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TENSOR PRODUCTS AND q-CHARACTERS OF HL-MODULES AND MONOIDAL CATEGORIFICATIONS

by Matheus Brito & Vyjayanthi Chari

Abstract. — We study certain monoidal subcategories (introduced by David Hernandez and Bernard Leclerc) of finite-dimensional representations of a quantum affine algebra of type A. We classify the set of prime representations in these subcategories and give necessary and sufficient conditions for a tensor product of two prime representations to be irreducible. In the case of a reducible tensor product we describe the prime decomposition of the simple factors. As a consequence we prove that these subcategories are monoidal categorifications of a cluster algebra of type A with coefficients.

Résumé (Produits tensoriels et q-caractères de HL-modules et catégorifications monoïdales)

Dans ce travail, nous étudions certaines sous-catégories monoïdales (introduites par David Hernandez et Bernard Leclerc) de représentations de dimension finie d'une algèbre affine de type A. Nous classifions l'ensemble des représentations premières de ces sous-catégories, et donnons des conditions nécessaires et suffisantes pour que le produit tensoriel des deux représentations premières soit irréductible. Dans le cas où le produit tensoriel est réductible, nous décrivons une factorisation en modules premiers des facteurs simples. En conséquence, nous prouvons que ces sous-catégories monoïdales sont des catégorifications monoïdales d'algèbres amassées de type A avec coefficients.

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Introduction

The study of the category $\mathscr{F}$ finite-dimensional representations of a quantum affine algebra goes back nearly thirty years and continues to be of significant interest. The irreducible objects in this category are indexed by elements of a free abelian monoid (denoted $\mathscr{P}^+$) with generators $\boldsymbol{\omega}_{i,a}$ where i varies over the index set for the simple roots and a varies over non-zero elements of the field of rational functions in a variable q. The category is not semisimple and there are many interesting indecomposable objects in it. In recent years, there has been a new insight in the study of $\mathscr{F}$ coming from connections with cluster algebras through the work of [19], [18], [21] and also from KLR algebras through the work of [20].

The category $\mathscr{F}$ is a monoidal tensor category and an interesting feature is that a tensor product of generic simple objects is simple. An obviously related notion is that of a prime simple object; this is one which cannot be written in a non-trivial way as a tensor product of objects of $\mathscr{F}$. An open and very difficult question is the following: classify prime simple objects in $\mathscr{F}$ and describe the factorization of an arbitrary simple object as a tensor product of primes. The answer to this question for $\mathfrak{sl}_2$ was given in [8] where it was also proved that the factorization was unique. In higher rank the question along with that of uniqueness remains unanswered. However, in [16] and [17] an important result was established which greatly simplifies the problem by reducing it to following: give a necessary and sufficient condition for the tensor product of a pair of prime simple objects to be simple.

In this paper we focus on this question for certain subcategories of $\mathscr{F}$ associated with quantum affine $\mathfrak{sl}_{n+1}$. These subcategories were introduced by David Hernandez and Bernard Leclerc ([19], [18]) and the definition has its roots in the theory of cluster algebras. The remarkable insight was that prime representations were analogous to cluster variables and the irreducibility of a tensor product of prime objects was analogous to the idea of two elements belonging to the same cluster. The role of the quiver in the theory of cluster algebras is played by the height function; a height function (of type A_n) is a function $\xi : [1,n] \to \mathbb{Z}$ satisfying the condition $|\xi(i) - \xi(i+1)| = 1$ for $1 \leqslant i \leqslant n-1$. Define $\mathscr{P}^+_\xi$ to be the submonoid of $\mathscr{P}^+$ generated by elements $\boldsymbol{\omega}_{i,q^{\xi(i)\pm 1}}$ and let $\mathscr{F}_\xi$ be the full subcategory of $\mathscr{F}$ consisting of objects whose Jordan-Holder constituents are indexed by elements of $\mathscr{P}^+_\xi$. It was proved in [18] that $\mathscr{F}_\xi$ is a monoidal tensor category and we let $\mathscr{K}_0(\mathscr{F}_\xi)$ be the Grothendieck ring of $\mathscr{F}_\xi$. In the case when ξ is the bipartite height function, i.e., $\xi(i-1) = \xi(i+1)$ for $2 \leqslant i \leqslant n-2$ or the monotonic function $\xi(i) = i$ they showed that $\mathscr{K}_0(\mathscr{F}_\xi)$ is isomorphic to a cluster algebra with coefficients of type A.

In this paper we prove the result for all height functions of type A by representation theoretic methods. We define a subset $\mathbf{Pr}_\xi$ of $\mathscr{P}^+_\xi$ such that the corresponding irreducible representations (which we call HL-modules) are prime. Working entirely in $\mathscr{F}_\xi$ we show that the HL-modules are precisely all the prime objects in this category. To do this, we establish necessary and sufficient conditions for a tensor product of HL-modules to be irreducible. In the case when the tensor product is reducible we describe the Jordan-Holder constituents and their factorization as a tensor product of HL-modules.

The connection with cluster algebras is then made as follows. We define a quiver Q_ξ associated with ξ; since we are working in the general case the quiver we use is a mutation of the quivers in [19] and [18]. This mutation allows us to map a non-frozen variable in the initial seed of the cluster algebra to the class of the irreducible module corresponding to either $\boldsymbol{\omega}_{i,\xi(i)+1}$ or $\boldsymbol{\omega}_{i,\xi(i)-1}$. The first mutation at any element of the initial seed is easily described; however is not necessarily of the form $\boldsymbol{\omega}_{i,\xi(i)\pm1}$. Our tensor product formulas now allow us to prove the existence of an algebra isomorphism between the cluster algebra with n frozen variables and $\mathscr{K}_0(\mathscr{F}_\xi)$. The isomorphism maps a cluster variable to an HL-module and we identify this module explicitly. We also show that the isomorphism maps cluster monomials to simple tensor products of HL-modules. As a consequence of this result we give an alternative proof for the product of a pair of cluster variables to be a cluster monomial; equivalently we give an alternative proof of the criterion for a pair of roots to be compatible. In Proposition 2.5 we give a closed formula for a cluster variable in terms of the original seed. In terms of representation theory this can be interpreted as giving a q-character formula for the prime representations in $\mathscr{F}_\xi$. It is useful to remark here that other explicit formulas for cluster variables can be found in the literature see for instance, [2], [4], [11], [14]. Not all these papers deal with frozen variables and even those that do impose conditions on the frozen variables which are not satisfied by the quivers considered in this paper. The role of the frozen variable in the connection with representation theory is important and motivates our formulas.

The paper is organized as follows. In Section 1 we recall the definition of the height function ξ and introduce the associated quiver Q_ξ. We then state and prove our main result modulo the key Propositions 1.5, 1.6 and 1.7. In Section 2 we prove Proposition 1.5 which gives a recursive formula for a cluster variable. This is done by a simple analysis of the quiver obtained by mutating at successive nodes. The answer we obtain is in a form which is well adapted to the representation theory of quantum affine algebras and can be viewed as an analog of Pieri's rule in classical representation theory. We then solve the recursion to give a closed formula for the cluster variable in terms of the initial cluster which includes the frozen variables. In Sections 3, 4 and 5 we provide sufficient and necessary conditions, for the tensor product of two HL-modules to be irreducible. We also analyze the Jordan-Holder series of a reducible tensor product of HL-modules. The proof of Propositions 1.6 and 1.7 can be found in Section 4.

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1. The main results

Throughout the paper we denote by $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ the set of complex numbers, integers, non-negative and positive integers respectively. For $i, j \in \mathbb{Z}_+$ with $i \leqslant j$ we let $[i, j] = \{i, i+1, \dots, j\}$. Given a commutative ring A we denote by $A[q]$ (resp. $A(q)$) the ring (resp. quotient field) of polynomials in an indeterminate q with coefficients in A.

1.1. The cluster algebra $\mathscr{A}(\boldsymbol{x}, Q_\xi)$. — Let $\xi : [1, n] \to \mathbb{Z}$ be a height function; namely a function which satisfies the conditions

$$|\xi(i) - \xi(i-1)| = 1, \qquad 2 \leqslant i \leqslant n.$$

It will be convenient to extend ξ to $[0, n+1]$ by setting $\xi(0) = \xi(2)$ and $\xi(n-1) = \xi(n+1)$.

Remark. — Although trivial, it is useful to note that

$$\{\xi(i+1), \xi(i-1)\} \subset \{\xi(i)+1, \xi(i)-1\}$$

and that the inclusion can be strict.

For $i \in [1, n-2]$, let $i_\diamond \in [i, n]$, $i_\diamond \geqslant i$ be minimal such that $\xi(i_\diamond) = \xi(i_\diamond + 2)$ and set $(n-1)_\diamond = (n-1)$ and $n_\diamond = n$. Let Q_ξ be a quiver with $2n$ vertices labeled $\{1, \dots, n, 1', \dots, n'\}$ and with the set of edges given as follows:

– there are no edges between the primed vertices; in other words the vertices $\{1', \dots, n'\}$ are frozen,

– if $1 \leqslant j \leqslant n-1$ and $\xi(j) = \xi(j+1) + 1$, the edges at j are:

$$\begin{tikzcd}
(j-1) & j \arrow[l] \arrow[r, bend left, "\delta_{j,j_\diamond}"] \arrow[dr, "1-\delta_{j,j_\diamond}"'] & (j+1) \arrow[l, bend left, "1-\delta_{j,j_\diamond}"] \\
 & j' \arrow[u] & (j+1)'
\end{tikzcd}$$

and the reverse orientations if $\xi(j) = \xi(j+1) - 1$, where $\delta_{j,j_\diamond}$ is the Kronecker delta function and we adopt the convention that a labeled edge exists iff the label is one,

– at the vertex n we have edges $(n-1) \to n \to n'$ if $\xi(n-1) = \xi(n) + 1$ and the reverse orientation otherwise.

Clearly j is a sink or source of Q_ξ (where we ignore the frozen vertices) iff $j = 1$ or $j = j_\diamond$. For $1 \leqslant j \leqslant 1_\diamond$ set $j_\bullet = 0$ and for $j > 1_\diamond$ let $j_\bullet$ be the maximal sink or source of Q_ξ satisfying $j_\bullet < j$.

Fix a set $\boldsymbol{x} = \{x_1, \dots, x_n, f_1, \dots, f_n\}$ of algebraically independent variables and let $\mathscr{A}(\boldsymbol{x}, Q_\xi)$ be the cluster algebra (with coefficients) with initial seed $(\boldsymbol{x}, Q_\xi)$. The definition of a cluster algebra is recalled briefly in Section 2.1; for the rest of this section we shall freely use the language of cluster algebras. Since the principal unfrozen part of Q_ξ is a quiver of type A_n, the set of non-frozen cluster variables in $\mathscr{A}(\boldsymbol{x}, Q_\xi)$

are indexed by the set $\Phi_{\geqslant -1}$ of almost positive roots of a root system of type A_n. In other words if we let $\{\alpha_i : 1 \leqslant i \leqslant n\}$, be a set of simple roots for A_n and set $\alpha_{i,j} = \alpha_i + \cdots + \alpha_j$, $1 \leqslant i \leqslant j \leqslant n$, then

$$\Phi_{\geqslant -1} = \{-\alpha_i,\ \alpha_{i,j} : 1 \leqslant i \leqslant j \leqslant n\},$$

and the cluster variables are denoted

$$\{x_i := x[-\alpha_i],\ x[\alpha_{i,j}],\ f_i :\ 1 \leqslant i \leqslant j \leqslant n\}.$$

Moreover, the cluster variable $x[\alpha_{i,j}]$ is obtained by applying the sequence $i, i+1, \ldots, j$ of mutations at the original cluster.

1.2. The category $\mathscr{F}_\xi$. — Let $\widehat{\mathbf{U}}_q$ be the quantum loop algebra over $\mathbb{C}(q)$ associated to $\mathfrak{sl}_{n+1}$ and let $\mathscr{F}$ be the monoidal tensor category whose objects are finite-dimensional representations of $\widehat{\mathbf{U}}_q$. Given a height function $\xi : [1, n] \to \mathbb{Z}$ we take $\mathscr{P}_\xi^+$ to be the free abelian monoid with generators $\{\boldsymbol{\omega}_{i,\xi(i)\pm 1} : i \in [1, n]\}$. It is known that $\mathscr{P}_\xi^+$ is the index set for a (sub)-family of isomorphism classes of irreducible objects of $\mathscr{F}$. We define $\mathscr{F}_\xi$ to be the full subcategory of $\mathscr{F}$ consisting of objects all of whose Jordan-Holder constituents are indexed by elements of $\mathscr{P}_\xi^+$. It was proved in [18] that $\mathscr{F}_\xi$ is a monoidal category and we let $\mathscr{K}_0(\mathscr{F}_\xi)$ be the corresponding Grothendieck ring. For $\boldsymbol{\omega} \in \mathscr{P}_\xi^+$ let $[\boldsymbol{\omega}] \in \mathscr{K}_0(\mathscr{F}_\xi)$ be the isomorphism class of the corresponding object in $\mathscr{F}_\xi$.

Remark. — It is important to keep in mind that the assignment $\boldsymbol{\omega} \to [\boldsymbol{\omega}]$ is not a morphism of monoids $\mathscr{P}_\xi^+ \to \mathscr{K}_0(\mathscr{F}_\xi)$, i.e., $[\boldsymbol{\omega}][\boldsymbol{\omega}']$ is not always equal to $[\boldsymbol{\omega}\boldsymbol{\omega}']$. One of the goals of this paper is to determine a necessary and sufficient condition for equality to hold.

For $i \in [1, n]$ set

$$\boldsymbol{f}_i = \boldsymbol{\omega}_{i,\xi(i)+1}\boldsymbol{\omega}_{i,\xi(i)-1}.$$

If $1 \leqslant i < j \leqslant n$, let $i_2 < \cdots < i_{k-1}$ be an ordered enumeration of the subset

$$\{p : i < p < j,\ \xi(p-1) = \xi(p+1)\},$$

$i_1 = i$, $i_k = j$ and define an element $\boldsymbol{\omega}(i, j) \in \mathscr{P}_\xi^+$ by:

$$\boldsymbol{\omega}(i, j) = \boldsymbol{\omega}_{i_1,a_1} \cdots \boldsymbol{\omega}_{i_k,a_k},$$

where $a_1 = \xi(i) \pm 1$ if $\xi(i+1) = \xi(i) \mp 1$ and $a_m = \xi(i_m) \pm 1$ if $\xi(i_m) = \xi(i_m - 1) \pm 1$, for $m \geqslant 2$. Set

$$\mathbf{Pr}_\xi = \{\boldsymbol{\omega}_{i,\xi(i)\pm 1},\ \boldsymbol{\omega}(i, j) : 1 \leqslant i \leqslant j \leqslant n,\ i \neq j\}.$$

Clearly the set $\mathbf{Pr}_\xi$ has the same cardinality as the set of unfrozen cluster variables in $\mathscr{A}(\boldsymbol{x}, Q_\xi)$.

Recall that an object of $\mathscr{F}$ is said to be prime if it cannot be written in a non-trivial way as a tensor product of objects of $\mathscr{F}$. The following is a special case of the main result of [6].

Lemma. — *The irreducible object of $\mathscr{F}_\xi$ associated to an element*

$$\boldsymbol{\omega} \in \mathbf{Pr}_\xi \cup \{\boldsymbol{f}_i : 1 \leqslant i \leqslant n\}$$

is prime. □

1.3. Main Theorem. — Recall that by definition $n = n_\diamond$ and $(n-1) = (n-1)_\diamond$ which in particular implies that $n_\bullet = n-1$. For $k \geqslant 1$, set

$$\overline{k} = (k+1)(1-\delta_{k,k_\diamond}) + (k_\bullet + 1)\delta_{k,k_\diamond}. \tag{1.1}$$

Theorem 1. — *Let $\xi : [1,n] \to \mathbb{Z}$ be a height function. The assignment*

$$\iota(x_i) = [\boldsymbol{\omega}_{i,\xi(i+1)}], \quad \iota(f_i) = [\boldsymbol{f}_i],$$

extends to an isomorphism of rings $\iota : \mathscr{A}(\boldsymbol{x}, Q_\xi) \to \mathscr{K}_0(\mathscr{F}_\xi)$ such that for $1 \leqslant i \leqslant k \leqslant n$,

$$\iota(x[\alpha_{i,i_\diamond}]) = [\boldsymbol{\omega}_{i,\xi(i+1)\pm 2}], \quad \xi(i) = \xi(i+1) \pm 1, \tag{1.2}$$

$$\iota(x[\alpha_{i,k}]) = [\boldsymbol{\omega}(i,\overline{k})], \quad k \neq i_\diamond, \tag{1.3}$$

$$\iota(f_p x[\alpha]) = [\boldsymbol{f}_p \boldsymbol{\omega}] \ \ p \in [1,n], \quad \alpha \in \Phi_{\geqslant -1}, \quad [\boldsymbol{\omega}] = \iota(x[\alpha]). \tag{1.4}$$

In particular ι maps cluster variable to a prime object of $\mathscr{F}_\xi$. Moreover, if $\beta_1, \beta_2 \in \Phi_{\geqslant -1}$ are such that $x[\beta_1]x[\beta_2]$ is a cluster monomial then

$$[\boldsymbol{\omega}_1\boldsymbol{\omega}_2] = [\boldsymbol{\omega}_1][\boldsymbol{\omega}_2], \quad [\boldsymbol{\omega}_s] = \iota(x[\beta_s]), \qquad s = 1,2.$$

Corollary. — *The homomorphism ι sends a cluster monomial to the equivalence class of an irreducible object of $\mathscr{F}_\xi$. In particular any irreducible prime object is of the form $[\boldsymbol{\omega}]$, $\boldsymbol{\omega} \in \mathbf{Pr}_\xi \cup \{\boldsymbol{f}_i : 1 \leqslant i \leqslant n\}$ and $\mathscr{F}_\xi$ is a monoidal categorification of $\mathscr{A}(\boldsymbol{x}, \xi)$.*

Proof of Corollary. — Let $x[\beta_1] \cdots x[\beta_r]$ be a cluster monomial for some $\beta_1, \dots, \beta_r \in \Phi_{\geqslant -1}$ and set $[\boldsymbol{\omega}_i] = \iota(x[\beta_i])$ for $1 \leqslant i \leqslant r$. Then the pairs $x[\beta_j]x[\beta_p]$, $1 \leqslant j \neq p \leqslant r$ are cluster monomials and hence using the Theorem 1 we have $\iota(x[\beta_j]x[\beta_p]) = [\boldsymbol{\omega}_j\boldsymbol{\omega}_p]$ for $1 \leqslant j \neq p \leqslant r$. It follows from the main result of [16] (see Section 3 of this paper for the statement) that $\iota(x[\beta_1] \cdots x[\beta_r]) = [\boldsymbol{\omega}_1 \cdots \boldsymbol{\omega}_r]$ and the first assertion of the corollary is established. Suppose that $[\boldsymbol{\omega}]$ is an irreducible object of $\mathscr{F}_\xi$. Then $\iota^{-1}[\boldsymbol{\omega}]$ is a sum of cluster monomials. Hence $[\boldsymbol{\omega}]$ can be written as a linear combination of elements $[\boldsymbol{\pi}]$ where each $\boldsymbol{\pi}$ is a product of elements from $\mathbf{Pr}_\xi \cup \{\boldsymbol{f}_i : 1 \leqslant i \leqslant n\}$. Since irreducible modules are a basis $\mathscr{K}_0(\mathscr{F}_\xi)$ it follows that $[\boldsymbol{\omega}]$ is a product of elements from $\mathbf{Pr}_\xi \cup \{\boldsymbol{f}_i : 1 \leqslant i \leqslant n\}$ and the corollary is established. □

Remark. — Suppose that ξ satisfies $\xi(i-1) = \xi(i+1)$ for all $1 \leqslant i \leqslant n$ or that $\xi(j) = \xi(i) + (j-i)$ for all $1 \leqslant i \leqslant j \leqslant n$. In these two cases the existence of ι was established in [19],[18] by very different methods. As was noted in [18] the categories $\mathscr{F}_\xi$ are not necessarily equivalent for different height functions.

1.4. — In Theorem 3 of this paper we give conditions for the equality $[\boldsymbol{\pi}\boldsymbol{\pi}'] = [\boldsymbol{\pi}][\boldsymbol{\pi}']$ when $\boldsymbol{\pi}, \boldsymbol{\pi}' \in \mathbf{Pr}_\xi$ to hold in $\mathscr{K}_0(\mathscr{F}_\xi)$. The translation to the language of cluster algebra gives the conditions for describing when two roots are compatible. Thus our theorem gives a proof of the following assertion (compare with the description in [18, §10.2.3] where a similar description in the case of the bipartite height function).

Assume that $i \leqslant j$, $k \leqslant \ell$ and $i \leqslant k$. If $j \neq i_\diamond$ the roots $\alpha_{i,j}$, $\alpha_{k,\ell}$ are compatible iff:

- $k = i$ or $k > j+1$,
- $j = j_\diamond$ and $j_\bullet + 1 \leqslant k \leqslant j$,
- $\ell \neq k_\diamond$ and
    - either $\overline{j} = \overline{\ell}$,
    - or $i < k < \overline{j} < \overline{\ell}$, and $\#\{k \leqslant m < \overline{j} - 1 : m = m_\diamond\} \in 2\mathbb{Z}_+ + 1$,
    - or $i < k < \overline{\ell} < \overline{j}$, and $\#\{k \leqslant m < \overline{\ell} - 1 : m = m_\diamond\} \in 2\mathbb{Z}_+$.

The roots $\alpha_{i,i_\diamond}$ and $\alpha_{k,\ell}$ with $i \leqslant k$ are compatible iff :

$$k_\bullet \neq k-1, \quad \text{or} \quad (k-1)_\bullet \geqslant i \quad \text{or} \quad \ell \neq k_\diamond \text{ and } i = k.$$

The roots $-\alpha_i$ and $\alpha_{k,\ell}$ are in the same cluster iff either $k > i$ or $\ell < i$.

In Theorem 4 we write down the Jordan-Holder series for a reducible tensor product of objects. This amounts to writing down all the non-trivial exchange relations for cluster variables including the frozen variables and is not hard to do using the analysis above.

1.5. — The proof of the theorem involves three principal steps. For $1 \leqslant j \leqslant n$, set

$$d_j = \delta_{j,j_\diamond} = \delta_{\xi(j),\xi(j+2)}.$$

The first step is the following proposition which gives a recursive formula for the cluster variables. We adopt the convention that $\alpha_{i,m} = \alpha_m,\ m \leqslant i$.

Proposition. — *For $1 \leqslant i < j \leqslant n$ we have*

$$\begin{aligned} x_i x[\alpha_i] &= f_i x_{i+1}^{1-d_i} + f_{i+1}^{1-d_i} x_{i-1} x_{i+1}^{d_i}, \\ x_j x[\alpha_{i,j}] &= f_j^{d_j - 1} x[\alpha_{i,j-1}] x_{j+1}^{1-d_j} + f_{j+1}^{1-d_j} x_{j+1}^{d_j} \Big((\delta_{i_\bullet, j_\bullet} + \delta_{i,j_\bullet}) f_i^{\delta_{i,j_\bullet}} x_{i-1}^{1-\delta_{i,j_\bullet}} \\ &\qquad + (1 - \delta_{i_\bullet, j_\bullet} - \delta_{i,j_\bullet}) f_{j_\bullet}^{d_{j_\bullet} - 1} x[\alpha_{i,j_\bullet - 1}] \Big). \end{aligned}$$

The proof of this proposition is in Section 2 where we also give a closed formula for $x[\alpha_{i,j}]$ as a Laurent polynomial in the variables $\{x_1, \ldots, x_n, f_1, \ldots, f_n\}$.

1.6. — The second step in the proof of the theorem is the following. We adopt the convention that we take $\boldsymbol{\omega}_{i,\xi(i+1)+2}$ if $\xi(i) = \xi(i+1)+1$ and we take $\boldsymbol{\omega}_{i,\xi(i+1)-2}$ if $\xi(i) = \xi(i+1)-1$.

Proposition. — *The following equalities hold in* $\mathscr{K}_0(\mathscr{F}_\xi)$ *for* $1 \leqslant i \leqslant j \leqslant n$.

