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COMMENSURATING ACTIONS OF BIRATIONAL GROUPS AND GROUPS OF PSEUDO-AUTOMORPHISMS

by Serge Cantat & Yves de Cornulier

Abstract. — Pseudo-automorphisms are birational transformations acting as regular automorphisms in codimension 1. We import ideas from geometric group theory to prove that a group of birational transformations that satisfies a fixed point property on CAT(0) cubical complexes, for example a discrete group with Kazhdan Property (T), is birationally conjugate to a group acting by pseudo-automorphisms on some non-empty Zariski-open subset. We apply this argument to classify groups of birational transformations of surfaces with this fixed point property up to birational conjugacy.

Résumé (Actions commensurantes de groupes birationnels et groupes de pseudo-automorphismes)

Les pseudo-automorphismes sont les transformations birationnelles qui sont régulières en codimension 1. On emploie des idées de théorie géométrique des groupes pour obtenir qu'un groupe de transformations birationnelles satisfaisant une propriété de point fixe sur les complexes cubiques CAT(0), par exemple un groupe ayant la propriété (T) de Kazhdan, est birationnellement conjugué à un groupe agissant par pseudo-automorphismes sur un ouvert de Zariski non vide. On utilise cet argument pour classifier, modulo conjugaison birationnelle, les groupes de transformations birationnelles de surfaces avec cette propriété de point fixe.

Contents

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1. Introduction

1.1. Birational transformations and pseudo-automorphisms. — Let X be a quasi-projective variety, over an algebraically closed field $\mathbf{k}$. Denote by $\mathsf{Bir}(X)$ the group of birational transformations of X and by $\mathsf{Aut}(X)$ the subgroup of (regular) automorphisms of X. For the affine space of dimension n, automorphisms are invertible transformations $f\colon \mathbb{A}^n_{\mathbf{k}} \to \mathbb{A}^n_{\mathbf{k}}$ such that both f and f^{-1} are defined by polynomial formulas in affine coordinates:

$$f(x_1,\dots,x_n) = (f_1,\dots,f_n),\;\; f^{-1}(x_1,\dots,x_n) = (g_1,\dots,g_n)$$

with $f_i,\, g_i \in \mathbf{k}[x_1,\dots,x_n]$. Similarly, birational transformations of $\mathbb{A}^n_{\mathbf{k}}$ are given by rational formulas, i.e., $f_i,\, g_i \in \mathbf{k}(x_1,\dots,x_n)$.

Birational transformations may contract hypersurfaces. *Pseudo-automorphisms* are birational transformations that act as automorphisms in codimension 1. Precisely, a birational transformation $f\colon X \dashrightarrow X$ is a pseudo-automorphism if there exist Zariski-open subsets $\mathscr{U}$ and $\mathscr{V}$ in X such that $X \smallsetminus \mathscr{U}$ and $X \smallsetminus \mathscr{V}$ have codimension $\geqslant 2$ and f induces an isomorphism from $\mathscr{U}$ to $\mathscr{V}$. The pseudo-automorphisms of X form a group, which we denote by $\mathsf{Psaut}(X)$. For instance, all birational transformations of Calabi-Yau manifolds are pseudo-automorphisms; and there are examples of such manifolds for which $\mathsf{Psaut}(X)$ is infinite while $\mathsf{Aut}(X)$ is trivial (see [12]). Pseudo-automorphisms are studied in Section 2.

Definition 1.1. — Let $\Gamma \subset \mathsf{Bir}(X)$ be a group of birational transformations of an irreducible projective variety X. We say that Γ is *pseudo-regularizable* if there exists a triple $(Y, \mathscr{U}, \varphi)$ where

(1) Y is a projective variety and $\varphi\colon Y \dashrightarrow X$ is a birational map;
(2) $\mathscr{U}$ is a dense Zariski open subset of Y;
(3) $\varphi^{-1} \circ \Gamma \circ \varphi$ yields an action of Γ by pseudo-automorphisms on $\mathscr{U}$.

More generally if $\alpha : \Gamma \to \mathsf{Bir}(X)$ is a homomorphism, we say that it is pseudo-regularizable if $\alpha(\Gamma)$ is pseudo-regularizable.

One goal of this article is to use rigidity properties of commensurating actions, a purely group-theoretic concept, to show that many group actions are pseudo-regularizable. In particular, we exhibit a class of groups for which all actions by birational transformations on projective varieties are pseudo-regularizable.

1.2. Property (FW). — The class of groups we shall be mainly interested in is characterized by a fixed point property appearing in several related situations, for instance for actions on $\mathrm{CAT}(0)$ cubical complexes. Here, we adopt the viewpoint of commensurated subsets. Let Γ be a group, and $\Gamma \times S \to S$ an action of Γ on a set S. Let A be a subset of S. One says that Γ *commensurates* A if the symmetric difference

$$\gamma(A) \triangle A = (\gamma(A) \smallsetminus A) \cup (A \smallsetminus \gamma(A))$$

is finite for every element γ of Γ. One says that Γ *transfixes* A if there is a subset B of S such that $A \triangle B$ is finite and B is Γ-invariant: $\gamma(B) = B$, for every γ in Γ.

A group Γ has *Property (FW)* if, given any action of Γ on any set S, all commensurated subsets of S are transfixed. For instance, the cyclic group $(\mathbf{Z},+)$ acts on itself by translation; this action commensurates $\mathbf{Z}_+$ but does not transfix it, hence $\mathbf{Z}$ does not have Property (FW). More generally, Property (FW) is not satisfied by non-trivial free groups. To get examples with Property (FW), recall that a countable group Γ has Kazhdan's *Property (T)* if every affine isometric action of Γ on a Hilbert space fixes a point: Property (T) implies (FW), so that all lattices in higher rank simple Lie groups have Property (FW), for instance $\mathsf{SL}_m(\mathbf{Z})$ when $m \geqslant 3$ (see [14] and Section 3.2). The group $\mathsf{SL}_2(\mathbf{Z}[\sqrt{5}])$ also has Property (FW) without satisfying Property (T) (see [14]).

Property (FW) is discussed in Section 3. Let us mention that among its various characterizations, one is: every combinatorial action of Γ on a CAT(0) cube complex fixes some cube. Another, for Γ finitely generated, is that all its infinite connected Schreier graphs are one-ended (see [14]).

1.3. PSEUDO-REGULARIZATIONS. — Let X be a projective variety. The group $\mathsf{Bir}(X)$ does not really act on X, because there are indeterminacy points; it does not act on the set of hypersurfaces either, because some of them may be contracted. As we shall see, one can introduce the set $\widetilde{\mathsf{Hyp}}(X)$ of all irreducible and reduced hypersurfaces in all birational models $X' \dashrightarrow X$ (up to a natural identification). Then, there is a natural action of the group $\mathsf{Bir}(X)$ on this set, given by strict transforms of hypersurfaces. Indeed, one rigorous construction of this action naturally follows from the action on the set of divisorial valuations. Since this action commensurates the subset $\mathsf{Hyp}(X)$ of hypersurfaces of X, this construction leads to the following result.

THEOREM 1. — *Let X be a projective variety over an algebraically closed field. Let Γ be a subgroup of $\mathsf{Bir}(X)$. If Γ has Property (FW), then Γ is pseudo-regularizable.*

There is also a relative version of Property (FW) for pairs of groups $\Lambda \leqslant \Gamma$, which leads to a similar pseudo-regularization theorem for the subgroup Λ: this is discussed in Section 5.4, with applications to distorted birational transformations.

REMARK 1.2. — Theorem 1 provides a triple $(Y, \mathscr{U}, \varphi)$ such that φ conjugates Γ to a group of pseudo-automorphisms on the open subset $\mathscr{U} \subset Y$. There are two extreme cases for the pair $(Y, \mathscr{U})$ depending on the size of the boundary $Y \smallsetminus \mathscr{U}$. If this boundary is empty, Γ acts by pseudo-automorphisms on a projective variety Y. If it is ample, its complement $\mathscr{U}$ is an affine variety; if $\mathscr{U}$ is smooth (or locally factorial) then Γ actually acts by regular automorphisms on $\mathscr{U}$ (see Section 2.4). Thus, in the study of groups of birational transformations, *pseudo-automorphisms of projective varieties and regular automorphisms of affine varieties deserve specific attention.*

1.4. CLASSIFICATION IN DIMENSION 2. — In dimension 2, pseudo-automorphisms do not differ much from automorphisms; for instance, $\mathsf{Psaut}(X)$ coincides with $\mathsf{Aut}(X)$ if X is a smooth projective surface. Thus, for groups with Property (FW), Theorem 1 can be used to reduce the study of birational transformations to the study of automorphisms of quasi-projective surfaces. Combining results of Danilov and Gizatullin on automorphisms of affine surfaces with a theorem of Farley and Hughes on groups of piecewise affine transformations of the circle, we prove the following theorem.

Theorem 2. — *Let X be a smooth, projective, and irreducible surface, over an algebraically closed field. Let Γ be an infinite subgroup of $\mathsf{Bir}(X)$. If Γ has Property (FW), there is a birational map $\varphi\colon Y \dashrightarrow X$ such that*

(1) *Y is the projective plane $\mathbb{P}^2$, a Hirzebruch surface $\mathbb{F}_m$ with $m \geqslant 1$, or the product of a curve C by the projective line $\mathbb{P}^1$. If the characteristic of the field is positive, Y is the projective plane $\mathbb{P}^2_{\mathbf{k}}$.*

(2) *The subgroup $\varphi^{-1} \circ \Gamma \circ \varphi$ is contained in $\mathsf{Aut}(Y)$.*

Remark 1.3. — There is an infinite subgroup of $\mathsf{Aut}(Y)$ with Property (FW) for all surfaces Y of Assertion (1). Namely, if the algebraically closed field $\mathbf{k}$ has characteristic zero, $\mathsf{Aut}(Y)$ contains $\mathsf{PGL}_2(\mathbf{k})$ or the quotient of $\mathsf{GL}_2(\mathbf{k})$ by a central cyclic subgroup in case Y is a Hirzebruch surface. Thus, there is a morphism $\mathsf{SL}_2(\mathbf{Z}[\sqrt{5}]) \to \mathsf{Aut}(Y)$ with finite kernel and, as mentioned in Section 1.2, $\mathsf{SL}_2(\mathbf{Z}[\sqrt{5}])$ has Property (FW). In characteristic $p > 0$, the only case is that of $\mathbb{P}^2_{\mathbf{k}}$, whose automorphism group contains the group $\mathrm{PSL}_3(\mathbf{F}_p[t])$, which has Kazhdan's Property (T).

Remark 1.4. — The group $\mathsf{Aut}(Y)$ has finitely many connected components for all surfaces Y of Assertion (1) in Theorem 2. Thus, changing Γ into a finite index subgroup, one gets a subgroup of $\mathsf{Aut}(Y)^0$; here $\mathsf{Aut}(Y)^0$ denotes the connected component of the identity: this is an algebraic group, acting algebraically on Y.

Example 1.5. — Groups with Kazhdan Property (T) satisfy Property (FW) (see Section 3). Also, if Y is a Hirzebruch surface or a product $C\times\mathbb{P}^1$ for some curve C, then $\mathsf{Aut}(Y)$ does not contain any group with Property (T), because the group $\mathsf{PGL}_2(\mathbf{k})$ does not contain such a group. Thus, Theorem 2 extends [10, Th. A], at least in the projective case, and the present article offers a new proof of that result. Theorem 2 can also be applied to the group $\mathsf{SL}_2(\mathbf{Z}[\sqrt{d}])$ when the integer $d \geqslant 2$ is not a perfect square: every action of this group on a projective surface by birational transformations is conjugate to an action by regular automorphisms on $\mathbb{P}^2_{\mathbf{k}}$, the product of a curve C by the projective line $\mathbb{P}^1_{\mathbf{k}}$, or a Hirzebruch surface. Theorem 9.1 provides a more precise result, based on Theorem 2 and Margulis' superrigidity theorem.

Remark 1.6. — Let X be a normal projective variety. One can ask whether $\mathsf{Bir}(X)$ transfixes $\mathsf{Hyp}(X)$, or equivalently is pseudo-regularizable (see Theorem 5.4). For surfaces, this holds precisely when X is not birationally equivalent to the product of the projective line with a curve. See Section 6.1 for more precise results.

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2. Pseudo-automorphisms

This preliminary section introduces useful notation for birational transformations and pseudo-automorphisms, and presents a few basic results.

2.1. Birational transformations. — Let X and Y be two irreducible and reduced algebraic varieties over an algebraically closed field $\mathbf{k}$. Let $f\colon X \dashrightarrow Y$ be a birational map. Choose dense Zariski open subsets $U \subset X$ and $V \subset Y$ such that f induces an isomorphism $f_{U,V}\colon U \to V$. Then the graph $\mathfrak{G}_f$ of f is defined as the Zariski closure of $\{(x, f_{U,V}(x)) : x \in U\}$ in $X \times Y$; it does not depend on the choice of U and V. The graph $\mathfrak{G}_f$ is an irreducible variety; both projections

$$u\colon \mathfrak{G}_f \longrightarrow X \quad \text{and} \quad v\colon \mathfrak{G}_f \longrightarrow Y$$

are birational morphisms and $f = v \circ u^{-1}$.

We shall denote by $\mathrm{Ind}(f)$ the indeterminacy set of the birational map f.

Theorem 2.1 ([28, Th. 2.17]). — *Let $f : X \dashrightarrow Y$ be a rational map, with X a normal variety and Y a projective variety. Then the indeterminacy set of f has codimension $\geqslant 2$.*

Example 2.2. — The transformation of the affine plane $(x, y) \mapsto (x, y/x)$ is birational, and its indeterminacy locus is the line $\{x = 0\}$: this set of codimension 1 is "mapped to infinity". If the affine plane is compactified by the projective plane, the transformation becomes $[x : y : z] \mapsto [x^2 : yz : xz]$, with two indeterminacy points.

The *total transform* of a closed subset $Z \subset X$ is denoted by $f_*(Z)$; by definition, $f_*(Z) = v(u^{-1}(Z))$. If Z is irreducible and is not contained in $\mathrm{Ind}(f)$, we denote by $f_\circ(Z)$ its *strict transform*, defined as the Zariski closure of $f(Z \smallsetminus \mathrm{Ind}(f))$. We say that an irreducible hypersurface $W \subset X$ is contracted if it is not contained in the indeterminacy set and the codimension of its strict transform is larger than 1; the *exceptional divisor* of f is the union of all contracted hypersurfaces.

We say that f is a *local isomorphism* near a point $x \in X$ if there are open subsets $U \subset X$ and $V \subset Y$ such that U contains x and f induces an isomorphism from U to V. The *exceptional set* of f is the subset of X along which f is not a local isomorphism; this set is Zariski closed, and is made of three parts: the indeterminacy locus, the exceptional divisor, and a residual part of codimension $\geqslant 2$.

2.2. Pseudo-isomorphisms. — A birational map $f\colon X \dashrightarrow Y$ is a *pseudo-isomorphism* if one can find Zariski open subsets $\mathscr{U} \subset X$ and $\mathscr{V} \subset Y$ such that

(i) f realizes a regular isomorphism from $\mathscr{U}$ to $\mathscr{V}$ and
(ii) $X \smallsetminus \mathscr{U}$ and $Y \smallsetminus \mathscr{V}$ have codimension $\geqslant 2$.

Pseudo-isomorphisms from X to itself are called *pseudo-automorphisms* (see § 1.2). *The set of pseudo-automorphisms of X is a subgroup* $\mathsf{Psaut}(X)$ *of* $\mathsf{Bir}(X)$.

Example 2.3. — Start with the standard birational involution $\sigma_n\colon \mathbb{P}^n_{\mathbf{k}} \dashrightarrow \mathbb{P}^n_{\mathbf{k}}$ which is defined in homogeneous coordinates by $\sigma_n[x_0 : \cdots : x_n] = [x_0^{-1} : \cdots : x_n^{-1}]$. Blow-up the $(n+1)$ vertices of the simplex $\Delta_n = \{[x_0 : \cdots : x_n];\ \prod x_i = 0\}$; this provides a smooth rational variety X_n together with a birational morphism $\pi\colon X_n \to \mathbb{P}^n_{\mathbf{k}}$. Then, $\pi^{-1}\circ\sigma_n\circ\pi$ is a pseudo-automorphism of X_n, and is an automorphism if $n \leqslant 2$.

Proposition 2.4. — *Let $f\colon X \dashrightarrow Y$ be a birational map between two (irreducible, reduced) normal algebraic varieties. Then, the following properties are equivalent:*

(1) *The birational maps f and f^{-1} do not contract any hypersurface, and their indeterminacy sets have codimension $\geqslant 2$ in X and Y respectively.*

(2) *The birational map f is a pseudo-isomorphism from X to Y.*

Proof. — Denote by g the inverse of f. The second assertion implies the first because any hypersurface intersects the complement of every closed subset of codimension $\geqslant 2$. Let us prove that the first assertion implies the second. Let $\mathscr{U}_0 \subset X$ (resp. $\mathscr{V}_0 \subset Y$) be the complement of the singular locus of X (resp. Y) and the indeterminacy locus of f (resp. g). Let $\mathscr{U}$ be the pre-image of $\mathscr{V}_0$ by the birational map $f_{\mathscr{U}_0}\colon \mathscr{U}_0 \dashrightarrow \mathscr{V}_0$; the complement of $\mathscr{U}$ in $\mathscr{U}_0$, and therefore in X too, has codimension $\geqslant 2$ because the codimension of $Y \smallsetminus \mathscr{V}_0$ is at least 2 and f does not contract any hypersurface. Define $\mathscr{V} \subset \mathscr{V}_0$ to be the pre-image of $\mathscr{U}$ by g (restricted to $\mathscr{V}_0$); the codimension of $Y \smallsetminus \mathscr{V}$ is also $\geqslant 2$. Then, the restriction $f_{\mathscr{U}}\colon \mathscr{U} \dashrightarrow \mathscr{V}$ is a regular isomorphism, with inverse $g_{\mathscr{V}}\colon \mathscr{V} \dashrightarrow \mathscr{U}$. □

Example 2.5. — Let X be a smooth projective variety with trivial canonical bundle K_X. Let Ω be a non-vanishing section of K_X, and let f be a birational transformation of X. Then, $f^*\Omega$ extends from $X \smallsetminus \mathrm{Ind}(f)$ to X and determines a new section of K_X; this section does not vanish identically because f is dominant, hence it does not vanish at all because K_X is trivial. As a consequence, f does not contract any hypersurface, because otherwise $f^*\Omega$ would vanish along this hypersurface. Since X is projective, the codimension of $\mathrm{Ind}(f)$ is $\geqslant 2$ (Theorem 2.1). Thus, f is a pseudo-automorphism of X, and $\mathsf{Bir}(X) = \mathsf{Psaut}(X)$. We refer to [12, 19] for families of Calabi-Yau varieties with an infinite group of pseudo-automorphisms.

2.3. Projective varieties

2.3.1. *Smooth varieties*. — Assume that X and Y are smooth. The *Jacobian determinant* $\mathsf{Jac}(f)(x)$ is defined in local coordinates as the determinant of the differential df_x; the rational function $\mathsf{Jac}(f)$ depends on the chosen coordinates (on X and Y), but its zero locus does not. The zeroes of $\mathsf{Jac}(f)$ form a hypersurface of $X \smallsetminus \mathrm{Ind}(f)$; the zero locus of $\mathsf{Jac}(f)$ will be defined as the Zariski closure of this hypersurface in X.

