Tomoyuki Arakawa, Cuipo Jiang, & Anne Moreau

Simplicity of vacuum modules and associated varieties

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SIMPLICITY OF VACUUM MODULES AND ASSOCIATED VARIETIES

by Tomoyuki Arakawa, Cuipo Jiang & Anne Moreau

Abstract. — In this note, we prove that the universal affine vertex algebra associated with a simple Lie algebra $g$ is simple if and only if the associated variety of its unique simple quotient is equal to $g^*$. We also derive an analogous result for the quantized Drinfeld-Sokolov reduction applied to the universal affine vertex algebra.

Résumé (Simplicité des algèbres vertex affines et variétés associées). — Dans cet article, nous démontrons que l’algèbre vertex affine universelle associée à une algèbre de Lie simple $g$ est simple si et seulement si la variété associée à son unique quotient simple est égale à $g^*$. Nous en déduisons un résultat analogue pour la réduction quantique de Drinfeld-Sokolov appliquée à l’algèbre vertex affine universelle.

Contents
1. Introduction..................................................................169
2. Universal affine vertex algebras and associated graded vertex Poisson algebras172
3. Proof of the main result......................................................178
4. $W$-algebras and proof of Theorem 1.3.........................................187
References.......................................................................189

1. Introduction

Let $V$ be a vertex algebra, and let

$$V \longrightarrow (\text{End } V)[z, z^{-1}], \quad a \mapsto a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1},$$

be the state-field correspondence. The Zhu $C_2$-algebra [Zhu96] of $V$ is by definition the quotient space $R_V = V/C_2(V)$, where $C_2(V) = \text{span}_C \{a(-2)b \mid a, b \in V \}$, equipped with the Poisson algebra structure given by

$$[a, b] = a(-1)b, \quad \{a, b\} = a(0)b.$$
for $a, b \in V$ with $\pi := a + C_2(V)$. The associated variety $X_V$ of $V$ is the reduced scheme $X_V = \text{Specm}(R_V)$ corresponding to $R_V$. It is a fundamental invariant of $V$ that captures important properties of the vertex algebra $V$ itself (see, for example, [BFM, Zhu96, ABD04, Miy04, Ara12a, Ara15a, Ara15b, AM18a, AM17, AK18]). Moreover, the associated variety $X_V$ conjecturally [BR18] coincides with the Higgs branch of a 4D $N = 2$ superconformal field theory $\mathcal{T}$, if $V$ corresponds to a theory $\mathcal{T}$ by the 4D/2D duality discovered in [BLL+15]. Note that the Higgs branch of a 4D $N = 2$ superconformal field theory is a hyperkähler cone, possibly singular.

In the case where $V$ is the universal affine vertex algebra $V^k(\mathfrak{g})$ at level $k \in \mathbb{C}$ associated with a complex finite-dimensional simple Lie algebra $\mathfrak{g}$, the variety $X_V$ is just the affine space $\mathfrak{g}^*$ with Kirillov-Kostant Poisson structure. In the case where $V$ is the unique simple graded quotient $L_k(\mathfrak{g})$ of $V^k(\mathfrak{g})$, the variety $X_V$ is a Poisson subscheme of $\mathfrak{g}^*$ which is $G$-invariant and conic, where $G$ is the adjoint group of $\mathfrak{g}$.

Note that if the level $k$ is irrational, then $L_k(\mathfrak{g}) = V^k(\mathfrak{g})$, and hence $X_{L_k}(\mathfrak{g}) = \mathfrak{g}^*$. More generally, if $L_k(\mathfrak{g}) = V^k(\mathfrak{g})$, that is, $V^k(\mathfrak{g})$ is simple, then obviously $X_{L_k}(\mathfrak{g}) = \mathfrak{g}^*$.

In this article, we prove that the converse is true.

**Theorem 1.1.** — The equality $L_k(\mathfrak{g}) = V^k(\mathfrak{g})$ holds, that is, $V^k(\mathfrak{g})$ is simple, if and only if $X_{L_k}(\mathfrak{g}) = \mathfrak{g}^*$.

It is known by Gorelik and Kac [GK07] that $V^k(\mathfrak{g})$ is not simple if and only if

$$r^\vee(k + h^\vee) \in \mathbb{Q}_{\geq 0} \setminus \{1/m \mid m \in \mathbb{Z}_{\geq 1}\},$$

where $h^\vee$ is the dual Coxeter number and $r^\vee$ is the lacing number of $\mathfrak{g}$. Therefore, Theorem 1.1 can be rephrased as

$$X_{L_k}(\mathfrak{g}) \subseteq \mathfrak{g}^* \iff (1.1) \text{ holds.}$$

Let us mention the cases when the variety $X_{L_k}(\mathfrak{g})$ is known for $k$ satisfying (1.1).

First, it is known [Zhu96, DM06] that $X_{L_k}(\mathfrak{g}) = \{0\}$ if and only if $L_k(\mathfrak{g})$ is integrable, that is, $k$ is a nonnegative integer. Next, it is known that if $L_k(\mathfrak{g})$ is admissible [KW89], or equivalently, if

$$k + h^\vee = \frac{p}{q}, \quad p, q \in \mathbb{Z}_{\geq 1}, \quad (p, q) = 1, \quad p \geq \begin{cases} h^\vee & \text{if } (r^\vee, q) = 1, \\ \frac{h}{r^\vee} & \text{if } (r^\vee, q) \neq 1, \end{cases}$$

where $h$ is the Coxeter number of $\mathfrak{g}$, then $X_{L_k}(\mathfrak{g})$ is the closure of some nilpotent orbit in $\mathfrak{g}$ ([Ara15a]). Further, it was observed in [AM18a, AM18b] that there are cases when $L_k(\mathfrak{g})$ is non-admissible and $X_{L_k}(\mathfrak{g})$ is the closure of some nilpotent orbit. In fact, it was recently conjectured in physics [XY19] that, in view of the 4D/2D duality, there should be a large list of non-admissible simple affine vertex algebras whose associated varieties are the closures of some nilpotent orbits. Finally, there are also cases [AM17] where $X_{L_k}(\mathfrak{g})$ is neither $\mathfrak{g}^*$ nor contained in the nilpotent cone $\mathcal{N}(\mathfrak{g})$ of $\mathfrak{g}$.

In general, the problem of determining the variety $X_{L_k}(\mathfrak{g})$ is wide open.
Now let us explain the outline of the proof of Theorem 1.1. First, Theorem 1.1 is known for the critical level \( k = -h^* \) ([FF92, FG04]). Therefore, since \( R_{V^k(g)} \) is a polynomial ring \( \mathbb{C}[\mathfrak{g}^*] \), Theorem 1.1 follows from the following fact.

**Theorem 1.2.** — Suppose that the level is non-critical, that is, \( k \neq -h^* \). The image of any nonzero singular vector \( v \) of \( V^k(g) \) in the Zhu \( C_2 \)-algebra \( R_{V^k(g)} \) is nonzero.

The symbol \( \sigma(w) \) of a singular vector \( w \) in \( V^k(g) \) is a singular vector in the corresponding vertex Poisson algebra \( \text{gr}V^k(g) \cong S(t^{-1}g[t^{-1}]) \cong \mathbb{C}[J_\infty \mathfrak{g}^*] \), where \( J_\infty \mathfrak{g}^* \) is the arc space of \( \mathfrak{g}^* \). Theorem 1.2 states that the image of \( \sigma(w) \) of a non-trivial singular vector \( w \) under the projection

\[
\mathbb{C}[J_\infty \mathfrak{g}^*] \longrightarrow \mathbb{C}[\mathfrak{g}^*] = R_{V^k(g)}
\]

is nonzero, provided that \( k \) is non-critical. Here the projection (1.3) is defined by identifying \( \mathbb{C}[\mathfrak{g}^*] \) with the Zhu \( C_2 \)-algebra of the commutative vertex algebra \( \mathbb{C}[J_\infty \mathfrak{g}^*] \).

Hence, Theorem 1.2 would follow if the image of any nontrivial singular vector in \( \mathbb{C}[J_\infty \mathfrak{g}^*] \) under the projection (1.3) is nonzero. However, this is false as there are singular vectors in \( \mathbb{C}[J_\infty \mathfrak{g}^*] \) that do not come from singular vectors of \( V^k(g) \) and that belong to the kernel of (1.3) (see Section 3.4). Therefore, we do need to make use of the fact that \( \sigma(w) \) is the symbol of a singular vector \( w \) in \( V^k(g) \). We also note that the statement of Theorem 1.2 is not true if \( k \) is critical (see Section 3.4).

For this reason the proof of Theorem 1.2 is divided roughly into two parts. First, we work in the commutative setting to deduce a first important reduction (Lemma 3.1). Next, we use the Sugawara construction – which is available only at non-critical levels – in the non-commutative setting in order to complete the proof.

Now, let us consider the \( W \)-algebra \( \mathcal{W}^k(g, f) \) associated with a nilpotent element \( f \) of \( \mathfrak{g} \) at the level \( k \) defined by the generalized quantized Drinfeld-Sokolov reduction [FF90, KRW03]:

\[
\mathcal{W}^k(g, f) = H^0_{\text{DS}, f}(V^k(g)).
\]

Here, \( H^0_{\text{DS}, f}(M) \) denotes the BRST cohomology of the generalized quantized Drinfeld-Sokolov reduction associated with \( f \in N(g) \) with coefficients in a \( V^k(g) \)-module \( M \).

