f = \varphi \circ \lambda$ , then  $t_\varphi$  as in (2.1.1) and  $r_f$  as in Proposition 3.1 are equal (well, up to switching the factors of  $E \times E$ ), because they are sections of the same  $\mathbb{G}_m$ -torsor over  $E$ , with the same value at 0. Therefore, Proposition 3.3 below proves Theorem 2.4.

The following Proposition gives the torsion property of  $r_f$  at the torsion points of  $A^\vee$ : it implies that for  $T \rightarrow S$  and  $x$  in  $A^\vee[n](T)$  we have  $n^2 r_f(x) = 1$ . (See Proposition 5.3 and Theorem 6.1 for other proofs of this equality.)

**PROPOSITION 3.3.** — *Let  $S, A, P, f, \alpha$  and  $r_f$  be as in Proposition 3.1. Let  $n \geq 1$ , let  $T$  be an  $S$ -scheme, and  $x \in A^\vee[n](T)$ . Then*

$$nr_f(x) = e_n(fx, x) \quad \text{in} \quad P(n\alpha x, x) = P(0, x) = \mathbb{G}_m(T),$$

with  $e_n: A[n](T) \times A^\vee[n](T) \rightarrow \mu_n(T)$  the Weil pairing (whose definition is recalled below).

*Proof.* — The base change  $T \rightarrow S$  reduces to the case where  $T = S$ . First we describe the Weil pairing in terms of  $P$ . Let  $z \in A[n](S)$  and  $y \in A^\vee[n](S)$ . We have the following canonical isomorphisms between  $\mathbb{G}_m$ -torsors on  $S$ ,

$$\begin{array}{ccc}
 \mathbb{G}_{mS} & \xlongequal{\quad} & P(z, 0) \xlongequal{\quad} P(z, ny) \xlongequal{+2} P(z, y)^{\otimes n} \\
 \downarrow e_n(z, y) & & \parallel \text{id} \\
 \mathbb{G}_{mS} & \xlongequal{\quad} & P(0, y) \xlongequal{\quad} P(nz, y) \xlongequal{+1} P(z, y)^{\otimes n}
 \end{array}$$

where the superscript  $+1$  means “induced by additivity in the first coordinate”, etc., and where  $P(z, y)^{\otimes n}$  is the contracted product of  $n$  copies of  $P(z, y)$ . As the diagram shows, we define  $e_n(z, y)$  to be the image of the section 1 of the top  $\mathbb{G}_{mS}$  in the bottom  $\mathbb{G}_{mS}$ . We claim that this is the usual Weil pairing: let  $P_y$  be the extension of  $A$  by  $\mathbb{G}_{mS}$  at  $y$ , then, as  $n \cdot y = 0$  in  $A^\vee(S)$ , the pullback of the extension

$$\mathbb{G}_{mS} \hookrightarrow P_y \twoheadrightarrow A$$

by  $n \cdot : A \rightarrow A$  splits (uniquely as for all extensions of abelian schemes by affine group schemes), and so there is a unique  $\tilde{n} : A \rightarrow P_y$  that lifts  $n \cdot : A \rightarrow A$ , and the restriction  $\tilde{n} : A[n] \rightarrow \mu_n$  sends  $z$  to  $e_n(z, y)$ .

The following commutative diagram relates  $nr_f(x)$  to  $e_n(fx, x)$  and  $e_n(x, f^\vee x)$ : going from bottom right to upper right and then upper left is multiplication by  $e_n(x, f^\vee x)$ , going from bottom right to middle right and then middle left and then upper left is  $nr_f(x)$  by (3.1.2), and from bottom right to upper left via bottom left is  $e_n(fx, x)$ .

$$(3.3.1) \quad \begin{array}{ccccc} \mathbb{G}_{mS} & \xlongequal{\text{id}} & \mathbb{G}_{mS} & \xlongequal{\text{id}} & \mathbb{G}_{mS} \\ \parallel & \text{b} & \parallel & \text{f} & \parallel \\ P(0, x) & \xlongequal{\quad} & (\sigma^*P)(0, f^\vee x) & \xlongequal{\quad} & P(f^\vee x, 0) \\ \parallel & \text{c} & \parallel & \text{g} & \parallel \\ \text{a} \ P(fx, x)^{\otimes n} & \xlongequal{\quad} & (\sigma^*P)(x, f^\vee x)^{\otimes n} & \xlongequal{\quad} & P(f^\vee x, x)^{\otimes n} \ \text{j} \\ \parallel & \text{d} & \parallel & \text{h} & \parallel \\ P(fx, 0) & \xlongequal{\quad} & (\sigma^*P)(x, 0) & \xlongequal{\quad} & P(0, x) \\ \parallel & \text{e} & \parallel & \text{i} & \parallel \\ \mathbb{G}_{mS} & \xlongequal{\text{id}} & \mathbb{G}_{mS} & \xlongequal{\text{id}} & \mathbb{G}_{mS} \end{array}$$

Here are arguments for the commutativity of all faces (a–j) in the diagram.

- (a) This is the definition of  $e_n(fx, x)$ .
- (b–e) This is because the equality signs in (3.1.1) are isomorphisms of biextensions on  $A_2^\vee \times_S A_2^\vee$ .
- (f–i) These follow directly from the definition of  $\sigma^*P$ .
- (j) This is the definition of  $e_n(x, f^\vee x)$ .

Let us remark that the commutativity of this diagram shows that  $f^\vee$  and  $f$  are adjoints for the  $e_n$ -pairing, and that when  $f^\vee = f$ ,  $e_n(fx, x) = 1$  for all  $x$  in  $A^\vee[n](S)$ , in particular, that the pairings attached to a polarisation are alternating.  $\square$

#### 4. THE POINCARÉ TORSOR AS MIXED SHIMURA VARIETY

In this section we describe the Poincaré torsor of the universal family of principally polarised complex abelian varieties of dimension  $d$  as a mixed Shimura variety, that is, as a moduli space for mixed Hodge structures. We recommend [24, §2] (and also [17] and [8]) as an introduction to mixed Hodge structures and (connected) mixed Shimura varieties, but we do not assume the reader to be familiar with these notions. In fact, we hope that the example treated here also provides a good introduction, and perhaps a motivation to read more. We find that the point of view of mixed Shimura varieties gives a simple and beautiful perspective on the uniformisation of the universal Poincaré torsor. The notion of 1-motives from [9] provides an algebraic description of the mixed Hodge structures that we encounter, but we will not use this.

4.1. PURE HODGE STRUCTURES. — For  $n$  in  $\mathbb{Z}$ , a  $\mathbb{Z}$ -Hodge structure of weight  $n$  is a finitely generated  $\mathbb{Z}$ -module  $M$  together with a decomposition (called Hodge decomposition) of the complex vector space  $M_{\mathbb{C}} := \mathbb{C} \otimes M$ :

$$M_{\mathbb{C}} = \bigoplus_{\substack{p,q \in \mathbb{Z} \\ p+q=n}} M^{p,q},$$

such that for all  $p, q$  in  $\mathbb{Z}$  with  $p + q = n$ :

$$M^{q,p} = \overline{M^{p,q}},$$

where  $\overline{M^{p,q}}$  is the image of  $M^{p,q}$  under the map  $M_{\mathbb{C}} \rightarrow M_{\mathbb{C}}$  that sends  $z \otimes m$  to  $\bar{z} \otimes m$ . A pure  $\mathbb{Z}$ -Hodge structure (also called split mixed  $\mathbb{Z}$ -Hodge structure) is a finitely generated  $\mathbb{Z}$ -module  $M$ , together with a direct sum decomposition

$$M/M_{\text{tors}} = \bigoplus_{n \in \mathbb{Z}} M_n,$$

and for each  $n$  a Hodge structure of weight  $n$ ,

$$M_{n,\mathbb{C}} = \bigoplus_{p+q=n} M^{p,q}.$$

For  $T \subset \mathbb{Z}^2$ ,  $M$  is said to be of type  $T$ , if, for all  $(p, q)$  not in  $T$ ,  $M^{p,q}$  is zero.

A morphism of pure  $\mathbb{Z}$ -Hodge structures

$$(M, (M^{p,q})_{p,q}) \longrightarrow (N, (N^{p,q})_{p,q})$$

is a morphism  $f: M \rightarrow N$  of  $\mathbb{Z}$ -modules such that for all  $(p, q)$  one has  $f_{\mathbb{C}}(M^{p,q}) \subset N^{p,q}$ .

For  $M$  and  $N$  pure  $\mathbb{Z}$ -Hodge structures,  $M^{\vee}$ ,  $M \otimes N$  are given pure  $\mathbb{Z}$ -Hodge structures as follows:

$$(M^{\vee})^{p,q} = (M^{-p,-q})^{\vee}, \quad (M \otimes N)^{p,q} = \bigoplus_{\substack{a+c=p \\ b+d=q}} (M^{a,b} \otimes N^{c,d}),$$

and this dictates the rule for  $\text{Hom}(M, N)$ :

$$\text{Hom}(M, N)^{p,q} = (M^{\vee} \otimes N)^{p,q} = \bigoplus_{\substack{-a+c=p \\ -b+d=q}} \text{Hom}(M^{a,b}, N^{c,d}).$$

It is convenient to define, for  $m$  in  $\mathbb{Z}$ , the  $\mathbb{Z}$ -Hodge structure  $\mathbb{Z}(m)$  of weight  $-2m$  as the sub- $\mathbb{Z}$ -module  $(2\pi i)^m \mathbb{Z}$  of  $\mathbb{C}$ , with  $\mathbb{Z}(m)_{\mathbb{C}} = \mathbb{Z}(m)^{-m,-m}$ . For  $M$  a pure  $\mathbb{Z}$ -Hodge structure, and  $m$  in  $\mathbb{Z}$ ,  $M(m)$  denotes  $M \otimes \mathbb{Z}(m)$ . The embedding  $(2\pi i)^m \mathbb{Z} \subset \mathbb{C}$  gives the isomorphisms  $\mathbb{Z}(m)_{\mathbb{C}} = \mathbb{C}$  and  $M(m)_{\mathbb{C}} = M_{\mathbb{C}}$ .

A polarisation on a pure  $\mathbb{Z}$ -Hodge structure  $M$  of weight  $n$  is a morphism of pure  $\mathbb{Z}$ -Hodge structures  $\Psi: M \otimes M \rightarrow \mathbb{Z}(-n)$  such that for every  $(p, q)$  with  $p + q = n$  the map

$$M^{p,q} \times M^{p,q} \longrightarrow \mathbb{C}, \quad (v, w) \longmapsto (-1)^p \Psi(v, \bar{w})$$

is a complex inner product (that is, for all  $(v, w)$ ,  $\Psi(w, \bar{v}) = \overline{\Psi(v, \bar{w})}$ , and, for all  $v \neq 0$ ,  $(-1)^p \Psi(v, \bar{v}) > 0$ ). The symmetry condition is equivalent to  $\Psi$  being symmetric if  $n$

is even and antisymmetric if  $n$  is odd. The symmetry and positivity conditions are equivalent to the restriction to  $M_{\mathbb{R}} \times M_{\mathbb{R}}$  of the  $\mathbb{C}$ -bilinear map

$$M_{\mathbb{C}} \times M_{\mathbb{C}} \longrightarrow \mathbb{C}, \quad (x, y) \longmapsto (2\pi i)^n \Psi(x \otimes i \cdot y),$$

with  $i$  acting on  $M^{p,q}$  as multiplication by  $i^{-p}\bar{i}^{-q}$  being  $\mathbb{R}$ -valued, symmetric and positive definite.

**4.2. PRINCIPALLY POLARISED ABELIAN VARIETIES.** — Let  $d$  be in  $\mathbb{Z}_{\geq 1}$ . Principally polarised complex abelian varieties of dimension  $d$  are conveniently described as follows. Their lattice is a free  $\mathbb{Z}$ -module  $M$  of rank  $2d$  with a Hodge structure  $M_{\mathbb{C}} = M^{-1,0} \oplus M^{0,-1}$ , and the polarisation  $\Psi: M \otimes M \rightarrow \mathbb{Z}(1) = 2\pi i\mathbb{Z}$  is anti-symmetric and induces an isomorphism  $M \rightarrow M^{\vee}(1)$ . The abelian variety is then  $M_{\mathbb{C}}/(M^{0,-1} + M)$ . Then  $M$  together with  $\Psi$  is isomorphic to  $\mathbb{Z}^{2d}$  with

$$\Psi: \mathbb{Z}^{2d} \otimes \mathbb{Z}^{2d} \longrightarrow \mathbb{Z}(1), \quad x \otimes y \longmapsto 2\pi i x^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y,$$

and such an isomorphism is unique up to composition with an element of  $\mathrm{Sp}(\Psi)(\mathbb{Z})$  (the stabiliser of  $\Psi$  in  $\mathrm{GL}_{2d}(\mathbb{Z})$ ). Let  $(e_1, \dots, e_{2d})$  be the standard basis of  $\mathbb{Z}^{2d}$ . The subspace  $M^{0,-1}$  of  $\mathbb{C}^{2d}$ , on which  $(v, w) \mapsto \Psi(v, \bar{w})$  is an inner product, has trivial intersection with the isotropic subspaces generated by  $e_1, \dots, e_d$  and  $e_{d+1}, \dots, e_{2d}$ , hence there is a unique  $\tau$  in  $\mathrm{GL}_d(\mathbb{C})$  such that  $M^{0,-1} = \{ \begin{pmatrix} \tau v \\ v \end{pmatrix} : v \in \mathbb{C}^d \}$ . As  $\Psi$  is a morphism of Hodge structures,  $M^{0,-1}$  is isotropic for  $\Psi$ , giving  $\tau^t = \tau$ . The positivity of the complex inner product on  $M^{0,-1}$  gives that  $\mathrm{Im}(\tau) = (\tau - \bar{\tau})/2i$  is positive definite. Conversely, for every  $\tau \in \mathrm{M}_d(\mathbb{C})$  with  $\tau^t = \tau$  and  $\mathrm{Im}(\tau)$  positive definite,  $\tau$  is in  $\mathrm{GL}_d(\mathbb{C})$  and  $M^{0,-1} := \{ \begin{pmatrix} \tau v \\ v \end{pmatrix} : v \in \mathbb{C}^d \}$  gives a Hodge structure on  $\mathbb{Z}^{2d}$  such that  $\Psi$  is a principal polarisation.