Proposition 2.6. — *Let $f\colon X \dashrightarrow Y$ be a birational transformation between two smooth varieties. Assume that $\mathrm{Ind}(f)$ and $\mathrm{Ind}(f^{-1})$ have codimension $\geqslant 2$. The following properties are equivalent.*

(1) *The Jacobian determinants of f and f^{-1} do not vanish.*

(2) *For every $q \in X \smallsetminus \mathrm{Ind}(f)$, f is an isomorphism from a neighborhood of q to a neighborhood of $f(q)$, and the same holds for f^{-1}.*

(3) *The birational map f is a pseudo-isomorphism from X to Y.*

Proof. — Denote by g the inverse of f. If the Jacobian determinant of f vanishes at some point of $X \setminus \mathrm{Ind}(f)$, then it vanishes along a hypersurface $V \subset X$. If (1) is satisfied, then f does not contract any positive dimensional subset of $X \setminus \mathrm{Ind}(f)$: f is a quasi-finite map from $X \setminus \mathrm{Ind}(f)$ to its image, and so is g. Zariski's main theorem implies that f realizes an isomorphism from $X \setminus \mathrm{Ind}(f)$ to $Y \setminus \mathrm{Ind}(g)$ (see [37, Prop. 8.57]). Thus, (1) implies (2) and (3). Since (3) implies (1), this concludes the proof. □

Proposition 2.7 (see [7]). — *Let $f\colon X \dashrightarrow Y$ be a pseudo-isomorphism between two smooth projective varieties. Then*

(1) *the total transform of* $\mathrm{Ind}(f)$ *by f is equal to* $\mathrm{Ind}(f^{-1})$*;*
(2) *f has no isolated indeterminacy point;*
(3) *if* $\dim(X) = 2$*, then f is a regular isomorphism.*

Proof. — Since X and Y are projective, $\mathrm{Ind}(f)$ and $\mathrm{Ind}(f^{-1})$ have codimension $\geqslant 2$: we can apply Propositions 2.4 and 2.6. Let $p \in X$ be an indeterminacy point of the pseudo-isomorphism $f\colon X \dashrightarrow Y$. Then f^{-1} contracts a subset $C \subset Y$ of positive dimension on p. Since f and f^{-1} are local isomorphisms on the complement of their indeterminacy sets, C is contained in $\mathrm{Ind}(f^{-1})$. The total transform of a point $q \in C$ by f^{-1} is a connected subset of X that contains p and has dimension $\geqslant 1$. This set D_q is contained in $\mathrm{Ind}(f)$ because f is a local isomorphism on the complement of $\mathrm{Ind}(f)$; since $p \in D_q \subset \mathrm{Ind}(f)$, p is not an isolated indeterminacy point. This proves Assertions (1) and (2). The third assertion follows from the second one because indeterminacy sets of birational transformations of projective surfaces are finite sets. □

2.3.2. *Divisors and Néron-Severi group*. — Let W be a hypersurface of X, and let $f\colon X \dashrightarrow Y$ be a pseudo-isomorphism. The divisorial part of the total transform $f_*(W)$ coincides with the strict transform $f_\circ(W)$. Indeed, $f_*(W)$ and $f_\circ(W)$ coincide on the open subset of Y on which f^{-1} is a local isomorphism, and this open subset has codimension $\geqslant 2$.

Recall that the Néron-Severi group $\mathsf{NS}(X)$ is the free abelian group of codimension 1 cycles modulo cycles which are numerically equivalent to 0. Its rank is finite and is called the Picard number of X.

Theorem 2.8. — *The action of pseudo-isomorphisms on Néron-Severi groups is functorial: $(g \circ f)_* = g_* \circ f_*$ for all pairs of pseudo-isomorphisms $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow Z$. If X is a normal projective variety, the group $\mathsf{Psaut}(X)$ acts linearly on the Néron-Severi group $\mathsf{NS}(X)$; this provides a morphism*

$$\mathsf{Psaut}(X) \longrightarrow \mathsf{GL}(\mathsf{NS}(X)).$$

The kernel of this morphism is contained in $\mathsf{Aut}(X)$ and contains $\mathsf{Aut}(X)^0$ as a finite index subgroup.

As a consequence, if X is projective the group $\mathsf{Psaut}(X)$ is an extension of a discrete linear subgroup of $\mathsf{GL}(\mathsf{NS}(X))$ by an algebraic group.

Proof. — The first statement follows from the equality $f_* = f_\circ$ on divisors. The second follows from the first.

For the last assertion, we shall need the following fact: *if $f\colon X \dashrightarrow Y$ is a pseudo-isomorphism between normal projective varieties such that $f_*(H_X) = H_Y$ for some pair of very ample divisors H_X and H_Y on X and Y, then, f is an isomorphism* (see [29, Exer. 5.6] and [36]). Indeed, $(f^{-1})_* = (f^{-1})_\circ$ maps the linear system $|H_Y|$ bijectively onto $|H_X|$; if f had an indeterminacy point, there would be a curve in its graph whose first projection would be a point $q \in X$ and second projection would be a curve $C \subset Y$: since all members of $|H_Y|$ intersect C, all members of $|H_X|$ should contain q, in contradiction with the very ampleness of H_X.

We can now study the kernel K of the representation $\mathsf{Psaut}(X) \to \mathsf{GL}(\mathsf{NS}(X))$. Fix an embedding $X \subset \mathbb{P}^m_{\mathbf{k}}$ and denote by H_X the polarization given by hyperplane sections. For every f in K, $f_*(H_X)$ is very ample because its class in $\mathsf{NS}(X)$ coincides with the class of H_X. Thus, by what has just been proven, f^* is an automorphism. To conclude, note that $\mathsf{Aut}(X)^0$ has finite index in the kernel of the action of $\mathsf{Aut}(X)$ on $\mathsf{NS}(X)$: see [35, Th. 6 in §11] and its extension to arbitrary projective varieties in [23, p. 268]; and see [31, Prop. 2.2] for compact Kähler manifold. □

2.4. Affine varieties. — The group $\mathsf{Psaut}(\mathbb{A}^n_{\mathbf{k}})$ coincides with the group $\mathsf{Aut}(\mathbb{A}^n_{\mathbf{k}})$ of polynomial automorphisms of the affine space $\mathbb{A}^n_{\mathbf{k}}$: this is a special case of the following proposition.

Proposition 2.9. — *Let Z be an affine variety. If Z is locally factorial, the group* $\mathsf{Psaut}(Z)$ *coincides with the group* $\mathsf{Aut}(Z)$.

Proof. — Fix an embedding $Z \to \mathbb{A}^m_{\mathbf{k}}$. Rational functions on Z are restrictions of rational functions on $\mathbb{A}^m_{\mathbf{k}}$. Thus, every birational transformation $f\colon Z \dashrightarrow Z$ is given by rational formulas $f(x_1, \dots, x_m) = (f_1, \dots, f_m)$ where each f_i is a rational function. To show that f is an automorphism, we only need to prove that f_i is in the local ring $\mathscr{O}_{Z,x}$ for every index i and every point $x \in Z$. Otherwise

$$f_i = \frac{p_i}{q_i},$$

where p_i and q_i are relatively prime elements of the local ring $\mathscr{O}_{Z,x}$, and q_i is not invertible. Fix an irreducible factor h of q_i, and an open neighborhood U of x on which p_i, q_i and h are defined. The hypersurfaces $W_U(p_i) = \{z \in U : \ p_i(z) = 0\}$ and $W_U(h) = \{z \in U : \ h(z) = 0\}$ have no common components, hence the latter would be mapped to infinity by f, and f would not be a pseudo-automorphism. This contradiction shows that all f_i are regular and f is an automorphism. □

Example 2.10. — Consider the affine quadric cone $Q \subset \mathbb{A}^3$ defined by the equation $z^2 = x^2 + y^2$; the origin is a singular point of Q, and it is not factorial at that point, because the relation $z^2 = (x+iy)(x-iy)$ shows that z^2 can be factorized in two distinct ways. Now, consider the affine variety $Z = Q \times \mathbb{A}^1 \subset \mathbb{A}^4$, with coordinates (x, y, z, t). The map $f(x, y, z, t) = (x, y, z, t + z/(x + iy))$ is a birational transformation of Z. The indeterminacy sets of f and f^{-1} coincide with the vertical line $\{(0, 0, 0)\} \times \mathbb{A}^1$ and f and f^{-1} do not contract any hypersurface, hence f is a pseudo-automorphism. But f is not an automorphism.

3. Groups with Property (FW)

3.1. Commensurated subsets and cardinal definite length functions (see [14])

Let G be a group, and $G \times S \to S$ an action of G on a set S. Let A be a subset of S. As in the introduction, one says that G *commensurates* A if the symmetric difference $A \triangle gA$ is finite for every element $g \in G$. One says that G *transfixes* A if there is a subset B of S such that $A \triangle B$ is finite and B is G-invariant: $gB = B$ for every g in G. If A is transfixed, then it is commensurated. Actually, A is transfixed if and only if the function $g \mapsto \#(A \triangle gA)$ is bounded on G.

A group G has *Property (FW)* if, given any action of G on a set S, all commensurated subsets of S are automatically transfixed. More generally, if H is a subgroup of G, then (G, H) has *relative Property (FW)* if every commensurating action of G is transfixing in restriction to H. This means that, if G acts on a set S and commensurates a subset A, then H transfixes automatically A. The case $H = G$ is Property (FW) for G.

We refer to [14] for a detailed study of Property (FW). The next paragraphs present the two main sources of examples for groups with Property (FW) or its relative version, namely Property (T) and distorted subgroups.

Remark 3.1. — Property (FW) should be thought of as a rigidity property. To illustrate this idea, consider a group K with Property (PW); by definition, this means that K admits a commensurating action on a set S, with a commensurating subset C such that the function $g \mapsto \#(C \triangle gC)$ has finite fibers. If G is a group with Property (FW), then, every homomorphism $G \to K$ has finite image.

3.2. Property (FW) and Property (T) (see [14]). — One can rephrase Property (FW) as follows: G has Property (FW) if and only if every isometric action on an "integral Hilbert space" $\ell^2(X, \mathbf{Z})$ has bounded orbits, for any discrete set X.

A group has Property (FH) if all its isometric actions on Hilbert spaces have fixed points. More generally, a pair (G, H) of a group G and a subgroup $H \subset G$ has relative Property (FH) if every isometric G-action on a Hilbert space has an H-fixed point. Thus, the relative Property (FH) implies the relative Property (FW).

By a theorem of Delorme and Guichardet, Property (FH) is equivalent to Kazhdan's Property (T) for countable groups; this is the viewpoint we used to describe Property (T) in the introduction (see [24] for other equivalent definitions). Thus, Property (T) implies Property (FW). Kazhdan's Property (T) is satisfied by lattices in semisimple Lie groups all of whose simple factors have Property (T), for instance if all simple factors have real rank $\geqslant 2$. For example, $\mathsf{SL}_3(\mathbf{Z})$ satisfies Property (T).

Property (FW) is actually conjectured to hold for all irreducible lattices in semisimple Lie groups of real rank $\geqslant 2$, such as $\mathsf{SL}_2(\mathbf{R})^k$ for $k \geqslant 2$. (here, irreducible means that the projection of the lattice *modulo* every simple factor is dense.) This is known in the case of a semisimple Lie group admitting at least one noncompact simple factor with Kazhdan's Property (T), for instance in $\mathsf{SO}(2,3) \times \mathsf{SO}(1,4)$, which admits irreducible lattices (see [13]).

3.3. Distortion. — Let G be a group. An element g of G is *distorted* in G if there exists a finite subset Σ of G generating a subgroup $\langle\Sigma\rangle$ containing g, such that $\lim_{n\to\infty} \frac{1}{n}|g^n|_\Sigma = 0$; here, $|g|_\Sigma$ is the length of g with respect to the set Σ. If G is finitely generated, this condition holds for some Σ if and only if it holds for every finite generating subset of G. For example, every finite order element is distorted.

Example 3.2. — Let K be a field. The distorted elements of $\mathsf{SL}_n(K)$ are exactly the virtually unipotent elements, that is, those elements whose eigenvalues are all roots of unity; in positive characteristic, these are elements of finite order. By results of Lubotzky, Mozes, and Raghunathan (see [34, 33]), the same characterization holds in $\mathsf{SL}_n(\mathbf{Z})$ when $n \geqslant 3$; it also holds in $\mathsf{SL}_n(\mathbf{Z}[\sqrt{d}])$ when $n \geqslant 2$ and $d \geqslant 2$ is not a perfect square. In contrast, in $\mathsf{SL}_2(\mathbf{Z})$, every element of infinite order is undistorted.

Lemma 3.3 (see [14]). — *Let G be a group, and H a finitely generated abelian subgroup of G consisting of distorted elements. Then, the pair (G,H) has relative Property (FW).*

This lemma provides many examples. For instance, if G is any finitely generated nilpotent group and G' is its derived subgroup, then (G,G') has relative Property (FH); this result is due to Houghton, in a more general formulation encompassing polycyclic groups (see [14]). Bounded generation by distorted unipotent elements can also be used to obtain nontrivial examples of groups with Property (FW), including the above examples $\mathsf{SL}_n(\mathbf{Z})$ for $n \geqslant 3$, and $\mathsf{SL}_n(\mathbf{Z}[\sqrt{d}])$. The case of $\mathsf{SL}_2(\mathbf{Z}[\sqrt{d}])$ is particularly interesting because it does not have Property (T).

3.4. Subgroups of $\mathsf{PGL}_2(\mathbf{k})$ with Property (FW). — If a group G acts on a tree T by graph automorphisms, then G acts on the set E of directed edges of T (T is non-oriented, so each edge gives rise to a pair of opposite directed edges). Let E_v be the set of directed edges pointing towards a vertex v. Then $E_v \triangle E_w$ is the set of directed edges lying in the segment between v and w; it is finite of cardinality $2d(v,w)$, where d is the graph distance. The group G commensurates E_v for every v, and $\#(E_v \triangle gE_v) = 2d(v,gv)$. Consequently, if G has Property (FW), then it has Property (FA) meaning that every action of G on a tree has bounded orbits. Combined with [14, Prop. 5.B.1], this argument leads to the following lemma.

Lemma 3.4 (See [14]). — *Let G be a group with Property (FW), then all finite index subgroups of G have Property (FW), and hence have Property (FA). Conversely, if a finite index subgroup of G has Property (FW), then so does G.*

On the other hand, Property (FA) is not stable by taking finite index subgroups.

Lemma 3.5. — *Let $\mathbf{k}$ be an algebraically closed field and Λ be a subgroup of $\mathsf{GL}_2(\mathbf{k})$.*

(1) *Λ has a finite orbit on the projective line if and only if it is virtually solvable, if and only if its Zariski closure does not contain SL_2.*

(2) *Assume that all finite index subgroups of Λ have Property (FA) (e.g., Λ has Property FW). If the action of Λ on the projective line preserves a non-empty, finite set, then Λ is finite.*

The proof of the first assertion is standard and omitted. The second assertion follows directly from the first one.

In what follows, we denote by $\overline{\mathbf{Z}} \subset \overline{\mathbf{Q}}$ the ring of algebraic integers (in some fixed algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$).

Theorem 3.6 (Bass [3]). — *Let $\mathbf{k}$ be an algebraically closed field.*

(1) *If $\mathbf{k}$ has positive characteristic, then $\mathsf{GL}_2(\mathbf{k})$ has no infinite subgroup with Property (FA).*

(2) *Suppose that $\mathbf{k}$ has characteristic zero and that $\Gamma \subset \mathsf{GL}_2(\mathbf{k})$ is a countable subgroup with Property (FA), and is not virtually abelian. Then Γ acts irreducibly on $\mathbf{k}^2$, and is conjugate to a subgroup of $\mathsf{GL}_2(\overline{\mathbf{Z}})$. If moreover $\Gamma \subset \mathsf{GL}_2(K)$ for some subfield $K \subset \mathbf{k}$ containing $\overline{\mathbf{Q}}$, then we can choose the conjugating matrix to belong to $\mathsf{GL}_2(K)$.*

On the proof. — According to [39, §6, Th. 15], a countable group with Property (FA) is finitely generated. Thus, if $\Gamma \subset \mathsf{GL}_2(\mathbf{k})$ has Property (FA) it is contained in $\mathsf{GL}_2(K)$ for some finitely generated subfield $K \subset \mathbf{k}$ (choose K to be the subfield generated by entries of a finite generating subset of Γ). Then, the first statement follows from [3, Cor. 6.6].

Now, assume that the characteristic of $\mathbf{k}$ is 0. Since a group with Property (FA) has no infinite cyclic quotient, and is not a non-trivial amalgam, [3, Th. 6.5] can be applied, giving the first assertion of (3.6) (see also the first Theorem in [4]). For the last assertion, we have $\Gamma \cup B\Gamma B^{-1} \subset \mathsf{GL}_2(K)$ for some $B \in \mathsf{GL}_2(\mathbf{k})$ such that $B\Gamma B^{-1} \subset \mathsf{GL}_2(\overline{\mathbf{Z}})$; we claim that this implies that $B \in \mathbf{k}^* \,\mathsf{GL}_2(K)$. First, since Γ is absolutely irreducible, this implies that $B\mathscr{M}_2(K)B^{-1} \subset \mathscr{M}_2(K)$. The conclusion follows from Lemma 3.7 below, which can be of independent interest. □

Lemma 3.7. — *Let $K \subset L$ be fields. Then the normalizer $\{B \in \mathsf{GL}_2(L) : B\mathscr{M}_2(K)B^{-1} \subset \mathscr{M}_2(K)\}$ is reduced to $L^* \,\mathsf{GL}_2(K) = \{\lambda A : \lambda \in L^*, A \in \mathsf{GL}_2(K)\}$.*

Proof. — Write

$$B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.$$

Since $BAB^{-1} \in \mathscr{M}_2(K)$ for the three elementary matrices $A \in \{E_{11}, E_{12}, E_{21}\}$, we deduce by a plain computation that $b_i b_j / b_k b_\ell \in K$ for all $1 \leqslant i, j, k, \ell \leqslant 4$ such that $b_k b_\ell \neq 0$. In particular, for all indices i and j such that b_i and b_j are nonzero, the quotient $b_i/b_j = b_i b_j / b_j^2$ belongs to K. It follows that $B \in L^* \mathsf{GL}_2(K)$. □

Corollary 3.8. — *Let $\mathbf{k}$ be an algebraically closed field. Let C be a projective curve over $\mathbf{k}$, and let $\mathbf{k}(C)$ be the field of rational functions on the curve C. Let Γ be an infinite subgroup of $\mathsf{PGL}_2(\mathbf{k}(C))$. If Γ has Property (FA), then*

(1) *the field $\mathbf{k}$ has characteristic 0;*

(2) *there is an element of $\mathsf{PGL}_2(\mathbf{k}(C))$ that conjugates Γ to a subgroup of $\mathsf{PGL}_2(\overline{\mathbf{Z}}) \subset \mathsf{PGL}_2(\mathbf{k}(C))$.* □

4. Divisorial valuations, hypersurfaces, and the action of $\mathsf{Bir}(X)$

The group of birational transformations $\mathsf{Bir}(X)$ acts on the function field $\mathbf{k}(X)$, hence also on the set of valuations of $\mathbf{k}(X)$. The subset of divisorial valuations is invariant, and the centers of those valuations correspond to irreducible hypersurfaces in various models of X. In this way, we obtain a natural action of $\mathsf{Bir}(X)$ on (reduced, irreducible) hypersurfaces in all models of X; this section presents this classical construction (we refer to [41, Chap. VI] and [40] for detailed references).

4.1. Divisorial valuations. — Consider a projective variety X over an algebraically closed field $\mathbf{k}$ and let $\mathbf{k}(X)$ be its function field. A discrete, rank 1, valuation v on $\mathbf{k}(X)$ is a function on the multiplicative group $\mathbf{k}(X)^*$ with values in the cyclic group $\mathbf{Z}$ such that

(i) $v(\varphi\psi) = v(\varphi) + v(\psi)$ and $v(\varphi+\psi) \geqslant \min(v(\varphi), v(\psi))$, $\forall\, \varphi,\ \psi \in \mathbf{k}(X)^*$,

(ii) v vanishes on the set of constant functions $\mathbf{k} \subset \mathbf{k}(X)$,

(iii) $v(\mathbf{k}(X)) = \mathbf{Z}$ (we assume that the value group is equal to $\mathbf{Z}$ in this article).

