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# CORRIGENDUM TO "HEIGHT GRADED RELATIVE HYPERBOLICITY AND QUASICONVEXITY"

by François Dahmani & Mahan Mj

Abstract. — There is an unfortunate mistake in the statement and the proof of Proposition 5.1 of [DM17]. This affects one direction of the implications of the main theorem. A correction is given, that states that given a quasi-convex subgroup of a hyperbolic (or relatively hyperbolic) group, the graded relative hyperbolic structure holds with respect to saturations of $i$-fold intersections, that are stabilizers of limit sets of $i$-fold intersections.

Résumé (Correction à « Hauteur, hyperbolicité relative graduée, et quasiconvexité »)

Une malencontreuse erreur entache la preuve, et l'énoncé, de la Proposition 5.1 de l'article mentionné en titre. Celle-ci affecte un sens d'implication du théorème principal. Nous en donnons ici une correction, qui indique que, étant donné un sous-groupe quasi-convexe d'un groupe hyperbolique, ou relativement hyperbolique, la collection des saturations des intersections multiples (et non pas des intersections multiples elles-mêmes) fournit une structure relativement hyperbolique graduée.

## Contents



There is an unfortunate mistake in the statement and the proof of Proposition 5.1 of [DM17]. This affects one direction of the implications of the main theorem 6.4 (also visible as 1.4). This also affects 5.2, 5.3, 5.4, and 5.5 that are essentially the different specific cases of the statement of 6.4. Unless explicitly mentioned the references to theorem numbers, etc. refer to [DM17]. We explain the problem, and give a correction.

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## 1. Main modifications and corrections

Proposition 5.1 of [DM17] must be replaced by Proposition 3 below.

The conclusions of Propositions 5.2, 5.3, 5.4 and 5.5 should be changed to: $(G, \{H\}, d)$ have the *saturated* (geometric) graded relative hyperbolicity (as defined below in Section 3).

Subsequently, in the conclusions of Theorems 1.4 and 6.4, the graded relative hyperbolicity should be changed to *saturated* graded relative hyperbolicity as stated in Section 6 below.

## 2. Setting of Proposition 5.1

Recall the setting of Proposition 5.1. Let $G$ be a group, $d$ a word metric on $G$ with respect to some generating set (not necessarily finite), such that $(G, d)$ is hyperbolic. Let $H$ be a subgroup of $G$. Assume that $H$ has finite geometric height (Definition 4.1) and uniform qi intersection property (Definition 3.9).

The original version of Proposition 5.1 then asserted that $(G, \{H\}, d)$ had the geometric graded relative hyperbolicity property. This is false in general. We propose here a correction of the statement and of the argument, which uses a natural operation, the saturation of a quasiconvex subgroup.

In order to guide the reader, we first discuss an example given by D. Osin, in the vocabulary of [DM17]. Consider a free group $F$ with basis $\{a, b\}$ and $H = \langle a^2 \rangle$. The height of $H$ in $F$ is two: it is at least two since $aH \neq H$ but $aHa^{-1} = H$, and it is at most two since $\langle a \rangle$ is malnormal in $F$. The two-fold intersections of essentially distinct conjugates of $H$ are the conjugates of $H$. Any three-fold intersection is trivial, hence the metric $d_3$ is the word metric on $F$ over $\{a, b\}$. However this metric is not relatively hyperbolic with respect to the two-fold intersections (i.e. the conjugates of $H$) as was incorrectly promised by Proposition 5.1, because $H$ has two cosets that remain at bounded distance of one another. It is nonetheless relatively hyperbolic with respect to the conjugates of the stabilizer of the limit set of $H$ (which is $\langle a \rangle$, and is its own normalizer). This last statement can be generalized to the setting of Proposition 5.1, and is the purpose of this corrigendum.

