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ON THE TOPOLOGY OF CLOSED MANIFOLDS WITH QUASI-CONSTANT SECTIONAL CURVATURE

by Louis Funar

Abstract. — We prove that closed manifolds admitting a generic metric whose sectional curvature is locally quasi-constant are graphs of space forms. In the more general setting of QC spaces where sets of isotropic points are arbitrary, under suitable positivity assumption and for torsion-free fundamental groups, they are still diffeomorphic to connected sums of spherical bundles over the circle.

Résumé (Sur la topologie des variété fermées à courbure sectionnelle quasi-constante)

Nous montrons que les variétés fermées admettant une métrique générique dont la courbure sectionnelle est localement quasi-constante sont des sommes graphées de variétés de courbure constante. Ensuite nous étendons ce résultat au cas des espaces QC dont l'ensemble des points isotropes pourrait être arbitraire en démontrant que, sous une condition de positivité et lorsque leurs groupes fondamentaux sont sans torsion, ils sont difféomorphes à des sommes connexes de fibrés en sphères sur le cercle.

Contents

1. Introduction . . . 367
2. Preliminaries . . . 373
3. The topology of locally 1-QC manifolds . . . 381
4. Proof of Theorem 1.8 . . . 405
5. More about the equivariant immersions of 1-QC manifolds . . . 418
References . . . 421

1. Introduction

The classification of (locally) conformally flat closed manifolds seems out of reach in full generality since M. Kapovich proved (see [34]) that an arbitrary finitely presented group G is a subgroup of a free amalgamated product of the form $G * H$ which is a fundamental group of a conformally flat closed manifold of dimension (at least) 4.

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On the other hand, simple infinite groups (e.g. Thompson's groups) do not occur among fundamental groups of conformally flat closed manifolds. Indeed, the holonomy representation would be injective, so the fundamental group has to be linear, but finitely generated linear groups are residually finite, according to classical results of Malcev, and hence they cannot be infinite and simple. Kamishima ([32]), improving previous work by Goldman ([23]) for virtually nilpotent groups, showed that a closed conformally flat n-manifold with virtually solvable fundamental group is conformally equivalent to a n-spherical space form, an Euclidean space-form or it is finitely covered by some Hopf manifold $S^1 \times S^{n-1}$. Further, Goldman and Kamishima in [24] classified up to a finite covering the conformally flat closed manifolds whose universal covering admits a complete conformal Killing vector field, by adding to the previous list the closed hyperbolic manifolds, their products with a circle and conformally homogeneous quotients of the form $(S^n \smallsetminus \lim \Gamma)/\Gamma$, where $\lim \Gamma \subset S^{n-2}$ is the limit set of the conformal holonomy group Γ.

The aim of this paper is to describe the topology of a family of conformally flat manifolds admitting a local vector field with respect to which the curvature is quasi-constant. This hypothesis is similar in spirit to that used by Goldman and Kamishima ([24]), but in our case the vector field is not a conformal Killing field in general, and the methods of the proof are rather different, coming from foliations and Cheeger-Gromov's theory.

The first family of spaces under consideration here are manifolds admitting a Riemannian metric of quasi-constant sectional curvature of dimension $n \geqslant 3$ (see the definition below). These spaces were first studied from a geometrical point of view by Boju and Popescu in [6] and coincide with the k-special conformally flat spaces considered earlier in a different context by B.-Y. Chen and K. Yano ([30, 14]). These were further analyzed by Ganchev and Mihova in [22]. It was noticed in [5, 30] and later in [22] that these are locally conformally flat manifolds for $n \geqslant 4$.

Definition 1.1. — Let ξ be a smooth line field on the Riemannian manifold M. We denote by $K(\sigma)$ the sectional curvature of the tangent 2-plane σ at $p \in M$ and by $\measuredangle(\sigma, \xi)$ the angle between σ and the line $\xi|_p$.

The metric of M is *globally* 1-QC with respect to the line field ξ if there exists some function $\theta : M \to (0, \pi/2)$ such that for any point $p \in M$, and any 2-planes σ_1, σ_2 tangent at p to M satisfying $\measuredangle(\sigma_1, \xi) = \measuredangle(\sigma_2, \xi) = \theta(p)$ we have:

$$K(\sigma_1) = K(\sigma_2).$$

In this case ξ is called a *distinguished* line field of the Riemannian manifold M. Further, we say that (the metric of) M is *globally* 1-QC if it is *globally* 1-QC with respect to some smooth line field ξ.

The original definition in [6, 22] (see also the comments of the last section in [6]) used a vector field instead of a line field and is too restrictive for our purposes.

It is proved in [6] that the sectional curvature of a globally 1-QC manifold is constant along any cone of 2-planes making a constant angle in $[0, \pi/2]$ with ξ, in every point.

Recall that a *line* field is a section of the projectivized tangent bundle. A line field ξ on M determines a real line bundle, still denoted ξ over M. Its first Stiefel-Whitney class $w_1(\xi) \in H^1(M, \mathbb{Z}/2\mathbb{Z})$ vanishes if and only if ξ has a section, namely a non-zero vector field $\widehat{\xi}$ defining the same line field. When this is the case, the line field will be called *orientable* and such a vector field $\widehat{\xi}$ will be called a *lift* of the line field. When ξ is non-orientable, its pull-back to the 2-fold cover $\widehat{M}$ of M associated to the class $w_1(\xi)$ is orientable and defines a vector field $\widehat{\xi}$ on M. We call $\widehat{\xi}$ the distinguished vector field associated to the globally 1-QC manifold with distinguished line field ξ.

Throughout this paper we will use the term *cylinder* in its topological sense, namely, a cylinder is a manifold diffeomorphic to the product of a compact manifold with an interval (which might be open or closed). More generally, a *twisted cylinder* is an interval bundle over a compact manifold, which might possibly be a non-trivial bundle.

The topology of globally 1-QC manifolds is rather simple, as we will see that they must be 2-fold covered by cylinders. Further, the geometry of globally 1-QC manifolds with orientable line field was completely described under an additional condition (namely, that the distinguished field be geodesic) in [22, §6]: these are precisely the warped product Riemannian manifolds between some closed space form and an interval which are called sub-projective spaces. Notice that the vector field ξ is (locally) conformally Killing if and only if it is Killing and if and only if it is geodesic.

In order to enlarge the number of possible topologies arising in the definition above we slightly weaken it as follows.

Definition 1.2. — The (metric of the) Riemannian manifold M is called

(1) *locally* 1-*QC*, if any point of M has an open neighborhood endowed with some line field ξ such that the induced metric is globally 1-QC with respect to ξ;

(2) 1-*QC* if any *non-isotropic* point for the sectional curvature function has an open neighborhood which is locally 1-QC.

Note that line fields do not necessarily define a global line field on the manifold M. Observe that a locally 1-QC manifold is a 1-QC manifold.

Recall that a point of a Riemannian manifold is called *isotropic* if all its tangent 2-planes have the same sectional curvature. Denote by C the set of isotropic points of M. In the case of a globally 1-QC manifold the sectional curvatures N and H of 2-planes containing and respectively orthogonal to the distinguished line field ξ will be called *vertical* and *horizontal* curvature functions, respectively.

Remark 1.3. — An open set of dimension at least 3 consisting of isotropic points has constant sectional curvature, by F. Schur's theorem ([50]).

It will be shown in the next section that both H and N extend smoothly from the globally 1-QC neighborhoods to all of M, when the manifold M is 1-QC. Moreover, the set of non-isotropic points carries an unique distinguished line field extending the distinguished line fields associated to various locally 1-QC neighborhoods.

For a 1-QC manifold M, the set C of its isotropic points then consists of those points $p \in M$ satisfying the equation:

$$H(p) = N(p).$$

Definition 1.4. — The metric on a 1-QC manifold is *generic* if the horizontal curvature function H is not eventually constant on $M \smallsetminus C$, namely, for every connected open subset $U \subset M \smallsetminus C$ such that the restriction of H to U is constant we have $\overline{U} \cap C = \varnothing$.

Our main issue in this paper is to give some insights on the topology of C and $\overline{M \smallsetminus C}$. Results of foliation theory give the description of $M \smallsetminus C$ as twisted cylinders. Further, we will prove that $\overline{M \smallsetminus C}$ and $\overline{\mathrm{int}(C)}$ can be approximated by manifolds with boundary when M is locally 1-QC.

Definition 1.5. — A *space form* is a compact Riemannian manifold with constant sectional curvature.

A *graph of space forms* is a closed manifold M which can be obtained from a finite set of compact space forms with totally umbilical boundary components by gluing isometrically interval bundles (twisted cylinders) along boundary components. The pieces will be accordingly called *vertex* manifolds and *edge* manifolds or *tubes*.

Note that the induced metric on the graph of space forms is not necessarily smooth and its diffeomorphism type depends on additional choices, like the gluing maps. In particular, fiber bundles over the circle whose fibers are closed space forms are such graphs of space forms.

The structure of conformally flat closed manifolds whose universal covering have a conformal vector field was described by Goldman and Kamishima (see [24]). This includes the case of globally 1-QC manifolds with geodesic distinguished field ξ. Motivated by this, our first result is a general description of the topology of the locally 1-QC spaces, under a mild genericity condition, as follows:

Theorem 1.6. — *Let M^n be a closed locally 1-QC n-manifold, $n \geqslant 3$, whose metric is generic and assumed conformally flat when $n = 3$. Assume that the rank of $H_1(M \smallsetminus C; \mathbb{Q})$ is finite. Then M is diffeomorphic to a graph of space forms.*

Remark 1.7. — The same result holds under the assumption that $H_1(M \smallsetminus C; \mathbb{Z}/2\mathbb{Z})$ is finite.

The *curvature leaves function* λ is defined at non-isotropic points (see also Section 2.5) by:

$$\lambda = H + \frac{\| \operatorname{grad} H\|^2}{4(H-N)^2}.$$

Our second result aims to extend Theorem 1.6 to the case of 1-QC manifolds under additional hypotheses.

Theorem 1.8. — *Let M^n, $n \geqslant 3$, be a closed n-manifold with infinite torsion-free $\pi_1(M)$, admitting a 1-QC metric with orientable distinguished line field, which is conformally flat when $n = 3$. Moreover, suppose $\lambda > 0$ on $M \smallsetminus C$ and that every tangent 2-plane at an isotropic point has positive sectional curvature. Then M is diffeomorphic to a connected sum of S^{n-1}-bundles over S^1 (possibly not orientable).*

By convention a connected sum of an empty set of n-manifolds is the n-sphere. The main issue is the description of degeneracies occurring when compactifying $M \smallsetminus C$. One key ingredient is the characterization of compact 1-QC manifolds as those Riemannian manifolds whose universal coverings have codimension-one isometric immersions into hyperbolic spaces (see Propositions 2.14 and 2.18). The topology of closed manifolds admitting such codimension-one immersions is known, as this problem already appeared in conformal geometry. Kulkarni has proved in [36] that closed orientable conformally flat manifolds which admit conformal embeddings as hypersurfaces in $\mathbb{R}^{n+1}$ are conformally equivalent to either the round n-sphere or some Hopf manifold $S^1 \times S^{n-1}$, if both the immersion and the metric are *analytic*. Moreover, in [8, 46] the authors proved that closed conformally flat manifolds which admit conformal embeddings as hypersurfaces in some hyperbolic of Euclidean space are diffeomorphic and respectively conformally equivalent to an n-sphere with several (possibly unoriented) 1-handles, i.e., connected sums of S^{n-1}-bundles the circle. We use the methods from [8, 46] suitably extended to our non-compact situation, in order to analyze $\overline{M \smallsetminus C}$. We will show that a suitable neighborhood of $\overline{\mathrm{int}(C)}$ in M can be obtained from a closed space form by removing finitely many disjoint disks. For this purpose we prove that one can cap off its boundary spheres by using spherical caps with controlled geometry so that the manifold obtained this way is diffeomorphic to a space form, according to a theorem of Nikolaev ([44]).

Remark 1.9. — A classical result of Kuiper states that a closed simply connected conformally flat manifold is conformally equivalent to the round sphere (see [35]). Kuiper worked under $\mathscr{C}^3$ differentiability assumptions, but this result is known to hold under the weaker $\mathscr{C}^1$-assumptions. If M is a closed 1-QC manifold with finite $\pi_1(M)$, then its universal covering $\widetilde{M}$ is therefore conformally equivalent to the round sphere.

Remark 1.10. — If $H, N > 0$, then classical results of Bochner, Lichnerowicz and Myers show that the manifold is a homology sphere with finite fundamental group. In particular manifolds with a positive number of 1-handles in theorem 1.8 cannot have metrics with positive N.

Remark 1.11. — If $(n-2)H + 2N \geqslant 0$ then M^n is conformally equivalent to a Kleinian quotient (see [49, Th. 4.5]). Moreover, if the inequality above is strict, then $\pi_j(M) = 0$, for j such that $2 \leqslant j \leqslant n/2$ (see [49, Th. 4.6]).

Remark 1.12. — Rational Pontryagin forms, and hence classes, vanish on conformally flat manifolds, see [15, 37]. Conversely, it seems unknown whether conformally flat manifolds admit conformally flat metrics for which the curvature tensor in a suitable basis has all its components with 3 and 4 distinct indices vanishing.

Throughout this paper (local) 1-QC manifold or space will mean a manifold which admits a Riemannian (local) 1-QC metric.

The structure of this paper is as follows. In Section 2 we collect a number of general results concerning 1-QC manifolds for further use. We show that the horizontal and vertical curvature functions extend smoothly to the entire manifold first in the locally 1-QC case (Lemma 2.2), then using the higher Weitzenböck curvatures in the general 1-QC case (Lemma 2.7). In dimensions $n \geqslant 4$, 1-QC manifolds are conformally flat, while this has to be added to our assumptions when $n = 3$ (see Lemma 2.9). The level hypersurfaces of the horizontal curvature define a partial foliation with complete leaves which are totally umbilical submanifolds of intrinsic constant curvature λ. We show that for locally 1-QC manifolds λ extends to the closure of the open set of non-isotropic points (Lemma 2.11). Eventually, we show that the 1-QC manifolds of dimension $n \geqslant 4$ are precisely those Riemannian manifolds whose universal coverings have codimension-one isometric immersions into the hyperbolic space, which are equivariant with respect to some holonomy homomorphism (Propositions 2.14–2.18).

Section 3 is devoted to the proof of Theorem 1.6. We start by describing the topology of globally 1-QC manifolds with non-empty set of isotropic points: they are twisted cylinders over compact space space forms (Proposition 3.1), since the curvature leaves are compact (Proposition 3.3). This is a consequence of deep results of Reckziegel ([47, 48]) concerning the completeness of curvature leaves and Haefliger's classical theory of foliations with compact leaves ([28]). In Section 3.8 we collect a few results to be used later about the geometry of the curvature leaves, by showing that their second fundamental form, injectivity radius and (local) normal injectivity are uniformly bounded. The existence of a well-defined distinguished vector field $\widehat{\xi}$ in a neighborhood of an isotropic point shows that curvature leaves have a controlled behavior when they approach the isotropic point, by limiting to some branched hypersurface (Proposition 3.29). We further need to analyze the degeneracies occurring in the closure of these components by using the branched hypersurfaces which arise as limit leaves (Proposition 3.34). We show that we can choose finitely many limit leaves such that the open isotropic components can be compactified to manifolds with boundary (Propositions 3.40 and 3.42). By gluing isometrically some cylinders endowed with a warped product metric along the boundary we construct space forms with umbilical boundary.

In Section 4 we are concerned with the proof of Theorem 1.8. The distinguished line field cannot be extended to isotropic points as in the previous section but the curvature leaves are now constrained to be diffeomorphic to spheres, as they have positive curvature. Every connected component of a saturated neighborhood of the set of isotropic points is a manifold whose boundary components are spheres (Proposition 4.9). Group

theory arguments involving Grushko's theorem prove that there are only finitely many spheres which do not bound a ball (Proposition 4.5). The intrinsic curvature of a spherical leaf provides control on the size of a spherical cap in the hyperbolic space which bounds that leaf (Lemmas 4.8 and 4.18). We then prove that we can realize the capping off the boundary spheres in the Riemannian setting and the closed manifold obtained this way is diffeomorphic to a space form, according to a theorem of Nikolaev ([44]) (see Proposition 4.17).

In Section 5 we complete the description of isometric immersions of 1-QC conformally flat manifolds with the case of dimension $n = 3$ (Proposition 5.1). Further, we note that a 1-QC metric admits an extension to a hyperbolic metric on the product $M \times [0, \infty)$ (Proposition 5.3).

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2. Preliminaries

2.1. Isotropic points of local 1-QC manifolds

Lemma 2.1. — *If M is locally 1-QC with respect to distinct distinguished line fields at p then p is an isotropic point.*

Proof. — Let X and Y be unit vectors representing the distinguished lines fields at p. According to [6] the curvature of a 2-plane σ is given by

$$K(\sigma) = H_X \sin^2 \measuredangle(X, \sigma) + N_X \cos^2 \measuredangle(X, \sigma) = H_Y \sin^2 \measuredangle(Y, \sigma) + N_Y \cos^2 \measuredangle(Y, \sigma),$$

where H_X, N_X and H_Y, N_Y are the vertical and horizontal curvatures associated to the respective lines fields.