(i) *We have*

$$[\boldsymbol{\omega}_{i,\xi(i+1)}][\boldsymbol{\omega}(i,i+1)]^{1-d_i}[\boldsymbol{\omega}_{i,\xi(i+1)\pm 2}]^{d_i} = [\boldsymbol{f}_i]\ [\boldsymbol{\omega}_{i+1,\xi(i+2)}]^{1-d_i} + [\boldsymbol{f}_{i+1}]^{1-d_i}[\boldsymbol{\omega}_{i+1,\xi(i+2)}]^{d_i}\ [\boldsymbol{\omega}_{i-1,\xi(i)}].$$

(ii) *If* $j_\bullet \leqslant i < j$ *then*

$$\begin{aligned}[\boldsymbol{\omega}_{j,\xi(j+1)}][\boldsymbol{\omega}(i,\bar{j})]^{1-\delta_{j,i_\diamond}}[\boldsymbol{\omega}_{i,\xi(i+1)\pm 2}]^{\delta_{j,i_\diamond}} &= [\boldsymbol{f}_j]^{d_{j-1}}\ [\boldsymbol{\omega}(i,j)]^{1-d_{j-1}}[\boldsymbol{\omega}_{i,\xi(i+1)\pm 2}]^{d_{j-1}}[\boldsymbol{\omega}_{j+1,\xi(j+2)}]^{1-d_j}\\ &\quad + [\boldsymbol{f}_{j+1}]^{1-d_j}[\boldsymbol{f}_i]^{\delta_{i,j_\bullet}}[\boldsymbol{\omega}_{j+1,\xi(j+2)}]^{d_j}[\boldsymbol{\omega}_{i-1,\xi(i)}]^{1-\delta_{i,j_\bullet}}.\end{aligned}$$

(iii) *If* $i < j_\bullet$ *choose* $z \in \{\xi(i)+1, \xi(i)-1\}$ *so that* $\boldsymbol{\omega}_{i,z}^{-1}\boldsymbol{\omega}(i,j_\bullet) \in \mathscr{P}_\xi^+$ *and set* $k = (j_\bullet)_\bullet$. *Then,*

$$\begin{aligned}[\boldsymbol{\omega}_{j,\xi(j+1)}][\boldsymbol{\omega}(i,\bar{j})] &= [\boldsymbol{f}_j]^{d_{j-1}}\ [\boldsymbol{\omega}_{j+1,\xi(j+2)}]^{1-d_j}\ [\boldsymbol{\omega}(i,\overline{j-1})]^{1-\delta_{(j-1)_\bullet,i_\bullet}}\ [\boldsymbol{\omega}_{i,z}]^{\delta_{(j-1)_\bullet,i_\bullet}}\\ &\quad + [\boldsymbol{f}_{j+1}]^{1-d_j}\ [\boldsymbol{f}_{j_\bullet}]^{d_{j_\bullet-1}}[\boldsymbol{\omega}_{j+1,\xi(j+2)}]^{d_j}\ [\boldsymbol{\omega}(i,\overline{j_\bullet-1})]^{1-\delta_{i_\bullet,k_\bullet}d_{j_\bullet-1}}\ [\boldsymbol{\omega}_{i,z}]^{\delta_{i_\bullet,k_\bullet}d_{j_\bullet-1}}.\end{aligned}$$

The proof of this proposition can be found in Section 4.

1.7. — Proposition 1.5 and Proposition 1.6 are enough to establish the existence of ι and to identify the image of a cluster variable. The third step needed to establish the theorem is to show that ι maps a cluster monomial to the isomorphism class of an irreducible representation. To do this we will need the following result.

Proposition. — *Let* $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathbf{Pr}_\xi$. *Then either* $[\boldsymbol{\omega}][\boldsymbol{\omega}'] = [\boldsymbol{\omega}\boldsymbol{\omega}']$ *or* $[\boldsymbol{\omega}][\boldsymbol{\omega}'] = [\boldsymbol{\omega}_1]+[\boldsymbol{\omega}_2]$ *where* $[\boldsymbol{\omega}_1]$ *and* $[\boldsymbol{\omega}_2]$ *are the images under* ι *of cluster monomials.*

A much more precise statement can be found in Theorem 3 and Theorem 4 in Sections 3 and 4. In the rest of this section we assume Proposition 1.5, Proposition 1.6, Proposition 1.7 and prove Theorem 1.

1.8. Existence of ι. — Recall [1] that an element of $\mathscr{A}(\boldsymbol{x}, Q_\xi)$ is said to be a standard monomial if it is a monomial in the elements $\{x_i, x[\alpha_i] : i \in [1,n]\}$ and does not involve any product of the form $x_i x[\alpha_i]$, $i \in [1,n]$. It was proved in [1] that standard monomials are a $\mathbb{Z}[f_i : i \in I]$-basis of $\mathscr{A}(\boldsymbol{x}, \xi)$.

On the other hand consider the quotient of the polynomial ring (with integer coefficients) in variables $X_i, X[\alpha_i], F_i$, $i \in [1,n]$ subject to the first relation in Proposition 1.5. It is not hard to show that this ring is the $\mathbb{Z}[F_i : i \in I]$ span of monomials in $X_i, X[\alpha_i]$, $i \in [1,n]$ which do not involve products of $X_i X[\alpha_i]$ for any $i \in [1,n]$. It follows that $\mathscr{A}(\boldsymbol{x}, Q_\xi)$ is is isomorphic to this quotient (compare with [19, Lem. 4.4]).

Using Proposition 1.6(i) we have

$$[\boldsymbol{\omega}_{i,\ \xi(i+1)}][\boldsymbol{\omega}(i,i+1)]^{1-d_i}[\boldsymbol{\omega}_{i,\xi(i+1)\pm 2}]^{d_i} = [\boldsymbol{f}_i][\boldsymbol{\omega}_{i+1,\xi(i+2)}]^{1-d_i} + [\boldsymbol{f}_{i+1}]^{1-d_i}[\boldsymbol{\omega}_{i+1,\xi(i+2)}]^{d_i}[\boldsymbol{\omega}_{i-1,\xi(i)}].$$

It is now immediate that the assignment

$$x_i \longrightarrow [\boldsymbol{\omega}_{i,\xi(i+1)}], \quad f_i \longrightarrow \boldsymbol{f}_i, \quad x[\alpha_i] \longrightarrow [\boldsymbol{\omega}_{i,\xi(i+1)\pm 2}]^{\delta_{i,i_\diamond}}[\boldsymbol{\omega}(i,i+1)]^{1-\delta_{i,i_\diamond}}$$

defines a homomorphism of rings $\iota : \mathscr{A}(\boldsymbol{x}, Q_\xi) \to \mathscr{K}_0(\mathscr{F}_\xi)$.

1.9. The elements $\iota(x[\alpha])$, $\alpha \in \Phi_{\geqslant -1}$. — The formulas given in (1.2) and (1.3) can be rewritten as follows:

$$\iota(x[\alpha_{i,j}]) = [\boldsymbol{\omega}(i,j+1)]^{1-d_j}\,[\boldsymbol{\omega}_{i,\xi(i+1)\pm 2}]^{\delta_{i_\bullet,j_\bullet}d_j}\,[\boldsymbol{\omega}(i,j_\bullet+1)]^{d_j(1-\delta_{i_\bullet,j_\bullet})}, \quad j \geqslant i. \tag{1.5}$$

We shall prove this reformulation by induction on $j-i$. Observe that induction begins when $j=i$ by definition. For the inductive step apply ι to both sides of the second equation in Proposition 1.5. We will show that the right hand side of this equation is the same as the right hand side of the equation in Proposition 1.6(ii), (iii). Hence the left hand sides must match up. The inductive step is immediate once we observe that $\mathscr{K}_0(\mathscr{F}_\xi)$ has no zero divisors.

To prove that the right hand sides are the same, suppose first that $j_\bullet \leqslant i$ (in particular $j_\bullet = i_\bullet$ or $j_\bullet = i$). Applying ι to both sides of the second equation in Proposition 1.5 gives

$$[\boldsymbol{\omega}_{j,\xi(j+1)}]\iota(x[\alpha_{i,j}]) = \boldsymbol{f}_j^{d_{j-1}}\iota(x[\alpha_{i,j-1}])[\boldsymbol{\omega}_{j+1,\xi(j+2)}]^{1-d_j} + \boldsymbol{f}_{j+1}^{1-d_j}\boldsymbol{f}_i^{\delta_{i,j_\bullet}}[\boldsymbol{\omega}_{j+1,\xi(j+2)}]^{d_j}[\boldsymbol{\omega}_{i-1,\xi(i)}]^{1-\delta_{i,j_\bullet}}.$$

The second term on the right hand side of the preceding equation is equal to the second term on the right hand side of the equation in Proposition 1.6(ii). To see that the first terms match up we use the inductive hypothesis for $\iota(x[\alpha_{i,j-1}])$ and see that it suffices to prove that

$$[\boldsymbol{\omega}_{i,\xi(i+1)\pm 2}]^{d_{j-1}} = \left([\boldsymbol{\omega}_{i,\xi(i+1)\pm 2}]^{\delta_{i_\bullet,(j-1)_\bullet}}[\boldsymbol{\omega}(i,(j-1)_\bullet+1)]^{1-\delta_{i_\bullet,(j-1)_\bullet}}\right)^{d_{j-1}}.$$

If $d_{j-1}=0$, then the preceding equality is obviously true. Since

$$d_{j-1} = 1 \implies (j-1) = (j-1)_\diamond = j_\bullet = i \implies i_\bullet = (j-1)_\bullet$$

and the equality follows.

If $i < j_\bullet$, then the result follows if we prove that,

$$\iota(x[\alpha_{i,j-1}]) = [\boldsymbol{\omega}(i,j)]^{1-d_{j-1}}\left([\boldsymbol{\omega}_{i,z}]^{\delta_{k,i_\bullet}}[\boldsymbol{\omega}(i,k+1)]^{1-\delta_{k,i_\bullet}}\right)^{d_{j-1}},$$
$$\iota(x[\alpha_{i,j_\bullet-1}]) = [\boldsymbol{\omega}(i,j_\bullet)]^{1-d_{j_\bullet-1}}\left([\boldsymbol{\omega}_{i,z}]^{\delta_{i_\bullet,k_\bullet}}[\boldsymbol{\omega}(i,k_\bullet+1)]^{1-\delta_{i_\bullet,k_\bullet}}\right)^{d_{j_\bullet-1}},$$

where we recall that $k=(j_\bullet)_\bullet$. If $d_{j-1}=0$ the first equality follows from the definition and the inductive hypothesis and if $d_{j-1}=1$ then $(j-1)=j_\bullet$ and so $(j-1)_\bullet = k$. The first equality again follows from the inductive hypothesis. The second equality is deduced in the same way from the inductive hypothesis.

1.10. — We prove now that ι is an isomorphism. Let $\{\omega_1, \dots, \omega_n\}$ which are dual to the simple roots of A_n and P^+ be their $\mathbb{Z}_+$-span. It is convenient to set $\omega_0 = \omega_{n+1} = 0$. Let $\leqslant$ be the usual partial order on P^+ given by $\mu \leqslant \lambda$ iff $\lambda - \mu$ is in the $\mathbb{Z}_+$-span of $\{\alpha_1, \dots, \alpha_n\}$.

Define a morphism of monoids $\mathrm{wt} : \mathscr{P}_\xi^+ \to P^+$ by setting $\mathrm{wt}\,\boldsymbol{\omega}_{i,a} = \omega_i$. Since $\mathscr{F}_\xi$ is a tensor category it is well-known that the following holds in $\mathscr{K}_0(\mathscr{F}_\xi)$; for $\boldsymbol{\omega} = \boldsymbol{\omega}_{i_1,a_1} \cdots \boldsymbol{\omega}_{i_k,a_k} \in \mathscr{P}_\xi^+$:

$$(1.6) \qquad [\boldsymbol{\omega}_{i_1,a_1}] \cdots [\boldsymbol{\omega}_{i_k,a_k}] = [\boldsymbol{\omega}] + \sum_{\substack{\boldsymbol{\pi} \in \mathscr{P}_\xi^+ \\ \mathrm{wt}\,\boldsymbol{\pi} < \mathrm{wt}\,\boldsymbol{\omega}}} r(\boldsymbol{\omega}, \boldsymbol{\pi})[\boldsymbol{\pi}], \quad \text{for some } r(\boldsymbol{\omega}, \boldsymbol{\pi}) \in \mathbb{Z}_+.$$

A straightforward induction on $\mathrm{wt}\,\boldsymbol{\omega}$ shows that $\mathscr{K}_0(\mathscr{F}_\xi)$ is generated as a ring by the elements $[\boldsymbol{\omega}_{i,\xi(i)\pm 1}]$. By Section 1.9 we see that $\iota(\{x[-\alpha_i], x[\alpha_{i,i_\diamond}]\}) = \{[\boldsymbol{\omega}_{i,\xi(i)+1}], [\boldsymbol{\omega}_{i,\xi(i)-1}]\}$ and hence it follows that ι is surjective.

We prove that ι is injective. Set

$$\mathrm{wt}_\ell\, x_i = \boldsymbol{\omega}_{i,\xi(i+1)}, \quad \mathrm{wt}_\ell\, f_i = \boldsymbol{f}_i, \ \mathrm{wt}_\ell\, x[\alpha_i] = \boldsymbol{\pi}, \quad \text{such that} \quad \iota(x[\alpha_i]) = [\boldsymbol{\pi}].$$

Extend wt_ℓ in the obvious way to the basis of $\mathscr{A}(\boldsymbol{x}, \xi)$; if

$$\boldsymbol{m} = x_1^{p_1} \cdots x_n^{p_n} x[\alpha_1]^{m_1} \cdots x[\alpha_n]^{m_n}$$

is a standard monomial in $\mathscr{A}(\boldsymbol{x}, \xi)$ and $\boldsymbol{f} = f_1^{r_1} \cdots f_n^{r_n} \in \mathbb{Z}[f_1^{\pm 1}, \dots, f_n^{\pm 1}]$ then

$$\mathrm{wt}_\ell\, \boldsymbol{f}\boldsymbol{m} = \prod_{i=1}^n \boldsymbol{f}_i^{r_i} \boldsymbol{\omega}_{i,\ \xi(i+1)}^{p_i} \Big(\boldsymbol{\omega}_{i,\ \xi(i+1)+2}^{\delta_{\xi(i),\xi(i+1)+1}} \boldsymbol{\omega}_{i,\xi(i+1)-2}^{\delta_{\xi(i),\xi(i+1)-1}} \boldsymbol{\omega}_{i+1,\ \xi(i+2)}^{1-d_i} \Big)^{m_i}.$$

Lemma. — *Let $\boldsymbol{m}, \boldsymbol{m}'$ be standard monomials in $\mathscr{A}(\boldsymbol{x}, Q_\xi)$ and $\boldsymbol{f}, \boldsymbol{f}'$ be monomials in $\{f_i : i \in [1, n]\}$. Then*

$$\mathrm{wt}_\ell\, \boldsymbol{f}\boldsymbol{m} = \mathrm{wt}_\ell\, \boldsymbol{f}'\boldsymbol{m}' \iff \boldsymbol{f} = \boldsymbol{f}' \text{ and } \boldsymbol{m} = \boldsymbol{m}'.$$

Proof. — Write

$$\boldsymbol{m} = x_1^{p_1} \cdots x_n^{p_n} x[\alpha_1]^{m_1} \cdots x[\alpha_n]^{m_n}, \quad \boldsymbol{f} = f_1^{r_1} \cdots f_n^{r_n},$$

and let $\boldsymbol{m}', \boldsymbol{f}'$ be defined similarly with p_i replaced by p_i' etc. If $p_1 > 0$ then $m_1 = 0$ and using the fact that $\mathscr{P}_\xi^+$ is a free abelian monoid we have

$$\boldsymbol{f}_1^{r_1} \boldsymbol{\omega}_{1,\xi(2)}^{p_1} = \boldsymbol{f}_1^{r_1'} \boldsymbol{\omega}_{1,\ \xi(2)}^{p_1'} \boldsymbol{\omega}_{1,\ \xi(2)+2}^{m_1'\delta_{\xi(1),\xi(2)+1}} \boldsymbol{\omega}_{1,\ \xi(2)-2}^{m_1'\delta_{\xi(1),\xi(2)-1}}.$$

Since $\boldsymbol{f}_1 = \boldsymbol{\omega}_{1,\xi(1)+1} \boldsymbol{\omega}_{i,\xi(1)-1}$, we get

$$r_1 + p_1 = r_1' + p_1', \quad r_1 = m_1' + r_1'.$$

If $m_1' \neq 0$ then $p_1' = 0$ and we have $r_1 > r_1'$ and $r_1' > r_1$ which is absurd. Hence $m_1' = 0$ and so $r_1' = r_1$ and $p_1' = p_1$. Writing $\boldsymbol{m} = x_1^{p_1}\boldsymbol{m}_1$ and $\boldsymbol{m}' = x_1^{p_1}\boldsymbol{m}_1'$ we see that $\boldsymbol{m}_1$ and $\boldsymbol{m}_1'$ are both standard monomials and

$$\mathrm{wt}_\ell\, f_2^{r_2} \cdots f_n^{r_n} \boldsymbol{m}_1 = \mathrm{wt}_\ell\, f_2^{r_2'} \cdots f_n^{r_n'} \boldsymbol{m}_1'.$$

An obvious iteration of the preceding argument proves the lemma. $\square$

Suppose that

$$\iota\Big(\sum_{r,s} c_{r,s}\boldsymbol{f}(s)\boldsymbol{m}_r\Big) = 0,$$

where $\boldsymbol{m}_r$ varies over standard monomials in $\mathscr{A}(\boldsymbol{x}, Q_\xi)$, and $\boldsymbol{f}(s)$ varies over monomials in f_i, $i \in [1,n]$ and $c_{r,s} \in \mathbb{Z}$ with only finitely many being non-zero. Assume for a contradiction that $c_{r,s} \neq 0$ for some r, s and let λ be a maximal element (with respect to the partial order on P^+) of the set $\{\mathrm{wt}(\mathrm{wt}_\ell\, \boldsymbol{f}(s)\boldsymbol{m}_r) : c_{r,s} \neq 0\}$. Using (1.6) we get

$$0 = \sum_{\mathrm{wt}(\mathrm{wt}_\ell\, \boldsymbol{f}(s)\boldsymbol{m}_r)=\lambda} c_{r,s}[\mathrm{wt}_\ell\, \boldsymbol{f}(s)\boldsymbol{m}_r] + \sum_{\mathrm{wt}\,\boldsymbol{\omega}\not>\lambda} n_{\boldsymbol{\omega}}[\boldsymbol{\omega}], \qquad n_{\boldsymbol{\omega}} \in \mathbb{Z}.$$

Since the elements $[\boldsymbol{\omega}]$, $\boldsymbol{\omega} \in \mathscr{P}_\xi^+$ are linearly independent elements of $\mathscr{K}_0(\mathscr{F}_\xi)$ we get

$$\sum_{\mathrm{wt}(\mathrm{wt}_\ell\, \boldsymbol{f}(s)\boldsymbol{m}_r)=\lambda} c_{r,s}[\mathrm{wt}_\ell\, \boldsymbol{f}(s)\boldsymbol{m}_r] = 0.$$

By Lemma 1.10 the elements $[\mathrm{wt}_\ell(\boldsymbol{f}(s)\boldsymbol{m}_r)]$ are all distinct and hence also linearly independent. This forces $c_{r,s} = 0$ contradicting our assumption and proves that ι is injective.

1.11. The elements $\iota(x[\beta_1]x[\beta_2])$. — We now prove the final assertion of the theorem. Write $[\boldsymbol{\omega}_s] = \iota([x[\beta_s]])$, $s = 1, 2$ and let $\boldsymbol{\omega} = \boldsymbol{\omega}_1\boldsymbol{\omega}_2$. Assuming that $[\boldsymbol{\omega}] \neq [\boldsymbol{\omega}_1][\boldsymbol{\omega}_2]$ we shall prove that $x[\alpha]x[\beta]$ is not a cluster monomial. By Proposition 1.7 we can write $[\boldsymbol{\omega}_1][\boldsymbol{\omega}_2]$ as the non-trivial sum of elements which are images under ι of cluster monomials. Since cluster monomials are linearly independent and ι is an isomorphism we see that $x[\beta_1]x[\beta_2]$ is not a cluster monomial and the proof of the main theorem is complete.

2. Proof of Proposition 1.5 and a q-character formula.

In this section we prove Proposition 1.5 which is a recursive formula for a cluster variable. We also solve this recursions and give a closed formula for the cluster variable in terms of the initial cluster and the frozen variables. In view of Section 1.9 this formula can also be viewed as giving the q-character of $[\boldsymbol{\omega}]$, $\boldsymbol{\omega} \in \mathbf{Pr}_\xi$ in terms of the local Weyl modules and Kirillov-Reshetikhin modules.

2.1. — We briefly recall the definition (see [13]) of a cluster algebra. Let Q be a quiver with $(n+m)$-vertices labeled $\{1, \dots, n, 1', \dots, m'\}$ and assume that the set of edges has no loops or 2-cycles. A mutation of Q at a vertex i is the quiver obtained by performing the following three operations.

– reverse all edges at i,
– given edges $j \to i \to k$ add a new edge $j \to k$,
– remove any two cycles that may have been created.

We shall assume that mutation is never allowed at the vertices labeled $\{1', \dots, m'\}$; these are called the frozen vertices. Suppose that $\boldsymbol{x} = \{x_1, \dots, x_n, f_1, \dots, f_m\}$ is an algebraically independent set and let $\mathbb{Q}(\boldsymbol{x})$ be the field of rational functions in these variables. The set $\boldsymbol{x}$ is called the initial cluster and $(\boldsymbol{x}, Q)$ is called the initial seed.

Corresponding to a mutation of Q at a vertex i define a new cluster

$$\boldsymbol{x}' = \{x'_1, \ldots, x'_n, f_1, \ldots, f_m\}$$

by

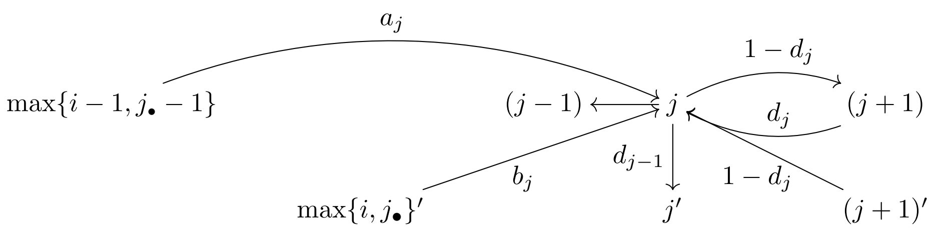
$$x'_j = x_j, \quad j \neq i, \quad x'_i x_i = \prod_{\substack{\exists\ j \to i \\ \text{in } Q}} f_j \prod_{\substack{\exists\ j \to i \\ \text{in } Q}} x_j + \prod_{\substack{\exists\ i \to k \\ \text{in } Q}} f_k \prod_{\substack{\exists\ i \to k \\ \text{in } Q}} x_k.$$

The new cluster again consists of algebraically independent elements and we have a new seed $(\boldsymbol{x}', Q')$ where Q' is the mutation of Q at i. Iterating this process defines a collection of new clusters and new seeds. An element of a given cluster is called a cluster variable. A cluster monomial is a product of cluster variables all belonging to the same cluster. The associated cluster algebra is the $\mathbb{Z}$ subring (of the field of rational function $\mathbb{Q}(x)$) generated by all the cluster variables.

2.2. The quiver $Q_\xi[i,j]$. — Given $1 \leqslant i \leqslant n-1$ set $Q_\xi = Q_\xi[i,j]$ if $j < i$ and let $Q_\xi[i,i]$ be obtained by mutating Q_ξ at i. Assume that we have defined $Q_\xi[i,j-1]$ for $j > i$ let $Q_\xi[i,j]$ be the quiver defined by mutating $Q_\xi[i,j-1]$ at j.

Proposition 1.5 is a simple inspection when $j = i$ and if $j > i$ then it is a consequence of the discussion in Section 2.1, the following lemma and an induction on $j - i$.

Lemma. — *Suppose that $j > i$ and that we have an arrow $(j-1) \to j$ in Q_ξ. In $Q_\xi[i, j-1]$ we have the following edges at the vertex j:*



where $a_j = 1 - \delta_{i,j_\bullet}$ and $b_j = \min\{1, (1-\delta_{j_\bullet, i_\bullet}) d_{j_\bullet - 1} + \delta_{j_\bullet, i}\}$.

Proof. — We proceed by induction on $j - i$. To see that induction begins when $j = i+1$ notice that

$$d_i = 1 \implies i = (i+1)_\bullet \implies a_{i+1} = 0, \quad b_{i+1} = 1,$$
$$d_i = 0 \implies i_\bullet = (i+1)_\bullet \implies a_{i+1} = 1, \quad b_{i+1} = 0.$$

On the other hand, in $Q_\xi[i,i]$ which is the mutation of Q_ξ at i, an inspection shows that the edges at $i+1$ are given as follows:

$$\begin{tikzcd}
i & (i+1) \arrow[l] \arrow[r, bend left, "1-d_{i+1}"] \arrow[d] & (i+2) \arrow[l, bend left, "d_{i+1}"] \\
i' \arrow[ur] & (i+1)' & (i+2)' \arrow[ul, "1-d_{i+1}"]
\end{tikzcd}\ , \quad d_i = 1,$$

$$\begin{tikzcd}
(i-1) \arrow[rr, bend left] & i & (i+1) \arrow[l] \arrow[r, bend left, "1-d_{i+1}"] & (i+2) \arrow[l, bend left, "d_{i+1}"] \\
 & & & (i+2)' \arrow[ul, "1-d_{i+1}"]
\end{tikzcd}\ , \quad d_i = 0,$$

and it follows that induction begins. For the inductive step we assume that the result holds for the edges at $j<n$ in $Q_\xi[i,j-1]$ for and prove that it holds for the node $j+1$ in $Q_\xi[i,j]$.