## 3. Definition of saturation

Let $(G, d)$ be a group with a hyperbolic word metric (over a possibly infinite generating set), let $\partial(G, d)$ be its Gromov boundary. If $A$ is a quasi-convex subgroup of infinite diameter for $d$, its limit set $\Lambda A$ in $\partial(G, d)$ is the intersection of $\overline{A}$, the closure of $A$ in $G \cup \partial(G, d)$, with $\partial(G, d)$. It is a closed $A$-invariant subset of $\partial(G, d)$. Since for all $g \in G$, the coset $gA$ remains at bounded distance to the subgroup $gAg^{-1}$, the limit set of $gAg^{-1}$ is the intersection of $\overline{gA}$ with $\partial(G, d)$, and so is the translate by $g$ of $\Lambda A$. In particular, if $g$ normalizes $A$ it preserves its limit set.

Let us also remark that if $A$ is quasi-convex and of infinite diameter, its limit set contains at least two points. To see this, we argue that we may find a hyperbolic isometry of $(G, d)$ in $A$ (which is sufficient to provide two different points in the limit set). Assume that there are only non-hyperbolic isometries in $A$. Recall Gromov's classification of actions on hyperbolic spaces (in particular [Gro87, Lem. 8.1.A]): either $A$ is bounded, or $\Lambda A$ is a single point at infinity, fixed by $A$, and all sequences in $A$ whose distance to 1 goes to infinity converge to this point. The first case is excluded by assumption on the diameter of $A$.

In the second case, consider two sequences of elements in $A$, $a_n$ and $a'_n$ such that the distance $d(a_n, a'_n)$ goes to infinity, and both converging to this point. By quasiconvexity, there is $M$ such that for all $n$, there is a $M$-quasi-geodesic in $A$ between them. After multiplication by an element of $A$ (near the midpoint of this quasigeodesic) we may assume that it passes uniformly close to 1 and that $d(1, a_n)$ and $d(1, a'_n)$ go to infinity. By Gromov's lemma [Gro87, Lem. 8.1.A], as recalled, the two sequences still converge to this unique point in the limit set, and therefore the Gromov product $(a_n \cdot a'_n)_1$ goes to infinity. But this contradicts that we have an $M$-quasigeodesic in $A$ between $a_n$ and $a'_n$ that passes uniformly close to 1.

If $A$ has infinite diameter, we define the saturation of $A$, denoted $A_s$, to be the stabilizer of the limit set $\Lambda A$ of $A$ in $\partial(G, d)$. If $A$ is a finite subgroup of $G$, we define $A_s$ to be $A_s = A$.

In the special case where $(G, d)$ is a relatively hyperbolic group, with the relative metric, we also consider the case where $A$ is infinite, of bounded diameter for $d$. In such case, it is classical that $A$ is necessarily contained in a parabolic subgroup (see for instance [DG18, Lem. 2.3]), and its saturation $A_s$ is the (unique) maximal parabolic subgroup containing $A$. In all three cases, $A$ is a subgroup of $A_s$.

We refrain from defining the saturation of an infinite bounded subgroup of $(G, d)$ in general, since apart from the relatively hyperbolic case, we will use this notion for convex cocompact subgroups of Mapping Class Groups and $\mathrm{Out}(F_n)$, which are never both infinite and bounded for the curve complex metric or the free factor complex metric.

Let us observe the following.

Lemma 1. — *Let $(G, d)$ be a group with a hyperbolic word metric. If a subgroup $A$ is quasi-convex, of infinite diameter in $(G, d)$, then its saturation $A_s$ is its own normalizer, contains $A$ and remains at bounded distance from $A$ in $(G, d)$.*

*If $(G, d)$ is relatively hyperbolic, and $A$ is an infinite bounded subgroup, the same conclusion holds.*

*Proof*. — The second assertion is actually a classical fact of relative hyperbolicity (as we mentioned, $A$ is then parabolic, and by definition, its saturation is the unique maximal parabolic subgroup containing $A$, which is its own normalizer in $G$, by [Osi06, Th. 1.4]). We focus on the proof of the first assertion. By our choices of definitions, $A$ is a subgroup of $A_s$.