Its valuation ring is the subring $R_v \subset \mathbf{k}(X)$ defined by $R_v = v^{-1}(\mathbf{Z}_+)$, where $\mathbf{Z}_+$ is the set of non-negative integers. This ring contains a unique maximal ideal, namely $\mathfrak{m}_v = v^{-1}(\mathbf{Z}_+^*)$, where $\mathbf{Z}_+^*$ is the set of positive integers. The residue field is the quotient field $\mathbf{k}(X)_v = R_v/\mathfrak{m}_v$; if its transcendence degree is equal to $\dim(X) - 1$, then v is said to be a *divisorial valuation* (see [41, §VI.14], [40, §10]). We shall denote by $\mathrm{DV}(X)$ the set of divisorial valuations on $\mathbf{k}(X)$.

Any birational map $f\colon X \dashrightarrow X'$ determines an isomorphism of function fields and transports divisorial valuations to divisorial valuations: if v is a divisorial valuation on $\mathbf{k}(X)$, then $f(v)(\varphi) := v(\varphi \circ f)$ defines a divisorial valuation on $\mathbf{k}(X')$. Indeed, the group of values is not modified by this action, and the residue fields $\mathbf{k}(X)_v$ and $\mathbf{k}(X)_{f(v)}$ are isomorphic. In this way, $\mathsf{Bir}(X)$ acts on $\mathrm{DV}(X)$.

4.2. Hypersurfaces. — We now work with normal and projective varieties; we shall use that their singular loci, and the indeterminacy loci of birational maps have codimension $\geqslant 2$ (in particular, the strict transform of any hypersurface is well defined).

Let $\pi\colon Y\to X$ be a birational morphism between normal projective varieties. Let E be a reduced, irreducible, hypersurface in Y. Since Y is normal, it is smooth at the generic point of E; thus, if φ is an element of $\mathbf{k}(X)^*$, we can define the order of vanishing $v_E(\varphi)$ of φ along E: $v_E(\varphi)=a\geqslant 0$ if $\varphi\circ\pi$ vanishes at order a along E, and $v_E(\varphi)=-a$ if $\varphi\circ\pi$ has a pole of order a along E. Then, v_E is a divisorial valuation, with residue field isomorphic to $\mathbf{k}(E)$. One says that v_E is the *geometric valuation* associated to E (or more precisely to (π,E)). A theorem of Zariski asserts that *every divisorial valuation is geometric* (see [41, §VI.14] or [40, §10]). Thus, one may define the set $\widetilde{\mathsf{Hyp}}(X)$ of *irreducible hypersurfaces* in all (normal) models of X as the set of divisorial valuations $\mathrm{DV}(X)$. Any reduced and irreducible hypersurface E in any model $Y\to X$ determines such a point $E\in\widetilde{\mathsf{Hyp}}(X)$; two divisors E and E' in two models $\pi\colon Y\to X$ and $\pi'\colon Y'\to X$ correspond to the same point in $\widetilde{\mathsf{Hyp}}(X)$ if and only if the two valuations v_E and $v_{E'}$ coincide, if and only if E is the strict transform of E' by the birational map $\pi^{-1}\circ\pi'\colon Y'\dashrightarrow Y$. The action of $\mathsf{Bir}(X)$ on valuations becomes an action by permutations on $\widetilde{\mathsf{Hyp}}(X)$, which we denote by

$$f_\bullet\colon E\in\widetilde{\mathsf{Hyp}}(X)\longmapsto f_\bullet(E); \tag{4.1}$$

it satisfies $f(v_{f_\bullet(E)})=v_E$. If E is a reduced and irreducible hypersurface in the model $\pi\colon Y\to X$, there is a birational morphism $\pi'\colon Y'\to X$ such that $\pi'^{-1}\circ f\circ\pi$ does not contract E; then, the strict transform of E by $\pi'^{-1}\circ f\circ\pi$ is a reduced, irreducible hypersurface E' in Y' that represents the point $f_\bullet(E)$ in $\widetilde{\mathsf{Hyp}}(X)$.

More generally, if $f\colon X\dashrightarrow X'$ is a birational map between normal projective varieties, we obtain a bijection $f_\bullet\colon\widetilde{\mathsf{Hyp}}(X)\to\widetilde{\mathsf{Hyp}}(X')$.

4.3. The subset $\mathsf{Hyp}(X)$. — Let $\mathsf{Hyp}(X)\subset\widetilde{\mathsf{Hyp}}(X)$ be the subset of all reduced, irreducible hypersurfaces of the normal variety X. Recall that a hypersurface is contracted by a birational map if its strict transform is a subset of codimension >1. Given a birational map $f: X\dashrightarrow X'$ between normal projective varieties, define

$$\mathsf{exc}(f)=\#\big\{S\in\mathsf{Hyp}(X):\ f\text{ contracts }S\big\}.$$

This is the *number of contracted hypersurfaces* $S\in\mathsf{Hyp}(X)$ by f. In the following proposition, $f_\circ$ denotes the strict transform and $f_\bullet$ the action on $\widetilde{\mathsf{Hyp}}(X)$.

Proposition 4.1. — *Let $f: X\dashrightarrow X'$ be a birational transformation between normal irreducible projective varieties. Let S be an element of $\mathsf{Hyp}(X)$.*

(1) *If $S\in(f^{-1})_\circ\mathsf{Hyp}(X')$, then $f_\bullet(S)=f_\circ(S)\in\mathsf{Hyp}(X')$.*

(2) *If $S\notin(f^{-1})_\circ\mathsf{Hyp}(X')$, then $f_\circ(S)$ has codimension $\geqslant 2$ (i.e., v contracts S), and $f_\bullet(S)$ is an element of $\widetilde{\mathsf{Hyp}}(X')\smallsetminus\mathsf{Hyp}(X')$.*

(3) *The symmetric difference $f_\bullet(\mathsf{Hyp}(X))\triangle\mathsf{Hyp}(X')$ contains $\mathsf{exc}(f)+\mathsf{exc}(f^{-1})$ elements.*

Proof. — Let U be the complement of $\mathrm{Ind}(f)$ in X'. Since, by Theorem 2.1, $\mathrm{Ind}(f)$ has codimension $\geqslant 2$, no $S\in\mathsf{Hyp}(X)$ is contained in $\mathrm{Ind}(f)$. Let us prove (1). This is clear when f is a birational morphism. To deal with the general case, write $f=g\circ h^{-1}$

where $h\colon Y\to X$ and $g\colon Y\to X'$ are birational morphisms from a normal variety Y. Since h is a birational morphism, $h^\bullet(S)=h^\circ(S)\subset \mathsf{Hyp}(Y)$; since S is not contracted by f, $g_\bullet(h^\circ(S))=g_\circ(h^\circ(S))\in \mathsf{Hyp}(X')$. Thus, $f_\bullet(S)=g_\bullet(h^\bullet(S))$ coincides with the strict transform $f_\circ(S)\in\mathsf{Hyp}(X')$.

Now let us prove (2), assuming thus that $S\notin (f^{-1})_\circ \mathsf{Hyp}(X')$. Let $S''\in\mathsf{Hyp}(Y)$ be the hypersurface $(h^{-1})_\bullet(S)=(h^{-1})_\circ(S)$. Then $h(S'')=S$. If $g_\circ(S'')$ is a hypersurface S', then $(f^{-1})_\circ(S')=S$, contradicting $S\notin (f^{-1})_\circ\mathsf{Hyp}(X')$. Thus, g contracts S'' onto a subset $S'\subset X'$ of codimension $\geqslant 2$. Since $S'=f_\circ(S)$, assertion (2) is proved.

Assertion (3) follows from the previous two assertions. □

Example 4.2. — Let g be a birational transformation of $\mathbb{P}^n_{\mathbf{k}}$ of degree d, meaning that g is defined by $n+1$ homogeneous polynomials of degree d without common factor of positive degree, or equivalently that $g^*(H)\simeq dH$ where H is any hyperplane of $\mathbb{P}^n_{\mathbf{k}}$. The exceptional set of g has degree $(n+1)(d-1)$; thus, $\mathsf{exc}_{\mathbb{P}^n_{\mathbf{k}}}(g)\leqslant (n+1)(d-1)$. More generally, if H is a polarization of X, then $\mathsf{exc}_X(g)$ is bounded from above by a function that depends only on the degree $\deg_H(g):=(g^*H)\cdot H^{\dim(X)-1}$.

Theorem 4.3. — *Let X be a normal projective variety. The group $\mathsf{Bir}(X)$ acts faithfully by permutations on the set $\widetilde{\mathsf{Hyp}}(X)$ via the homomorphism $g\mapsto g_\bullet$ from $\mathsf{Bir}(X)$ to $\mathsf{Perm}(\widetilde{\mathsf{Hyp}}(X))$. This action commensurates the subset $\mathsf{Hyp}(X)$ of $\widetilde{\mathsf{Hyp}}(X)$: for every $g\in\mathsf{Bir}(X)$, $|g_\bullet(\mathsf{Hyp}(X))\triangle\mathsf{Hyp}(X)|=\mathsf{exc}(g)+\mathsf{exc}(g^{-1})$.*

It remains only to prove that the homomorphism $f\in\mathsf{Bir}(X)\mapsto f_\bullet\in\mathsf{Perm}(\widetilde{\mathsf{Hyp}}(X))$ is injective. An element of its kernel satisfies $f_\circ(W)=W$ for every hypersurface W of X. Embedding X in some projective space $\mathbb{P}^m_{\mathbf{k}}$, every point of $X(\mathbf{k})$ is the intersection of finitely many irreducible hyperplane sections of X: since all these sections are fixed by f, every point is fixed by f, and f is the identity.

4.4. Products of varieties. — Let X and Y be irreducible, normal projective varieties. Consider the embedding of $\mathsf{Bir}(X)$ into $\mathsf{Bir}(X\times Y)$ given by the action $f\cdot(x,y)=(f(x),y)$ for $f\in\mathsf{Bir}(X)$. The injection j_Y of $\mathsf{Hyp}(X)$ into $\mathsf{Hyp}(X\times Y)$ given by $j_Y(S)=S\times Y$ extends to an injection of $\widetilde{\mathsf{Hyp}}(X)$ into $\widetilde{\mathsf{Hyp}}(X\times Y)$; this inclusion is $\mathsf{Bir}(X)$-equivariant. The following result will be applied to Corollary 5.7.

Proposition 4.4. — *Let a group Γ act on X by birational transformations. Then Γ transfixes $\mathsf{Hyp}(X)$ in $\widetilde{\mathsf{Hyp}}(X)$ if and only if it transfixes $\mathsf{Hyp}(X\times Y)$ in $\widetilde{\mathsf{Hyp}}(X\times Y)$. More precisely, the subset $\mathsf{Hyp}(X\times Y)\smallsetminus j_Y(\mathsf{Hyp}(X))$ is $\mathsf{Bir}(X)$-invariant.*

Proof. — The reverse implication is immediate. The direct one follows from the latter statement, which we now prove. The projection of a hypersurface $S\in\mathsf{Hyp}(X\times Y)\smallsetminus j_Y(\mathsf{Hyp}(X))$ on X is surjective. For $f\in\mathsf{Bir}(X)$, f induces an isomorphism between dense open subsets U and V of X, and hence between $U\times Y$ and $V\times Y$; in particular, f does not contract S. This shows that f stabilizes $\mathsf{Hyp}(X\times Y)\smallsetminus j_Y(\mathsf{Hyp}(X))$. □

5. Pseudo-regularization of birational transformations

In this section, the action of $\mathsf{Bir}(X)$ on $\widetilde{\mathsf{Hyp}}(X)$ is used to characterize and study groups of birational transformations that are pseudo-regularizable, in the sense of Definition 1.1. As before, $\mathbf{k}$ is an algebraically closed field.

5.1. An example. — Consider the birational transformation $f(x, y) = (x + 1, xy)$ of $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$. The vertical curves $C_i = \{x = -i\}$, $i \in \mathbf{Z}_+$, are exceptional curves for the cyclic group $\Gamma = \langle f\rangle$: each of these curves is contracted by an element of Γ onto a point, namely $f^{i+1}_\circ(C_i) = (1, 0)$. Let $\varphi\colon Y \dashrightarrow \mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ be a birational map, and let $\mathscr{U}$ be a non-empty open subset of Y. Consider the subgroup $\Gamma_Y := \varphi^{-1} \circ \Gamma \circ \varphi$ of $\mathsf{Bir}(Y)$. If i is large enough, $\varphi^{-1}_\circ(C_i)$ is an irreducible curve $C'_i \subset Y$, and these curves C'_i are pairwise distinct, so that most of them intersect $\mathscr{U}$. For positive integers m, f^{i+m} maps C_i onto $(m, 0)$, and $(m, 0)$ is not an indeterminacy point of φ^{-1} if m is large. Thus, $\varphi^{-1} \circ f^m \circ \varphi$ contracts C'_i, and $\varphi^{-1} \circ f^m \circ \varphi$ is not a pseudo-automorphism of $\mathscr{U}$. This argument proves the following lemma.

Lemma 5.1. — *Let X be the surface $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$. Let $f\colon X \dashrightarrow X$ be defined by $f(x, y) = (x+1, xy)$, and let Γ be the subgroup generated by f^ℓ, for some $\ell \geqslant 1$. Then the cyclic group Γ is not pseudo-regularizable.*

This shows that Theorem 1 requires an assumption on Γ. More generally, a subgroup $\Gamma \subset \mathsf{Bir}(X)$ cannot be pseudo-regularized if

(a) Γ contracts a family of hypersurfaces $W_i \subset X$ whose union is Zariski dense

(b) the union of all strict transforms $f_\circ(W_i)$, for $f \in \Gamma$ contracting W_i, is a subset of X whose Zariski closure has codimension at most 1.

5.2. Characterization of pseudo-isomorphisms. — Recall that $f_\bullet$ denotes the bijection $\widetilde{\mathsf{Hyp}}(X) \to \widetilde{\mathsf{Hyp}}(X')$ which is induced by a birational map $f\colon X \dashrightarrow X'$. Also, for any nonempty open subset $U \subset X$, we define $\mathsf{Hyp}(U) = \{H \in \mathsf{Hyp}(X) : H \cap U \neq \varnothing\}$; its complement in $\mathsf{Hyp}(X)$ is finite.

Proposition 5.2. — *Let $f : X \dashrightarrow X'$ be a birational map between normal projective varieties. Let $U \subset X$ and $U' \subset X'$ be two dense open subsets. Then, f induces a pseudo-isomorphism $U \dashrightarrow U'$ if and only if $f_\bullet(\mathsf{Hyp}(U)) = \mathsf{Hyp}(U')$.*

Proof. — If f restricts to a pseudo-isomorphism $U \dashrightarrow U'$, then f maps every hypersurface of U to a hypersurface of U' by strict transform. And $(f^{-1})_\circ$ is an inverse for $f_\circ\colon \mathsf{Hyp}(U) \to \mathsf{Hyp}(U')$. Thus, $f_\bullet(\mathsf{Hyp}(U)) = f_\circ(\mathsf{Hyp}(U) = \mathsf{Hyp}(U')$.

Now, assume that $f_\bullet(\mathsf{Hyp}(U)) = \mathsf{Hyp}(U')$. Since X and X' are normal, $\mathrm{Ind}(f)$ and $\mathrm{Ind}(f^{-1})$ have codimension $\geqslant 2$ (Theorem 2.1).

Let $f_{U,U'}$ be the birational map from U to U' which is induced by f. The indeterminacy set of $f_{U,U'}$ is contained in the union of the set $\mathrm{Ind}(f) \cap U$ and the set of points $x \in U \smallsetminus \mathrm{Ind}(f)$ which are mapped by f in the complement of U'; this second part of $\mathrm{Ind}(f_{U,U'})$ has codimension 2, because otherwise there would be an irreducible hypersurface W in U which would be mapped in $X' \smallsetminus U'$, contradicting the equality

$f_\bullet(\mathsf{Hyp}(U)) = \mathsf{Hyp}(U')$. Thus, the indeterminacy set of $f_{U,U'}$ has codimension $\geqslant 2$. Changing f in its inverse f^{-1}, we see that the indeterminacy set of $f^{-1}_{U',U}\colon U' \dashrightarrow U'$ has codimension $\geqslant 2$ too.

If $f_{U,U'}$ contracted an irreducible hypersurface $W \subset U$ onto a subset of U' of codimension $\geqslant 2$, then $f_\bullet(W)$ would not be contained in $\mathsf{Hyp}(U')$ (it would correspond to an element of $\widetilde{\mathsf{Hyp}}(X') \smallsetminus \mathsf{Hyp}(X')$ by Proposition 4.1). Thus, $f_{U,U'}$ satisfies the first property of Proposition 2.4 and, therefore, is a pseudo-isomorphism. □

5.3. Characterization of pseudo-regularization. — Let X be a (reduced, irreducible) normal projective variety. Let Γ be a subgroup of $\mathsf{Bir}(X)$. Assume that the action of Γ on $\widetilde{\mathsf{Hyp}}(X)$ fixes (globally) a subset $A \subset \widetilde{\mathsf{Hyp}}(X)$ such that

$$|A \triangle \mathsf{Hyp}(X)| < +\infty.$$

In other words, A is obtained from $\mathsf{Hyp}(X)$ by removing finitely many hypersurfaces $W_i \in \mathsf{Hyp}(X)$ and adding finitely many hypersurfaces $W'_j \in \widetilde{\mathsf{Hyp}}(X) \smallsetminus \mathsf{Hyp}(X)$. Each W'_j comes from an irreducible hypersurface in some model $\pi_j\colon X_j \to X$, and there is a model $\pi\colon Y \to X$ that covers all of them (i.e., $\pi \circ \pi_j^{-1}$ is a morphism from Y to X_j for every j). Then, $\pi^\circ(A)$ is a subset of $\mathsf{Hyp}(Y)$. Changing X into Y, A into $\pi^\circ(A)$, and Γ into $\pi^{-1} \circ \Gamma \circ \pi$, we may assume that

(1) $A = \mathsf{Hyp}(X) \smallsetminus \{E_1, \ldots, E_\ell\}$ where the E_i are ℓ distinct irreducible hypersurfaces of X,

(2) the action of Γ on $\widetilde{\mathsf{Hyp}}(X)$ fixes the set A.

In what follows, we denote by $\mathscr{U}$ the Zariski open subset $X \smallsetminus \cup_i E_i$ and by ∂X the set $X \smallsetminus \mathscr{U} = E_1 \cup \cdots \cup E_\ell$, considered as the boundary of the compactification X of $\mathscr{U}$.

Lemma 5.3. — *The group Γ acts by pseudo-automorphisms on the open subset $\mathscr{U}$. If $\mathscr{U}$ is smooth (or locally factorial) and there is an ample divisor D whose support coincides with ∂X, then Γ acts by automorphisms on $\mathscr{U}$.*

In this statement, we say that the support of a divisor D coincides with ∂X if $D = \sum_i a_i E_i$ with $a_i > 0$ for every $1 \leqslant i \leqslant \ell$.

Proof. — Since $A = \mathsf{Hyp}(\mathscr{U})$ is Γ-invariant, Proposition 5.2 shows that Γ acts by pseudo-automorphisms on $\mathscr{U}$. Since D is an ample divisor, some positive multiple mD is very ample, and the complete linear system $|mD|$ provides an embedding of X in a projective space. The divisor mD corresponds to a hyperplane section of X in this embedding, and the open subset $\mathscr{U}$ is an affine variety because the support of D is equal to ∂X. Proposition 2.9 concludes the proof of the lemma. □

By Theorem 4.3, every subgroup of $\mathsf{Bir}(X)$ acts on $\widetilde{\mathsf{Hyp}}(X)$ and commensurates $\mathsf{Hyp}(X)$. If Γ transfixes $\mathsf{Hyp}(X)$, there is an invariant subset A of $\widetilde{\mathsf{Hyp}}(X)$ for which $A \triangle \mathsf{Hyp}(X)$ is finite. Thus, one gets the following characterization of pseudo-regularizability (the converse being immediate).