By the Jacobson-Morosov theorem, \( f \) embeds into an \( \mathfrak{sl}_2 \)-triple \((e, h, f)\). The Slodowy slice \( \mathcal{S}_f \) at \( f \) is the affine space \( \mathcal{S}_f = f + \mathfrak{g}^e \), where \( \mathfrak{g}^e \) is the centralizer of \( e \) in \( \mathfrak{g} \). It has a natural Poisson structure induced from that of \( \mathfrak{g}^* \) (see [GG02]), and we have [DSK06, Ara15a] a natural isomorphism \( R_{\mathcal{W}^k(g, f)} \cong \mathbb{C}[\mathcal{S}_f] \) of Poisson algebras, so that

\[
X_{\mathcal{W}^k(g, f)} = \mathcal{S}_f.
\]

The natural surjection \( V^k(g) \twoheadrightarrow L_k(g) \) induces a surjection \( \mathcal{W}^k(g, f) \twoheadrightarrow H^0_{\text{DS}, f}(L_k(g)) \) of vertex algebras ([Ara15a]). Hence the variety \( X_{H^0_{\text{DS}, f}(L_k(g))} \) is a \( \mathbb{C}^* \)-invariant Poisson subvarieties of the Slodowy slice \( \mathcal{S}_f \).

Conjecturally [KRW03, KW08], the vertex algebra \( H^0_{\text{DS}, f}(L_k(g)) \) coincides the unique simple (graded) quotient \( \mathcal{W}_k(g, f) \) of \( \mathcal{W}^k(g, f) \) provided that \( H^0_{\text{DS}, f}(L_k(g)) \neq 0 \). (This conjecture has been verified in many cases [Ara05, Ara07, Ara11, AvE19].)
As a consequence of Theorem 1.1, we obtain the following result.

Theorem 1.3. — Let $f$ be any nilpotent element of $\mathfrak{g}$. The following assertions are equivalent:

1. $V^k(\mathfrak{g})$ is simple,
2. $\mathcal{W}^k(\mathfrak{g}, f) = H^0_{\mathcal{D}, f}(L_k(\mathfrak{g}))$,
3. $X_{H^0_{\mathcal{D}, f}(L_k(\mathfrak{g}))} = S_f$.

Note that Theorem 1.3 implies that $V^k(\mathfrak{g})$ is simple if $\mathcal{W}^k(\mathfrak{g}, f) = S_f$ and $H^0_{\mathcal{D}, f}(L_k(\mathfrak{g})) \neq 0$ since $X_{H^0_{\mathcal{D}, f}(L_k(\mathfrak{g}))} \supset X_{\mathcal{W}^k(\mathfrak{g}, f)}$.

The remainder of the paper is structured as follows. In Section 2 we set up notation in the case of affine vertex algebras that will be the framework of this note. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we have compiled some known facts on Slodowy slices, $W$-algebras and their associated varieties. Theorem 1.3 is proved in this section.

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2. Universal affine vertex algebras and associated graded vertex Poisson algebras

Let $\widehat{\mathfrak{g}}$ be the affine Kac-Moody algebra associated with $\mathfrak{g}$, that is,

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K,$$

where the commutation relations are given by

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m(x|y)\delta_{m+n,0}K, \quad [K, \widehat{\mathfrak{g}}] = 0,$$

for $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. Here,

$$
\langle \cdot, \cdot \rangle = \frac{1}{2\hbar} \times \text{Killing form of } \mathfrak{g}
$$

is the usual normalized inner product. For $x \in \mathfrak{g}$ and $m \in \mathbb{Z}$, we shall write $x(m)$ for $x \otimes t^m$.

2.1. Universal affine vertex algebras. — For $k \in \mathbb{C}$, set

$$V^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes U(\mathfrak{g}[t] \oplus \mathbb{C}K) \mathbb{C}_k,$$

where $\mathbb{C}_k$ is the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which $K$ acts as multiplication by $k$ and $\mathfrak{g} \otimes \mathbb{C}[t]$ acts trivially.

By the Poincaré-Birkhoff-Witt Theorem, the direct sum decomposition, we have

$$V^k(\mathfrak{g}) \cong U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) = U(t^{-1}\mathfrak{g}[t^{-1}]).$$

The space $V^k(\mathfrak{g})$ is naturally graded,

$$V^k(\mathfrak{g}) = \bigoplus_{\Delta \in \mathbb{Z}_{\geq 0}} V^k(\mathfrak{g})_\Delta,$$
where the grading is defined by
\[
\deg(x^1(-n_1) \cdots x^r(-n_r)1) = \sum_{i=1}^r n_i, \quad r \geq 0, \quad x^i \in \mathfrak{g},
\]
with 1 the image of 1 \(\otimes 1\) in \(V^k(\mathfrak{g})\). We have \(V^k(\mathfrak{g})_0 = \mathbb{C}1\), and we identify \(\mathfrak{g}\) with \(V^k(\mathfrak{g})_1\) via the linear isomorphism defined by \(x \mapsto x(-1)1\).

It is well-known that \(V^k(\mathfrak{g})\) has a unique vertex algebra structure such that 1 is the vacuum vector,
\[
x(z) := Y(x \otimes t^{-1}, z) = \sum_{n \in \mathbb{Z}} x(n)z^{-n-1},
\]
and
\[
[T, x(z)] = \partial_z x(z)
\]
for \(x \in \mathfrak{g}\), where \(T\) is the translation operator. Here, \(x(n)\) acts on \(V^k(\mathfrak{g})\) by left multiplication, and so, one can view \(x(n)\) as an endomorphism of \(V^k(\mathfrak{g})\). The vertex algebra \(V^k(\mathfrak{g})\) is called the universal affine vertex algebra associated with \(\mathfrak{g}\) at level \(k\) [FZ92, Zhu96, LL04].

The vertex algebra \(V^k(\mathfrak{g})\) is a vertex operator algebra, provided that \(k + h^\vee \neq 0\), by the Sugawara construction. More specifically, set
\[
S = \frac{1}{2} \sum_{i=1}^d x_i(-1)x^i(-1)1,
\]
where \(\{x_i \mid i = 1, \ldots, d\}\) is the dual basis of a basis \(\{x^i \mid i = 1, \ldots, \dim \mathfrak{g}\}\) of \(\mathfrak{g}\) with respect to the bilinear form \((\cdot \mid \cdot)\), with \(d = \dim \mathfrak{g}\). Then for \(k \neq -h^\vee\), the vector \(\omega = S/(k + h^\vee)\) is a conformal vector of \(V^k(\mathfrak{g})\) with central charge
\[
c(k) = \frac{k \dim \mathfrak{g}}{k + h^\vee}.
\]
Note that, writing \(\omega(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}\), we have
\[
L_0 = \frac{1}{2(k + h^\vee)} \left( \sum_{i=1}^d x_i(0)x^i(0) + \sum_{n=1}^\infty \sum_{i=1}^d (x_i(-n)x^i(n) + x^i(-n)x_i(n)) \right),
\]
\[
L_n = \frac{1}{2(k + h^\vee)} \left( \sum_{m=1}^\infty \sum_{i=1}^d x_i(-m)x^i(m + n) + \sum_{m=0}^\infty \sum_{i=1}^d x^i(-m + n)x_i(m) \right), \quad \text{if } n \neq 0.
\]

Lemma 2.1 ([Kac90]). We have
\[
[L_n, x(m)] = -nx(m + n), \quad \text{for } x \in \mathfrak{g}, \ m, n \in \mathbb{Z},
\]
and \(L_n 1 = 0\) for \(n \geq -1\).

We have \(V^k(\mathfrak{g})_\Delta = \{v \in V^k(\mathfrak{g}) \mid L_0 v = \Delta v \}\) and \(T = L_{-1}\) on \(V^k(\mathfrak{g})\), provided that \(k + h^\vee \neq 0\).

Any graded quotient of \(V^k(\mathfrak{g})\) as \(\hat{\mathfrak{g}}\)-module has the structure of a quotient vertex algebra. In particular, the unique simple graded quotient \(L_k(\mathfrak{g})\) is a vertex algebra, and is called the simple affine vertex algebra associated with \(\mathfrak{g}\) at level \(k\).
2.2. Associate graded vertex Poisson algebras of affine vertex algebras

It is known by Li [Li05] that any vertex algebra $V$ admits a canonical filtration $F^*V$, called the Li filtration of $V$. For a quotient $V$ of $V^k(g)$, $F^*V$ is described as follows. The subspace $F^pV$ is spanned by the elements

$$y_1(-n_1 - 1) \cdots y_r(-n_r - 1)1$$

with $y_i \in g$, $n_i \in \mathbb{Z}_{\geq 0}$, $n_1 + \cdots + n_r \geq p$. We have

$$V = F^0V \supset F^1V \supset \cdots, \quad \bigcap_p F^pV = 0,$$

$T F^pV \subset F^{p+1}V$,

$$a(n)F^qV \subset F^{p+q-n-1}V \text{ for } a \in F^pV, \quad n \in \mathbb{Z},$$

$$a(n)F^qV \subset F^{p+q-n}V \text{ for } a \in F^pV, \quad n \geq 0.$$  

(2.2)

Here we have set $F^pV = V$ for $p < 0$.

Let $gr^F V = \bigoplus_p F^pV/F^{p+1}V$ be the associated graded vector space. The space $gr^F V$ is a vertex Poisson algebra by

$$\sigma_p(a)\sigma_q(b) = \sigma_{p+q}(a(-1)b),$$

$$T\sigma_p(a) = \sigma_{p+1}(Ta),$$

$$\sigma_p(a)\sigma_q(b) = \sigma_{p+q-n}(a(n)b)$$

for $a,b \in V$, $n \geq 0$, where $\sigma_p: F^pV \to F^pV/F^{p+1}V$ is the principal symbol map. In particular, $gr^F V$ is a $g[t]$-module by the correspondence

$$g[t] \ni x(n) \mapsto \sigma_0(x)(n) \in \text{End}(gr^F V)$$

for $x \in g$, $n \geq 0$.

The filtration $F^*V$ is compatible with the grading: $F^pV = \bigoplus_{\Delta \in \mathbb{Z}_{\geq 0}} F^p V_{\Delta}$, where $F^p V_{\Delta} := V_{\Delta} \cap F^p V$.

Let $U_*(t^{-1}g[t^{-1}])$ be the PBW filtration of $U(t^{-1}g[t^{-1}])$, that is, $U_*(t^{-1}g[t^{-1}])$ is the subspace of $U(t^{-1}g[t^{-1}])$ spanned by monomials $y_1 y_2 \cdots y_r$ with $y_i \in g$, $r \leq p$.