Case 1. — If $d_j = 1$ then j is a sink of Q_ξ by assumption and so we have an edge $(j+1)\to j$ in Q_ξ. Hence by the inductive hypothesis the edges at j and $(j+1)$ in $Q_\xi[i,j-1]$ are

$$\begin{tikzcd}
\max\{i-1, j_\bullet - 1\} \arrow[rr, bend left, "a_j"] & (j-1) & j \arrow[l] \arrow[d, "d_{j-1}"'] & (j+1) \arrow[l] \arrow[r, bend left, "d_{j+1}"] \arrow[dr, "1-d_{j+1}"] & (j+2) \arrow[l, bend left, "1-d_{j+1}"] \\
 & \max\{i, j_\bullet\}' \arrow[ur, "b_j"] & j' & (j+1)' \arrow[u] & (j+2)'.
\end{tikzcd}$$

Mutating at j we see that the edges at $(j+1)$ are

$$\begin{tikzcd}
(j-1) \arrow[r] & j \arrow[r] & (j+1) \arrow[ll, bend right] \arrow[dl, "d_{j-1}"'] \arrow[r, bend left, "d_{j+1}"] \arrow[dr, "1-d_{j+1}"] & (j+2) \arrow[l, bend left, "1-d_{j+1}"] \\
 & j' & (j+1)' \arrow[u] & (j+2)'
\end{tikzcd}\ .$$

The inductive step follows since $d_j = 1 \implies (j+1)_\bullet = j$ and so

$$\max\{i-1,(j+1)_\bullet - 1\} = j-1, \quad \max\{i,(j+1)_\bullet\}' = j', \quad a_{j+1} = 1 = d_j, \quad b_{j+1} = d_{j-1}.$$

Case 2. — If $d_j = 0$ or equivalently $j_\diamond \neq j$ then in Q_ξ we have an edge $j \to j+1$. By the induction hypothesis, the edges at j and $(j+1)$ in $Q_\xi[i, j-1]$ are

$$\begin{tikzcd}
\max\{i-1, j_\bullet - 1\} \arrow[rrr, bend left, "a_j"] & & (j-1) & j \arrow[l] \arrow[r] \arrow[d, "d_{j-1}"'] & (j+1) \arrow[r, bend left, "1-d_{j+1}"] \arrow[d, "1-d_{j+1}"] & (j+2) \arrow[l, bend left, "d_{j+1}"'] \\
& \max\{i, j_\bullet\}' \arrow[urr, "b_j"] & & j' & (j+1)' \arrow[ul] & (j+2)' \arrow[ul]
\end{tikzcd} .$$

Mutating at j we obtain

$$\begin{tikzcd}
\max\{i-1, j_\bullet - 1\} \arrow[rrr, bend left, "a_j"] & & j & (j+1) \arrow[l] \arrow[r, bend left, "1-d_{j+1}"] & (j+2) \arrow[l, bend left, "d_{j+1}"'] \\
& \max\{i, j_\bullet\}' \arrow[urr, "b_j"] & & & (j+2)' \arrow[ul, "1-d_{j+1}"]
\end{tikzcd}$$

The inductive step follows from the fact that $d_j = 0 \implies (j+1)_\bullet = j_\bullet < j$ and so

$$\max\{i-1, (j+1)_\bullet - 1\} = \max\{i-1, j_\bullet - 1\}, \quad \max\{i, (j+1)_\bullet\}' = \max\{i, j_\bullet\}',$$

and

$$a_{j+1} = a_j, \quad b_{j+1} = b_j, \quad d_j = 0.$$

The proof of the lemma is complete. $\square$

2.3. The set $\Gamma_{i,j}$. — We continue to set $d_m = \delta_{m,m_\diamond}$ for $1 \leqslant m \leqslant n$. For $i, j \in [1, n]$ define sets $\Gamma_{i,j}$ as follows: $\Gamma_{i,j} = \{0\}$ if $j < i$ and if $i \leqslant j$ then $\Gamma_{i,j}$ is the subset of $\mathbb{Z}_+^{j-i+2}$ consisting of elements $\boldsymbol{\varepsilon} = (\varepsilon_i, \ldots, \varepsilon_{j+1})$ satisfying the following conditions: for $r, m \in [i, j]$ with $r \leqslant m$ and $\sigma_{r,m}(\boldsymbol{\varepsilon}) = \varepsilon_r + \cdots + \varepsilon_m$, we have

$$\varepsilon_{j+1} = 1 + (d_j - 1)\sigma_{\max\{i, j_\bullet + 1\}, j}(\boldsymbol{\varepsilon}), \tag{2.1}$$

$$\sigma_{\max\{i, j_\bullet + 1\}, j}(\boldsymbol{\varepsilon}) \leqslant 1, \tag{2.2}$$

$$\sigma_{i, i_\diamond}(\boldsymbol{\varepsilon}) \leqslant 1 \leqslant \sigma_{i, i_\diamond + 1}(\boldsymbol{\varepsilon}) \quad \text{if} \quad i_\diamond \leqslant j, \tag{2.3}$$

$$\sigma_{m+1, (m+1)_\diamond}(\boldsymbol{\varepsilon}) \leqslant 1 \leqslant \sigma_{m+1, (m+1)_\diamond + 1}(\boldsymbol{\varepsilon}), \quad \text{if} \quad i_\diamond \leqslant m = m_\diamond < j_\bullet. \tag{2.4}$$

Clearly, $\varepsilon_m \in \{0, 1\}$ for $i \leqslant m \leqslant j+1$. For $i \leqslant j$ let

$$\Gamma^1_{i,j} = \{\boldsymbol{\varepsilon} \in \Gamma_{i,j} : \sigma_{\max\{i, j_\bullet + 1\}, j}(\boldsymbol{\varepsilon}) = 1\}, \quad \Gamma^0_{i,j} = \{\boldsymbol{\varepsilon} \in \Gamma_{i,j} : \sigma_{\max\{i, j_\bullet + 1\}, j}(\boldsymbol{\varepsilon}) = 0\}.$$

The condition in (2.2) shows that

$$\Gamma_{i,j} = \Gamma^1_{i,j} \sqcup \Gamma^0_{i,j}.$$

We shall use the following freely:

$$d_{m-1} = 0 \iff (m-1)_\bullet = m_\bullet, \quad d_{m-1} = 1 \iff m_\bullet = m - 1. \tag{2.5}$$

Lemma. — *For $j>i$ the assignments*

$$(\varepsilon_i,\dots,\varepsilon_j)\longrightarrow(\varepsilon_i,\dots,\varepsilon_j,d_j),$$
$$(\varepsilon_i,\dots,\varepsilon_{j_\bullet})\longrightarrow(\varepsilon_i+\delta_{i,j_\bullet},\dots,\varepsilon_{j_\bullet},0,\dots,0,1),$$

define bijections $\iota_{j-1}:\Gamma_{i,j-1}\to\Gamma^1_{i,j}$ and $\iota_{j_\bullet-1}:\Gamma_{i,j_\bullet-1}\to\Gamma^0_{i,j}$ respectively.

Proof. — For the first assertion of the lemma we must prove that

$$\widetilde{\boldsymbol{\varepsilon}}=(\varepsilon_i,\dots,\varepsilon_j)\in\Gamma_{i,j-1}\iff\boldsymbol{\varepsilon}=(\varepsilon_i,\dots,\varepsilon_j,d_j)\in\Gamma^1_{i,j}.$$

Clearly we have $\sigma_{m,r}(\boldsymbol{\varepsilon})=\sigma_{m,r}(\widetilde{\boldsymbol{\varepsilon}})$ for all $i\leqslant m\leqslant r\leqslant j$. Using (2.5) we see that

$$(*)\qquad \varepsilon_j=1+(d_{j-1}-1)\sigma_{\max\{i,(j-1)_\bullet+1\},j-1}(\widetilde{\boldsymbol{\varepsilon}})\iff\sigma_{\max\{i,j_\bullet+1\},j}(\boldsymbol{\varepsilon})=1.$$

It follows that $\widetilde{\boldsymbol{\varepsilon}}$ satisfies (2.1) if $\boldsymbol{\varepsilon}\in\Gamma^1_{i,j}$. It also proves that $\boldsymbol{\varepsilon}$ satisfies (2.1) and (2.2) if $\widetilde{\boldsymbol{\varepsilon}}\in\Gamma_{i,j-1}$. To see that $\widetilde{\boldsymbol{\varepsilon}}$ satisfies (2.2) if $\boldsymbol{\varepsilon}\in\Gamma^1_{i,j}$ we note that this is clear if $(j-1)_\bullet=j_\bullet$ and if $j_\bullet=j-1$ it follows from the fact that $\boldsymbol{\varepsilon}$ satisfies (2.4) with $m=(j-1)_\bullet$.

It is obvious that $\widetilde{\boldsymbol{\varepsilon}}$ satisfies (2.3) (resp. (2.4)) if $\boldsymbol{\varepsilon}\in\Gamma^1_{i,j}$; it is also obvious that $\boldsymbol{\varepsilon}$ satisfies these inequalities if $\widetilde{\boldsymbol{\varepsilon}}\in\Gamma_{i,j-1}$ as long as $i_\diamond\leqslant j-1$ (resp. $i_\diamond\leqslant m<(j-1)_\bullet$)). If $i_\diamond=j$ then $d_j=1$ and $j_\bullet=i_\bullet<i$. Using $(*)$ and the fact that we have already proved that $\boldsymbol{\varepsilon}$ satisfies (2.1) we get

$$\sigma_{\max\{i,j_\bullet+1\},j}(\boldsymbol{\varepsilon})=1\leqslant\max\sigma_{\max\{i,j_\bullet+1\},j+1}(\boldsymbol{\varepsilon})=2,$$

proving that (2.3) holds for $\boldsymbol{\varepsilon}$. If $(j-1)_\bullet\leqslant m=m_\diamond<j_\bullet$ then we must have $m=(j-1)_\bullet$, and $j_\bullet=j-1=(m+1)_\diamond$. It follows that $d_{j-1}=1$, $\varepsilon_j=1$ and so we have

$$\sigma_{(j-1)_\bullet+1,j-1}(\widetilde{\boldsymbol{\varepsilon}})=\sigma_{(j-1)_\bullet+1,j-1}(\boldsymbol{\varepsilon})\leqslant1=\varepsilon_j\leqslant\sigma_{(j-1)_\bullet+1,j}(\boldsymbol{\varepsilon}),$$

proving that $\boldsymbol{\varepsilon}$ satisfies (2.4). The proof of the first assertion is complete.

We prove the second assertion of the lemma; note that if $\boldsymbol{\varepsilon}\in\Gamma^0_{i,j}$ then we must have $\varepsilon_m=0$ for $j_\bullet+1\leqslant m\leqslant j$ and hence by (2.1) we also have $\varepsilon_{j+1}=1$. Since

$$j_\bullet\leqslant i\implies\Gamma_{i,j_\bullet-1}=\{0\}\quad\text{and}\quad\Gamma^0_{i,j}=\{(\delta_{i,j_\bullet},\dots,0,1)\},$$

the result is trivially true in this case. We assume from now on that $j_\bullet>i$ (in particular $j_\bullet\geqslant i_\diamond$) and let

$$\widetilde{\boldsymbol{\varepsilon}}=(\varepsilon_i,\dots,\varepsilon_{j_\bullet})\qquad\boldsymbol{\varepsilon}=(\varepsilon_i,\dots,\varepsilon_{j_\bullet},0,\dots,0,1).$$

Suppose that $\widetilde{\boldsymbol{\varepsilon}}\in\Gamma_{i,j_\bullet-1}$. It is obvious that $\boldsymbol{\varepsilon}$ satisfies (2.1) and (2.2) and (2.3) and for $i_\diamond\leqslant m<(j_\bullet-1)_\bullet$ that $\boldsymbol{\varepsilon}$ satisfies (2.4). If $(j_\bullet-1)_\bullet\leqslant m=m_\diamond\leqslant j_\bullet-1$ then either $m=(j_\bullet-1)_\bullet$ or $m=j_\bullet-1$. In the first case the first inequality in (2.4) for $\boldsymbol{\varepsilon}$ is just (2.2) for $\widetilde{\boldsymbol{\varepsilon}}$ while the second inequality follows from (2.1) for $\widetilde{\boldsymbol{\varepsilon}}$. If $m=m_\diamond=j_\bullet-1$, then (2.1) forces $\varepsilon_{j_\bullet}=1$ and hence we have $\varepsilon_{j_\bullet}\leqslant1\leqslant\varepsilon_{j_\bullet}+\varepsilon_{j_\bullet+1}$. This proves that (2.4) holds for $\boldsymbol{\varepsilon}$ and so $\boldsymbol{\varepsilon}\in\Gamma^0_{i,j}$.

Next we assume that $\boldsymbol{\varepsilon} \in \Gamma^0_{i,j}$ and prove that $\widetilde{\boldsymbol{\varepsilon}} \in \Gamma_{i,j_\bullet-1}$. To prove that (2.1) holds for $\widetilde{\boldsymbol{\varepsilon}}$ it suffices to observe that if $j_\bullet = i_\diamond$ (resp. $(j_\bullet)_\bullet \geqslant i_\diamond$) then (2.3) (resp. (2.4)) for $\boldsymbol{\varepsilon}$ gives

$$\sigma_{\max\{i,(j_\bullet)_\bullet+1\},j_\bullet}(\widetilde{\boldsymbol{\varepsilon}}) = \sigma_{\max\{i,(j_\bullet)_\bullet+1\},j_\bullet}(\boldsymbol{\varepsilon}) = 1.$$

If $d_{j_\bullet-1} = 1$ then $(j_\bullet)_\bullet = j_\bullet - 1$ and so the preceding equality is $\varepsilon_{j_\bullet} = 1$ as needed. If $d_{j_\bullet-1} = 0$ then $(j_\bullet - 1)_\bullet = (j_\bullet)_\bullet$ and again the preceding equality is a reformulation of (2.1) for $\widetilde{\boldsymbol{\varepsilon}}$. The fact that $\widetilde{\boldsymbol{\varepsilon}}$ satisfies (2.2) follows by using (2.3) for $\boldsymbol{\varepsilon}$ if $(j_\bullet)_\bullet < i_\diamond$ and using (2.4) for $\boldsymbol{\varepsilon}$ otherwise. It is clear that (2.3) and (2.4) hold for $\widetilde{\boldsymbol{\varepsilon}}$ since they are the same as the corresponding ones for $\boldsymbol{\varepsilon}$ and the proof of the lemma is complete. □

2.4. The sets $\Gamma'_{i,j}$. — For $i \leqslant j$ define a map

$$p_{ij} : \Gamma_{i,j} \longrightarrow \mathbb{Z}^{(j-i+2)}, \quad p_{i,j}(\varepsilon_i, \ldots, \varepsilon_{j+1}) = (\varepsilon'_i, \ldots, \varepsilon'_{j+1}),$$

as follows:

– $\varepsilon'_{j+1} = (1-d_j)\sigma_{\max\{i,j_\bullet+1\},j}(\boldsymbol{\varepsilon}) + d_j(1-\sigma_{\max\{i,j_\bullet+1\},j}(\boldsymbol{\varepsilon}))$,

– if $i_\bullet = m_\bullet$ or $\sigma_{\max\{i,(m_\bullet)_\bullet+1\},m_\bullet} = 1$ then,

$$\varepsilon'_m = \begin{cases} (d_m - 1)\varepsilon_{m+1} - d_m, & \sigma_{\max\{i,m_\bullet+1\},m}(\boldsymbol{\varepsilon}) = 0, \\ d_m - (\varepsilon_m + \varepsilon_{m+1}), & \sigma_{\max\{i,m_\bullet+1\},m}(\boldsymbol{\varepsilon}) = 1, \end{cases}$$

– if $m_\bullet \geqslant i$ and $\sigma_{\max\{i,(m_\bullet)_\bullet+1,m_\bullet\}}(\boldsymbol{\varepsilon}) = 0$ then $\varepsilon'_m = d_m(1-\varepsilon_{m+1})$.

It is easily seen that $\varepsilon'_m \in \{-1,0,1\}$ for $i \leqslant m \leqslant j$. For $i \leqslant j$ let $\Gamma'_{i,j}$ be the image of p_{ij} and set $\Gamma'_{i,j} = \{0\}$ if $i > j$.

Lemma. — *Let $1 \leqslant i \leqslant j \leqslant n$.*

(i) *If $\widetilde{\boldsymbol{\varepsilon}} = (\varepsilon_i, \ldots, \varepsilon_j) \in \Gamma_{i,j-1}$ then*

$$p_{ij-1}(\widetilde{\boldsymbol{\varepsilon}}) = (\varepsilon'_i, \ldots, \varepsilon'_j) \implies p_{ij}(\iota_{j-1}(\widetilde{\boldsymbol{\varepsilon}})) = (\varepsilon'_i, \ldots, \varepsilon'_{j-1}, -1+\varepsilon'_j, 1-d_j).$$

(ii) *If $\widetilde{\boldsymbol{\varepsilon}} = (\varepsilon_i, \ldots, \varepsilon_{j_\bullet}) \in \Gamma_{i,j_\bullet-1}$ then*

$$p_{ij_\bullet-1}(\widetilde{\boldsymbol{\varepsilon}}) = (\varepsilon'_i, \ldots, \varepsilon'_{j_\bullet}) \implies p_{ij}(\iota_{j_\bullet-1}(\widetilde{\boldsymbol{\varepsilon}})) = (\varepsilon'_i, \ldots, \varepsilon'_{j_\bullet}, 0\cdots, 0, -1, d_j).$$

Proof. — Let $\boldsymbol{\varepsilon} = (\varepsilon_i, \ldots, \varepsilon_j, d_j) = \iota_{i,j-1}(\widetilde{\boldsymbol{\varepsilon}})$ and let $p_{ij}(\boldsymbol{\varepsilon}) = (\varepsilon''_i, \ldots, \varepsilon''_{j+1})$. Since

$$\sigma_{m,r}(\widetilde{\boldsymbol{\varepsilon}}) = \sigma_{m,r}(\boldsymbol{\varepsilon}), \quad m \leqslant r \leqslant j,$$

it is clear from the definition that $\varepsilon'_m = \varepsilon''_m$ if $m \leqslant j-1$.

By Lemma 2.3 we have $\boldsymbol{\varepsilon} \in \Gamma^1_{i,j}$ and hence $\sigma_{\max\{i,j_\bullet+1\},j}(\boldsymbol{\varepsilon}) = 1$. It is immediate from the definition of p_{ij} that $\varepsilon''_{j+1} = 1 - d_j$. We now prove that $\varepsilon''_j = -1 + \varepsilon'_j$; using the definition of ε'_j this is equivalent to proving

$$\begin{aligned} \varepsilon''_j &= -1 + (1-d_{j-1})\sigma_{\max\{i,(j-1)_\bullet+1\},j-1}(\widetilde{\boldsymbol{\varepsilon}}) \\ &\qquad\qquad + d_{j-1}(1-\sigma_{\max\{i,(j-1)_\bullet+1,\},j-1}(\widetilde{\boldsymbol{\varepsilon}})) \\ &= -1 + (1-d_{j-1})\sigma_{\max\{i,j_\bullet+1\},j-1}(\boldsymbol{\varepsilon}) + d_{j-1}(1-\sigma_{\max\{i,(j_\bullet)_\bullet+1\},j_\bullet}(\boldsymbol{\varepsilon})). \end{aligned} \tag{2.6}$$

If $j_\bullet \geqslant i$ and $\sigma_{\max\{i,(j_\bullet)_\bullet+1\},j_\bullet}(\boldsymbol{\varepsilon}) = 0$ then by (2.4) we have $\varepsilon_{j_\bullet+1} = 1$ and so $\sigma_{j_\bullet+1,j-1}(\boldsymbol{\varepsilon}) = 1$. This means that the right hand side of (2.6) is zero. Since by definition $\varepsilon''_j = d_j(1-d_j) = 0$ the result is proved in this case.

If $i_\bullet = j_\bullet$ then $d_{j-1} = 0$ and since $\sigma_{\max\{i,j_\bullet+1\},j}(\boldsymbol{\varepsilon}) = 1$ it follows that the right hand side of (2.6) is $-\varepsilon'_j$ which is precisely the value of ε''_j in this case.

Suppose that $\sigma_{\max\{i,(j_\bullet)_\bullet+1\},j_\bullet}(\boldsymbol{\varepsilon}) = 1$ and that $j_\bullet \geqslant i$. This means that the second term on the right hand side of (2.6) is zero. Since $\sigma_{\max\{i,j_\bullet+1\},j}(\boldsymbol{\varepsilon}) = 1$ by definition we have $\varepsilon''_j = -\varepsilon_j$. Recalling that $\varepsilon_j = 1 + (d_{j-1} - 1)\sigma_{\max\{i,(j-1)_\bullet+1\},j-1}$ we see that the right hand side of (2.6) is also $-\varepsilon'_j$. The proof of part (i) is now complete.

We prove part (ii). Let
$$\boldsymbol{\varepsilon} = \iota_{i,j_\bullet-1}(\widetilde{\boldsymbol{\varepsilon}}) \in \Gamma^0_{i,j}, \quad p_{ij}(\boldsymbol{\varepsilon}) = (\varepsilon''_i, \ldots, \varepsilon''_{j+1}).$$
Since $\sigma_{m,r}(\widetilde{\boldsymbol{\varepsilon}}) = \sigma_{m,r}(\boldsymbol{\varepsilon})$ for all $m \leqslant r \leqslant j_\bullet - 1$ it is clear from the definition that $\varepsilon'_m = \varepsilon''_m$ if $m \leqslant j_\bullet - 1$. Since $d_m = 0$ if $j_\bullet + 1 \leqslant m \leqslant j - 1$ a simple inspection also shows that
$$\varepsilon''_m = 0, \quad j_\bullet + 1 \leqslant m \leqslant j - 1, \quad \varepsilon''_{j+1} = d_j.$$
It remains to prove that $\varepsilon''_{j_\bullet} = \varepsilon'_{j_\bullet}$ and that $\varepsilon''_j = -1$.

If $j_\bullet = i_\bullet$ then $\widetilde{\boldsymbol{\varepsilon}} = \{0\}$, $\Gamma_{i,j_\bullet-1} = \{0\}$ and $\boldsymbol{\varepsilon} = (0, \ldots, 0, 1)$. By definition $p_{i,j}(\boldsymbol{\varepsilon}) = (0, \ldots, 0, -1, d_j)$ and we are done in this case. If $i = j_\bullet$ then $\Gamma_{i,j_\bullet-1} = \{0\}$ and $\boldsymbol{\varepsilon} = (1, 0, \ldots, 1)$. and one checks easily that $\varepsilon'_i = 0$. On the other hand, by definition $\varepsilon''_i = d_i - (\delta_{i,j_\bullet} + \varepsilon_{i+1}) = 0$. The fact that $\varepsilon''_j = -1$ is a straightforward checking from the definition.

Suppose that $j_\bullet > i$. Since $\varepsilon_{j_\bullet+1} = 0$ we have $\sigma_{\max\{i,j_\bullet+1\},j}(\boldsymbol{\varepsilon}) = 0$ and by using (2.4) that $\sigma_{\max\{i,(j_\bullet)_\bullet+1\},j_\bullet}(\boldsymbol{\varepsilon}) = 1$. Since $\varepsilon_{j+1} = 1$ it follows by definition that $\varepsilon''_j = -1$ as needed.

Finally, to show $\varepsilon''_{j_\bullet} = \varepsilon'_{j_\bullet}$, we write $m = j_\bullet$ and see that we must prove
$$\begin{aligned}\varepsilon''_m &= (1 - d_{m-1})\sigma_{\max\{i,(m-1)_\bullet+1\},m-1}(\widetilde{\boldsymbol{\varepsilon}}) \\ &\qquad\qquad + d_{m-1}(1 - \sigma_{\max\{i,(m-1)_\bullet+1\},m-1}(\widetilde{\boldsymbol{\varepsilon}})) \\ &= (1 - d_{m-1})\sigma_{\max\{i,m_\bullet+1\},m-1}(\boldsymbol{\varepsilon}) + d_{m-1}(1 - \sigma_{\max\{i,(m_\bullet)_\bullet+1\},m_\bullet}(\boldsymbol{\varepsilon})) \\ &= (1 - d_{m-1})(1 - \varepsilon_m) + d_{m-1}(1 - \sigma_{\max\{i,(m_\bullet)_\bullet+1\},m_\bullet}(\boldsymbol{\varepsilon})).\end{aligned} \tag{2.7}$$
If $\sigma_{\max\{i,(m_\bullet)_\bullet+1\},m_\bullet}(\boldsymbol{\varepsilon})) = 1$ then $\varepsilon''_m = 1 - \varepsilon_m$ by definition. By (2.4) we have $\varepsilon_m = 1$ if $d_{m-1} = 1$ and hence $(1 - \varepsilon_m) = (1 - d_{m-1})(1 - \varepsilon_m)$ and (2.7) is proved. If $\sigma_{\max\{i,(m_\bullet)_\bullet+1\},m_\bullet}(\boldsymbol{\varepsilon}) = 0$ then by definition $\varepsilon''_m = 1$. Hence we must prove that
$$1 = (1 - d_{m-1})(1 - \varepsilon_m) + d_{m-1}.$$
If $d_{m-1} = 1$ this is clear from the preceding computation. If $d_{m-1} = 0$ then $m_\bullet + 1 < m$ and (2.4) forces $\varepsilon_{m_\bullet+1} = 1$; in particular it follows that $\varepsilon_m = 0$ and (2.7) and is completely proved. □

2.5

Proposition. — *For $1 \leqslant i \leqslant j \leqslant n$ we have*
$$x[\alpha_{i,j}] = \sum_{\boldsymbol{\varepsilon} \in \Gamma_{i,j}} f^{\boldsymbol{\varepsilon}}_{i,j} m^{\boldsymbol{\varepsilon}}_{i,j},$$
where
$$m^{\boldsymbol{\varepsilon}}_{i,j} = x^{1-\varepsilon_i}_{i-1} x^{\varepsilon'_i}_i \cdots x^{\varepsilon'_j}_j x^{\varepsilon'_{j+1}}_{j+1}, \quad f^{\boldsymbol{\varepsilon}}_{i,j} = f^{\varepsilon_i}_i \cdots f^{\varepsilon_j}_j f^{(1-d_j)\varepsilon_{j+1}}_{j+1},$$
with $\boldsymbol{\varepsilon} = (\varepsilon_i, \ldots, \varepsilon_{j+1})$ and $p_{i,j}(\boldsymbol{\varepsilon}) = (\varepsilon'_i, \ldots, \varepsilon'_{j+1})$.