Theorem 5.4. — *Let X be a normal projective variety over an algebraically closed field $\mathbf{k}$. Let Γ be a subgroup of $\mathsf{Bir}(X)$. Then Γ transfixes the subset $\mathsf{Hyp}(X)$ of $\widetilde{\mathsf{Hyp}}(X)$ if and only if Γ is pseudo-regularizable. More precisely, if Γ transfixes $\mathsf{Hyp}(X)$, then there is a birational morphism $\pi\colon Y\to X$ and a dense open subset $\mathscr{U}\subset Y$ such that $\pi^{-1}\circ\Gamma\circ\pi$ acts by pseudo-automorphisms on $\mathscr{U}$.*

This theorem applies directly when $\Gamma\subset\mathsf{Bir}(X)$ has property (FW) because Theorem 4.3 shows that Γ commensurates $\mathsf{Hyp}(X)$. This proves Theorem 1.

Remark 5.5. — Assuming $\mathrm{char}(\mathbf{k})=0$, we may apply the resolution of singularities and work in the category of smooth varieties. As explained in Remark 1.2 and Lemma 5.3, there are two extreme cases, corresponding to an empty or an ample boundary $B=\cup_i E_i$. If $\mathscr{U}=Y$, Γ acts by pseudo-automorphisms on the projective model Y. As explained in Theorem 2.8, $\mathsf{Psaut}(Y)$ is an extension of a subgroup of $\mathsf{GL}(\mathsf{NS}(Y))$ by an algebraic group which contains $\mathsf{Aut}(Y)^0$ as a finite index subgroup. If $\mathscr{U}$ is affine, Γ acts by automorphisms on $\mathscr{U}$. The group $\mathsf{Aut}(\mathscr{U})$ may be huge ($\mathscr{U}$ could be the affine space), but there are techniques to study groups of automorphisms that are not available for birational transformations; for instance Γ is residually finite and virtually torsion-free if Γ is a group of automorphisms generated by finitely many elements (see [5]).

5.4. Distorted elements. — Theorem 5.4 may be applied when Γ has Property (FW), or for pairs (Λ,Γ) with relative Property (FW). Here is one application:

Corollary 5.6. — *Let X be an irreducible projective variety. Let Γ be a distorted cyclic subgroup of $\mathsf{Bir}(X)$. Then Γ is pseudo-regularizable.*

The contraposition is useful to show that some elements of $\mathsf{Bir}(X)$ are undistorted. Let us state it in a strong "stable" way.

Corollary 5.7. — *Let X be a normal irreducible projective variety and let f be an element of $\mathsf{Bir}(X)$ such that the cyclic group $\langle f\rangle$ does not transfix $\mathsf{Hyp}(X)$ (i.e., f is not pseudo-regularizable). Then $\langle f\rangle$ is undistorted in $\mathsf{Bir}(X)$; more generally the cyclic subgroup $\langle f\times \mathrm{Id}_Y\rangle$ is undistorted in $\mathsf{Bir}(X\times Y)$ for every irreducible projective variety Y.*

The latter consequence indeed follows from Proposition 4.4. This can be applied to various examples, such as those in Example 6.9.

6. Illustrating results

6.1. Surfaces whose birational group is transfixing. — If X is a projective curve, $\mathsf{Bir}(X)$ always transfixes $\mathsf{Hyp}(X)$, since it acts by automorphisms on a smooth model of X. We now consider the same problem for surfaces.

Proposition 6.1. — *Let X be a normal irreducible variety of positive dimension over an algebraically closed field $\mathbf{k}$. Then $\mathsf{Bir}(X\times\mathbb{P}^1)$ does not transfix $\mathsf{Hyp}(X\times\mathbb{P}^1)$.*

Proof. — We can suppose that X is affine and work in the model $X \times \mathbb{A}^1$. For φ a nonzero regular function on X, define a regular self-map f of $X \times \mathbb{A}^1$ by $f(x,t) = (x, \varphi(x)t)$. Denoting by $Z(\varphi)$ the zero set of φ, we remark that f induces an automorphism of the open subset $(X \smallsetminus Z(\varphi)) \times \mathbb{A}^1$. In particular, it induces a permutation of $\mathsf{Hyp}((X \smallsetminus Z(\varphi)) \times \mathbb{A}^1)$. Set $M = \mathsf{Hyp}(X \times \mathbb{A}^1)$. Since f contracts the complement $Z(\varphi) \times \mathbb{A}^1$ to the subset $Z(\varphi) \times \{0\}$, which has codimension $\geqslant 2$, its action on $\widetilde{\mathsf{Hyp}}(X \times \mathbb{A}^1)$ maps the codimension 1 components of $Z(\varphi) \times \mathbb{A}^1$ outside M. Therefore $M \smallsetminus f^{-1}(M)$ is the set of irreducible components of $Z(\varphi) \times \mathbb{A}^1$. Its cardinal is equal to the number of irreducible components of $Z(\varphi)$. When φ varies, this number is unbounded; hence, $\mathsf{Bir}(X \times \mathbb{A}^1)$ does not transfix $\mathsf{Hyp}(X \times \mathbb{A}^1)$. □

Varieties which are birational to the product of a variety and the projective line are said to be *ruled*. Proposition 6.1 states that $\mathsf{Bir}(Y)$ does not transfix $\mathsf{Hyp}(Y)$ when Y is ruled and of dimension $\geqslant 2$. The converse holds for surfaces:

Theorem 6.2. — *Let* $\mathbf{k}$ *be an algebraically closed field. Let* X *be an irreducible normal projective surface over* $\mathbf{k}$*. The following are equivalent:*

(1) $\mathsf{Bir}(X)$ *does not transfix* $\mathsf{Hyp}(X)$*;*
(2) *the Kodaira dimension of* X *is* $-\infty$*;*
(3) X *is ruled;*
(4) *there is no projective surface* Y *that is birationally equivalent to* X *and satisfies* $\mathsf{Bir}(Y) = \mathsf{Aut}(Y)$*.*

Proof. — The equivalence between (2) and (3) is classical (see [2] and [1, 32]). The group $\mathsf{Aut}(Y)$ fixes $\mathsf{Hyp}(Y) \subset \widetilde{\mathsf{Hyp}}(Y)$, hence (1) implies (4). If the Kodaira dimension of X is $\geqslant 0$, then X has a unique minimal model X_0, and $\mathsf{Bir}(X_0) = \mathsf{Aut}(X_0)$. Thus, (4) implies (2). Finally, Proposition 6.1 shows that (3) implies (1). □

Theorem 6.3. — *Let* X *be an irreducible projective surface over an algebraically closed field* $\mathbf{k}$*. The following are equivalent:*

(1) *some finitely generated subgroup of* $\mathsf{Bir}(X)$ *does not transfix* $\mathsf{Hyp}(X)$*;*
(2) *some cyclic subgroup of* $\mathsf{Bir}(X)$ *does not transfix* $\mathsf{Hyp}(X)$*;*
(3)
- $\mathbf{k}$ *has characteristic* 0*, and* X *is birationally equivalent to the product of the projective line with a curve of genus 0 or 1, or*
- $\mathbf{k}$ *has positive characteristic, and* X *is a rational surface.*

Example 6.4. — Let $\mathbf{k}$ be an algebraically closed field that is not algebraic over a finite field. Let t be an element of infinite order in the multiplicative group $\mathbf{k}^*$. Then the birational transformation g of $\mathbb{P}^2_{\mathbf{k}}$ given, in affine coordinates, by $(x,y) \mapsto (tx+1, xy)$ does not transfix $\mathsf{Hyp}(\mathbb{P}^2_{\mathbf{k}})$. Indeed, it is easy to show that the hypersurface $C = \{x = 0\}$ satisfies, for $n \in \mathbf{Z}$, $f^n_\bullet(C) \in \mathsf{Hyp}(\mathbb{P}^2_{\mathbf{k}})$ if and only if $n \leqslant 0$.

Example 6.5. — Example 6.4 works under a small restriction on $\mathbf{k}$. Here is an example over an arbitrary algebraically closed field $\mathbf{k}$. Let L and L' be two lines in $\mathbb{P}^2_{\mathbf{k}}$

intersecting transversally at a point q. Let f be a birational transformation of $\mathbb{P}^2_{\mathbf{k}}$ that contracts L' onto q and fixes L. For instance, in affine coordinates, the monomial map $(x,y)\mapsto(x,xy)$ contracts the y-axis onto the origin, and fixes the x-axis. Assume that there is an open neighborhood $\mathscr{U}$ of q such that f does not contract any curve in $\mathscr{U}$ except the line L'. Let C be an irreducible curve that intersects L and L' transversally at q. Then, for every $n\geqslant 1$, the strict transform $f_\circ^n(C)$ is an irreducible curve, and its order of tangency with L goes to infinity with n. Thus, the degree of $f_\circ^n(C)$ goes to infinity, and the $f_\circ^n(C)$ form an infinite sequence in $\mathsf{Hyp}(\mathbb{P}^2_{\mathbf{k}})$.

Now, assume that C is contracted by f^{-1} onto a point p, $p\notin \mathrm{Ind}(f)$, and p is fixed by f^{-1}. Then, for every $m\geqslant 1$, $f_\bullet^{-m}(C)$ is not in $\mathsf{Hyp}(\mathbb{P}^2_{\mathbf{k}})$. This shows that the orbit of C under the action of $f_\bullet$ intersects $\mathsf{Hyp}(\mathbb{P}^2_{\mathbf{k}})$ and its complement $\widetilde{\mathsf{Hyp}}(\mathbb{P}^2_{\mathbf{k}})\smallsetminus\mathsf{Hyp}(\mathbb{P}^2_{\mathbf{k}})$ on the infinite sets $\{f_\circ^n(C) : n\geqslant 1\}$ and $\{f_\bullet^{-m}(C) : m\geqslant 1\}$. In particular, f does not transfix $\mathsf{Hyp}(\mathbb{P}^2_{\mathbf{k}})$.

Since such maps exist over every algebraically closed field $\mathbf{k}$, this example shows that property (2) of Theorem 6.3 is satisfied for every rational surface X.

Proof. — Trivially (2) implies (1). Suppose that (3) holds and let us prove (2). The case $X=\mathbb{P}^1\times\mathbb{P}^1$ is already covered by Lemma 5.1 in characteristic zero, and by the previous example in positive characteristic. The case $X=C\times\mathbb{P}^1$ in characteristic zero, where C is an elliptic curve, is similar. To see it, fix a point $t_0\in C$ and a rational function φ on C that vanishes at t_0. Then, since $\mathbf{k}$ has characteristic zero, one can find a translation s of C of infinite order such that the orbit $\{s^n(t_0) : n\in\mathbf{Z}\}$ does not contain any other zero or pole of φ (here we use that the characteristic of $\mathbf{k}$ is 0). Consider the birational transformation $f\in\mathsf{Bir}(X)$ given by $f(t,x)=(s(t),\varphi(t)x)$. Let H be the hypersurface $\{t_0\}\times C$. Then for $n\in\mathbf{Z}$, we have $(f_\bullet)^nH\in\mathsf{Hyp}(X)$ if and only if $n\leqslant 0$. Hence the action of the cyclic group $\langle f\rangle$ does not transfix $\mathsf{Hyp}(X)$.

Let us now prove that (1) implies (3). Applying Theorem 6.2, and changing X to a birationally equivalent surface if necessary, we assume that $X=C\times\mathbb{P}^1$ for some (smooth irreducible) curve C. We may now assume that the genus of C is $\geqslant 2$, or $\geqslant 1$ in positive characteristic, and we have to show that every finitely generated group Γ of $\mathsf{Bir}(X)$ transfixes $\mathsf{Hyp}(X)$. Since the genus of C is $\geqslant 1$, the group $\mathsf{Bir}(X)$ preserves the fibration $X\to C$; this gives a surjective homomorphism $\mathsf{Bir}(X)\to\mathsf{Aut}(C)$. Now let us fully use the assumption on C: if its genus is $\geqslant 2$, then $\mathsf{Aut}(C)$ is finite; if its genus is 1 and $\mathbf{k}$ has positive characteristic, then $\mathsf{Aut}(C)$ is locally finite (every finitely generated subgroup is finite), and in particular the projection of Γ on $\mathsf{Aut}(C)$ has a finite image. Thus the kernel of this homomorphism intersects Γ in a finite index subgroup Γ_0. It now suffices to show that Γ_0 transfixes $\mathsf{Hyp}(X)$. Every $f\in\Gamma_0$ has the form $f(t,x)=(t,\varphi_t(x))$ for some rational map $t\mapsto\varphi_t$ from C to PGL_2; define $U_f\subset C$ as the open and dense subset on which φ_γ is regular: by definition, f restricts to an automorphism of $U_f\times\mathbb{P}^1$. Let S be a finite generating subset of Γ_0, and let U_S be the intersection of the open subsets U_g, for $g\in S$. Then Γ_0 acts by automorphisms on $U_S\times\mathbb{P}^1$ and its action on $\mathsf{Hyp}(X)$ fixes the subset $\mathsf{Hyp}(U_S)$. Hence Γ transfixes $\mathsf{Hyp}(X)$. $\square$

It would be interesting to obtain characterizations of the same properties in dimension 3 (see Question 10.2).

6.2. Transfixing Jonquières twists. — Let X be an irreducible normal projective surface and π a morphism onto a smooth projective curve C with connected rational fibers. Let $\mathsf{Bir}_\pi(X)$ be the subgroup of $\mathsf{Bir}(X)$ permuting the fibers of π. Since C is a smooth projective curve, the group $\mathsf{Bir}(C)$ coincides with $\mathsf{Aut}(C)$ and we get a canonical homomorphism $\mathrm{r}_C\colon\ \mathsf{Bir}_\pi(X)\to\mathsf{Aut}(C)$.

The main examples to keep in mind are provided by $\mathbb{P}^1\times\mathbb{P}^1$, Hirzebruch surfaces, and $C\times\mathbb{P}^1$ for some genus 1 curve C, π being the first projection.

Let $\mathsf{Hyp}_\pi(X)$ denote the set of irreducible curves which are contained in fibers of π, and define $\widetilde{\mathsf{Hyp}}_\pi(X)=\mathsf{Hyp}_\pi(X)\sqcup(\widetilde{\mathsf{Hyp}}(X)\smallsetminus\mathsf{Hyp}(X))$, so that $\widetilde{\mathsf{Hyp}}(X)=\widetilde{\mathsf{Hyp}}_\pi(X)\sqcup(\mathsf{Hyp}(X)\smallsetminus\mathsf{Hyp}_\pi(X))$. An irreducible curve $H\subset X$ is an element of $\mathsf{Hyp}(X)\smallsetminus\mathsf{Hyp}_\pi(X)$ if and only if its projection $\pi(H)$ coincides with C; this curves are said to be transverse to π.

Proposition 6.6. — *The decomposition* $\widetilde{\mathsf{Hyp}}(X)=\widetilde{\mathsf{Hyp}}_\pi(X)\sqcup(\mathsf{Hyp}(X)\smallsetminus\mathsf{Hyp}_\pi(X))$ *is* $\mathsf{Bir}_\pi(X)$*-invariant.*

Proof. — Let $H\subset X$ be an irreducible curve which is transverse to π. Since $\mathsf{Bir}_\pi(X)$ acts by automorphisms on C, H can not be contracted by any element of $\mathsf{Bir}_\pi(X)$; more precisely, for every $g\in\mathsf{Bir}_\pi(X)$, $g_\bullet(H)$ is an element of $\mathsf{Hyp}(X)$ which is transverse to π. Thus the set of transverse curves is $\mathsf{Bir}_\pi(X)$-invariant. □

This proposition and the proof of Theorem 6.3 lead to the following corollary.

Corollary 6.7. — *Let G be a subgroup of* $\mathsf{Bir}_\pi(X)$*. If π maps the set of indeterminacy points of the elements of G into a finite subset of C, then G transfixes* $\mathsf{Hyp}(X)$*.*

In the case of cyclic subgroups, we establish a converse under the mild assumption of algebraic stability. Recall that a birational transformation f of a smooth projective surface is *algebraically stable* if the forward orbit of $\mathrm{Ind}(f^{-1})$ does not intersect $\mathrm{Ind}(f)$. By [16], given any birational transformation f of a surface X, there is a birational morphism $u\colon Y\to X$, with Y a smooth projective surface, such that $f_Y:=u^{-1}\circ f\circ u$ is algebraically stable. If $\pi\colon X\to C$ is a fibration, as above, and f is in $\mathsf{Bir}_\pi(X)$, then f_Y preserves the fibration $\pi\circ u$. Thus, we may always assume that X is smooth and f is algebraically stable after a birational conjugacy.

Proposition 6.8. — *Let X be a smooth projective surface, and $\pi\colon X\to C$ a rational fibration. If $f\in\mathsf{Bir}_\pi(X)$ is algebraically stable, then f transfixes* $\mathsf{Hyp}(X)$ *if, and only if the orbit of* $\pi(\mathrm{Ind}(f))$ *under the action of* $\mathrm{r}_C(f)$ *is finite.* □

For $X=\mathbb{P}^1\times\mathbb{P}^1$, the reader can check (e.g., conjugating a suitable automorphism) that the proposition fails without the algebraic stability assumption.

Proof. — Denote by $A \subset \mathsf{Aut}(C)$ the subgroup generated by $\mathrm{r}_C(f)$. Consider a fiber $F \simeq \mathbb{P}^1$ which is contracted to a point q by f. Then, there is a unique indeterminacy point p of f on F. If the orbit of $\pi(q)$ under the action of A is infinite, the orbit of q under the action of f is infinite too. Set $q_n = f^{n-1}(q)$ for $n \geqslant 1$ (so that $q_1 = q$); this sequence of points is well defined because f is algebraically stable: for every $n \geqslant 1$, f is a local isomorphism from a neighborhood of q_n to a neighborhood of q_{n+1}. Then, the image of F in $\widetilde{\mathsf{Hyp}}(X)$ under the action of f^n is an element of $\widetilde{\mathsf{Hyp}}(X) \smallsetminus \mathsf{Hyp}(X)$: it is obtained by a finite number of blow-ups above q_n. Since the points q_n form an infinite set, the images of F form an infinite subset of $\widetilde{\mathsf{Hyp}}(X) \smallsetminus \mathsf{Hyp}(X)$. Together with the previous corollary, this argument proves the proposition. □

Example 6.9. — Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$, with $\pi(x, y) = x$ (using affine coordinates). Start with $f_a(x, y) = (ax, xy)$, for some non-zero parameter $a \in \mathbf{k}$. The action of $\mathrm{r}_C(f_a)$ on $C = \mathbb{P}^1$ fixes the images 0 and ∞ of the indeterminacy points of f_a. Thus, f_a transfixes $\widetilde{\mathsf{Hyp}}(X)$ by Corollary 6.7. Now, consider $g_a(x, y) = (ax, (x+1)y)$. Then, the orbit of -1 under multiplication by a is finite if and only if a is a root of unity; thus, if a is not a root of unity, g_a does not transfix $\mathsf{Hyp}(X)$. Section 5.1 provides more examples of that kind.

7. Birational transformations of surfaces I

From now on, we work in dimension 2. We shall repeatedly use two specific features of surfaces. First, the resolution of singularities is available in all characteristics, so that we can always assume the varieties to be smooth. Hence X, Y, and Z will be smooth projective surfaces over the algebraically closed field $\mathbf{k}$. Second, smooth rational curves of self-intersection -1, also called exceptional curves of the first kind or (-1)-curves, can be blown down onto a smooth point. And if a curve is contracted by a birational morphism $\pi\colon Y \to X$, then the contraction can be down by successively contracting (-1)-curves.

7.1. Regularization.

— In this section, we refine Theorem 5.4, in order to apply results of Danilov and Gizatullin. Recall that a curve C in a smooth surface Y has *normal crossings* if each of its singularities is a simple node with two transverse tangents. In the complex case, this means that C is locally analytically equivalent to $\{xy = 0\}$ (two branches intersecting transversally) in an analytic neighborhood of each of its singularities.