We prove that $A_s$ remains at bounded distance from $A$. Assume that a sequence of elements $\alpha_n$ of $A_s$ diverges from $A$ in $(G,d)$. By multiplying on the left by elements of $A$ (which does not change its distance to $A$), we may assume that the distance to $A$ is realized by $d(1,\alpha_n)$. By quasi-convexity of $A$, there is $M$ such that the Gromov product $(\alpha_n \cdot \xi)_1$ remains bounded by $M$, for all $\xi$ in $\Lambda A$. Since $A_s$ preserves $\Lambda A$, for all such $\xi$, $\alpha_n \xi \in \Lambda A$, and therefore, any $(\delta)$-quasigeodesic ray from $\alpha_n$ to $\alpha_n\xi$ must fellow-travel the segment $[\alpha_n, 1]$ for at least $d(1,\alpha_n) - M - 20\delta$. Translating this to the base point 1, it means that the quasigeodesic rays from 1 to any point in $\Lambda A$ must fellow travel the same segment $[1, \alpha_n^{-1}]$ for such a length. Therefore for all $\xi, \xi'$ in $\Lambda A$, the Gromov product $(\xi \cdot \xi')_1$ is at least $d(1,\alpha_n) - M - 10\delta$. Letting $n$ go to infinity yields that $\Lambda A$ is a single point, a contradiction with our previous observation on limit sets.

As a consequence, the limit set of $A_s$ cannot be larger than that of $A$, and therefore $\Lambda A_s = \Lambda A$.

As was noticed earlier, any element in the normalizer of $A_s$ has to preserve $\Lambda A_s$. Hence by definition of $A_s$, and the last equality $\Lambda A_s = \Lambda A$, it has to be in $A_s$. □

Let us define the saturated (geometric) graded relative hyperbolicity property, as a variation of Definition 4.3, as follows. Note that only the second condition is changed compared to the original definition 4.3.

Let $G$ be a group, $d$ the word metric with respect to some (not necessarily finite) generating set and $\mathscr{H}$ a finite collection of subgroups. Let $\mathscr{H}_i$ be the collection of all $i$-fold conjugates of $\mathscr{H}$. Let $(\mathscr{H}_i)_0$ be a choice of conjugacy representatives and $\mathscr{C}\mathscr{H}_i$ the set of left cosets of elements of $(\mathscr{H}_i)_0$.

Let also $\mathscr{S}\mathscr{H}_i$ be the collection of saturations of the groups in $\mathscr{H}_i$ for the metric $d$. Let $(\mathscr{S}\mathscr{H}_i)_0$ be a choice of conjugacy representatives of these groups, and $\mathscr{C}\mathscr{S}\mathscr{H}_i$ the family of left cosets of $(\mathscr{S}\mathscr{H}_i)_0$. Let $d_i^s$ be the metric on $G$ obtained from $d$ by electrifying the elements of $\mathscr{C}\mathscr{S}\mathscr{H}_i$. Let $\mathscr{C}\mathscr{S}\mathscr{H}_{\mathbb{N}}$ be the graded family $(\mathscr{C}\mathscr{S}\mathscr{H}_i)_{i\in\mathbb{N}}$. We say that $(G,d)$ has saturated (geometric) graded relative hyperbolicity with respect to $\mathscr{C}\mathscr{S}\mathscr{H}_{\mathbb{N}}$ if

(1) $\mathscr{H}$ has (geometric) height $n$ for some $n \in \mathbb{N}$, and for each $i$ there are finitely many orbits of $i$-fold intersections,

(2) for all $i$, $\mathscr{C}\mathscr{S}\mathscr{H}_i$ coarsely hyperbolically embeds in $(G, d_i^s)$,

(3) there is $D_i$ such that all items of $\mathscr{C}\mathscr{H}_i$ are $D_i$-coarsely path connected in $(G,d)$.