Proof. — The proof of the proposition proceeds by an induction on $j-i$. To see that induction begins recall from Proposition 1.5 that

$$x[\alpha_i]=\delta_{i,i_\diamond}\left(f_ix_i^{-1}+x_{i-1}x_{i+1}x_i^{-1}\right)+(1-\delta_{i,i_\diamond})\big(x_i^{-1}\left(f_ix_{i+1}+x_{i-1}f_{i+1}\right)\big).$$

Since

$$i=i_\diamond\implies\Gamma_{i,i}=\{(1,1),(0,1)\},\quad \Gamma'_{i,i}=\{(-1,0),(-1,1)\}$$

and

$$i\neq i_\diamond\implies\Gamma_{i,i}=\{(0,1),(1,0)\},\quad \Gamma'_{i,i}=\{(-1,0),(-1,1)\}$$

we see that induction begins. For the inductive step, Proposition 1.5 asserts that

$$\begin{aligned}x_jx[\alpha_{i,j}]&=f_j^{d_{j-1}}x[\alpha_{i,j-1}]x_{j+1}^{1-d_j}\\&\quad+f_{j+1}^{1-d_j}x_{j+1}^{d_j}\Big((\delta_{i_\bullet,j_\bullet}+\delta_{i,j_\bullet})f_i^{\delta_{i,j_\bullet}}x_{i-1}^{1-\delta_{i,j_\bullet}}+(1-\delta_{i_\bullet,j_\bullet}-\delta_{i,j_\bullet})f_{j_\bullet}^{d_{j_\bullet}-1}x[\alpha_{i,j_\bullet-1}]\Big).\end{aligned}$$

Let $\widetilde{\boldsymbol{\varepsilon}}=(\varepsilon_i,\dots,\varepsilon_j)\in\Gamma_{i,j-1}$. By Lemmas 2.3 and 2.4 we have

$$m_{i,j}^{\boldsymbol{\varepsilon}}f_{i,j}^{\boldsymbol{\varepsilon}}=m_{i,j-1}^{\widetilde{\boldsymbol{\varepsilon}}}f_{i,j}^{\widetilde{\boldsymbol{\varepsilon}}}f_j^{d_{j-1}}x_{j+1}^{1-d_j}x_j^{-1},\quad \boldsymbol{\varepsilon}=\iota_{i,j-1}(\widetilde{\boldsymbol{\varepsilon}}),$$

once we notice that $(1-d_{j-1})\varepsilon_j+d_{j-1}=\varepsilon_j$. Hence using the inductive hypothesis we get

$$f_j^{d_{j-1}}x[\alpha_{i,j-1}]x_{j+1}^{1-d_j}x_j^{-1}=\sum_{\boldsymbol{\varepsilon}\in\Gamma^1_{i,j}}f_{i,j}^{\boldsymbol{\varepsilon}}m_{i,j}^{\boldsymbol{\varepsilon}}.$$

Similarly, let $\widetilde{\boldsymbol{\varepsilon}}=(\varepsilon_i,\dots,\varepsilon_{\max\{i,j_\bullet+1\}})\in\Gamma_{i,j_\bullet-1}$ and $\boldsymbol{\varepsilon}=\iota_{i,j_\bullet-1}(\widetilde{\boldsymbol{\varepsilon}})$. Then $\varepsilon_{j+1}=1-d_j$ by definition and $1-\varepsilon_i=1-\delta_{i,j_\bullet}$ if $j_\bullet\leqslant i$ and we get

$$m_{i,j}^{\boldsymbol{\varepsilon}}f_{i,j}^{\boldsymbol{\varepsilon}}=\begin{cases}f_{j+1}^{1-d_j}f_i^{\delta_{i,j_\bullet}}x_{i-1}^{1-\delta_{i,j_\bullet}}x_j^{-1}x_{j+1}^{d_j}, & j_\bullet\leqslant i\\ f_{i,j_\bullet-1}^{\widetilde{\boldsymbol{\varepsilon}}}f_{j+1}^{1-d_j}f_{j_\bullet}^{d_\bullet-1}m_{i,j_\bullet-1}^{\widetilde{\boldsymbol{\varepsilon}}}x_j^{-1}x_{j+1}^{d_j}, & i<j_\bullet.\end{cases}$$

The inductive step follows from the inductive hypothesis and the fact that $\Gamma_{i,j}=\Gamma^0_{i,j}\sqcup\Gamma^1_{i,j}$. □

3. Irreducible tensor products.

In this section we give a sufficient condition (see Section 3.6) for the equality $[\boldsymbol{\pi}_1][\boldsymbol{\pi}_2]=[\boldsymbol{\pi}_1\boldsymbol{\pi}_2]$ to hold when $\boldsymbol{\pi}_1,\boldsymbol{\pi}_2\in\mathbf{Pr}_\xi$. We shall see in later sections that the conditions are necessary as well. We shall often need to work in the monoidal category $\mathscr{F}_\xi$ rather than its Grothendieck ring; by abuse of notation we shall use the symbol $[\boldsymbol{\omega}]$ to also denote an irreducible module in $\mathscr{F}_\xi$ with label $\boldsymbol{\omega}$. To emphasize that we are working in the category we shall write $[\boldsymbol{\omega}]\otimes[\boldsymbol{\omega}']$ for the tensor product of the corresponding objects.

3.1. — We collect some well-known facts on the category $\mathscr{F}_\xi$. An object of $\mathscr{F}_\xi$ is said to be *ℓ-highest weight with highest weight* $\boldsymbol{\omega}$ if it has $[\boldsymbol{\omega}]$ as its unique irreducible quotient. Clearly any quotient of an ℓ-highest weight module is also ℓ-highest weight with the same irreducible quotient. Given $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathscr{P}_\xi^+$ the module $[\boldsymbol{\omega}_1\boldsymbol{\omega}_2]$ occurs in the Jordan-Holder series of $[\boldsymbol{\omega}_1]\otimes[\boldsymbol{\omega}_2]$ with multiplicity one. In particular if $[\boldsymbol{\omega}_1]\otimes[\boldsymbol{\omega}_2]$ is an ℓ-highest weight module then $[\boldsymbol{\omega}_1\boldsymbol{\omega}_2]$ is its unique irreducible quotient and hence $[\boldsymbol{\omega}_1] \otimes [\boldsymbol{\omega}_2]$ is irreducible iff $[\boldsymbol{\omega}_1] \otimes [\boldsymbol{\omega}_2] \cong [\boldsymbol{\omega}_2] \otimes [\boldsymbol{\omega}_1] \cong [\boldsymbol{\omega}_1\boldsymbol{\omega}_2]$.

The following results from [16], [17] play an important role in this section.

Theorem 2. — *Let $\boldsymbol{\omega}_s \in \mathscr{P}_\xi^+$ for $1 \leqslant s \leqslant r$. Then $[\boldsymbol{\omega}_1] \otimes \cdots \otimes [\boldsymbol{\omega}_s]$ is ℓ-highest weight if every pair $[\boldsymbol{\omega}_s] \otimes [\boldsymbol{\omega}_p]$ with $1 \leqslant s < p \leqslant r$ is ℓ-highest weight. Moreover if $[\boldsymbol{\omega}_s] \otimes [\boldsymbol{\omega}_p] \cong [\boldsymbol{\omega}_s\boldsymbol{\omega}_p]$ for all $1 \leqslant s < p \leqslant r$ then*

$$[\boldsymbol{\omega}_1] \otimes \cdots \otimes [\boldsymbol{\omega}_r] \cong [\boldsymbol{\omega}_1 \cdots \boldsymbol{\omega}_r]. \qquad \square$$

3.2. — Given $\boldsymbol{\omega}_{i,a}, \boldsymbol{\omega}_{j,b} \in \mathscr{P}_\xi^+$ it was shown in [5] that the module $[\boldsymbol{\omega}_{i,a}] \otimes [\boldsymbol{\omega}_{j,b}]$ is ℓ-highest weight (resp. irreducible) if

$$(b-a) \notin \{2p+2-i-j : \max\{i,j\} < p+1 \leqslant \min\{n+1, i+j\},$$

$$(\text{resp.} \quad \pm(b-a) \notin \{2p+2-i-j : \max\{i,j\} < p+1 \leqslant \min\{n+1, i+j\}).$$

The next proposition is a simple calculation using the preceding criterion and the fact that $|\xi(j)-\xi(i)| \leqslant |j-i|$ for all $i,j \in [1,n]$.

Proposition. — *Let $\boldsymbol{\omega}_{i,a}, \boldsymbol{\omega}_{j,b} \in \mathscr{P}_\xi^+$. Then $[\boldsymbol{\omega}_{i,a}] \otimes [\boldsymbol{\omega}_{j,b}]$ (resp. $[\boldsymbol{\omega}_{j,b}] \otimes [\boldsymbol{\omega}_{i,a}]$) is ℓ-highest weight if $a = \xi(i)+1$ (resp. $a = \xi(i)-1$). Moreover,*

$$\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,b} \notin \mathbf{Pr}_\xi \cup \{\boldsymbol{f}_i\} \implies [\boldsymbol{\omega}_{i,a}] \otimes [\boldsymbol{\omega}_{j,b}] \cong [\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,b}]. \qquad \square$$

3.3. — Let ξ^* be the height function defined by $\xi^*(i) = \xi(n+1-i)$. The assignment

$$\boldsymbol{\omega}_{i,\xi(i)\pm 1} \longrightarrow \boldsymbol{\omega}_{n+1-i,\xi^*(n+1-i)\pm 1}$$

extends to an isomorphism $\mathscr{P}_\xi^+ \cong \mathscr{P}_{\xi^*}^+$, and if $\boldsymbol{\omega} = \boldsymbol{\omega}_{i_1,a_1} \cdots \boldsymbol{\omega}_{i_k,a_k} \in \mathscr{P}_\xi^+$ we set

$$\boldsymbol{\omega}^* = \boldsymbol{\omega}_{n+1-i_1,a_1} \cdots \boldsymbol{\omega}_{n+1-i_k,a_k} \in \mathscr{P}_{\xi^*}^+.$$

It was proved in [7] that if $[\boldsymbol{\omega}_1] \otimes [\boldsymbol{\omega}_2]$ and $[\boldsymbol{\omega}_2^*] \otimes [\boldsymbol{\omega}_1^*]$ are both ℓ-highest weight then they are both irreducible with the converse being trivially true.

Say that an ordered triple of elements $(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3)$ from $\mathscr{P}_\xi^+$ is ξ-admissible if:

– $[\boldsymbol{\omega}_s] \otimes [\boldsymbol{\omega}_3]$ is irreducible for $s = 1, 2$,
– either $[\boldsymbol{\omega}_1] \otimes [\boldsymbol{\omega}_2]$ or $[\boldsymbol{\omega}_2] \otimes [\boldsymbol{\omega}_1]$ is ℓ-highest weight.

Lemma. — *If $(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3)$ is ξ-admissible and $(\boldsymbol{\omega}_1^*, \boldsymbol{\omega}_2^*, \boldsymbol{\omega}_3^*)$ is ξ^*-admissible then $[\boldsymbol{\omega}_1\boldsymbol{\omega}_2] \otimes [\boldsymbol{\omega}_3] \cong [\boldsymbol{\omega}_1\boldsymbol{\omega}_2\boldsymbol{\omega}_3]$.*

Proof. — Suppose that $(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3)$ is ξ-admissible and that $[\boldsymbol{\omega}_1]\otimes[\boldsymbol{\omega}_2]$ is ℓ-highest weight. Then Theorem 2 shows that the modules $[\boldsymbol{\omega}_1]\otimes[\boldsymbol{\omega}_2]\otimes[\boldsymbol{\omega}_3]$ and $[\boldsymbol{\omega}_3]\otimes[\boldsymbol{\omega}_1]\otimes[\boldsymbol{\omega}_2]$ are ℓ-highest weight. Hence the corresponding quotients $[\boldsymbol{\omega}_1\boldsymbol{\omega}_2]\otimes[\boldsymbol{\omega}_3]$ and $[\boldsymbol{\omega}_3]\otimes[\boldsymbol{\omega}_1\boldsymbol{\omega}_2]$ are ℓ-highest weight. Working with ξ^* we see similarly that $[\boldsymbol{\omega}_1^*\boldsymbol{\omega}_2^*] \otimes [\boldsymbol{\omega}_3^*]$ and

$[\boldsymbol{\omega}_3^*]\otimes[\boldsymbol{\omega}_1^*\boldsymbol{\omega}_2^*]$ are ℓ-highest weight. The irreducibility of the four quotient modules follows from the discussion preceding the statement of the lemma. $\square$

3.4. — Recall the map $\mathrm{wt} : \mathscr{P}_\xi^+ \to P^+$ given by extending $\mathrm{wt}\,\boldsymbol{\omega}_{i,a} = \omega_i$ to a morphism of monoids; for $\boldsymbol{\pi} \in \mathscr{P}_\xi^+$ with $\mathrm{wt}\,\boldsymbol{\pi} = \sum_{i=1}^n r_i\omega_i$, set

$$\mathrm{ht}\,\boldsymbol{\pi} = \sum_{i=1}^n r_i, \quad \min\boldsymbol{\pi} = \min\{i \in I : r_i \neq 0\}, \quad \max\boldsymbol{\pi} = \max\{i \in I : r_i \neq 0\}.$$

If $\boldsymbol{\pi} \in \mathbf{Pr}_\xi$ and $b \in \{\xi(j)+1, \xi(j)-1\}$, $j \in [1,n]$ are such that $\boldsymbol{\omega}_{j,b}^{-1}\boldsymbol{\pi} \in \mathbf{Pr}_\xi$ then $j \in \{\min\boldsymbol{\pi}, \max\boldsymbol{\pi}\}$.

Proposition. — *Let ξ be an arbitrary height function and let $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathbf{Pr}_\xi$.*

(i) *Suppose that $\boldsymbol{\omega}\boldsymbol{\omega}' \in \mathbf{Pr}_\xi$ and write $\boldsymbol{\omega} = \boldsymbol{\omega}_1\boldsymbol{\omega}_{i,a}$ with $\boldsymbol{\omega}_1 \in \mathbf{Pr}_\xi$ and $\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}' \in \mathbf{Pr}_\xi$. If $a = \xi(i)+1$ then $[\boldsymbol{\omega}]\otimes[\boldsymbol{\omega}']$ is ℓ-highest weight and otherwise $[\boldsymbol{\omega}']\otimes[\boldsymbol{\omega}]$ is ℓ-highest weight.*

(ii) *If $i = \max\boldsymbol{\omega} < j = \min\boldsymbol{\omega}'$ and $|\xi(i)-\xi(j)| \neq j-i$, then the module $[\boldsymbol{\omega}]\otimes[\boldsymbol{\omega}']$ is irreducible.*

Proof. — The proof of both parts is by an induction on $p = \mathrm{ht}\,\boldsymbol{\omega} + \mathrm{ht}\,\boldsymbol{\omega}'$ with Proposition 3.2 showing that induction begins when $p = 2$. For the inductive step, assume that we have proved both parts for $p' < p$ and also assume without loss of generality that $\mathrm{ht}\,\boldsymbol{\omega}' \geqslant 2$.

For part (i) write

$$\boldsymbol{\omega}' = \boldsymbol{\omega}_{j,b}\boldsymbol{\omega}'', \quad \text{with} \quad \boldsymbol{\omega}'' \in \mathbf{Pr}_\xi,\ \boldsymbol{\omega}\boldsymbol{\omega}_{j,b} \in \mathbf{Pr}_\xi$$

and observe that if $k \in \{\min\boldsymbol{\omega}'', \max\boldsymbol{\omega}''\}$ then $|\xi(i)-\xi(k)| \neq |k-i|$. The inductive hypothesis applies to the pairs $(\boldsymbol{\omega}_{j,b}, \boldsymbol{\omega}'')$ and to $(\boldsymbol{\omega}, \boldsymbol{\omega}_{j,b})$; hence either both of $[\boldsymbol{\omega}_{j,b}]\otimes[\boldsymbol{\omega}'']$ and $[\boldsymbol{\omega}_{j,b}]\otimes[\boldsymbol{\omega}]$ or both of $[\boldsymbol{\omega}'']\otimes[\boldsymbol{\omega}_{j,b}]$ and $[\boldsymbol{\omega}]\otimes[\boldsymbol{\omega}_{j,b}]$ are ℓ-highest weight. The inductive hypothesis from part (ii) applies to the pair $(\boldsymbol{\omega}, \boldsymbol{\omega}'')$ and so $[\boldsymbol{\omega}]\otimes[\boldsymbol{\omega}'']$ is irreducible. It follows from Theorem 2 that the module $[\boldsymbol{\omega}_{j,b}]\otimes[\boldsymbol{\omega}'']\otimes[\boldsymbol{\omega}]$ is ℓ-highest weight (or $[\boldsymbol{\omega}]\otimes[\boldsymbol{\omega}'']\otimes[\boldsymbol{\omega}_{j,b}]$ is ℓ-highest weight). Hence the quotient $[\boldsymbol{\omega}']\otimes[\boldsymbol{\omega}]$ (or $[\boldsymbol{\omega}]\otimes[\boldsymbol{\omega}']$) is ℓ-highest weight and the inductive step for (i) is proved.

For part (ii) we continue to write $\boldsymbol{\omega}' = \boldsymbol{\omega}_{j,b}\boldsymbol{\omega}''$ and observe that the inductive hypothesis applies to the pairs $(\boldsymbol{\omega}, \boldsymbol{\omega}_{j,b})$ and $(\boldsymbol{\omega}, \boldsymbol{\omega}'')$ and gives that $[\boldsymbol{\omega}]\otimes[\boldsymbol{\omega}_{j,b}]$ and $[\boldsymbol{\omega}]\otimes[\boldsymbol{\omega}'']$ are irreducible. Since the inductive step has been proved for part (i) it applies to the pair $(\boldsymbol{\omega}_{j,b}, \boldsymbol{\omega}'')$ and so we see that $(\boldsymbol{\omega}'', \boldsymbol{\omega}_{j,b}, \boldsymbol{\omega})$ is ξ-admissible. The conditions of the proposition obviously hold for ξ iff they hold for ξ^*; hence it follows from Lemma 3.3 that $[\boldsymbol{\omega}]\otimes[\boldsymbol{\omega}']$ is irreducible. $\square$

3.5. — The next proposition is essential to prove our main result.

Proposition. — *Suppose that $1 \leqslant j_1 \leqslant j_2 \leqslant j_3 \leqslant j_4 \leqslant n$ are such that*

$$|\xi(j_s) - \xi(j_{s+1})| = j_{s+1} - j_s$$

for all $1 \leqslant s \leqslant 3$.

(i) *Let* $j_1 < j_2$ *and* $\boldsymbol{\omega}_{j_3,a} \in \mathbf{Pr}_\xi$. *The module* $[\boldsymbol{\omega}(j_1,j_2)] \otimes [\boldsymbol{\omega}_{j_3,a}]$ *is irreducible if*
$$\boldsymbol{\omega}(j_1,j_2)\boldsymbol{\omega}_{j_3,a} \notin \{\boldsymbol{\omega}(j_1,j_3), \quad \boldsymbol{f}_3\boldsymbol{\omega}_{j_1,\xi(j_1)\pm 1}\}.$$
An analogous statement holds if $j_2 < j_3$ *and* $\boldsymbol{\omega}_{j_1,a} \in \mathbf{Pr}_\xi$.

(ii) *Let* $j_1 < j_2 < j_3$. *The following modules are irreducible :*
- $[\boldsymbol{\omega}(j_1,j_2)] \otimes [\boldsymbol{\omega}(j_1,j_2)]$,
- $[\boldsymbol{\omega}(j_1,j_2)] \otimes [\boldsymbol{\omega}(j_1,j_3)]$ *if* $\xi(j_2-1) \neq \xi(j_2+1)$,
- $[\boldsymbol{\omega}(j_1,j_2)] \otimes [\boldsymbol{\omega}(j_2,j_3)]$ *if* $\xi(j_2-1) = \xi(j_2+1)$.

(iii) *Assume that* $j_1 < j_2 < j_3 < j_4$. *Then the following are irreducible:*
- $[\boldsymbol{\omega}(j_1,j_4)] \otimes [\boldsymbol{\omega}(j_2,j_3)]$ *if* $|\xi(j_4)-\xi(j_1)| = j_4 - j_1$,
- $[\boldsymbol{\omega}(j_1,j_2)] \otimes [\boldsymbol{\omega}(j_3,j_4)]$ *if* $\boldsymbol{\omega}(j_1,j_2)\boldsymbol{\omega}(j_3,j_4) \neq \boldsymbol{\omega}(j_1,j_4)$.

Proof. — Part (i) was proved in [18] if $|\xi(j_3)-\xi(j_1)| = j_3 - j_1$. If $|\xi(j_3)-\xi(j_1)| \neq j_3 - j_1$writing $\boldsymbol{\omega}(j_1,j_2) = \boldsymbol{\omega}_{j_1,a_1}\boldsymbol{\omega}_{j_2,a_2}$, our assumptions force,
$$j_2 \neq j_3, \quad \xi(j_2-1) = \xi(j_2+1), \quad \boldsymbol{\omega}_{j_2,a_2}\boldsymbol{\omega}_{j_3,a} \neq \boldsymbol{\omega}(j_2,j_3).$$
Proposition 3.2 now shows that $(\boldsymbol{\omega}_{j_1,a_1}, \boldsymbol{\omega}_{j_2,a_2}, \boldsymbol{\omega}_{j_3,a})$ is a ξ-admissible triple. It also proves that $(\boldsymbol{\omega}_{n+1-j_1,a_1}, \boldsymbol{\omega}_{n+1-j_2,a_2}, \boldsymbol{\omega}_{n+1-j_3,a})$ is ξ^*-admissible and the hence Lemma 3.3 gives the result. The proof of the analogous statement for $\boldsymbol{\omega}_{j_1,a}$ is entirely similar.

The first two assertions in part (ii) were proved in [18]. Suppose that $j_2 < j_3$ and $\xi(j_2-1) = \xi(j_2+1)$ and write
$$\boldsymbol{\omega}(j_1,j_2) = \boldsymbol{\omega}_{j_1,a_1}\boldsymbol{\omega}_{j_2,a_2}, \quad \boldsymbol{\omega}(j_2,j_3) = \boldsymbol{\omega}_{j_2,a_2}\boldsymbol{\omega}_{j_3,a_3}.$$
Then
$$a_1 = \xi(j_1) \pm 1 \iff a_2 = \xi(j_2) \mp 1, \quad a_3 = \xi(j_3) \pm 1.$$
Assuming that $a_1 = \xi(j_1)+1$ we use Proposition 3.4, Theorem 2 and part (i) of this proposition to see that
$$[\boldsymbol{\omega}_{j_3,a_3}] \otimes [\boldsymbol{\omega}_{j_2,a_2}] \otimes [\boldsymbol{\omega}(j_1,j_2)]$$
is ℓ-highest weight and hence so is the quotient $[\boldsymbol{\omega}(j_2,j_3)] \otimes [\boldsymbol{\omega}(j_1,j_2)]$. Similarly, working with
$$[\boldsymbol{\omega}_{j_1,a_1}] \otimes [\boldsymbol{\omega}_{j_2,a_2}] \otimes [\boldsymbol{\omega}(j_2,j_3)],$$
we see that $[\boldsymbol{\omega}(j_1,j_2)] \otimes [\boldsymbol{\omega}(j_2,j_3)]$ is ℓ-highest weight. Repeating the argument with ξ^* proves the irreducibility and proves the third assertion of part (ii).

The first assertion in (iii) was proved in [18]. If $|\xi(j_4)-\xi(j_1)| \neq j_4 - j_1$ then either $\xi(j_2-1) = \xi(j_2+1)$ or $\xi(j_3-1) = \xi(j_3+1)$. Write
$$\boldsymbol{\omega}(j_1,j_2) = \boldsymbol{\omega}_{j_1,a_1}\boldsymbol{\omega}_{j_2,a_2}, \qquad \boldsymbol{\omega}(j_3,j_4) = \boldsymbol{\omega}_{j_3,a_3}\boldsymbol{\omega}_{j_4,a_4},$$
and observe that since $j_2 < j_3$ and $\boldsymbol{\omega}(j_1,j_2)\boldsymbol{\omega}(j_3,j_4) \neq \boldsymbol{\omega}(j_1,j_4)$ we get
$$(a_1,a_2) = (\xi(j_1) \mp 1, \xi(j_2) \pm 1) \iff (a_3,a_4) = (\xi(j_3) \pm 1, \xi(j_4) \mp 1).$$
If $\xi(j_3-1) = \xi(j_3+1)$ then $|\xi(j_4)-\xi(j_2)| \neq (j_4 - j_2)$. Using parts (i) and (ii) of the current proposition and Proposition 3.4 we see that $(\boldsymbol{\omega}_{j_3,a_3}, \boldsymbol{\omega}_{j_4,a_4}, \boldsymbol{\omega}(j_1,j_2))$ is ξ-admissible. If $\xi(j_2-1) = \xi(j_2+1)$ then an identical argument shows that

$(\boldsymbol{\omega}_{j_1,a_1},\boldsymbol{\omega}_{j_2,a_2},\boldsymbol{\omega}(j_3,j_4))$ is ξ-admissible. Since the analogous equalities hold for ξ^* Lemma 3.3 now proves the irreducibility of $[\boldsymbol{\omega}(j_1,j_2)]\otimes[\boldsymbol{\omega}(j_3,j_4)]$. $\square$

3.6. — In the rest of the section we shall prove the following theorem.