Theorem 7.1. — *Let X be a smooth projective surface, defined over an algebraically closed field $\mathbf{k}$. Let Γ be a subgroup of $\mathsf{Bir}(X)$ that transfixes the subset $\mathsf{Hyp}(X)$ of $\widetilde{\mathsf{Hyp}}(X)$. There exists a smooth projective surface Z, a birational map $\varphi\colon Z \dashrightarrow X$ and a dense open subset $\mathscr{U} \subset Z$ such that, writing the boundary $\partial Z := Z \smallsetminus \mathscr{U}$ as a finite union of irreducible components $E_i \subset Z$, $1 \leqslant i \leqslant \ell$, the following properties hold:*

(1) *The boundary ∂Z is a curve with normal crossings.*

(2) *The subgroup* $\Gamma_Z := \varphi^{-1} \circ \Gamma \circ \varphi \subset \mathsf{Bir}(Z)$ *acts by automorphisms on the open subset* $\mathscr{U}$.

(3) *For all* $i \in \{1, \ldots, \ell\}$ *and* $g \in \Gamma_Z$*, the strict transform of* E_i *under the action of* g *on* Z *is contained in* ∂Z*: either* $g_\circ(E_i)$ *is a point of* ∂Z *or* $g_\circ(E_i)$ *is an irreducible component* E_j *of* ∂Z.

(4) *For all* $i \in \{1, \ldots, \ell\}$*, there exists an element* $g \in \Gamma_Z$ *that contracts* E_i *to a point* $g_\circ(E_i) \in \partial Z$*. In particular,* E_i *is a rational curve.*

(5) *The pair* $(Z, \mathscr{U})$ *is minimal for the previous properties, in the following sense: if one contracts a smooth curve of self-intersection* -1 *in* ∂Z*, then the boundary stops to be a normal crossing divisor.*

Before starting the proof, note that the boundary ∂Z may a priori contain an irreducible rational curve E with a node.

Proof. — We apply Theorem 5.4, and get a smooth surface Y_0, a birational morphism $\varphi_0 \colon Y_0 \to X$, and an open subset $\mathscr{U}_0$ of Y_0 such that $\Gamma_0 := \varphi_0^{-1} \circ \Gamma \circ \varphi_0$ acts by pseudo-automorphisms on $\mathscr{U}_0$ and $\partial Y_0 := Y_0 \smallsetminus \mathscr{U}_0$ is a curve. The action of Γ_0 on $\mathscr{U}_0$ is not yet by automorphisms; we shall progressively modify the triple $(Y_0, \mathscr{U}_0, \varphi_0)$ to obtain a surface Z with properties (1) to (5).

Step 1. — First, we blow-up the singularities of the curve ∂Y_0 which are not simple nodes to get a boundary that is a normal crossing divisor. This replaces the surface Y_0 by a new one, still denoted Y_0. This modification adds new components to the boundary ∂Y_0 but does not change the fact that Γ_0 acts by pseudo-automorphisms on $\mathscr{U}_0$. Let ℓ_0 be the number of irreducible components of $Y_0 \smallsetminus \mathscr{U}_0$.

Step 2. — Consider a point q in $\mathscr{U}_0$, and assume that there is an irreducible component E_i of ∂Y_0 that is contracted to q by an element $g \in \Gamma_0$; fix such a g, and denote by D the union of the irreducible components E_j such that $g_\circ(E_j) = q$. By construction, g is a pseudo-automorphism of $\mathscr{U}_0$. The curve D does not intersect the indeterminacy set of g, since otherwise there would be a curve C containing q that is contracted by g^{-1}. And D is a connected component of ∂Y_0, because otherwise g maps one of the irreducible components of ∂Y_0 to a curve that intersects $\mathscr{U}_0$. Thus, there are open neighborhoods $\mathscr{W}$ of D and $\mathscr{W}'$ of q such that $\mathscr{W} \cap \partial Y_0 = D$ and g realizes an isomorphism from $\mathscr{W} \smallsetminus D$ to $\mathscr{W}' \smallsetminus \{q\}$, contracting D onto the smooth point $q \in Y_0$. In particular, $\mathscr{W}$ can be contracted onto a smooth point (by a succession of contractions of exceptional curves of the first kind). As a consequence, there is a birational morphism $\pi_1 \colon Y_0 \to Y_1$ such that

(1) Y_1 is smooth;
(2) π_1 contracts D onto a point $q_1 \in Y_1$;
(3) π_1 is an isomorphism from $Y_0 \smallsetminus D$ to $Y_1 \smallsetminus \{q_1\}$.

In particular, $\pi_1(\mathscr{U}_0)$ is an open subset of Y_1 and $\mathscr{U}_1 = \pi_1(\mathscr{U}_0) \cup \{q_1\}$ is an open neighborhood of q_1 in Y_1.

Then, $\Gamma_1 := \pi_1 \circ \Gamma_0 \circ \pi_1^{-1}$ acts birationally on Y_1, and by pseudo-automorphisms on $\mathscr{U}_1$. The boundary $\partial Y_1 = Y_1 \smallsetminus \mathscr{U}_1$ contains ℓ_1 irreducible components, with $\ell_1 < \ell_0$ (the difference is the number of components of D), and is a normal crossing divisor because D is a connected component of ∂Y_0.

Repeating this process, we construct a sequence of surfaces $\pi_k \colon Y_{k-1} \to Y_k$ and open subsets $\pi_k(\mathscr{U}_{k-1}) \subset \mathscr{U}_k \subset Y_k$ such that the number of irreducible components of $\partial Y_k = Y_k \smallsetminus \mathscr{U}_k$ decreases. After a finite number of steps (at most ℓ_0), we may assume that $\Gamma_k \subset \mathsf{Bir}(Y_k)$ does not contract any boundary curve onto a point of the open subset $\mathscr{U}_k$. On such a model, Γ_k acts by automorphisms on $\mathscr{U}_k$.

We fix such a model, which we denote by the letters Y, $\mathscr{U}$, ∂Y, φ. The new birational map $\varphi \colon Y \dashrightarrow X$ is the composition of φ_0 with the inverse of the morphism $Y_0 \to Y_k$. On such a model, properties (1) and (2) are satisfied. Moreover, (3) follows from (2). We now modify Y further to get property (4).

Step 3. — Assume that the irreducible component $E_i \subset Y \smallsetminus \mathscr{U}$ is not contracted by Γ. Let F be the orbit of E_i: $F = \cup_{g\in\Gamma} g_\circ(E_i)$; by property (3), this curve is contained in the boundary ∂Y of the open subset $\mathscr{U}$. Let $\overline{\partial Y \smallsetminus F}$ denote the Zariski closure of $\partial Y \smallsetminus F$, and set

$$\mathscr{U}' = \mathscr{U} \cup (F \smallsetminus \overline{\partial Y \smallsetminus F}).$$

The group Γ also acts by pseudo-automorphisms on $\mathscr{U}'$. This operation decreases the number ℓ of irreducible components of the boundary. Thus, combining steps 2 and 3 finitely many times, we reach a model that satisfies Properties (1) to (4). We continue to denote it by Y.

Step 4. — If the boundary ∂Y contains a smooth (rational) curve E_i of self-intersection -1, it can be blown down to a smooth point q by a birational morphism $\pi \colon Y \to Y'$; the open subset $\mathscr{U}$ is not affected, but the boundary $\partial Y'$ has one component less. If E_i was a connected component of ∂Y, then $\mathscr{U}' = \pi(\mathscr{U}) \cup \{q\}$ is a neighborhood of q and one replaces $\mathscr{U}$ by $\mathscr{U}'$, as in step 2. Now, two cases may happen. If the boundary $\partial Y'$ ceases to be a normal crossing divisor, we come back to Y and do not apply this surgery. If $\partial Y'$ has normal crossings, we replace Y by this new model. In a finite number of steps, looking successively at all (-1)-curves and iterating the process, we reach a new surface Z on which all five properties are satisfied. $\square$

Remark 7.2. — One may also remove property (5) and replace property (1) by

(1') The E_i are rational curves, and none of them is a smooth rational curve with self-intersection -1.

But doing so, we may lose the normal crossing property. To get property (1'), apply the theorem and argue as in step 4.

7.2. Constraints on the boundary. — We now work on the new surface Z given by Theorem 7.1. Thus, Z is the surface, Γ the subgroup of $\mathsf{Bir}(Z)$, $\mathscr{U}$ the open subset on which Γ acts by automorphisms, and ∂Z the boundary of $\mathscr{U}$.

Proposition 7.3 (Gizatullin, [20, §4]). — *There are four possibilities for the geometry of the boundary* $\partial Z = Z \setminus \mathscr{U}$.

(1) ∂Z *is empty.*

(2) ∂Z *is a cycle of rational curves.*

(3) ∂Z *is a chain of* ℓ *rational curves and if* $\ell = 1$ *it is a smooth rational curve of positive self intersection.*

(4) ∂Z *is the disjoint union of finitely many smooth rational curves of self-intersection* 0.

Moreover, in cases (2) *and* (3)*, the open subset* $\mathscr{U}$ *is the blow-up of an affine surface.*

Thus, there are four possibilities for ∂Z, which we study successively. We shall start with (1) and (4) in Sections 7.3 and 7.4. Then case (3) is dealt with in Section 7.5. Case (2) is more involved: it is treated in Section 8.

Before that, let us explain how Proposition 7.3 follows from [20, §5]. First, we describe the precise meaning of the statement, and then we explain how the original results of [20] apply to our situation.

The boundary and its dual graph. — Consider the dual graph $\mathscr{G}_Z$ of the boundary ∂Z. The vertices of $\mathscr{G}_Z$ are in one to one correspondence with the irreducible components E_i of ∂Z. The edges correspond to singularities of ∂Z: each singular point q gives rise to an edge connecting the components E_i that determine the two local branches of ∂Z at q. When the two branches correspond to the same irreducible component, one gets a loop of the graph $\mathscr{G}_Z$.

We say that ∂Z is a *chain* of rational curves if the dual graph is of type A_ℓ: ℓ is the number of components, and the graph is linear, with ℓ vertices. Chains are also called *zigzags* by Danilov and Gizatullin.

We say that ∂Z is a *cycle* if the dual graph is isomorphic to a regular polygon with ℓ vertices. There are two special cases: when ∂Z is reduced to one component, this curve is a rational curve with one singular point and the dual graph is a loop (one vertex, one edge); when ∂Z is made of two components, these components intersect in two distinct points, and the dual graph is made of two vertices with two edges between them. For $\ell = 3, 4, \ldots$, the graph is a triangle, a square, etc.

Gizatullin's original statement. — To describe Gizatullin's article, let us introduce some vocabulary. Let S be a projective surface, and $C \subset S$ be a curve; C is a union of irreducible components, which may have singularities. Assume that S is smooth in a neighborhood of C. Let S_0 be the complement of C in S, and let $\iota\colon S_0 \to S$ be the natural embedding of S_0 in S. Then, S is a *completion* of S_0: this completion is marked by the embedding $\iota\colon S_0 \to S$, and its boundary is the curve C. Following [20] and [21, 22], we only consider completions of S_0 by curves (i.e., $S \setminus \iota(S_0)$ is of pure dimension 1), and we always assume S to be smooth in a neighborhood of the boundary. Such a completion is

(i) *simple* if the boundary C has normal crossings;

(ii) *minimal* if it is simple and minimal for this property: if $C_i \subset C$ is an exceptional curve of the first kind then, contracting C_i, the image of C is not a normal crossing divisor anymore. Equivalently, C_i intersects at least three other components of C. Equivalently, if $\iota' \colon S_0 \to S'$ is another simple completion, and $\pi \colon S \to S'$ is a birational morphism such that $\pi \circ \iota = \iota'$, then π is an isomorphism.

If S is a completion of S_0, one can blow-up boundary points to obtain a simple completion, and then blow-down some of the boundary components C_i to reach a minimal completion.

Now, consider the group of automorphisms of the open surface S_0. This group $\mathsf{Aut}(S_0)$ acts by birational transformations on S. An irreducible component E_i of the boundary C is *contracted* if there is an element g of $\mathsf{Aut}(S_0)$ that contracts E_i: $g_\circ(E_i)$ is a point of C. Let E be the union of the contracted components. In [20, Cor. 4 & Prop. 5 of §5], Gizatullin proves that E satisfies one of the four properties stated in Proposition 7.3; moreover, in cases (2) and (3), E contains an irreducible component E_i with $E_i^2 \geqslant 0$; note that (4) contains the case of a unique rational curve of self-intersection 0 (a different choice is made in [20]).

Thus, Proposition 7.3 follows from the properties of the pair $(Z, \mathscr{U}, \Gamma)$: the open subset $\mathscr{U}$ plays the role of S_0, and Z is the completion S; the boundary ∂Z is the curve C: it is a normal crossing divisor, and it is minimal by construction. Since every component of ∂Z is contracted by at least one element of $\Gamma \subset \mathsf{Aut}(\mathscr{U})$, ∂Z coincides with Gizatullin's curve E. The only thing we have to prove is the last sentence of Proposition 7.3, concerning the structure of the open subset $\mathscr{U}$; thus, we assume that we are in cases (2) or (3) of Proposition 7.3.

First, let us show that $E = \partial Z$ supports an effective divisor D such that $D^2 > 0$ and $D \cdot F \geqslant 0$ for every irreducible curve. If ∂Z is irreducible, then it is a curve of positive self intersection (by convention in case (3), and by [20, Cor. 4 in §4]). Thus, we may assume that ∂Z is a chain or a loop of length $\ell \geqslant 2$. To construct D, fix an irreducible component E_0 of ∂Z with $E_0^2 \geqslant 0$; as said above such a curve exists by Gizatullin's results ([20, Cor. 4 of §5]). Assume that ∂Z is a cycle, and list cyclically the other irreducible components: E_1, E_2, …, up to E_m, with E_1 and E_m intersecting E_0 (and $m = \ell - 1$). If $m = 1$, we set $D = a_0 E_0 + E_1$; then $D \cdot E_0 = 2$ and $D \cdot E_1 = 2a_0 + E_1^2$ are positive if a_0 is large enough. If $m \geqslant 2$, we consider $D_1 = a_1 E_0 + E_1$. Then $D_1 \cdot E_0 = 1$ and $D_1 \cdot E_1 = a_1 + E_1^2$ are both positive if a_1 is large enough; moreover, $D_1 \cdot E_2 = 1$ and $D_1 \cdot E_m = a_1$ if $m \geqslant 3$, or $D_1 \cdot E_2 = a_1 + 1$ if $m = 2$. Then, set $D_2 = a_2 D_1 + E_2$, …, up to $D_m = a_m D_{m-1} + E_m$. If the a_j are large enough, all intersections $D_m \cdot E_j$ are positive, for all $0 \leqslant j \leqslant m$. We choose such a sequence of integers a_i, and set $D = D_m$. Then D intersects every irreducible curve F non-negatively and $D^2 > 0$. Thus, D is big and nef (see [30, §2.2]). A similar proof applies when ∂Z is a zigzag. Let $[D]^\perp$ be the subspace of $\mathsf{NS}(X)$ spanned by classes of irreducible curves F with $D \cdot F = 0$.

Now, consider the linear system $|mD|$ for a large divisible integer $m > 0$, and decompose it into a mobile part $|M_m|$ and a fixed part $|R_m|$, where M_m and R_m are effective divisors with

$$mD = M_m + R_m.$$

Note that the irreducible curves F with $[F] \in [D]^\perp$ are characterized by the property $F \cdot M_m = 0 = F \cdot R_m$. By definition, $|M_m|$ has only finitely many base points. Thus, changing m into some large multiple if necessary, and applying Fujita-Zariski theorem (see [30, 2.1.32, p. 132]), we may assume that

(i) M_m is big (because so is D);
(ii) M_m is nef (because M_m is mobile);
(iii) M_m is free of base point (by Fujita-Zariski theorem).

Then, the linear system $|M_m|$ gives a birational morphism (see [30, 2.1.27, p. 129])

$$\varphi\colon Z \longrightarrow Z' \subset \mathbb{P}^N$$

onto a normal, projective surface Z' such that M_m coincides with the pullback of a hyperplane section H of Z'. In particular, $H^1(Z, dM_m) = 0$ for large values of d. Now, let us show that $R_m = 0$ for some adequate choice of mD. If not, some curve E_j of the boundary ∂Z appears as a component of R_m but not as a component of M_m; since $\partial Z = \mathrm{Support}(D)$ is connected, we can choose such an E_j that intersects M_m in at least one point. Thus, $(dM_m + a_jE_j) \cdot E_j > 0$ for any a_j and every large d. Consider the exact sequence of sheaves $\mathscr{O}(dM_m) \to \mathscr{O}(dM_m + E_j) \to \mathscr{O}_{E_j}((dM_m + E_j)_{|E_j})$, and the associated long exact sequence in cohomology. By the vanishing of $H^1(Z, dM_m)$ we get

$$H^0(Z, dM_m) \longrightarrow H^0(Z, dM_m + E_j) \longrightarrow H^0(E_j, dM_m + E_j) \longrightarrow 0.$$

If E_j were part of the base locus of the linear system $|dM_m + E_j|$, then the second arrow in this sequence would vanish, so that $H^0(E_j, dM_m + E_j) = 0$. But this would be a contradiction because E_j is a rational curve and $dM_m + E_j$ has positive degree on E_j. Thus, E_j is not in the base locus of $|dM_m + E_j|$: we may now assume $R_m = 0$. From this, we deduce that an irreducible curve $C \subset Z$ is contracted by φ if and only if $C \cdot M_m = 0$, if and only if $[C] \in [D]^\perp$, if and only if C does not intersect the boundary curve ∂Z; and that φ induces a birational morphism from $Z \smallsetminus \partial Z$ to the affine surface $Z' \smallsetminus H$. This concludes the proof of the proposition.

7.3. Projective surfaces and automorphisms. — In this section, we (almost always) assume that Γ acts by regular automorphisms on a projective surface X. This corresponds to case (1) in Proposition 7.3. Our goal is the special case of Theorem 2 which is stated below as Theorem 7.8. We shall assume that Γ has property (FW) in some of the statements (this was not a hypothesis in Theorem 7.1). We may, and shall, assume that X is smooth. We refer to [2, 6, 25] for the classification of surfaces and the main notions attached to them.

7.3.1. *Action on the Néron-Severi group*. — The intersection form is a non-degenerate quadratic form q_X on the Néron-Severi group $\mathsf{NS}(X)$, and Hodge index theorem asserts that its signature is $(1, \rho(X) - 1)$, where $\rho(X)$ denotes the Picard number, i.e., the rank of the lattice $\mathsf{NS}(X) \simeq \mathbf{Z}^\rho$.

The action of $\mathsf{Aut}(X)$ on the Néron-Severi group $\mathsf{NS}(X)$ provides a linear representation preserving the intersection form q_X. This gives a homomorphism

$$\mathsf{Aut}(X) \longrightarrow \mathsf{O}(\mathsf{NS}(X); q_X).$$

Fix an ample class a in $\mathsf{NS}(X)$ and consider the hyperboloid

$$\mathbb{H}_X = \{u \in \mathsf{NS}(X) \otimes_{\mathbf{Z}} \mathbf{R} : \ q_X(u,u) = 1 \text{ and } q_X(u,a) > 0\}.$$

This set is one of the two connected components of $\{u : q_X(u,u) = 1\}$. With the riemannian metric induced by $(-q_X)$, it is a copy of the hyperbolic space of dimension $\rho(X) - 1$; the group $\mathsf{Aut}(X)$ acts by isometries on this space (see [11]).