Define

$$G_p V = U_p(t^{-1}g[t^{-1}])1.$$

Then $G_\bullet V$ defines an increasing filtration of $V$. We have

$$F^p V_{\Delta} = G_{\Delta-p} G_{\Delta},$$

(2.4)

where $G_p V_{\Delta} := G_p V \cap V_{\Delta}$, see [Ara12a, Prop. 2.6.1]. Therefore, the graded space $gr^G V = \bigoplus_{p \in \mathbb{Z}_{\geq 0}} G_p V/G_{p-1} V$ is isomorphic to $gr^F V$. In particular, we have

$$gr V^k(g) \cong gr U_*(t^{-1}g[t^{-1}]) \cong S(t^{-1}g[t^{-1}]).$$

The action of $g[t]$ on $gr V^k(g) = S(t^{-1}g[t^{-1}])$ coincides with the one induced from the action of $g[t]$ on $g[t, t^{-1}]/g[t] \cong t^{-1}g[t^{-1}]$. More precisely, the element $x(m)$, for $x \in g$.
and $m \in \mathbb{Z}_{\geq 0}$, acts on $S(t^{-1}g[t^{-1}])$ as follows:

$$x(m) \cdot 1 = 0,$$

(2.5) $$x(m) \cdot v = \sum_{j=1}^{r} \sum_{n_j > 0} y_1(-n_1) \cdots [x, y_j](m - n_j) \cdots y_r(-n_r),$$

if $v = y_1(-n_1) \cdots y_r(-n_r)$ with $y_i \in g$, $n_1, \ldots, n_r \in \mathbb{Z}_{> 0}$.

2.3. Zhu’s $C_2$-algebras and associated varieties of affine vertex algebras

We have [Li05, Lem. 2.9]

$$F^pV = \text{span}_\mathbb{C}\{a_{(-i-1)}b \mid a \in V, i \geq 1, b \in F^{p-1}V\}$$

for all $p \geq 1$. In particular,

$$F^1V = C_2(V),$$

where $C_2(V) = \text{span}_\mathbb{C}\{a_{(-2)}b \mid a, b \in V\}$. Set

$$R_V = V/C_2(V) = F^0V/F^1V \subset \text{gr}F^V.$$

It is known by Zhu [Zhu96] that $R_V$ is a Poisson algebra. The Poisson algebra structure can be understood as the restriction of the vertex Poisson structure of $\text{gr}F^V$. It is given by

$$\pi \cdot \overline{b} = \overline{a_{(-1)}b}, \quad \{\pi, \overline{b}\} = \overline{a_{(0)}b},$$

for $a, b \in V$, where $\pi = a + C_2(V)$.

By definition [Ara12a], the associated variety of $V$ is the reduced scheme

$$X_V := \text{Specm}(R_V).$$

It is easily seen that

$$F^1V^k(g) = C_2(V^k(g)) = t^{-2}g[t^{-1}]V^k(g).$$

The following map defines an isomorphism of Poisson algebras

$$\mathbb{C}[g^*] \cong S(g) \rightarrow R_{V^k(g)},$$

$$g \ni x \mapsto x(-1)1 + t^{-2}g[t^{-1}]V^k(g).$$

Therefore, $R_{V^k(g)} \cong \mathbb{C}[g^*]$ and so, $X_{V^k(g)} \cong g^*$.

More generally, if $V$ is a quotient of $V^k(g)$ by some ideal $N$, then we have

(2.6) $$R_V \cong \mathbb{C}[g^*]/IN$$

as Poisson algebras, where $IN$ is the image of $N$ in $R_{V^k(g)} = \mathbb{C}[g^*]$. Then $X_V$ is just the zero locus of $IN$ in $g^*$. It is a closed $G$-invariant conic subset of $g^*$.

Identifying $g^*$ with $g$ through the bilinear form $(\cdot | \cdot)$, one may view $X_V$ as a subvariety of $g$. 

J.É.P. – M., 2021, tome 8
2.4. PBW basis. — Let $\Delta_+ = \{\beta_1, \ldots, \beta_q\}$ be the set of positive roots for $\mathfrak{g}$ with respect to a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $q = (d - \ell)/2$ and $\ell = \text{rk}(\mathfrak{g})$.

Form now on, we fix a basis
\[
\{u^i, e_{\beta_j}, f_{\beta_j} \mid i = 1, \ldots, \ell, j = 1, \ldots, q\}
\]
of $\mathfrak{g}$ such that $\{u^i \mid i = 1, \ldots, \ell\}$ is an orthonormal basis of $\mathfrak{h}$ with respect to $(\mid \mid)$ and $(e_{\beta_i}, f_{\beta_j}) = 1$ for $i = 1, 2, \ldots, q$. In particular, $[e_{\beta_i}, f_{\beta_j}] = \beta_i$ for $i = 1, \ldots, q$ (see, for example, [Hum72, Prop. 8.3]), where $\mathfrak{h}^*$ and $\mathfrak{h}$ are identified through $(\mid \mid)$. One may also assume that $h_t(\beta_i) \leq h_t(\beta_j)$ for $i < j$, where $h_t(\beta_i)$ stands for the height of the positive root $\beta_i$.

We define the structure constants $c_{\alpha, \beta}$ by
\[
[e_{\alpha}, e_{\beta}] = e_{\alpha, \beta} e_{\alpha + \beta},
\]
provided that $\alpha$, $\beta$, and $\alpha + \beta$ are in $\Delta$. Our convention is that $e_{-\alpha}$ stands for $f_\alpha$ if $\alpha \in \Delta_+$. If $\alpha$, $\beta$ and $\alpha + \beta$ are in $\Delta_+$, then from the equalities,

\[
c_{-\alpha, \alpha + \beta} = (f_{\beta}[[e_{\alpha}, e_{\alpha + \beta}]] = -(f_{\beta}[[e_{\alpha + \beta}, f_{\alpha}]] = -(f_{\beta} [[e_{\alpha + \beta}], f_{\alpha}]) = -c_{-\beta, \alpha + \beta},
\]

we get

\[
c_{-\alpha, \alpha + \beta} = -c_{-\beta, \alpha + \beta}.
\]

By (2.1), the above basis of $\mathfrak{g}$ induces a basis of $V^k(\mathfrak{g})$ consisted of $1$ and the elements of the form

\[
z = z^{(+)} z^{(-)} z^{(0)} 1,
\]
with

\[
z^{(+)} := e_{\beta_1} (-1)^{a_{1,1}} \cdots e_{\beta_1} (-r_1)^{a_{1,r_1}} \cdots e_{\beta_q} (-1)^{a_{q,1}} \cdots e_{\beta_q} (-r_q)^{a_{q,r_q}},
\]
\[
z^{(-)} := f_{\beta_1} (-1)^{b_{1,1}} \cdots f_{\beta_1} (-s_1)^{b_{1,s_1}} \cdots f_{\beta_q} (-1)^{b_{q,1}} \cdots f_{\beta_q} (-s_q)^{b_{q,s_q}},
\]
\[
z^{(0)} := u^1 (-1)^{c_{1,1}} \cdots u^1 (-t_1)^{c_{1,t_1}} \cdots u^\ell (-1)^{c_{\ell,1}} \cdots u^\ell (-t_\ell)^{c_{\ell,t_\ell}},
\]

where $r_1, \ldots, r_q, s_1, \ldots, s_q, t_1, \ldots, t_\ell$ are positive integers, and $a_{i,m}, b_{i,n}, c_{i,j}$, for $l = 1, \ldots, q$, $m = 1, \ldots, r_i$, $n = 1, \ldots, s_i$, $i = 1, \ldots, \ell$, $j = 1, \ldots, t_i$ are nonnegative integers such that at least one of them is nonzero.

**Definition 2.2.** — Each element $x$ of $V^k(\mathfrak{g})$ is a linear combination of elements in the above PBW basis, each of them will be called a **PBW monomial** of $x$.

**Definition 2.3.** — For a PBW monomial $v$ as in (2.8), we call the integer

\[
\text{depth}(v) = \sum_{i=1}^q \left( \sum_{j=1}^{r_i} a_{i,j} (j - 1) + \sum_{j=1}^{s_i} b_{i,j} (j - 1) \right) + \sum_{i=1}^{\ell} \sum_{j=1}^{t_i} c_{i,j} (j - 1)
\]

the **depth** of $v$. In other words, a PBW monomial $v$ has depth $p$ means that $v \in F^p V^k(\mathfrak{g})$ and $v \not\in F^{p+1} V^k(\mathfrak{g})$. By convention, depth$(1) = 0$. 

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*J.É.P. — M., 2004, tome 8*
For a PBW monomial \( v \) as in (2.8), we call degree of \( v \) the integer

\[
\deg(v) = \sum_{i=1}^{q} \left( \sum_{j=1}^{r_i} a_{i,j} + \sum_{j=1}^{s_i} b_{i,j} \right) + \sum_{i=1}^{\ell} \sum_{j=1}^{e_{i,j}} c_{i,j},
\]

In other words, \( v \) has degree \( p \) means that \( v \in G_p V^k(\mathfrak{g}) \) and \( v \not\in G_{p-1} V^k(\mathfrak{g}) \) since the PBW filtration of \( V^k(\mathfrak{g}) \) coincides with the standard filtration \( G_* V^k(\mathfrak{g}) \). By convention, \( \deg(1) = 0 \).

Recall that a singular vector of a \( \mathfrak{g}[t] \)-representation \( M \) is a vector \( m \in M \) such that \( c_\alpha(0) \cdot m = 0 \), for all \( \alpha \in \Delta_+ \), and \( f_\theta(1) \cdot m = 0 \), where \( \theta \) is the highest positive root of \( \mathfrak{g} \). From the identity

\[
L_{-1} = \frac{1}{k + h^*} \left( \sum_{i=1}^{\ell} \sum_{m=0}^{\infty} u^i(-1 - m)u^i(m) + \sum_{\alpha \in \Delta_+} \sum_{m=0}^{\infty} (e_\alpha(-1 - m)f_\alpha(m) + f_\alpha(-1 - m)e_\alpha(m)) \right),
\]

we deduce the following easy observation, which will be useful in the proof of the main result.