We say that $(G,d)$ has saturated graded relative hyperbolicity with respect to $\mathscr{C}\mathscr{S}\mathscr{H}_{\mathbb{N}}$ if (2) and (3) are true, and (1) is true for the algebraic height.

## 4. Uniform qi intersection property passes to saturations

Recall that uniform qi intersection property was defined in Definition 3.9. Let us first notice the following, that essentially states that this property passes to saturations.

Lemma 2. — *Assume that $(G,d)$ is hyperbolic. If $H$ has uniform qi intersection property in $(G,d)$ and if $\mathscr{CSH}_{\mathbb{N}}$ is obtained as in the definition above, then for all $i$ there exists $C_i^s$ such that each element of $\mathscr{CSH}_i$ is quasi-convex in $(G,d)$, and such that if $A_0, B_0$ are in $(\mathscr{SH}_i)_0$ and if $\Pi_{B_0}(gA_0)$ has diameter larger than $C_i^s$ for the metric $d$, then $gA_0g^{-1}\cap B_0$ has diameter larger than $C_i^s$ for $d$.*

*Proof*. — We first prove the uniform quasi-convexity of elements of $(\mathscr{SH}_i)_0$. The groups in $(\mathscr{H}_i)_0$ are uniformly quasi-convex in $(G,d)$, by the first point of Definition 3.9. Each group $A_s$ in $(\mathscr{SH}_i)_0$ has a subgroup $A$ in $(\mathscr{H}_i)_0$ which is co-bounded in $A_s$ (for the metric of $(G,d)$). By hyperbolicity of $(G,d)$ any geodesic between two points of $A_s$ is close to a geodesic between two points of $A$, which itself remains at bounded distance from $A$ hence from $A_s$. This proves the first point.

For the second point, assume the contrary: for all $C_i$ we can find $A_s, B_s$ in $(\mathscr{SH}_i)_0$ and $g$ (all depending on $C_i$) such that $\Pi_{B_s}(gA_s)$ has diameter larger than $C_i$ for the metric $d$, but $gA_sg^{-1}\cap B_s$ has diameter smaller than $C_i$ for $d$. Consider elements $A, B$ in $(\mathscr{H}_i)_0$ of which $A_s$ and $B_s$ are the saturations. Of course $gAg^{-1}\cap B$ has diameter smaller than $C_i$ for $d$. On the other hand, we can see that the diameter of $\Pi_B(gA)$ must go to infinity with $C_i$. Indeed, take pairs of points $a_0, a_1$ in $gA_s$ and $b_0, b_1$ in $B_s$ realizing the shortest point projection of $a_0, a_1$ respectively, and such that $d(b_0, b_1)$ is larger than $C_i$. Then, we may find $a_0', a_1'$ in $gA$ close to $a_0, a_1$ (say at distance at most $D$), and consider their shortest point projection on $B$, say $b_0', b_1'$. Approximate the octagon $(b_0, a_0, a_0', b_0', b_1', a_1', a_1, b_1)$ by a tree, by hyperbolicity. Because both $B$ and $B_s$ are quasi-convex (with uniform constant over $(\mathscr{SH}_i)_0$), the Gromov products of the consecutive sides at the vertices $b_0, b_1, b_0', b_1'$ are uniformly bounded. One can then deduce that the central subsegment of $[b_0, b_1]$ of length at least $C_i$ minus a universal constant, remains close to $[b_0', b_1']$. Thus, the diameter of $\Pi_B(gA)$ is larger than this quantity.

By uniform qi-intersection property for $\mathscr{H}_i$, we then have a contradiction. □

## 5. Correction to Proposition 5.1

Then we can show a correct version of Proposition 5.1.