Proposition 7.4. — *Let X be a smooth projective surface. Let Γ be a subgroup of $\mathsf{Aut}(X)$. If Γ has Property (FW), then its action on $\mathsf{NS}(X)$ fixes a very ample class, the image of Γ in $\mathsf{O}(\mathsf{NS}(X); q_X)$ is finite, and a finite index subgroup of Γ is contained in $\mathsf{Aut}(X)^0$.*

Proof. — The image Γ^* of Γ is contained in the arithmetic group $\mathsf{O}(\mathsf{NS}(X); q_X)$. The Néron-Severi group $\mathsf{NS}(X)$ is a lattice $\mathbf{Z}^\rho$ and q_X is defined over $\mathbf{Z}$. If ρ is odd, one can change $\mathsf{NS}(X)$ into a $(\rho+1)$-dimensional lattice $\mathsf{NS}(X) \oplus \mathbf{Ze}$ and change q_X into the quadratic form defined by $q(\mathbf{u}+m\mathbf{e}) = q_X(\mathbf{u}) - m^2$ for all $\mathbf{u}+m\mathbf{e}$ in $\mathsf{NS}(X) \oplus \mathbf{Ze}$. After such a change, Γ^* embeds into the orthogonal group $\mathsf{O}(\mathbf{Z}^r; q)$ for some even $r \in \{\rho, \rho+1\}$ and some integral quadratic form of signature $(1, r-1)$. It is proved by Bergeron, Haglund, and Wise that such a lattice acts properly on some CAT(0) cube complex (see Theorem 6.1 and the paragraph before Theorem 6.2 in [9]; see [8] for the case of uniform lattices). But if a group with Property (FW) acts by isometries on such a complex, it has a fixed point (see [14]). Thus, by properness of the action, the image Γ^* of Γ in $\mathsf{O}(\mathsf{NS}(X); q_X)$ is finite.

The kernel $K \subset \mathsf{Aut}(X)$ of the action on $\mathsf{NS}(X)$ contains $\mathsf{Aut}(X)^0$ as a finite index subgroup. Thus, if Γ has Property (FW), it contains a finite index subgroup that is contained in $\mathsf{Aut}(X)^0$ (see Theorem 2.8). □

7.3.2. *Non-rational surfaces*. — Here, the surface X is not rational. The following proposition classifies subgroups of $\mathsf{Bir}(X)$ with Property (FW); in particular, such a group is finite if the Kodaira dimension of X is non-negative (resp. if the characteristic of $\mathbf{k}$ is positive). Recall that $\overline{\mathbf{Z}} \subset \overline{\mathbf{Q}}$ is the ring of algebraic integers.

Proposition 7.5. — *Let X be a smooth, projective, and non-rational surface, over the algebraically closed field $\mathbf{k}$. Let Γ be an infinite subgroup of $\mathsf{Bir}(X)$ with Property (FW). Then $\mathbf{k}$ has characteristic 0, and there is a birational map $\varphi\colon X \dashrightarrow C \times \mathbb{P}^1_{\mathbf{k}}$ that conjugates Γ to a subgroup of $\mathsf{Aut}(C \times \mathbb{P}^1_{\mathbf{k}})$. Moreover, there is a finite index subgroup Γ_0 of Γ such that $\varphi \circ \Gamma_0 \circ \varphi^{-1}$, is a subgroup of $\mathsf{PGL}_2(\overline{\mathbf{Z}})$, acting on $C \times \mathbb{P}^1_{\mathbf{k}}$ by linear projective transformations on the second factor.*

Proof. — Assume, first, that the Kodaira dimension of X is non-negative. Let $\pi\colon X \to X_0$ be the projection of X on its (unique) minimal model (see [25, Th. V.5.8]). The group $\mathsf{Bir}(X_0)$ coincides with $\mathsf{Aut}(X_0)$; thus, after conjugacy by π, Γ becomes a subgroup of $\mathsf{Aut}(X_0)$, and Proposition 7.4 provides a finite index subgroup $\Gamma_0 \leqslant \Gamma$ that is contained in $\mathsf{Aut}(X_0)^0$. Note that Γ_0 inherits Property (FW) from Γ.

If the Kodaira dimension of X is equal to 2, the group $\mathsf{Aut}(X_0)^0$ is trivial; hence $\Gamma_0 = \{\mathrm{Id}_{X_0}\}$ and Γ is finite. If the Kodaira dimension is equal to 1, $\mathsf{Aut}(X_0)^0$ is either trivial, or isomorphic to an elliptic curve, acting by translations on the fibers of the Kodaira-Iitaka fibration of X_0 (this occurs, for instance, when X_0 is the product of an elliptic curve with a curve of higher genus). If the Kodaira dimension is 0, then $\mathsf{Aut}(X_0)^0$ is also an abelian group (it is an abelian variety of dimension $\leqslant 2$). Since abelian groups with Property (FW) are finite, the group Γ_0 is finite, and so is Γ.

We may now assume that the Kodaira dimension $\mathsf{kod}(X)$ is negative. Since X is not rational, then X is birationally equivalent to a product $S = C \times \mathbb{P}^1_{\mathbf{k}}$, where C is a curve of genus $\mathrm{g}(C) \geqslant 1$. Denote by $\mathbf{k}(C)$ the field of rational functions on the curve C. The semi-direct product $\mathsf{Aut}(C) \ltimes \mathsf{PGL}_2(\mathbf{k}(C))$ acts on S by birational transformations of the form

$$(x, y) \in C \times \mathbb{P}^1_{\mathbf{k}} \longmapsto \Big(f(x), \frac{a(x)y + b(x)}{c(x)y + d(x)}\Big);$$

here f is an automorphism of C, and a, b, c, and d are elements of $\mathbf{k}(C)$ such that $ad - bc$ is not identically 0. Moreover, $\mathsf{Bir}(S)$ coincides with this group $\mathsf{Aut}(C) \ltimes \mathsf{PGL}_2(\mathbf{k}(C))$ because the first projection $\pi\colon S \to C$ is equivariant under the action of $\mathsf{Bir}(S)$ (this follows from the fact that every rational map $\mathbb{P}^1_{\mathbf{k}} \to C$ is constant).

Since $\mathrm{g}(C) \geqslant 1$, $\mathsf{Aut}(C)$ is virtually abelian. Property (FW) implies that there is a finite index, normal subgroup $\Gamma_0 \leqslant \Gamma$ that is contained in $\mathsf{PGL}_2(\mathbf{k}(C))$. By Corollary 3.8, every subgroup of $\mathsf{PGL}_2(\mathbf{k}(C))$ with Property (FW) is conjugate to a subgroup of $\mathsf{PGL}_2(\overline{\mathbf{Z}})$ or a finite group if the characteristic of the field $\mathbf{k}$ is positive.

We may assume now that the characteristic of $\mathbf{k}$ is 0 and that $\Gamma_0 \subset \mathsf{PGL}_2(\overline{\mathbf{Z}})$ is infinite. Consider an element g of Γ; it acts as a birational transformation on the surface $S = C \times \mathbb{P}^1_{\mathbf{k}}$, and it normalizes Γ_0:

$$g \circ \Gamma_0 = \Gamma_0 \circ g.$$

Since Γ_0 acts by automorphisms on S, the finite set $\mathrm{Ind}(g)$ is Γ_0-invariant. But a subgroup of $\mathsf{PGL}_2(\mathbf{k})$ with Property (FW) preserving a non-empty, finite subset of $\mathbb{P}^1(\mathbf{k})$ is a finite group (by Lemma 3.5(2)). Thus, $\mathrm{Ind}(g)$ must be empty. This shows that Γ is contained in $\mathsf{Aut}(S)$. □

7.3.3. *Rational surfaces*. — Now, we assume that X is a smooth rational surface, that $\Gamma \leqslant \mathsf{Bir}(X)$ is an infinite subgroup with Property (FW), and that Γ contains a finite index, normal subgroup Γ_0 that is contained in $\mathsf{Aut}(X)^0$. Recall that a smooth surface Y is minimal if it does not contain any smooth rational curve of the first kind, i.e., with self-intersection -1. Every exceptional curve of the first kind in X is determined by its class in $\mathsf{NS}(X)$ and is therefore invariant under the action of $\mathsf{Aut}(X)^0$. The following lemma is obtained by contracting such (-1)-curves one by one.

Lemma 7.6. — *There is a birational morphism $\pi\colon X\to Y$ onto a minimal rational surface Y that is equivariant under the action of Γ_0; Y does not contain any exceptional curve of the first kind and Γ_0 becomes a subgroup of $\mathsf{Aut}(Y)^0$.*

Let us recall the classification of minimal rational surfaces and describe their groups of automorphisms. First, we have the projective plane $\mathbb{P}^2_{\mathbf{k}}$, with $\mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}})=\mathsf{PGL}_3(\mathbf{k})$ acting by linear projective transformations. Then comes the quadric $\mathbb{P}^1_{\mathbf{k}}\times\mathbb{P}^1_{\mathbf{k}}$, with $\mathsf{Aut}(\mathbb{P}^1_{\mathbf{k}}\times\mathbb{P}^1_{\mathbf{k}})^0=\mathsf{PGL}_2(\mathbf{k})\times\mathsf{PGL}_2(\mathbf{k})$ acting by linear projective transformations on each factor; the group of automorphisms of the quadric is the semi-direct product of $\mathsf{PGL}_2(\mathbf{k})\times\mathsf{PGL}_2(\mathbf{k})$ with the group of order 2 generated by the permutation of the two factors, $\eta(x,y)=(y,x)$. Then, for each integer $m\geqslant 1$, the Hirzebruch surface $\mathbb{F}_m$ is the projectivization of the rank 2 bundle $\mathscr{O}\oplus\mathscr{O}(m)$ over $\mathbb{P}^1_{\mathbf{k}}$; it may be characterized as the unique ruled surface $Z\to\mathbb{P}^1_{\mathbf{k}}$ with a section C of self-intersection $-m$. Its group of automorphisms is connected and preserves the ruling. This provides a homomorphism $\mathsf{Aut}(\mathbb{F}_m)\to\mathsf{PGL}_2(\mathbf{k})$ that describes the action on the base of the ruling, and it turns out that this homomorphism is surjective. If we choose coordinates for which the section C intersects each fiber at infinity, the kernel J_m of this homomorphism acts by transformations of type

$$(x,y)\longmapsto(x,\alpha y+\beta(x)),$$

where $\beta(x)$ is a polynomial function of degree $\leqslant m$. In particular, J_m is solvable. In other words, $\mathsf{Aut}(\mathbb{F}_m)$ is isomorphic to the group

$$(\mathsf{GL}_2(\mathbf{k})/\mu_m)\ltimes W_m,$$

where W_m is the linear representation of $\mathsf{GL}_2(\mathbf{k})$ on homogeneous polynomials of degree m in two variables, and μ_m is the kernel of this representation: it is the subgroup of $\mathsf{GL}_2(\mathbf{k})$ given by scalar multiplications by roots of unity of order dividing m.

Lemma 7.7. — *Given the above conjugacy $\pi\colon X\to Y$, the subgroup $\pi\circ\Gamma\circ\pi^{-1}$ of $\mathsf{Bir}(Y)$ is contained in $\mathsf{Aut}(Y)$.*

Proof. — Assume that the surface Y is the quadric $\mathbb{P}^1_{\mathbf{k}}\times\mathbb{P}^1_{\mathbf{k}}$. Then, according to Theorem 3.6, Γ_0 is conjugate to a subgroup of $\mathsf{PGL}_2(\overline{\mathbf{Z}})\times\mathsf{PGL}_2(\overline{\mathbf{Z}})$. If g is an element of Γ, its indeterminacy locus is a finite subset $\mathrm{Ind}(g)$ of $\mathbb{P}^1_{\mathbf{k}}\times\mathbb{P}^1_{\mathbf{k}}$ that is invariant under the action of Γ_0, because g normalizes Γ_0. Since Γ_0 is infinite and has Property (FW), this set $\mathrm{Ind}(g)$ is empty (Lemma 3.5). Thus, Γ is contained in $\mathsf{Aut}(\mathbb{P}^1_{\mathbf{k}}\times\mathbb{P}^1_{\mathbf{k}})$.

The same argument applies for Hirzebruch surfaces. Indeed, Γ_0 is an infinite subgroup of $\mathsf{Aut}(\mathbb{F}_m)$ with Property (FW). Thus, up to conjugacy, its projection in $\mathsf{PGL}_2(\mathbf{k})$ is contained in $\mathsf{PGL}_2(\overline{\mathbf{Z}})$. If it were finite, a finite index subgroup of Γ_0 would be contained in the solvable group J_m, and would therefore be finite too by Property (FW); this would contradict $|\Gamma_0|=\infty$. Thus, the projection of Γ_0 in $\mathsf{PGL}(\overline{\mathbf{Z}})$ is infinite. If g is an element of Γ, $\mathrm{Ind}(g)$ is a finite, Γ_0-invariant subset, and by looking at the projection of this set in $\mathbb{P}^1_{\mathbf{k}}$ one deduces that it is empty (Lemma 3.5). This proves that Γ is contained in $\mathsf{Aut}(\mathbb{F}_m)$.

Let us now assume that Y is the projective plane. Fix an element g of Γ, and assume that g is not an automorphism of $Y = \mathbb{P}^2$; the indeterminacy and exceptional sets of g are Γ_0 invariant. Consider an irreducible curve C in the exceptional set of g, together with an indeterminacy point q of g on C. Changing Γ_0 in a finite index subgroup, we may assume that Γ_0 fixes C and q; in particular, Γ_0 fixes q, and permutes the tangent lines of C through q. But the algebraic subgroup of $\mathsf{PGL}_3(\mathbf{k})$ preserving a point q and a line through q does not contain any infinite group with Property (FW) (Lemma 3.5). Thus, again, Γ is contained in $\mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}})$. □

7.3.4. *Conclusion, in Case* (1). — Putting everything together, we obtain the following particular case of Theorem 2.

Theorem 7.8. — *Let X be a smooth projective surface over an algebraically closed field $\mathbf{k}$. Let Γ be an infinite subgroup of $\mathsf{Bir}(X)$ with Property (FW). If a finite index subgroup of Γ is contained in $\mathsf{Aut}(X)$, there is a birational morphism $\varphi\colon X \to Y$ that conjugates Γ to a subgroup Γ_Y of $\mathsf{Aut}(Y)$, with Y in the following list:*

(1) *Y is the product of a curve C by $\mathbb{P}^1_{\mathbf{k}}$, the field $\mathbf{k}$ has characteristic 0, and a finite index subgroup Γ'_Y of Γ_Y is contained in $\mathsf{PGL}_2(\overline{\mathbf{Z}})$, acting by linear projective transformations on the second factor;*

(2) *Y is $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$, the field $\mathbf{k}$ has characteristic 0, and Γ_Y is contained in $\mathsf{PGL}_2(\overline{\mathbf{Z}}) \times \mathsf{PGL}_2(\overline{\mathbf{Z}})$;*

(3) *Y is a Hirzebruch surface $\mathbb{F}_m$ and $\mathbf{k}$ has characteristic 0;*

(4) *Y is the projective plane $\mathbb{P}^2_{\mathbf{k}}$.*

In particular, $Y = \mathbb{P}^2_{\mathbf{k}}$ if the characteristic of $\mathbf{k}$ is positive.

Remark 7.9. — Denote by $\varphi\colon X \to Y$ the birational morphism given by the theorem. Changing Γ in a finite index subgroup, we may assume that it acts by automorphisms on both X and Y.

If $Y = C \times \mathbb{P}^1$, then φ is in fact an isomorphism. To prove this fact, denote by ψ the inverse of φ. The indeterminacy set $\mathrm{Ind}(\psi)$ is Γ_Y invariant because both Γ and Γ_Y act by automorphisms. From Lemma 3.5, applied to $\Gamma'_Y \subset \mathsf{PGL}_2(\mathbf{k})$, we deduce that $\mathrm{Ind}(\psi)$ is empty and ψ is an isomorphism. The same argument implies that the conjugacy is an isomorphism if $Y = \mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ or a Hirzebruch surface $\mathbb{F}_m$, $m \geqslant 1$.

Now, if Y is $\mathbb{P}^2_{\mathbf{k}}$, φ is not always an isomorphism. For instance, $\mathsf{SL}_2(\mathbf{C})$ acts on $\mathbb{P}^2_{\mathbf{k}}$ with a fixed point, and one may blow up this point to get a new surface with an action of groups with Property (FW). But this is the only possible example, i.e., X is either $\mathbb{P}^2_{\mathbf{k}}$, or a single blow-up of $\mathbb{P}^2_{\mathbf{k}}$ (because $\Gamma \subset \mathsf{PGL}_3(\mathbf{C})$ can not preserve more than one base point for φ^{-1} without loosing Property (FW)).

7.4. Invariant fibrations. — We now assume that Γ has Property (FW) and acts by automorphisms on $\mathscr{U} \subset X$, and that the boundary $\partial X = X \smallsetminus \mathscr{U}$ is the union of $\ell \geqslant 1$ pairwise disjoint rational curves E_i; each of them has self-intersection $E_i^2 = 0$ and is contracted by at least one element of Γ. This corresponds to the fourth possibility in Gizatullin's Proposition 7.3. Since $E_i \cdot E_j = 0$, the Hodge index theorem implies that the classes $e_i = [E_i]$ span a unique line in $\mathsf{NS}(X)$, and that $[E_i]$ intersects non-negatively every curve.

From Section 7.3.2, we may, and do assume that X is a rational surface. In particular, the Euler characteristic of the structural sheaf is equal to 1: $\chi(\mathscr{O}_X) = 1$, and Riemann-Roch formula gives

$$h^0(X, E_1) - h^1(X, E_1) + h^2(X, E_1) = \frac{E_1^2 - K_X \cdot E_1}{2} + 1.$$

The genus formula implies $K_X \cdot E_1 = -2$, and Serre duality shows that $h^2(X, E_1) = h^0(X, K_X - E_1) = 0$ because otherwise $-2 = (K_X - E_1) \cdot E_1$ would be non-negative (because E_1 intersects non-negatively every curve). From this, we obtain

$$h^0(X, E_1) = h^1(X, E_1) + 2 \geqslant 2.$$

If F is a member of the complete linear system $|E_1|$, then $F \cdot E_1 = E_1 \cdot E_1 = 0$, and F is disjoint from the smooth irreducible curve E_1. Thus, $|E_1|$ is base point free, and $|E_1|$ determines a fibration $\pi\colon X \to B$ onto a curve B; in fact $B = \mathbb{P}^1_{\mathbf{k}}$ because X is a rational surface, and $H^0(X, E_1) = 2$ because $\mathscr{O}(E_1)$ is the pull back of an ample line bundle on B (see [30, Th. 2.1.27]). The curve E_1, as well as the E_i for $i \geqslant 2$, are fibers of π.

If f is an automorphism of $\mathscr{U}$ and $F \subset \mathscr{U}$ is a fiber of π, then $f(F)$ is a (complete) rational curve. Its projection $\pi(f(F))$ is contained in the affine curve $\mathbb{P}^1_{\mathbf{k}} \smallsetminus \bigcup_i \pi(E_i)$ and must therefore be reduced to a point. Thus, $f(F)$ is a fiber of π and f preserves the fibration. This proves the following lemma.

Lemma 7.10. — *There is a fibration $\pi\colon X \to \mathbb{P}^1_{\mathbf{k}}$ such that*

(1) *every component E_i of ∂X is a fiber of π, and $\mathscr{U} = \pi^{-1}(\mathscr{V})$ for an open subset $\mathscr{V} \subset \mathbb{P}^1_{\mathbf{k}}$;*

(2) *the general fiber of π is a smooth rational curve;*

(3) Γ *permutes the fibers of π: there is a morphism $\rho\colon \Gamma \to \mathsf{PGL}_2(\mathbf{k})$ such that $\pi \circ f = \rho(f) \circ \pi$ for every $f \in \Gamma$.*

The open subset $\mathscr{V} \subsetneq \mathbb{P}^1_{\mathbf{k}}$ is invariant under the action of $\rho(\Gamma)$; hence $\rho(\Gamma)$ is finite by Property (FW) and Lemma 3.5. Let Γ_0 be the kernel of this morphism. Let $\varphi\colon X \dashrightarrow \mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ be a birational map that conjugates the fibration π to the first projection $\tau\colon \mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}} \to \mathbb{P}^1_{\mathbf{k}}$. Then, Γ_0 is conjugate to a subgroup of $\mathsf{PGL}_2(\mathbf{k}(x))$ acting on $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ by linear projective transformations of the fibers of τ. From Corollary 3.8, a new conjugacy by an element of $\mathsf{PGL}_2(\mathbf{k}(x))$ changes Γ_0 in an infinite subgroup of $\mathsf{PGL}_2(\overline{\mathbf{Z}})$. Then, as in Sections 7.3.2 and 7.3.3 we conclude that Γ becomes a subgroup of $\mathsf{PGL}_2(\overline{\mathbf{Z}}) \times \mathsf{PGL}_2(\overline{\mathbf{Z}})$, with a finite projection on the first factor.