We conclude: the set  $D_{\Psi}$  of Hodge structures of type  $\{(-1, 0), (0, -1)\}$  on  $\mathbb{Z}^{2d}$  for which  $\Psi$  is a polarisation is in bijection with the Siegel half space  $\mathbb{H}_d$  of symmetric  $\tau \in \mathrm{M}_d(\mathbb{C})$  with  $\mathrm{Im}(\tau)$  positive definite, via  $\tau \mapsto M_{\tau}^{0,-1} := \{ \begin{pmatrix} \tau v \\ v \end{pmatrix} : v \in \mathbb{C}^d \}$ . Note that  $\mathbb{H}_d$  is a convex open subset of the set of symmetric  $d$  by  $d$  complex matrices. The action of  $\mathrm{Sp}(\Psi)(\mathbb{Z})$  describes the moduli of complex principally polarised abelian varieties of dimension  $d$ : the quotients by suitable congruence subgroups give fine moduli spaces, and the stacky quotient by  $\mathrm{Sp}(\Psi)(\mathbb{Z})$  gives the stack of complex principally polarised abelian varieties of dimension  $d$ . Let us write more explicitly the abelian variety  $A_{\tau} := \mathbb{C}^{2d}/(M_{\tau}^{0,-1} + \mathbb{Z}^{2d})$  at  $\tau$  in  $\mathbb{H}_d$ . The  $\mathbb{C}$ -linear map  $\mathbb{C}^{2d} \rightarrow \mathbb{C}^d$ ,  $\begin{pmatrix} w \\ v \end{pmatrix} \mapsto w - \tau v$  is surjective and has kernel  $M^{0,-1}$ . So  $A_{\tau}$  is the cokernel of  $(1_d - \tau) \cdot \mathbb{Z}^{2d} \rightarrow \mathbb{C}^d$ ,  $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x - \tau y$ , that is,  $A_{\tau}$  is the quotient of  $\mathbb{C}^d$  by the lattice generated by  $\mathbb{Z}^d$  and the columns of  $\tau$ .

For all  $M^{0,-1}$  in  $D_{\Psi}$  and all  $g$  in  $\mathrm{GL}_{2d}(\mathbb{R})$ ,  $gM^{0,-1}$  is a Hodge structure of type  $\{(-1, 0), (0, -1)\}$  for which  $g\Psi$  is a polarisation, where, for all  $x, y$  in  $\mathbb{R}^{2d}$ ,

$$(g\Psi)(x \otimes y) = \Psi((g^{-1}x) \otimes (g^{-1}y)).$$

Hence  $\mathrm{Sp}(\Psi)(\mathbb{R})$ , the subgroup of  $\mathrm{GL}_{2d}(\mathbb{R})$  that preserves  $\Psi$ , acts on  $D_{\Psi}$ .

The following argument shows that this action is transitive. Let  $M^{0,-1}$  be in  $D_{\Psi}$ , and let  $v_1, \dots, v_d$  be an orthonormal basis for  $M^{0,-1}$ . Then the  $2d$  elements of  $\mathbb{R}^{2d}$ ,

$\operatorname{Re}(v_1), \dots, \operatorname{Re}(v_d), \operatorname{Im}(v_1), \dots, \operatorname{Im}(v_d)$ , form an  $\mathbb{R}$ -basis of  $\mathbb{R}^{2d}$  with respect to which  $M^{0,-1}$  and  $\Psi$  do not depend on  $M^{0,-1}$ : indeed,  $M^{0,-1} \subset \mathbb{C}^{2d}$  is the  $\mathbb{C}$ -subspace generated by  $\operatorname{Re}(v_1) + i\operatorname{Im}(v_1), \dots, \operatorname{Re}(v_d) + i\operatorname{Im}(v_d)$ , and, for every  $j$ , we have that  $\Psi(\operatorname{Re}(v_j), \operatorname{Im}(v_j)) = i/2$  and all other  $\Psi(\operatorname{Re}(v_j), \operatorname{Im}(v_k))$  and  $\Psi(\operatorname{Re}(v_j), \operatorname{Re}(v_k))$  and  $\Psi(\operatorname{Im}(v_j), \operatorname{Im}(v_k))$  are zero.

In fact a slightly bigger group acts on  $D_\Psi$ . We view  $\Psi$  as an element of the  $\mathbb{R}$ -vector space  $(\mathbb{R}^{2d} \otimes_{\mathbb{R}} \mathbb{R}^{2d})^\vee \otimes_{\mathbb{R}} \mathbb{R}(1)$ , on which the group  $\operatorname{GL}_{2d}(\mathbb{R}) \times \mathbb{R}^\times$  acts. An element  $(g, \lambda)$  acts as  $(g^{-1} \otimes g^{-1})^\vee \otimes \lambda$ . Then  $(g, \lambda)$  fixes  $\Psi$  if and only if for all  $x, y \in \mathbb{R}^{2d}$ ,  $\Psi(gx, gy) = \lambda\Psi(x, y)$ . We let  $\operatorname{GSp}_\Psi(\mathbb{R})$  be the group of such  $(g, \lambda)$ , and  $\operatorname{GSp}_\Psi(\mathbb{R})^+$  the subgroup of the  $(g, \lambda)$  with  $\lambda > 0$ . Then  $\operatorname{GSp}_\Psi(\mathbb{R})^+$  acts on  $D_\Psi$  via  $M^{0,-1} \mapsto g \cdot M^{0,-1}$ .

**4.3. MIXED HODGE STRUCTURES.** — A *mixed Hodge structure* on a finitely generated  $\mathbb{Z}$ -module  $M$  is the data of an increasing filtration  $(W_n M)_{n \in \mathbb{Z}}$  (called the weight filtration) with  $W_n M = M_{\text{tors}}$  for  $n$  small enough and  $W_n M = M$  for  $n$  large enough, with all  $M/W_n M$  torsion free, and a decreasing filtration  $(F^p M_{\mathbb{C}})_{p \in \mathbb{Z}}$  of the  $\mathbb{C}$ -vector space  $M_{\mathbb{C}}$ , with  $F^p M_{\mathbb{C}} = M_{\mathbb{C}}$  for small enough  $p$  and  $F^p M_{\mathbb{C}} = 0$  for large enough  $p$ , such that for each  $n$  in  $\mathbb{Z}$  the filtration induced by  $F$  on  $(\operatorname{Gr}_n^W M)_{\mathbb{C}} := ((W_n M)/(W_{n-1} M))_{\mathbb{C}}$  is a Hodge structure of weight  $n$ :

$$(\operatorname{Gr}_n^W M)_{\mathbb{C}} = \bigoplus_{p+q=n} (\operatorname{Gr}_n^W M)_{\mathbb{C}}^{p,q},$$

with

$$(\operatorname{Gr}_n^W M)_{\mathbb{C}}^{p,q} = F^p (\operatorname{Gr}_n^W M)_{\mathbb{C}} \cap \overline{F^q (\operatorname{Gr}_n^W M)_{\mathbb{C}}}.$$

Let us determine all mixed Hodge structures on  $M := \mathbb{Z} \cdot e_1 \oplus \mathbb{Z} \cdot e_2$ , with  $W_{-3}(M) = 0$ ,  $W_{-2}(M) = W_{-1}(M) = \mathbb{Z} \cdot e_1$  and  $W_0(M) = M$ , of type  $\{(-1, -1), (0, 0)\}$ , that is, extensions of  $\mathbb{Z}(0)$  by  $\mathbb{Z}(1)$ . Then  $F^{-1} M_{\mathbb{C}} = M_{\mathbb{C}}$ ,  $F^1 M_{\mathbb{C}} = 0$ , and  $F^0 M_{\mathbb{C}} \cap \mathbb{C} \cdot e_1 = 0$  and under the quotient map  $q: M_{\mathbb{C}} \rightarrow M_{\mathbb{C}}/W_{-1} M_{\mathbb{C}} = \mathbb{C} \cdot e_2$ ,  $F^0 M_{\mathbb{C}}$  is mapped surjectively. So  $F^0 M_{\mathbb{C}}$  is a line, of the form  $L_a := \mathbb{C} \cdot (e_2 + ae_1)$  for a unique  $a$  in  $\mathbb{C}$ , giving a bijection from  $\mathbb{C}$  to the set  $D_W$  of mixed Hodge structures of the type we consider.

Let  $P_W(\mathbb{R})$  be the subgroup of  $\operatorname{GL}_2(\mathbb{R}) \times \operatorname{GL}(\mathbb{R}(1)) \times \operatorname{GL}(\mathbb{R}(0))$  that fixes

$$\begin{array}{lll} \mathbb{R}(1) \longrightarrow \mathbb{R}^2, & \mathbb{R}^2 \longrightarrow \mathbb{R}(0), & \mathbb{R}(0) \otimes \mathbb{R}(0) \longrightarrow \mathbb{R}(0). \\ 2\pi i \longmapsto e_1 & (x, y) \longmapsto y & x \otimes y \longmapsto xy \end{array}$$

Then

$$P_W(\mathbb{R}) = \left\{ \left( \begin{pmatrix} \lambda & x \\ 0 & 1 \end{pmatrix}, \lambda, 1 \right) : \lambda \in \mathbb{R}^\times, x \in \mathbb{R} \right\}.$$

By definition  $P_W(\mathbb{R})$  acts on  $D_W$ , and transported to  $\mathbb{C}$  this action is given by  $a \mapsto \lambda a + x$ . This action has two orbits:  $\mathbb{R}$  and  $\mathbb{C} - \mathbb{R}$ . We would like to have a transitive action (in order to get a “connected mixed Shimura datum” as in [24, Def. 2.1]). To get that, we allow  $x$  to be complex, that is, we let  $U_W(\mathbb{C})$  be the subgroup of  $\operatorname{GL}_2(\mathbb{C})$  of unipotent matrices  $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$  with  $x \in \mathbb{C}$ , and let

$$P_W(\mathbb{R})U_W(\mathbb{C}) = \left\{ \left( \begin{pmatrix} \lambda & x \\ 0 & 1 \end{pmatrix}, \lambda, 1 \right) : \lambda \in \mathbb{R}^\times, x \in \mathbb{C} \right\}$$

act on  $D_W$ . The action of  $P_W(\mathbb{Z})$  on  $\mathbb{C}$  describes the moduli of mixed  $\mathbb{Z}$ -Hodge structures that are extension of  $\mathbb{Z}(0)$  by  $\mathbb{Z}(1)$ . The coarse moduli space is the quotient

$$\mathbb{C} \longrightarrow \mathbb{C}^\times \longrightarrow \mathbb{C}, \quad a \longmapsto \exp(2\pi ia) \longmapsto \exp(2\pi ia) + \exp(-2\pi ia).$$

4.4. THE UNIVERSAL POINCARÉ TORSOR AS MODULI SPACE OF MIXED HODGE STRUCTURES

Let  $d$  be in  $\mathbb{Z}_{\geq 1}$  and

$$M := \mathbb{Z}(1) \oplus \mathbb{Z}^{2d} \oplus \mathbb{Z},$$

with standard basis  $2\pi ie_0, e_1, \dots, e_{2d+1}$ , and with the following filtration:

$$\begin{aligned} W_{-3}M &= \{0\}, & W_{-2}M &= \mathbb{Z} \cdot 2\pi ie_0, \\ W_{-1}M &= \mathbb{Z} \cdot 2\pi ie_0 \oplus \dots \oplus \mathbb{Z} \cdot e_{2d}, & W_0M &= M. \end{aligned}$$

Let  $D$  be the set of filtrations  $F$  on  $M_{\mathbb{C}}$  such that  $(M, W, F)$  is a mixed  $\mathbb{Z}$ -Hodge structure of type  $\{(-1, -1), (-1, 0), (0, -1), (0, 0)\}$ , and such that  $\Psi: (x, y) \mapsto 2\pi ix^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y$  is, via the given bases, a polarisation on  $\text{Gr}_{-1}^W M$ . For  $F$  in  $D$  we have  $F^{-1}M_{\mathbb{C}} = M_{\mathbb{C}}$ , and  $F^1M_{\mathbb{C}} = \{0\}$ , so  $F$  is given by  $F^0M_{\mathbb{C}}$ . We get a map from  $D$  to the set  $D_\Psi$  (see Section 4.2) by sending  $F^0$  to  $F^0(\text{Gr}_{-1}^W M_{\mathbb{C}})$ . Recall that we have a bijection  $\mathbb{H}_d \rightarrow D_\Psi$  that sends  $\tau$  to  $M_\tau^{0,-1} = \begin{pmatrix} \tau \\ 1_d \end{pmatrix} \mathbb{C}^d \subset \mathbb{C}^{2d}$ .