Take σ to be the span of the orthogonal unit vectors X and Z. If g denotes the metric on M, then:

$$K(\sigma) = H_Y + (g(Y, X)^2 + g(Y, Z)^2)(N_Y - H_Y) = H_X.$$

If $H_Y = N_Y$ then p is isotropic. Otherwise $g(Y, Z)^2$ should not depend on the choice of Z orthogonal to X and hence $g(Y, Z) = 0$, so that the lines defined by Y and X coincide, contradicting our hypothesis. □

Lemma 2.2. — *If M is a compact locally 1-QC manifold then the functions H and N are well-defined smooth functions on M.*

Proof. — For any $p \in M$ there is a line field ξ_U defined on an open neighborhood U of p such that the metric is globally 1-QC on U relative to ξ_U. This provides the smooth curvature functions H_U and N_U on U. We extract a finite covering $\{U_i\}$ of M by such open sets.

If $U_i \cap U_j \neq \varnothing$ we set

$$C_{U_iU_j} = \{p \in U_i \cap U_j \ ; \ \xi_{U_i}|_p \neq \xi_{U_j}|_p\}.$$

Then, by Lemma 2.1, every point $p \in C_{U_iU_j}$ is isotropic for the sectional curvature. This implies that

$$H_{U_i}(p) = N_{U_i}(p) = H_{U_j}(p) = N_{U_j}(p).$$

Further, if $p \in U_i \cap U_j \smallsetminus C_{U_iU_j}$, then $\xi_{U_i}|_p = \xi_{U_j}|_p$ and thus $H_{U_i}(p) = H_{U_j}(p)$, $N_{U_i}(p) = N_{U_j}(p)$.

Therefore, we have a finite collection of smooth functions H_{U_i} and N_{U_i} whose restrictions to intersections $U_i \cap U_j$ coincide. Thus they define smooth functions on M. □

Recall that C denotes the closed subset of M of isotropic points for the sectional curvature.

Lemma 2.3. — *If M is a compact locally 1-QC manifold then the distinguished line field ξ is well-defined on an open set $\Omega \supset \overline{M \smallsetminus C}$.*

Proof. — Consider a finite open cover of M by globally 1-QC subsets $\{U_i\}$, as above. We take a refined open cover $\{V_j\}$ such that every $\overline{V_j}$ is contained into some U_i. If $p \in (V_i \cap V_j) \smallsetminus C$, then $\xi_{V_i}|_p = \xi_{V_j}|_p$, by Lemma 2.1. This closed condition is therefore valid for any point $p \in \overline{(V_i \cap V_j) \smallsetminus C}$. It follows that the line fields ξ_{V_i} agree at any point $p \in \overline{M \smallsetminus C}$. They define therefore a distinguished line field on $\overline{M \smallsetminus C}$.

When the line field is orientable, it can be extended to an open set Ω containing $\overline{M \smallsetminus C}$, by a classical partition of unity argument. Otherwise, we construct an extension on the 2-fold cover of an open neighborhood of $\overline{M \smallsetminus C}$ associated to the first Stiefel-Whitney class of ξ and then project it down. □

2.2. Weitzenböck curvatures. — The m-th Weitzenböck curvatures were explicitly computed for manifolds with vanishing Weyl tensor by Guan and Wang ([27]), as follows.

Let Δ be the Hodge-de Rham Laplacian $\Delta = dd^* + d^*d$ and $\nabla^*\nabla$ be the Bochner Laplacian of the Riemannian manifold M. Recall that $\nabla : \Omega^m(M) \to T^*(M) \otimes \Omega^m(M)$ is the covariant derivative induced by the Levi-Civita connection on the space of m-forms $\Omega^m(M)$ and ∇^* its adjoint, namely

$$\nabla^*\omega(X_1, \ldots, X_m) = -\sum_{i=1}^{n} \nabla_{e_i}\omega(e_i, X_1, \ldots, X_m),$$

for any vector fields X_j and an orthonormal local basis $\{e_i\}$.

If ω is a m-form, then the Weitzenböck formula states that:

$$\Delta\omega = \nabla^*\nabla + \mathscr{R}_m\omega.$$

Here, $\mathscr{R}_m$ is the m-th Weitzenböck curvature operator, namely a degree zero pseudo-differential operator determined by the Riemann tensor:

$$\mathscr{R}_m\omega = \sum_{i,j=1}^{n} (R(e_i, X_j)\omega)(X_1, \dots, X_{j-1}, e_i, X_{j+1} \dots, X_m)$$

The Riemannian connection induces a metric connection in the bundle $\Omega^m(M)$ and its curvature R used above is a section of $\Omega^2(M) \otimes \Omega^m(M)$.

Let now consider the real function defined on M by the formula:

$$G_m = \left((n-p) \sum_{i=1}^{p} \lambda_i + p \sum_{i=p+1}^{n} \lambda_i \right),$$

where $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ denote the eigenvalues of the Schouten operator S defined as

$$S(X,Y) = \frac{1}{n-2}\Big(\mathrm{Ric}(X,Y) - \frac{1}{2(n-1)}\, Rg(X,Y)\Big)$$

in terms of the Ricci tensor Ric and the scalar curvature R. Assume that the Weyl tensor of M vanishes so that the Riemann tensor satisfies the following identity (see [42]):

$$R(X,Y,Z,T) = g(Y,Z)S(X,T) - g(X,Z)S(Y,T) + g(X,T)S(Y,Z) - g(Y,T)S(X,Z).$$

It follows by direct calculation (see [27, App.]) that $\mathscr{R}_m\omega = G_m\omega$. In particular, we obtain the following:

Lemma 2.4. — *Let M be a Riemannian n-manifold, $n \geqslant 3$, whose Weyl tensor vanishes. Then the function $G_m : M \to \mathbb{R}$ is smooth.*

2.3. Isotropic points of 1-QC manifolds. — In this section M is a 1-QC manifold and C denotes the closed set of its isotropic points. The first observation is:

Lemma 2.5. — *If M is a compact 1-QC then $M \smallsetminus C$ is globally 1-QC. In particular, H and N are smooth real functions defined on $M \smallsetminus C$.*

Proof. — According to the definition, $M \smallsetminus C$ is locally 1-QC, hence it admits a covering by open sets $U_i \subset M \smallsetminus C$ which are globally 1-QC with respect to line fields ξ_{U_i}. However, points $p \in U_i \cap U_j$ where the corresponding line fields do not agree should be isotropic, following Lemma 2.1 and the proof of Lemma 2.2, which contradicts $p \in M \smallsetminus C$. Thus the distinguished line field is well-defined and $M \smallsetminus C$ is a globally 1-QC manifold. □

Lemma 2.6. — *The Weyl tensor of M vanishes.*

Proof. — This is known for points of $M \smallsetminus C$ (see [5, 22]) and is classical for isotropic points. □

Let $\widehat{\xi}$ be a locally defined unit vector field which determines the line field ξ. Moreover, we denote by the same letter $\widehat{\xi}$ an arbitrary extension (not necessarily continuous) to all of M.

Lemma 2.7. — *If M is a compact 1-QC manifold then the functions H and N have a canonical smooth extension to M.*

Proof. — Let R denote the scalar curvature and *Ric* the Ricci curvature. We set then:

$$H(p) = N(p) = \frac{1}{n(n-1)}\, R(p), \quad \text{if } p \in C.$$

The formulas for the Ricci tensor from [5, 6] imply that

$$\mathrm{Ric}(X,Y) = ((n-2)H + N)g(X,Y) + (n-2)(N-H)g(X,\widehat{\xi})g(Y,\widehat{\xi})$$

at all points $p \in M \smallsetminus C$. Note that $g(X,\widehat{\xi})g(Y,\widehat{\xi})$ is well-defined on $M \smallsetminus C$ and independent on the choice of the vector field $\widehat{\xi}$. Nevertheless, this formula makes sense and holds at points $p \in C$, as well.

The Ricci operator has eigenvalues $(n-2)H + N$ and $(n-1)N$ with multiplicities $n-1$ and 1, respectively at points of $M \smallsetminus C$ and $(n-2)H + N = (n-1)N$ with multiplicity n at points of C.

The Schouten tensor S is from above:

$$S(X,Y) = \frac{H}{2}\, g(X,Y) + (N-H)g(X,\widehat{\xi})g(Y,\widehat{\xi}).$$

The eigenvalues of the Schouten operator are therefore $H/2$ and $N - H/2$, with multiplicities $n-1$ and 1, respectively at points of $M \smallsetminus C$ and $H/2 = N - H/2$ with multiplicity n at points of C.

According to Lemmas 2.4 and 2.6 the function G_m is smooth on M. An explicit calculation shows that:

$$\begin{aligned} G_1 &= N + (n-2)\min(H,N), \quad \text{for } n \geqslant 3, \\ G_2 &= 2N + (n-2)H + (n-4)\min(H,N) \\ &= \frac{R}{n-1} + (n-4)\min(H,N), \quad \text{for } n \geqslant 3, \\ G_3 &= 3N + 2(n-3)H + (n-6)\min(H,N), \quad \text{for } n \geqslant 4. \end{aligned}$$

If $n \neq 4$, since R, G_1 and G_2 are smooth, we derive that $\min(H,N)$ and then N and H are smooth functions on M. When $n = 4$, $G_1 + G_3 = 2H + 4N$ and $R/3 = 2N + 2H$ are smooth, so H, N and $\min(H,N)$ are smooth. □

Remark 2.8. — The above extensions of H and N are the unique continuous extensions to $\overline{M \smallsetminus C}$.

2.4. Conformal flatness. — Let M be a compact 1-QC manifold. We define the *horizontal distribution* $\mathscr{D}$ to be the orthogonal complement to the distinguished line field ξ. By Lemmas 2.5 and 2.3, the distribution $\mathscr{D}$ is well-defined on $M \smallsetminus C$. One important ingredient used in this paper is the following easy observation (see also [5, 22]):

Lemma 2.9

(1) *1-QC manifolds of dimension $n \geqslant 4$ are conformally flat.*

(2) 1-*QC manifolds of dimension* $n = 3$ *are conformally flat if and only if the horizontal distribution* $\mathscr{D}$ *is completely integrable on* $M \smallsetminus C$.

Proof. — This is already known when the dimension $n \geqslant 4$ because the Weyl tensor vanishes (see [5, p. 379] and [22]).

By the Weyl-Schouten theorem (see [42, §C, Th. 9]) the 3-manifold M is conformally flat iff the so-called Cotton-Schouten tensor vanishes, namely if:

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = 0.$$

Given $p \in M \smallsetminus C$, let U be a disk neighborhood of p within $M \smallsetminus C$. Define the 1-form η on U by $\eta(X) = g(\widehat{\xi|_U}, X)$, where $\widehat{\xi|_U}$ is a vector field generating $\xi|_U$. Then the horizontal distribution $\mathscr{D}$ is the kernel of η on U. The vanishing of the Cotton-Schouten tensor on U is equivalent to:

$$\begin{aligned} dH(X)g(Y,Z) - dH(Y)g(X,Z) + 2(N-H)d\eta(X,Y)\eta(Z) \\ + 2(N-H)\left(\eta(Y)(\nabla_X\eta)Z - \eta(X)(\nabla_Y\eta)Z\right) \\ + 2\left(d(N-H)(X)\eta(Y) - d(N-H)(Y)\eta(X)\right)\eta(Z) = 0. \end{aligned}$$

Using all possibilities for choosing X, Y, Z among the elements of a basis of the horizontal distribution $\mathscr{D}$ or $\widehat{\xi|_U}$ it follows that this is equivalent to the system of the following three equations:

$$\begin{aligned} dH &= \widehat{\xi|_U}(H)\eta; \\ \nabla_X\widehat{\xi|_U} &= \frac{\widehat{\xi|_U}(H)}{2(N-H)}\,X, \quad \text{if } X \in \mathscr{D}; \\ d\eta(X,Y) &= 0, \quad \text{if } X, Y \in \mathscr{D}. \end{aligned}$$

The first two equations are always verified by using the method of [6, 22] in all dimensions $n \geqslant 3$. The third one is equivalent to the fact that the horizontal distribution $\mathscr{D}$ is completely integrable on U. Thus $\mathscr{D}$ is integrable if M is conformally flat.

Conversely, if $\mathscr{D}$ is integrable, the Cotton-Schouten tensor vanishes on $M \smallsetminus C$, and hence on $\overline{M \smallsetminus C}$. On the other hand $M \smallsetminus \overline{M \smallsetminus C}$ coincides with the set $\mathrm{int}(C)$ of interior points of C, namely those points of C having an open neighborhood in M which is contained in C. By Schur's theorem $H = N$ are constant in any connected component of $\mathrm{int}(C)$ and in particular the Cotton-Schouten tensor vanishes. This proves that M is conformally flat. □

Remark 2.10. — One cannot dispose of the integrability condition for $\mathscr{D}$ when $n = 3$. In fact, any closed 3-dimensional nilmanifold is covered by the Heisenberg group, endowed with a left invariant Riemannian metric. Then the tangent space at each point is identified to the nilpotent Lie algebra generated by X, Y, ξ with relations $[X,Y] = \xi$, $[X,\xi] = [Y,\xi] = 0$. The horizontal distribution corresponds to the subspace generated by X, Y and is well-known to be non-integrable. However, the left invariant Riemannian metric on the Heisenberg group is globally 1-QC, as we can compute $H = -1/4$ and $N = 3/4$. This furnishes the typical example of a closed 1-QC 3-manifold which

is not conformally flat. In fact closed nilmanifolds are not conformally flat (see [23]) unless they are finitely covered by a sphere, a torus or $S^1 \times S^2$.

2.5. THE LEAF CURVATURE FUNCTION λ. — Let M be a 1-QC manifold, which is assumed to be conformally flat, when $n = 3$. Then the distribution $\mathscr{D}$ is integrable and hence defines a foliation of $M \smallsetminus C$. The leaves of the foliation will be called *curvature leaves*. According to [6, 22] every curvature leaf is a totally umbilical submanifold of M of intrinsic constant curvature equal to:

$$\lambda = H + \frac{\widehat{\xi}(H)^2}{4(N-H)^2}.$$

Observe that the right hand side above is independent on the choice of a local distinguished unit vector field $\widehat{\xi}$ which lifts ξ.

Before we proceed, recall that a real function f defined on a compact subset K of a manifold M is called *smooth* if for every point $p \in K$ there exists an open neighborhood $U \subset M$ containing p and a real smooth function f_U defined on U such that $f|_{K\cap U} = f_U|_K$.

LEMMA 2.11. — *Let M be a locally 1-QC manifold and ξ denote a distinguished line field extension to $\Omega \supset \overline{M \smallsetminus C}$. Then λ admits a well-defined smooth extension to $\overline{M \smallsetminus C}$.*

Proof. — For every point p of M we choose an open neighborhood $V \subset M$ which is contractible so that the distinguished line field ξ_V is defined and can be lifted to a vector field $\widehat{\xi}_V$. Let X_V be a unit vector field on V orthogonal to $\widehat{\xi}_V$. According to [6, (3.11)] and [22, (3.5)] we have:

$$\left.\frac{\widehat{\xi}_V(H)}{2(N-H)}\right|_p = g(\nabla_{X_V}\widehat{\xi}_V, X_V)|_p, \quad \text{for } p \in V \cap (M \smallsetminus C).$$

The function $(g(\nabla_{X_V}\widehat{\xi}_V, X_V))^2$ is smooth and well-defined on V, namely independent on the choice of the lift $\widehat{\xi}_V$. We set therefore

$$\lambda_V(p) = \begin{cases} \lambda(p), & \text{if } p \in V \cap (M \smallsetminus C); \\ H + (g(\nabla_{X_V}\widehat{\xi}_V, X_V))^2, & \text{if } p \in V \cap C, \end{cases}$$

where X_V is an arbitrary smooth unit vector field on V orthogonal to $\widehat{\xi}_V$. Then λ_V is a smooth function on V.

Consider a finite open covering $\{U_i\}$ of M by open sets as above. Let $\Omega \supset \overline{M \smallsetminus C}$ be an open set on which the distinguished vector field ξ is well-defined, as provided by Lemma 2.3. If U_i, U_j are two such open sets and $p \in U_i \cap U_j \cap \overline{M \smallsetminus C}$, then Lemma 2.3 implies that $\widehat{\xi}_{U_i}|_p = \pm\widehat{\xi}_{U_j}|_p$, so that

$$(g(\nabla_{X_{U_i}}\widehat{\xi}_{U_i}, X_{U_i}))^2|_p = (g(\nabla_{X_{U_j}}\widehat{\xi}_{U_j}, X_{U_j}))^2|_p, \quad \text{if } p \in U_i \cap U_j \cap \overline{M \smallsetminus C}.$$

Therefore the collection λ_{U_i} defines a smooth extension $\lambda : \overline{M \smallsetminus C} \to \mathbb{R}$. □

REMARK 2.12. — Lemma 2.11 shows that $\lambda(p)$ remains bounded as p approaches C.

Remark 2.13. — Recall also (see [6, 22]) that

$$|\widehat{\xi}(H)| = \|\operatorname{grad} H\|$$

and the function λ is constant on the level hypersurfaces of H.

2.6. Isometric immersions of 1-QC manifolds. — One important tool which will be used throughout this paper is the codimension-one immersability of simply connected 1-QC manifolds. Let $\mathbb{H}^m_\kappa$ denote the m-dimensional hyperbolic space of constant sectional curvature $-\kappa$, when $\kappa > 0$ and the Euclidean space when $\kappa = 0$, respectively. For a 1-QC manifold M^n we denote

$$\kappa = \begin{cases} 1 - \inf_{p\in M} H(p), & \text{when } \inf_{p\in M} H(p) \leqslant 0; \\ 0, & \text{when } \inf_{p\in M} H(p) > 0. \end{cases}$$

The main result of this section is:

Proposition 2.14. — *Assume that M^n, $n \geqslant 3$, admits a 1-QC metric which is conformally flat when $n = 3$. Then there exists an equivariant isometric immersion, namely an isometric immersion $f : \widetilde{M} \to \mathbb{H}^{n+1}_\kappa$, and a group homomorphism $\rho : \pi_1(M) \to \operatorname{Iso}(\mathbb{H}^{n+1}_\kappa)$ such that*

$$f(\gamma \cdot x) = \rho(\gamma) \cdot f(x), \quad x \in \widetilde{M},\ \gamma \in \pi_1(M),$$

where the action on the right side is the action by isometries of $\operatorname{Iso}(\mathbb{H}^{n+1}_\kappa)$ on $\mathbb{H}^{n+1}_\kappa$ while the left side action is by deck transformations on the universal covering.