Proposition 7.11. — *Let Γ be an infinite group with Property (FW), with $\Gamma \subset \mathsf{Aut}(\mathscr{U})$, and $\mathscr{U} \subset Z$ as in case (4) of Proposition 7.3. There exists a birational map $\psi\colon Z \dashrightarrow \mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ that conjugates Γ to a subgroup of $\mathsf{PGL}_2(\overline{\mathbf{Z}}) \times \mathsf{PGL}_2(\overline{\mathbf{Z}})$, with a finite projection on the first factor.*

7.5. Completions by zigzags. — Two cases remain to be studied: ∂Z can be a chain of rational curves (a zigzag in Gizatullin's terminology) or a cycle of rational curves (a loop in Gizatullin's terminology). Cycles are considered in Section 8. In this section, we rely on difficult results of Danilov and Gizatullin to treat the case of chains of rational curves (i.e., case (3) in Proposition 7.3). Thus, in this section

(i) ∂X is a chain of smooth rational curves E_i

(ii) $\mathscr{U} = X \smallsetminus \partial X$ is an affine surface (singularities are allowed)

(iii) every irreducible component E_i is contracted to a point of ∂X by at least one element of $\Gamma \subset \mathsf{Aut}(\mathscr{U}) \subset \mathsf{Bir}(X)$.

In [21, 22], Danilov and Gizatullin introduce a set of "standard completions" of the affine surface $\mathscr{U}$. As in Section 7.2, a completion (or more precisely a "marked completion") is an embedding $\iota\colon \mathscr{U} \to Y$ into a complete surface such that $\partial Y = Y \smallsetminus \iota(\mathscr{U})$ is a curve (this boundary curve may be reducible). Danilov and Gizatullin only consider completions for which ∂Y is a chain of smooth rational curves and Y is smooth in a neighborhood of ∂Y; the surface X provides such a completion. Two completions $\iota\colon \mathscr{U} \to Y$ and $\iota'\colon \mathscr{U} \to Y'$ are isomorphic if the birational map $\iota' \circ \iota^{-1}\colon Y \to Y'$ is an isomorphism; in particular, the boundary curves are identified by this isomorphism. The group $\mathsf{Aut}(\mathscr{U})$ acts by pre-composition on the set of isomorphism classes of (marked) completions.

Among all possible completions, Danilov and Gizatullin distinguish a class of "standard (marked) completions", for which we refer to [21] for a definition. There are elementary links (corresponding to certain birational mappings $Y \dashrightarrow Y'$) between standard completions, and one can construct a graph $\Delta_{\mathscr{U}}$ whose vertices are standard completions; there is an edge between two completions if one can pass from one to the other by an elementary link.

Example 7.12. — A completion is m-standard, for some $m \in \mathbf{Z}$, if the boundary curve ∂Y is a chain of $n+1$ consecutive rational curves E_0, E_1, $\ldots$, E_n ($n \geqslant 1$) such that

$$E_0^2 = 0, \quad E_1^2 = -m, \quad \text{and} \quad E_i^2 = -2 \ \text{ if } \ i \geqslant 2.$$

Blowing-up the intersection point $q = E_0 \cap E_1$, one creates a new chain starting by E_0' with $(E_0')^2 = -1$; blowing down E_0', one creates a new $(m+1)$-standard completion. This is one of the elementary links.

Standard completions are defined by constraints on the self-intersections of the components E_i. Thus, the action of $\mathsf{Aut}(\mathscr{U})$ on completions permutes the standard completions; this action determines a homomorphism from $\mathsf{Aut}(\mathscr{U})$ to the group of isometries (or automorphisms) of the graph $\Delta_{\mathscr{U}}$ (see [21]):

$$\mathsf{Aut}(\mathscr{U}) \longrightarrow \mathsf{Iso}(\Delta_{\mathscr{U}}).$$

Theorem 7.13 (Danilov and Gizatullin, [21, 22]). — *The graph $\Delta_{\mathscr{U}}$ of all isomorphism classes of standard completions of $\mathscr{U}$ is a tree. The group $\mathsf{Aut}(\mathscr{U})$ acts by isometries of this tree. The stabilizer of a vertex $\iota\colon \mathscr{U} \to Y$ is the subgroup $G(\iota)$ of automorphisms* *of the complete surface Y that fix the curve ∂Y. This group is an algebraic subgroup of* $\mathsf{Aut}(Y)$.

The last property means that $G(\iota)$ is an algebraic group that acts algebraically on Y. It coincides with the subgroup of $\mathsf{Aut}(Y)$ fixing the boundary ∂Y; the fact that it is algebraic follows from the existence of a $G(\iota)$-invariant, big and nef divisor which is supported on ∂Y (see the last sentence of Proposition 7.3). The crucial assertion in this theorem is that $\Delta_{\mathscr{U}}$ is a simplicial tree (typically, infinitely many edges emanate from each vertex). There are sufficiently many links to assure connectedness, but not too many in order to prevent the existence of cycles in the graph $\Delta_{\mathscr{U}}$.

Corollary 7.14. — *If Γ is a subgroup of $\mathsf{Aut}(\mathscr{U})$ that has the fixed point property on trees, then Γ is contained in $G(\iota) \subset \mathsf{Aut}(Y)$ for some completion $\iota\colon \mathscr{U} \to Y$.*

If Γ has Property (FW), it has Property (FA) (see Section 3.4). Thus, if it acts by automorphisms on $\mathscr{U}$, Γ is conjugate to the subgroup $G(\iota)$ of $\mathsf{Aut}(Y)$, for some zigzag-completion $\iota\colon \mathscr{U} \to Y$. Theorem 7.8 of Section 7.3.3 implies that the action of Γ on the initial surface X is conjugate to a regular action on $\mathbb{P}^2_{\mathbf{k}}$, $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ or $\mathbb{F}_m$, for some Hirzebruch surface $\mathbb{F}_m$. This action preserves a curve, namely the image of the zigzag into the surface Y. The following examples list all possibilities, and conclude the proof of Theorem 2 in the case of zigzags (i.e., case (3) in Proposition 7.3).

Example 7.15. — Consider the projective plane $\mathbb{P}^2_{\mathbf{k}}$, together with an infinite subgroup $\Gamma \subset \mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}})$ that preserves a curve C and has Property (FW). Then, C must be a smooth rational curve: either a line, or a smooth conic. Indeed, if the genus of C is positive, or if C is rational but is not smooth, then the action of Γ on C factors through a finite quotient of Γ (see Lemma 3.5); but then the image of Γ in $\mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}})$ would be virtually solvable, hence finite by Property (FW). Now, if C is the line "at infinity", then Γ acts by affine transformations on the affine plane $\mathbb{P}^2_{\mathbf{k}} \setminus C$. If C is the conic $x^2 + y^2 + z^2 = 0$, Γ becomes a subgroup of $\mathsf{PO}_3(\mathbf{k})$.

Example 7.16. — When Γ is a subgroup of $\mathsf{Aut}(\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}})$ that preserves a curve C and has Property (FW), then C must be a smooth curve because Γ has no finite orbit (Lemma 3.5). Similarly, the two projections $C \to \mathbb{P}^1_{\mathbf{k}}$ being equivariant with respect to the morphisms $\Gamma \to \mathsf{PGL}_2(\mathbf{k})$, they have no ramification points. Thus, C is a smooth rational curve, and its projections onto each factor are isomorphisms. In particular, the action of Γ on C and on each factor are conjugate. These conjugacies show that Γ is conjugate to a diagonal embedding

$$\gamma \in \Gamma \longmapsto (\rho(\gamma), \rho(\gamma)) \in \mathsf{PGL}_2(\mathbf{k}) \times \mathsf{PGL}_2(\mathbf{k}).$$

Example 7.17. — Similarly, the group $\mathsf{SL}_2(\mathbf{k})$ acts on the Hirzebruch surface $\mathbb{F}_m$, preserving the zero section of the fibration $\pi\colon \mathbb{F}_m \to \mathbb{P}^1_{\mathbf{k}}$. This gives examples of groups with Property (FW) acting on $\mathbb{F}_m$ and preserving a big and nef curve C.

Starting with one of the above examples, one can blow-up points on the invariant curve C, and then contract C, to get examples of zigzag completions Y on which Γ acts and contracts the boundary ∂Y.

8. Birational transformations of surfaces II

In this section, $\mathscr{U}$ is a (normal, singular) affine surface with a completion X by a cycle of ℓ rational curves. Every irreducible component E_i of the boundary $\partial X = X \setminus \mathscr{U}$ is contracted by at least one automorphism of $\mathscr{U}$. Our goal is to classify subgroups Γ of $\mathsf{Aut}(\mathscr{U}) \subset \mathsf{Bir}(X)$ that are infinite and have Property (FW): in fact, we shall show that no such group exists. This ends the proof of Theorem 2 since the other possibilities of Proposition 7.3 have been dealt with in the previous section.

Remark 8.1. — The proof is based on the fact that $\mathsf{Aut}(\mathscr{U})$ acts in a piecewise $\mathsf{PGL}(2, \mathbf{Z})$ way on a circle whose rational points correspond to divisors at infinity in various compactifications of $\mathscr{U}$. To describe this action, our presentation is similar to the one in [26]. Another equivalent, more precise, but slightly longer route is to consider the set of valuations on the ring of regular functions on $\mathscr{U}$ which are centered on ∂X. The circle we are looking for corresponds to a certain set of valuations with log-discrepancy 0; this approach is described in a particular case in [15]; to study the log-discrepancy in our context, one could refer to [17] (in order to construct a regular 2-form on $\mathscr{U}$ with poles exactly along ∂X after compactification). Also, we use both Farey and dyadic partitions of the circle because the Farey viewpoint is used by algebraic geometers, while dyadic partitions are often used in group theory (see [38, §1.5]); these are just two equivalent viewpoints.

Example 8.2. — Let $(\mathbb{A}^1_{\mathbf{k}})^*$ denote the complement of the origin in the affine line $\mathbb{A}^1_{\mathbf{k}}$; it is isomorphic to the multiplicative group $\mathbb{G}_m$ over $\mathbf{k}$. The surface $(\mathbb{A}^1_{\mathbf{k}})^* \times (\mathbb{A}^1_{\mathbf{k}})^*$ is an open subset in $\mathbb{P}^2_{\mathbf{k}}$ whose boundary is the triangle of coordinate lines $\{[x : y : z] : \ xyz = 0\}$. Thus, the boundary is a cycle of length $\ell = 3$. The group of automorphisms of $(\mathbb{A}^1_{\mathbf{k}})^* \times (\mathbb{A}^1_{\mathbf{k}})^*$ is the semi-direct product $\mathsf{GL}_2(\mathbf{Z}) \ltimes (\mathbb{G}_m(\mathbf{k}) \times \mathbb{G}_m(\mathbf{k}))$; it does not contain any infinite group with Property (FW).

8.1. Resolution of indeterminacies.

— Let us order cyclically the irreducible components E_i of ∂X, so that $E_i \cap E_j \neq \varnothing$ if and only if $i - j = \pm 1 (\mathrm{mod}\, \ell)$. Blowing up finitely many singularities of ∂X, we may assume that $\ell = 2^m$ for some integer $m \geqslant 1$; in particular, every curve E_i is smooth. (With such a modification, one may a priori create irreducible components of ∂X that are not contracted by the group Γ.)

Lemma 8.3. — *Let f be an automorphism of $\mathscr{U}$ and let f_X be the birational extension of f to the surface X. Then*

(1) *every indeterminacy point of f_X is a singular point of ∂X, i.e., one of the intersection points $E_i \cap E_{i+1}$;*

(2) *indeterminacies of f_X are resolved by inserting chains of rational curves.*

Property (2) means that there exists a resolution of the indeterminacies of f_X, given by two birational morphisms $\varepsilon\colon Y\to X$ and $\pi\colon Y\to X$ with $f\circ\varepsilon=\pi$, such that $\pi^{-1}(\partial X)=\varepsilon^{-1}(\partial X)$ is a cycle of rational curves. Some of the singularities of ∂X have been blown-up into chains of rational curves to construct Y.

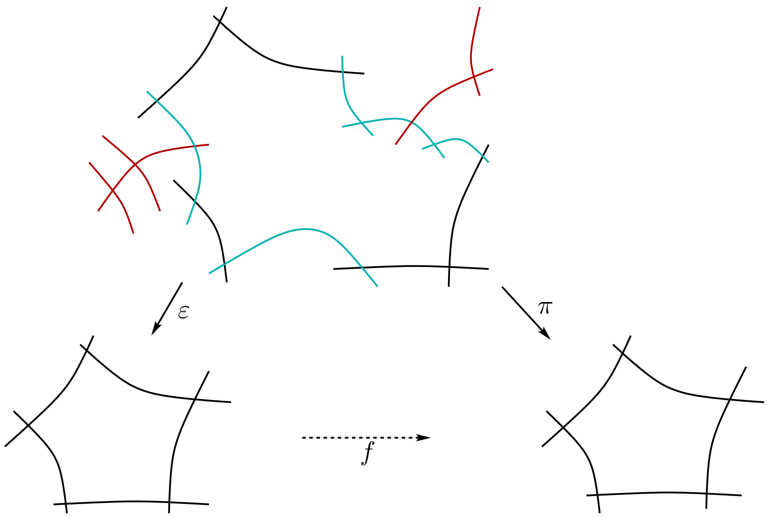

Figure 8.1. A blow-up sequence creating two (red) branches. No branch of this type appears for minimal resolution.

Proof. — Consider a minimal resolution of the indeterminacies of f_X. It is given by a finite sequence of blow-ups of the base points of f_X, producing a surface Y and two birational morphisms $\varepsilon\colon Y\to X$ and $\pi\colon Y\to X$ such that $f_X=\pi\circ\varepsilon^{-1}$. Since the indeterminacy points of f_X are contained in ∂X, all necessary blow-ups are centered on ∂X.

The total transform $F=\varepsilon^*(\partial X)$ is a union of rational curves: it is made of a cycle, together with branches emanating from it. One of the assertions (1) and (2) fails if and only if F is not a cycle; in that case, there is at least one branch.

Each branch is a tree of smooth rational curves, which may be blown-down onto a smooth point; indeed, these branches come from smooth points of the main cycle that have been blown-up finitely many times. Thus, there is a birational morphism $\eta\colon Y\to Y_0$ onto a smooth surface Y_0 that contracts the branches (and nothing more).

The morphism π maps F onto the cycle ∂X, so that all branches of F are contracted by π. Thus, both ε and π induce (regular) birational morphisms $\varepsilon_0\colon Y_0\to X$ and $\pi_0\colon Y_0\to X$. This contradicts the minimality of the resolution. □

Let us introduce a family of surfaces

$$\pi_k\colon X_k\longrightarrow X.$$

First, $X_1 = X$ and π_1 is the identity map. Then, X_2 is obtained by blowing-up the ℓ singularities of ∂X_1; X_2 is a compactification of $\mathscr{U}$ by a cycle ∂X_2 of $2\ell = 2^{m+1}$ smooth rational curves. Then, X_3 is obtained by blowing up the singularities of ∂X_2, and so on. In particular, ∂X_k is a cycle of $2^{k-1}\ell = 2^{m+k-1}$ curves.

Denote by $\mathscr{D}_k$ the *dual graph* of ∂X_k: vertices of $\mathscr{D}_k$ correspond to irreducible components E_i of ∂X_k and edges to intersection points $E_i \cap E_j$. A simple blow-up (of a singular point) modifies both ∂X_k and $\mathscr{D}_k$ locally as follows

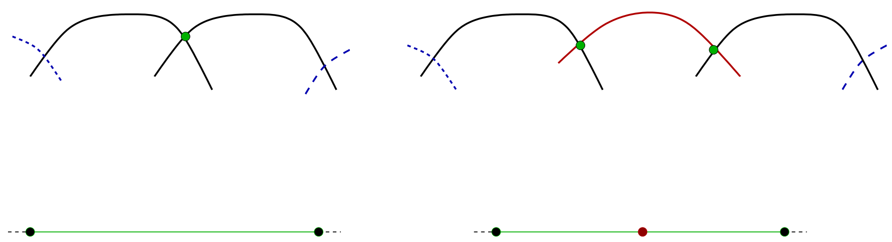

Figure 8.2. Blowing-up one point.

The group $\mathsf{Aut}(\mathscr{U})$ acts on $\widetilde{\mathsf{Hyp}}(X)$ and Lemma 8.3 shows that its action stabilizes the subset $\mathscr{B}$ of $\widetilde{\mathsf{Hyp}}(X)$ defined as

$$\mathscr{B} = \Big\{ C \in \widetilde{\mathsf{Hyp}}(X) :\ \exists\, k \geqslant 1, C \text{ is an irreducible component of } \partial X_k \Big\}.$$

In what follows, we shall parametrize $\mathscr{B}$ in two distinct ways by rational numbers.

8.2. Farey and dyadic parametrizations. — Consider an edge of the graph $\mathscr{D}_1$, and identify this edge with the unit interval $[0,1]$. Its endpoints correspond to two adjacent components E_i and E_{i+1} of ∂X_1, and the edge corresponds to their intersection q. Blowing-up q creates a new vertex (see Figure 8.2). The edge is replaced by two adjacent edges of $\mathscr{D}_2$ with a common vertex corresponding to the exceptional divisor and the other vertices corresponding to (the strict transforms of) E_i and E_{i+1}; we may identify this part of $\mathscr{D}_2$ with the segment $[0,1]$, the three vertices with $\{0, 1/2, 1\}$, and the two edges with $[0,1/2]$ and $[1/2,1]$.

Subsequent blow-ups may be organized in two different ways by using either a dyadic or a Farey algorithm (see Figure 8.3).

In the dyadic algorithm, the vertices are labeled by dyadic numbers $m/2^k$. The vertices of $\mathscr{D}_{k+1}$ coming from an initial edge $[0,1]$ of $\mathscr{D}_1$ are the 2^k+1 points $\{n/2^k :\ 0 \leqslant n \leqslant 2^k\}$ of the segment $[0,1]$. We denote by $\mathrm{Dyad}(k)$ the set of dyadic numbers $n/2^k \in [0,1]$; thus, $\mathrm{Dyad}(k) \subset \mathrm{Dyad}(k+1)$. We say that an interval $[a,b]$ is a *standard dyadic* interval if a and b are two consecutive numbers in $\mathrm{Dyad}(k)$ for some k.

In the Farey algorithm, the vertices correspond to rational numbers p/q. Adjacent vertices of $\mathscr{D}_k$ coming from the initial segment $[0,1]$ correspond to pairs of rational numbers $(p/q, r/s)$ with $ps - qr = \pm 1$; two adjacent vertices of $\mathscr{D}_k$ give birth to a new, middle vertex in $\mathscr{D}_{k+1}$: this middle vertex is $(p+r)/(q+s)$ (in the dyadic algorithm, the middle vertex is the "usual" euclidean middle). We shall say that an interval $[a,b]$ is a *standard Farey* interval if $a=p/q$ and $b=r/s$ with $ps-qr=-1$. We denote by $\mathrm{Far}(k)$ the finite set of rational numbers $p/q\in[0,1]$ that is given by the k-th step of Farey algorithm; thus, $\mathrm{Far}(0)=\{0,1\}$ and $\mathrm{Far}(k)$ is a set of 2^k+1 rational numbers p/q with $0\leqslant p\leqslant q$. (One can check that $1\leqslant q\leqslant \mathrm{Fib}(k+2)$, with $\mathrm{Fib}(k)$ the k-th term in the Fibonacci sequence $\mathrm{Fib}(0)=0$, $\mathrm{Fib}(1)=1$, $\mathrm{Fib}(m+1)=\mathrm{Fib}(m)+\mathrm{Fib}(m-1)$.)