For  $m$  and  $n$  in  $\mathbb{Z}_{\geq 0}$  we denote by  $M_{m,n}(\mathbb{C})$  the set of complex  $m$  by  $n$  matrices.

PROPOSITION 4.5. — *There is a bijection  $\mathbb{H}_d \times M_{1,d}(\mathbb{C}) \times M_{d,1}(\mathbb{C}) \times \mathbb{C} \rightarrow D$ ,*

$$(\tau, u, v, w) \longmapsto \begin{pmatrix} u & w \\ \tau & v \\ 1_d & 0 \\ 0 & 1 \end{pmatrix} \mathbb{C}^{d+1} \subset M_{\mathbb{C}} = \bigoplus_{j=0}^{2d+1} \mathbb{C}e_j.$$

*Proof.* — Let  $\tau$  be in  $\mathbb{H}_d$ . The  $F^0(W_{-1}(M)_{\mathbb{C}})$  in the fibre over  $\tau$  are the subspaces of  $W_{-1}(M)_{\mathbb{C}}$  that are mapped isomorphically to the subspace  $M_\tau^{0,-1}$  of  $\text{Gr}_{-1}^W(M)_{\mathbb{C}}$  in the short exact sequence

$$0 \longrightarrow W_{-2}(M)_{\mathbb{C}} \longrightarrow W_{-1}(M)_{\mathbb{C}} \longrightarrow \text{Gr}_{-1}^W(M)_{\mathbb{C}} \longrightarrow 0.$$

This accounts for the first  $d$  columns in the matrix above. We take these columns as the first  $d$  elements of our basis of  $F^0M_{\mathbb{C}}$ .

Each  $F^0(M_{\mathbb{C}})$  in  $D$  that restricts to  $F^0(W_{-1}M_{\mathbb{C}})$  given by a  $(\tau, u)$  has a unique  $(d+1)$ th basis vector  $\sum a_i e_i$  ending with  $d$  zeros and then a 1. This accounts for the last column. □

Let  $P$  be the subgroup scheme of  $\text{GL}(M) \times \text{GL}(\mathbb{Z}(1))$  that fixes  $W, \mathbb{Z}(1) \rightarrow W_{-2}(M), 2\pi ia \mapsto 2\pi ia e_0, \mathbb{Z}(0) \rightarrow \text{Gr}_0^W(M), a \mapsto a e_{2d+1}$ , and  $\Psi: \text{Gr}_{-1}^W(M) \otimes \text{Gr}_{-1}^W(M) \rightarrow \mathbb{Z}(1)$ . Then, for any  $\mathbb{Z}$ -algebra  $R$  (we will only use  $\mathbb{Z}, \mathbb{R}$  and  $\mathbb{C}$ ), we have

$$(4.5.1) \quad P(R) = \left\{ \begin{pmatrix} \mu(g) & x & z \\ 0 & g & y \\ 0 & 0 & 1 \end{pmatrix} : \begin{array}{l} (g, \mu(g)) \in \text{GSp}(\Psi)(R), \\ x \in M_{1,2d}(R), y \in M_{2d,1}(R), z \in R \end{array} \right\},$$

where the matrices are with respect to the  $\mathbb{Z}$ -basis  $2\pi i e_0, e_1, \dots, e_{2d+1}$  of  $M$ . We let  $U$  be the subgroup scheme of  $P$  given by

$$U(R) = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in R \right\}.$$

We also let  $P^u$  be the unipotent radical of  $P$ , that is,

$$P^u(R) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x \in M_{1,2d}(R), y \in M_{2d,1}(R), z \in R \right\},$$

also known as the Heisenberg group. Then  $P^u$  is a central extension of the vector group  $P^u/U$  by  $\mathbb{G}_a$ . The commutator pairing on  $P^u/U$  sends  $((x, y), (x', y'))$  to  $xy' - x'y$ .

For  $R$  a subring of  $\mathbb{C}$ , the matrix with respect to the  $\mathbb{C}$ -basis  $e_0, \dots, e_{2d+1}$  of  $M_{\mathbb{C}}$  of the element of  $P(R)$  above is

$$(4.5.2) \quad \begin{pmatrix} \mu(g) & 2\pi i x & 2\pi i z \\ 0 & g & y \\ 0 & 0 & 1 \end{pmatrix}.$$

By definition,  $P(\mathbb{R})^+U(\mathbb{C})$  acts on  $D$ . We make this explicit for elements of  $P^u(\mathbb{R})U(\mathbb{C})$ , with respect to the  $\mathbb{C}$ -basis  $e_0, \dots, e_{2d+1}$ , writing  $2\pi i x = (2\pi i x_1 \ 2\pi i x_2)$  and  $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ :

$$(4.5.3) \quad \begin{aligned} & \begin{pmatrix} 1 & 2\pi i x_1 & 2\pi i x_2 & 2\pi i z \\ 0 & 1_d & 0 & y_1 \\ 0 & 0 & 1_d & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u & w \\ \tau & v \\ 1_d & 0 \\ 0 & 1 \end{pmatrix} \mathbb{C}^{d+1} \\ &= \begin{pmatrix} u + 2\pi i x_1 \tau + 2\pi i x_2 w + 2\pi i x_1 v + 2\pi i z \\ \tau & v + y_1 \\ 1_d & y_2 \\ 0 & 1 \end{pmatrix} \mathbb{C}^{d+1} \\ &= \begin{pmatrix} u + 2\pi i x_1 \tau + 2\pi i x_2 w + 2\pi i x_1 v + 2\pi i z \\ \tau & v + y_1 \\ 1_d & y_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1_d & -y_2 \\ 0 & 1 \end{pmatrix} \mathbb{C}^{d+1} \\ &= \begin{pmatrix} u + 2\pi i x_1 \tau + 2\pi i x_2 w + 2\pi i x_1 v + 2\pi i z - (u + 2\pi i x_1 \tau + 2\pi i x_2 y_2) \\ \tau & v + y_1 - \tau y_2 \\ 1_d & 0 \\ 0 & 1 \end{pmatrix} \mathbb{C}^{d+1}. \end{aligned}$$



As the action of  $\mathrm{Sp}_{\Psi}(\mathbb{R})$  on  $D_{\Psi}$  is transitive, we conclude that the action of  $P(\mathbb{R})^+U(\mathbb{C})$  on  $D$  is transitive. We also write out the action of  $\mathrm{GSp}_{\Psi}(\mathbb{R})^+$  on  $D$ :

$$\begin{aligned}
 (4.5.4) \quad \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u & w \\ \tau & v \\ 1_d & 0 \\ 0 & 1 \end{pmatrix} \mathbb{C}^{d+1} &= \begin{pmatrix} \mu u & \mu w \\ a\tau + b & av \\ c\tau + d & cv \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (c\tau + d)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \mathbb{C}^{d+1} \\
 &= \begin{pmatrix} \mu u(c\tau + d)^{-1} & \mu w \\ (a\tau + b)(c\tau + d)^{-1} & av \\ 1_d & cv \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1_d & -cv \\ 0 & 1 \end{pmatrix} \mathbb{C}^{d+1} \\
 &= \begin{pmatrix} \mu u(c\tau + d)^{-1} & \mu w - \mu u(c\tau + d)^{-1}cv \\ (a\tau + b)(c\tau + d)^{-1} & av - (a\tau + b)(c\tau + d)^{-1}cv \\ 1_d & 0 \\ 0 & 1 \end{pmatrix} \mathbb{C}^{d+1}.
 \end{aligned}$$

**PROPOSITION 4.6.** — *The quotient  $P^u(\mathbb{Z})\backslash D$  is the universal Poincaré torsor over  $\mathbb{H}_d$ .*

*Proof.* — We prove this by showing that the universal extension of the universal abelian variety over  $\mathbb{H}_d$  by  $\mathbb{C}^{\times}$  is uniformised in exactly the same way when we express everything in terms of matrices. We view  $M_{1,d}(\mathbb{C})$  and  $M_{d,1}(\mathbb{C})$  as duals via the matrix multiplication (row times column).

Let us first consider a complex torus  $A = V/L$ , and an extension of complex Lie groups

$$0 \longrightarrow \mathbb{C}^{\times} \longrightarrow E \longrightarrow A \longrightarrow 0.$$

Passing to universal covers gives us an extension of  $\mathbb{C}$ -vector spaces

$$0 \longrightarrow \mathbb{C} \longrightarrow \tilde{E} \longrightarrow V \longrightarrow 0,$$

mapping to the previous sequence by exponential maps. The kernels of these maps form an extension

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow M \longrightarrow L \longrightarrow 0.$$

The extensions of  $V$  by  $\mathbb{C}$  and of  $L$  by  $\mathbb{Z}(1)$  admit splittings, and these are unique up to  $V^{\vee} := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$  and  $\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}(1)) = L^{\vee}(1)$ . It follows that all extensions of  $A$  by  $\mathbb{C}^{\times}$  are obtained as cokernels of maps

$$\begin{aligned}
 (4.6.1) \quad \mathbb{Z}(1) \oplus L &\longrightarrow \mathbb{C} \oplus V, \\
 (2\pi in, m) &\longmapsto (2\pi in - \alpha(m), m), \quad \text{with } \alpha \in \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}) = L_{\mathbb{C}}^{\vee}.
 \end{aligned}$$

Our reason for choosing  $2\pi in - \alpha(m)$  in the line above, and not  $2\pi in + \alpha(m)$ , is to avoid a sign in the isomorphism under construction between our universal extension here and that given by  $P^u(\mathbb{Z})\backslash D$ ; see the term  $-uy_2$  in the upper right coefficient in the last matrix in (4.5.3).

More explicitly, over  $L_{\mathbb{C}}^{\vee}$  we have a family of extensions, with fibre at  $\alpha$  the cokernel above. This family is universal for extensions with given splitting of their tangent spaces at 0 and given splitting of the kernel of the exponential map. On it, we have

actions of  $V^\vee$  and  $L^\vee(1)$ , the quotient by which gives us the universal extension of  $A$  by  $\mathbb{C}^\times$ , with base  $L_{\mathbb{C}}^\vee/(V^\vee + L^\vee(1))$ , which is therefore the dual complex torus. The family itself is the quotient of  $L_{\mathbb{C}}^\vee \times V \times \mathbb{C}$  by a joint action of  $V^\vee$ ,  $L^\vee(1)$ ,  $L$  and  $\mathbb{Z}(1)$ . By “joint action” we mean that the actions of the individual elements of these four groups taken in this order induce a group structure on  $V^\vee \times L^\vee(1) \times L \times \mathbb{Z}(1)$  and an action by that group on  $L_{\mathbb{C}}^\vee \times V \times \mathbb{C}$ . We make this more explicit for the family over  $\mathbb{H}_d$ .

Let  $\tau$  be in  $\mathbb{H}_d$ . As in Section 4.2 we have

$$\begin{aligned} A_\tau &= \mathbb{C}^{2d}/((\begin{smallmatrix} \tau \\ 1_d \end{smallmatrix})\mathbb{C}^d + \mathbb{Z}^{2d}) = \mathbb{C}^d/((1_d - \tau)\mathbb{Z}^{2d}) \\ &= M_{d,1}(\mathbb{C})/((1_d - \tau)M_{2d,1}(\mathbb{Z})). \end{aligned}$$

The universal extension of  $A_\tau$  by  $\mathbb{C}^\times$  is the quotient of  $M_{1,2d}(\mathbb{C}) \times M_{d,1}(\mathbb{C}) \times \mathbb{C}$  by the joint actions of the groups  $M_{1,d}(\mathbb{C})$ ,  $M_{2d,1}(\mathbb{Z})$ ,  $M_{1,2d}(\mathbb{Z}(1))$ , and  $\mathbb{Z}(1)$ . We admit that this is not the same order as a few lines above, but the rest of the proof shows that once the quotient by  $M_{1,d}(\mathbb{C})$  has been taken, the remaining three groups match the corresponding pieces of the Heisenberg group, and therefore the order in which we consider their actions is irrelevant.