Proof. — The universal covering $\widetilde{M}$ is a 1-QC manifold with the induced metric by the covering $\pi : \widetilde{M} \to M$.

Every $p \in M \smallsetminus C$ has an open contractible neighborhood such that the line field ξ can be lifted to a vector field $\widehat{\xi}_U$. We derive a smooth tensor $h'_U(X,Y) = g(\widehat{\xi}_U, X)g(\widehat{\xi}_U, Y)$ on U. If $p \in U \cap V$ then $\widehat{\xi}_U|_p = \pm\widehat{\xi}_V|_p$ and thus $h'_U(X,Y)|_p = h'_V(X,Y)|_p$, for any $p \in U \cap V$. Thus the collection h'_U, where $\{U\}$ is a covering of $M \smallsetminus C$ defines a smooth tensor field h' on $M \smallsetminus C$.

Define now the following smooth symmetric (1,1) tensor field on M pointwise, by means of the formula:

$$h(X,Y)|_p = \begin{cases} \sqrt{H+\kappa} \cdot g(X,Y)|_p + \dfrac{N-H}{\sqrt{H+\kappa}} \cdot h'(X,Y)|_p, & \text{if } p \in M \smallsetminus C; \\ \sqrt{H+\kappa} \cdot g(X,Y)|_p, & \text{if } p \in C, \end{cases}$$

and also denote by h the pull-back to $\widetilde{M}$.

One verifies immediately the Gauss and Codazzi-Mainardi equations associated to the tensor h, namely:

$$\begin{aligned} R(X,Y,Z,W) = -\kappa\,(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)) \\ + h(Y,Z)h(X,W) - h(X,Z)h(Y,W), \\ \nabla_X h(Y,Z) = \nabla_Y h(X,Z). \end{aligned}$$

This follows from the second Bianchi identity of the curvature tensor when $n \geqslant 4$, and from the vanishing of the Cotton-Schouten tensor in dimension 3. The second equality is treated as in the proof of Lemma 2.9 by noticing that only the last condition discussed there has to be verified, the other ones being automatic.

Therefore, by the fundamental theorem for submanifolds of constant curvature space, (see [52, Chap. 7, §C, Th. 21]) $\widetilde{M}$ admits an isometric immersion $f : \widetilde{M} \to \mathbb{H}^{n+1}_\kappa$, whose second fundamental form is h.

The second fundamental form h on $\widetilde{M}$ is invariant by the action of the deck group, by construction. This means that there exist open neighborhoods of x and $\gamma \cdot x$ which are isometric. Let $\rho(\gamma)_x$ be the global isometry sending one neighborhood onto the other. Now $\rho(\gamma)_x$ is locally constant, as a function of x, and hence it is independent on x. Furthermore $\rho(\gamma)$ is uniquely defined by the requirement to send $n+1$ points x_i in general position in $\mathbb{H}^{n+1}_\kappa$, which belong to a small open neighborhood of x into their respective images $\gamma \cdot x_i$. Therefore, if the image of the immersion is not contained in a hyperbolic hyperplane then $\rho(\gamma)$ is unique and this also implies that ρ is a group homomorphism. In the remaining case M is hyperbolic and the result follows again. □

Remark 2.15. — The immersability result holds also when M^n is not necessarily compact, provided $\inf_{p\in M} H(p)$ exists.

Contrasting with the 2-dimensional case, but in accordance with Schur's theorem, the space of such isometric immersion is discrete. The first and simplest case is:

Proposition 2.16. — *Suppose that $n \geqslant 4$. If κ is such that $H + \kappa > 0$ on M, then there exists a unique isometric immersion $f : \widetilde{M} \to \mathbb{H}^{n+1}_\kappa$, up to an isometry (possibly reversing the orientation) of $\mathbb{H}^{n+1}_\kappa$.*

Proof. — If h is the second fundamental form of such an immersion, then linear algebra computations show that the Gauss equation already determine h up to a sign. A hyperbolic isometry reversing orientation will change the sign of h. Now an isometric immersion is uniquely determined by its second fundamental form. □

Remark 2.17. — The same argument shows that there does not exist any isometric immersion in $\mathbb{H}^{n+1}_{\kappa'}$ around a point where $H + \kappa' \leqslant 0$.

Conversely we have:

Proposition 2.18. — *If the conformally flat manifold M^n, $n \geqslant 4$, admits an equivariant isometric immersion $f : \widetilde{M} \to \mathbb{H}^{n+1}_\kappa$, and a group homomorphism $\rho : \pi_1(M) \to \mathrm{Iso}(\mathbb{H}^{n+1}_\kappa)$ such that*

$$f(\gamma \cdot x) = \rho(\gamma) \cdot f(x), \quad x \in \widetilde{M},\ \gamma \in \pi_1(M),$$

then M is 1-QC.

Proof. — According to Cartan (see [10] and [42, Th. 4]) at each point $p \in \widetilde{M}$ there exists a principal curvature μ (i.e., an eigenvalue of the second fundamental form) of the immersed manifold of multiplicity at least $n-1$. Let $\mathscr{N}$ be the open set of non-umbilical points, namely those for which μ has multiplicity exactly $n-1$. The distribution $\mathscr{D}$ is the eigenspace for the shape operator corresponding to the multiple eigenvalue. From Gauss' equations we obtain that $\mathscr{N}$ is globally 1-QC, with distinguished line field given by the normal line field to $\mathscr{D}$. The complement of $\mathscr{N}$ is totally umbilical in $\mathbb{H}^{n+1}_\kappa$ and hence it has constant curvature. Since the immersion is equivariant these properties descend to M. □

Remark 2.19. — Note that the statement of Proposition 2.18 is not true when $n = 3$ as Cartan's theorem does not extend to $n = 3$ (see [42]).

3. The topology of locally 1-QC manifolds

3.1. Globally 1-QC manifolds with orientable line field. — The purpose of this section is to prove that the globally 1-QC manifolds have a rather simple topological structure. Throughout this section M is a conformally flat 1-QC manifold of dimension $n \geqslant 3$. Let E be a connected component of $M \smallsetminus C$. By Lemma 2.5 the open set $M \smallsetminus C$ is globally 1-QC. In order to have a uniform approach we denote by $\widehat{E}$ the 2-fold cover determined by the first Stiefel-Whitney class of $\xi|_E$ when $\xi|_E$ is non-orientable, and E itself otherwise.

Moreover, $M \smallsetminus C$ is endowed with a smooth foliation $\mathscr{D}$, which is orthogonal to the line field ξ. Following [48] the leaves of the foliation $\mathscr{D}$ will be called *curvature leaves*, as the principal curvatures are constant along them.

Proposition 3.1. — *If $\widehat{E}$ is closed then it is diffeomorphic to a fibration over S^1. If $\widehat{E}$ is non-compact then $\widehat{E}$ is diffeomorphic to $\widehat{P} \times \mathbb{R}$, where $\widehat{P}$ is a closed space form.*

Corollary 3.2. — *If E is a connected component of $M \smallsetminus C$ then $\lambda(p)$ is either positive, or negative, or else vanishes for all $p \in E$.*

Proof. — With respect to the pull-back metric to the cover the slices $\widehat{P} \times \{a\}$ of $\widehat{E}$ have constant curvature λ. But a closed manifold $\widehat{P}$ cannot support constant curvature metrics with different signs. In fact $\pi_1(\widehat{P})$ is either hyperbolic and not virtually cyclic, virtually abelian or finite, according to whether λ is negative, null or positive. □

The main step is to establish the following:

Proposition 3.3. — *Leaves of the curvature foliation on $\widehat{E}$ are compact.*

Proof of Proposition 3.1 *assuming Proposition* 3.3. — The strong form of the global Reeb stability theorem states that a codimension-one transversely orientable foliation whose leaves are all compact is locally a fibration and the holonomy groups are finite (see [43, 18]). □

Outline of the proof of Proposition 3.3. — We use a theorem of Reckziegel to prove that the foliation has complete leaves with respect to the induced metric in Section 3.2. Since H is constant along the curvature leaves we derive that those leaves on which the value of H is non-critical should be compact in Section 3.5. This can be improved to those leaves which can be approached by non-critical leaves by using a Theorem of Haefliger (Section 3.3) which states that the union of closed leaves is closed. It remains then to analyze the leaves for which H is constant on an open neighborhood. Such a maximal saturated open set on which H is constant admits a metric completion whose general structure was described by Dippolito (see Section 3.4). We will first show that the completion is transversely compact and further that the connected components of its boundary are compact, as they can be approached by non-critical and hence compact leaves, in Section 3.6. We derive then that the completion is a foliated product and hence all leaves are compact.

3.2. Completeness of curvature leaves. — Let $f : N \to P$ be an isometric immersion of the Riemannian manifold N into a Riemannian manifold P with constant curvature and dimension strictly larger than that of N. Set $\nu(f)$ for the normal bundle of f, $\nu^*(f)$ for its dual and $\mathrm{II} : TN \times TN \to \nu(f)$ for the second fundamental tensor of f. The shape operator $A_\eta : TN \to TN$ is defined by the identity $g(A_\eta X, Y) = g_P(\mathrm{II}(X,Y), \eta)$, for all sections X, Y of TN and η of $\nu(f)$, where g_P and g are the metric tensors on TP and $\nu(f)$, respectively.

A *principal curvature function* is a continuous section μ of $\nu^*(f)$ for which the eigenspace $\mathscr{E}(\mu(p))$ has dimension $\dim \mathscr{E}(\mu(p)) \geqslant 1$ for each $p \in N$, where we put

$$\mathscr{E}(\mu(p)) = \{v \in T_pN \ ; \ A_\eta v = \mu(\eta)v, \text{ for any } \eta \in \nu_p(f)\}.$$

Let U be an open subset of N for which $\dim \mathscr{E}(\mu(p))$ is constant, say equal to k, for every $p \in U$.

A key ingredient of the proof of Proposition 3.3 is the following result of Reckziegel ([48, Th. 1.1, Rem. 1, 2]).

Theorem 3.4

(1) *The principal curvature function μ is smooth on U.*

(2) *The vector spaces $\mathscr{E}(\mu(p))$, for $p \in U$, form a subbundle $\mathscr{E}(\mu)$ of $TN|_U$ which is integrable.*

(3) *If L is a leaf of the foliation defined by the subbundle $\mathscr{E}(\mu)$ then L is a totally umbilical submanifold of N of dimension k and $f|_L : L \to P$ is a totally umbilical immersion.*

(4) *If $k \geqslant 2$, for any geodesic $c : [0, \ell) \to L$ with $\ell < \infty$ for which $\lim_{s \to \ell} c(x) = q$ exists in N, then $\dim \mathscr{E}(\mu(q)) = k$.*

(5) *If $k \geqslant 2$, N is complete and U is the subset of those points $p \in N$ for which $\dim \mathscr{E}(\mu(p))$ is minimal (U is open since this $\dim \mathscr{E}(\mu(p))$ is upper semicontinuous), then every leaf L of the foliation of U defined by $\mathscr{E}(\mu)$ is a complete metric space.*

Consider now a conformally flat 1-QC manifold M of dimension $n \geqslant 3$. Proposition 2.14 shows that there exists an equivariant isometric immersion $f : \widetilde{M} \to \mathbb{H}^{n+1}_\kappa$. Set $\pi : \widetilde{M} \to M$ for the universal covering projection. Observe now that one principal curvature μ of the immersion f is of multiplicity $n-1$ precisely along the open set $U = \pi^{-1}(M \smallsetminus C)$. Moreover, the pull-back $\pi^*\mathscr{D}$ of the distribution $\mathscr{D}$ by π coincides with the subbundle $\mathscr{E}(\mu)$ on U.

Reckziegel's Theorem 3.4 in the case $N = \widetilde{M}$ and $P = \mathbb{H}^{n+1}_\kappa$ therefore has the following corollary.

Lemma 3.5. — *The leaves of the foliation determined by $\pi^*\mathscr{D}$ on the subset $U = \pi^{-1}(M \smallsetminus C)$ of $\widetilde{M}$ are complete.*

3.3. Haefliger's theorem about the set of compact leaves. — One more ingredient needed in the sequel is the following theorem due to Haefliger (see [28, Th. 3.2, p. 386]).

Theorem 3.6. — *Let V be a connected manifold with countable basis such that the rank of $H_1(V;\mathbb{Q})$ is finite. Consider a codimension-one foliation of V. Then the union of closed leaves of the foliation is a closed subset of V. Moreover, if V is also an open subset of a compact manifold, then the union of compact leaves of the foliation is a closed subset of V.*

Remark 3.7. — The proof given by Haefliger works as well under the assumption that $H_1(V;\mathbb{Z}/2\mathbb{Z})$ is finite.

3.4. Open saturated subsets after Dippolito. — The structure of open saturated subsets U of a foliated manifold W was described by Dippolito ([17]) in the case when W is compact and it was extended to relatively compact saturated subsets U of open manifolds in [7]. Specifically, let $\widehat{U}$ denote the completion of U with respect to the induced Riemannian metric from W. Note that, in general, $\widehat{U}$ is non-compact. Then the inclusion $i : U \to W$ induces an immersion $\widehat{i} : \widehat{U} \to W$, by extending i to limit points of trajectories of a transverse foliation.

Dippolito's nucleus Theorem still holds in the open case as explained in [7, §4]:

Proposition 3.8. — *There exists a compact connected manifold with piecewise smooth boundary $K \subset \widehat{U}$ called a* nucleus *and finitely many $V_i \subset \widehat{U}$ such that:*

(1) *$\widehat{U} = K \cup V_1 \cup \cdots \cup V_r$;*

(2) *V_i is diffeomorphic to $B_i \times [0,1]$, where $B_i \times \{0\}$ and $B_i \times \{1\}$ are complete connected submanifolds of the part of $\partial\widehat{U}$ which is tangent to $\mathscr{D}$;*

(3) *for each $z \in B_i$, $\{z\} \times [0,1]$ is a trajectory of $\widehat{\xi}$;*

(4) *if $K \neq \varnothing$, then $V_i \cap K$ is non-empty and connected;*

(5) *if $i \neq j$ then $V_i \cap V_j = \varnothing$;*

(6) *if L is a component of $\partial\widehat{U}$ which is tangent to $\mathscr{D}$ and K is non-empty, then $K \cap L$ is non-empty and connected.*

The nucleus is not unique in general. If we can take $K = \varnothing$, then $\widehat{U} = L \times [0,1]$ is a *foliated product*, foliated by the leaves of the foliation diffeomorphic to L and also transversally by closed intervals.

We will need in the sequel the following result stated by Dippolito [17, Prop. 2] in the compact case and by Cantwell and Conlon [7, Lem. 4.2] in the open case.

Lemma 3.9. — *Let $U \subset W$ be an open connected saturated set. Then $\partial\widehat{U}$ has only finitely many connected components tangent to $\mathscr{D}$.*

Note that the number of such boundary components is bounded by twice the number of so-called biregular charts needed to cover $\overline{U}$, which is finite since $\overline{U}$ is compact (see [17]).

3.5. Proof of Proposition 3.3 for everywhere non-constant H. — The function $H : M \to \mathbb{R}$ is smooth and proper, because M is compact. Then, its critical values are nowhere dense and the preimage of any non-critical value is a codimension-one compact submanifold of M. A curvature leaf L will be called *non-critical* if $H(L)$ is a non-critical value. We want to stress that a curvature leaf L is contained in $M \smallsetminus C$.

Lemma 3.10. — *A non-critical curvature leave L is a connected component of $H^{-1}(H(L))$, and in particular it is compact.*

Proof. — Recall that H is constant on curvature leaves. This follows from [6, (3.5)] and [22, (3.4-3.5)(i)], which state that

$$X(H) = 0, \quad \text{for } X \in \mathscr{D}.$$

If $\pi : \widetilde{M} \to M$ is the universal covering projection, then each component of the preimage $\pi^{-1}(L)$ of L in $\widetilde{M}$ is complete, following Reckziegel's Lemma 3.5. This implies that the leaf L itself should be geodesically complete, as otherwise we could adjoin additional limit points to $\pi^{-1}(L)$ contradicting its completeness. Thus L is complete with respect to its intrinsic metric.

As a word of warning the projection π is not a closed map, if the covering has infinite degree. In particular, L is not necessarily a complete subspace of M and in particular it might not be a closed subset of M.

If $H(L)$ is a non-critical value of H then $H^{-1}(H(L))$ is a smooth hypersurface of $M \smallsetminus C$; since H is proper, $H^{-1}(H(L))$ is closed. Then L is a codimension zero submanifold of the closed manifold $H^{-1}(H(L))$. The level hypersurface $H^{-1}(H(L))$ has an induced Riemannian metric from M whose restriction to its open subset L coincides with the intrinsic metric of L. Therefore L is a complete subspace of the compact space $H^{-1}(H(L))$ and hence it has also to be closed. This implies that L must be a connected component of the level hypersurface $H^{-1}(H(L))$ and hence compact. □

Note that, as $\mathscr{D}$ is an integrable distribution on $M \smallsetminus C$, all curvature leaves, in particular critical leaves, are nonsingular.