Theorem 3

(a) *For all* $\boldsymbol{\pi}\in\mathbf{Pr}_\xi$ *and* $k\in[1,n]$ *we have* $[\boldsymbol{\pi}][\boldsymbol{f}_k]=[\boldsymbol{\pi}\boldsymbol{f}_k]$.

(b) *Let* $\boldsymbol{\pi}_1,\boldsymbol{\pi}_2\in\mathbf{Pr}_\xi$. *The equality* $[\boldsymbol{\pi}_1\boldsymbol{\pi}_2]=[\boldsymbol{\pi}_1][\boldsymbol{\pi}_2]$ *holds in* $\mathscr{K}_0(\mathscr{F}_\xi)$ *if one of the following conditions is satisfied:*

(i) $\boldsymbol{\pi}_1\boldsymbol{\pi}_2\notin\mathbf{Pr}_\xi$ *and* $\max\boldsymbol{\pi}_s<\min\boldsymbol{\pi}_m$ *for some* $s,m\in[1,2]$,

(ii) *there exists* $1\leqslant i\leqslant n$ *and* $a\in\{\xi(i)+1,\xi(i)-1\}$ *such that* $\boldsymbol{\omega}_{i,a}^{-1}\boldsymbol{\pi}_s\in\mathbf{Pr}_\xi$ *for* $s=1,2$,

(iii) *there exists* $s,m\in\{1,2\}$ *such that*

(a) *either* $\min\boldsymbol{\pi}_s<i=\min\boldsymbol{\pi}_m<j=\max\boldsymbol{\pi}_s<\max\boldsymbol{\pi}_m$ *and* $\operatorname{ht}\boldsymbol{\omega}(i,j)$ *is odd,*

(b) *or* $\min\boldsymbol{\pi}_s<i=\min\boldsymbol{\pi}_m<j=\max\boldsymbol{\pi}_m<\max\boldsymbol{\pi}_s$ *and* $\operatorname{ht}\boldsymbol{\omega}(i,j)$ *is even.*

3.7. Proof of Theorem 3. — Notice that the hypotheses of the theorem hold for the pair $(\boldsymbol{\pi}_1,\boldsymbol{\pi}_2)$ if and only if they hold for the pair $(\boldsymbol{\pi}_1^*,\boldsymbol{\pi}_2^*)$ of elements in $\mathbf{Pr}_{\xi^*}$. In particular, if we show that the conditions imply that we can write $\boldsymbol{\pi}_1=\boldsymbol{\omega}_1\boldsymbol{\omega}_2$ so that $(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\boldsymbol{\pi}_2)$ is ξ-admissible, then the triple $(\boldsymbol{\omega}_1^*,\boldsymbol{\omega}_2^*,\boldsymbol{\pi}_2^*)$ is ξ^*-admissible. Lemma 3.3 then proves that $[\boldsymbol{\pi}_1]\otimes[\boldsymbol{\pi}_2]$ is irreducible. A similar comment applies to the pair $(\boldsymbol{f}_p,\boldsymbol{\pi})$. This observation will be frequently used without further mention in the proof of the theorem.

We proceed by induction on $\operatorname{ht}\boldsymbol{\pi}$. If $\boldsymbol{\pi}=\boldsymbol{\omega}_{i,a}$ and $|\xi(i)-\xi(k)|=|k-i|$ the result was proved in [18]. If $|\xi(i)-\xi(k)|\neq|k-i|$ then Proposition 3.4 shows that the triple $(\boldsymbol{\omega}_{k,\xi(k)+1},\boldsymbol{\omega}_{k,\xi(k)-1},\boldsymbol{\omega}_{i,a})$ is ξ-admissible, proving that $[\boldsymbol{\omega}_{i,a}]\otimes[\boldsymbol{f}_k]$ is irreducible. If $\operatorname{ht}\boldsymbol{\pi}>1$, write $\boldsymbol{\pi}=\boldsymbol{\omega}\boldsymbol{\omega}_{i,a}$ with $i=\max\boldsymbol{\pi}$ and $\boldsymbol{\omega}\in\mathbf{Pr}_\xi$. The inductive hypothesis and Proposition 3.4 show that the triple $(\boldsymbol{\omega},\boldsymbol{\omega}_{i,a},\boldsymbol{f}_k)$ is ξ-admissible and the inductive step is proved.

All three assertions in part (b) are proved by an induction on $p=\operatorname{ht}\boldsymbol{\pi}_1+\operatorname{ht}\boldsymbol{\pi}_2$. Proposition 3.5 shows that induction begins when $p\leqslant 3$. It also shows that the results hold when $\operatorname{ht}\boldsymbol{\pi}_1=\operatorname{ht}\boldsymbol{\pi}_2=2$. Hence for the inductive step we assume that the results hold for all p' such that $3\leqslant p'<p$ and that either $\operatorname{ht}\boldsymbol{\pi}_1>2$ or $\operatorname{ht}\boldsymbol{\pi}_2>2$.

To prove the inductive step for (i), assume without loss of generality that $\operatorname{ht}\boldsymbol{\pi}_1>2$ and write $\boldsymbol{\pi}_1=\boldsymbol{\omega}_1\boldsymbol{\omega}_{j_1,a_1}\boldsymbol{\omega}_{j_2,a_2}$ with $\boldsymbol{\omega}_1\in\mathbf{Pr}_\xi$ such that one of the following holds:

$$\max\boldsymbol{\pi}_1<\min\boldsymbol{\pi}_2,\quad \max\boldsymbol{\omega}_1<j_1<j_2=\max\boldsymbol{\pi}_1\quad\text{and}\quad \xi(j_1-1)=\xi(j_1+1),$$
$$\max\boldsymbol{\pi}_2<\min\boldsymbol{\pi}_1,\quad \min\boldsymbol{\pi}_1=j_1<j_2<\min\boldsymbol{\omega}_1\quad\text{and}\quad \xi(j_2-1)=\xi(j_2+1).$$

It follows that $\boldsymbol{\omega}_1\boldsymbol{\pi}_2\notin\mathbf{Pr}_\xi$ and since $\boldsymbol{\pi}_1\boldsymbol{\pi}_2\notin\mathbf{Pr}_\xi$ we also have $\boldsymbol{\omega}_{j_1,a_1}\boldsymbol{\omega}_{j_2,a_2}\boldsymbol{\pi}_2\notin\mathbf{Pr}_\xi$. Hence $[\boldsymbol{\omega}_{j_1,a_1}\boldsymbol{\omega}_{j_2,a_2}]\otimes[\boldsymbol{\pi}_2]$ and $[\boldsymbol{\omega}_1]\otimes[\boldsymbol{\pi}_2]$ are irreducible by the inductive hypothesis. Proposition 3.4 shows that either $[\boldsymbol{\omega}_1]\otimes[\boldsymbol{\omega}_{j_1,a_1}\boldsymbol{\omega}_{j_2,a_2}]$ or $[\boldsymbol{\omega}_{j_1,a_1}\boldsymbol{\omega}_{j_2,a_2}]\otimes[\boldsymbol{\omega}_1]$ is

ℓ-highest weight, proving that the triple $(\boldsymbol{\omega}_1, \boldsymbol{\omega}_{j_1,a_1}\boldsymbol{\omega}_{j_2,a_2}, \boldsymbol{\pi}_2)$ is ξ-admissible. The proof of the inductive step for (i) is complete.

To prove the inductive step for (ii), notice that the conditions on $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ imply that one of the following holds: $\max\boldsymbol{\pi}_1 = \min\boldsymbol{\pi}_2 = i$ and $\xi(i-1) = \xi(i+1)$ or $\min\boldsymbol{\pi}_1 = \min\boldsymbol{\pi}_2 = i$ or $\max\boldsymbol{\pi}_1 = \max\boldsymbol{\pi}_2 = i$. Assume first that $\max\boldsymbol{\pi}_1 = \min\boldsymbol{\pi}_2 = i$. If $\operatorname{ht}\boldsymbol{\pi}_1 \geqslant 3$, write $\boldsymbol{\pi}_1 = \boldsymbol{\omega}_{k,c}\boldsymbol{\omega}_1$ with $\max\boldsymbol{\omega}_1 = i$; otherwise $\operatorname{ht}\boldsymbol{\pi}_2 \geqslant 3$ write $\boldsymbol{\pi}_2 = \boldsymbol{\omega}_{k,c}\boldsymbol{\omega}_2$ and $\min\boldsymbol{\omega}_2 = i$. In the first case, Proposition 3.4 and the inductive hypothesis show that the triple $(\boldsymbol{\omega}_{k,c}, \boldsymbol{\omega}_1, \boldsymbol{\pi}_2)$ is ξ-admissible while in the second case we get that $(\boldsymbol{\omega}_{k,c}, \boldsymbol{\omega}_2, \boldsymbol{\pi}_1)$ is ξ-admissible. In either case the irreducibility of $[\boldsymbol{\pi}_1]\otimes[\boldsymbol{\pi}_2]$ follows from Lemma 3.3. If $\min\boldsymbol{\pi}_1 = \min\boldsymbol{\pi}_2$, assume without loss of generality that $\operatorname{ht}\boldsymbol{\pi}_1 \leqslant \operatorname{ht}\boldsymbol{\pi}_2$ and let $k = \max\boldsymbol{\pi}_1$. Write $\boldsymbol{\pi}_2 = \boldsymbol{\omega}\boldsymbol{\omega}'$ with $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathbf{Pr}_\xi$ satisfying:

$$\min\boldsymbol{\omega} = \min\boldsymbol{\pi}_1,\ \max\boldsymbol{\omega} < k,\ \min\boldsymbol{\omega}' \geqslant k \text{ and } \min\boldsymbol{\omega}' = k \text{ if } \xi(k-1) = \xi(k+1).$$

The inductive hypothesis applies to $(\boldsymbol{\pi}_1, \boldsymbol{\omega})$, it also applies to $(\boldsymbol{\pi}_1, \boldsymbol{\omega}')$ if $\xi(k-1) = \xi(k+1)$ and otherwise $\boldsymbol{\pi}_1\boldsymbol{\omega}' \notin \mathbf{Pr}_\xi$ and part (b)(i) applies and shows that the corresponding tensor products are irreducible. Since Proposition 3.4 applies to $(\boldsymbol{\omega}, \boldsymbol{\omega}')$, we have now shown that $(\boldsymbol{\omega}, \boldsymbol{\omega}', \boldsymbol{\pi}_1)$ is ξ-admissible and the inductive step is proved in this case

Finally, we prove the inductive step for (iii). This amounts to proving the following: if $1 \leqslant i_1 < i_2 < i_3 < i_4 \leqslant n$ then the tensor product $[\boldsymbol{\omega}(i_1,i_3)]\otimes[\boldsymbol{\omega}(i_2,i_4)]$ is irreducible if $\operatorname{ht}\boldsymbol{\omega}(i_2,i_3)$ is odd and $[\boldsymbol{\omega}(i_1,i_4)]\otimes[\boldsymbol{\omega}(i_2,i_3)]$ is irreducible if $\operatorname{ht}\boldsymbol{\omega}(i_2,i_3)$ is even. It is simple to see that $p = \operatorname{ht}\boldsymbol{\omega}(i_1,i_3) + \operatorname{ht}\boldsymbol{\omega}(i_2,i_4) = \operatorname{ht}\boldsymbol{\omega}(i_1,i_4) + \operatorname{ht}\boldsymbol{\omega}(i_2,i_3)$. Since Proposition 3.5 shows that the result holds when $p = 4$ it means that it holds when $\operatorname{ht}\boldsymbol{\omega}(i_1,i_4) = 2$. Hence for the inductive step we may assume $\operatorname{ht}\boldsymbol{\omega}(i_1,i_4) \geqslant 3$.

Suppose that $\operatorname{ht}\boldsymbol{\omega}(i_2,i_3) = 2$. Using Proposition 3.4, the inductive hypothesis and parts (b)(i),(ii) of this theorem we see that one of the following holds:

– there exists $i_1 < m \leqslant i_2$ and $i_4 > p \geqslant i_3$ such that $\boldsymbol{\omega}(i_1,i_4) = \boldsymbol{\omega}(i_1,m)\boldsymbol{\omega}(p,i_4)$ and $(\boldsymbol{\omega}(i_1,m), \boldsymbol{\omega}(p,i_4), \boldsymbol{\omega}(i_2,i_3))$ is ξ-admissible,

– there exists $b \in \{\xi(i_4)+1, \xi(i_4)-1\}$ with $\boldsymbol{\omega}(i_1,i_4) = \boldsymbol{\omega}(i_1,i_2)\boldsymbol{\omega}_{i_4,b}$ and $(\boldsymbol{\omega}(i_1,i_2), \boldsymbol{\omega}_{i_4,b}, \boldsymbol{\omega}(i_2,i_3))$ is ξ-admissible,

– there exists $a \in \{\xi(i_1)+1, \xi(i_1)-1\}$ with $\boldsymbol{\omega}(i_1,i_4) = \boldsymbol{\omega}_{i_1,a}\boldsymbol{\omega}(i_3,i_4)$ and $(\boldsymbol{\omega}(i_3,i_4), \boldsymbol{\omega}_{i_1,a}, \boldsymbol{\omega}(i_2,i_3))$ is ξ-admissible.

In all cases the irreducibility of $[\boldsymbol{\omega}(i_1,i_4)]\otimes[\boldsymbol{\omega}(i_2,i_3)]$ is proved.

Suppose that $\operatorname{ht}\boldsymbol{\omega}(i_2,i_3) \geqslant 3$ and let $i_2 < p < i_3$ be minimal such that $|\xi(p)-\xi(i_2)| = p - i_2$ with $\xi(p-1) = \xi(p+1)$. Similarly, let $i_2 < m < i_3$ be maximal so that $|\xi(i_3)-\xi(m)| = i_3 - m$ and $\xi(m-1) = \xi(m+1)$. Then Proposition 3.4, parts (b)(i) and (b)(ii) and the inductive hypothesis show that one of the following holds:

– if $\xi(i_2-1) = \xi(i_2+1)$, then

$$\operatorname{ht}\boldsymbol{\omega}(i_2,i_3) \text{ odd} \implies (\boldsymbol{\omega}(i_1,i_2), \boldsymbol{\omega}(p,i_3), \boldsymbol{\omega}(i_2,i_4)) \text{ is } \xi\text{-admissible},$$
$$\operatorname{ht}\boldsymbol{\omega}(i_2,i_3) \text{ even} \implies (\boldsymbol{\omega}(i_1,i_2), \boldsymbol{\omega}(p,i_4), \boldsymbol{\omega}(i_2,i_3)) \text{ is } \xi\text{-admissible},$$

– if $\xi(i_3-1)=\xi(i_3+1)$, then

$$\operatorname{ht}\boldsymbol{\omega}(i_2,i_3)\ \text{odd} \implies (\boldsymbol{\omega}(i_2,m),\boldsymbol{\omega}(i_3,i_4),\boldsymbol{\omega}(i_1,i_3))\ \text{is } \xi\text{-admissible}$$
$$\operatorname{ht}\boldsymbol{\omega}(i_2,i_3)\ \text{even} \implies \boldsymbol{\omega}(i_1,m),\boldsymbol{\omega}(i_3,i_4),\boldsymbol{\omega}(i_2,i_3))\ \text{is } \xi\text{-admissible},$$

– $\xi(i_j+1)\neq\xi(i_j-1)$ for $j=2,3$.

If $p=m$, there exists $b\in\mathbb{C}(q)^\times$ such that $\boldsymbol{\omega}_{i_4,b}^{-1}\boldsymbol{\omega}(i_2,i_4)\in\mathbf{Pr}_\xi$ and the triple $(\boldsymbol{\omega}(i_2,p),\boldsymbol{\omega}_{i_4,b},\boldsymbol{\omega}(i_1,i_3))$ is ξ-admissible.

If $p\neq m$, let $p<p'\leqslant m$ be minimum such that $\xi(p'-1)=\xi(p'+1)$. Then

– $(\boldsymbol{\omega}(i_2,p),\boldsymbol{\omega}(p',i_4),\boldsymbol{\omega}(i_1,i_3))$ is ξ-admissible if $\operatorname{ht}\boldsymbol{\omega}(i_2,i_3)$ is odd

– and otherwise $(\boldsymbol{\omega}(i_2,p),\boldsymbol{\omega}(p',i_3),\boldsymbol{\omega}(i_1,i_4))$ is ξ-admissible.

In all cases the inductive step follows and the proof of the theorem is complete.

4. Identities in $\mathscr{K}_0(\mathscr{F}_\xi)$

In this section we establish Proposition 1.6 and Proposition 1.7.

4.1. — We will need the converse of Theorem 3(b). The most elementary case is the following well-known result. Namely, let $i\leqslant j$ satisfy $\xi(i)-\xi(j)=\pm(j-i)$; then the following equality holds in $\mathscr{K}_0(\mathscr{F}_\xi)$:

$$[\boldsymbol{\omega}_{i,\xi(i)\pm1}][\boldsymbol{\omega}_{j,\xi(j)\mp1}]=[\boldsymbol{\omega}_{i,\xi(i)\pm1}\boldsymbol{\omega}_{j,\xi(j)\mp1}]+[\boldsymbol{\omega}_{i-1,\xi(i)}][\boldsymbol{\omega}_{j+1,\xi(j)}].\tag{4.1}$$

Given $\boldsymbol{\pi}=\boldsymbol{\omega}\boldsymbol{\omega}_{i,a}\in\mathbf{Pr}_\xi$ with $\boldsymbol{\omega}\in\mathbf{Pr}_\xi$, set

$$\boldsymbol{\pi}'=\boldsymbol{\omega}\boldsymbol{\omega}_{i-1,\xi(i)},\quad i=\max\boldsymbol{\pi},\quad {}'\boldsymbol{\pi}=\boldsymbol{\omega}_{i+1,\xi(i)}\boldsymbol{\omega},\quad i=\min\boldsymbol{\pi}.$$

In the remaining cases the converse is most conveniently stated as follows.

Theorem 4

(i) *Suppose that* $\boldsymbol{\pi}_1\boldsymbol{\pi}_2\in\mathbf{Pr}_\xi$ *and* $\max\boldsymbol{\pi}_1<\min\boldsymbol{\pi}_2$. *Then*

$$[\boldsymbol{\pi}_1][\boldsymbol{\pi}_2]=[\boldsymbol{\pi}_1\boldsymbol{\pi}_2]+[\boldsymbol{\pi}_1'][{}'\boldsymbol{\pi}_2].\tag{4.2}$$

(ii) *Suppose that* $\boldsymbol{\omega}(m,p)\in\mathbf{Pr}_\xi$ *and for* $m<i<p$, *write*

$$\boldsymbol{\omega}(m,i)=\boldsymbol{\omega}_1\boldsymbol{\omega}_{i,a},\quad \boldsymbol{\omega}(i,p)=\boldsymbol{\omega}_{i,b}\boldsymbol{\omega}_2.$$

If $a\neq b$ *then*

$$[\boldsymbol{\omega}(m,p)][\boldsymbol{\omega}_{i,a}]=[\boldsymbol{\omega}(m,i)][\boldsymbol{\omega}_2]+[\boldsymbol{\omega}_1'][{}'\boldsymbol{\omega}(i,p)],\tag{$*$}$$
$$[\boldsymbol{\omega}(m,p)][\boldsymbol{\omega}_{i,b}]=[\boldsymbol{\omega}_1][\boldsymbol{\omega}(i,p)]+[\boldsymbol{\omega}(m,i)'][{}'\boldsymbol{\omega}_2].\tag{$**$}$$

If $a=b$ *then*

$$\begin{gathered}[\boldsymbol{\omega}(m,p)][\boldsymbol{\omega}_{i,a'}]=[\boldsymbol{\omega}_1][\boldsymbol{f}_i][\boldsymbol{\omega}_2]+[\boldsymbol{\omega}(m,i)'][{}'\boldsymbol{\omega}(i,p)],\\ \{a,a'\}=\{\xi(i)+1,\xi(i)-1\},\end{gathered}\tag{$\dagger$}$$
$$[\boldsymbol{\omega}(m,p)][\boldsymbol{\omega}_{i,a}]=[\boldsymbol{\omega}(m,i)][\boldsymbol{\omega}(i,p)]+[\boldsymbol{\omega}_1'][\boldsymbol{f}_i][{}'\boldsymbol{\omega}_2].\tag{$\dagger\dagger$}$$

Finally if $\boldsymbol{\pi}_1=\boldsymbol{\omega}_1\boldsymbol{\omega}_{i,a}$ *and* $\boldsymbol{\pi}_2=\boldsymbol{\omega}_{i,b}\boldsymbol{\omega}_2$ *are in* $\mathbf{Pr}_\xi$ *with* $\max\boldsymbol{\pi}_1=\min\boldsymbol{\pi}_2$ *and* $a\neq b$ *then*

$$[\boldsymbol{\pi}_1][\boldsymbol{\pi}_2]=[\boldsymbol{\omega}_1][\boldsymbol{f}_i][\boldsymbol{\omega}_2]+[\boldsymbol{\pi}_1'][{}'\boldsymbol{\pi}_2].\tag{4.3}$$

(iii) *Assume that* $i_1 < i_2 < i_3 < i_4$ *and write*

$$\boldsymbol{\omega}(i_1, i_2) = \boldsymbol{\omega}_1 \boldsymbol{\omega}_{i_2,a}, \quad \boldsymbol{\omega}(i_2, i_3) = \boldsymbol{\omega}_{i_2,b} \boldsymbol{\omega} \boldsymbol{\omega}_{i_3,c}, \quad \boldsymbol{\omega}(i_3, i_4) = \boldsymbol{\omega}_{i_3,d} \boldsymbol{\omega}_2.$$

Then $(-1)^{\mathrm{ht}(\boldsymbol{\omega}(i_2,i_3))} \left([\boldsymbol{\omega}(i_1, i_3)][\boldsymbol{\omega}(i_2, i_4)] - [\boldsymbol{\omega}(i_1, i_4)][\boldsymbol{\omega}(i_2, i_3)]\right)$ *is equal to*

$$[\boldsymbol{\omega}_1']^{\delta_{a,b}} [\boldsymbol{\omega}(i_1, i_2)')]^{1-\delta_{a,b}} \left(\prod_{s=i_2}^{i_3} [\boldsymbol{f}_s]^{\delta_{\xi(s-1),\xi(s+1)}} \right) ['\boldsymbol{\omega}_2]^{\delta_{c,d}} ['\boldsymbol{\omega}(i_3, i_4))]^{1-\delta_{c,d}}.$$

From now on we freely use (often without mention) the results of Theorem 3. We deduce Proposition 1.6 and Proposition 1.7 before proving Theorem 4.

4.2. Proof of Proposition 1.6. — The proposition is obviously a special case of equation (4.1) and Theorem 4(i),(ii). However the translation from the formulation in this section to the one in Section 1 which is adapted to cluster algebras needs some clarification, which we provide for the readers convenience. We recall that $d_j = \delta_{\xi(j),\xi(j+2)} = \delta_{j,j_\diamond}$.

For part (i) of Proposition 1.6 we take

$$\boldsymbol{\pi}_1 = \boldsymbol{\omega}_{i,\xi(i+1)}, \qquad \boldsymbol{\pi}_2 = \boldsymbol{\omega}(i, i+1)^{1-d_i} \boldsymbol{\omega}_{i,\xi(i+1)\pm 2}^{d_i} = \boldsymbol{\omega}_{i,\xi(i+1)\pm 2} \boldsymbol{\omega}_{i+1,\xi(i+2)}^{1-d_i},$$

where the second formula for $\boldsymbol{\pi}_2$ uses the fact that $\xi(i+2) = \xi(i+1) \mp 1 = \xi(i) \mp 2$ if $d_i = 0$. Theorem 3 gives

$$[\boldsymbol{\pi}_1 \boldsymbol{\pi}_2] = [\boldsymbol{f}_i][\boldsymbol{\omega}_{i+1,\xi(i+2)}]^{1-d_i}, \qquad [\boldsymbol{\omega}_{i+1,\xi(i)} \boldsymbol{\omega}_{i+1,\xi(i+2)}^{1-d_i}] = [\boldsymbol{f}_{i+1}]^{1-d_i} [\boldsymbol{\omega}_{i+1,\xi(i)}]^{d_i},$$

Using either (4.1) or (4.3) we get

$$[\boldsymbol{\pi}_1][\boldsymbol{\pi}_2] = [\boldsymbol{f}_i][\boldsymbol{\omega}_{i+1,\xi(i+2)}]^{1-d_i} + [\boldsymbol{\omega}_{i-1,\xi(i)}][\boldsymbol{f}_{i+1}]^{1-d_i} [\boldsymbol{\omega}_{i+1,\xi(i)}]^{d_i}$$

as needed.