An element  $\ell$  in  $M_{1,d}(\mathbb{C})$  acts by postcomposing the embedding of  $\mathbb{Z}(1) \oplus M_{2d,1}(\mathbb{Z})$  in  $\mathbb{C} \oplus M_{d,1}(\mathbb{C})$  as in (4.6.1) with

$$\begin{pmatrix} w \\ v \end{pmatrix} \mapsto \begin{pmatrix} 1 & \ell \\ 0 & 1_d \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} w + \ell v \\ v \end{pmatrix}$$

giving the embedding

$$\begin{pmatrix} 2\pi i n \\ m_1 \\ m_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & \ell \\ 0 & 1_d \end{pmatrix} \begin{pmatrix} 1 - \alpha_1 & -\alpha_2 \\ 0 & 1_d & -\tau \end{pmatrix} \cdot \begin{pmatrix} 2\pi i n \\ m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 1 - \alpha_1 + \ell & -\alpha_2 - \ell\tau \\ 0 & 1_d & -\tau \end{pmatrix} \cdot \begin{pmatrix} 2\pi i n \\ m_1 \\ m_2 \end{pmatrix}.$$

The two displayed formulas above give the actions of  $\ell$  on  $(v, w)$  in  $M_{d,1}(\mathbb{C}) \times \mathbb{C}$  and on  $(\alpha_1, \alpha_2)$  in  $M_{1,2d}(\mathbb{C})$ , and therefore the action on  $M_{1,2d}(\mathbb{C}) \times M_{d,1}(\mathbb{C}) \times \mathbb{C}$

$$\ell : (\alpha_1, \alpha_2, v, w) \mapsto (\alpha_1 - \ell, \alpha_2 + \ell\tau, v, w + \ell(v)).$$

We make a quotient map for this action as follows. For every  $(\alpha_1, \alpha_2, v, w)$  there is a unique  $\ell$ , namely,  $\alpha_1$ , that brings it to the subset of all  $(0, \alpha_2, v, w)$ . This gives us the quotient map

$$\begin{aligned} q : M_{1,2d}(\mathbb{C}) \times M_{d,1}(\mathbb{C}) \times \mathbb{C} &\longrightarrow M_{1,d}(\mathbb{C}) \times M_{d,1}(\mathbb{C}) \times \mathbb{C}, \\ (\alpha_1, \alpha_2, v, w) &\longmapsto (\alpha_1\tau + \alpha_2, v, w + \alpha_1 v), \end{aligned}$$

whose target is the source at  $\tau$  of the bijection in Proposition 4.5. Now we consider the other actions and push them to this quotient.

At the point  $(\alpha_1 \ \alpha_2)$  in  $M_{1,2d}(\mathbb{C})$  the embedding of  $\mathbb{Z}(1) \oplus M_{2d,1}(\mathbb{Z})$  in  $\mathbb{C} \oplus M_{d,1}(\mathbb{C})$  is

$$\begin{pmatrix} 2\pi i n \\ m_1 \\ m_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -\alpha_1 & -\alpha_2 \\ 0 & 1_d & -\tau \end{pmatrix} \cdot \begin{pmatrix} 2\pi i n \\ m_1 \\ m_2 \end{pmatrix},$$

and therefore  $(2\pi in, \begin{pmatrix} m_1 \\ m_2 \end{pmatrix})$  in  $\mathbb{Z}(1) \times M_{2d,1}(\mathbb{Z})$  acts on  $M_{1,2d}(\mathbb{C}) \times M_{d,1}(\mathbb{C}) \times \mathbb{C}$  by the translations

$$(\alpha_1, \alpha_2, v, w) \xrightarrow{(2\pi in, \begin{pmatrix} m_1 \\ m_2 \end{pmatrix})} (\alpha_1, \alpha_2, v + m_1 - \tau m_2, w + 2\pi in - \alpha_1 m_1 - \alpha_2 m_2).$$

It follows that  $2\pi in$  and  $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$  act on  $M_{1,d}(\mathbb{C}) \times M_{d,1}(\mathbb{C}) \times \mathbb{C}$  by

$$(4.6.2) \quad \begin{aligned} 2\pi in &: (u, v, w) \mapsto (u, v, w + 2\pi in), \\ \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} &: (u, v, w) \mapsto (u, v + m_1 - \tau m_2, w - um_2). \end{aligned}$$

An element  $2\pi i(n_1 \ n_2)$  in  $M_{1,2d}(\mathbb{Z}(1))$  acts by precomposing the embedding

$$\mathbb{Z}(1) \oplus M_{2d,1}(\mathbb{Z}) \hookrightarrow \mathbb{C} \oplus M_{d,1}(\mathbb{C})$$

with

$$\begin{pmatrix} 2\pi in \\ m_1 \\ m_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -2\pi in_1 & -2\pi in_2 \\ 0 & 1_d & 0 \\ 0 & 0 & 1_d \end{pmatrix} \cdot \begin{pmatrix} 2\pi in \\ m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 2\pi i(n - n_1 m_1 - n_2 m_2) \\ m_1 \\ m_2 \end{pmatrix}.$$

where we have introduced a factor  $-1$  because we want a left action. This gives the embedding

$$\begin{aligned} \begin{pmatrix} 2\pi in \\ m_1 \\ m_2 \end{pmatrix} &\mapsto \begin{pmatrix} 1 & -\alpha_1 & -\alpha_2 \\ 0 & 1_d & -\tau \end{pmatrix} \cdot \begin{pmatrix} 1 & -2\pi in_1 & -2\pi in_2 \\ 0 & 1_d & 0 \\ 0 & 0 & 1_d \end{pmatrix} \cdot \begin{pmatrix} 2\pi in \\ m_1 \\ m_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\alpha_1 - 2\pi in_1 - \alpha_2 - 2\pi in_2 \\ 0 & 1_d & -\tau \end{pmatrix} \cdot \begin{pmatrix} 2\pi in \\ m_1 \\ m_2 \end{pmatrix}. \end{aligned}$$

So the identity on  $\mathbb{C} \oplus M_{d,1}(\mathbb{C})$  and the inverse of the action of  $2\pi i(n_1 \ n_2)$  on  $\mathbb{Z}(1) \oplus M_{2d,1}(\mathbb{Z})$  induce an isomorphism from the extension at  $(\alpha_1, \alpha_2)$  to the extension at  $(\alpha_1 + 2\pi in_1, \alpha_2 + 2\pi in_2)$ . Therefore the action of  $2\pi i(n_1 \ n_2)$  in  $M_{1,2d}(\mathbb{Z}(1))$  on  $M_{1,2d}(\mathbb{C}) \times M_{d,1}(\mathbb{C}) \times \mathbb{C}$  is by the translations

$$2\pi i(n_1 \ n_2): (\alpha_1, \alpha_2, v, w) \mapsto (\alpha_1 + 2\pi in_1, \alpha_2 + 2\pi in_2, v, w).$$

Pushing this to the quotient gives

$$(4.6.3) \quad 2\pi i(n_1 \ n_2): (u, v, w) \mapsto (u + 2\pi in_1 \tau + 2\pi in_2, v, w + 2\pi in_1 v).$$

By inspection, one sees that the bijection in Proposition 4.5 is equivariant for the actions on its source by  $M_{2d,1}(\mathbb{Z})$ ,  $M_{1,2d}(\mathbb{Z}(1))$ , and  $\mathbb{Z}(1)$  given in (4.6.2) and (4.6.3) and the action on its target by  $P^u(\mathbb{Z})$  given in (4.5.3), where  $2\pi in$  in  $\mathbb{Z}(1)$ ,  $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$  in  $M_{2d,1}(\mathbb{Z})$  and  $2\pi i(n_1 \ n_2)$  in  $M_{1,2d}(\mathbb{Z}(1))$  respectively correspond to

$$(4.6.4) \quad \begin{pmatrix} 1 & 0 & 2\pi in \\ 0 & 1_{2d} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1_d & 0 & m_1 \\ 0 & 0 & 1_d & m_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2\pi in_1 & 2\pi in_2 & 0 \\ 0 & 1_d & 0 & 0 \\ 0 & 0 & 1_d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This finishes our identification of  $P^u(\mathbb{Z}) \setminus D$  with the universal Poincaré torsor over the Siegel space  $\mathbb{H}_d$ .  $\square$

4.7. DUALITY AND THE POINCARÉ TORSOR. — Proposition 4.6 together with the equations (4.5.3) give us an explicit description of the Poincaré torsor over  $\mathbb{H}_d$ . Let  $\tau$  be in  $\mathbb{H}_d$ . Then we have (as in Section 4.2)  $A_\tau = M_{d,1}(\mathbb{C}) / (1_d - \tau) \cdot M_{2d,1}(\mathbb{Z})$  (see the second column of the last matrix in (4.5.3), and  $B_\tau = M_{1,d}(\mathbb{C}) / M_{1,2d}(\mathbb{Z}(1)) \cdot \begin{pmatrix} \tau \\ 1_d \end{pmatrix}$  (consider the first row), and the Poincaré torsor  $P_\tau$  on  $A_\tau \times B_\tau$  that is the universal extension of  $A_\tau$  by  $\mathbb{C}^\times$  and of  $B_\tau$  by  $\mathbb{C}^\times$ , giving isomorphisms  $B_\tau = \text{Ext}^1(A_\tau, \mathbb{C}^\times) = A_\tau^\vee$  and  $A_\tau = \text{Ext}^1(B_\tau, \mathbb{C}^\times) = B_\tau^\vee$ .

Let now  $f: B_\tau \rightarrow A_\tau$  be a morphism of abelian varieties. Then  $f$  is given by a complex linear map

$$M_{1,d}(\mathbb{C}) \longrightarrow M_{d,1}(\mathbb{C}), \quad u \longmapsto f_{\mathbb{C}} \cdot u^t, \quad \text{with } f_{\mathbb{C}} \text{ in } M_d(\mathbb{C}),$$

and a  $\mathbb{Z}$ -linear map

$$M_{1,2d}(\mathbb{Z}(1)) \longrightarrow M_{2d,1}(\mathbb{Z}), \quad 2\pi i \begin{pmatrix} n_1 & n_2 \end{pmatrix} \longmapsto f_{\mathbb{Z}} \cdot \begin{pmatrix} n_1^t \\ n_2^t \end{pmatrix},$$

with

$$f_{\mathbb{Z}} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_{2d}(\mathbb{Z}).$$

The fact that these form a commutative diagram

$$\begin{array}{ccc} M_{1,2d}(\mathbb{Z}(1)) & \xrightarrow{\cdot \begin{pmatrix} \tau \\ 1_d \end{pmatrix}} & M_{1,d}(\mathbb{C}) \\ f_{\mathbb{Z}} \downarrow & & \downarrow f_{\mathbb{C}} \\ M_{2d,1}(\mathbb{Z}) & \xrightarrow{(1_d - \tau)} & M_{d,1}(\mathbb{C}) \end{array}$$

gives us

$$(4.7.1) \quad 2\pi i f_{\mathbb{C}} \tau^t = \alpha - \tau \gamma, \quad \text{and} \quad 2\pi i f_{\mathbb{C}} = \beta - \tau \delta.$$

The morphism  $f: B_\tau \rightarrow A_\tau$  gives us the dual  $f^\vee: B_\tau \rightarrow A_\tau$ . We want to know what  $(f^\vee)_{\mathbb{C}}$  and  $(f^\vee)_{\mathbb{Z}}$  are. The following proposition answers this question.

PROPOSITION 4.8. — *In the situation above, we have*

$$(f^\vee)_{\mathbb{Z}} = -(f_{\mathbb{Z}})^t, \quad \text{and} \quad (f^\vee)_{\mathbb{C}} = \frac{1}{2\pi i} (-\gamma^t + \tau \delta^t).$$

*Proof.* — Let  $b \in B_\tau$ . By the rigidity of extensions of abelian varieties by  $\mathbb{G}_m$ ,  $f^\vee(b)$  is the unique  $a \in A_\tau$  such that there is a morphism of extensions

$$\begin{array}{ccccc} \mathbb{C}^\times & \hookrightarrow & P_{\tau,b} & \twoheadrightarrow & A_\tau \\ \parallel & & \uparrow & & \uparrow f \cdot \\ \mathbb{C}^\times & \hookrightarrow & P_{\tau,a} & \twoheadrightarrow & B_\tau \end{array}$$

Let  $u \in M_{1,d}(\mathbb{C})$  be an element that maps to  $b$ . Then we are looking for a  $v$  in  $M_{d,1}(\mathbb{C})$  (mapping to  $a$ ),  $b_1$  and  $b_2$  in  $M_{d,1}(\mathbb{Z})$ , and  $y$  in  $M_{d,1}(\mathbb{C})$  such that the diagram

$$\begin{array}{ccc}
 (2\pi i n, 2\pi i (n_1 \ n_2)) & \longmapsto & \left( 2\pi i (n + (n_1 \ n_2) \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}), f_{\mathbb{Z}} \cdot \begin{pmatrix} n_1^t \\ n_2^t \end{pmatrix} \right) \\
 \\
 \mathbb{Z}(1) \oplus M_{1,2d}(\mathbb{Z}(1)) & \longrightarrow & \mathbb{Z}(1) \oplus M_{2d,1}(\mathbb{Z}) \\
 \cdot \begin{pmatrix} 1 & 0 \\ v & \tau \\ 0 & 1_d \end{pmatrix} \downarrow & & \downarrow \begin{pmatrix} 1 & 0 & -u \\ 0 & 1_d & -\tau \end{pmatrix} \\
 \mathbb{C} \oplus M_{1,d}(\mathbb{C}) & \longrightarrow & \mathbb{C} \oplus M_{d,1}(\mathbb{C}) \\
 \\
 (z, x) & \longmapsto & (z + x \cdot y, f_{\mathbb{C}} \cdot x^t)
 \end{array}$$

is commutative. This commutativity is equivalent to: for all  $n_1$  and  $n_2$  in  $M_{1,d}(\mathbb{Z})$

$$2\pi i (n_1 \cdot (v + \tau \cdot y) + n_2 \cdot y) = 2\pi i (n_1 \cdot b_1 + n_2 \cdot b_2) - u \cdot (\gamma \cdot n_1^t + \delta \cdot n_2^t),$$

which in turn is equivalent to:

$$2\pi i (v + \tau \cdot y) = 2\pi i b_1 - \gamma^t \cdot u^t \quad \text{and} \quad 2\pi i y = 2\pi i b_2 - \delta^t \cdot u^t.$$