Suppose now that H is *everywhere non-constant*, namely that there is no open subset $U \subset M \smallsetminus C$ for which the restriction of H to U is constant. We want to apply Haefliger's result to the foliation by curvature leaves of the open manifold $M \smallsetminus C$. Assume that the set of non-compact leaves is non-empty. By Lemma 3.6 the union U of non-compact leaves is an open subset of $M \smallsetminus C$. Let $p \in U$, so that L_p is a non-compact leaf and $H(L_p) = c$ is a critical value of H. Consider a germ of an integral curve $\gamma \subset U$ of the line field ξ. Then the restriction of H to γ is non-constant, as otherwise the union of leaves intersecting γ would contain an open set on which H is constant, contradicting the fact that H is everywhere non-constant. It follows that $H(\gamma)$ contains a small interval and hence c can be approached by a sequence of regular values $c_i \in H(\gamma)$. Take points $p_i \in \gamma$ with $H(p_i) = c_i$. Therefore the leaves L_{p_i} are non-critical leaves and hence compact leaves by above. This contradicts the fact that U is the union of non-compact leaves. This ends the proof of Proposition 3.3 for the case of everywhere non-constant H.

3.6. Proof of Proposition 3.3 for generic but not everywhere non-constant H

By the previous Section 3.5 it remains to show that leaves passing through an open set on which H is constant are also compact. Let $p \in M$ be a point such that H is constant on an open neighborhood U of p in $M \smallsetminus C$. Since the union of leaves intersecting an open set is an open set we can suppose that U is saturated, namely union of leaves. Moreover, as H is smooth, H is constant on the closure $\overline{U}$. Curvature leaves L_p arising in this way will be called *super-critical*.

Lemma 3.11. — *A super-critical curvature leaf L_p is totally geodesic.*

Proof. — Since H is constant in a neighborhood of p, we have $dH|_p = 0$. Since $p \in M \smallsetminus C$, we can write $dH = \widehat{\xi}(H)\eta$, for some locally defined vector field $\widehat{\xi}$ lifting the line field ξ. Thus $\nabla_X \widehat{\xi} = \alpha X = (\widehat{\xi}(H)/(N-H))X$ vanishes at p, for any horizontal vector field X. In particular $\alpha(p) = 0$ and $\lambda(p) = H(p)$, according to the proof of Lemma 2.11. Since λ is constant on the level hypersurfaces of H it follows that $\lambda(q) = H(q) = c_0$, for any $q \in L_p$ and thus the curvature leaf L_p is totally geodesic. □

Let E be the connected component of $M \smallsetminus C$ containing p, $\widehat{E}$ the cover of E determined by $w_1(\xi|_E)$ and $\widehat{\xi}$ be a vector field lift of the line field. The cover $\widehat{E}$ is endowed with a pull-back metric and a completely integrable distribution still denoted g and $\mathscr{D}$, respectively. We keep the notation U for the lift of U and work henceforth on $\widehat{E}$ in this section. Note that $\widehat{E}$ is contained in $\widehat{\Omega}$ and has compact closure there.

We want to construct a *compact* submanifold $N \subset \widehat{E}$ containing U which is foliated by curvature leaves. Possibly enlarging U we can assume that U is the *maximal* open connected saturated subset of E containing p such that $H|_U = H(p)$. Note that U is the connected component of $\mathrm{int}(H^{-1}(H(p)) \cap \widehat{E})$ containing p. Since H is not eventually constant, the closure $\overline{U}$ in M is contained in $\widehat{E}$, so that U is relatively compact in $\widehat{E}$.

For every $p \in U$ we consider the *flow* $\varphi_p(t)$ determined by $\widehat{\xi}$ with initial condition $\varphi(0) = p$. Then $\varphi_p(t)$ is not necessarily defined on all of $\mathbb{R}$, since ξ was only defined on an open neighborhood Ω of $\overline{M \smallsetminus C}$. Let $I_p \ni 0$ be the maximal interval consisting of $t \in \mathbb{R}$ for which $\varphi_p(t) \in \widehat{E}$. Set further $J_p \subset I_p$ for the maximal interval on which $H \circ \varphi_p$ is constant, namely such that $0 \in J_p$ and all $t \in J_p$ satisfy $H(\varphi_p(t)) = H(p)$. Then J_p is non-empty, because it contains the connected component of 0 in $\varphi_p^{-1}(U)$.

Lemma 3.12. — *The maximal interval J_p on which $H \circ \varphi_p$ is constant is a finite proper sub-interval of I_p.*

Proof. — If I_p has a finite supremum (or infimum), then $\varphi_p(t)$ reaches C in finite time. In this case $\sup J_p < \sup I_p$. Otherwise H would be constant on the set U', which is the union of all leaves intersecting $\varphi_p(I_p)$ (actually a small open neighborhood of it), while $\overline{U'} \cap C \neq \varnothing$. This would contradict the genericity of H.

If I is unbounded, say it contains $[0, \infty)$, we have two cases: either the forward limit set $\omega(\varphi_p)$ intersects C, or else $\omega(\varphi_p) \subset \widehat{E}$. Recall that the *forward limit* (or *ω-limit*) set of the flow φ_p is the set of points obtained as limits of sequences $\varphi_p(s_i)$, where $s_i \to \infty$. In particular, $\omega(\varphi_p)$ is a compact set invariant by the flow.

In the first case, if $\sup J_p = \infty$, then again H is constant on the union U' of leaves intersecting $\varphi(I_p)$, while $\overline{U'} \cap C \neq \varnothing$, contradicting the genericity of H.

Consider now the second case, when $\omega(\varphi_p) \subset \widehat{E}$ and assume that $J_p \supset [0, \infty)$. Recall that $H|_{\widehat{E}}$ is non-constant and thus there exist non-critical values $c \neq H(p)$. Let then L_q be a non-critical leaf through a point $q \in E$, with $H(L_q) = c$. By Lemma 3.10, L_q is a compact leaf.

Further $\omega(\varphi_p) \cap L_q = \varnothing$, since $H(\omega(\varphi_p)) = H(p)$ and $H(L_q) = c \neq H(p)$. Since both sets are compact, the distance between $\omega(\varphi_p)$ and L_q is non-zero. Let then $\beta : [0, 1] \to \widehat{E}$, with $\beta(0) \in \omega(\varphi_p)$, $\beta(1) \in L_q$ be a minimal geodesic in M realizing the distance between them. Since $\omega(\varphi_p)$ is invariant by the one-parameter flow generated by $\widehat{\xi}$ the integral orbit $\mathscr{O}$ of $\beta(0)$ under this one-parameter flow is contained in $\omega(\varphi_p)$. Standard variational arguments show that β must be orthogonal to both the orbit $\mathscr{O}$ and the leaf L_q, namely that

$$g(\dot{\beta}(0), \widehat{\xi}|_{\beta(0)}) = 0, \quad g(\dot{\beta}(1), \mathscr{D}|_{\beta(1)}) = 0.$$

The first equation means that $\dot{\beta(0)}$ is tangent to the leaf $L_{\beta(0)}$ passing through $\beta(0) \in \widehat{E}$.

Write then $\beta(0) = \lim \varphi_p(s_i)$, where $s_i \to \infty$. All leaves $L_{\varphi_p(s_i)}$ are super-critical (by our assumption on J) and hence $\alpha(x) = 0$, for every $x \in L_{\varphi_p(s_i)}$. Since α is smooth, we derive that $\alpha(x) = 0$ for all $x \in L_{\beta(0)}$, which means that $L_{\beta(0)}$ is a totally geodesic leaf.

Eventually, a minimal geodesic in M whose direction is tangent to a totally geodesic leaf must be contained in that leaf. On the other hand the second equation above shows that β is orthogonal to the leaf L_q at $\beta(1)$, which is a contradiction. □

Note that Lemma 3.12 already shows that $\widehat{U}$ is foliated transversely by closed intervals.

Lemma 3.13. — *Every component of $\delta(\widehat{U}) = \widehat{i}(\partial\widehat{U})$ is a compact leaf.*

Proof. — The image $\widehat{i}(\widehat{U})$ of the completion $\widehat{U}$ is the subset of $\widehat{\Omega}$ obtained by completing every transverse trajectory of $\widehat{\xi}$, namely replacing every open interval $\varphi_p(J_p)$ by the corresponding closed interval (or circle) $\varphi_p(\overline{J_p})$.

In general, one can have part of the border tangent to the foliation $\mathscr{D}$ and part of it tangent to some trajectory of $\widehat{\xi}$ (see [7, §4]). However, the second case cannot occur here because $\widehat{E}$ is relatively compact within $\widehat{\Omega}$, more precisely it is contained in the pull-back of $\overline{E}$ to the double cover $\widehat{\Omega}$. Thus every point of $\delta(\widehat{U})$ is of the form $\varphi_p(\sup J_p)$, or $\varphi_p(\inf J_p)$, for some $p \in U$ and $\delta(\widehat{U})$ is also the union of the corresponding curvature leaves passing through these points.

As $\overline{J_p}$ is a closed proper interval within I_p, the restriction $H|_{\varphi_p([\sup J_p, \sup J_p + \varepsilon])}$ is not constant and hence there exists a sequence of non-critical values of the form $H(\varphi_p(\sup J_p + t_i))$ approaching $H(p)$, for some decreasing $t_i \to 0$. Since non-critical leaves are compact, Haefliger's Theorem 3.6 implies that the leaves $L_{\varphi_p(\sup J_p)}$ and $L_{\varphi_p(\inf J_p)}$ are compact. □

There are only finitely many components of $\partial\widehat{U}$ and hence $\delta(\widehat{U})$, by Lemma 3.9. It follows that $\widehat{U}$ is a compact manifold. Now, every pair of points $p, q \in \partial\widehat{U}$ having the same image by $\widehat{i}$, give raise to two leaves which are tangent at $\widehat{i}$ so that they should coincide. Therefore $N = \widehat{i}(\widehat{U})$ is a codimension zero compact submanifold of $\widehat{E}$. Specifically, N is the closure of the connected component of $\mathrm{int}(H^{-1}(H(p)) \cap \widehat{E})$ containing p.

Proposition 3.3 will then follow from the following:

Proposition 3.14. — *The submanifold N is diffeomorphic to a cylinder foliated as a product.*

Proof. — Lemmas 3.12 and 3.13 and Dippolito's description from Section 3.4, show that U has an empty nucleus, and hence $\widehat{U}$ is a foliated product by closed intervals.

Here is a direct proof. Each component of ∂N has a sign determined by the fact that $\widehat{\xi}$ points either inward or outward with respect to N. This is well-defined because $\widehat{\xi}$ is a smooth vector field on $\widehat{E}$ orthogonal to ∂N, so that if $\widehat{\xi}$ points inward at some point of ∂N then it points inward at all points of its connected boundary component.

Let $\partial^+ N$ be the union of those components of ∂N for which $\widehat{\xi}$ is inward pointing and $\partial^- N$ be its complementary. Observe that N cannot be a closed manifold without boundary since it is contained in $\widehat{E}$. By the symmetry of the situation we can assume that $\partial^+ N$ is non-empty. We claim now that:

Lemma 3.15. — *Every integral trajectory of $\widehat{\xi}$ starting at a point $x \in \partial^+ N$ should cross $\partial^- N$.*

Proof. — Assume the contrary. As the trajectory cannot cross again $\partial^+ N$, necessarily in outward direction, because this would contradict the choice of $\partial^+ N$, it will remain forever in N. Thus its forward limit set is a compact invariant subset of N. The argument used in the proof of Lemma 3.12 shows that this is impossible. □

Let then denote by $\ell(x)$ the length of the integral trajectory φ_x of $\widehat{\xi}$ between x and its endpoint on $\partial^- N$. We consider then $\phi : \partial^+ N \times [0,1] \to N$ defined by:

$$\phi(x,t) = \varphi_x \left(t/\ell(x) \right).$$

Then ϕ is a diffeomorphism on its image. The only obstruction for the surjectivity of ϕ is the existence of closed trajectories of $\widehat{\xi}$. Again, the argument used in the proof of Lemma 3.12 shows that there are no closed orbits of $\widehat{\xi}$ contained in $\mathrm{int}(N)$. This proves Proposition 3.14. □

3.7. Another proof of Proposition 3.3. — We first recall that two points of a foliated manifold are *Novikov equivalent* if they belong to the same leaf or to a closed transversal. An equivalence class of points is called a *Novikov component* of the foliation. The following classical result is due to Novikov.

Lemma 3.16 ([45]). — *Assume that we have a foliation of a compact manifold. Then every Novikov component is either a compact leaf which does not admit any closed transversal or else an open codimension zero submanifold whose closure has finitely many boundary components consisting of compact leaves.*

An alternative way to complete the proof of Proposition 3.3, which does not use Proposition 3.14 is as follows. If H is generic and not everywhere non-constant, we proved above that we have an induced foliation of the compact submanifold $N \subset \widehat{E}$. If some super critical leaf L_p were not compact, then L_p would be contained within some open Novikov component $V \subset N$. Recall that a Novikov component V is a saturated open submanifold such that every leaf contained in V is non-compact and approaches some boundary leaf of its closure $\overline{V}$. Moreover, from the proof of Lemma 3.11 all leaves in V, including the boundary ones in $\overline{V}$ should be totally geodesic. We are in position to apply now the following result of Carrière and Ghys ([9]) on totally geodesic foliations:

Lemma 3.17. — *The foliated manifold $\overline{V}$ is diffeomorphic to a cylinder foliated as a product.*

Proof. — The codimension-one foliation of $\overline{V}$ is transversely orientable and totally geodesic. According to [9, Prop. II.2] such a foliation with a compact leaf is a fibration over an interval. □

But this leads us to a contradiction, as we supposed V to be a Novikov component. This ends the proof of Proposition 3.3 in the case when H is generic and not everywhere non-constant.

Remark 3.18. — The arguments in the proof of Lemma 3.11 also show that critical leaves L_p for which $\nabla H|_p = 0$ are totally geodesic.

Remark 3.19. — The present proof does not extend to the case when H is eventually constant. There exist for instance totally geodesic foliations all whose leaves are non-compact and have constant curvature, as the ones described in [9] in dimension 3.

3.8. Uniform geometry of curvature leaves. — In this section we restrict ourselves to locally 1-QC manifolds. With notation from above we set:

$$\alpha_V(p) = g(\nabla_{X_V}\widehat{\xi}_V, X_V), \quad \text{for } p \in V \cap (M \smallsetminus C).$$

Then curvature leaves are totally umbilical with principal curvatures equal to α_V. Note that if $\xi|_{L_p}$ is orientable then the curvature leaf L_p is 2-sided and there exists a coherent choice of the lifts $\widehat{\xi}_V$ such that the value of $\alpha_V(x)$ is independent on V, for any $x \in L_p$. However, if $\xi|_{L_p}$ is non-orientable only the absolute value $|\alpha_V(x)|$ is well-defined globally, as the curvature leaf is 1-sided. Moreover, from [6, (3.5)] and [22, (3.4-3.5)(i)] $\alpha_V(x)$ and $|\alpha_V(x)|$ are constant along a curvature leaf L_p if $\xi|_{L_p}$ is orientable and non-orientable, respectively.

Recall that the *focal radius* of L at a point $p \in L$ is the smallest r for which $r\widehat{\xi}|_p$ is a critical point of the exponential map and the *focal radius* $f_{L\subset M}$ is the smallest focal radius at p among all $p \in L$.

Lemma 3.20. — *Let M be a compact locally 1-QC manifold. Then there exists a positive constant $\gamma = \gamma(A) > 0$ such that given a manifold M with curvature bounded by A any codimension 1 submanifold $L \subset M$ satisfying $\|\mathrm{II}_L\| \leqslant A$ has both injectivity radius i_L and focal radius $f_{L\subset M}$ (i.e., the distance to the closest focal point of L) bounded from below by γ.*

Proof. — The first claim is the content of [20, Lem. 3]. The second claim is well-known (see [21, Lem. 2.3]). An explicit lower bound for the focal radius of L was provided by Warner in [53, Cor. 4.2], namely the focal radius of the totally umbilical hypersurface of principal curvature $\min \alpha$ within the space of constant curvature $\max(H, N)$. □

For every $p \in M \smallsetminus C$ we consider the metric disk $D_\gamma(p) \subset L_p$ of radius γ within the leaf L_p endowed with the induced metric. Let $i_{p,\gamma}$ denote the *normal injectivity radius* of $D_\gamma(p)$ inside M, i.e., the largest r such that the restriction of the exponential map to the radius r disk subbundle of the normal bundle of L_p (restricted to $D_\gamma(p)$) provides a diffeomorphism onto its image.

We say that an arc ζ joining $p, q \in L$ is an *orthogeodesic* if it is a minimal geodesic in M between p and q which is orthogonal to L at its endpoints. We denote by $o_{D_\gamma(p)\subset M}$ the smallest orthogeodesic length between points of $D_\gamma(p)$. It follows from [29, Th. 4.2] that the normal injectivity radius $i_{p,\gamma}$ is given by

$$i_{p,\gamma} = \min(f_{D_\gamma(p)\subset M}, \tfrac{1}{2} o_{D_\gamma(p)\subset M}).$$

Our next goal is to show that $i_{p,\gamma}$ is bounded from below. We will closely follow the reasoning of Corlette from [16].