For Proposition 1.6(ii), using the definition of $\overline{j}$ we can rewrite its left hand side as

$$[\boldsymbol{\omega}(i, \overline{j})]^{1-\delta_{j,i_\diamond}} [\boldsymbol{\omega}_{i,\xi(i+1)\pm 2}]^{\delta_{j,i_\diamond}} = [\boldsymbol{\omega}(i, j+1)]^{1-d_j} \left([\boldsymbol{\omega}_{i,\xi(i+1)\pm 2}]^{1-\delta_{i,j_\bullet}} [\boldsymbol{\omega}(i, j_\bullet + 1)]^{\delta_{i,j_\bullet}} \right)^{d_j}.$$

It is easiest to verify the four cases given by $d_j \in \{0,1\}$ and $\delta_{j_\bullet,i} \in \{0,1\}$ separately. If $d_j = 1$ the left hand side of Proposition 1.6(ii) is $[\boldsymbol{\pi}_1][\boldsymbol{\pi}_2]$, where

$$\boldsymbol{\pi}_1 = \boldsymbol{\omega}_{j,\xi(j+1)}, \quad \boldsymbol{\pi}_2 = \boldsymbol{\omega}_{i,\xi(i+1)\pm 2}^{1-\delta_{i,j_\bullet}} \boldsymbol{\omega}(i, i+1)^{\delta_{i,j_\bullet}} = \boldsymbol{\omega}_{i,\xi(i+1)\pm 2} \boldsymbol{\omega}_{i+1,\xi(i+2)\mp 2}^{\delta_{i,j_\bullet}}$$

and the right hand side is

$$[\boldsymbol{f}_j]^{d_{j-1}} \; [\boldsymbol{\omega}(i,j)]^{1-d_{j-1}} [\boldsymbol{\omega}_{i,\xi(i+1)\pm 2}]^{d_{j-1}} + [\boldsymbol{f}_i]^{\delta_{i,j_\bullet}} [\boldsymbol{\omega}_{j+1,\xi(j+2)}][\boldsymbol{\omega}_{i-1,\xi(i)}]^{1-\delta_{i,j_\bullet}}.$$

Since $\delta_{j_\bullet,i} = 0 \Rightarrow j_\bullet < i \leqslant j-1 \Rightarrow d_{j-1} = 0$, and $\boldsymbol{\omega}(i,j) = \boldsymbol{\omega}_{i,\xi(i+1)\pm 2} \boldsymbol{\omega}_{j,\xi(j+1)}$, we see that the right hand side of Proposition 1.6(ii) is precisely the right hand side of (4.1) and we are done. Otherwise

$$\delta_{j_\bullet,i} = 1 \quad \text{and either} \quad j_\bullet = j-1 \quad \text{or} \quad j_\bullet < j-1.$$

In the first case $d_{j-1} = 1$ and $i+1 = j$ and so the result follows from (4.3); in the second case we have $i+1 < j$ and $d_{j-1} = 0$. Since $i = j_\bullet = i_\diamond$ we also have

$\xi(i) = \xi(i+2)$ which implies that $\boldsymbol{\omega}(i,i+1)\boldsymbol{\omega}_{j,\xi(j+1)} = \boldsymbol{\omega}(i,j)$, The result follows from Theorem 4(i).

If $d_j = 0$ then the left hand side of Proposition 1.6 is $[\boldsymbol{\pi}_1][\boldsymbol{\pi}_2]$, where

$$\boldsymbol{\pi}_1 = \boldsymbol{\omega}_{j,\xi(j)+1}, \quad \boldsymbol{\pi}_2 = \boldsymbol{\omega}(i,j+1) = \boldsymbol{\omega}_{i,\xi(i+1)\pm 2}\boldsymbol{\omega}^{d_{j-1}}_{j,\xi(j+1)\mp 2}\boldsymbol{\omega}_{j+1,\xi(j+2)}$$

and the right hand side of Proposition 1.6 is

$$[\boldsymbol{f}_j]^{d_{j-1}}\ [\boldsymbol{\omega}(i,j)]^{1-d_{j-1}}[\boldsymbol{\omega}_{i,\xi(i+1)\pm 2}]^{d_{j-1}}[\boldsymbol{\omega}_{j+1,\xi(j+2)}] + [\boldsymbol{f}_{j+1}][\boldsymbol{f}_i]^{\delta_{i,j_\bullet}}[\boldsymbol{\omega}_{i-1,\xi(i)}]^{1-\delta_{i,j_\bullet}}.$$

If $d_{j-1} = 1$ then $i = j-1 = j_\bullet$ and the result follows from (†) in Theorem 4(ii). If $d_{j-1} = 0$, then the result follows from equation $(*)$ in Theorem 4(ii).

The proof of part (iii) is a similar detailed analysis. Note that $[\boldsymbol{\omega}(i,\overline{j})] = [\boldsymbol{\omega}(i,j_\bullet+1)]^{d_j}[\boldsymbol{\omega}(i,j+1)]^{1-d_j}$. If $d_j = 1$, we take

$$\boldsymbol{\pi}_1 = \boldsymbol{\omega}_{j,\xi(j+1)}, \quad \boldsymbol{\pi}_2 = \boldsymbol{\omega}(i,j_\bullet+1)$$

and use Theorem 4(i) if $j_\bullet+1 < j$ and (4.3) if $j_\bullet+1 = j$. If $d_j = 0$ we take

$$\boldsymbol{\pi}_1 = \boldsymbol{\omega}_{j,\xi(j+1)}, \quad \boldsymbol{\pi}_2 = \boldsymbol{\omega}(i,j+1).$$

Note that $\boldsymbol{\omega}(i,j_\bullet+1)\boldsymbol{\omega}_{j,\xi(j+1)} = \boldsymbol{\omega}(i,j) \in \mathbf{Pr}_\xi$ if $j_\bullet+1 < j$ and that if we write $k = (j_\bullet)_\bullet$ then

$$\boldsymbol{\pi}_2\boldsymbol{\omega}_{j,\xi(j+1)} = \begin{cases} (\boldsymbol{\omega}^{\delta_{k,i_\bullet}}_{i,\xi(i+1)\pm 2}\boldsymbol{\omega}(i,k+1)^{1-\delta_{k,i_\bullet}})\boldsymbol{f}_j\boldsymbol{\omega}_{j+1,\xi(j+2)}, & d_{j-1} = 1,\\ \boldsymbol{\omega}(i,j_\bullet+1)\boldsymbol{\omega}_{j+1,\xi(j+2)}, & d_{j-1} = 0.\end{cases}$$

An application of Theorem 4 as in the other cases completes the proof. □

4.3. Proof of Proposition 1.7. — Let $\boldsymbol{\omega},\boldsymbol{\omega}' \in \mathbf{Pr}_\xi$ be such that $[\boldsymbol{\omega}\boldsymbol{\omega}'] = [\boldsymbol{\omega}][\boldsymbol{\omega}']$. By Section 1.9 we can choose $\alpha,\beta \in \Phi_{\geqslant -1}$ such that

$$[\boldsymbol{\omega}] = \iota(x[\alpha]), \quad [\boldsymbol{\omega}'] = \iota(x[\beta]).$$

We claim that $x[\alpha]x[\beta]$ is a cluster monomial. If not, we can write

$$x[\alpha]x[\beta] = m_1x[\gamma]x[\eta] + m_2x[\gamma']x[\eta'],$$

where $m_1,m_2 \in \mathbb{Z}_{\geqslant 0}[f_i : i \in I]$ and $\gamma,\gamma',\eta,\eta'$ are in $\Phi_{\geqslant -1}$. Applying ι to both sides of the equation we get

$$[\boldsymbol{\omega}\boldsymbol{\omega}'] = [\boldsymbol{\omega}][\boldsymbol{\omega}'] = \iota(x[\alpha]x[\beta]) = \iota(m_1)\iota(x[\gamma])\iota(x[\eta]) + \iota(m_2)\iota(x[\gamma'])\iota(x[\eta']).$$

Since $\iota(m_1),\iota(m_2) \in \mathbb{Z}_{\geqslant 0}[\boldsymbol{f}_i : i \in I]$, this means that we can write $[\boldsymbol{\omega}\boldsymbol{\omega}']$ as a non-trivial linear combination of elements $\{[\boldsymbol{\pi}] : \boldsymbol{\pi} \in \mathscr{P}^+_\xi\}$ which is absurd.

Suppose now that $\boldsymbol{\pi},\boldsymbol{\pi}' \in \mathbf{Pr}_\xi$ are such that $[\boldsymbol{\pi}][\boldsymbol{\pi}'] \neq [\boldsymbol{\pi}\boldsymbol{\pi}']$. Theorem 3 and Theorem 4 imply that $[\boldsymbol{\pi}\boldsymbol{\pi}'] = [\boldsymbol{f}][\boldsymbol{\varpi}_1]$ and $[\boldsymbol{\pi}][\boldsymbol{\pi}'] - [\boldsymbol{\pi}\boldsymbol{\pi}'] = [\boldsymbol{f}'][\boldsymbol{\varpi}_2]$, where $\boldsymbol{f}$ and $\boldsymbol{f}'$ are products of $\boldsymbol{f}_i$, and $[\boldsymbol{\varpi}_j]$ is a product of elements $[\boldsymbol{\varpi}_j] = [\boldsymbol{\omega}^j_1]\cdots[\boldsymbol{\omega}^j_{p_j}]$ with $\boldsymbol{\omega}^j_s \in \mathbf{Pr}_\xi$, $1 \leqslant s \leqslant p_j$, $j = 1,2$. By the previous discussion it follows that such products are the images of cluster monomials. Hence the inverse image under ι of $[\boldsymbol{\pi}][\boldsymbol{\pi}']$ is a positive linear combination of certain cluster monomials; in particular the inverse image is not a cluster monomial and the proof is complete. □

4.4. — In the rest of the paper we prove Theorem 4. The crucial step is the following proposition whose proof we postpone to the next section.

Proposition. — *Let $\boldsymbol{\omega}_{i,a}, \boldsymbol{\pi}$ be elements of $\mathbf{Pr}_\xi$ with $i < \min \boldsymbol{\pi} = j$ or $i > \max \boldsymbol{\pi} = k$ and $\boldsymbol{\omega}_{i,a}\boldsymbol{\pi} \in \mathbf{Pr}_\xi$. Let $b, c \in \mathbb{C}(q)$ be such that $\boldsymbol{\omega}_{j,b}^{-1}\boldsymbol{\pi}$ and $\boldsymbol{\omega}_{k,c}^{-1}\boldsymbol{\pi}$ are elements of $\mathbf{Pr}_\xi$. We have*

$$[\boldsymbol{\omega}_{i,a}][\boldsymbol{\pi}] - [\boldsymbol{\omega}_{i,a}\boldsymbol{\pi}] = [\boldsymbol{\omega}_{i-1,\xi(i)}][{}'\boldsymbol{\pi}], \quad i < j,$$
$$[\boldsymbol{\omega}_{i,a}][\boldsymbol{\pi}] - [\boldsymbol{\omega}_{i,a}\boldsymbol{\pi}] = [\boldsymbol{\omega}_{i+1,\xi(i)}][\boldsymbol{\pi}'], \quad i > k.$$

4.5. Proof of Theorem 4(i). — We need the following consequence of Proposition 4.4.

Lemma. — *Let $\boldsymbol{\omega}_{i,a}, \boldsymbol{\omega}_{i,b}\boldsymbol{\pi} \in \mathbf{Pr}_\xi$ and assume that $a \neq b$ and $\min \boldsymbol{\pi} > i$ (resp. $\max \boldsymbol{\pi} < i$). Then*

$$[\boldsymbol{\omega}_{i,a}][\boldsymbol{\omega}_{i,b}\boldsymbol{\pi}] = [\boldsymbol{f}_i][\boldsymbol{\pi}] + [\boldsymbol{\omega}_{i-1,\xi(i)}][\boldsymbol{\omega}_{i+1,\xi(i)}\boldsymbol{\pi}],$$
$$(\textit{resp. } [\boldsymbol{\omega}_{i,a}][\boldsymbol{\omega}_{i,b}\boldsymbol{\pi}] = [\boldsymbol{f}_i][\boldsymbol{\pi}] + [\boldsymbol{\omega}_{i+1,\xi(i)}][\boldsymbol{\omega}_{i-1,\xi(i)}\boldsymbol{\pi}]).$$

Proof. — Proceed by induction on $\operatorname{ht} \boldsymbol{\omega}_{i,b}\boldsymbol{\pi}$. If $\operatorname{ht} \boldsymbol{\omega}_{i,b}\boldsymbol{\pi} = 1$ then the result is well-known (see for instance [15]). Assume that we have proved the result if $\operatorname{ht} \boldsymbol{\omega}_{i,b}\boldsymbol{\pi} < r$. Write $\boldsymbol{\pi} = \boldsymbol{\omega}_{m,c}\boldsymbol{\omega}$ with $m = \min \boldsymbol{\pi}$ and note that

$$a = \xi(i) \mp 1 \iff b = \xi(i) \pm 1 \iff \xi(i+1) \pm 1 = \xi(i) \text{ and } c = \xi(m) \mp 1.$$

It follows that the pair $(\boldsymbol{\omega}_{i+1,\xi(i)}, \boldsymbol{\omega}_{m,c}\boldsymbol{\omega})$ satisfies the conditions of Proposition 4.4 if $i+1 \neq m$ and the inductive hypothesis of this lemma if $i+1 = m$, and so we have

$$(*) \qquad [\boldsymbol{\omega}_{i+1,\xi(i)}][\boldsymbol{\omega}_{m,c}\boldsymbol{\omega}] = [\boldsymbol{\omega}_{i+1,\xi(i)}\boldsymbol{\omega}_{m,c}\boldsymbol{\omega}] + [\boldsymbol{\omega}_{i,a}][\boldsymbol{\omega}_{m+1,\xi(m)}\boldsymbol{\omega}].$$

The inductive hypothesis and Proposition 4.4 also give

$$\begin{aligned}[\boldsymbol{\omega}_{i,a}][\boldsymbol{\omega}_{i,b}][\boldsymbol{\pi}] &= [\boldsymbol{\omega}_{i,a}]\left([\boldsymbol{\omega}_{i,b}\boldsymbol{\pi}] + [\boldsymbol{\omega}_{i-1,\xi(i)}][\boldsymbol{\omega}\boldsymbol{\omega}_{m+1,\xi(m)}]\right)\\ &= \left([\boldsymbol{f}_i] + [\boldsymbol{\omega}_{i-1,\xi(i)}][\boldsymbol{\omega}_{i+1,\xi(i)}]\right)[\boldsymbol{\pi}]\\ &= [\boldsymbol{f}_i][\boldsymbol{\pi}] + [\boldsymbol{\omega}_{i-1,\xi(i)}]\big([\boldsymbol{\omega}_{i+1,\xi(i)}\boldsymbol{\omega}_{m,c}\boldsymbol{\omega}] + [\boldsymbol{\omega}_{i,a}][\boldsymbol{\omega}_{m+1,\xi(m)}\boldsymbol{\omega}]\big).\end{aligned}$$

Equating the first and third terms on the right hand side and using $(*)$ gives

$$[\boldsymbol{f}_i][\boldsymbol{\pi}] + [\boldsymbol{\omega}_{i-1,\xi(i)}][\boldsymbol{\omega}_{i+1,\xi(i)}\boldsymbol{\omega}_{m,c}\boldsymbol{\omega}] = [\boldsymbol{\omega}_{i,a}][\boldsymbol{\omega}_{i,b}\boldsymbol{\pi}],$$

which establishes the inductive step. □

The proof of Theorem 4(i) proceeds by an induction on $\operatorname{ht} \boldsymbol{\pi}_1$ with Proposition 4.4, showing that induction begins when $\operatorname{ht} \boldsymbol{\pi}_1 = 1$. For the inductive step, recall that $\boldsymbol{\pi}_1 = \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_1$ and $\boldsymbol{\pi}_2 = \boldsymbol{\omega}_{j,b}\boldsymbol{\omega}_2$ with $\max \boldsymbol{\pi}_1 = i < j = \min \boldsymbol{\pi}_2$. Since $\boldsymbol{\pi}_1\boldsymbol{\pi}_2 \in \mathbf{Pr}_\xi$, we see that Proposition 4.4 applies to the pairs $(\boldsymbol{\omega}_1, \boldsymbol{\omega}_{i,a})$, $(\boldsymbol{\omega}_{i,a}, \boldsymbol{\pi}_2)$ and also to the pairs $(\boldsymbol{\omega}_{i+1,\xi(i)}, \boldsymbol{\pi}_2)$ and $(\boldsymbol{\omega}_1, \boldsymbol{\omega}_{i-1,\xi(i)})$ if $i+1 \neq j$ and $i-1 \neq \min \boldsymbol{\omega}_1$. If $i+1 = j$ (resp. $i-1 = \min \boldsymbol{\omega}_1$) then Lemma 4.5 applies to $(\boldsymbol{\omega}_{i+1,\xi(i)}, \boldsymbol{\pi}_2)$ (resp.$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_{i-1,\xi(i)})$).

Together with the inductive hypothesis which applies to $(\boldsymbol{\omega}_1, \boldsymbol{\omega}_{i,a}\boldsymbol{\pi}_2)$ we get the following series of equalities:

$$\begin{aligned}
&[\boldsymbol{\pi}_1][\boldsymbol{\pi}_2] + [\boldsymbol{\omega}_1']\big([\boldsymbol{\omega}_{i+1,\xi(i)}\boldsymbol{\pi}_2] + [\boldsymbol{\omega}_{i,\xi(i+1)}][{}'\boldsymbol{\pi}_2]\big)\\
&\qquad = \big([\boldsymbol{\pi}_1 + [\boldsymbol{\omega}_1'][\boldsymbol{\omega}_{i+1,\xi(i)}]\big)\,[\boldsymbol{\pi}_2] = [\boldsymbol{\omega}_1][\boldsymbol{\omega}_{i,a}][\boldsymbol{\pi}_2]\\
&\qquad = [\boldsymbol{\omega}_1]\,\big([\boldsymbol{\omega}_{i,a}\boldsymbol{\pi}_2] + [\boldsymbol{\omega}_{i-1,\xi(i)}][{}'\boldsymbol{\pi}_2]\big)\\
&\qquad = [\boldsymbol{\pi}_1\boldsymbol{\pi}_2] + [\boldsymbol{\omega}_1'][\boldsymbol{\omega}_{i+1,\xi(i)}\boldsymbol{\pi}_2] + [\boldsymbol{\omega}_1\boldsymbol{\omega}_{i-1\xi(i)}][{}'\boldsymbol{\pi}_2] + [\boldsymbol{\omega}_1'][\boldsymbol{\omega}_{i,\xi(i-1)}][{}'\boldsymbol{\pi}_2].
\end{aligned}$$

Equating the first and the fifth terms gives the inductive step since $\xi(i-1) = \xi(i+1)$ and part (i) is proved.

4.6. Proof of Theorem 4(ii). — Suppose that $a \neq b$ which means that $\xi(i-1) \neq \xi(i+1)$ and hence $\boldsymbol{\omega}(m,p) = \boldsymbol{\omega}_1\boldsymbol{\omega}_2$. We prove equation $(**)$, the proof of $(*)$ being an obvious modification. Using Theorem 3(b)(i) gives that $[\boldsymbol{\omega}(m,p)\boldsymbol{\omega}_{i,b}] = [\boldsymbol{\omega}_1][\boldsymbol{\omega}(i,p)]$ and we have to prove that

$$[\boldsymbol{\omega}(m,p)][\boldsymbol{\omega}_{i,b}] - [\boldsymbol{\omega}_1][\boldsymbol{\omega}(i,p)] = [\boldsymbol{\omega}(m,i)'][{}'\boldsymbol{\omega}_2].$$

For this we calculate $[\boldsymbol{\omega}_1][\boldsymbol{\omega}_{i,b}][\boldsymbol{\omega}_2]$ in two ways by using Proposition 4.4 on $(\boldsymbol{\omega}_{i,b}, \boldsymbol{\omega}_2)$ and part (i) of the theorem on $(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$. This gives

$$\begin{aligned}
[\boldsymbol{\omega}_1][\boldsymbol{\omega}_{i,b}][\boldsymbol{\omega}_2] &= [\boldsymbol{\omega}_1][\boldsymbol{\omega}(i,p)] + [\boldsymbol{\omega}_1][\boldsymbol{\omega}_{i-1,\xi(i)}][{}'\boldsymbol{\omega}_2]\\
&= [\boldsymbol{\omega}_{i,b}][\boldsymbol{\omega}(m,p)] + [\boldsymbol{\omega}_{i,b}][\boldsymbol{\omega}_1'][{}'\boldsymbol{\omega}_2].
\end{aligned}$$

Equating we see that we must prove that

$$[\boldsymbol{\omega}_1][\boldsymbol{\omega}_{i-1,\xi(i)}] - [\boldsymbol{\omega}_1'][\boldsymbol{\omega}_{i,b}] = [\boldsymbol{\omega}_1\boldsymbol{\omega}_{i-1,\xi(i)}] = [\boldsymbol{\omega}(m,i)']. \tag{4.4}$$

This follows since Proposition 4.4 applies if $\min\boldsymbol{\omega}_1 < i-1$ (and Lemma 4.5 if $\min\boldsymbol{\omega}_1 = i-1$) to the pair $(\boldsymbol{\omega}_1, \boldsymbol{\omega}_{i-1,\xi(i)})$.

If $a = b$ then $\xi(i-1) = \xi(i+1)$ and hence Theorem 3 shows that $[\boldsymbol{\omega}(m,i)][\boldsymbol{\omega}(i,p)] = [\boldsymbol{\omega}(m,p)\boldsymbol{\omega}_{i,a}]$ and $[\boldsymbol{\omega}(m,p)\boldsymbol{\omega}_{i,a'}] = [\boldsymbol{\omega}_1][\boldsymbol{f}_i][\boldsymbol{\omega}_2]$. To prove (†) we use part (i) of the theorem on the pair $(\boldsymbol{\omega}_1, \boldsymbol{\omega}(i,p))$ and Lemma 4.5 on the pair $(\boldsymbol{\omega}_{i,a'}, \boldsymbol{\omega}(i,p))$ to get

$$\begin{aligned}
[\boldsymbol{\omega}_1][\boldsymbol{\omega}_{i,a'}][\boldsymbol{\omega}(i,p)] &= [\boldsymbol{\omega}_{i,a'}][\boldsymbol{\omega}(m,p)] + [\boldsymbol{\omega}_{i,a}][\boldsymbol{\omega}_1'][{}'\boldsymbol{\omega}(i,p)]\\
&= [\boldsymbol{\omega}_1][\boldsymbol{f}_i][\boldsymbol{\omega}_2] + [\boldsymbol{\omega}_1][\boldsymbol{\omega}_{i-1,\xi(i)}][{}'\boldsymbol{\omega}(i,p)].
\end{aligned}$$

Equating the right hand sides and using (4.4) gives the result. The proof of (††) is similar; we calculate $[\boldsymbol{\omega}_{i,a}][\boldsymbol{\omega}_1][\boldsymbol{\omega}(i,p)]$ in two ways by using Proposition 4.4 on $(\boldsymbol{\omega}_{i,a}, \boldsymbol{\omega}_1)$ and part (i) of the theorem on $(\boldsymbol{\omega}_1, \boldsymbol{\omega}(i,p))$. This gives

$$\begin{aligned}
[\boldsymbol{\omega}_{i,a}][\boldsymbol{\omega}_1][\boldsymbol{\omega}(i,p)] &= [\boldsymbol{\omega}(m,i)][\boldsymbol{\omega}(i,p)] + [\boldsymbol{\omega}_1'][\boldsymbol{\omega}_{i+1,\xi(i)}][\boldsymbol{\omega}(i,p)]\\
&= [\boldsymbol{\omega}_{i,a}][\boldsymbol{\omega}(m,p)] + [\boldsymbol{\omega}_{i,a}][\boldsymbol{\omega}_1'][{}'\boldsymbol{\omega}(i,p)].
\end{aligned}$$

We then observe that $\boldsymbol{\omega}_{i+1,\xi(i)}\boldsymbol{\omega}_2 \in \mathbf{Pr}_\xi$ if $\xi(i) \neq \xi(i+2)$ and $\boldsymbol{\omega}_{i+1,\xi(i)}\boldsymbol{\omega}_2 = \boldsymbol{f}_{i+1}\boldsymbol{\omega}'$, with $\boldsymbol{\omega}' \in \mathbf{Pr}_\xi$, if $\xi(i+1) = \xi(i+3)$. Then we can apply the results proved above of part (ii) of this theorem to the pair $(\boldsymbol{\omega}_{i+1,\xi(i)}, \boldsymbol{\omega}(i,p))$, and hence either by $(**)$ or by (†) we get

$$[\boldsymbol{\omega}_{i+1,\xi(i)}][\boldsymbol{\omega}(i,p)] = [\boldsymbol{\omega}_{i,a}][\boldsymbol{\omega}_{i+1,\xi(i)}\boldsymbol{\omega}_2] + [{}'\boldsymbol{\omega}_2][\boldsymbol{f}_i]$$

Equation (††) now follows by a substitution, recalling that $[{}'\boldsymbol{\omega}(i,p)] = [\boldsymbol{\omega}_{i+1,\xi(i)}\boldsymbol{\omega}_2]$, by definition.