**Lemma 2.4.** If \( w \) is a singular vector of \( V^k(\mathfrak{g}) \), then

\[
L_{-1}w = \frac{1}{k + h^*} \left( \sum_{i=1}^{\ell} u^i(-1)u^i(0) + \sum_{\alpha \in \Delta_+} e_\alpha(-1)f_\alpha(0) \right) w.
\]

**2.5. Basis of associated graded vertex Poisson algebras.** Note that \( \operatorname{gr} V^k(\mathfrak{g}) = S(t^{-1}\mathfrak{g}[t^{-1}]) \) has a basis consisting of \( 1 \) and elements of the form (2.8). Similarly to Definition 2.2, we have the following definition.

**Definition 2.5.** Each element \( x \) of \( S(t^{-1}\mathfrak{g}[t^{-1}]) \) is a linear combination of elements in the above basis, each of them will be called a monomial of \( x \).

As in the case of \( V^k(\mathfrak{g}) \), the space \( S(t^{-1}\mathfrak{g}[t^{-1}]) \) has two natural gradations. The first one is induced from the degree of elements as polynomials. We shall write \( \deg(v) \) for the degree of a homogeneous element \( v \in S(t^{-1}\mathfrak{g}[t^{-1}]) \) with respect to this gradation.

The second one is induced from the Li filtration via the isomorphism \( S(t^{-1}\mathfrak{g}[t^{-1}]) \cong \operatorname{gr}^L V^k(\mathfrak{g}) \). The degree of a homogeneous element \( v \in S(t^{-1}\mathfrak{g}[t^{-1}]) \) with respect to the gradation induced by Li filtration will be called the depth of \( v \), and will be denoted by \( \operatorname{depth}(v) \).

Notice that any element \( v \) of the form (2.8) is homogeneous for both gradations. By convention, \( \deg(1) = \operatorname{depth}(1) = 0 \).

As a consequence of (2.5), we get that

\[
\deg(x(m) \cdot v) = \deg(v) \quad \text{and} \quad \operatorname{depth}(x(m) \cdot v) = \operatorname{depth}(v) - m,
\]

for \( m \geq 0 \), \( x \in \mathfrak{g} \), and any homogeneous element \( v \in S(t^{-1}\mathfrak{g}[t^{-1}]) \) with respect to both gradations.
In the sequel, we will also use the following notation, for \( v \) of the form (2.8), viewed either as an element of \( V^k(\mathfrak{g}) \) or of \( S(t^{-1}\mathfrak{g}[t^{-1}]) \):

\[
\text{deg}^{(0)}(v) := \sum_{j=1}^\ell c_{j,1},
\]

which corresponds to the degree of the element obtained from \( v^{(0)} \) by keeping only the terms of depth 0, that is, the terms \( w^i(1), i = 1, \ldots, \ell \).

Notice that a nonzero depth-homogeneous element of \( S(t^{-1}\mathfrak{g}[t^{-1}]) \) has depth 0 if and only if its image in

\[
R_{V^k(\mathfrak{g})} = V^k(\mathfrak{g})/t^{-2}\mathfrak{g}[t^{-1}]V^k(\mathfrak{g})
\]

is nonzero.

3. PROOF OF THE MAIN RESULT

This section is devoted to the proof of Theorem 1.1.

3.1. Strategy. — Let \( N_k \) be the maximal graded submodule of \( V^k(\mathfrak{g}) \), so that \( L_k(\mathfrak{g}) = V^k(\mathfrak{g})/N_k \). Our aim is to show that if \( V^k(\mathfrak{g}) \) is not simple, that is, \( N_k \neq \{0\} \), then \( X_{L_k(\mathfrak{g})} \) is strictly contained in \( \mathfrak{g}^* \cong \mathfrak{g} \), that is, the image \( I_k := I_{N_k} \) of \( N_k \) in \( R_{V^k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*] \) is nonzero.

For \( k = -h^\vee \), it follows from [FG04] that \( I_k \) is the defining ideal of the nilpotent cone \( N(\mathfrak{g}) \) of \( \mathfrak{g} \), and so \( X_{L_k(\mathfrak{g})} = N(\mathfrak{g}) \) (see [Ara12b] or Section 3.4 below). Hence, there is no loss of generality in assuming that \( k + h^\vee \neq 0 \).

Henceforth, we suppose that \( k + h^\vee \neq 0 \) and that \( V^k(\mathfrak{g}) \) is not simple, that is, \( N_k \neq \{0\} \). Then there exists at least one non-trivial (that is, nonzero and different from 1) singular vector \( w \) in \( V^k(\mathfrak{g}) \). Theorem 1.2 states that the image of \( w \) in \( I_k \) is non-zero, and this proves Theorem 1.1. The rest of this section is devoted to the proof of Theorem 1.2.

Let \( w \) be a nontrivial singular vector of \( V^k(\mathfrak{g}) \). One can assume that \( w \in F^pV^k(\mathfrak{g}) \setminus F^{p+1}V^k(\mathfrak{g}) \) for some \( p \in \mathbb{Z}_{\geq 0} \).

The image

\[
\overline{w} := \sigma(w)
\]

of this singular vector in \( S(t^{-1}\mathfrak{g}[t^{-1}]) \cong \text{gr} F V^k(\mathfrak{g}) \) is a nontrivial singular vector of \( S(t^{-1}\mathfrak{g}[t^{-1}]) \). Here \( \sigma : V^k(\mathfrak{g}) \to \text{gr} F V^k(\mathfrak{g}) \) stands for the principal symbol map. It follows from (2.9) that one can assume that \( \overline{w} \) is homogeneous with respect to both gradations on \( S(t^{-1}\mathfrak{g}[t^{-1}]) \). In particular \( \overline{w} \) has depth \( p \). It is enough to show that \( p = 0 \), that is, \( \overline{w} \) has depth zero. Write

\[
w = \sum_{j \in J} \lambda_j w^j,
\]

where \( J \) is a finite index set, \( \lambda_j \) are nonzero scalar for all \( j \in J \), and \( w_j \) are pairwise distinct PBW monomials of the form (2.8). Let \( I \subset J \) be the subset of \( i \in J \) such that
depth $\varpi^i = p = \text{depth } \varpi$. Since $w \in FpV^k(\mathfrak{g}) \times Fp^{+1}V^k(\mathfrak{g})$, the set $I$ is nonempty. Here, $\varpi^i$ stands for the image of $w^i$ in $\text{gr}FV^k(\mathfrak{g}) \cong S(t^{-1}\mathfrak{g}[t^{-1}])$. More specifically, for any $j \in I$, write

\begin{align*}
(3.1) \quad w^j &= (w^j)^{(+)}(w^j)^{(-)}(w^j)^{(0)} \mathbf{1},
\end{align*}

with

\begin{align*}
(w^j)^{(+)} &= e_{\beta_1}(-1)^{a_{1,j}} \cdots e_{\beta_1}(-1)^{a_{r_1,j}} \cdots e_{\beta_q}(-1)^{a_{q,j}} \cdots e_{\beta_q}(-r_q)^{a_{q,j}}, \\
(w^j)^{(-)} &= f_{\beta_1}(-r_1)^{b_{1,j}} \cdots f_{\beta_1}(-s_1)^{b_{1,j}} \cdots f_{\beta_q}(-s_q)^{b_{q,j}} \cdots f_{\beta_q}(-1)^{b_{q,j}}, \\
(w^j)^{(0)} &= u^1(-1)^{c_{1,j}} \cdots u^l(-1)^{c_{l,j}} \cdots u^\ell(-1)^{c_{\ell,j}} \cdots u^\ell(-t_j)^{c_{\ell,j}},
\end{align*}

where $r_1, \ldots, r_q, s_1, \ldots, s_q, t_1, \ldots, t_\ell$ are nonnegative integers, and $a_{i,m}^{(j)}, b_{i,n}^{(j)}, c_{i,p}^{(j)}$, for $l = 1, \ldots, q$, $m = 1, \ldots, r_i$, $n = 1, \ldots, s_i$, $i = 1, \ldots, \ell$, $p = 1, \ldots, t_i$, are nonnegative integers such that at least one of them is nonzero.

The integers $r_i$’s, for $l = 1, \ldots, q$, are chosen so that at least one of the $a_{i,m}^{(j)}$’s is nonzero for $j$ running through $J$ if for some $j \in J$, $(w^j)^{(+) \neq 1)$. Otherwise, we just set $(w^j)^{(+) = 1)$. Similarly are defined the integers $s_i$’s and $t_m$’s, for $l = 1, \ldots, q$ and $m = 1, \ldots, \ell$. By our assumption, note that for all $i \in I$,

\begin{align*}
\sum_{n=1}^{q} \left( \sum_{l=1}^{r_n} a_{n,l}^{(i)} + \sum_{l=1}^{s_n} b_{n,l}^{(i)} \right) + \sum_{n=1}^{\ell} \sum_{l=1}^{t_n} c_{n,l}^{(i)} = \deg(\varpi) \\
\sum_{n=1}^{q} \left( \sum_{l=1}^{r_n} a_{n,l}^{(i)}(l-1) + \sum_{l=1}^{s_n} b_{n,l}^{(i)}(l-1) \right) + \sum_{n=1}^{\ell} \sum_{l=1}^{t_n} c_{n,l}^{(i)}(l-1) = \text{depth}(\varpi) = p.
\end{align*}

3.2. **Technical lemma.** — In this paragraph we remain in the commutative setting, and we only deal with $\varpi \in S(t^{-1}\mathfrak{g}[t^{-1}])$ and its monomials $\varpi^i$’s, for $i \in I$.