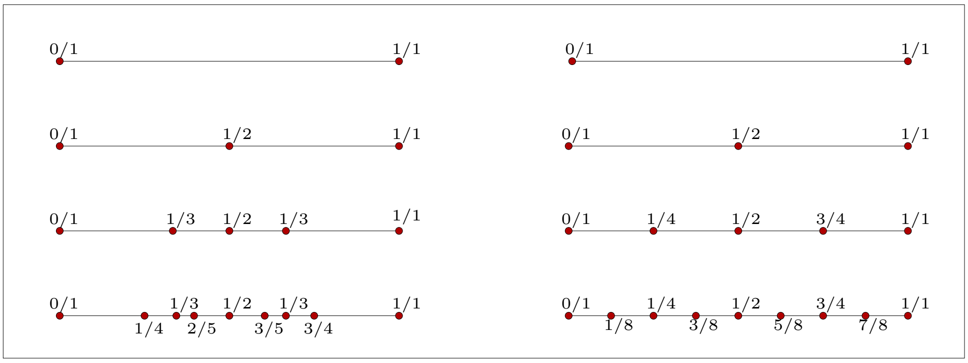

Figure 8.3. On the left, the Farey algorithm. On the right, the dyadic one. Here $k=0$ (top), to $k=3$ (bottom).

By construction, the graph $\mathscr{D}_1$ has $\ell=2^m$ edges. The edges of $\mathscr{D}_1$ are in one-to-one correspondence with the singularities q_j of ∂X_1. Each edge determines a subset $\mathscr{B}_j$ of $\mathscr{B}$; the elements of $\mathscr{B}_j$ are the curves $C\subset\partial X_k$ $(k\geqslant 1)$ such that $\pi_k(C)$ contains the singularity q_j determined by the edge. Using the dyadic algorithm (resp. Farey algorithm), the elements of $\mathscr{B}_j$ are in one-to-one correspondence with dyadic (resp. rational) numbers in $[0,1]$. Gluing these segments cyclically one gets a circle $\mathbb{S}^1$, together with a nested sequence of subdivisions in ℓ, 2ℓ, …, $2^{k-1}\ell$, … intervals; each interval is a standard dyadic (or Farey) interval of one of the initial edges.

Since there are $\ell=2^m$ initial edges, we may identify the graph $\mathscr{D}_1$ with the circle $\mathbb{S}^1=\mathbf{R}/\mathbf{Z}=[0,1]/_{0\simeq 1}$ and the initial vertices with the dyadic numbers in $\mathrm{Dyad}(m)$ modulo 1 (resp. the elements of $\mathrm{Far}(m)$ modulo 1). The vertices of $\mathscr{D}_k$ are in one to one correspondence with the dyadic numbers in $\mathrm{Dyad}(k+m-1)$.

Remark 8.4

(a) By construction, the interval $[p/q,r/s]\subset[0,1]$ is a standard Farey interval if and only if $ps-qr=-1$, iff it is delimited by two adjacent elements of $\mathrm{Far}(m)$ for some m.

(b) If $h\colon[x,y]\to[x',y']$ is a homeomorphism between two standard Farey intervals mapping rational numbers to rational numbers and standard Farey intervals to standard Farey intervals, then h is the restriction to $[x,y]$ of a unique linear projective transformation with integer coefficients:

$$h(t) = \frac{at+b}{ct+d}, \quad \text{for some element } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ of } \mathsf{PGL}_2(\mathbf{Z}).$$

(c) Similarly, if h is a homeomorphism mapping standard dyadic intervals to intervals of the same type, then h is the restriction of an affine dyadic map

$$h(t) = 2^m t + \frac{u}{2^n}, \quad \text{with } m, n \in \mathbf{Z}.$$

In what follows, we denote by $\mathrm{G}_{\mathrm{Far}}$ the group of self-homeomorphisms of $\mathbb{S}^1 = \mathbf{R}/\mathbf{Z}$ that are piecewise $\mathsf{PGL}_2(\mathbf{Z})$ mapping with respect to a finite decomposition of the circle in standard Farey intervals $[p/q, r/s]$. In other words, if f is an element of $\mathrm{G}_{\mathrm{Far}}$, there are two partitions of the circle into consecutive intervals I_i and J_i such that the I_i are intervals with rational endpoints, h maps I_i to J_i, and the restriction $f\colon I_i \to J_i$ is the restriction of an element of $\mathsf{PGL}_2(\mathbf{Z})$ (see [38, §1.5.1]).

Theorem 8.5. — *Let $\mathscr{U}$ be an affine surface with a compactification $\mathscr{U} \subset X$ such that $\partial X := X \smallsetminus \mathscr{U}$ is a cycle of smooth rational curves. In the Farey parametrization of the set $\mathscr{B} \subset \widetilde{\mathsf{Hyp}}(X)$ of boundary curves, the group $\mathsf{Aut}(\mathscr{U})$ acts on $\mathscr{B}$ as a subgroup of $\mathrm{G}_{\mathrm{Far}}$.*

Remark 8.6. — There is a unique orientation preserving self-homeomorphism of the circle that maps $\mathrm{Dyad}(k)$ to $\mathrm{Far}(k)$ for every k. This self-homeomorphism conjugates $\mathrm{G}_{\mathrm{Far}}$ to the group $\mathrm{G}_{\mathrm{Dya}}$ of self-homeomorphisms of the circle that are piecewise affine with respect to a dyadic decomposition of the circle, with slopes in $\pm 2^{\mathbf{Z}}$, and with translation parts in $\mathbf{Z}[1/2]$. Using the parametrization of $\mathscr{B}$ by dyadic numbers, the image of $\mathsf{Aut}(\mathscr{U})$ becomes a subgroup of $\mathrm{G}_{\mathrm{Dya}}$.

Proof. — Lemma 8.3 is the main ingredient. Consider the action of the group $\mathsf{Aut}(\mathscr{U})$ on the set $\mathscr{B}$. Let f be an element of $\mathsf{Aut}(\mathscr{U}) \subset \mathsf{Bir}(X)$. Consider an irreducible curve $E \in \mathscr{B}$, and denote by F its image: $F = f_\bullet(E)$ is an element of $\mathscr{B}$ by Lemma 8.3. There are integers k and l such that $E \subset \partial X_k$ and $F \subset \partial X_l$. Replacing X_k by a higher blow-up $X_m \to X$, we may assume that $f_{lm} := \pi_l^{-1} \circ f \circ \pi_m$ is regular on a neighborhood of the curve E (Lemma 8.3). Let q_k be one of the two singularities of ∂X_m that are contained in E, and let E' be the second irreducible component of ∂X_m containing q. If E' is blown down by f_{lm}, its image is one of the two singularities of ∂X_l contained in F (by Lemma 8.3). Consider the smallest integer $n \geqslant l$ such that ∂X_n contains the strict transform $F' = f_\bullet(E')$; in X_n, the curve F' is adjacent to the strict transform of F (still denoted F), and f is a local isomorphism from a neighborhood of q in X_m to a neighborhood of $q' := F \cap F'$ in X_n.

Now, if one blows-up q, the exceptional divisor D is mapped by $f_\bullet$ to the exceptional divisor D' obtained by a simple blow-up of q: f lifts to a local isomorphism from a neighborhood of D to a neighborhood of D', the action from D to D' being given by the differential df_q. The curve D contains two singularities of ∂X_{m+1}, which can be blown-up too: again, f lifts to a local isomorphism if one blow-ups the singularities of $\partial X_{n+1} \cap D'$. We can repeat this process indefinitely. Let us now phrase this remark differently. The point q determines an edge of $\mathscr{D}_m$, hence a standard Farey interval $I(q)$. The point q' determines an edge of $\mathscr{D}_n$, hence another standard Farey interval $I(q')$. Then, the points of $\mathscr{B}$ that are parametrized by rational numbers in $I(q)$ are mapped by $f_\bullet$ to rational numbers in $I(q')$ and this map respects the Farey order: if we identify $I(q)$ and $I(q')$ to $[0,1]$, $f_\bullet$ is the restriction of a monotone map that sends $\mathrm{Far}(k)$ to $\mathrm{Far}(k)$ for every k. Thus, on $I(q)$, $f_\bullet$ is the restriction of a linear projective transformation with integer coefficients (see Remark 8.4-(b)). This shows that $f_\bullet$ is an element of $\mathrm{G}_{\mathrm{Far}}$. ☐

8.3. Conclusion. — Consider the group ${\mathrm{G}_{\mathrm{Dya}}}^*$ of self-homeomorphisms of the circle $\mathbb{S}^1 = \mathbf{R}/\mathbf{Z}$ that are piecewise affine with respect to a finite partition of $\mathbf{R}/\mathbf{Z}$ into dyadic intervals $[x_i, x_{i+1}[$ with x_i in $\mathbf{Z}[1/2]/\mathbf{Z}$ for every i, and satisfy $h(t) = 2^{m_i}t + a_i$ with $m_i \in \mathbf{Z}$ and $a_i \in \mathbf{Z}[1/2]$ for every i. This group is known as the Thompson group of the circle, and is isomorphic to the group ${\mathrm{G}_{\mathrm{Far}}}^*$ of orientation-preserving self-homeomorphisms in $\mathrm{G}_{\mathrm{Far}}$ (defined in Section 8.2).

Theorem 8.7 (Farley, Hughes [18, 27]). — *Every subgroup of the Thompson group* ${\mathrm{G}_{\mathrm{Dya}}}^*$ *(and hence of* ${\mathrm{G}_{\mathrm{Far}}}^*$*) with Property (FW) is a finite cyclic group.*

Indeed fixing a gap in an earlier construction of Farley [18][(1)], Hughes proved [27] that $\mathrm{G}_{\mathrm{Far}}$ has Property PW, in the sense that it admits a commensurating action whose associated length function is a proper map (see also Navas' book [38]). This implies the conclusion, because every finite group of orientation-preserving self-homeomorphisms of the circle is cyclic.

Thus, if Γ is a subgroup of $\mathsf{Aut}(\mathscr{U})$ with Property (FW), it contains a finite index subgroup Γ_0 that acts trivially on the set $\mathscr{B} \subset \widetilde{\mathsf{Hyp}}(X)$. This means that Γ_0 extends as a group of automorphisms of X fixing the boundary ∂X. Since ∂X supports a big and nef divisor, Γ_0 contains a finite index subgroup Γ_1 that is contained in $\mathsf{Aut}(X)^0$.

Note that Γ_1 has Property (FW) because it is a finite index subgroup of Γ. It preserves every irreducible component of the boundary curve ∂X, as well as its singularities. As such, it must act trivially on ∂X. When we apply Theorem 7.8 to Γ_1, the conjugacy $\varphi\colon X \to Y$ can not contract ∂X, because the boundary supports an ample divisor. Thus, Γ_1 is conjugate to a subgroup of $\mathsf{Aut}(Y)$ that fixes a curve pointwise. This is not possible if Γ_1 is infinite (see Theorem 7.8 and the remarks following it).

We conclude that Γ is finite in case (2) of Proposition 7.3.

9. Birational actions of $\mathsf{SL}_2(\mathbf{Z}[\sqrt{d}])$

We develop here Example 1.5. If $\mathbf{k}$ is an algebraically closed field of characteristic 0, therefore containing $\overline{\mathbf{Q}}$, we denote by σ_1 and σ_2 the distinct embeddings of $\mathbf{Q}(\sqrt{d})$ into $\mathbf{k}$. Let j_1 and j_2 be the resulting embeddings of $\mathsf{SL}_2(\mathbf{Z}[\sqrt{d}])$ into $\mathsf{SL}_2(\mathbf{k})$, and $j = j_1 \times j_2$ the compound embedding into $\mathsf{G} = \mathsf{SL}_2(\mathbf{k}) \times \mathsf{SL}_2(\mathbf{k})$.

(1) The gap in Farley's argument lies in Prop. 2.3 and Th. 2.4 of [18].

Theorem 9.1. — *Let Γ be a finite index subgroup of $\mathsf{SL}_2(\mathbf{Z}[\sqrt{d}])$. Let X be an irreducible projective surface over an algebraically closed field $\mathbf{k}$. Let $\alpha:\Gamma\to\mathsf{Bir}(X)$ be a homomorphism with infinite image. Then $\mathbf{k}$ has characteristic zero, and there exist a finite index subgroup Γ_0 of Γ and a birational map $\varphi:Y\dashrightarrow X$ such that*

(1) *Y is the projective plane $\mathbb{P}^2$, a Hirzebruch surface $\mathbb{F}_m$, or $C\times\mathbb{P}^1$ for some curve C;*

(2) *$\varphi^{-1}\alpha(\Gamma)\varphi\subset\mathsf{Aut}(Y)$;*

(3) *there is a unique algebraic homomorphism $\beta:\mathsf{G}\to\mathsf{Aut}(Y)$ such that $\beta(j(\gamma))=\varphi^{-1}\alpha(\gamma)\varphi$ for every $\gamma\in\Gamma_0$.*

To prove this result, assume first that $\mathbf{k}$ has positive characteristic. Theorem 2 ensures that Y is the projective plane, and the Γ-action is given by a homomorphism into $\mathsf{PGL}_3(\mathbf{k})$. Then remark that every homomorphism $\tau:\Gamma\to\mathsf{GL}_n(\mathbf{k})$ has finite image; indeed, it is well-known that $\mathsf{GL}_n(\mathbf{k})$ has no infinite order distorted element: elements of infinite order have some transcendental eigenvalue and the conclusion easily follows. Since Γ has an exponentially distorted cyclic subgroup, the kernel of τ is infinite, and by the Margulis normal subgroup theorem the image of τ is finite.

Now, assume that the characteristic of $\mathbf{k}$ is 0. From Theorem 2, Assertions (1) and (2) are satisfied. If Y is $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$, or a Hirzebruch surface $\mathbb{F}_m$, then $\mathsf{Aut}(Y)$ is a linear algebraic group. If Y is a product $C\times\mathbb{P}^1$, with $g(C)\geqslant 1$, the projection onto C gives a Γ-equivariant morphism; since $g(C)\geqslant 1$, the automorphism group of C is virtually abelian, and a finite index subgroup Γ_1 of Γ acts trivially on C. Thus, the action of Γ_1 on Y preserves the projection onto $\mathbb{P}^1$ and acts via an embedding into the linear algebraic group $\mathsf{Aut}(\mathbb{P}^1)=\mathsf{PGL}_2(\mathbf{k})$. Then, the proof of Theorem 9.1 follows from the next lemma.

Lemma 9.2. — *Let $\mathbf{k}$ be a field containing $\mathbf{Q}(\sqrt{d})$. Consider the compound embedding j of $\mathsf{SL}_2(\mathbf{Z}[\sqrt{d}])$ into $G=\mathsf{SL}_2(\mathbf{k})\times\mathsf{SL}_2(\mathbf{k})$. For every linear algebraic group H and homomorphism $f:\mathsf{SL}_2(\mathbf{Z}[\sqrt{d}])\to H(\mathbf{k})$, there exists a unique homomorphism $\widetilde{f}:G\to H$ of $\mathbf{k}$-algebraic groups such that the homomorphisms f and $\widetilde{f}\circ j$ coincide on some finite index subgroup of Γ.*

Sketch of proof. — The uniqueness is a consequence of the Zariski density of the image of j. Let us prove the existence. The Zariski density allows to reduce to the case when $H=\mathsf{SL}_n$. In the case $\mathbf{k}=\mathbf{R}$, one first remarks that the image of $\mathsf{SL}_2(\mathbf{Z}[\sqrt{d}])$ in $\mathsf{SL}_n(\mathbf{R})$ is not contained in a compact group because $\mathsf{SL}_2(\mathbf{Z}[\sqrt{d}])$ contains exponentially distorted elements. Then, Margulis' superrigidity and the fact that every continuous real representation of $\mathsf{SL}_2(\mathbf{R})$ is algebraic prove the lemma. The case of fields containing $\mathbf{R}$ immediately follows, and in turn it follows for subfields of overfields of $\mathbf{R}$ (as soon as they contain $\mathbf{Q}(\sqrt{d})$). □

10. Open problems

Question 10.1. — Let Γ be a group with Property (FW). Is every birational action of Γ regularizable ? Here regularizable is defined in the same way as pseudo-regularizable, but assuming that the action on $\mathscr{U}$ is by automorphisms (instead of pseudo-automorphisms).

A particular case is given by Calabi-Yau varieties (simply connected complex projective manifolds X with trivial canonical bundle and $h^{d,0}(X) = 0$ for all d such that $0 < d < \dim(X)$). For such a variety, $\mathsf{Bir}(X)$ coincides with $\mathsf{Psaut}(X)$. One can then ask (1) whether every subgroup Γ of $\mathsf{Psaut}(X)$ with property (FW) is regularizable on some birational model Y of X (without restricting the action to a dense open subset), and (2) what are the possibilities for such a group Γ.

Question 10.2. — For which irreducible projective varieties X

(1) $\mathsf{Bir}(X)$ does not transfix $\mathsf{Hyp}(X)$?
(2) some finitely generated subgroup of $\mathsf{Bir}(X)$ does not transfix $\mathsf{Hyp}(X)$?
(3) some cyclic subgroup of $\mathsf{Bir}(X)$ does not transfix $\mathsf{Hyp}(X)$.

We have the implications: X is ruled $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). In dimension 2, we have: ruled $\Leftrightarrow$ (1) $\not\Rightarrow$(2) $\Leftrightarrow$ (3) (see Section 6.1). It would be interesting to find counterexamples to these equivalences in higher dimension, and settle each of the problems raised in Question 10.2 in dimension 3.

The group of affine transformations of $\mathbb{A}^3_{\mathbf{C}}$ contains $\mathsf{SL}_3(\mathbf{C})$, and this group contains many subgroups with Property (FW). For surfaces, Theorem 2 shows that groups of birational transformations with Property (FW) are contained in algebraic groups, up to conjugacy. The following question asks whether this type of theorem may hold for $\mathsf{Aut}(\mathbb{A}^3_{\mathbf{C}})$.

Question 10.3. — Does there exist an infinite subgroup of $\mathsf{Aut}(\mathbb{A}^3_{\mathbf{C}})$ with Property (FW) that is not conjugate to a group of affine transformations of $\mathbb{A}^3_{\mathbf{C}}$?

Recall that a length function ℓ on a group G is a function $\ell\colon G \to \mathbf{R}_+$ such that $\ell(g) = 0$ if and only if g is the neutral element, $\ell(g) = \ell(g^{-1})$, and $\ell(gh) \leqslant \ell(g) + \ell(h)$ for every pair of elements g and h in G. A length function is *quasi-geodesic* if there exists $M > 0$ such that for every $n \geqslant 1$ and every $g \in G$ with $\ell(g) \leqslant n$, there exist $1 = g_0, g_1, \ldots, g_n = g$ in G such that $\ell(g_{i-1}^{-1} g_i) \leqslant M$ for all i. Equivalently G, endowed with the distance $(g, h) \mapsto \ell(g^{-1}h)$, is quasi-isometric to a connected graph.

Question 10.4. — Given an irreducible variety X, is the length function

$$g \in \mathsf{Bir}(X) \longmapsto |\,\mathsf{Hyp}(X) \triangle g\,\mathsf{Hyp}(X)|$$

quasi-geodesic? In particular, what about $X = \mathbb{P}^2$ and the Cremona group $\mathsf{Bir}(\mathbb{P}^2)$?

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