Proposition 3. — *Let $G$ be either a relatively hyperbolic group with a relative word metric $d$, or a Mapping class group with a word metric $d$ equivariantly quasi-isometric to the curve complex, or $\mathrm{Out}(F_n)$ with a word metric $d$ equivariantly quasi-isometric to the free factor complex.*

*Let $H$ be a subgroup of $G$. If $\{H\}$ has finite (geometric) height for $d$ and has the uniform qi-intersection property, then $(G,\{H\},d)$ has the saturated (geometric) graded relative hyperbolicity property with respect to $\mathscr{CSH}_{\mathbb{N}}$.*

*Proof*. — By Lemma 2, the elements $\mathscr{CSH}_i$ also are $C_1^s$-quasi convex in $(G,d)$ for some uniform constant $C_1^s$. By Proposition 2.11, all the elements of $\mathscr{CSH}_i$ are uniformly quasi-convex in $(G, d_{i+1})$.

We now need to show that the elements of $\mathscr{CSH}_i$ are mutually cobounded for $d_{i+1}^s$ (a property which fails in general for $\mathscr{CH}_n$).

Assume, by contradiction, that the elements (cosets) of $\mathscr{CSH}_i$ are not mutually co-bounded for the metric $d^s_{i+1}$. For all $D$ there exist two essentially different cosets $A_s$ (which can be assumed in $(\mathscr{SH}_i)_0$) and $gB_s$ (for $B_s \in (\mathscr{SH}_i)_0$ as well), that have projection larger than $D$ on one another for the metric $d^s_{i+1}$. Recall that essentially different means that either $A_s \neq B_s$ or $g \notin B_s$.

By Lemma 2, for $D$ large enough, $A_s \cap gB_sg^{-1}$ has diameter larger than $D - 2C_n$ for $d^s_{i+1}$.

By definition of saturation, $A_s$ and $B_s$ are either bounded, or equal the stabilizers of their respective limit sets $\Lambda A_s, \Lambda B_s$ in the hyperbolic metric $d$.

In the case where both $A_s$ and $B_s$ are bounded, yet infinite (or even of sufficiently large cardinality), then we are in the relatively hyperbolic case, and these groups must be parabolic, and as saturations, they are equal to maximal parabolic group containing $A$ and $B$. Their intersection cannot be larger than a universal constant for $d$, contrary to our assumption.

In the case $A_s$ and $B_s$ are unbounded, we first observe that in that metric,

$$\Lambda A_s \cap \Lambda gB_sg^{-1} = \Lambda(A_s \cap gB_sg^{-1});$$

for proving this we distinguish whether $G$ is a relatively hyperbolic group or not. This is a result of Yang [Yan12, Th. 1.1] for relatively hyperbolic groups.

For a convex cocompact subgroup of a Mapping Class Group, (respectively of $\mathrm{Out}(F_n)$), this observation can be derived from the fellow traveling property of a thick part of Teichmüller space (Rafi [Raf14]) (respectively Dowdall-Taylor [DT18, DT17]), and that the weak hull of the group remains in a thick part. We sketch an argument for Mapping Class Groups. If $\xi$ is limit of $(a_n)$ (sequence in $A_s$) and of $(gb_ng^{-1})$ (in $gB_sg^{-1}$), then after possible re-indexing of subsequences, and choosing a base point $x_0$ in Teichmüller space, $a_nx_0$ and $gb_ng^{-1}x_0$ remain at bounded distance. Thus infinitely often, $a_i^{-1}a_j = gb_i^{-1}b_jg^{-1}$, hence the intersection accumulates on $\xi$.

Now, taking $A$ and $B$ in $\mathscr{H}_i$ of which $A_s, B_s$ are the saturations, $\Lambda A_s = \Lambda A$, and $\Lambda gB_sg^{-1} = \Lambda gBg^{-1}$. Therefore, the saturation of $(A \cap gBg^{-1})$ contains $A_s \cap gB_sg^{-1}$, hence it has diameter larger than $D - 2C_n$ for $d^s_{i+1}$.