Recall from (2.10) that,

\[ \text{deg}_{-1}(w^i) = \sum_{j=1}^{\ell} c_{j,1}^{(i)} \]

for $i \in I$. Set

\[ d_{-1}^{(0)}(I) := \max\{\text{deg}_{-1}(w^i) \mid i \in I\}, \]

and

\[ I_{-1}^{(0)} := \{i \in I \mid \text{deg}_{-1}(w^i) = d_{-1}^{(0)}(I)\}. \]

If $(w^i)^{(0)} = 1$ for all $i \in I$, we just set $d_{-1}^{(0)}(I) = 0$ and then $I_{-1}^{(0)} = I$.

**Lemma 3.1.** — If $i \in I_{-1}^{(0)}$, then $(\varpi^i)^{(+) = 1)$. In other words, for $i \in I_{-1}^{(0)}$, we have $\varpi^i = (\varpi^i)^{(+) \mathbf{1)}$. \]

**Proof.** — Suppose the assertion is false. Then for some positive roots $\beta_{j_1}, \ldots, \beta_{j_t} \in \Delta_+$, one can write for any $i \in I_{-1}^{(0)}$,

\begin{align*}
(3.2) \quad (\varpi^i)^{(+) = f_{\beta_{j_1}}(-1)^{b_{1,j_1}} \cdots f_{\beta_{j_1}}(-s_{j_1})^{b_{1,j_1}} \cdots f_{\beta_{j_1}}(-1)^{b_{1,j_1}} \cdots f_{\beta_{j_1}}(-s_{j_1})^{b_{1,j_1}},}
\end{align*}
so that for any $l \in \{1, \ldots, t\}$,
\[
\{b_{j_1,s_{j_1}}^{(l)} \mid i \in I_1^{(l)}\} \neq \{0\}.
\]
Set
\[
K_{-1}^{(0)} = \{i \in I_1^{(0)} \mid b_{j_1,s_{j_1}}^{(i)} > 0\}.
\]
Since $\overline{w}$ is a singular vector of $S(t^{-1}g[t^{-1}])$ and $s_{j_1} - 1 \in \mathbb{Z}_{\geq 0}$, we have
\[
e_{\beta_{j_1}}(s_{j_1} - 1) \cdot \overline{w} = 0.
\]
On the other hand, using the action of $g[t]$ on $S(t^{-1}g[t^{-1}])$ as described by (2.5), we see that
\[
0 = e_{\beta_{j_1}}(s_{j_1} - 1) \cdot \overline{w} = \sum_{i \in K_{-1}^{(0)}} \lambda_i b_{j_1,s_{j_1}}^{(i)} v^i + v,
\]
where for $i \in K_{-1}^{(0)},$
\[
v^i := (\overline{w})^{(0)} b_{j_1}^{(i)} (-1)^{s_{j_1} - 1} f_{\beta_{j_1}} (-1)^{s_{j_1}} \cdots f_{\beta_{j_1}} (-s_{j_1})^{b_{j_1,s_{j_1}}^{(i)} - 1} \cdots f_{\beta_{j_1}} (-1)^{s_{j_1} - 1} f_{\beta_{j_1}} (-s_{j_1})^{b_{j_1,s_{j_1}}^{(i)} (w^i)^{(+)} 1},
\]
and $v$ is a linear combination of monomials $x$ such that
\[
\deg_{-1}^{(0)}(x) \leq d_{-1}^{(0)}(I).
\]
Indeed, for $i \in K_{-1}^{(0)}$, it is clear that
\[
e_{\beta_{j_1}}(s_{j_1} - 1) \cdot w^i = b_{j_1,s_{j_1}}^{(i)} v^i + y^i,
\]
where $y^i$ is a linear combination of monomials $y$ such that $\deg_{-1}^{(0)}(y) \leq d_{-1}^{(0)}(I)$ because $ht(\beta_{j_1}) \leq ht(\beta_{j_1})$ for all $l \in \{1, \ldots, t\}$. Next, for $i \in I_1^{(0)} \setminus K_{-1}^{(0)}$, $e_{\beta_{j_1}}(s_{j_1} - 1) \cdot \overline{w}$ is a linear combination of monomials $x$ such that $\deg_{-1}^{(0)}(x) \leq d_{-1}^{(0)}(I)$ because $b_{j_1,s_{j_1}}^{(i)} = 0$.
Finally, for $i \in I \setminus I_1^{(0)}$, we have $\deg_{-1}^{(0)}(\overline{w}) < d_{-1}^{(0)}(I)$ and, hence, $e_{\beta_{j_1}}(s_{j_1} - 1) \cdot \overline{w}$ is a linear combination of monomials $z$ such that $\deg_{-1}^{(0)}(z) \leq d_{-1}^{(0)}(I)$ as well.

Now, note that for each $i \in K_{-1}^{(0)},$
\[
\deg_{-1}^{(0)}(v^i) = \deg_{-1}^{(0)}(\overline{w}) + 1 = d_{-1}^{(0)}(I) + 1.
\]
Hence by (3.3) we get a contradiction because all monomials $v^i$, for $i$ running through $K_{-1}^{(0)}$, are linearly independent while $\lambda_i b_{j_1,s_{j_1}}^{(i)} \neq 0$, for $i \in K_{-1}^{(0)}$. This concludes the proof of the lemma. \hfill \Box

3.3. Use of Sugawara operators. — Recall that $w = \sum_{j \in J} \lambda_j w^j$. Let $J_1 \subseteq J$ be such that for $i \in J_1$, $(w^i)^\wedge = 1$. Then by Lemma 3.1,
\[
\emptyset \neq I_1^{(0)} \subseteq J_1.
\]
So $J_1 \neq \emptyset$. Set
\[
d_{-1}^{(0)} := d_{-1}^{(0)}(J_1) = \max\{\deg_{-1}^{(0)}(w^i) \mid i \in J_1\},
\]
and
\[ J_{1}^{0} := \{ i \in J_{1} \mid \deg_{-1}^{(0)}(w^{i}) = d_{-1}^{(0)} \}. \]
Then \( d_{-1}^{(0)}(I) \leq d_{-1}^{(0)} \). Set
\[ d^{+} := \max\{\deg(w^{i})(+) \mid i \in J_{1}^{(0)}\} \]
and let
\[ J^{+} = \{ i \in J_{1}^{(0)} \mid \deg(w^{i})(+) = d^{+} \} \subseteq J_{1}^{(0)}. \]
Our next aim is to show that for \( i \in J^{+} \), \( w^{i} \) has depth zero, whence \( p = 0 \) since \( p \) is by definition the smallest depth of the \( w^{i} \)'s, and so the image of \( w \) in \( R_{V^{k}(g)} = F^{0}V^{k}(g)/F^{1}V^{k}(g) \) is nonzero.

This will be achieved in this paragraph through the use of the Sugawara construction.

Recall that by Lemma 2.4,
\[ L_{-1}w = \tilde{L}_{-1}w \]
thus \( w \) is a singular vector of \( V^{k}(g) \), where
\[ \tilde{L}_{-1} := \frac{1}{k + h'} \left( \sum_{i=1}^{t} u^{i}(-1)u^{i}(0) + \sum_{\alpha \in \Delta_{+}} e_{\alpha}(-1)f_{\alpha}(0) \right). \]

**Lemma 3.2.** — Let \( z \) be a PBW monomial of the form (2.8). Then \( \tilde{L}_{-1}z \) is a linear combination of PBW monomials \( x \) satisfying all the following conditions:

(a) \( \deg(x^{(+)}) \leq \deg(z^{(+)}) + 1 \) and \( \deg(x^{(0)}) \leq \deg(z^{(0)}) + 1 \),

(b) \( \text{if } z^{(-)} \neq 1 \), then \( x^{(-)} \neq 1 \).

(c) \( \text{if } x^{(-)} = z^{(-)} \), then either \( \deg(x^{(0)}) = \deg(z^{(0)}) + 1 \), or \( x^{(0)} = z^{(0)} \).

(d) \( \text{if } \deg(x^{(0)}) = \deg(z^{(0)}) + 1 \), then \( x^{(-)} = z^{(-)} \) and \( \deg(x^{(+)}) \leq \deg(z^{(+)}) \).

**Proof.** — Parts (a)–(c) are easy to see. We only prove (d). Assume that \( \deg(x^{(0)}) = \deg(z^{(0)}) + 1 \). Either \( x \) comes from the term \( \sum_{i=1}^{t} u^{i}(-1)u^{i}(0)z \), or it comes from a term \( e_{\alpha}(-1)f_{\alpha}(0)z \) for some \( \alpha \in \Delta_{+} \).

If \( x \) comes from the term \( \sum_{i=1}^{t} u^{i}(-1)u^{i}(0)z \), then it is obvious that \( x^{(-)} = z^{(-)} \) and \( x^{(+) } = z^{(+) } \).

Assume that \( x \) comes from \( e_{\alpha}(-1)f_{\alpha}(0)z \) for some \( \alpha \in \Delta_{+} \). We have
\[ e_{\alpha}(-1)f_{\alpha}(0)z = e_{\alpha}(-1)[f_{\alpha}(0), z^{(+)}]z^{(-)}z^{(0)}1 + e_{\alpha}(-1)z^{(+)}[f_{\alpha}(0), z^{(-)}]z^{(0)}1 \]
\[ + e_{\alpha}(-1)z^{(+)}z^{(-)}[f_{\alpha}(0), z^{(0)}]1, \]
Clearly, any PBW monomials \( x \) from
\[ e_{\alpha}(-1)z^{(+)}[f_{\alpha}(0), z^{(-)}]z^{(0)}1 \quad \text{or} \quad e_{\alpha}(-1)z^{(+)}z^{(-)}[f_{\alpha}(0), z^{(0)}]1 \]
satisfies that \( \deg(x^{(0)}) \leq \deg(z^{(0)}) \). Then it is enough to consider PBW monomials in
\[ e_{\alpha}(-1)[f_{\alpha}(0), z^{(+)}]z^{(-)}z^{(0)}1. \]

The only possibility for a PBW monomial \( x \) in \( e_{\alpha}(-1)[f_{\alpha}(0), z^{(+)}]z^{(-)}z^{(0)}1 \) to satisfy \( \deg(x^{(0)}) = \deg(z^{(0)}) + 1 \) is that it comes from a term \( [f_{\alpha}(0), e_{\alpha}(-n)] = -\alpha(-n) \) for
some $n \in \mathbb{Z}_{>0}$, where $e_\alpha(-n)$ is a term in $z^{(+)}$. But then, for PBW monomials $x$ in $e_\alpha(-1)[f_\alpha(0), z^{(+)}]z^{(0)}1$ such that $\deg(x^{(0)}) = \deg(z^{(0)}) + 1$, we have $x^{(-)} = z^{(-)}$ and $\deg(x^{(+)}) \leq \deg(z^{(+)}). \ \Box$

We now consider the action of $L_{-1}$ on particular PBW monomials.