Let d_L denote the Riemannian distance on the leaf L and d_M the Riemannian metric on M. We denote by $D_r^M(p) \subset M$ the radius r metric disk on M with center at p. It will be convenient for us to change the metric into a flat metric in a small neighborhood of a given point. The exponential map $\exp_p : T_pM \to M$ restricts to a diffeomorphism from the tangent metric ball $\{v \in T_pM \ ; \ \|v\| \leqslant \gamma\}$ onto $D_\gamma^M(p)$. Let $g^\flat$ denote the pull-back by $\exp_p^{-1}$ of the flat Riemannian metric on T_pM and $d_M^\flat$ the associated distance. According to [16, Lem. 2.1] and the lines after its proof p. 157, by choosing γ sufficiently small, we can arrange that the distorsion between $g^\flat$ and the initial metric g be uniformly bounded:

Lemma 3.21 ([16, Lem. 2.1]). — *For any $\lambda > 1$, $\delta > 0$ there exists some $R(\lambda, \delta)$ (smaller than the injectivity radius) with the property that:*

(1) *for any $p \in M$ the identity map between $(D_\gamma^M(p), d_M)$ and $(D_\gamma^M(p), d_M^\flat)$ is λ-Lipschitz, for any $\gamma \leqslant R(\lambda, \delta)$;*

(2) *the difference between the two Levi-Civita connections on these two metric balls is bounded by δ (in the $\mathscr{C}^0$ topology).*

In particular, if $J \subset D_\gamma^M(p)$ is a submanifold with $\|\mathrm{II}_J\| < A$ in the initial metric g, then $\|\mathrm{II}_J^\flat\| < C(A, \lambda, \delta)$ in the flat metric $g^\flat$, where $C(A, \lambda, \delta)$ is a continuous function depending only on A, λ, δ such that $\lim_{\lambda \to 1, \delta \to 0} C(A, \lambda, \delta) = A$.

The second ingredient needed is the following version of a theorem of Schwarz, as stated by Corlette in every dimension:

Theorem 3.22 ([16, Th. 2.3]). — *Let c be a curve in the Euclidean space joining the points q and q' and whose second fundamental form is bounded in norm by $1/r$, where r is at least half the distance between q and q'. Consider the circle of curvature $1/r$ which passes through q and q', which is separated by the pair of points into two arcs, say of lengths $\ell_1 \leqslant \ell_2$. Then, either the length of c is smaller than ℓ_1 or else it is larger than ℓ_2.*

We are ready to prove now:

Lemma 3.23. — *Let M be a compact locally 1-QC manifold. Then there exists some γ depending only on A and the sectional curvature bounds such that the normal injectivity radius $i_{p,\gamma}$ of curvature leaves in E is uniformly bounded from below.*

Proof. — Choose λ, δ such that $C(A, \lambda, \delta) < 2A$ and $\lambda < 1.1$. Further, change γ into $\min(\gamma, 1/2A, R(\lambda, \delta)/3)$.

Consider an orthogeodesic β of length $2t$ joining the points $q, q' \in D_\gamma(p)$. It is enough to consider that β is the shortest one, namely the midpoint u of β has $d_M(u, x) \geqslant t$, for every $x \in D_\gamma(p)$. Let c be a geodesic on L joining q and q', which is therefore contained in $D_\gamma(p) \subset D_\gamma^M(p)$. Observe that the second fundamental form of the geodesic c is bounded in norm by A. Now, $D_\gamma^M(p) \subset D_{R(\lambda,\delta)}^M(u)$.

Consider next the flat metric $g^\flat$ on $D^M_{R(\lambda,\delta)}(u)$. Then the metric sphere $\partial D^M_t(u)$ becomes an ordinary sphere of radius t in the Euclidean space. The second fundamental form of c in the flat metric $g^\flat$ is bounded by $2A$, by Lemma 3.21. The shortest arc of circle joining q to q' of curvature equal to $2A$ has length $\ell_1 = \arcsin(2At)/A$.

Suppose that $100t < \gamma$. Then $2At < 1/100$ and so $\ell_1 < 2.2\,t$. On the other hand the curve c lies outside the sphere of radius t and hence its length is at least the length of a great arc of circle joining q and q' on the sphere, namely πt.

From the Schwarz Theorem 3.22 it follows that the length of c must be at least the length $\ell_2 = (\pi/A) - \ell_1$ of the longest arc of circle joining q to q' of curvature equal to $2A$. Observe now that:

$$\ell_2 > \frac{\pi}{A} - 2.2\,t > \Big(\pi - \frac{1}{40}\Big)\frac{1}{A} > \frac{1.1}{A} > 2.2\,\gamma.$$

Therefore the curve c in the flat metric should have length greater than $2.2\,\gamma$, contradicting the fact that $d_L(q,q') \leqslant 2\gamma$ and the Lipschitz constant is $\lambda < 1.1$.

We derive that $t \geqslant \frac{1}{100}\,\gamma$ and hence

$$i_{p,\gamma} = \min(f_{D_\gamma(p)\subset M}, \tfrac{1}{2} o_{D_\gamma(p)\subset M}) \geqslant \frac{1}{100}\,\gamma. \qquad \square$$

Lemma 3.24. — *Let M be a compact locally 1-QC manifold. Then the curvature leaves have uniformly bounded metric distorsion at small scales: there exists some constant $B > 0$ such that for any leaf L and any two points $p, q \in L$ with $d_L(p,q) < \gamma$, we have $d_M(p,q) \geqslant B d_L(p,q)$.*

Proof. — This follows directly from Lemma 3.23 and Weinstein's estimate of the distorsion for submanifolds with normal injectivity radius bounded from below (see [54, Cor. 3.5]). $\square$

Lemma 3.25. — *Assume that M is locally 1-QC and the metric is generic.*

(1) *Let E be a connected component of $M \smallsetminus C$ with the property that $\lambda|_E$ is not identically zero. Then both the volume and the diameter of a curvature leaf $L \subset E$ are uniformly bounded from above and from below away from zero.*

(2) *Let E be a connected component of $M \smallsetminus C$ with the property that $\lambda|_E \equiv 0$. Then the volume of a curvature leaf $L \subset E$ is uniformly bounded from below away from zero.*

Proof. — By Proposition 3.1, E is diffeomorphic to an interval bundle over some closed space form P. If $\lambda > 0$ or $\lambda < 0$ and dimension $n \geqslant 4$, then constant curvature metrics on P are unique up to isometry. In particular, the volume of L is a constant times $\lambda^{-n/2}$ and the diameter is also a constant times $\lambda^{-1/2}$. As λ is bounded, the first claim follows. By Corollary 3.2 the only other possibility is that $\lambda \equiv 0$ on E. Moreover, if this is the case, then by Lemmas 3.20 the injectivity radius is bounded from below by γ and hence the volume of L is at least the volume of the Euclidean ball of radius γ. $\square$

The next step is to construct uniform cylindrical neighborhoods around point of M. For this purpose, for every $p \in M \smallsetminus C$ we now consider the relative metric neighborhood $N_\gamma(D_\gamma(p))$, which is the image of the set $\{(q, \rho\xi)\ ;\ q \in D_\gamma(p),\ |\rho| \leqslant \gamma\} \subset TM$ by the exponential map. Since the exponential map is a diffeomorphism on its image, the later is a relative regular neighborhood with respect to its boundary. Let now $q \in M$ at distance $\gamma/20$ from $D_\gamma(p)$ the point which belongs to the orthogonal geodesic θ_p to L_p issued from p in the direction given by a lift $\widehat{\xi}$.

Lemma 3.26. — *There exists some constants Q, R such that for $\gamma < R$ the metric disk $D^M(q, Q\gamma)$ is contained within $N_\gamma(D_\gamma(p))$.*

Proof. — We change the metric in $D^M(p, 4\gamma)$, with γ small enough, to the flat metric such that the second fundamental form is bounded by $2A$, using Lemma 3.23. We can also assume that the line field ξ is defined on $D^M(p, 4\gamma)$ and that in the flat metric the angle between $\xi|_p$ and $\xi|_y$ is smaller than $\pi/10$, for any $y \in D^M(p, 4\gamma)$. Let p' be a point of the metric sphere $\partial D_\gamma(p) \subset L_p$ and θ be the orthogonal geodesic arc of length γ issued from p'. By Lemma 3.23, $N_\gamma(D_\gamma(p))$ is diffeomorphic to a cylinder $D_\gamma(p) \times [0, \gamma]$ whose lateral surface corresponding to $\partial D_\gamma(p) \times [0, \gamma]$ is the union of all arcs $\theta_{p'}$. Since the second fundamental form of $D_\gamma(p)$ is bounded, we can suppose from Lemma 3.24, by passing to a small γ if needed, that $d_M(p, p') \geqslant Bd(p, p')$. In the flat metric the geodesics arcs θ_p and $\theta_{p'}$ are line segments and their angle is smaller than $\pi/10$. Then elementary geometry arguments show that $d_M(q, \theta_{p'}) > \frac{9}{10}\, d_M(p, \theta_{p'})$, $d_M(p, \theta_{p'}) \geqslant \cos(\pi/10)\, d_M(p, p')$ and so $d_M(q, \theta_{p'}) > \frac{B}{2}\, d(p, p')$. This implies that the distance between q and the lateral surface of $\partial N_\gamma(D_\gamma(p))$ is at least $\frac{B}{2}\,\gamma$. On the other hand a similar argument shows that the distance between q and the upper cap of the cylinder corresponding to $D_\gamma(p) \times \{\gamma\}$ is also at least $\frac{B}{2}\,\gamma$, which implies our claim for $Q = B/2$. □

3.9. Limit leaves for locally 1-QC manifolds

Definition 3.27. — A smooth *branched* hypersurface in M is a codimension 1 immersion $f : X \to M$ with the following properties:

(1) the source X is the union of closed manifolds, namely compact manifolds with empty boundary;

(2) f has no triple points;

(3) if Y is any (topological) connected component of the manifold X, then the restriction $f|_Y : Y \to M$ is an embedding;

(4) double points are tangencies between images of distinct connected components of X. Specifically, if $(x_1, x_2) \in X \times X$ and $f(x_1) = f(x_2)$, then there exist two distinct connected components $X_1, X_2 \subset X$ of X such that $x_1 \in X_1$, $x_2 \in X_2$ and $f(X_1)$ and $f(X_2)$ are tangent at $f(x_1) = f(x_2)$.

The connected components of X are called the *branches* at the source, their images in M are the branches of $f(X)$ and the set of double points in M is the *branch* set of f.

We extend this definition to smooth *branched hypersurfaces with self-tangencies* in M by allowing the restrictions $f|_Y : Y \to M$ to be immersions with self-tangencies instead of embeddings.

By abuse of language we will also refer to the image $f(X)$ as a branched hypersurface. With this convention the tangent space to a branched hypersurface is well-defined also at branch points.

Recall that any smooth hypersurface could be defined locally as the graph of some smooth function. This permits to define limits of hypersurfaces as follows.

Definition 3.28. — A sequence of compact hypersurfaces $H_n \subset M$ *converges* to a smooth hypersurface H_∞ if there is an open finite cover $\{U_i\}$ of M by cylinders $U_i = \mathrm{int}(D_i) \times (0,1)$ such that for every i and large enough n the hypersurfaces $\overline{U_i} \cap H_n$ are either empty or the graphs of some smooth functions $\psi_{i,n} : D_i \to [0,1]$ with the property that $\psi_{i,n}$ converges in the C^k-topology to $\psi_{i,\infty}$ for all $k \geqslant 2$.

Further, the sequence of submanifolds H_n *limits to* the finite union $\bigcup_{i=1}^m H_{\infty,i}$ of submanifolds $H_{\infty,i}$ if $\mathbb{Z}_+ = \bigcup_{i=1}^m J_i$ such that each sub-sequence $(H_n)_{n\in J_i}$ converges to $H_{\infty,i}$.

We know that the distribution $\mathscr{D}$ is integrable on $M \smallsetminus C$. Our aim is to show that a weaker version of integrability holds on the closure $\overline{M \smallsetminus C}$. Our main result in this section is:

Proposition 3.29. — *Let M be a locally 1-QC manifold, which is assumed to be conformally flat when $n = 3$ and let ξ denote a distinguished line field extension to an open set containing $\overline{M \smallsetminus C}$.*

(1) *Then for every $q \in \overline{M \smallsetminus C} \cap C$ there exists a limit leaf L_q for $\mathscr{D}$.*

(2) *If ξ is orientable then the limit leaf L_q passing through $q \in \overline{M \smallsetminus C} \cap C$ is a branched hypersurface. Moreover, each branch of the limit leaf L_q is a totally umbilical compact submanifold of constant intrinsic sectional curvature λ tangent to (the extension to $\overline{M \smallsetminus C}$ of) the horizontal distribution $\mathscr{D}$.*

(3) *When ξ is non-orientable, the limit leaf L_q is a branched hypersurface as above, possibly with self-tangencies.*

(4) *Every limit leaf L_q is contained within $\overline{M \smallsetminus C} \cap C$.*

Outline of the proof of Proposition 3.29. — We have to analyze the behaviour of leaves L_{p_i}, when p_i approach a point $q \in C \cap \overline{M \smallsetminus C}$. We first consider the problem locally. Using the bounded geometry of leaves from Section 3.8 we shall consider specific neighborhoods which are products of a disk in a leaf and an interval. The size of these neighborhoods is bounded from below and chosen small enough such that leaves intersections are graphs of functions. The Arzelà-Ascoli argument provides the existence of a lower and an upper limit for the corresponding sequence of functions, whose associated graphs form the two local branches passing through q. An uniform lower bound for the normal injectivity radius of leaves permits to show that each leaf L_{p_i} intersects only once (or twice in the non-orientable case) a specific

neighborhood. This permits to pass from a local limit leaf to a global limit leaf which should be compact since it can be approached everywhere by some sequence of compact leaves L_{p_i}.

Proof. — We use notation from the proof of Lemma 2.11. Since smooth, α_V is bounded along with all its derivatives on any open subset V' with closure $\overline{V'}$ contained in V.

Further there exist two finite open ball coverings $\{V_i\}$ and $\{V_i'\}$ of $\overline{M \smallsetminus C}$ such that $\overline{V_i'} \subset V_i$, for every i. Then $\alpha_{V_i'}$ are all bounded, so that there exists some constant A such that for every i:

$$|\alpha_{V_i'}| \leqslant A.$$

We can take A large enough such that $\max(|H(x)|, |N(x)|) \leqslant A$, for all $x \in M$. Since every curvature leaf $L \subset M \smallsetminus C$ is totally umbilical of principal curvatures $|\alpha|$, the previous inequality shows that the second fundamental form II_L has uniformly bounded norm:

$$\|\mathrm{II}_L\| \leqslant A.$$

By hypothesis there exists a local conformal change of the metric to a flat metric, in particular we can assume that there is a flat metric on each V_j. Changing the original metric on V_j' to the flat one amounts to changing the constant A above. Now, we can therefore compare tangent vectors at different points of V_j. As $\xi|_{V_j'}$ is a smooth vector field defined on V_j', there exists $\delta(\varepsilon) > 0$ such that whenever $x, y \in V_j'$ are at distance at most δ apart then the angle between $\xi|_x$ and $\xi|_y$ is at most ε.

Our key ingredient is the following lemma which seems to be well-known to the specialists.

Lemma 3.30. — *Suppose that $C \neq \varnothing$. Let $q \in \overline{M \smallsetminus C} \cap C$ and $p_i \in M \smallsetminus C$ such that $\lim_{i\to\infty} p_i = q$. Then there exists an open neighborhood U_q of q in M with the following properties.*

(1) *The size of $\overline{U_q}$ is uniformly bounded from below, namely it contains an interval bundle over a metric disk having both radius and height larger than some constant independent on q.*

(2) *The intersection of the curvature leaf L_{p_i} with U_q is connected if ξ is orientable and has at most two connected components, if the restriction of ξ is to the connected component of $M \smallsetminus C$ containing p_i is non-orientable.*

(3) *Let $(L_{p_i} \cap \overline{U_q})^\circ$ denote the connected component of $L_{p_i} \cap \overline{U_q}$ containing p_i. Then the sequence $(L_{p_i} \cap \overline{U_q})^\circ$ limits to a branched hypersurface $(L_q \cap \overline{U_q})^\circ$, called a piece of the local limit leaf at q. Moreover, $(L_q \cap \overline{U_q})^\circ$ is the union of two codimension 1 submanifolds of M passing through q which are tangent to $\mathscr{D}$ which are called (local) branches.*

(4) *For any sequence of points $p_i \in M \smallsetminus C$ with $\lim p_i = q$, the sequence $(L_{p_i} \cap \overline{U_q})^\circ$ limits to $(L_q \cap \overline{U_q})^\circ$ or a branch of it.*

(5) *Each piece $(L_q \cap \overline{U_q})^\circ$ of the local limit leaf at q is contained in $(\overline{M \smallsetminus C}) \cap C$.*

(6) *The sequence of curvature leaves $L_{p_i} \cap \overline{U_q}$ limits to a local limit leaf $L_q \cap \overline{U_q}$. Moreover, the local limit leaf $L_q \cap \overline{U_q}$ is the union of at most two pieces of local limit leaves.*

Proof of Lemma 3.30. — According to Lemmas 3.20 and 3.23 there is γ such that any leaf L has both injectivity radius i_L and focal radius $f_{L\subset M}$ bounded from below by γ while the normal injectivity radius $i_{p,\gamma}$ is bounded from below. Now, as $\|\mathrm{II}_L\| \leqslant A$ and each leaf is of constant curvature α, uniformly bounded, it follows that each the embedding of each leaf $L \to M$ has uniformly bounded local metric distorsion. Change now γ into $\min(\gamma, i_{p,\gamma})$. It follows that $i_{p,\gamma} \geqslant \gamma$.

Consider now the image $N_\gamma(D_\gamma(p))$ of $\{(q, \rho\xi)\ ;\ q \in D_\gamma(p), |\rho| < \gamma\} \subset TM$ by the exponential map. By above the exponential map is a diffeomorphism on its image, which is a relative regular neighborhood with respect to its boundary. Moreover, as $D_\gamma(p)$ is diffeomorphic to a $(n-1)$-dimensional disk, $\xi|_{D_\gamma(p)}$ is a trivial line bundle and hence $N_\gamma(D_\gamma(p))$ is diffeomorphic to $D_\gamma(p) \times [-\gamma, \gamma]$.