Finally we prove (4.3). If $\operatorname{ht}\boldsymbol{\pi}_1 = 1$ or $\operatorname{ht}\boldsymbol{\pi}_2 = 1$ this was proved in Lemma 4.5. Hence we may assume that $\boldsymbol{\pi}_1 = \boldsymbol{\omega}(m,i)$ and $\boldsymbol{\pi}_2 = \boldsymbol{\omega}(i,p)$ for some $m < i < p$. Since $a \neq b$ we use Theorem 3 to see that $[\boldsymbol{\omega}(m,i)\boldsymbol{\omega}(i,p)] = [\boldsymbol{f}_i][\boldsymbol{\omega}(m,p)]$ and we prove that

$$[\boldsymbol{\omega}(m,i)][\boldsymbol{\omega}(i,p)] - [\boldsymbol{f}_i][\boldsymbol{\omega}(m,p)] = [\boldsymbol{\omega}(m,i)'][{}'\boldsymbol{\omega}(i,p)]. \tag{4.5}$$

For this, we note that $\boldsymbol{\omega}_1\boldsymbol{\omega}_2 = \boldsymbol{\omega}(m,p)$ and hence, using part (i) of the theorem to the pair $(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$ and Proposition 4.4 or Lemma 4.5 to the pairs $(\boldsymbol{\omega}_1, \boldsymbol{\omega}_{i-1,\xi(i)})$ and $(\boldsymbol{\omega}_{i,a}, \boldsymbol{\omega}(i,p))$ we get

$$\begin{aligned}[\boldsymbol{f}_i]([\boldsymbol{\omega}(m,p)] + [\boldsymbol{\omega}_1'][{}'\boldsymbol{\omega}_2]) + ([\boldsymbol{\omega}(m,i)'] + [\boldsymbol{\omega}_1'][\boldsymbol{\omega}_{i,b}])\,[{}'\boldsymbol{\omega}(i,p)]&\\ = [\boldsymbol{\omega}_1][\boldsymbol{f}_i][\boldsymbol{\omega}_2] + [\boldsymbol{\omega}_1][\boldsymbol{\omega}_{i-1,\xi(i)}][{}'\boldsymbol{\omega}(i,p)] &= [\boldsymbol{\omega}_1][\boldsymbol{\omega}_{i,a}][\boldsymbol{\omega}(i,p)]\\ = [\boldsymbol{\omega}(m,i)][\boldsymbol{\omega}(i,p)] + [\boldsymbol{\omega}_1'][\boldsymbol{\omega}_{i+1,\xi(i)}][\boldsymbol{\omega}(i,p)].&\end{aligned}$$

Equating the first and last terms we see that (4.5) follows if we prove that

$$[\boldsymbol{f}_i][\boldsymbol{\omega}_2'] + [\boldsymbol{\omega}_{i,b}][{}'\boldsymbol{\omega}(i,p)] = [\boldsymbol{\omega}_{i+1,\xi(i)}][\boldsymbol{\omega}(i,p)].$$

But this follows from the cases of part (ii) of this theorem proved above. This completes the proof of part (ii).

4.7. Proof of Theorem 4(iii). — We proceed by induction on $N = \operatorname{ht}\boldsymbol{\omega}(i_1,i_3) + \operatorname{ht}\boldsymbol{\omega}(i_2,i_4)$ with [18] showing that induction begins when $N = 4$. Recall that

$$\boldsymbol{\omega}(i_1,i_2) = \boldsymbol{\omega}_1\boldsymbol{\omega}_{i_2,a}, \quad \boldsymbol{\omega}(i_2,i_3) = \boldsymbol{\omega}_{i_2,b}\boldsymbol{\omega}\boldsymbol{\omega}_{i_3,c}, \quad \boldsymbol{\omega}(i_3,i_4) = \boldsymbol{\omega}_{i_3,d}\boldsymbol{\omega}_2.$$

Set

$$[\boldsymbol{f}_{i_2,i_3}] = \prod_{s=i_2}^{i_3} [\boldsymbol{f}_s]^{\delta_{\xi(s-1),\xi(s+1)}},$$

and note that $\boldsymbol{\omega}(i_2,i_4) = \boldsymbol{\omega}_{i_2,b}\boldsymbol{\omega}\boldsymbol{\omega}_{i_3,c}^{\delta_{c,d}}\boldsymbol{\omega}_2$.

Case 1: $a = b$ or $c = d$. — Suppose that $a = b$; the proof is similar when $c = d$. Then $\boldsymbol{\omega}_1\boldsymbol{\omega}(i_2,i_s) \in \mathbf{Pr}_\xi$ for $s = 3,4$. Hence (4.2) gives

$$\begin{aligned}[\boldsymbol{\omega}_1][\boldsymbol{\omega}(i_2,i_4)][\boldsymbol{\omega}(i_2,i_3)] &= ([\boldsymbol{\omega}(i_1,i_4)] + [\boldsymbol{\omega}_1'][{}'\boldsymbol{\omega}(i_2,i_4)])\,[\boldsymbol{\omega}(i_2,i_3)]\\ &= ([\boldsymbol{\omega}(i_1,i_3)] + [\boldsymbol{\omega}_1'][{}'\boldsymbol{\omega}(i_2,i_3)]))\,[\boldsymbol{\omega}(i_2,i_4)].\end{aligned}$$

The result follows if we prove that

$$\begin{aligned}&[{}'\boldsymbol{\omega}(i_2,i_3)][\boldsymbol{\omega}(i_2,i_4)] - [{}'\boldsymbol{\omega}(i_2,i_4)][\boldsymbol{\omega}(i_2,i_3)]\\ &\qquad= (-1)^{\operatorname{ht}\boldsymbol{\omega}(i_2,i_3)}[\boldsymbol{f}_{i_2,i_3}][{}'\boldsymbol{\omega}_2]^{\delta_{c,d}}[{}'\boldsymbol{\omega}(i_3,i_4))]^{1-\delta_{c,d}}.\end{aligned}$$

Note that we have the following possibilities for the pair $({}'\boldsymbol{\omega}(i_2,i_3), {}'\boldsymbol{\omega}(i_2,i_4))$:

$$(\boldsymbol{\omega}(i_2+1,i_3), \boldsymbol{\omega}(i_2+1,i_4)), \quad (\boldsymbol{f}_{i_2+1}\boldsymbol{\omega}(m,i_3), \boldsymbol{f}_{i_2+1}(\boldsymbol{\omega}(m,i_4)),$$
$$(\boldsymbol{f}_{i_2+1}\boldsymbol{\omega}_{i_3,c},\ \boldsymbol{f}_{i_2+1}\boldsymbol{\omega}_{i_3,c}^{\delta_{c,d}}\boldsymbol{\omega}_2), \quad (\boldsymbol{f}_{i_2+1}, (\boldsymbol{f}_{i_2+1}\boldsymbol{\omega}_2)^{\delta_{c,d}}\boldsymbol{\omega}(i_2+1,i_4)^{1-\delta_{c,d}}).$$

In the first case, $\operatorname{ht}\boldsymbol{\omega}(i_2,i_3)=\operatorname{ht}\boldsymbol{\omega}(i_2+1,i_3)$, $\operatorname{ht}\boldsymbol{\omega}(i_2,i_3)<\operatorname{ht}\boldsymbol{\omega}(i_1,i_3)$ the inductive hypothesis applies to $i_2<i_2+1<i_3<i_4$ and gives the result. In the second case the inductive hypothesis applies to $i_2<m<i_3<i_4$ and gives the result. In the third case we use equations $(*)$ and $(\dagger\dagger)$ of Theorem 4(ii) to get the result. In the fourth case we use Theorem 4(i) if $c=d$ and (4.3) if $c\neq d$.

Case 2. — Assume that $a\neq b$ and $c\neq d$. Since $N\geqslant 5$ we may assume without loss of generality that $\operatorname{ht}\boldsymbol{\omega}(i_1,i_3)\geqslant 3$. If $\operatorname{ht}\boldsymbol{\omega}(i_1,i_2)\geqslant 3$ let $i_1<j<i_2$ be minimal with $\xi(j-1)=\xi(j+1)$. We choose $z\in\mathbb{C}(q)^\times$ so that $\boldsymbol{\omega}(i_1,i_2)\boldsymbol{\omega}_{i_1,z}^{-1}\in\mathbf{Pr}_\xi$ and calculate $[\boldsymbol{\omega}_{i_1,z}]\left([\boldsymbol{\omega}(j,i_4)][\boldsymbol{\omega}(i_2,i_3)]-[\boldsymbol{\omega}(j,i_3)][\boldsymbol{\omega}(i_2,i_4)]\right)$ in two ways to get two expressions for it; the first one by using the inductive hypothesis which shows that it is equal to

$$(-1)^{\operatorname{ht}\boldsymbol{\omega}(i_2,i_3)}[\boldsymbol{\omega}_{i_1,z}][\boldsymbol{\omega}(j,i_2)'][\boldsymbol{f}_{i_2,i_3}][{}'\boldsymbol{\omega}(i_3,i_4)]$$

and the second one by using Theorem 4(i) on the pairs $(\boldsymbol{\omega}_{i_1,z},\boldsymbol{\omega}(j,i_s))$, $s=3,4$, which gives that it is equal to

$$\begin{aligned}&[\boldsymbol{\omega}(i_1,i_4)][\boldsymbol{\omega}(i_2,i_3)]-[\boldsymbol{\omega}(i_1,i_3)][\boldsymbol{\omega}(i_2,i_4)]\\ &\qquad+[\boldsymbol{\omega}_{i_1-1,\xi(i_1)}]([{}'\boldsymbol{\omega}(j,i_4)][\boldsymbol{\omega}(i_2,i_3)]-[{}'\boldsymbol{\omega}(j,i_3)][\boldsymbol{\omega}(i_2,i_4)].\end{aligned}$$

Hence the inductive step follows if we prove that

$$\begin{aligned}&\left([\boldsymbol{\omega}_{i_1,z}][\boldsymbol{\omega}(j,i_2)']-[\boldsymbol{\omega}(i_1,i_2)']\right)[\boldsymbol{f}_{i_2,i_3}][{}'\boldsymbol{\omega}(i_3,i_4)]\\ &\qquad=(-1)^{\operatorname{ht}\boldsymbol{\omega}(i_2,i_3)}[\boldsymbol{\omega}_{i-1,\xi(i)}]\big([{}'\boldsymbol{\omega}(j,i_4)][\boldsymbol{\omega}(i_2,i_3)]-[{}'\boldsymbol{\omega}(j,i_3)][\boldsymbol{\omega}(i_2,i_4)]\big).\end{aligned}$$

This is proved by noting that

$$\boldsymbol{\omega}(j,i_2)'=\boldsymbol{\omega}(j,i_2-1)^{(1-\delta_{j,i_2-1})(1-\delta_{\xi(i_2),\xi(i_2)-2})}\big(\boldsymbol{\omega}_3^{(1-\delta_{j,i_2-1})}\boldsymbol{f}_{i_2-1}\big)^{\delta_{\xi(i_2),\xi(i_2)-2}},$$

where $\boldsymbol{\omega}_1=\boldsymbol{\omega}_3\boldsymbol{\omega}_{i_2-1,\xi(i_2)\pm 2}$ if $\xi(i_2)=\xi(i_2-2)$ and considering the different cases. In each case, Theorem 4(i) applies to the left hand side while the induction hypothesis or Theorem 4(ii) applies to the right hand side and gives the answer. As an example, suppose that $j=i_2-1$ and $\xi(i_2-2)=\xi(i_2)$. Then $\boldsymbol{\omega}(j,i_2)'=\boldsymbol{f}_j$ and the minimality of j shows that $\boldsymbol{\omega}(i_1,i_2)'=\boldsymbol{\omega}_{i_1,z}\boldsymbol{f}_j$ and hence the left hand side is zero. On the right hand side, since $\xi(i_2-1)\neq\xi(i_2+1)$ by assumption, we get ${}'\boldsymbol{\omega}(j,i_s)=\boldsymbol{\omega}(i_2,i_s)$ and so the right hand side is zero as well. We omit the details in the other cases.

Finally suppose that $j>i_2$ and let $b'\in\mathbb{C}(q)$ be such that

$$\{b,b'\}=\{\xi(i_2)+1,\xi(i_2)-1\};$$

we have the following series of equalities.

$$\begin{aligned}&\big([\boldsymbol{\omega}(i_1,i_4)]+[\boldsymbol{\omega}_{i_1-1,\xi(i_1)}][{}'\boldsymbol{\omega}(j,i_4)]\big)[\boldsymbol{\omega}(i_2,i_3)]+[\boldsymbol{\omega}(i_1,i_2)'][{}'\boldsymbol{\omega}(j,i_3)][\boldsymbol{\omega}(j,i_4)]\\ &\quad=\left([\boldsymbol{\omega}_{i_1,a}][\boldsymbol{\omega}(i_2,i_3)]+[\boldsymbol{\omega}(i_1,i_2)'][{}'\boldsymbol{\omega}(j,i_3)]\right)[\boldsymbol{\omega}(j,i_4)]\\ &\quad=[\boldsymbol{\omega}_{i_2,b'}][\boldsymbol{\omega}(i_1,i_3)][\boldsymbol{\omega}(j,i_4)]\\ &\quad=\big([\boldsymbol{\omega}(i_2,i_4)]+[\boldsymbol{\omega}_{i_2-1,\xi(i_2)}][{}'\boldsymbol{\omega}(j,i_4)]\big)[\boldsymbol{\omega}(i_1,i_3)]\\ &\quad=[\boldsymbol{\omega}(i_2,i_4)][\boldsymbol{\omega}(i_1,i_3)]+[{}'\boldsymbol{\omega}(j,i_4)]\big([\boldsymbol{\omega}(i_1,i_2)'][\boldsymbol{\omega}(j,i_3)]+[\boldsymbol{\omega}_{i_1-1,\xi(i_1)}][\boldsymbol{\omega}(i_2,i_3)]\big),\end{aligned}$$

where the first and third equality follow from applying (4.2) to the pairs $(\boldsymbol{\omega}_{i_1,a}, \boldsymbol{\omega}(j,i_4))$ and $(\boldsymbol{\omega}_{i_2,b'}, \boldsymbol{\omega}(j,i_4))$, respectively, and the second and fourth equality follow busing $(*), (**)$ of Theorem 4(ii) to $(\boldsymbol{\omega}_{i_2,b'}, \boldsymbol{\omega}(i_1,i_3))$ and $(\boldsymbol{\omega}_{i_2-1,\xi(i_2)}, \boldsymbol{\omega}(i_1,i_3))$. The inductive step follows by establishing

$$(-1)^{\operatorname{ht}\boldsymbol{\omega}(i_2,i_3)}[\boldsymbol{f}_{i_2,i_3}]['\boldsymbol{\omega}(i_3,i_4)] = [\boldsymbol{\omega}(j,i_3)]['\boldsymbol{\omega}(j,i_4)] - ['\boldsymbol{\omega}(j,i_3)][\boldsymbol{\omega}(j,i_4)]. \tag{4.6}$$

The calculations are similar to the ones done so far and we omit further details. $\square$

5. Proof of Proposition 4.4

In this section we prove Proposition 4.4 when $i < j$; the proof in the case $i > k$ is identical. We recall the statement of the proposition for the readers convenience.

Proposition. — *Suppose that* $\boldsymbol{\omega}_{ia}\boldsymbol{\omega}_{j,b}\boldsymbol{\omega} \in \mathbf{Pr}_\xi$ *with* $i < j < \min\boldsymbol{\omega}$ *and set* $\boldsymbol{\pi} = \boldsymbol{\omega}_{j,b}\boldsymbol{\omega}$. *We have*

$$[\boldsymbol{\omega}_{i,a}][\boldsymbol{\pi}] - [\boldsymbol{\omega}_{i,a}\boldsymbol{\pi}] = [\boldsymbol{\omega}_{i-1,\xi(i)}][\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega}].$$

We make some preliminary remarks about the proof. Recall from Lemma 1.2 that for all $\boldsymbol{\omega} \in \mathbf{Pr}_\xi$ the module $[\boldsymbol{\omega}]$ is prime, i.e., that it cannot be written as a tensor product of non-trivial finite-dimensional representations of $\widehat{\mathbf{U}}_q$. It follows that the module $[\boldsymbol{\omega}_{i,a}\boldsymbol{\pi}]$ is a proper subquotient of $[\boldsymbol{\omega}_{i,a}] \otimes [\boldsymbol{\pi}]$.

We claim that $[\boldsymbol{\omega}_{i-1,\xi(i)}] \otimes [\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega}]$ is irreducible. The condition $\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,b}\boldsymbol{\omega} \in \mathbf{Pr}_\xi$ forces $\xi(j-1) = \xi(j+1)$ and hence

$$j+1 < \min\boldsymbol{\omega} \implies \boldsymbol{\omega}_{i-1,\xi(i)}\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega} \notin \mathbf{Pr}_\xi,$$

and the claim follows from Theorem 3(b)(i). Otherwise, we have

$$j+1 = \min\boldsymbol{\omega} \implies \boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega} = \boldsymbol{f}_{j+1}\boldsymbol{\omega}', \quad \boldsymbol{\omega}' \in \mathbf{Pr}_\xi \cup \{1\},$$

Theorem 3(a) gives

$$[\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega}'] = [\boldsymbol{\omega}'] \otimes [\boldsymbol{f}_{j+1}], \quad [\boldsymbol{\omega}_{i-1,\xi(i)}] \otimes [\boldsymbol{f}_{j+1}] = [\boldsymbol{\omega}_{i-1,\xi(i)}\boldsymbol{f}_{j+1}],$$

while by (b)(i),

$$[\boldsymbol{\omega}_{i-1,\xi(i)}] \otimes [\boldsymbol{\omega}'] = [\boldsymbol{\omega}_{i-1,\xi(i)}\boldsymbol{\omega}'].$$

An application of Theorem 2 now proves the claim in this case.

In the first part of this section we shall show that $[\boldsymbol{\omega}_{i-1,\xi(i)}] \otimes [\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega}]$ is also a subquotient of $[\boldsymbol{\omega}_{i,a}] \otimes [\boldsymbol{\omega}_{j,b}\boldsymbol{\omega}]$; in particular,

$$\dim[\boldsymbol{\omega}_{i,a}]\dim[\boldsymbol{\omega}_{j,b}\boldsymbol{\omega}] \geqslant \dim[\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,b}\boldsymbol{\omega}] + \dim[\boldsymbol{\omega}_{i-1,\xi(i)}]\dim[\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega}].$$

The proposition clearly follows if we prove the reverse inequality. This is done by using a presentation of the graded limit of the modules $[\boldsymbol{\pi}]$, $\boldsymbol{\pi} \in \mathbf{Pr}_\xi$ given in [3] along with some additional results in the representation theory of current algebras.

5.1. — The proof of the next result is an elementary application of q-character theory for quantum affine algebras.

Lemma. — *The module* $[\boldsymbol{\omega}_{i-1,\xi(i)}\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega}]$ *occurs in the Jordan-Holder series of* $[\boldsymbol{\omega}_{i,a}]\otimes[\boldsymbol{\omega}_{j,b}\boldsymbol{\omega}]$.

Proof. — It suffices to show that there exists an ℓ-highest weight vector with ℓ-highest weight $\boldsymbol{\omega}_{i-1,\xi(i)}\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega}$ in $[\boldsymbol{\omega}_{i,a}]\otimes[\boldsymbol{\pi}]$ (resp. $[\boldsymbol{\pi}]\otimes[\boldsymbol{\omega}_{i,a}]$) if $a=\xi(i)+1$ (resp. $a=\xi(i)-1$).

But this is true by a routine argument using q-characters. Namely one observes that the element $\boldsymbol{\omega}_{i-1,\xi(i)}\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega}$ is an ℓ-weight of $[\boldsymbol{\omega}_{i,a}]\otimes[\boldsymbol{\pi}]$ but not of $[\boldsymbol{\omega}_{i,a}\boldsymbol{\pi}]$. It is then elementary to see that the corresponding eigenvector is necessarily highest weight. We omit the details. □

5.2. — We need some standard notation from the theory of simple Lie algebras. Thus, $\mathfrak{h}$ denotes a Cartan subalgebra of $\mathfrak{sl}_{n+1}$, $\{\alpha_i : 1\leqslant i\leqslant n\}$ a set of simple roots for $(\mathfrak{sl}_{n+1},\mathfrak{h})$ and $R^+=\{\alpha_{i,j}:=\alpha_i+\cdots+\alpha_j : 1\leqslant i\leqslant j\leqslant n\}$ the corresponding set of positive roots. Fix a Chevalley basis $x^{\pm}_{i,j}$, $1\leqslant i\leqslant j\leqslant n$, and h_j, $1\leqslant j\leqslant n$, for $\mathfrak{sl}_{n+1}$. Set $x^{\pm}_{j,j}=x^{\pm}_j$ and $h_{i,j}=h_i+\cdots+h_j$ for all $1\leqslant i\leqslant j\leqslant n$.

As in the earlier sections P^+ will be the set of dominant integral weights corresponding to a set $\{\omega_i : 1\leqslant i\leqslant n\}$ of fundamental weights and we set

$$P^+(1)=\{\lambda\in P^+ : \lambda(h_i)\leqslant 1,\ 1\leqslant i\leqslant n\}.$$

For $\lambda\in P^+$ let $V(\lambda)$ be an irreducible finite dimensional $\mathfrak{sl}_{n+1}$ with highest weight λ.

Let t be an indeterminate and $\mathbb{C}[t]$ the corresponding polynomial ring with complex coefficients. Denote by $\mathfrak{sl}_{n+1}[t]$ the Lie algebra with underlying vector space $\mathfrak{sl}_{n+1}\otimes\mathbb{C}[t]$ and commutator given by

$$[a\otimes f,b\otimes g]=[a,b]\otimes fg,\quad a,b\in\mathfrak{sl}_{n+1},\quad f,g\in\mathbb{C}[t].$$

Then $\mathfrak{sl}_{n+1}[t]$ and its universal enveloping algebra admit a natural $\mathbb{Z}_+$-grading given by declaring a monomial $(a_1\otimes t^{r_1})\cdots(a_p\otimes t^{r_p})$ to have grade $r_1+\cdots+r_p$, where $a_s\in\mathfrak{sl}_{n+1}$ and $r_s\in\mathbb{Z}_+$ for $1\leqslant s\leqslant p$.

5.3. — We shall be interested in the category of $\mathbb{Z}_+$-graded modules for $\mathfrak{sl}_{n+1}[t]$. An object of this category is a module V for $\mathfrak{sl}_{n+1}[t]$ which admits a compatible $\mathbb{Z}$-grading, i.e.,

$$V=\bigoplus_{s\in\mathbb{Z}}V[s],\quad (x\otimes t^r)V[s]\subset V[r+s],\quad x\in\mathfrak{sl}_{n+1},\quad r\in\mathbb{Z}_+.$$

For any $p\in\mathbb{Z}$ we let τ_p^*V be the graded module which given by shifting the grades up by p and leaving the action of $\mathfrak{sl}_{n+1}[t]$ unchanged. The morphisms between graded modules are $\mathfrak{sl}_{n+1}[t]$- maps of grade zero. A $\mathfrak{sl}_{n+1}$-module M will be regarded as an object (denoted ev_0^*M) of this category by placing M in degree zero and requiring that

$$(a\otimes t^r)m=\delta_{r,0}am,\quad a\in\mathfrak{sl}_{n+1},\quad m\in M,\quad r\in\mathbb{Z}_+.$$

For $\lambda \in P^+$, the local Weyl module $W_{\mathrm{loc}}(\lambda)$ is the $\mathfrak{sl}_{n+1}[t]$-module generated by an element w_λ with graded defining relations:

$$(5.1)\qquad (x_i^+ \otimes 1)w_\lambda = 0, \quad (h \otimes t^r)w_\lambda = \delta_{r,0}\lambda(h)w_\lambda, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}w_\lambda = 0,$$

where $1 \leqslant i \leqslant n$ and $r \in \mathbb{Z}_+$. Define a grading on $W_{\mathrm{loc}}(\lambda)$ by requiring $\mathrm{gr}\, w_\lambda = 0$. It is straightforward to see that

$$W_{\mathrm{loc}}(\omega_i) \cong_{\mathfrak{sl}_{n+1}} V(\omega_i), \quad 1 \leqslant i \leqslant n.$$

In general $W_{\mathrm{loc}}(\lambda)$ has a unique graded irreducible quotient which is isomorphic to $\mathrm{ev}_0^* V(\lambda)$. It is obtained by imposing the additional relation $(x_\alpha^- \otimes t)w_\lambda = 0$ for all $\alpha \in R^+$.