**Lemma 3.3.** Let $z$ be a PBW monomial of the form (2.8) such that $z^{(-)} = 1$ and $\deg(z^{(+)}) = 0$, that is, either $z^{(+) = 1}$, or for some $j_1, \ldots, j_t \in \{1, \ldots, q\}$ (with possible repetitions),

$$z = e_{\beta_1}(-1)e_{\beta_2}(-1) \cdots e_{\beta_t}(-1)z^{(0)}1.$$

Then $L_{-1}z$ is a linear combination of PBW monomials $y$ satisfying one of the following conditions:

1. $y^{(-)} = 1$, $\deg(y^{(+)}) \geq 1$, $\deg(y^{(+)}) \leq \deg(z^{(+)})$, $y^{(0)} = z^{(0)}$,
2. $y^{(-)} = 1$, $\deg(y^{(+)}) = 0$, $\deg(y^{(+)}) \leq \deg(z^{(+)}) - 1$, and $\deg(y^{(0)}) > \deg(z^{(0)})$,
3. $y^{(-)} = 1$, $\deg(y^{(+)}) > 1$, $\deg(y^{(+)}) \leq \deg(z^{(+)}) - 1$, and $\deg_{-1}(y) = \deg_{-1}(z) + 1$,
4. $y^{(-)} \neq 1$.

**Proof.** First, we have

$$\sum_{i=1}^{t} u_i^{(-1)}u_i^{(0)}z = \sum_{r=1}^{t} e_{\beta_1}(-1) \cdots \left[ \sum_{i=1}^{r} u_i^{(-1)}u_i^{(0)}, e_{\beta_i}(-1) \right] \cdots e_{\beta_1}(-1)z^{(0)}1,$$

and

$$\sum_{i=1}^{t} u_i^{(-1)}u_i^{(0)}, e_{\beta_r}(-1) = \sum_{i=1}^{t} (u_i^{(-1)}[u_i^{(0)}, e_{\beta_r}(-1)] + [u_i^{(-1)}, e_{\beta_r}(-1)]u_i^{(0)}) = \beta_r(-1)e_{\beta_r}(-1) + e_{\beta_r}(-2)\beta_r(0).$$

So

$$\sum_{i=1}^{t} u_i^{(-1)}u_i^{(0)}z = \sum_{i=1}^{t} e_{\beta_1}(-1) \cdots (\beta_r(-1)e_{\beta_r}(-1) + e_{\beta_r}(-2)\beta_r(0)) \cdots e_{\beta_1}(-1)z^{(0)}1.$$ 

Second, we have

$$\sum_{\alpha \in \Delta_+} e_{\alpha}(-1)f_{\alpha}(0)z = \sum_{\alpha \in \Delta_+} \sum_{r=1}^{t} e_{\alpha}(-1)e_{\beta_1}(-1) \cdots [f_{\alpha}(0), e_{\beta_r}(-1)] \cdots e_{\beta_1}(-1)z^{(0)}1 + \sum_{\alpha \in \Delta_+} e_{\alpha}(-1)e_{\beta_1}(-1)e_{\beta_2}(-1) \cdots e_{\beta_t}(-1)[f_{\alpha}(0), z^{(0)}]1.$$

It is clear that any PBW monomial $y$ in

$$\sum_{\alpha \in \Delta_+} e_{\alpha}(-1)e_{\beta_1}(-1)e_{\beta_2}(-1) \cdots e_{\beta_t}(-1)[f_{\alpha}(0), z^{(0)}]1$$

satisfies

$$y^{(-)} \neq 1.$$
We now consider
\[ u_r := \sum_{\alpha \in \Delta_+} c_\alpha (-1) e_{\beta_{i_1}} (-1) \cdots [f_{\alpha_0} (0), e_{\beta_{i_r}} (-1)] \cdots e_{\beta_{i_t}} (-1) z^{(0)} 1, \text{ for } 1 \leq r \leq t. \]

- If \( \beta_{j_r} = \alpha + \beta \) for some \( \alpha, \beta \in \Delta_+ \), then there is a partial sum of two terms in \( u_r \):
\[
c_{-\alpha, \alpha+\beta} c_\alpha (-1) e_{\beta_{i_1}} (-1) \cdots e_{\beta_{i_r}} (-1) z^{(0)} 1 \]
\[
+ c_{-\beta, \alpha+\beta} e_\beta (-1) e_{\beta_{i_1}} (-1) \cdots e_{\beta_{i_r}} (-1) z^{(0)} 1. \]

Rewriting the above sum to a linear combination of PBW monomials, and noticing due to (2.7), we deduce that it is a linear combination of PBW monomials which satisfies one of the following:
\[ y(-) = z(-) = 1, \quad y^{(0)} = z^{(0)}, \quad \text{depth}(y^{(+)}) \geq 1, \quad \text{deg}(y^{(+)}) \leq \text{deg}(z^{(+)}) , \]
where \( c_{-\alpha, \alpha+\beta}, c_{-\beta, \alpha+\beta}, c_{\alpha+\beta} \in \mathbb{R}^* \).

- If \( \alpha - \beta_{j_r} \in \Delta_+ \) for some \( \alpha \in \Delta_+ \), then there is a term in \( u_r \):
\[ c_{-\alpha, \beta_{j_r}} e_\alpha (-1) e_{\beta_{i_1}} (-1) \cdots e_{\beta_{j_r-1}} (-1) f_{\alpha - \beta_{j_r}} (-1) e_{\beta_{j_r+1}} (-1) \cdots e_{\beta_{j_t}} (-1) z^{(0)} 1. \]

It is easy to see that (3.7) is a linear combination of PBW monomials \( y \) such that \( y \) satisfies one of the following:
\[ y(-) = 1, \quad \text{depth}(y^{(+)}) \geq 1, \quad \text{deg}(y^{(+)}) \leq \text{deg}(z^{(+)}) , \quad y^{(0)} = z^{(0)}, \]
\[ y(-) = 1, \quad \text{depth}(y^{(+)}) = 0, \quad \text{deg}(y^{(+)}) \leq \text{deg}(z^{(+)}) - 1, \quad \text{deg}(y^{(0)}) > \text{deg}(z^{(0)}), \quad \text{deg}_{-1} (y) = \text{deg}_{-1} (z), \]
\[ y(-) \neq 1. \]

Notice also that with \( \alpha = \beta_{j_r} \), there is a term in \( u_r \):
\[ -e_{\beta_{j_r}} (-1) e_{\beta_{i_1}} (-1) \cdots e_{\beta_{j_r-1}} (-1) \beta_{j_r} (-1) e_{\beta_{j_r+1}} (-1) \cdots e_{\beta_{j_t}} (-1) z^{(0)} 1. \]

Together with (3.4), we see that
\[ \sum_{i=1}^t u_i^{(i)} u_i^{(0)} z + \sum_{r=1}^t e_{\beta_{i_1}} (-1) e_{\beta_{i_2}} (-1) \cdots [f_{\beta_{i_r}} (0), e_{\beta_{i_r}} (-1)] \cdots e_{\beta_{i_t}} (-1) z^{(0)} 1 \]
\[ = \sum_{r=1}^t e_{\beta_{i_1}} (-1) \cdots (\beta_{j_r} (-1) e_{\beta_{i_r}} (-1) + e_{\beta_{i_r}} (-2) \beta_{j_r} (0)) \cdots e_{\beta_{i_t}} (-1) z^{(0)} 1 \]
\[ - \sum_{r=1}^t \sum_{s=1}^{r-1} e_{\beta_{i_1}} (-1) \cdots [e_{\beta_{i_r}} (-1), e_{\beta_{i_r}} (-1)] \cdots e_{\beta_{i_r}} (-1) \beta_{j_r} (-1) e_{\beta_{i_r+1}} (-1) \cdots e_{\beta_{i_t}} (-1) z^{(0)} 1 \]
\[ - \sum_{r=1}^t e_{\beta_{i_1}} (-1) \cdots e_{\beta_{j_r-1}} (-1) e_{\beta_{j_r}} (-1) \beta_{j_r} (-1) e_{\beta_{j_r+1}} (-1) \cdots e_{\beta_{j_t}} (-1) z^{(0)} 1. \]
is a linear combination of PBW monomials $y$ satisfying one of the following:

\[(3.11) \quad y^{(-)} = 1, \text{ depth}(y^{(+)}) \geq 1, \deg(y^{(+)}) \leq \deg(z^{(+)}) - 1, \quad y^{(0)} = z^{(0)},\]

\[(3.12) \quad y^{(-)} = 1, \text{ depth}(y^{(+)}) \geq 1, \deg(y^{(+)}) \leq \deg(z^{(+)}) - 1, \quad \deg_{-1}(y) = \deg_{-1}(z) + 1.\]

Then the lemma follows from (3.5), (3.6), (3.8)–(3.12).