We solve this by taking

$$b_1 = 0, \quad b_2 = 0, \quad y = -(2\pi i)^{-1} \delta^t \cdot u^t, \quad v = (2\pi i)^{-1} (-\gamma^t \cdot u^t + \tau \cdot \delta^t \cdot u^t).$$

We conclude that  $f^\vee : B_\tau \rightarrow A_\tau$  is given by

$$M_{1,d}(\mathbb{C}) \longrightarrow M_{d,1}(\mathbb{C}), \quad u \longmapsto (f^\vee)_{\mathbb{C}} \cdot u^t,$$

with

$$(f^\vee)_{\mathbb{C}} = (2\pi i)^{-1} (-\gamma^t + \tau \cdot \delta^t).$$

The fact that  $(f^\vee)_{\mathbb{Z}}$  is as claimed follows from the commutativity of the diagram

$$\begin{array}{ccc}
 2\pi i (n_1 \ n_2) & \longmapsto & - \begin{pmatrix} \alpha^t & \gamma^t \\ \beta^t & \delta^t \end{pmatrix} \cdot \begin{pmatrix} n_1^t \\ n_2^t \end{pmatrix} \\
 \curvearrowleft & & \curvearrowright \\
 M_{1,2d}(\mathbb{Z}(1)) & \longrightarrow & M_{2d,1}(\mathbb{Z}) \\
 \cdot \begin{pmatrix} \tau \\ 1_d \end{pmatrix} \downarrow & & \downarrow (1_d \ -\tau) \\
 M_{1,d}(\mathbb{C}) & \longrightarrow & M_{d,1}(\mathbb{C}) \\
 \curvearrowright & & \curvearrowleft \\
 2\pi i (n_1 \cdot \tau + n_2) & \longmapsto & (-\gamma^t + \tau \cdot \delta^t) \cdot (\tau^t \cdot n_1^t + n_2^t).
 \end{array}$$

To establish this commutativity one uses (4.7.1). □

To finish this section, we include the polarisation

$$\Psi : M_{2d,1}(\mathbb{Z}) \otimes M_{2d,1}(\mathbb{Z}) \longrightarrow \mathbb{Z}(1), \quad x \otimes y \longmapsto 2\pi i x^t \cdot \begin{pmatrix} 0 & -1_d \\ 1_d & 0 \end{pmatrix} \cdot y$$

in the present discussion (up to here we have not used it, and the results above are valid for  $\tau$  in  $M_d(\mathbb{C})$  whose imaginary part is invertible). Fixing the second variable in  $\Psi$  gives us the isomorphism

$$\Psi_1 : M_{2d,1}(\mathbb{Z}) \longrightarrow M_{2d,1}(\mathbb{Z})^\vee(1), \quad y \longmapsto (x \mapsto \Psi(x \otimes y))$$

of  $\mathbb{Z}$ -Hodge structures (at  $\tau$  in  $\mathbb{H}_d$ ), and therefore an isomorphism of complex tori

$$\begin{aligned} \lambda_\tau : A_\tau = M_{2d,1}(\mathbb{C}) / (M_\tau^{0,-1} + M_{2d,1}(\mathbb{Z})) \\ \longrightarrow M_{2d,1}(\mathbb{C})^\vee / ((M^\vee)_\tau^{0,-1} + M_{2d,1}(\mathbb{Z})^\vee(1)) = B_\tau, \end{aligned}$$

where the identification with  $B_\tau$  is via universal extensions as in the proof of Proposition 4.6.

**PROPOSITION 4.9.** — *With the notation above, the  $\mathbb{C}$ -linear and  $\mathbb{Z}$ -linear maps corresponding to  $\lambda_\tau$  are*

$$(\lambda_\tau)_\mathbb{C} : M_{d,1}(\mathbb{C}) \longrightarrow M_{1,d}(\mathbb{C}), \quad v \longmapsto 2\pi i v^t$$

and

$$\begin{aligned} (\lambda_\tau)_\mathbb{Z} : M_{2d,1}(\mathbb{Z}) &\longrightarrow M_{1,2d}(\mathbb{Z})(1), \\ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &\longmapsto 2\pi i y^t \cdot \begin{pmatrix} 0 & 1_d \\ -1_d & 0 \end{pmatrix} = 2\pi i \begin{pmatrix} -y_2^t & y_1^t \end{pmatrix}. \end{aligned}$$

*Proof.* — For  $(\lambda_\tau)_\mathbb{Z}$ , this follows directly from the proof of Proposition 4.6. For  $(\lambda_\tau)_\mathbb{C}$ , it follows from the commutativity of the diagram

$$\begin{array}{ccc} M_{2d,1}(\mathbb{Z}) & \longrightarrow & M_{1,2d}(\mathbb{Z})(1) & \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longmapsto 2\pi i \begin{pmatrix} -y_2^t & y_1^t \end{pmatrix} \\ \downarrow (1_d - \tau) \cdot & & \downarrow \cdot (1_d) & \\ M_{d,1}(\mathbb{C}) & \longrightarrow & M_{1,d}(\mathbb{C}) & v \longmapsto 2\pi i v^t. \end{array}$$

Here one uses that  $\tau^t = \tau$ . □

It is reassuring to see, using Proposition 4.8, that , as  $(\lambda_\tau)_\mathbb{Z} = \begin{pmatrix} 0 & 1_d \\ -1_d & 0 \end{pmatrix}$  is antisymmetric  $\lambda_\tau^\vee = \lambda_\tau$ .

### 5. RIBET VARIETIES ARE SPECIAL SUBVARIETIES

We recall that in Section 3 we had an abelian scheme  $A \rightarrow S$  and a morphism  $f : A^\vee \rightarrow A$ , and  $\alpha := f - f^\vee : A^\vee \rightarrow A$ , hence  $\alpha^\vee = -\alpha$ , and a section  $r_f$  of the Poincaré torsor over the graph of  $\alpha$ . Now we describe this in the present context, over  $\mathbb{C}$ , in the principally polarised case.

Let  $M := \mathbb{Z}(1) \oplus \mathbb{Z}^{2d} \oplus \mathbb{Z}$ ,  $W$ ,  $D$ , and  $P$  be as in Section 4.4, and recall the notation  $B_\tau$  from the beginning of Section 4.7. Let  $\tau_0$  be in  $\mathbb{H}_d$ ,  $f : B_{\tau_0} \rightarrow A_{\tau_0}$  a morphism, and  $\alpha := f - f^\vee : B_{\tau_0} \rightarrow A_{\tau_0}$ . Then  $\alpha$  gives (and is given by) the  $\mathbb{Z}$ -linear map

$$(5.0.1) \quad \begin{aligned} \text{Gr}_{-1}^W(M)^\vee(1) = M_{1,2d}(\mathbb{Z}(1)) &\longrightarrow M_{2d,1}(\mathbb{Z}) = \text{Gr}_{-1}^W(M), \\ 2\pi i(n_1 \ n_2) &\longmapsto \alpha_{\mathbb{Z}} \cdot \begin{pmatrix} n_1^t \\ n_2^t \end{pmatrix}, \end{aligned}$$

with  $\alpha_{\mathbb{Z}} \in M_{2d}(\mathbb{Z})$ . By Proposition 4.8,

$$\alpha_{\mathbb{Z}} = f_{\mathbb{Z}} - (f^{\vee})_{\mathbb{Z}} = f_{\mathbb{Z}} + (f_{\mathbb{Z}})^t.$$

Hence  $\alpha_{\mathbb{Z}}$  is symmetric and the quadratic form

$$M_{1,2d}(\mathbb{Z}) \longrightarrow \mathbb{Z}, \quad x \longmapsto \frac{1}{2} x \cdot \alpha_{\mathbb{Z}} \cdot x^t = x \cdot f_{\mathbb{Z}} \cdot x^t$$

is  $\mathbb{Z}$ -valued. Just for completeness, we include that the endomorphism  $\beta := \alpha \circ \lambda_{\tau_0}$  of  $A_{\tau_0}$  is anti-symmetric for the Rosati involution:

$$\lambda_{\tau_0}^{-1} \circ \beta^{\vee} \circ \lambda_{\tau_0} = \lambda_{\tau_0}^{-1} \circ (\alpha \circ \lambda_{\tau_0})^{\vee} \circ \lambda_{\tau_0} = \lambda_{\tau_0}^{-1} \circ \lambda_{\tau_0}^{\vee} \circ \alpha^{\vee} \circ \lambda_{\tau_0} = -\alpha \circ \lambda_{\tau_0} = -\beta.$$

Now, everything is in place to introduce the connected mixed Shimura subvariety of the universal Poincaré-torsor  $P^u(\mathbb{Z}) \backslash D$  over  $\mathbb{H}_d$  (quotiented by a suitable congruence subgroup of  $\mathrm{GSp}(\Psi)(\mathbb{Z})$ ) that is dictated by the map in (5.0.1) being a morphism of Hodge structures. Concretely, we let  $P_{\alpha}$  and  $G_{\alpha}$  be the connected components of identity of the stabilisers of (5.0.1) in  $P$  and in  $\mathrm{GSp}(\Psi)$ . As the action of  $P$  on  $\mathrm{Gr}_{-1}^W(M)$  factors through  $\mathrm{GSp}(\Psi)$ ,  $P_{\alpha}$  is the inverse image in  $P$  of  $G_{\alpha}$ , and the unipotent radical  $P_{\alpha}^u$  of  $P_{\alpha}$  is equal to  $P^u$ , hence contains  $U$ . In  $D$  and  $\mathbb{H}_d$ , we consider the orbits

$$(5.0.2) \quad D_{\alpha} := P_{\alpha}(\mathbb{R})^+ U(\mathbb{C}) \cdot \tilde{\tau}_0 \subset D \quad \text{and} \quad \mathbb{H}_{d,\alpha} := G_{\alpha}(\mathbb{R})^+ \cdot \tau_0 \subset \mathbb{H}_d,$$

where  $\tilde{\tau}_0$  is the element of  $D$  that corresponds to  $(\tau_0, 0, 0, 0)$  under the bijection of Proposition 4.5. More intrinsically:  $\tilde{\tau}_0$  is the mixed Hodge structure on  $M$  in which the weight filtration is split over  $\mathbb{Z}$  by the given  $\mathbb{Z}$ -basis, and which induces that given by  $\tau_0$  on  $\mathrm{Gr}_{-1}^W M$ . Here, it does not matter which lift of  $\tau_0$  we take, but it will matter further on when we describe the Ribet section in  $D_{\alpha}$ .

Deligne’s group theoretical description of Shimura varieties shows that  $\mathbb{H}_{d,\alpha}$  is the connected component containing  $\tau_0$  of the set of  $\tau \in \mathbb{H}_d$  where (5.0.1) is a morphism of Hodge structures (equivalently: where it induces a morphism  $\alpha: B_{\tau} \rightarrow A_{\tau}$ ). Let us explain in a few lines how this works; for details, see [20, §2.4] and [10, §1.1.12]. Pure Hodge structures on an  $\mathbb{R}$ -vector space correspond to  $\mathbb{R}$ -algebraic actions of  $\mathbb{C}^{\times}$ . For a connected linear algebraic group  $G$  over  $\mathbb{R}$ , the set of  $\mathbb{R}$ -morphisms  $\mathrm{Hom}(\mathbb{C}^{\times}, G(\mathbb{R}))$  is the set of  $\mathbb{R}$ -points of a smooth  $\mathbb{R}$ -scheme, which is the disjoint union of  $G$ -orbits (for  $G$  acting by composition with inner automorphisms). The  $G(\mathbb{R})^+$ -orbits in  $\mathrm{Hom}(\mathbb{C}^{\times}, G(\mathbb{R}))$  are the connected components for the Archimedean topology. References in [28] (SGA 3): Exp. IX, Cor. 3.3, and Exp. XI, Cor. 4.2.

The pairs  $(P_{\mathbb{Q}}, D)$ ,  $(G_{\alpha,\mathbb{Q}}, \mathbb{H}_{d,\alpha})$  and  $(P_{\alpha,\mathbb{Q}}, D_{\alpha})$  are connected mixed Shimura data as in [24, Def. 2.1], and we have the diagram of morphisms of connected mixed Shimura data

$$(5.0.3) \quad \begin{array}{ccc} (P_{\alpha,\mathbb{Q}}, D_{\alpha}) & \hookrightarrow & (P_{\mathbb{Q}}, D) \\ \downarrow & & \downarrow \\ (G_{\alpha,\mathbb{Q}}, \mathbb{H}_{d,\alpha}) & \hookrightarrow & (\mathrm{GSp}(\Psi)_{\mathbb{Q}}, \mathbb{H}_d). \end{array}$$

The careful reader will have noticed that we must show that  $D$  is a  $P(\mathbb{R})^+U(\mathbb{C})$ -orbit in  $\text{Hom}(\mathbb{C}^\times \times \mathbb{C}^\times, P(\mathbb{C}))$  and  $D_\alpha$  is a  $P_\alpha(\mathbb{R})^+U(\mathbb{C})$ -orbit in  $\text{Hom}(\mathbb{C}^\times \times \mathbb{C}^\times, P_\alpha(\mathbb{C}))$ . For the fact that the natural maps from these orbits to  $D$  and  $D_\alpha$  are isomorphisms we refer to Propositions 1.18 and 1.16(c) in [23] (the surjectivity is clear because source and target are orbits for the same group, for the injectivity one has to show that the stabilisers are the same).