We can assume that $q \in \overline{M \smallsetminus C} \cap C \cap V_j'$, $p_i \in (M \smallsetminus C) \cap V_j'$ and $d(p_i, q) < \gamma/20$, for all i, so that $q \in \mathrm{int}(N_\gamma(D_\gamma(p_1))$. Consider now the cylindrical neighborhood $\overline{U_q} = N_\gamma(D_\gamma(p_1))$ of p_1 which is diffeomorphic to $D_\gamma(p_1) \times [-\gamma, \gamma]$. Note that there is a projection on the first factor $\pi : \overline{U_q} \to D_\gamma(p_1) \subset L_{p_1}$. Specifically, $\pi(z) \in D_\gamma(p_1)$ is the closest point to z. By our choice of γ, the map π is well-defined (see Lemma 3.23). Moreover, we can suppose that $q, p_i \in \mathrm{int}(\overline{U_q})$, for all i and that $\overline{U_q} \subset V_j'$.

We will suppose henceforth that ξ is defined on a γ-metric neighborhood Ω of $\overline{M \smallsetminus C}$, and in particular on every $\overline{U_q}$, for $q \in \overline{M \smallsetminus C} \cap C$.

By taking a smaller γ if needed, we can assume that $\overline{U_q}$ is included in a metric ball of radius $\delta(\pi/10)$, namely that $\xi|_x$ has only a small angle variation for all $x \in \overline{U_q}$ and moreover $d(p_i, p_1) < \gamma/10$. Now, U_q satisfies the requirements of the first claim in the lemma, by construction.

From Proposition 3.1 every connected component E of $M \smallsetminus C$ is an interval bundle over a compact space form P, as $\widehat{E}$ is a cylinder. Therefore the complement of the lift of every curvature leaf in $\widehat{E}$ has two connected components. Then [28, Lem. 2, p. 385] shows that under these conditions any transversal arc will cut a leaf of $\widehat{E}$ in at most one point. On the other hand, note that a geodesic arc issued from a point of $D_\gamma(p_1)$ and orthogonal to L_{p_1} should be transversal to all leaves that it intersects in $\overline{U_q}$, because of the small angle variation assumption. If ξ is orientable, then the intersection of curvature leaves L_{p_i} with $\overline{U_q}$ to should be connected, as otherwise orthogonal geodesics will intersect twice the same leaf. If ξ is non-orientable, the same argument works for the pull-back of $\overline{U_q}$ to $\widehat{E}$ should be connected and hence the intersection of curvature leaves L_{p_i} with $\overline{U_q}$ have at most two connected components. This proves the second item of our Lemma.

We further claim that the connected component of $(L_{p_i} \cap \overline{U_q})^\circ$ is the graph of a smooth function $y_i : D_\gamma(p_1) \to [-\gamma, \gamma]$. This is clear for $(L_{p_1} \cap \overline{U_q})^\circ$ and hence for all leaves $(L_{p_i} \cap \overline{U_q})^\circ$ which are close enough to it. We have to prove that this holds for all points p_i with $d(p_i, p_1) < \gamma/10$.

A compact hypersurface $L \cap \overline{U_q}$ is the graph of some smooth function $D_\gamma(p_1) \to [-\gamma, \gamma]$ if and only if the projection $\pi : \overline{U_q} \to D_\gamma(p_1)$ restricts to a diffeomorphism $\pi|_L : L \cap \overline{U_q} \to D_\gamma(p_1)$. The key point is to show that $\pi|_L$ is a bijection, as the smoothness of its inverse will follow from the smoothness of $L \cap \overline{U_q}$.

If $\pi|_{L_{p_i}}$ were not surjective, then there are two possibilities:

– either $(L_{p_i} \cap \overline{U_q})^\circ$ would intersect the top or the base of the cylinder $\overline{U_q}$, namely the image of $D_\gamma(p_1) \times \{-\gamma, \gamma\}$;

– or else a boundary point of the image $\pi|_{L_{p_i}}((L_{p_i} \cap \overline{U_q})^\circ)$ would be a critical value of $\pi|_{L_{p_i}}$ lying in $\mathrm{int}(D_\gamma(p_1))$.

In the first case we can find a curve contained in $(L_{p_i} \cap \overline{U_q})^\circ$ joining p_i to some point $z \in D_\gamma(p_1) \times \{-\gamma, \gamma\}$. By the mean value theorem there is some point r on this curve where the angle between $\xi|_{p_1}$ and $\xi|_r$ is at least $\arcsin\big((\sqrt{199}-1)/20\big) > \pi/10$, contradicting our hypothesis on the small variation of the angle in the (flat) chart U_q.

In the second case $(L_{p_i} \cap \overline{U_q})^\circ$ only intersects non-trivially the lateral surface $\partial\overline{U_q} \smallsetminus D_\gamma(p_1) \times \{-\gamma, \gamma\}$ of the boundary $\partial\overline{U_q}$. Now, if x is a critical point of $\pi|_{L_{p_i}}$ then $\xi|_x$ should be parallel to the tangent space $T_{\pi(x)}D_\gamma(p_1)$ and hence orthogonal to $\xi|_{\pi(x)}$, contradicting our assumptions on the small variation of the angle. We conclude that $\pi|_{L_{p_i}}$ is surjective.

Next, if $\pi|_{L_{p_i}}$ were not injective when restricted to a connected component of $(L_{p_i} \cap \overline{U_q})^\circ$, then there would exist a smooth path in $(L_{p_i} \cap \overline{U})^\circ$ joining two points with the same image under $\pi|_{L_{p_i}}$. The restriction of $\pi|_{L_{p_i}}$ to this path must then have a critical point and, as above, this would provide a point x of $(L_{p_i} \cap \overline{U_q})^\circ$ for which $\xi|_x$ is orthogonal to $\xi|_{\pi(x)}$. This would contradict our assumption on the small variation of the angle. Our claim follows.

Observe that $y_i : D_\gamma(p_1) \to \mathbb{R}$ are smooth bounded functions, as $|y_i| \leqslant \gamma$. Their derivatives are also bounded as $\xi|_x$ is close to $\xi|_q$, for any $x \in \overline{U_q}$, by the small angle variation assumption. Specifically, assume that we are in a flat chart containing $\overline{U_q}$ obtained by a conformal change of metric. We pick up coordinates $\{x_1, x_2, \dots, x_{n-1}, y\}$, the first $n-1$ coordinates corresponding to $D_\gamma(p_1)$. If the point $x \in \overline{U_q}$ belongs to the image of y_i, then we have:

$$\sin^2 \measuredangle(\xi|_x, \xi|_q) = \frac{\sum_{j=1}^{n-1} |\partial y_i/\partial x_j|^2}{1 + \sum_{j=1}^{n-1} |\partial y_i/\partial x_j|^2} < \sin^2(\pi/10) < 0.1,$$

from which we derive the following upper bound for the first derivatives of y_i:

$$\sum_{j=1}^{n-1} |\partial y_i/\partial x_j|^2 < \frac{\sin^2(\pi/10)}{1 + \sin^2(\pi/10)} < 0.1.$$

In the same flat chart the second fundamental form of the hypersurface given by the image of y_i has the matrix:

$$\mathrm{II}_{jk} = \frac{1}{1 + \sum_{j=1}^{n-1} |\partial y_i/\partial x_j|^2} \cdot \frac{\partial^2 y_i}{\partial x_j \partial x_k}.$$

Since the norm of the second fundamental form of $L_{p_i} \cap \overline{U_q}$ is uniformly bounded by A, we derive the following bound for the second derivatives of y_i:

$$\sum_{j,k=1}^{n-1} \left|\partial^2 y_i/\partial x_j \partial x_k\right|^2 < 1.1 \cdot A.$$

Alternatively, recall that II is diagonal in the 1-QC metric, because leaves are totally umbilical and the principal curvatures equal α. The entries of II in the flat chart can be expressed in terms of the entries of II in the 1-QC metric (namely 0 and α) and the real function specifying the conformal change in the metric. Since all derivatives of α are bounded in $\overline{U_q}$ higher derivatives of y_i are also bounded.

By the Arzelà-Ascoli theorem, the sequence y_i has a sub-sequence which converges to a smooth function $y_\infty : D_\gamma(p_1) \to [-\gamma, \gamma]$ in the C^k topology, for every $k \geqslant 2$.

Now, the leaves L_{p_i} are pairwise disjoint and hence there is a total order among the functions y_i. Therefore, there are only two possible limits of sub-sequences of y_i, namely $y_\infty^+ = \limsup y_i$ and $y_\infty^- = \liminf y_i$. We say that the sequence $(L_{p_i} \cap \overline{U_q})^\circ$ is increasing or decreasing if the sequence of functions y_i is increasing or decreasing, respectively.

Denote by $(L_q^\pm \cap \overline{U_q})^\circ$ the graph of the limit function $y_\infty^\pm$ and call them the *local branches* of the local limit leaf. We say that the sequence $L_{p_i}^\circ \cap \overline{U_q}$ is *below* q if the sequence y_i is increasing and *above* q if the sequence y_i is decreasing. Each local branch is a smooth totally umbilical hypersurface of principal curvature $\alpha(q)$. Since ξ is everywhere orthogonal at L_{p_i}, by passing to the limit we find that each $(L_q^\pm \cap U_q)^\circ$ is a codimension 1 submanifold which is tangent to the distribution $\mathscr{D}$ at each point $x \in (L_q^\pm \cap \overline{U_q})^\circ$.

Set $(L_q \cap \overline{U_q})^\circ = (L_q^+ \cap \overline{U_q})^\circ \cup (L_q^- \cap \overline{U_q})^\circ$ for the union of the two local branches. The local branch set is the set of tangency points between the two local branches $(L_q^\pm \cap \overline{U_q})^\circ$. The branch set could be all of $(L_q \cap \overline{U_q})^\circ$, in which case the two local branches coincide, or just one point.

Eventually, observe that $(L_{p_i} \cap \overline{U_q})^\circ$ limits to $(L_q \cap \overline{U_q})^\circ$ since the later is the union of the graphs of the functions $\limsup y_i$ and $\liminf y_i$.

We obtained above a piece $(L_q \cap \overline{U_q})^\circ$ of the limit leaf at q. We will need now to show that the piece $(L_q \cap \overline{U_q})^\circ$ does not depend on the choice of the sequence $p_i \in M \smallsetminus C$ with $\lim p_i = q$. It is enough to consider the case when the sequences p_i and p_i' are both increasing with respect to the order defined in the proof of Lemma 3.30. As their limit is the same the sequences of functions y_i and y_i' associated to the sequences p_i and p_i', respectively, are comparable, namely for every i_1 there exists some i_2, j_1, j_2 such that $y_{i_1} \leqslant y'_{j_1} \leqslant y_{i_2} \leqslant y'_{j_2}$. Then $\lim y_i = \lim y_i'$, which implies our claim and hence item (4) of our Lemma.

Further, if $q' \in M \smallsetminus C \cap (L_q \cap \overline{U_q})^\circ$, then $(L_q \cap \overline{U_q})^\circ = (L_{q'} \cap \overline{U_q})^\circ$, so that $q \in L_{q'}$. But the curvature leaf L_q' is complete with its intrinsic metric, according to Reckziegel's Theorem 3.4 (see Lemma 3.5) and $L_{q'} \subset M \smallsetminus C$. This leads to a contradiction. Thus $(L_q \cap \overline{U_q})^\circ \subset C$. On the other hand every point of $(L_q \cap \overline{U_q})^\circ$ is a limit point of $(L_{p_i} \cap \overline{U_q})^\circ$, so that $(L_q \cap \overline{U_q})^\circ \subset \overline{M \smallsetminus C}$.

We can define now the local limit leaf $L_q \cap \overline{U_q}$, in the case when $L_{p_i} \cap \overline{U_q}$ is not connected, for large enough i. It suffices to consider the set $p_i' \in M \smallsetminus C \cap U_q$ with the property that $L_{p_i} \cap \overline{U_q} = (L_{p_i} \cap \overline{U_q})^\circ \sqcup (L_{p_i'} \cap \overline{U_q})^\circ$ and $\lim p_i' = q' \in C$. Then we set $L_q \cap \overline{U_q} = (L_q \cap \overline{U_q})^\circ \cup (L_{q'} \cap \overline{U_q})^\circ$. It follows from above that $L_{p_i} \cap \overline{U_q}$ limits to $L_q \cap \overline{U_q}$. Such a local limit leaf $L_q \cap \overline{U_q}$ is obviously unique and independent on the choice of the sequence p_i. Moreover, by uniqueness if $q' \in C$ is such that q' is a limit point of $L_{p_i} \cap U_q$, then $(L_{q'} \cap U_{q'}) \cap U_q = (L_q \cap U_q) \cap U_{q'}$. □

Lemma 3.31. — *Assume that M is locally 1-QC. Let E be a component of $M \smallsetminus C$ for which the restriction $\xi|_E$ is orientable. Then the normal injectivity radius of every curvature leaf $L \subset E$ is bounded below by some constant γ depending on A and the bounds on the sectional curvature.*

Proof. — Consider a normal orthogeodesic joining $p, q \in L_p$. We consider the metric neighborhood $N_\gamma(D_\gamma(p))$ of p. Then the argument used above to prove Lemma 3.30 (2) shows that $L_p \cap N_\gamma(D_\gamma(p))$ is connected. In particular $d_M(p,q) \geqslant \gamma$. This provides an uniform lower bounds for the length of an orthogeodesic. Using [29, Th. 4.2] and the lower bound for the focal radius provided by Lemma 3.20 we obtain the claim. □

We now define an equivalence relation, called *confluence*, for branches of local limit leaves as follows. First, if $U_q \cap U_{q'} \neq \varnothing$ we say that $L_{q'}^+ \cap U_{q'}$ and $L_q^+ \cap U_q$ *have a common extension* if $(L_q^+ \cap U_q) \cap U_{q'} = (L_{q'}^+ \cap U_{q'}) \cap U_q$. Then $L_{q'}^+ \cap U_{q'}$ and $L_q^+ \cap U_q$ are called *confluent* if there exists a finite sequence of branches of local limit leaves starting at $L_q^+ \cap U_q$ and ending with $L_{q'}^+ \cap U_{q'}$, where consecutive terms have a common extension.

We now define *the branch L_q^+ of the limit leaf passing through q* as the union of all branches of local limit leaves confluent with $L_q^+ \cap U_q$. By definition L_q^+ is an immersed smooth codimension-one submanifold of M with possible self-tangencies, which is contained within $\overline{M \smallsetminus C} \cap C$. Eventually, we define *the limit leaf L_q* as the union of all branches of all local limit leaves confluent to some branch of $L_q \cap U_q$.

Lemma 3.32. — *Each branch L_q^+ of a limit leaf is compact.*

Proof. — We want to show that limit leaves can be approached globally by leaves. The following property of the confluence relation will be the key ingredient of the proof. We suppose that:

(1) $L_{p_i} \cap U_q$ limits to $L_q^+ \cap U_q$, where p_i are below q, and $L_{p_i'} \cap U_{q'}$ limits to $L_{q'}^+ \cap U_{q'}$;
(2) $U_q \cap U_{q'} \neq \varnothing$;
(3) $L_q^+ \cap U_q$ and and $L_{q'}^+ \cap U_{q'}$ have a common extension.

Then, we claim that there is a sub-sequence $L_{p_i''}$ of the sequence of leaves $L_{p_i}, L_{p_i'}$ with the property that $L_{p_i''} \cap (U_q \cup U_{q'})$ limits to $(L_q^+ \cap U_q) \cup (L_{q'}^+ \cap U_{q'})$.

Observe that p_i are below q and p_i' are below q'. Our claim is a consequence of an analogous result on sequences of real functions y_i, having the property that $y_{i+1} - y_i$ have constant sign. Specifically, assume we have two increasing sequences of real functions $y_i, y_i' : U \cup V \to \mathbb{R}$ such that their restrictions to the open relatively

compact sets U and V, with $U \cap V \neq \varnothing$, converge to the restrictions of a function $z : U \cup V \to \mathbb{R}$ defined on $U \cup V$, namely $y_i|_U \to z|_U$ and $y'_i|_V \to z|_V$. Moreover, suppose that $y_i - y'_j$ have constant sign, for all i, j. Then there exists a sub-sequence y''_i of the sequence obtained by putting together y_i, y'_i such that $y''_i|_{U\cup V} \to z|_{U\cup V}$. In fact it suffices to take $y''_i = \max(y_i, y'_i)$, which coincides with either y_i or y'_i on $U \cup V$. To allow a description of general hypersurfaces which might not be globally graphs we should extend this result to multi-valued functions. Consider now that $y_i|_U$, $y'_i|_V$, $z|_U$ and $z|_V$ are now usual functions, but $y_i|_V$ and $y'_i|_U$ are only required to be multi-valued functions whose graphs correspond to some smooth hypersurface. The argument above then shows that the graphs of the possibly multi-valued functions $\max(y_i, y'_i)|_{U\cup V}$ converge to the graph of $z|_{U\cup V}$ in the $\mathscr{C}^0$-topology: if $y_i|_{U\cap V} > y'_i|_{U\cap V}$, then $\max(y_i, y'_i) = y_i$ and $y'_i|_V < y_i|_V < z_V$. Thus the limit of any sub-sequence $y_i|_V$ as above is $z|_V$. However, if we assume that all derivatives of $y'_i|_V$ and $y_i|_V$ are continuous then $y_i|_V$ should be also usual functions, as otherwise their first derivative would be infinite in some direction. In our case of branches of limit leaves this assumption is satisfied since all leaves intersecting a open distinguished neighborhood U_q have almost horizontal tangent space.