**Lemma 3.4.** — Let $z$ be a PBW monomial of the form (2.8) such that $z^{(-)} = 1$. Then

\[
\tilde{L}_{-1} z = c z^{(+)}(\gamma - \sum_{j=1}^{q} a_{j,1} \beta_j)(-1)z^{(0)} + y^1,
\]

where $c$ is a nonzero constant, $\gamma = \sum_{j=1}^{q} \sum_{s=1}^{r_j} a_{j,s} \beta_j$, and $y^1$ is a linear combination of PBW monomials $y$ such that

\[
\deg_{-1}(y) = \deg_{-1}(z) + 1, \quad \deg(y^{(+)}) \leq \deg(z^{(+)}) - 1,
\]

or

\[
\deg_{-1}(y) \leq \deg_{-1}(z).
\]

**Proof.** — Since the proof is similar to that of Lemma 3.3, we left the verification to the reader.

**Lemma 3.5.** — For $i \in J^+$, we have that $\text{depth}((w^i)^{(+)}) = 0$.

**Proof.** — First we have

\[
w = \sum_{j \in J^+} \lambda_j w^j + \sum_{j \in J^+ \setminus J_1^+} \lambda_j w^j + \sum_{j \in J_1 \setminus J^+} \lambda_j w^j + \sum_{j \in J \setminus J_1} \lambda_j w^j.
\]

Then by Lemma 3.2(b) and Lemma 3.4, we have

\[
(k + h^+)\tilde{L}_{-1} w = \sum_{i \in J^+} (w^i)^{(+)}
\]

\[
= \sum_{i \in J_1^+} \left( \gamma_i - \sum_{j=1}^{q} a_{j,1} \beta_j \right)(-1)(w^i)^{(0)} + \sum_{i \in J_1 \setminus J^+} (w^i)^{(+)}
\]

\[
= \sum_{i \in J_1} \left( \gamma_i - \sum_{j=1}^{q} a_{j,1} \beta_j \right)(-1)(w^i)^{(0)} + y^1,
\]

where $\gamma_i = \sum_{j=1}^{q} \sum_{s=1}^{r_{j,i}} a_{j,i}^{(s)} \beta_i$, for $i \in J_1$, and $y^1$ is a linear combination of PBW monomials $y$ satisfying one of the following conditions:

\[
\deg_{-1}(y) = d_{-1} + 1, \quad \deg(y^{(+)}) \leq d^* - 1,
\]

\[
\deg_{-1}(y) \leq d_{-1},
\]

\[
y^{(-)} \neq 1.
\]

On the other hand, by Lemma 2.4

\[
L_{-1} w = \tilde{L}_{-1} w.
\]
By Lemma 2.1, there is no PBW monomial $y$ in $L_{-1}w$ such that $\deg(y^+) = d^+$, $y^(-) = 1$, and $\deg_{-1}(y) = d_{-1} + 1$. Then we deduce that
\[
\sum_{i \in J^+} (w^i)^{(+) \gamma_i} \left( \sum_{j=1}^{q} a_{j,1}^{(i)} \beta_j \right) (-1)(w^i)^{(0)} = 0,
\]
which means that $(\gamma_i - \sum_{j=1}^{q} a_{j,1}^{(i)} \beta_j) = 0$, for $i \in J^+$, that is, $\text{depth}((w^i)^{(+)}) = 0$. \hfill \Box

As explained at the beginning of §3.3, Theorem 1.1 will be a consequence of the following lemma.

**Lemma 3.6.** — For each $i \in J^+$, we have $\text{depth}(w^i) = 0$.

**Proof.** — By definition, for $i \in J^+$, $(w^i)^{(0)} = 1$. Moreover, by Lemma 3.5, $\text{depth}((w^i)^{(+)}) = 0$. Hence it suffices to prove that for $i \in J^+$,
\[
(w^i)^{(0)} = u^l(-1)^{\epsilon_i^0} \cdots u^l(-1)^{\epsilon_i^0}.
\]
Suppose the contrary. Then there exists $i \in J^+$ such that
\[
w^i = e_{\beta_1}(-1)^{\epsilon_1^0} \cdots e_{\beta_\ell}(-1)^{\epsilon_\ell^0} u^l(-1)^{\epsilon_1^0} \cdots u^l(-m_1)^{\epsilon_{1,m_1}},
\]
with at least one of the $m_j$’s, for $j = 1, \ldots, \ell$, strictly greater than 1 and $c_{1,m_1}^{(\ell)} \neq 0$ for such a $j$. Without loss of generality, one may assume that $1 \in J^+$, that
\[m_1 = \max \{ m_j \mid j = 1, \ldots, \ell \} \quad \text{and} \quad 0 \neq c_{1,m_1}^{(\ell)} \geq c_{1,m_1}^{(i)}, \text{ for } i \in J^+.
\]
Writing $L_{-1}w$ as
\[
L_{-1}w = \sum_{i \in J^+} L_{-1}w^i + \sum_{i \in J_{-1} \setminus J^+} L_{-1}w^i + \sum_{i \in J_1 \setminus J_{-1}} L_{-1}w^i + \sum_{i \in J \setminus J_1} L_{-1}w^i,
\]
we see by Lemma 2.1 that
\[
L_{-1}w = \lambda_1 m_1 c_{1,m_1}^{(\ell)} u^1 + \sum_{i \in J^+ \setminus i \neq 1} \lambda_i m_i c_{1,m_i}^{(i)} u^i + v + v',
\]
where for $i \in J^+$, $v^i$ is the PBW monomial defined by:
\[
(v^i)^{(-)} = (w^i)^{(-)} = 1,
\]
\[
(v^i)^{(+)} = (w^i)^{(+)} = e_{\beta_1}(-1)^{q_{i,1}} \cdots e_{\beta_\ell}(-1)^{q_{i,\ell}},
\]
\[
(v^i)^{(0)} = u^1(-1)^{c_{1,1}} \cdots u^1(-m_1)^{c_{1,m_1}} \cdots u^1(-m_1 - 1)^{c_{1,m_1}} u^l(-m_1)^{c_{1,m_1}},
\]
and so, by definition of $J^+ \subset J_{-1}^{(0)}$,
\[
\text{deg}_{-1}(v^i) = d_{-1},
\]
v is a linear combination of PBW monomials $x$ such that
\[
x^{(0)} = u^1(-1)^{c_{m_1}} \cdots u^1(-m_1)^{c_{m_1}} u^l(-1)^{c_{1,m_1}} \cdots u^l(-m_1)^{c_{1,m_1}}
\]
Lemma 3.2(c), either
\[ n_1(x) \leq m_1, \quad \text{or } \deg(x^{(i)}) \leq d^+ - 1, \quad \text{or } \deg_{-1}(x) \leq d_{-1} - 1, \]
and \( v' \) is a linear combination of PBW monomials \( x \) such that \( x^{(-)} \neq 1 \). Note that the assumption that \( m_1 \geq 2 \) makes sure that (3.17) holds, and that \( \text{depth}(v') = \text{depth}(w') + 1 \) for all \( i \in J^+ \).

On the other hand, by Lemma 2.4,
\[ L_{-1}w = \tilde{L}_{-1}w, \]
since \( w \) is a singular vector of \( V^k(q) \). Hence \( v^1 \) must be a PBW monomial of \( \tilde{L}_{-1}w \).

Our strategy to obtain the expected contradiction is to show that there is no PBW monomial \( v^1 \) in \( \tilde{L}_{-1}w \) for each \( i \in J \).

– Assume that \( i \in J^+ \), and suppose that \( v^1 \) is a PBW monomial in \( \tilde{L}_{-1}w \).

First of all, \( \deg((w^i)^{(+)}) = d^+ \) because \( i \in J^+ \). Moreover, by the definition of \( J_1 \) and Lemma 3.5, we have \( (w^i)^{(-)} = 1 \) and \( \text{depth}((w^i)^{(+)}) = 0 \). Hence by Lemma 3.3(2),
\[ \deg((v^1)^{(+)}) < \deg((w^i)^{(+)}) = d^+ \]
because \( (v^1)^{(-)} = 1 \) and \( \text{depth}((v^1)^{(+)}) = 0 \) by (3.14) and (3.15). But \( d^+ = \deg((v^1)^{(+)}) \) by (3.15), whence a contradiction.

– Assume that \( i \in J_{-1} \setminus J^+ \). By the definition of \( J^+ \) and (3.15),
\[ \deg((w^i)^{(+)}) < d^+ = \deg((v^1)^{(+)}) \]
Suppose that \( v^1 \) is a PBW monomial in \( \tilde{L}_{-1}w \). Then
\[ (w^i)^{(-)} = 1 = (v^1)^{(-)} \]
by Lemma 3.1 since \( i \in J_{-1}^{(0)} \). The last equality follows from (3.14). Then by Lemma 3.2(c), either \( \deg((v^1)^{(0)}) = \deg((w^i)^{(0)}) + 1 \), or \( (v^1)^{(0)} = (w^i)^{(0)} \). But it is impossible that \( \deg((v^1)^{(0)}) = \deg((w^i)^{(0)}) + 1 \), by (d) of Lemma 3.2 because \( \deg((v^1)^{(+)}) > \deg((w^i)^{(+)}) \). Therefore,
\[ (v^1)^{(0)} = (w^i)^{(0)}. \]
Computing \( \tilde{L}_{-1}w^i \), we deduce from
\[ (v^1)^{(+) = e_(\infty; i \cdot \cdot \cdot e_\infty(-1)^{a_{ii}} \cdot \cdot \cdot } \]
that
\[ (w^i)^{(+) = e_(\infty; i \cdot \cdot \cdot e_\infty(-1)^{a_{ii}} \cdot \cdot \cdot } \]
Since \( (v^1)^{(-)} = (w^i)^{(-)} = 1 \), it results from Lemma 3.3 that \( \deg((v^1)^{(+) \leq \deg((w^i)^{(+) \), which contradicts (3.18).

– Assume that \( i \in J_1 \setminus J_{-1}^{(0)} \). Then
\[ \deg_{-1}^{(0)}(v^1) < d_{-1}^{(0)} = \deg_{-1}^{(0)}(v^1) \]
by (3.17). Suppose that \( v^1 \) is a PBW monomial in \( \tilde{L}_{-1}w \). By Lemma 3.2(b) and (c),
\[ (w^i)^{(-)} = 1, \quad \deg_{-1}(v^1) = \deg_{-1}(w^i) + 1, \]
because \((v^1)^{-} = 1\) by (3.14). Remember that

\[(v^1)^{(+)} = e_{\beta_1}(-1)^{a_{11}} \cdots e_{\beta_\ell}(-1)^{a_{\ell 1}}.
\]

Computing \(\tilde{L}_{-1} w^t\), we deduce that

\[\partial^{\alpha} p_i = e_{\beta_1}(-1)^{a_{11}} \cdots e_{\beta_\ell}(-1)^{a_{\ell 1}}.
\]

Since \(v^{-} = 1\) and \(\deg_{-1}^{(0)}(v^1) = \deg_{-1}^{(0)}(w^t) + 1\), it results from Lemma 3.3(3) that \(\text{depth}(v^1)^{(1)} \geq 1\), which contradicts (3.22).