**PROPOSITION 5.1.** — *The quotient  $P_\alpha^u(\mathbb{Z}) \backslash D_\alpha$  is the universal Poincaré torsor over  $\mathbb{H}_{d,\alpha}$ . The quotient of  $D_\alpha$  by  $P_\alpha^u(\mathbb{Z})U(\mathbb{C})$  is the universal family of  $A_\tau \times B_\tau$ 's over  $\mathbb{H}_{d,\alpha}$ . The quotient of  $D_\alpha$  by  $P_\alpha^u(\mathbb{Z})M_{1,2d}(\mathbb{R})U(\mathbb{C})$  is the universal family of  $A_\tau$ 's over  $\mathbb{H}_{d,\alpha}$ , and the quotient of  $D_\alpha$  by  $P_\alpha^u(\mathbb{Z})M_{2d,1}(\mathbb{R})U(\mathbb{C})$  is the universal family of  $B_\tau$ 's over  $\mathbb{H}_{d,\alpha}$ .*

*Proof.* — One easily deduces this from Proposition 4.6 and parts of its proof.  $\square$

Now we proceed directly to the Ribet section, by revealing the tensor that defines it, namely, the map (encoded by a matrix  $\widetilde{\alpha}_{\mathbb{Z}}$ )

$$(5.1.1) \quad \begin{aligned} M^\vee(1) = \mathbb{Z} \oplus M_{1,2d}(\mathbb{Z}(1)) \oplus \mathbb{Z}(1) &\longrightarrow \mathbb{Z}(1) \oplus M_{2d,1}(\mathbb{Z}) \oplus \mathbb{Z} = M \\ x = (k_1 \quad 2\pi i n \quad 2\pi i k_2) &\longmapsto \begin{pmatrix} -2\pi i k_2 \\ \alpha_{\mathbb{Z}} \cdot n^t \\ -k_1 \end{pmatrix} = \widetilde{\alpha}_{\mathbb{Z}} \cdot x^t, \end{aligned}$$

where

$$(5.1.2) \quad \widetilde{\alpha}_{\mathbb{Z}} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \alpha_{\mathbb{Z}} & 0 \\ -1 & 0 & 0 \end{pmatrix} \text{ in } M_{2d+2}(\mathbb{Z}).$$

This tensor was already described in [26], see also [2, Lem. 6]. We let  $P_{\widetilde{\alpha}}$  be the stabiliser in  $P$  of this map (5.1.1), as a group scheme over  $\mathbb{Z}$ . Then, for any  $\mathbb{Z}$ -algebra  $R$  and for any  $p$  in  $P(R)$  we have  $p \in P_{\widetilde{\alpha}}(R)$  if and only if  $p \cdot \widetilde{\alpha}_{\mathbb{Z}} = \mu(p) \widetilde{\alpha}_{\mathbb{Z}} \cdot p^{-1,t}$  in  $M_{2d+2}(R)$ , which is equivalent to  $p \cdot \widetilde{\alpha}_{\mathbb{Z}} \cdot p^t = \mu(p) \widetilde{\alpha}_{\mathbb{Z}}$ . A direct computation then shows, for any  $\mathbb{Z}$ -algebra  $R$  in which multiplication by 2 is injective:

$$(5.1.3) \quad P_{\widetilde{\alpha}}(R) = \left\{ \begin{pmatrix} \mu(g) & x & \mu(g)^{-1} x f_{\mathbb{Z}} x^t \\ 0 & g & \mu(g)^{-1} g \alpha_{\mathbb{Z}} x^t \\ 0 & 0 & 1 \end{pmatrix} : (g, \mu(g)) \in G_\alpha(R), x \in M_{1,2d}(R) \right\},$$

where the matrices are with respect to the  $\mathbb{Z}$ -basis  $2\pi i e_0, e_1, \dots, e_{2d+1}$  of  $M$ . We note that for  $R$  on which multiplication by 2 is injective,  $P_{\widetilde{\alpha}}(R)$  is the semi-direct product

$$(5.1.4) \quad P_{\widetilde{\alpha}}(R) = M_{1,2d}(R) \rtimes G_\alpha(R) = \left\{ \begin{pmatrix} 1 & x & x f_{\mathbb{Z}} x^t \\ 0 & 1_{2d} & \alpha_{\mathbb{Z}} x^t \\ 0 & 0 & 1 \end{pmatrix} \right\} \cdot \left\{ \begin{pmatrix} \mu(g) & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$



where  $x$  ranges over  $M_{1,2d}(R)$  and  $g$  over  $G_\alpha(R)$ . In particular, the unipotent radical (over  $\mathbb{Z}[1/2]$ ) of  $P_{\tilde{\alpha}}$  is a vector group scheme, and the weight  $-2$  part of its Lie algebra is zero. We define

$$(5.1.5) \quad D_{\tilde{\alpha}} := P_{\tilde{\alpha}}(\mathbb{R})^+ \cdot \tilde{\tau}_0 \subset D_\alpha \subset D.$$

Then we have the following diagram of connected mixed Shimura data

$$(5.1.6) \quad \begin{array}{ccc} (P_{\tilde{\alpha},\mathbb{Q}}, D_{\tilde{\alpha}}) & \hookrightarrow & (P_{\alpha,\mathbb{Q}}, D_\alpha) \\ & \searrow & \downarrow \\ & & (G_{\alpha,\mathbb{Q}}, \mathbb{H}_{d,\alpha}). \end{array}$$

**THEOREM 5.2.** — *The quotient  $P_{\tilde{\alpha}}^u(\mathbb{Z}) \backslash D_{\tilde{\alpha}}$  is the image of a section  $r_f^{\text{Sh}}$  in  $P_\alpha^u(\mathbb{Z}) \backslash D_\alpha$  (the universal Poincaré torsor over  $\mathbb{H}_{d,\alpha}$ , see Proposition 5.1) over the family of  $B_\tau$  with  $\tau$  ranging over  $\mathbb{H}_{d,\alpha}$ . In particular, the image of  $r_f^{\text{Sh}}$  is a special subvariety. This section  $r_f^{\text{Sh}}$  is equal, in this setting, to the section  $r_f$  of Proposition 3.1.*

*Proof.* — It is sufficient to verify the claim at each  $\tau \in \mathbb{H}_{d,\alpha}$ . So let  $\tau$  be such. The description in (4.5.3) of the action of  $P_\alpha^u(\mathbb{R})U(\mathbb{C}) = P^u(\mathbb{R})U(\mathbb{C})$  on  $D$  shows that it is free and transitive on the fibre  $D_{\alpha,\tau}$  of  $D_\alpha \rightarrow \mathbb{H}_{d,\alpha}$  at  $\tau$ . This gives us the bijection

$$(5.2.1) \quad P_\alpha^u(\mathbb{R})U(\mathbb{C}) \xrightarrow{\cong} D_{\alpha,\tau} \quad p \mapsto p \cdot \tilde{\tau},$$

where  $\tilde{\tau}$  is the element of  $D$  that corresponds to  $(\tau, 0, 0, 0)$  under the bijection of Proposition 4.5. For  $g$  in  $G_\alpha(\mathbb{R})^+$  with  $g \cdot \tau_0 = \tau$ , we have  $g \in P_{\tilde{\alpha}}(\mathbb{R})^+$  via (5.1.4), and  $\tilde{\tau} = g \cdot \tilde{\tau}_0$  (use (4.5.4)), hence  $D_{\tilde{\alpha},\tau} = P_{\tilde{\alpha}}^u(\mathbb{R}) \cdot \tilde{\tau}$ . Via the bijection (5.2.1), the inclusion  $D_{\tilde{\alpha},\tau} \subset D_{\alpha,\tau}$  corresponds to the inclusion  $P_{\tilde{\alpha}}^u(\mathbb{R}) \subset P_\alpha^u(\mathbb{R})U(\mathbb{C})$ , and the  $P_{\tilde{\alpha}}^u(\mathbb{Z})$ -action on  $D_{\tilde{\alpha}}$  corresponds to the action by left-multiplication on  $P_{\tilde{\alpha}}^u(\mathbb{R})$ . By (5.1.4),  $P_{\tilde{\alpha}}^u(\mathbb{R}) = M_{1,2d}(\mathbb{R})$ , and (5.2.1) identifies this with  $M_{1,d}(\mathbb{C})$ , sending  $(x_1, x_2)$  to  $2\pi i(x_1\tau + x_2)$  by (4.5.3). Hence  $P_{\tilde{\alpha}}^u(\mathbb{Z}) \backslash D_{\tilde{\alpha},\tau} = B_\tau$ . Proposition 5.1 together with the description (5.1.4) of  $P_\alpha^u$  show that  $P_\alpha^u(\mathbb{Z}) \backslash D_{\tilde{\alpha}}$  is the image of a section  $r_f^{\text{Sh}}$  of the Poincaré torsor over the graph of  $\alpha: B_\tau \rightarrow A_\tau$  (equivalently, over  $B_\tau$ ). This section differs from  $r_f$  by multiplication by a global regular function on  $B_\tau$ , hence by a constant factor in  $\mathbb{C}^\times$ . As both sections have value 1 at  $0 \in B_\tau$ , they are equal.  $\square$

**PROPOSITION 5.3.** — *Let  $\tau$  be an element of  $\mathbb{H}_{d,\alpha}$ . The subset  $M_{1,2d}(\mathbb{Z}) \backslash M_{1,2d}(\mathbb{R}) \cdot 1$  of  $P_\alpha^u(\mathbb{Z}) \backslash P_\alpha^u(\mathbb{R})U(\mathbb{C})$  corresponds, under the bijection in (5.2.1), to the unit section over  $B_\tau$  of the Poincaré torsor  $P_\alpha^u(\mathbb{Z}) \backslash D_{\alpha,\tau}$  on  $A_\tau \times B_\tau$  (see Proposition 5.1). For  $x$  in  $M_{1,2d}(\mathbb{R})$  and  $\bar{x}$  its image in  $B_\tau$ , the extension  $E_{\tau,\bar{x}}$  of  $A_\tau$  by  $\mathbb{C}^\times$  corresponding to  $\bar{x}$  is, as real Lie group,  $(\mathbb{C}/2\pi i\mathbb{Z}) \times (M_{2d,1}(\mathbb{R})/M_{2d,1}(\mathbb{Z}))$ , and  $r_f(\bar{x})$  is given by  $(2\pi i x f_{\mathbb{Z}} x^t, \alpha_{\mathbb{Z}} x^t)$ . If  $\bar{x}$  is of order  $n$  in  $B_\tau$ , then  $r_f(\bar{x})$  in  $E_{\tau,\bar{x}}$  is killed by  $n^2$ .*

*Proof.* — Consider (5.2.1) and (4.5.3). Let  $x = (x_1, x_2) \in M_{1,2d}(\mathbb{R})$ . This gives the elements

$$p_x := \begin{pmatrix} 1 & 2\pi i x_1 & 2\pi i x_2 & 0 \\ 0 & 1_d & 0 & 0 \\ 0 & 0 & 1_d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in P_\alpha^u(\mathbb{R}), \quad \text{and} \quad p_x \cdot \tilde{\tau} = \begin{pmatrix} 2\pi i(x_1\tau + x_2) & 0 \\ \tau & 0 \\ 1_d & 0 \\ 0 & 1 \end{pmatrix} \mathbb{C}^{d+1} \in D_{\alpha,\tau}.$$

This proves the first claim of the proposition. To describe  $E_{\tau,\bar{x}}$ , let, for  $z$  in  $\mathbb{C}$  and  $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$  in  $M_{2d,1}(\mathbb{R})$ ,

$$p_{z,y} := \begin{pmatrix} 1 & 0 & 0 & 2\pi iz \\ 0 & 1_d & 0 & y_1 \\ 0 & 0 & 1_d & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and then

$$p_{z,y} \cdot p_x \cdot \tilde{\tau} = \begin{pmatrix} 2\pi i(x_1\tau + x_2) & 2\pi i(z - (x_1\tau + x_2)y_2) \\ \tau & y_1 - \tau y_2 \\ 1_d & 0 \\ 0 & 1 \end{pmatrix} \mathbb{C}^{d+1}.$$

Now observe that  $2\pi i(z - (x_1\tau + x_2)y_2)$  and  $y_1 - \tau y_2$  are  $\mathbb{R}$ -linear in  $z$ ,  $y_1$  and  $y_2$ , and that  $2\pi i(x_1\tau + x_2)$  does not depend on  $z$ ,  $y_1$  and  $y_2$ . Hence the  $\mathbb{R}$ -vector space structure on  $\{2\pi i(x_1\tau + x_2)\} \times M_{d,1}(\mathbb{C}) \times \mathbb{C}$  in  $D_{\alpha,\tau}$  corresponds to the  $\mathbb{R}$ -vector space structure on  $M_{2d,1}(\mathbb{R}) \times \mathbb{C}$  on the left, and therefore the same holds for the group structures. The left-action by the  $p_{z,y}$  with  $z \in \mathbb{Z}$  and  $y \in M_{2d,1}(\mathbb{Z})$  on these 2 real vector spaces then gives the description of  $E_{\tau,\bar{x}}$ . The description of  $P_\alpha^u$  in (5.1.4) proves the last two claims in the proposition.  $\square$

**REMARK 5.4.** — Assume that  $\alpha$  is an isogeny.

(1) The tensor  $\tilde{\alpha}$  in (5.1.1) that defines the Ribet variety as an irreducible component of its Hodge locus is a selfduality of mixed  $\mathbb{Q}$ -Hodge structures. It is interesting to see that on the underlying  $\mathbb{Z}$ -module  $M$  it is a symmetric  $\mathbb{Z}(1)$ -valued pairing. Algebraically this can be described as a self-duality of 1-motives with  $\mathbb{Q}$ -coefficients, see [26] and also [2].