5.4. — Given $\mu \in P^+(1)$, set

$$\min \mu = \min\{i : \mu(h_i) = 1\},$$
$$R^+(\mu) = \{\alpha_{i,j} \in R^+ : 1 \leqslant i < j \leqslant n,\ \mu(h_i) = 1 = \mu(h_j) \text{ and } \mu(h_{i,j}) = 2\}.$$

Given $\lambda = 2\lambda_0 + \lambda_1 \in P^+$ with $\lambda_0 \in P^+$ and $\lambda_1 \in P^+(1)$, and $0 \leqslant i < \min \lambda_1$, define $M(\omega_i, \lambda)$ to be the graded $\mathfrak{sl}_{n+1}[t]$-module generated by an element $m_{i,\lambda}$ of grade zero satisfying the graded relations in (5.1) and

$$(5.2)\qquad (x_p^- \otimes t^{(\lambda_0+\lambda_1+\omega_i)(h_p)})m_{i,\lambda} = 0 = (x_\alpha^- \otimes t^{\lambda_0(h_\alpha)+1})m_{i,\lambda}, \qquad 1 \leqslant p \leqslant n,\ \alpha \in R^+(\lambda_1).$$

Clearly $M(\omega_i, \lambda)$ is a graded quotient of $W_{\mathrm{loc}}(\lambda)$ and

$$M(0, \omega_i) \cong_{\mathfrak{sl}_{n+1}[t]} W_{\mathrm{loc}}(\omega_i) \cong_{\mathfrak{sl}_{n+1}} V(\omega_i).$$

If $\lambda_1 \neq 0$ and $i_1 = \min \lambda_1$, then $R^+(\lambda_1 + \omega_i) = R^+(\lambda_1) \cup \{\alpha_{i,i_1}\}$ and it is simple to check that the assignment $m_{i,\lambda} \to m_{0,\lambda+\omega_i}$ gives rise to the following short exact sequence of $\mathfrak{sl}_{n+1}[t]$-modules

$$(5.3)\qquad 0 \longrightarrow \mathbf{U}(\mathfrak{g}[t])(x_{i,i_1}^- \otimes t^{\lambda_0(h_{i,i_1})+1})m_{i,\lambda} \longrightarrow M(\omega_i, \lambda) \longrightarrow M(0, \lambda + \omega_i) \longrightarrow 0.$$

The modules $M(0, \lambda)$, $\lambda \in P^+$ are examples of level two Demazure modules; the latter have been studied extensively and are usually denoted as $D(2, \lambda)$ in the literature. We now state a result which relates modules for the quantum affine algebra which are defined over $\mathbb{C}(q)$ and modules for $\mathfrak{sl}_{n+1}[t]$ which are defined over $\mathbb{C}$. Denote by $\dim_{\mathbb{C}(q)} V$ the dimension of a module V for the quantum affine algebra and by $\dim M$ the dimension over $\mathbb{C}$ of a module M for $\mathfrak{sl}_{n+1}[t]$.

Part (i) of the following result was proved in [9, Th. 1] and parts (ii) and (iii) were proved in [3, Th. 1].

Theorem 5

(i) *Let $\mu \in P^+(1)$, $\nu_1, \nu \in P^+$ with $\nu - \nu_1 \in P^+$. Then*

$$\dim M(0, 2\nu) \dim M(0, \mu) = \dim M(0, 2\nu + \mu) = \dim M(0, 2\nu_1) \dim M(0, 2(\nu - \nu_1) + \mu).$$

(ii) *Let $\xi : I \to \mathbb{Z}$ be an arbitrary height function and $\boldsymbol{\pi} \in \mathbf{Pr}_\xi$. We have*

$$\dim M(0, \operatorname{wt} \boldsymbol{\pi}) = \dim_{\mathbb{C}(q)}[\boldsymbol{\pi}].$$

(iii) *For all $1 \leqslant p \leqslant n$ we have $\dim M(0, 2\omega_p) = \dim_{\mathbb{C}(q)}[\boldsymbol{f}_p]$.* □

Corollary. — *Let $\boldsymbol{\omega}_{j,b}\boldsymbol{\omega} \in \mathbf{Pr}_\xi$ with $j < k = \min \boldsymbol{\omega}$. We have*

$$\dim_{\mathbb{C}(q)}[\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega}] = \dim M(0, \omega_{j+1} + \operatorname{wt} \boldsymbol{\omega}).$$

Proof. — If $j+1 \neq k$ then $\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega} \in \mathbf{Pr}_\xi$ and the corollary is immediate from Theorem 5(ii). Suppose that $j+1 = k$. If $\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega} = \boldsymbol{f}_{j+1}$ then the assertion of the corollary is just Theorem 5(iii). Otherwise

$$\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega} = \boldsymbol{f}_{j+1}\boldsymbol{\omega}', \quad \boldsymbol{\omega}' \in \mathbf{Pr}_\xi, \quad \operatorname{wt} \boldsymbol{\omega}' = \operatorname{wt} \boldsymbol{\omega} - \omega_{j+1}.$$

Theorem 3(a) gives $[\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega}] = [\boldsymbol{f}_{j+1}][\boldsymbol{\omega}']$. Together with parts (ii) and (iii) of Theorem 5 we get

$$\begin{aligned}\dim_{\mathbb{C}(q)}[\boldsymbol{\omega}_{j+1,\xi(j)}\boldsymbol{\omega}] &= \dim_{\mathbb{C}(q)}[\boldsymbol{f}_{j+1}] \dim_{\mathbb{C}(q)}[\boldsymbol{\omega}'] \\ &= \dim M(0, 2\omega_{j+1}) \dim M(0, \operatorname{wt} \boldsymbol{\omega} - \omega_{j+1}).\end{aligned}$$

Now, using part (i) of the theorem we see that the right hand side is equal to $\dim M(0, \omega_{j+1} + \operatorname{wt} \boldsymbol{\omega})$ and the corollary is established. □

Along with Section 5.1 we have now established the following inequality. Let $\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,b}\boldsymbol{\omega} \in \mathbf{Pr}_\xi$ with $i < j < \min \boldsymbol{\omega}$. Then

$$\begin{aligned}\dim M(\omega_i, 0) \dim M(0, \omega_j + \operatorname{wt} \boldsymbol{\omega}) &\geqslant \dim M(0, \operatorname{wt} \boldsymbol{\omega} + \omega_i + \omega_j) \\ &\quad + \dim M(0, \omega_{i-1}) \dim M(0, \omega_{j+1} + \operatorname{wt} \boldsymbol{\omega}),\end{aligned} \tag{5.4}$$

and Proposition 5 follows if we prove that the preceding inequality is actually an equality. This is done in the rest of the section.

5.5. — We deduce a consequence of the preceding discussion.

Lemma. — *Let $\lambda_0 \in P^+$, $\lambda_1 \in P^+(1)$, $\lambda = 2\lambda_0 + \lambda_1$ and $1 \leqslant i < i_1 = \min \lambda_1$. Then*

$$\begin{aligned}&\dim M(\omega_i, 0) \dim M(0, \lambda) \\ &\qquad \geqslant \dim M(0, \lambda + \omega_i) + \dim M(0, \omega_{i-1}) \dim M(0, \omega_{i_1+1} + \lambda - \omega_{i_1}).\end{aligned}$$

Proof. — By Theorem 5(i) we see that for $\mu \in \{\lambda, \lambda + \omega_i, \lambda + \omega_{i_1+1} - \omega_{i_1}\}$ we can write

$$\dim M(0, \mu) = \dim M(0, 2\lambda_0) \dim M(0, \mu - 2\lambda_0).$$

Hence the lemma follows if we prove that

$$\begin{aligned}&\dim M(\omega_i, 0) \dim M(0, \lambda_1) \\ &\qquad \geqslant \dim M(0, \lambda_1 + \omega_i) + \dim M(0, \omega_{i-1}) \dim M(0, \omega_{i_1+1} + \lambda_1 - \omega_{i_1}).\end{aligned}$$

Comparing this with (5.4), we see that it suffices to prove that we can find a height function ξ such that there exists an element $\boldsymbol{\omega}_{i,a}\boldsymbol{\pi} \in \mathbf{Pr}_\xi$ with $\lambda_1 = \operatorname{wt}\boldsymbol{\pi}$. Writing $\lambda_1 = \omega_{i_1} + \cdots + \omega_{i_k}$ take $\xi : I \to \mathbb{Z}_+$ such that

$$\xi(m) = m, \quad 1 \leqslant m \leqslant i_1, \qquad \xi(i_k + j) = \xi(i_k) + (-1)^k j, \quad 1 \leqslant j \leqslant n - i_k,$$

and
$$\xi(i_{j+1}) - \xi(i_j) = (-1)^j (i_{j+1} - i_j), \quad 1 \leqslant j \leqslant k-1.$$

If $k=1$ then $\boldsymbol{\omega}_{i,i-1}\boldsymbol{\omega}_{i_1,i_1+1} \in \mathbf{Pr}_\xi$ and otherwise $\boldsymbol{\omega}_{i,i-1}\boldsymbol{\omega}(i_1,i_k) \in \mathbf{Pr}_\xi$ and the lemma is proved. □

5.6. — Given a module V for $\mathfrak{sl}_{n+1}[t]$ and $z \in \mathbb{C}$, denote by V^z the $\mathfrak{sl}_{n+1}[t]$-module with underlying vector space V and action given by

$$(x \otimes t^r)w = (x \otimes (t+z)^r)w, \quad x \in \mathfrak{sl}_{n+1}, \quad r \in \mathbb{Z}_+, \; w \in V.$$

Suppose that V_1, V_2 are cyclic finite-dimensional $\mathfrak{sl}_{n+1}[t]$-modules with cyclic vectors v_1 and v_2 respectively. It was proved in [12] that if z_1, z_2 are distinct complex numbers, then the tensor product $V_1^{z_1} \otimes V_2^{z_2}$ is a cyclic $\mathfrak{sl}_{n+1}[t]$-module generated by $v_1 \otimes v_2$. Further, this module admits a filtration by the non-negative integers: the r-th filtered piece of $V_1^{z_1} \otimes V_2^{z_2}$ is spanned by elements of the form $(y_1 \otimes t^{s_1}) \cdots (y_m \otimes t^{s_m})(v_1 \otimes v_2)$ where $m \geqslant 0$, $y_1, \ldots, y_m \in \mathfrak{sl}_{n+1}$, $s_1, \ldots, s_m \in \mathbb{Z}_+$ and $s_1 + \cdots + s_m \leqslant r$. The associated graded space is called a fusion product and is denoted $V_1^{z_1} * V_2^{z_2}$. It admits a canonical $\mathfrak{sl}_{n+1}[t]$-module structure and is generated by the image of $v_1 \otimes v_2$ and

$$\dim (V_1^{z_1} * V_2^{z_2}) = \dim V_1 \dim V_2.$$

PROPOSITION. — *Let* $\lambda_0 \in P^+$, $\lambda_1 \in P^+(1)$, $\lambda = 2\lambda_0 + \lambda_1$, $1 \leqslant i < \min \lambda_1$ *and* $z_1 \neq z_2 \in \mathbb{C}$. *There exists a surjective map of* $\mathfrak{sl}_{n+1}[t]$*-modules*

$$M(\omega_i, \lambda) \longrightarrow M^{z_1}(0,\lambda) * M^{z_2}(0,\omega_i), \qquad m_{i,\lambda} \longrightarrow m_{0,\lambda} * m_{0,\omega_i}.$$

In particular, $\dim M(\omega_i,\lambda) \geqslant \dim M(0,\lambda) \dim M(0,\omega_i)$.

Proof. — The proposition follows if we prove that the element $\boldsymbol{m} := m_{0,\lambda} * m_{0,\omega_i}$ (which generates $M^{z_1}(0,\lambda) * M^{z_2}(0,\omega_i)$) satisfies the same relations as $m_{i,\lambda}$. We first prove that $\boldsymbol{m}$ satisfies the three relations in (5.1). The first relation in that equation is true in the tensor product $M^{z_1}(0,\lambda) \otimes M^{z_2}(0,\omega_i)$ and hence hold in the fusion product as well. For the second relation, we use the definition of $M^{z_1}(0,\lambda)$ and $M^{z_2}(0,\omega_i)$ to see that

$$\begin{aligned}(h \otimes (t-z_1)^r)&(m_{0,\lambda} \otimes m_{0,\omega_i}) \\ &= ((h \otimes t^r)m_{0,\lambda}) \otimes m_{0,\omega_i} + m_{0,\lambda} \otimes (h \otimes (t + z_2 - z_1)^r)m_{0,\omega_i},\end{aligned}$$

where the action on the right hand side is in $M(0,\lambda) \otimes M(0,\omega_i)$. If $r=0$ the relation holds in the tensor product and we are done. If $r \geqslant 1$, the first term on the right hand side is zero and the second term is $\omega_i(h)(z_2 - z_1)^r (m_{0,\lambda} \otimes m_{0,\omega_i})$. Hence

$$(h \otimes (t-z_1)^r)(m_{0,\lambda} \otimes m_{0,\omega_i}) \in \mathbf{U}(\mathfrak{sl}_{n+1}[t])[0],$$

and so in the associated graded space we get

$$(h\otimes t^r)\boldsymbol{m}=(h\otimes(t-z_1)^r)\boldsymbol{m}=0,\quad r\geqslant 1.$$

The third relation in (5.1) is immediate from the finite-dimensional representation theory of $\mathfrak{sl}_{n+1}$. Next, a straightforward calculation gives

$$(x_p^-\otimes(t-z_1)^{(\lambda_0+\lambda_1)(h_p)}(t-z_2)^{\omega_i(h_p)})(m_{0,\lambda}\otimes m_{0,\omega_i})=0,\quad 1\leqslant p\leqslant n,$$

and

$$(x_\alpha^-\otimes(t-z_1)^{\lambda_0(h_\alpha)}(t-z_2))(m_{0,\lambda}\otimes m_{0,\omega_i})=0,\quad \alpha\in R^+(\lambda_1).$$

This means that in the fusion product we have

$$(x_p^-\otimes t^{(\lambda_0+\lambda_1)(h_p)+\omega_i(h_p)})\boldsymbol{m}=0,\quad 1\leqslant p\leqslant n,$$

and

$$(x_\alpha^-\otimes t^{\lambda_0(h_\alpha)+1})\boldsymbol{m}=0,\quad \alpha\in R^+(\lambda_1),$$

which proves that $\boldsymbol{m}$ satisfies the relations in (5.2). This completes the proof of the proposition. □

5.7. — We deduce some additional relations satisfied by $m_{i,\lambda}$. Note that by the second relation in (5.1) we get, for $\alpha\in R^+$, $r\in\mathbb{Z}_+$,

$$(x_\alpha^\pm\otimes t^r)m_{i,\lambda}=0 \implies (h_\alpha\otimes t^p)(x_\alpha^\pm\otimes t^r)m_{i,\lambda}=0 \implies (x_\alpha^\pm\otimes t^{r+p})m_{i,\lambda}=0,\ \ p\in\mathbb{Z}_+.$$

Together with the first relation in (5.2) we have by a simple induction on $k-j$ that for all $1\leqslant j\leqslant k\leqslant n$,

$$(x_{j,k}^-\otimes t^r)m_{i,\lambda}=0,\quad \text{if}\quad r\geqslant(\lambda_0+\lambda_1+\omega_i)(h_{j,k}). \tag{5.5}$$

Since $(x_j^-\otimes t^{\lambda_0(h_j)+1})m_{i,\lambda}=0$, $1\leqslant j\leqslant n$, a simple calculation (see [10] for instance) shows that

$$0=(x_j^+\otimes t)^{2\lambda_0(h_j)}(x_j^-\otimes 1)^{2\lambda_0(h_j)+2}m_{i,\lambda}=(x_j^-\otimes t^{\lambda_0(h_j)})^2m_{i,\lambda}.$$

If $\alpha_{j,k}\in R^+(\lambda_1)$ then by using the preceding two relations we get

$$\begin{aligned}0&=(x_{j+1,k}^-\otimes t^{\lambda_0(h_{j+1,k})+1})(x_j^-\otimes t^{\lambda_0(h_j)})^2m_{i,\lambda}\\&=(x_{j,k}^-\otimes t^{\lambda_0(h_{j,k})+1})(x_j^-\otimes t^{\lambda_0(h_j)})m_{i,\lambda}.\end{aligned}\tag{5.6}$$

Proposition. — *Suppose that $\lambda=2\lambda_0+\lambda_1$ with $\lambda_1\in P^+(1)$ and let $i<i_1=\min\lambda$. There exists a right exact sequence of $\mathfrak{sl}_{n+1}[t]$-modules*

$$M(\omega_{i-1},\lambda-\omega_{i_1}+\omega_{i_1+1})\longrightarrow M(\omega_i,\lambda)\longrightarrow M(0,\lambda+\omega_i)\longrightarrow 0.$$

Proof. — Set

$$s=(\lambda_0+\lambda_1)(h_{i,i_1}),\quad \min(\lambda_1-\omega_{i_1})=i_2,$$
$$\lambda_2=\lambda-\omega_{i_1}+\omega_{i_1+1}=2(\lambda_0+\delta_{i_1+1,i_2}\omega_{i_1+1})+\lambda_1-\omega_{i_1}+(1-\delta_{i_1+1,i_2})\omega_{i_1+1}.$$

In view of the short exact sequence in (5.3) it suffices to prove that the assignment

$$m_{i-1,\lambda_2}\longrightarrow(x_{i,i_1}^-\otimes t^s)m_{i,\lambda}$$

extends to a well-defined morphism $M(\omega_{i-1},\lambda_2)\to M(\omega_i,\lambda)$ of $\mathfrak{g}[t]$-modules. In other words it is enough to check that the element $m=(x^-_{i,i_1}\otimes t^s)m_{i,\lambda}$ satisfies the defining relations of $M(\omega_{i-1},\lambda_2)$. This is a tedious but straightforward checking. The first thing to check is that m satisfies the defining relations of $W_{\mathrm{loc}}(\lambda_2+\omega_{i-1})$. For this, we observe that for $1\leqslant j\leqslant n$,

$$\begin{aligned}(x^+_j\otimes 1)m &= [(x^+_j\otimes 1),(x^-_{i,i_1}\otimes t^s)]m_{i,\lambda}\\ &= (A\delta_{j,i}(x^-_{i+1,i_1}\otimes t^s)+B\delta_{j,i_1}(x^-_{i,i_1-1}\otimes t^s))m_{i,\lambda},\end{aligned}$$

for some $A,B\in\mathbb{C}$. It follows from (5.5) that the right hand side is zero once we note that

$$s=(\lambda_0+\lambda_1)(h_{i,i_1})\geqslant\max\big\{(\lambda_0+\lambda_1)(h_{i+1,i_1}),\ (\lambda_0+\lambda_1)(h_{i,i_1-1})\big\}.$$

For the second relation in (5.1) we observe

$$(h\otimes t^r)m=[h\otimes t^r,x^-_{i,i_1}\otimes t^s]m_{i,\lambda}=-(\delta_{r,0}\lambda-\alpha_{i,i_1})(h)(x^-_{i,i_1}\otimes t^{s+r})m_{i,\lambda}.$$

If $r\geqslant 1$ then $s+r\geqslant(\lambda_0+\lambda_1+\omega_i)(h_{i,i_1})$ and hence the right hand side is zero by (5.5). The final relation in (5.1) holds by the standard representation theory of $\mathfrak{sl}_{n+1}$. Next, we check that m satisfies the relations in (5.2). We first show that

$$(x^-_p\otimes t^{r_p})m=0,\quad r_p=(\lambda_0+\lambda_1-\omega_{i_1}+\omega_{i_1+1}+\omega_{i-1})(h_p),\quad 1\leqslant p\leqslant n.$$

If $p\in\{i,i_1\}$ this follows from (5.6). Assume that $p\notin\{i,i_1\}$. Then

$$(x^-_p\otimes t^{r_p})m=\begin{cases}(x^-_{i,i_1}\otimes t^s)(x^-_p\otimes t^{r_p})m_{i,\lambda}, & p\notin\{i-1,i_1+1\},\\ (x^-_{i-1,i_1}\otimes t^{s+r_{i-1}})m_{i,\lambda}, & p=i-1,\\ (x^-_{p,i_1+1}\otimes t^{s+r_{i_1+1}})m_{i,\lambda}, & p=i_1+1<i_2,\\ [x^-_{i,i_1-1}\otimes t^s,x^-_{i_1,i_1+1}\otimes t^{r_{i_1+1}}]m_{i,\lambda}, & p=i_1+1=i_2.\end{cases}$$

If $p\notin\{i-1,i_1+1\}$ then $r_p=(\lambda_0+\lambda_1+\omega_i)(h_p)$ and $(x^-_p\otimes t^{r_p})m_{i,\lambda}=0$. If $p=i-1$, then

$$s+r_{i-1}=(\lambda_0+\lambda_1)(h_{i,i_1})+(\lambda_0+\lambda_1+\omega_{i-1})(h_{i-1})=(\lambda_0+\lambda_1+\omega_i)(h_{i-1,i_1})$$

and (5.5) gives $(x^-_{i-1,i_1}\otimes t^{s+r_{i-1}})m_{i,\lambda}=0$. If $p=i_1+1$ and $i_1+1\neq i_2$ a similar argument shows that $(x^-_{i,i_1+1}\otimes t^{s+r_{i_1+1}})m_{i,\lambda}=0$. If $p=i_1+1=i_2$, then one checks

$$(x^-_{i,i_1-1}\otimes t^s)m_{i,\lambda}=0=(x^-_{i_1,i_1+1}\otimes t^{r_{i_1+1}})m_{i,\lambda}.$$

In all cases the first relation in (5.2) is now established. The second relation in (5.2) follows if we prove that

$$\begin{aligned}&(x^-_\alpha\otimes t^{(\lambda_0+\delta_{i_1+1,i_2}\omega_{i_1+1})(h_\alpha)+1})(x^-_{i,i_1}\otimes t^s)m_{i,\lambda}=0,\\ &\qquad\qquad\alpha\in R^+(\lambda_1-\omega_{i_1}+(1-2\delta_{i_1+1,i_2})\omega_{i_1+1}).\end{aligned}$$

If $i_1+1=i_2$, then

$$R^+(\lambda_1-\omega_{i_1}+(1-2\delta_{i_1+1,i_2})\omega_{i_1+1})\subset R^+(\lambda_1)-\{\alpha_{i_1,i_1+1}\}$$

and if $i_1+1<i_2$ then

$$R^+(\lambda_1-\omega_{i_1}+(1-2\delta_{i_1+1,i_2})\omega_{i_1+1})=(R^+(\lambda_1)-\{\alpha_{i_1,i_2}\})\cup\{\alpha_{i_1+1,i_2}\}.$$

If $\alpha \neq \alpha_{i_1+1,i_2}$ then $[x_\alpha \otimes t^r, x_{i,i_1} \otimes t^s] = 0$, for each $r \in \mathbb{Z}_+$ and hence we get

$$(x_\alpha^- \otimes t^{\lambda_0(h_\alpha)+1})(x_{i,i_1}^- \otimes t^s)m_{i,\lambda} = (x_{i,i_1}^- \otimes t^s)(x_\alpha^- \otimes t^{\lambda_0(h_\alpha)+1})m_{i,\lambda} = 0.$$

If $\alpha = \alpha_{i_1+1,i_2}$ then $i_1 + 1 < i_2$ and so by the defining relations of $M(\omega_i, \lambda)$ we have

$$(x_{i_1,i_2}^- \otimes t^{\lambda_0(h_{i_1,i_2})+1})m_{i,\lambda} = 0 = (x_{i,i_1-1}^- \otimes t^{\lambda_0(h_{i,i_1-1})+1})m_{i,\lambda} = 0,$$

and so

$$(x_{i,i_2}^- \otimes t^{\lambda_0(h_{i,i_2})+2})m_{i,\lambda} = 0.$$

It follows that

$$(x_{i_1+1,i_2}^- \otimes t^{\lambda_0(h_{i_1+1,i_2})+1})(x_{i,i_1}^- \otimes t^s)m_{i,\lambda} = A(x_{i,i_2}^- \otimes t^{\lambda_0(h_{i,i_2})+2})m_{i,\lambda} = 0, \quad A \in \mathbb{C},$$

which completes the proof of (5.2) and so also of the proposition. $\square$

5.8. — The proof of Proposition 5 is completed in the course of establishing the following claim: for $\lambda = 2\lambda_0 + \lambda_1 \in P^+$ and $i < \min \lambda_1$, we have

$$\dim M(\omega_i, \lambda) = \dim M(\omega_i, 0) \dim M(0, \lambda). \tag{5.7}$$

The claim is proved by an induction on i. Induction begins at $i = 0$ when there is nothing to prove since $M(0,0) \cong \mathbb{C}$. Otherwise, using Proposition 5.7 we have

$$\dim M(\omega_i, \lambda) \leqslant \dim M(\omega_{i-1}, \lambda - \omega_{i_1} + \omega_{i_1+1}) + \dim M(0, \lambda + \omega_i).$$

The following equality is clear if $i = 1$, and otherwise holds by the inductive hypothesis:

$$\dim M(\omega_{i-1}, \lambda - \omega_{i_1} + \omega_{i_1+1}) = \dim M(0, \omega_{i-1}) \dim M(0, \lambda - \omega_{i_1} + \omega_{i_1+1}),$$

and hence

$$\dim M(\omega_i, \lambda) \leqslant \dim M(0, \omega_{i-1}) \dim M(0, \lambda - \omega_{i_1} + \omega_{i_1+1}) + \dim M(0, \lambda + \omega_i).$$

By Proposition 5.6 we have $\dim M(\omega_i, 0) \dim M(0, \lambda) \leqslant \dim M(\omega_i, \lambda)$ and hence we get

$$\dim M(\omega_i, 0) \dim M(0, \lambda) \leqslant \dim M(\omega_{i-1}) \dim M(0, \lambda - \omega_{i_1} + \omega_{i_1+1}) + \dim M(0, \lambda + \omega_i).$$

Lemma 5.5 now shows that all the inequalities are actually equalities and the proof of the inductive step is complete. Notice that we have also proved that the inequality in (5.4) is an equality and so the proof of Proposition 5 is also complete. $\square$

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