By induction on the number of local branches we obtain that, up to extracting a sub-sequence (and possibly changing the initial sequence p_i by another sequence with the same properties) L_{p_i} limits from below to L^+_q.

Each branch L^+_q is complete with respect to the induced metric. To see that, it suffices to note that the metric disk $D_{Q\gamma}(q)$ within the branch L^+_q is contained within $L^+_q \cap U_q$, for every q, according to Lemma 3.26. To prove that L^+_q is compact, it is therefore enough to show that it is a proper leaf.

We assume that in our construction above the height γ of each U_q is smaller than $i_{p,\gamma}/3$. If $U_q = N_\gamma(D_\gamma(p_1))$ we denote by U'_q the corresponding metric neighborhood of double height, namely $N_{2\gamma}(D_\gamma(p_1)$. Since L^+_q is covered by some union $\bigcup U_{q'}$, its closure $\overline{L^+_q}$ in M is covered by the corresponding union $\bigcup U'_{q'}$. By compactness we can extract a finite covering U'_{q_i} of $\overline{L^+_q}$. If L^+_q were non-compact, then there would exist some neighborhood $U'_{q'} = N_{2\gamma}(D_\gamma(p'_1))$ which intersects infinitely many times L^+_q. But $U'_{q'}$ has the shape of a cylinder, so that there are geodesics issued from $D_\gamma(p'_1) \subset L_{p'_1}$ orthogonal to $L_{p'_1}$ which intersect infinitely many times L^+_q. Recall from above that every point of L^+_q can be approached arbitrarily closely by the leaves L_{p_i}. It follows that there exist leaves L_{p_i} which intersect more than 100 times some orthogonal geodesic within $U'_{q'}$. This contradicts the fact that any transverse arc in $M \smallsetminus C$ should intersect at most twice every leaf of our foliation within a small neighborhood (see [28, Lem. 2, p. 385] and Lemma 3.30 (2) above). □

From above L_q is the image of a codimension 1 immersion of some union X of closed manifolds, with only tangency or self-tangency points. The norm of the second fundamental form of every branch is also bounded by A, as this bound is valid for any local branch $L^\pm_q \cap \overline{U_q}$.

When ξ is orientable, then every branch is embedded, as the normal injectivity radius is bounded from below by Lemma 3.31 and hence there are not self-tangency points. Thus L_q is a branched hypersurfaces. Further $L_q \subset \overline{M \smallsetminus C} \cap C$ and each branch is a totally umbilical hypersurface with constant sectional curvature $\lambda(q)$. When ξ is non-orientable, then the lift L_q to the double cover $\widehat{\Omega}$ is a branched hypersurface, as claimed. This proves Proposition 3.29. □

An immediate corollary of the proof is the following:

Lemma 3.33. — *Let $q \in \overline{M \smallsetminus C} \cap C$ and $p_i \in M \smallsetminus C$ such that $\lim_{i\to\infty} p_i = q$. Then, up to passing to a sub-sequence, the limit of L_{p_i} in the Hausdorff topology is some branch of the limit leaf L_q. More specifically, if p_i are below q then the limit of L_{p_i} is the branch L_q^+.*

Recall that the *Hausdorff distance* between the non-empty compact subsets $X, Y \subset M$ is given by $\max(\sup_{x\in X} \inf_{y\in Y} d(x,y), \sup_{y\in Y} \inf_{x\in X} d(x,y))$, where d is the distance in M. It is well-known that the space of compact subsets of M equipped with the Hausdorff metric is complete.

3.10. Closure of non-compact globally 1-QC components

We denote by $\mathrm{Fr}(E) = \mathrm{Fr}(\overline{E}) = \overline{E} \smallsetminus E$ the *frontier* of a subset $E \subset M$. An *end* of a topological space $X = \bigcup_{n=1}^{\infty} K_n$ which is the ascending union of compact subspaces K_n, is an infinite sequence $V_1 \supset V_2 \supset V_3 \supset \cdots$ with $\bigcap_n V_n = \varnothing$, where each V_n is a connected component of $X \smallsetminus K_n$. When X is a path-connected CW-complex ends correspond to equivalence classes of proper continuous maps $\mathbb{R}_+ \to M$, up to proper homotopy of their restrictions to the subset of positive integers. For example $\mathbb{R}$ has two ends, $\mathbb{R}^2$ has one end while the infinite complete binary tree has uncountably many ends.

Proposition 3.29 shows that limit leaves are branched surfaces hence union of embedded (or immersed) compact hypersurfaces. Here we will obtain a more precise statement by showing that the number of branches in the frontier of a connected component E of $M \smallsetminus C$ is either one or two.

Proposition 3.34. — *Let M be a locally 1-QC manifold and E a non-compact connected component of $M \smallsetminus C$. Assume that $\xi|_E$ is orientable. Then $\overline{E}$ is either diffeomorphic to a space form cylinder or else homeomorphic to the result of identifying two subsets of the two boundary components of a space form cylinder.*

Proof. — By Proposition 3.1 E is diffeomorphic to a cylinder $P \times \mathbb{R}$ over some closed space form P. It makes sense then to say that a sequence of curvature leaves which goes to infinity belongs to one or another of the ends of $\mathbb{R}$.

Let $q \in \mathrm{Fr}(E)$. By Proposition 3.29 there exists a limit leaf L_q passing through q, which is a compact branched hypersurface everywhere tangent to $\mathscr{D}$. Now $L_q \cap \overline{E} \subset \mathrm{Fr}(E)$ as $L_q \cap (M \smallsetminus C) = \varnothing$. Note that an arbitrarily small backward trajectory of $\widehat{\xi}$ issued from q cannot be contained in L_q, as $\widehat{\xi}$ is orthogonal to L_q. Then, according to

Lemma 3.33, any sequence of curvature leaves belonging to a given end of E converges to a branch of $L_q \cap \overline{E}$. Thus L_q has at most two branches, corresponding to the two ends of E. Therefore $\mathrm{Fr}(\overline{E})$ is a compact branched hypersurface with at most two branches.

If there is only one branch for each end, then $\overline{E}$ is a compact manifold endowed with a foliation with compact leaves and hence diffeomorphic to a cylinder. Otherwise, $\overline{E}$ is homeomorphic to the result of gluing together the two ends of the cylinder along the branch locus, namely the quotient of $P \times [0,1]$ by the equivalence relation $(x,0) \sim (x,1)$, for every branch point x. □

Proposition 3.35. — *If E is a non-compact connected component of $M \smallsetminus C$ and $\xi|_E$ is non-orientable, then $\overline{E}$ is diffeomorphic to the image of an interval bundle with one boundary component by an immersion with self-tangencies along the boundary.*

Proof. — Let ξ be defined on the open set containing an open subset $\Omega \supset \overline{M \smallsetminus C}$ with compact closure. Let then $\widehat{\overline{E}}$ be the closure of $\widehat{E}$ within $\widehat{\Omega}$. By Proposition 3.34, $\widehat{\overline{E}}$ is the image of a cylinder $\widehat{P} \times [0,1]$ by an immersion with possible self-tangencies between the two boundary components, where $\widehat{P}$ is a closed manifold. This identification is canonical, as trajectories of $\widehat{\xi}$ are horizontal, i.e., of the form $\{p\} \times [0,1]$. The action of the deck group $\mathbb{G} = \mathbb{Z}/2\mathbb{Z}$ of the cover $\widehat{\overline{E}} \to \overline{E}$ is covered by a diffeomorphism $\varphi : \widehat{P} \times [0,1] \to \widehat{P} \times [0,1]$. Then φ preserves globally the set of boundary points. To describe $\overline{E}$ it suffices to understand the action of φ.

First, if $\varphi(\widehat{P} \times \{i\}) = \widehat{P} \times \{i\}$, for $i \in \{0,1\}$, then $\mathrm{Fr}(E)$ consists of the union of the images of $(\widehat{P} \times \{i\})/\mathbb{G}$, for $i \in \{0,1\}$. Each oriented trajectory of $\widehat{\xi}$ descends to a trajectory of the line field ξ. However, we can orient each trajectory of ξ, by declaring that it is issued from $(\widehat{P} \times \{0\})/\mathbb{G}$ and arrives to $(\widehat{P} \times \{1\})/\mathbb{G}$. This provides a lift of ξ to a vector field on $\overline{E}$, contradicting its non-orientability. It follows that φ must exchange the two boundary components of $\widehat{P} \times [0,1]$.

Lemma 3.36. — *Assume that φ exchanges the two components $\widehat{P} \times \{i\}$, for $i \in \{0,1\}$. Then φ induces a free action of $\mathbb{G}$ on $\widehat{P}$.*

Proof. — Let $p_E : \widehat{\overline{E}} \to \overline{E}$ denote the projection. We can write $\varphi(x,0) = (\phi(x),1)$, for any $x \in \widehat{P}$, where $\phi : \widehat{P} \to \widehat{P}$ is a diffeomorphism. As ξ is locally orientable, we can consider the trajectory γ of ξ in $\overline{E}$ issued from the point $p_E((x,0)) = p_E((\phi(x),1))$. Then $p_E^{-1}(\gamma)$ is the union of two arcs, say $\widehat{\gamma}_{(x,0)}$ and $\widehat{\gamma}_{(\phi(x),1)}$ which are respectively issued from $(x,0)$ and $(\phi(x),1)$. However, the pull-backs of ξ-trajectories are now trajectories of $\widehat{\xi}$, in particular they are horizontal, i.e., of the form $\{q\} \times [0,1]$. Now, the maximal extensions of these two $\widehat{\xi}$ trajectories are the arcs $\{x\} \times [0,1]$ and $\{\phi(x)\} \times [0,1]$.

If $x = \phi(x)$, then the restriction $p_E : p_E^{-1}(\gamma) = \{x\} \times [0,1] \to \gamma$ cannot be a 2-fold cover, which is a contradiction. Thus $\phi(x) \neq x$, for any x and $p_E^{-1}(\gamma) = \{x\} \times [0,1] \sqcup \{\phi(x)\} \times [0,1]$. This implies that γ intersects again $\mathrm{Fr}(E)$, in $p_E((\phi(x),0))$. If we travel backward along γ starting from $p_E((\phi(x),0))$ we have to end at $p_E((\phi,0))$; we derive

that $\varphi(\phi(x),0)) = (x,1)$, and thus $\phi(\phi(x)) = x$. Moreover, as $\widehat{\xi}$ is a unit vector field we have more generally that $p_E((x,t)) = p_E((\phi(x),1-t))$.

This shows that $\phi : \widehat{P} \to \widehat{P}$ defines a fixed point free involution. □

Set $P = \widehat{P}/\mathbb{G}$, which is a closed manifold by Lemma 3.36 and let $p_P : \widehat{P} \to P$ be the quotient map.

Given a class $\alpha \in H^1(P;\mathbb{Z}/2\mathbb{Z})$ we have associated an interval bundle $P \times_\alpha [0,1]$ which is a fibration over P by intervals, whose monodromy homomorphism $\pi_1(P) \to \mathbb{Z}/2\mathbb{Z}$ (obtained after the identification $\pi_0(\mathrm{Homeo}([0,1])) = \{\pm 1\}$) is given by α.

Lemma 3.37. — *There is a homeomorphism between $\overline{E}$ and the image of $P \times_{w_1(\xi|_E)} [0,1]$ by an immersion with possible self-tangencies on the boundary.*

Proof. — We have an exact sequence

$$1 \longrightarrow \pi_1(\widehat{P}) \longrightarrow \pi_1(P) \xrightarrow{\alpha} \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

and an element $j \in \pi_1(P)$ with $\alpha(j) \neq 1$ is represented by the projection of a loop in $\widehat{P}$ joining x to $\phi(x)$.

The projection map $\widehat{P} \times [0,1] \to P$ sending (x,t) into $p_P(x)$ factors through a map $\left(\widehat{P} \times [0,1]/_{(x,t)\sim(\varphi(x),1-t))}\right) \to P$. The latter induces a well-defined map $\overline{E} \to \overline{P}$, where $\overline{P}$ denotes the image of P by the immersion in M. The preimage of a point $p_P(x) \in P$ by the induced map $\overline{E} \to \overline{P}$ is the maximal trajectory arc $\gamma_x \subset \overline{E}$ joining the points x and $\phi(x)$ from $\mathrm{Fr}(E)$. It is immediate that the map $\overline{E} \to \overline{P}$ is a fibration with fiber $[0,1]$, which is covered by an interval fiber bundle $\left(\widehat{P} \times [0,1]/_{(x,t)\sim(\varphi(x),1-t))}\right) \to P$. The monodromy action of j sends the fiber γ_x into itself by exchanging its endpoints, while elements from $\pi_1(\widehat{P})$ act as identity. It follows that $\overline{E} \to \overline{P}$ is the twisted product $\overline{P} \times_\alpha [0,1]$, image by an immersion (and its equivariant lift) of the twisted product $P \times_\alpha [0,1]$, where α is a class in $H^1(P;\mathbb{Z}/2\mathbb{Z})$.

On the other hand $\widehat{\overline{E}} \to \overline{E}$ is a 2-fold cover of class $w_1(\xi) \in H^1(\overline{E};\mathbb{Z}/2\mathbb{Z})$, by hypothesis. This cover is the image of an immersion of the 2-fold cover $\widehat{P} \times [0,1] \to P \times_\alpha [0,1]$. Therefore its class in $H^1(P \times_\alpha [0,1])$ is determined by the exact sequence:

$$1 \longrightarrow \pi_1(\widehat{P} \times [0,1]) \longrightarrow \pi_1(P \times_\alpha [0,1]) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

and thus it can be identified with α, so that α is the pull-back of $w_1(\xi|_{\overline{E}})$ to $H^1(P;\mathbb{Z}/2\mathbb{Z})$. Recall that E is diffeomorphic an open interval bundle over P. Thus this class can be identified geometrically with $w_1(\xi|_E) \in H^1(P;\mathbb{Z}/2\mathbb{Z})$. □

This settles Proposition 3.35. □

3.11. Limit leaves subspace. — The *limit leaves subspace* of a 1-QC manifold is the compact set $\mathscr{L} = \mathrm{Fr}(M \setminus C) = \overline{M \setminus C} \cap C$.

Components E of $M \setminus C$ which are not cylinders will be called *topologically non-trivial*.

Lemma 3.38. — *There are only finitely many topologically non-trivial components E of $M \setminus C$.*

Proof. — Let E be a topologically non-trivial component of $M \smallsetminus C$. If $\overline{E}$ is a cylinder with its ends glued along a non-empty subset of the boundary, then the class of a loop which follows the vertical line contributes a non-trivial factor $\mathbb{Z}$ to $H_1(M)$. If E is a non-trivially twisted cylinder then it contributes a non-trivial factor $\mathbb{Z}/2\mathbb{Z}$ to $H_1(M)$. Since M is compact $H_1(M)$ is of finite type and hence we cannot have but finitely many such components. □

Consider a maximal set B of cylindrical components $E_i = P_i \times \mathbb{R}$ of $M \smallsetminus C$ such that $M \smallsetminus \bigcup_i E_i$ is connected. Then each E_i contributes a non-trivial factor $\mathbb{Z}$ in $H^1(M)$ which is of finite type and thus B is finite. If E is another cylindrical component $E = P \times \mathbb{R}$, then cutting $M \smallsetminus \bigcup_i E_i$ along P will disconnect it. In particular the class of P is a sum of several classes among the P_i in $H_{n-1}(M)$, or by duality in $H^1(M)$. Thus, the set of homology classes of cores P of non-essential cylindrical components is finite. We pick up a set $\mathscr{B}$ of such representatives. It follows that for any cylindrical component $E = P \times \mathbb{R}$ in M, either P bounds or there exists some $Q \in \mathscr{B}$ which is homologous to P and hence $P \cup Q$ bounds a codimension zero submanifold Z.

Lemma 3.39. — *Every set of disjoint codimension zero submanifolds Z of M with the property that ∂Z is a connected curvature leaf should be finite.*

Proof. — In the proof of Proposition 3.29 we noticed that there exists some $\gamma > 0$ such that the γ-neighborhood $N_\gamma(L)$ is diffeomorphic to a twisted product for any leaf L.

Consider now a leaf L bounding a submanifold Z and a point $p \in Z$ at distance $\gamma/2$ from L. Such a point exists as $N_\gamma(L)$ is embedded in M. Then the metric ball of radius $\gamma/2$ with center p is diffeomorphic to a ball and contained in Z. By the Rauch comparison theorem there exists some constant v depending on M such that this metric ball has volume at least v and hence the volume of Z is bounded below by v.

Thus there are only finitely many disjoint submanifolds Z with boundary a leaf. □

We add to the set $\mathscr{B}$ a maximal set of leaves which bound disjoint submanifolds.

Proposition 3.40. — *There exists $\varepsilon > 0$ such that for any leaves $Q \in \mathscr{B}$ and P cobounding with Q a codimension zero submanifold of M, such that $\min_{p \in P, q \in Q} d(p,q) < \varepsilon$, then Z is diffeomorphic to a cylinder.*

Proof. — We prove first a slightly weaker assertion, as follows.