(2) Let  $\Gamma_\alpha(3)$  be the kernel of  $G_\alpha(\mathbb{Z}) \rightarrow G_\alpha(\mathbb{F}_3)$ . Then  $\Gamma_\alpha(3)$  acts on the whole situation of Theorem 5.2, and freely on  $\mathbb{H}_{d,\alpha}$ . The quotient  $\Gamma_\alpha(3) \backslash D_\alpha$  is then the Poincaré torsor  $P$  for the abelian scheme  $A := \Gamma_\alpha(3) \backslash (P_\alpha^u(\mathbb{Z})M_{1,2d}(\mathbb{R})U(\mathbb{C}) \backslash D_\alpha)$  over the pure Shimura variety  $S := \Gamma_\alpha(3) \backslash \mathbb{H}_{d,\alpha}$ , with the image of the Ribet section  $r_f$  as a special subvariety of a family of semi-abelian varieties. As a generalisation of Lemma 2.3, we will now prove that this special subvariety is not a torsion translate of a family of algebraic subgroups. Let  $\tau$  be in  $\mathbb{H}_{d,\alpha}$  and  $x = (x_1, \dots, x_{2d})$  be in  $M_{1,2d}(\mathbb{R})$  such that  $x_1, \dots, x_{2d}, x\alpha_{\mathbb{Z}}x^t$  in  $\mathbb{R}$  are  $\mathbb{Q}$ -linearly independent. Then the coordinates of  $\alpha_{\mathbb{Z}} \cdot x^t$  and  $x\alpha_{\mathbb{Z}}x^t$  are  $\mathbb{Q}$ -linearly independent. By Proposition 5.3, the subgroup of  $E_{\tau,\bar{x}}$  generated by  $r_f(\bar{x})$  is dense, for the Archimedean topology, in  $(i\mathbb{R}/2\pi i\mathbb{Z}) \times (M_{2d,1}(\mathbb{R})/M_{2d,1}(\mathbb{Z}))$ . This shows that the union of the images of the  $nr_f$ ,

with  $n \in \mathbb{Z}$ , is dense, for the Archimedean topology, in a circle bundle of real codimension 1 in  $\mathcal{P}$ . The fibres of this circle bundle are the maximal compact subgroups of the corresponding complex analytic semi-abelian varieties.

(3) The example just given (the image of  $r_f$ ) now supports Pink's Conjecture 1.3 of [25]: indeed, it is a subvariety  $Y$  of  $\mathcal{P}$  containing a Zariski dense set of special points (i.e., special subvarieties of maximal codimension in  $\mathcal{P}$ ), and it is itself a special subvariety of  $\mathcal{P}$ . For further verifications in this context of [25], Conjecture 1.3, see [3] and [4].

(4) Let us now clarify what is wrong in the proof of Theorem 6.3 of [25]. The error is in the statement "Since the special subvarieties of  $A$  that dominate  $S$  are precisely the translates of semiabelian subschemes by torsion points,..."; we have just seen that this is not true. Similarly, note the sentence "Conversely, for any special subvariety  $T \subset A$ , every irreducible component of  $T \cap A_s$  is a translate of a semiabelian subvariety of  $A_s$  by a torsion point." in the proof of Theorem 5.7 of [25].

The essential difference between the case of Kuga varieties (Shimura families of abelian varieties over pure Shimura varieties), where the statement is correct ([24, Prop. 4.6]), and the case of Shimura families of tori over Kuga varieties is as follows. In the first case the morphism of mixed Shimura varieties  $A \rightarrow S$  is induced by a morphism of Shimura data  $(P, D_P) \rightarrow (G, D_G)$  with  $G$  reductive, and  $P \rightarrow G$  surjective, split, with kernel  $V$  a  $\mathbb{Q}$ -vector space. Then the special subvarieties  $Z$  of  $A$  that surject to  $S$  are given by morphisms of sub-Shimura data  $(Q, D_Q)$  of  $(P, D_P)$ , with  $Q \rightarrow G$  surjective. Then  $Q$  is an extension of  $G$  by  $Q \cap V$ , a sub- $\mathbb{Q}$ -vector space of  $V$ . This extension is split because  $H^2(G, Q \cap V) = 0$ , and the splitting is unique up to conjugation by  $Q \cap V$  because  $H^1(G, Q \cap V) = 0$ . So indeed such special subvarieties come from subfamilies  $B \rightarrow S$  of  $A \rightarrow S$  and Hecke correspondences that account for translations by torsion points. In the second case, say  $T \rightarrow A$ , these arguments no longer apply because the group  $P$  in the Shimura datum for  $A$  (such as  $P_\alpha/U$  as above) is not necessarily reductive (and indeed the extension  $P_\alpha$  of  $P_\alpha/U$  by  $U$  is not split).

## 6. THE ELLIPTIC CURVE EXAMPLE, VIA GENERALISED JACOBIANS

In this section we give a description of the example in Section 2 in terms of the generalised jacobian of a family of singular curves. Our reason to include it is that this description is more elementary than the one using the Poincaré bundle, and that it is more explicit in terms of divisors, rational functions, Weil pairing, and is a nice application of Weil reciprocity.

We return to the situation as in Section 2, except that now we let  $k$  be an arbitrary algebraically closed field. Let  $E$  be an elliptic curve over  $k$ . Here we will view  $E \times E$  as a family of elliptic curves over  $E$  via the 2nd projection  $\text{pr}_2: E_E = E \times E \rightarrow E$ ,  $(x, y) \mapsto y$ .

In our construction, we will remove a finite number of points of the base curve  $E$ , and denote the complement by  $U$ . This  $U$  will be shrunk a few times.

The diagonal morphism  $\Delta: E \rightarrow E_E, x \mapsto (x, x)$ , is a section, and the group law of  $E_E$  over  $E$  gives us a second section  $2\Delta, x \mapsto (2x, x)$ . The sections  $\Delta$  and  $2\Delta$  are disjoint over the open subset  $U := E - \{0\}$ .

We let  $C \rightarrow U$  be the singular curve over  $U$  obtained by identifying the disjoint sections  $2\Delta$  and  $\Delta$ . As a set, it is the quotient of  $E_U$  by the equivalence relation generated by  $(2x, x) \sim (x, x)$  with  $x$  ranging over  $U$ . The topology on  $C$  is the finest one for which the quotient map  $\text{quot}: E_U \rightarrow C$  is continuous: a subset  $V$  of  $C$  is open if and only if  $\text{quot}^{-1}V$  is open in  $E_U$ . The regular functions on an open set  $V$  of  $C$  are the regular functions  $f$  on  $\text{quot}^{-1}V$  such that  $f(2x, x) = f(x, x)$  whenever  $\text{quot}(x, x)$  is in  $V$ . It is proved in Theorem 5.4 of [12] that this topological space with sheaf of rings is indeed an algebraic variety over  $k$ . In the category of varieties over  $k$ ,  $\text{quot}: E_U \rightarrow C$  is the co-equaliser of the pair of morphisms  $(2\Delta, \Delta)$  from  $U$  to  $E_U$ :

$$U \begin{array}{c} \xrightarrow{2\Delta} \\ \xrightarrow{\Delta} \end{array} E_U \xrightarrow{\text{quot}} C.$$

The curve  $C \rightarrow U$  is a family of singular curves, each with an ordinary double point; it is semi-stable of genus 2 (see [5, 9.2/6, 9.2/8]). Its normalisation is  $\text{quot}: E_U \rightarrow C$ . Its generalised jacobian

$$G := \text{Pic}_{C/U}^0$$

is described in [5], 8.1/4, 8.2/7, 9.2, 9.4/1, and in more direct terms in this specific situation in [14]. As  $C \rightarrow U$  has a section (for example  $\bar{\Delta} := \text{quot} \circ \Delta$ ), we have, for every  $T \rightarrow U$ , that  $G(T)$  is equal to  $\text{Pic}^0(C_T/T)/\text{Pic}(T)$ , where  $\text{Pic}^0(C_T/T)$  is the group of isomorphism classes of invertible  $\mathcal{O}$ -modules on  $C_T$  that have degree zero on the fibres of  $C_T \rightarrow T$ . The group  $\text{Pic}(T)$  is contained as direct summand in  $\text{Pic}^0(C_T/T)$  via pullback by the projection  $C_T \rightarrow T$  and a chosen section. In particular, a divisor  $D$  on  $C$  that is finite over  $U$ , disjoint from  $\bar{\Delta}(U)$  and of degree zero after restriction to the fibres of  $C \rightarrow U$  gives the invertible  $\mathcal{O}_C$ -module  $\mathcal{O}_C(D)$  that has degree zero on the fibres and therefore gives an element denoted  $[D]$  in  $G(U)$ . An alternative and very useful description, given in detail in [14], of  $\text{Pic}(C_T)$  is the set of isomorphism classes of  $(L, \sigma)$ , with  $L$  an invertible  $\mathcal{O}$ -module on  $E_T$  and  $\sigma: (2\Delta)^*L \rightarrow \Delta^*L$  an isomorphism of  $\mathcal{O}$ -modules on  $T$ , where an isomorphism from  $(L, \sigma)$  to  $(L', \sigma')$  is an isomorphism  $f: L \rightarrow L'$  such that  $(\Delta^*f) \circ \sigma = \sigma' \circ (2\Delta)^*f$ .

For  $x$  in  $U$ , the fibre  $G_x$  is, as abelian group, the group  $\text{Pic}^0(C_x)$ . In terms of divisors this is the quotient of the group  $\text{Div}^0(C_x)$  of degree zero divisors with support outside  $\{\bar{\Delta}(x)\}$  by the subgroup of principal divisors  $\text{div}(f)$  for nonzero rational functions  $f$  in  $k(C_x)^\times$  that are regular and invertible at  $\bar{\Delta}(x)$ . As  $C_x - \{\bar{\Delta}(x)\}$  is the same as  $E - \{2x, x\}$ ,  $\text{Div}^0(C_x)$  is the group of degree zero divisors on  $E$  with support outside  $\{2x, x\}$ . An element  $f$  of  $k(C_x)^\times$  that is regular at  $\bar{\Delta}(x)$  is an element of  $k(E)^\times$  that is regular at  $2x$  and  $x$  and satisfies  $f(2x) = f(x)$ . This gives us a useful description of  $G_x$ .

The normalisation map  $\text{quot}: E_U \rightarrow C$  induces a morphism of group schemes over  $U$

$$\pi: G = \text{Pic}_{C/U}^0 \longrightarrow \text{Pic}_{E_U/U}^0 = E_U,$$

and identifies  $G$  with the extension of  $E$  by  $\mathbb{G}_m$  given by the section  $\Delta \in E_U(U)$ . For  $x$  in  $U$  and  $D \in \text{Div}^0(C_x)$ , the class  $[D]$  in  $G_x$  lies in the kernel  $k^\times$  of  $\pi_x$  if and only if there exists  $f \in k(E)^\times$  such that  $D = \text{div}(f)$  on  $E$ , and it is then a torsion point in  $k^\times$  if and only if the quotient  $f(2x)/f(x) \in k^\times$ , which does not depend on the choice of  $f$ , is a root of unity.

We recall that for  $u$  in  $\text{End}(E)$ , the pullback map  $u^*$  on  $\text{Div}(E)$  induces  $u^\vee$  in  $\text{End}(E^\vee)$ , the dual of  $u$ , and then  $\bar{u} := \lambda^{-1}u^\vee\lambda$  in  $\text{End}(E)$  is called the Rosati-dual of  $u$ , where  $\lambda$  is the standard polarisation as in Section 2. The map  $\text{End}(E) \rightarrow \text{End}(E)$ ,  $u \mapsto \bar{u}$  is a anti-morphism of rings, in fact an involution. It is characterised by the property that in  $\text{End}(E)$  we have  $\bar{u}u = \text{deg}(u) = \text{deg}(\bar{u})$  and  $u + \bar{u} \in \mathbb{Z}$ . Also, the pushforward map  $u_*$  on  $\text{Div}(E)$  induces an element still denoted  $u_*$  in  $\text{End}(E^\vee)$  such that  $\lambda u = u_*\lambda$  in  $\text{Hom}(E, E^\vee)$ , and  $u_*u^* = \text{deg}(u)$  in  $\text{End}(E^\vee)$ . Hence  $u_*$  and  $u^*$  are each other's Rosati duals. For  $f$  a nonzero rational function on  $E$  and  $u \neq 0$  we have  $u^*\text{div}(f) = \text{div}(f \circ u)$ , and  $u_*\text{div}(f) = \text{div}(\text{Norm}_u(f))$ , where  $\text{Norm}_u: k(E)^\times \rightarrow k(E)^\times$  is the norm map along  $u$ .