It follows that $(A \cap gBg^{-1})$ is not an essential $(i+1)$-fold intersection of conjugates of $H$. Writing $A$ and $B$ as $i$-fold intersection themselves as $A = X_1 \cap \cdots \cap X_i$, and $gBg^{-1} = Y_1 \cap \cdots \cap Y_i$, then $A \cap gBg^{-1} = X_1 \cap \cdots \cap X_i \cap Y_1 \cap \cdots \cap Y_i$, which can only contain $i$ essentially distinct conjugates. We must conclude that after permutation of indices, $X_j = Y_j$ for all $j$, and that $gBg^{-1} = A$. Therefore, at the level of saturations, $gB_sg^{-1} = A_s$. Since $A_s, B_s$ are in $(\mathscr{SH}_i)_0$ which is a set of conjugacy representatives, we have that $A_s = B_s$. And therefore $g$ normalizes $B_s$.

This is precisely where the mistake was: had we taken $B$ in $(\mathscr{H}_i)_0$, we could not have concluded that $g \in B$. But now, $B_s \in (\mathscr{SH}_i)_0$, which consists of saturated subgroups, which are equal to their own normalizers. Thus, indeed, we can conclude that $g \in B_s$. This contradicts our original choice that $A_s$ and $gB_s$ are essentially different.

The end of the proof is now the use of Proposition 2.10, and 2.23, as in the original version of [DM17]. $\square$

## 6. Conclusion

We can proceed and correct statements 5.2 to 5.5. In each of them only the conclusion is changed to: $(G,\{H\},d)$ have the *saturated* geometric graded relative hyperbolicity.

The syntactical modification of the proof is straightforward, using the corrected proposition above in place of Proposition 5.1.

The main results Theorems 1.4 and 6.4 need to be corrected as follows.

Theorem 4

(1) *Let $G$ be a hyperbolic group and $d$ the word metric with respect to a finite generating set $S$.*

– *If a subgroup $H$ is quasiconvex then $(G,\{H\})$ has saturated graded geometric relative hyperbolicity.*

– *If $(G,\{H\})$ has graded geometric relative hyperbolicity, then $H$ is quasiconvex.*

(2) *Let $G$ be a finitely generated group, hyperbolic relative to $\mathscr{P}$, $S$ a finite relative generating set, and $d$ the word metric with respect to $S\cup\mathscr{P}$.*

– *If a subgroup $H$ is relatively quasiconvex then $(G,\{H\},d)$ has saturated graded geometric relative hyperbolicity.*

– *If $(G,\{H\},d)$ has graded geometric relative hyperbolicity, then $H$ is relatively quasiconvex.*

(3) *Let $G$ be a mapping class group $\mathrm{Mod}(S)$ and $d$ the metric obtained by electrifying the subgraphs corresponding to sub mapping class groups so that $(G,d)$ is quasi-isometric to the curve complex $\mathrm{CC}(S)$.*

– *If a subgroup $H$ is convex cocompact in $\mathrm{Mod}(S)$ then $(G,\{H\},d)$ has saturated graded geometric relative hyperbolicity.*

– *If $(G,\{H\},d)$ has graded geometric relative hyperbolicity and the action of $H$ on the curve complex is uniformly proper, then $H$ is convex cocompact in $\mathrm{Mod}(S)$.*

(4) *Let $G$ be $\mathrm{Out}(F_n)$ and $d$ the metric obtained by electrifying the subgroups corresponding to subgroups that stabilize proper free factors so that $(G,d)$ is quasi-isometric to the free factor complex $\mathscr{F}_n$.*

– *If a subgroup $H$ is convex cocompact in $\mathrm{Out}(F_n)$, then $(G,\{H\},d)$ has saturated graded geometric relative hyperbolicity.*

– *If $(G,\{H\},d)$ has graded geometric relative hyperbolicity and the action of $H$ on the free factor complex is uniformly proper, then $H$ is convex cocompact in $\mathrm{Out}(F_n)$.*

*Proof*. — The forward implications of quasiconvexity to graded geometric relative hyperbolicity in the first 3 cases are proved by the corrections above of Propositions 5.2, 5.3, 5.4, and 5.5. and case 4 by the correction of Proposition 5.4.

Reverse implications are those of Theorem 6.4, their proof is unchanged. □

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