– Finally, if \(j \in J \setminus J_1\), then by Lemma 3.2(b), any PBW monomial \(y\) in \(\tilde{L}_{-1} w^t\) satisfies that \(y^{(c)} \neq 1\). So \(v^1\) cannot be a PBW monomial in \(\tilde{L}_{-1} w^t\).

This concludes the proof of the lemma. □

As already explained, Lemma 3.6 implies that \(w\) has zero depth and so its image in \(R_{V^\vee\rho}(q)\) is nonzero, achieving the proof of Theorem 1.1.

3.4. Remarks. — The statement of Theorem 1.2 is not true at the critical level. Also, it is not true that the depth of a depth-homogeneous singular vector of \(S(\mathfrak{g}[t^{-1}] t^{-1})\) is always zero. Indeed, the \(\mathfrak{g}[t]\)-module \(S(\mathfrak{g}[t^{-1}] t^{-1})\) can be naturally identified with \(\mathbb{C}[J_\infty \mathfrak{g}^*]\), where \(J_\infty X\) is the arc space of \(X\), and so \(S(\mathfrak{g}[t^{-1}] t^{-1})^q \cong \mathbb{C}[J_\infty \mathfrak{g}^*]^{J_\infty G}\). It is known [RT92, BD, EF01] that

\[\mathbb{C}[J_\infty \mathfrak{g}^*]^{J_\infty G} \cong \mathbb{C}[J_\infty (\mathfrak{g}^*/G)].\]

This means that the invariant ring is a polynomial ring with infinitely many variables \(\partial^j p_i, i = 1, \ldots, \ell, j \geq 0\), where \(p_1, \ldots, p_\ell\) is a set of homogeneous generators of \(S(\mathfrak{g})^G\) considered as elements of \(S(\mathfrak{g}[t^{-1}] t^{-1})\) via the embedding \(S(\mathfrak{g}) \hookrightarrow S(\mathfrak{g}[t^{-1}] t^{-1})\), \(\mathfrak{g} \ni x \mapsto x(-1)\). We have \(\text{depth}(\partial^j p_i) = j\) although each \(\partial^j p_i\) is a singular vector of \(S(\mathfrak{g}[t^{-1}] t^{-1})\).

For \(k = -h^\vee\), the maximal submodule \(N_k\) of \(V^k(\mathfrak{g})\) is generated by Feigin-Frenkel center ([FG04]). Hence [FF92, Fre05], \(\text{gr} N_k\) is exactly the argumentation ideal of \(S(\mathfrak{g}[t^{-1}] t^{-1})^q\). Therefore, the above argument shows that the statement of Theorem 1.2 is false at the critical level.

4. \(W\)-algebras and proof of Theorem 1.3

Let \(f\) be a nilpotent element of \(\mathfrak{g}\). By the Jacobson-Morosov theorem, it embeds into an \(\mathfrak{sl}_2\)-triple \((e, h, f)\) of \(\mathfrak{g}\). Recall that the Slodowy slice \(\mathcal{Y}_f\) is the affine space \(f + \mathfrak{g}^e\), where \(\mathfrak{g}^e\) is the centralizer of \(e\) in \(\mathfrak{g}\). It has a natural Poisson structure induced from that of \(\mathfrak{g}^*\) ([GG02]).

The embedding \(\text{span}_C\{e, h, f\} \cong \mathfrak{sl}_2 \hookrightarrow \mathfrak{g}\) exponentiates to a homomorphism \(\text{SL}_2 \rightarrow G\). By restriction to the one-dimensional torus consisting of diagonal matrices, we obtain a one-parameter subgroup \(\rho: \mathbb{C}^\times \rightarrow G\). For \(t \in \mathbb{C}^\times\) and \(x \in \mathfrak{g}\), set

\[\bar{\rho}(t)x := t^2 \rho(t)(x)\].
We have $\tilde{\rho}(t)f = f$, and the $\mathbb{C}^*$-action of $\tilde{\rho}$ stabilizes $\mathcal{S}_f$. Moreover, it is contracting to $f$ on $\mathcal{S}_f$, that is, for all $x \in \mathfrak{g}^e$,

$$\lim_{t \to 0} \tilde{\rho}(t)(f + x) = f.$$ 

The following proposition is well-known. Since its proof is short, we give below the argument for the convenience of the reader.

**Proposition 4.1 ([Slo80, Pre02, CM16]). — The morphism**

$$\theta_f: G \times \mathcal{S}_f \rightarrow \mathfrak{g}, \quad (g, x) \mapsto g \cdot x$$

**is smooth onto a dense open subset of** $\mathfrak{g}^*$. 

**Proof. —** Since $\mathfrak{g} = \mathfrak{g}^e + [f, \mathfrak{g}]$, the map $\theta_f$ is a submersion at $(1, f)$. Therefore, $\theta_f$ is a submersion at all points of $G \times (f + \mathfrak{g}^e)$ because it is $G$-equivariant for the left multiplication in $G$, and

$$\lim_{t \to \infty} \rho(t) \cdot x = f$$

for all $x$ in $f + \mathfrak{g}^e$. So, by [Har77, Ch.III, Prop.10.4], the map $\theta_f$ is a smooth morphism onto a dense open subset of $\mathfrak{g}$, containing $G \cdot f$. □

As in the introduction, let $\mathcal{W}^k(\mathfrak{g}, f)$ be the affine $\mathcal{W}$-algebra associated with a nilpotent element $f$ of $\mathfrak{g}$ defined by the generalized quantized Drinfeld-Sokolov reduction:

$$\mathcal{W}^k(\mathfrak{g}, f) = H^0_{\text{DS}, f}(V^k(\mathfrak{g})).$$

Here, $H^0_{\text{DS}, f}(M)$ denotes the BRST cohomology of the generalized quantized Drinfeld-Sokolov reduction associated with $f \in N(\mathfrak{g})$ with coefficients in a $V^k(\mathfrak{g})$-module $M$. Recall that we have [DSK06, Ara15a] a natural isomorphism $R_{\mathcal{W}^k(\mathfrak{g}, f)} \cong \mathbb{C}[\mathcal{S}_f]$ of Poisson algebras, so that

$$X_{\mathcal{W}^k(\mathfrak{g}, f)} = \mathcal{S}_f.$$ 

We write $\mathcal{W}^k(\mathfrak{g}, f)$ for the unique simple (graded) quotient of $\mathcal{W}^k(\mathfrak{g}, f)$. Then $X_{\mathcal{W}^k(\mathfrak{g}, f)}$ is a $\mathbb{C}^*$-invariant Poisson subvariety of the Slodowy slice $\mathcal{S}_f$.

Let $\mathcal{O}_k$ be the category $\mathcal{O}$ of $\hat{\mathfrak{g}}$ at level $k$. We have a functor

$$\mathcal{O}_k \rightarrow \mathcal{W}^k(\mathfrak{g}, f)\text{-Mod}, \quad M \mapsto H^0_{\text{DS}, f}(M),$$

where $\mathcal{W}^k(\mathfrak{g}, f)$-Mod denotes the category of $\mathcal{W}^k(\mathfrak{g}, f)$-modules.

The full subcategory of $\mathcal{O}_k$ consisting of objects $M$ on which $\mathfrak{g}$ acts locally finitely will be denoted by $\mathcal{K}_L$. Note that both $V^k(\mathfrak{g})$ and $L_k(\mathfrak{g})$ are objects of $\mathcal{K}_L$.

**Theorem 4.2 ([Ara15a])**

1. $H_{\text{DS}, f}(M) = 0$ for all $i \neq 0$, $M \in \mathcal{K}_L$. In particular, the functor

$$\mathcal{K}_L \rightarrow \mathcal{W}^k(\mathfrak{g}, f)\text{-Mod}, \quad M \mapsto H^0_{\text{DS}, f}(M),$$

is exact.
(2) For any quotient $V$ of $V^k(g)$, 

$$X_{H^{0}_{DS,f}}(V) = X_V \cap \mathcal{J}_f.$$ 

In particular $H^{0}_{DS,f}(V) \neq 0$ if and only if $\overline{G \cdot f} \subset X_V$.

By Theorem 4.2(1), $H^{0}_{DS,f}(L_k(g))$ is a quotient vertex algebra of $\mathcal{W}^k(g,f)$ if it is nonzero. Conjecturally [KRW03, KW08], we have

$$\mathcal{W}_k(g,f) \cong H^{0}_{DS,f}(L_k(g))$$

provided that $H^{0}_{DS,f}(L_k(g)) \neq 0$.

(This conjecture has been verified in many cases [Ara05, Ara07, Ara11, AvE19].)

Proof of Theorem 1.3. — The directions $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are obvious. Let us show that $(3)$ implies $(1)$. So suppose that $X_{H^{0}_{DS,f}}(L_k(g)) = \mathcal{J}_f$. By Theorem 1.1, it is enough to show that $X_{L_k(g)} = g^*$. Assume the contrary. Then $X_{L_k(g)}$ is contained in a proper $G$-invariant closed subset of $g$. On the other hand, by Theorem 4.2 and our hypothesis, we have

$$\mathcal{J}_f = X_{H^{0}_{DS,f}}(L_k(g)) = X_{L_k(g)} \cap \mathcal{J}_f.$$ 

Hence, $\mathcal{J}_f$ must be contained in a proper $G$-invariant closed subset of $g$. But this contradicts Proposition 4.1. The proof of the theorem is completed. \hfill $\square$

References