We will use Weil reciprocity: for  $f$  and  $g$  nonzero rational functions on  $E$  such that  $\text{div}(f)$  and  $\text{div}(g)$  have disjoint supports, one has  $f(\text{div}(g)) = g(\text{div}(f))$ , where for  $D = \sum_P D(P) \cdot P$  a divisor on  $E$  one defines  $f(D) = \prod_P f(P)^{D(P)}$ , cf. [27, III, Prop. 7].

We will also use the Weil pairing. For  $n$  a positive integer and  $P$  and  $Q$  in  $E[n]$  the element  $e_n(P, Q)$  in  $\mu_n(k)$  is defined as follows. Let  $D_P$  and  $D_Q$  in  $\text{Div}^0(E)$  be disjoint divisors representing  $\lambda(P)$  and  $\lambda(Q)$ . Let  $f$  and  $g$  be in  $k(E)^\times$  such that  $nD_P = \text{div}(f)$  and  $nD_Q = \text{div}(g)$ . Then  $e_n(P, Q) = f(D_Q)/g(D_P)$ . For  $n$  invertible in  $k$  this pairing  $e_n$  is a perfect alternating pairing, see [15, Chap. 12, Rem. 3.7].

We assume that  $\varphi$  is an endomorphism of  $E$  such that  $\alpha := \varphi - \bar{\varphi} \neq 0$ . We set

$$(6.0.1) \quad D_\varphi := \varphi_*((\Delta) - (2\Delta)) - \varphi^*((\Delta) - (2\Delta)) \quad \text{in } \text{Div}^0(E_U).$$

Note that  $(\Delta) - (2\Delta)$  is linearly equivalent to  $(0) - (\Delta)$ , and that, under  $\lambda: E \rightarrow E^\vee$ ,  $\Delta$  in  $E(E)$  is mapped to  $[(0) - (\Delta)]$ . We want the support of  $D_\varphi$  to be disjoint from  $\Delta$  and  $2\Delta$ , and this becomes true by removing from  $U$  the kernels of  $2(\varphi - 1)$ , of  $2\varphi - 1$  and of  $\varphi - 2$  (as  $\bar{\varphi} \neq \varphi$ , only a finite set is removed). We can now also view  $D_\varphi$  as element of  $\text{Div}^0(C)$ , and we set:

$$(6.0.2) \quad t_\varphi^J := [D_\varphi] \quad \text{in } G(U).$$

Combining Parts 2 and 4 of the following theorem provides a new proof in the elliptic case of Proposition 3.3, while Part 3 sharpens Theorem 2.4.

**THEOREM 6.1**

(1) *The image  $\pi(t_\varphi^J)$  of  $t_\varphi^J$  equals*

$$(\alpha, \text{id}_U): U \longrightarrow E \times U = E_U.$$

(2) *Let  $n$  be a positive integer and  $x$  in  $E$  with  $nx = 0$ . Then  $n^2t_\varphi^J(x) = 0$  in  $G_x$ , and  $nt_\varphi^J(x) = e_n(\varphi(x), x)$ .*

(3) Let  $n$  be a positive odd integer that is prime to  $\deg(\alpha)$ , invertible in  $k$ , and that divides none among  $\deg(2(\varphi - 1))$ ,  $\deg(2\varphi - 1)$  and  $\deg(\varphi - 2)$ . Then there is an  $x \in U$  of order  $n$ , such that the order of  $t_\varphi^J(x)$  is equal to  $n^2$ .

(4) The extension  $G$  of  $E_U$  by  $\mathbb{G}_{mU}$  is uniquely isomorphic to the restriction to  $E_U$  of the Poincaré torsor  $\mathcal{P}$  as in Section 2 (up to a switch of the factors of  $E \times E$ ), and under this isomorphism,  $t_\varphi^J$  equals the Ribet section  $t_\varphi$ .

*Proof.* — We prove part (1). The image  $\pi(t_\varphi^J)$  in  $E_U(U)$  of  $t_\varphi^J$  is the class of the divisor  $D_\varphi$  on  $E_U$ , hence we have, denoting by  $\simeq$  linear equivalence on  $\text{Div}^0(E_U)$ :

$$\begin{aligned} D_\varphi &\simeq \varphi_*((\Delta) - (2\Delta)) - \bar{\varphi}_*((\Delta) - (2\Delta)) \\ &= ((\varphi(\Delta)) - (2\varphi(\Delta))) - ((\bar{\varphi}(\Delta)) - (2\bar{\varphi}(\Delta))) \\ &\simeq ((0) - (\varphi(\Delta))) - ((0) - (\bar{\varphi}(\Delta))) \\ &\simeq ((0) - ((\varphi - \bar{\varphi})(\Delta))) = ((0) - (\alpha(\Delta))). \end{aligned}$$

Under the principal polarisation  $\lambda: E \rightarrow E^\vee$ ,  $x \mapsto [(0) - (x)]$ , this corresponds to  $\alpha(\Delta)$  in  $E(U)$ . This proof of part (1) is finished.

We prove part (2). So, let  $n$  be a positive integer, and let  $x \in U$  be a point such that  $nx = 0$  in  $E$ . As  $nx = 0$ , we have  $n\pi t_\varphi^J(x) = n\alpha(x) = 0$  in  $E$ . This means that  $nD_{\varphi,x}$  is a principal divisor on  $E$ . Let  $f \in k(E)^\times$  be such that  $\text{div}(f) = n(x) - n(2x)$  in  $\text{Div}(E)$ . Then we have, on  $E$ :

$$\begin{aligned} \text{div}(f \circ \varphi) &= \varphi^* \text{div}(f) = \varphi^*(n(x) - n(2x)), \\ \text{div}(\text{Norm}_\varphi(f)) &= \varphi_* \text{div}(f) = \varphi_*(n(x) - n(2x)). \end{aligned}$$

We define:

$$g_\varphi := \text{Norm}_\varphi(f)/(f \circ \varphi) \quad \text{in } k(E)^\times.$$

Then we have:

$$nD_{\varphi,x} = \text{div}(\text{Norm}_\varphi(f)) - \text{div}(f \circ \varphi) = \text{div}(g_\varphi) \quad \text{on } E.$$

This means that  $nt_\varphi^J(x)$  in  $G_x$  is the element  $g_\varphi(x)/g_\varphi(2x)$  of  $k^\times$ . By the construction of  $U$ , the divisor of  $f$  has support disjoint from that of  $g_\varphi$  and of  $\varphi^* \text{div}(f)$  and  $\varphi_* \text{div}(f)$ , and Weil reciprocity gives us:

$$\begin{aligned} \left(\frac{g_\varphi(x)}{g_\varphi(2x)}\right)^n &= g_\varphi(\text{div}(f)) = f(\text{div}(g_\varphi)) = f(\text{div}(\text{Norm}_\varphi(f)) - \text{div}(f \circ \varphi)) \\ &= \frac{f(\text{div}(\text{Norm}_\varphi(f)))}{f(\text{div}(f \circ \varphi))} = \frac{f(\varphi_* \text{div}(f))}{(f \circ \varphi)(\text{div}(f))} = \frac{f(\varphi_* \text{div}(f))}{f(\varphi_* \text{div}(f))} = 1. \end{aligned}$$

So, indeed  $n^2 t_\varphi^J(x) = 0$  in  $G_x$ . Let us also prove the equality  $nt_\varphi^J(x) = e_n(\varphi(x), x)$ . We have

$$\lambda(x) = [(x) - (2x)] \text{ in } E^\vee, \quad n((x) - (2x)) = \text{div}(f) \text{ in } \text{Div}(E),$$

and

$$\lambda(\varphi(x)) = [\varphi_*(x) - \varphi_*(2x)] \quad \text{in } E^\vee,$$

and

$$n(\varphi_*(x) - \varphi_*(2x)) = \operatorname{div}(\operatorname{Norm}_\varphi(f)) \quad \text{in } \operatorname{Div}(E).$$

So, by the description above of the Weil pairing,

$$\begin{aligned} e_n(\varphi(x), x) &= \frac{(\operatorname{Norm}_\varphi(f))((x) - (2x))}{f(\varphi_*((x) - (2x)))} = g_\varphi((x) - (2x)) \\ &= \frac{g_\varphi(x)}{g_\varphi(2x)} = nt_\varphi^J \quad \text{in } k^\times. \end{aligned}$$

We prove part (3). Let  $n$  be a positive odd integer that is prime to  $\deg(\alpha)$ , invertible in  $k$ , and that divides none among  $\deg(2(\varphi - 1))$ ,  $\deg(2\varphi - 1)$  and  $\deg(\varphi - 2)$ . To prove that there is a  $x$  in  $U$  such that the order of  $x$  is  $n$  and the order of  $t_\varphi^J(x)$  is  $n^2$ , it is sufficient to show that there is an  $x$  in  $U$  of order  $n$  such that  $e_n(\varphi(x), x)$  is of order  $n$ . As  $n$  does not divide  $\deg(2(\varphi - 1))$ ,  $\deg(2\varphi - 1)$ , and  $\deg(\varphi - 2)$ , each  $x$  in  $E$  of order  $n$  is in  $U$ .

Let now  $p$  be a prime number dividing  $n$ . Then  $p$  is odd, and  $p$  is invertible in  $k$ , hence  $E[p]$  is of dimension two as  $\mathbb{F}_p$ -vector space, with the symmetric bilinear form

$$E[p] \times E[p] \longrightarrow k^\times, \quad (x, y) \longmapsto e_p(\alpha(x), y).$$

As  $p$  does not divide  $\deg(\alpha)$ , this form is perfect. Therefore, there is an  $x_p$  in  $E[p]$  such that  $e_p(\alpha(x_p), x_p)$  is of order  $p$ . Then  $e_p(\varphi(x_p), x_p)$  is also of order  $p$ , as  $e_p(\alpha(x_p), x_p) = e_p(\varphi(x_p), x_p)^2$ . Let  $n_p$  be the exponent of  $p$  in the factorisation of  $n$ , and  $x'_p \in E$  such that  $x_p = p^{n_p-1}x'_p$ , then  $x'_p$  is in  $E[n]$ , and the order of  $e_n(\varphi(x'_p), x'_p)$  is  $p^{n_p}$ .

Taking for  $x$  the sum of the  $x'_p$  for  $p$  dividing  $n$  gives an  $x$  as desired. We have now finished the proof of part (3).

We prove part (4). The two families of extensions of  $E$  by  $\mathbb{G}_m$  are fibrewise isomorphic by construction, hence there is a unique isomorphism of extensions between them as  $\operatorname{Hom}(E, \mathbb{G}_m)$  is trivial. The sections  $t_\varphi$  and  $t_\varphi^J$  lie above the graph of  $\alpha: E \rightarrow E$ . We will show that  $t_\varphi^J$  extends from  $U$  to  $E$ , and that  $t_\varphi(0) = t_\varphi^J(0)$ . Then there is a unique  $c \in k^\times$  such that  $t_\varphi^J = ct_\varphi$ , and the  $c$  equals 1 because of the values at 0.

We show that  $t_\varphi^J$  extends from  $U$  to  $E$  by viewing as explained above, for  $T \rightarrow U$ ,  $\operatorname{Pic}(C_T)$  as the group of isomorphism classes of  $(L, \sigma)$ , with  $L$  an invertible  $\mathcal{O}$ -module on  $E_T$  and  $\sigma: \Delta^*L \rightarrow (2\Delta)^*L$  an isomorphism of  $\mathcal{O}$ -modules on  $T$ . This description extends as such to all  $T \rightarrow E$ , hence gives us an extension over all of  $E$  of the extension  $G$  of  $E_U$  by  $\mathbb{G}_{mU}$ . Now we show that  $t_\varphi^J$  extends over  $E$ . It suffices to take  $T = E$ , and show that the divisor  $\Delta^*(D_\varphi) - (2\Delta)^*(D_\varphi)$  on  $E$  is principal, and that the restriction  $D_{\varphi,0}$  of  $D_\varphi$  to  $E \times \{0\}$  is principal.

Definition (6.0.1) shows that  $D_{\varphi,0}$  is zero, as divisor on  $E$ . We claim that also  $\Delta^*(D_\varphi) - (2\Delta)^*(D_\varphi)$  is zero, as divisor on  $E$ . We give the computation. Let  $R$  be

any  $k$ -algebra. Then

$$\begin{aligned}(\Delta^*(\varphi_*(\Delta)))(R) &= \{x \in E(R) : \varphi(x) = x\} \\(\Delta^*(\varphi_*(2\Delta)))(R) &= \{x \in E(R) : 2\varphi(x) = x\} \\(\Delta^*(\varphi^*(\Delta)))(R) &= \{x \in E(R) : \varphi(x) = x\} \\(\Delta^*(\varphi^*(2\Delta)))(R) &= \{x \in E(R) : \varphi(x) = 2x\},\end{aligned}$$

and

$$\begin{aligned}((2\Delta)^*(\varphi_*(\Delta)))(R) &= \{x \in E(R) : \varphi(x) = 2x\} \\((2\Delta)^*(\varphi_*(2\Delta)))(R) &= \{x \in E(R) : 2\varphi(x) = 2x\} \\((2\Delta)^*(\varphi^*(\Delta)))(R) &= \{x \in E(R) : 2\varphi(x) = x\} \\((2\Delta)^*(\varphi^*(2\Delta)))(R) &= \{x \in E(R) : 2\varphi(x) = 2x\}.\end{aligned}$$

A little bit of bookkeeping shows that the balance is zero.  $\square$

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