Lemma 3.41. — *Let Q, P be the cores of two leaves belonging to cylindrical components of $M \smallsetminus C$. There exists $\delta > 0$ such that for any leaves $Q \in \mathscr{B}$ and P cobounding with Q a codimension zero submanifold of M, such that $P \subset N_\delta(Q)$, then Z is a cylinder.*

Proof. — We use the finite cover $\{U_i\}$ by cylinders of the form $\mathrm{int}(\overline{U})$ from the proof of Proposition 3.29 above. Then $P \cap \overline{U_i}$ and $Q \cap \overline{U_i}$ are both disjoint graphs of functions defined on $\overline{U_i}$, for every i. Therefore they are the upper and bottom cap of

a cylinder on $\overline{U_i}$. The union of all these cylinders is a cylinder over Q, since $\xi|_Q$ must be orientable. □

It remains to prove that for every $\delta > 0$ there exists $\varepsilon > 0$ such that for any leaves $Q \in \mathscr{B}$ and P cobounding with Q a codimension zero submanifold of M, such that $\min_{p\in P, q\in Q} d(p,q) < \varepsilon$, then $P \subset N_\delta(Q)$. Assume the contrary. Then we have a sequence of leaves Q_i cobounding with P such that $\lim_i \min_{p\in P, q\in Q_i} d(p,q) = 0$, but $P \not\subset N_\delta(Q_i)$. This provides a sub-sequence Q_i such that Q_i converges to P but then $P \not\subset N_\delta(Q_i)$ for all large enough i, contradicting Proposition 3.29 and Lemma 3.33. □

Proposition 3.42. — *If M is a compact locally 1-QC manifold, then there exists a compact branched hypersurface $L \subset \mathscr{L}$ such that connected components of $M \smallsetminus L$ are either components of* $\mathrm{int}(C)$ *or twisted cylinders.*

Proof. — By Proposition 3.29, $\mathscr{L}$ is the union of all limit leaves L_q. Note that two limit leaves $L_q, L_p \in \mathscr{L}$ are either disjoint or coincide. In fact they cannot intersect transversely, as both are limits of sequences of foliations leaves and if they are tangent somewhere then they should coincide.

In the proof of Proposition 3.29 we noticed that there exists some $\gamma > 0$ such that the γ-neighborhood $N_\gamma(L_p)$ is diffeomorphic to a twisted product for any leaf L_p. This continues to hold for any branch L_q^* of some limit leaf L_q.

We consider a maximal set $\mathscr{F}$ of leaves which are cores of cylindrical components E such that no two of them belong to the same component and $\min_{p\in P, q\in Q} d(p,q) > \varepsilon/2$, for every two leaves P and Q. By above there are only finitely many such leaves. Consider the collection of branched hypersurfaces associated to components E which are either topologically non-trivial or else they have leaves from $\mathscr{F}$ and denote by L their union. We claim that L verifies the claim. In fact the closures of the connected components of $M \smallsetminus L$ are either twisted cylinders or closures of components of $\mathrm{int}(C)$. In fact two cylinder components are separated by a component of $\mathrm{int}(C)$ or by a limit leaf. In the former case we need that the distance between boundaries be at least ε or otherwise the union of the three components will be a cylinder. In particular there are only finitely many such components of $\mathrm{int}(C)$. □

Remark 3.43. — Note that $\mathscr{L}$ might have infinitely many connected components, for instance an infinite union of compact leaves accumulating on a compact leaf. Nevertheless these leaves are contained in $\mathrm{int}(\overline{M \smallsetminus C})$.

3.12. End of proof of Theorem 1.6. — According to Proposition 3.42, M is the union of some interval bundles and finitely many pieces $\overline{X_i}$ where X_i are connected components of $\mathrm{int}(C)$. Each X_i is an open manifold with a constant curvature metric. Moreover, the curvature of each point of the closure $\overline{X_i} \subset M$ is also constant. The frontier $\mathrm{Fr}(X_i)$ consists in finitely many components, each one being either a totally umbilical constant curvature submanifold, the union of two such submanifolds of M which are glued together along some subset in such a way that the tangent space

is well-defined at any point of $\mathrm{Fr}(X_i)$ or else a totally umbilical constant curvature immersed hypersurface with self-tangencies.

Suppose that there exists a branched hypersurface $L^+ \cup L^-$ in the frontier $\mathrm{Fr}(X)$ of some X as above. Since $L^+ \cup L^-$ is a limit leaf, there exist arbitrarily small one-sided neighborhoods of every $L \in \{L^+, L^-\}$ which are diffeomorphic to cylinders, the other boundary components being curvature leaves. Thus, by adjoining finitely many cylinders to X we obtain a manifold with boundary $X^{\mathrm{ext}} \supset \overline{X}$. We claim that X^{ext} also admits a constant curvature metric with totally umbilical boundary components.

We first claim that there exists a smooth $f : [0,1] \to \mathbb{R}_+$ such that the warped product metric on the cylinder $Q = L \times [0,1]$ with rescaling function f has constant curvature H. Recall that L has constant intrinsic curvature $H + \alpha^2$ and is totally umbilical. The curvature $k_Q(\sigma)$ of a tangent plane σ spanned by the vectors $x + v$ and w, where v and w are tangent to L, x to the $[0,1]$ factor and $\|x\|^2 + \|v\|^2 = 1$ (in the warped product metric) is given by:

$$k_Q(\sigma) = -\frac{f''(t)}{f(t)}\,\|x\|^2 + \frac{k_L(v,w) - f'^2(t)}{f^2(t)}\,\|v\|^2,$$

where k_L denotes the sectional curvature of L. It follows that for every values of H and $H + \alpha^2$ there exists some scaling function f defined on a small interval containing 0 such that $f(0) = 1$ and $k_Q(\sigma) = H$, for any σ (see e.g. [4, Cor. 7.10] when $H + \alpha^2 \leqslant 0$).

Now we can glue together the Riemannian manifold obtained by cutting M open along L^+ and gluing the cylinder Q^+ with its warped product metric of constant curvature H along L^+. The result is a smooth Riemannian metric because the corresponding second fundamental forms of L^+ within M and Q^+ agree. We do a similar construction on Q^- and further for every branched leaf in $\mathrm{Fr}(X)$.

Eventually, note that the warped metric formula also makes sense for the case where L^+ is an immersed hypersurface in M. As we only have self-tangencies along L^+ we can perform the gluing procedure as above.

The result is a Riemannian manifold diffeomorphic to X^{ext} with a metric of constant curvature and totally umbilical boundary components.

Therefore M is a graph of space forms. This proves Theorem 1.6. □

4. Proof of Theorem 1.8

4.1. Outline. — The positivity of λ implies that connected components of the set of non-isotropic points consist of spherical cylinders. In the second step we describe the closures of these connected components, which are not necessarily manifolds. To simplify the topology we want to excise the cores of these tubes, although their number might be infinite. A key argument is Grushko's theorem in group theory which translates here into the finiteness of the number of factors in a free amalgamated product presentation of a finitely presented group. This shows that one could carry out surgery for only finitely many tubes.

It remains to understand the topology of the remaining pieces, which are neighborhoods of the set of isotropic points. We show there is a way to cap off the boundary

components of each piece, by means of hypersphere caps in the hyperbolic space whose geometry is controlled. A deep result of Nikolaev (see [44] and Proposition 4.20 for a precise statement) states that a manifold with bounded volume and diameter whose integral anisotropy is small enough should be diffeomorphic to a space form. This procedure constructs closed manifolds out of regular neighborhoods of isotropic points, whose geometry is controlled and whose integral anisotropy is arbitrarily small. Then Nikolaev's result shows that these are diffeomorphic to space forms.

4.2. The topology of $M \smallsetminus C$ when $\lambda > 0$. — Let M^n be a closed n-manifold, $n \geqslant 3$, admitting a 1-QC metric with orientable distinguished line field, which is assumed conformally flat when $n = 3$. Suppose that λ is positive and $\pi_1(M)$ is infinite torsion-free.

Proposition 4.1. — *Any non-compact connected component E of $M \smallsetminus C$ is diffeomorphic to $S^{n-1} \times \mathbb{R}$.*

Proof. — Since $\lambda|_E > 0$ curvature leaves are diffeomorphic to spheres quotients.

By Proposition 2.14 there exists an isometric immersion $\Phi : \widetilde{M} \to \mathbb{H}^{n+1}_\kappa$. Note that each curvature leaf is mapped isometrically onto some round $(n-1)$-sphere by the immersion in $\mathbb{H}^{n+1}_\kappa$ (see e.g. [47]). Recall that *round n-spheres* (sometimes also called geodesic or extrinsic n-spheres) are intersections of a hypersphere with $\mathbb{H}^n_\kappa \subset \mathbb{H}^{n+1}_\kappa$ (see [47, §2]).

Now, any curvature leaf L in M is covered by a disjoint union of spheres in $\widetilde{M}$ and thus L is diffeomorphic to a spherical space forms. The stabilizer of one sphere in $\widetilde{M}$ is finite and obviously contained in $\pi_1(M)$. Thus it must be trivial as $\pi_1(M)$ was assumed to be torsion-free and hence the curvature leaf L is a sphere. Therefore, E is diffeomorphic to $S^{n-1} \times \mathbb{R}$. □

When $C = \varnothing$ then, M is diffeomorphic to a spherical bundle over the circle.

4.3. Closures of globally 1-QC components within 1-QC manifolds. — Suppose now that there exist isotropic points, so that $C \neq \varnothing$.

Lemma 4.2. — *Let E be a connected component of $M \smallsetminus C$ such that $\xi|_E$ is orientable. We assume that $\lambda \geqslant c > 0$ on E. Then $\overline{E}$ is obtained from E by adjoining two boundary spheres, which might possibly be degenerate or tangent at one point or along a codimension-one sphere.*

Proof. — The proof is similar to [8, Lem. 2.7]. Let $q \in \overline{E} \cap C$. Choose a component E_0 of $\pi^{-1}(E) \subset \widetilde{M}$, which is diffeomorphic to a spherical cylinder since E is simply connected. Let then $\widetilde{q}$ be a lift of q which belongs to $\overline{E}$.

Any connected component $L_{\widetilde{p}}$ of the preimage $\pi^{-1}(L_p) \subset \widetilde{M}$ of a curvature leaf through p is sent by the isometric immersion Φ onto an embedded round sphere $S_{\lambda(p)}$ of curvature $\lambda(p)$ in $\mathbb{H}^{n+1}_\kappa$. Now, L_p is a quotient of the sphere $L_{\widetilde{p}}$ by a finite subgroup of $\pi_1(M)$. Since $\pi_1(M)$ is torsion-free L_p is also a sphere, which is isometric to $S_{\lambda(p)}$.

Let $p_i \in E$ be a sequence converging in M to q and $\widetilde{p}_i \in E_0$ lifts converging to $\widetilde{q}$ in $\widetilde{M}$. The curvature leaves L_{p_i} lift to curvature leaves $L_{\widetilde{p}_i}$ in $\widetilde{M}$ which are round spheres of radius R_i. Since λ_i is bounded from below, R_i are bounded. The sphere $L_{\widetilde{p}_i}$ lives in a geodesic n-plane $\mathbb{H}^n_\kappa$ orthogonal at the lift of the unit vector field $\xi|_{\widetilde{p}_i}$. By a compactness argument and passing to a sub-sequence we can assume that R_i and $\xi|_{\widetilde{p}_i}$ converge to R_∞ (which might be 0) and ξ_∞, respectively. Then the spheres $L_{\widetilde{p}_i}$ converge to the round sphere L_∞ of radius R_∞ living in the n-plane $\mathbb{H}^n_\kappa$ orthogonal to ξ_∞ and passing through $\widetilde{q}$.

The argument from [8, Lem. 2.7] applies to show that for any sequence $\widetilde{p'_i} \in E_0$ converging to $\widetilde{q}$ in $\widetilde{M}$ the spheres $L_{\widetilde{p'_i}}$ converge to the round sphere L_∞. Indeed, if we had another limit sphere L'_∞ passing through $\widetilde{q}$, then a small enough connected neighborhood V of some point $p \in L_\infty \smallsetminus L'_\infty$ would intersect all leaves $L_{\widetilde{p}_i}$ and would miss all $L_{\widetilde{p'_i}}$. Since any curvature leaf in E_0 disconnects E_0, this would imply that V is not connected, contradicting our choice.

Therefore $\overline{E_0}$ is obtained from E_0 by adjoining two boundary (possibly degenerate) round spheres corresponding to the ends. Since $\overline{E_0}$ is compact and π is a covering $\pi(\overline{E_0}) = \overline{E}$. Then $\overline{E}$ is obtained from E by adjoining two round spheres. They cannot intersect transversely, as otherwise close enough leaves of E should intersect. Thus the boundary spheres should be either disjoint or tangent. Lifting both of them to $\widetilde{M}$ we obtain two round $(n-1)$-spheres in $\mathbb{H}^{n+1}_\kappa$. They bound two convex n-balls whose intersection is convex. Therefore the spheres intersection is either one point or a $(n-2)$-sphere and in particular it is connected. □

If one only assumes that the metric is 1-QC the curvature leaves function λ might have poles at C, contrasting with the situation described in Lemma 2.11 for locally 1-QC metrics. The behavior of λ is directly related with the behavior, possibly wild, of the distinguished line field ξ in a neighborhood of an isotropic point. Consider a spherical cylinder with a ball adjoined to it, having the form of a rotationally invariant cap whose north pole is the only isotropic point. Then curvature leaves are spheres whose intrinsic curvature λ explodes when approaching the isotropic point. In fact, this is the only phenomenon which can arise:

Lemma 4.3. — *Assume that $\lambda > 0$ on $M \smallsetminus C$ and set*

$$C_\infty = \{q \in \overline{M \smallsetminus C} \cap C \ ; \ \limsup_{p \to q} \lambda(p) = \infty\}.$$

Then,

(1) *either there exists a connected E component of $M \smallsetminus C$ with the property that $q \in \overline{E}$ and $\overline{E}$ is diffeomorphic to a ball containing q,*

(2) *or else there exists a sequence E_i of components of $M \smallsetminus C$ with the property that $q \in \overline{\bigcup_{i=1}^\infty E_i}$, but $q \notin \overline{E_i}$, for any i. Moreover, $\overline{E_i}$ are embedded annuli with disjoint interiors.*

Proof. — Given a compact Riemannian M there exists some constant c_1 such that any embedded sphere S of diameter $d \leqslant c_1$ bounds an embedded ball D of diameter $f_1(d)$

and volume $f_2(d)$, such that $\lim_{d\to 0} f_i(d) = 0$, $i = 1, 2$. It suffices to cover M by finitely many open charts diffeomorphic to balls and to choose then c_1 so that any metric ball on M of radius c_1 is contained within some chart.

Now, the diameter of a round sphere S_λ goes to 0, when $\lambda \to \infty$. The same holds for the diameter of L_λ in the Riemannian metric of M. Therefore, if λ is larger than some c_2 then L_λ bounds an embedded ball D_λ in M.

Consider λ_1 such that the intrinsic diameter of S_{λ_1} is smaller than c_1. There exists a curvature leaf L_1 of curvature $2\lambda_1$ in E, which therefore bounds a ball $D_1 \subset M$. Let $L_2 \subset E \cap (M \smallsetminus D_1)$ be a curvature leaf whose curvature is $\lambda > \lambda_1$ and thus it bounds also a ball D_2 in M. The leaves L_1 and L_2 bound a submanifold E_0. From Proposition 4.1 and the strong form of the global Reeb stability theorem (see [43, 18]) E_0 is compact and diffeomorphic to a spherical cylinder.

Observe that there is an unique embedded ball D bounded by an embedded codimension-one sphere in M, as otherwise M would be homeomorphic to a sphere, contradicting the fact that $\pi_1(M)$ is infinite. Hence $D_2 = E_0 \cup D_1$.

If $q \notin D_1$, as $q \in C_\infty$ we could choose λ large enough so that $f_2(\lambda) < \mathrm{vol}(D_1)$. On the other hand, $D_2 \subset D_1$ and hence the volume of the unique embedded ball bounding L_2 would be larger than $\mathrm{vol}(D_1)$, contradicting our choice of λ. Thus $q \in D_1$.

Note that, when $q \in C_\infty$ is not as above, then there exists an infinite sequence E_i of connected components of $M \smallsetminus C$ such that $q \in \overline{\bigcup_i E_i}$, but $q \notin \overline{E_i}$, for all i. These components should have disjoint interiors and they are diffeomorphic to spherical cylinders. The size of $\overline{E_i}$ (namely the radius of a boundary circle and the width) should converge to 0. Moreover, for every $\varepsilon > 0$ all but finitely many $\overline{E_i}$ are contained in a metric ball of radius ε around q. They are therefore null-homotopic. □

4.4. Construction of neighborhoods of isotropic points

Lemma 4.4. — *Suppose that $\lambda > 0$ on $M \smallsetminus C$. We set*

$$V_\varepsilon(C) = \{p \in M \ ; \ |H(p) - N(p)| < \varepsilon\}.$$

Then there exist arbitrarily small saturated neighborhoods of C, namely for every $\varepsilon > 0$ small enough there exists an open neighborhood $W_\varepsilon \subset V_\varepsilon(C)$ such that $W_\varepsilon \smallsetminus C$ is a union of leaves and $W_\varepsilon \supset V_\delta(C)$.

Proof. — This is a classical result in codimension-one foliations with compact leaves. From Lemmas 4.2 and 4.3 for each $p \in C \cap \overline{M \smallsetminus C}$ there exists a (possibly not unique) limit curvature leaf $L_p \subset C \cap \overline{M \smallsetminus C}$ which passes through p. This limit curvature leaf might degenerate to a single point.

Assume now that $V_\varepsilon(C)$ does not contain a saturated neighborhood. Then we can find a sequence of pairs of points p_i, q_i, each pair belonging to the same curvature leaf in $M \smallsetminus C$ such that $p_i \to p \in C$ but $q_i \notin V_\varepsilon(C)$. If q is an accumulation point of q_i then it cannot belong to C since $|H(q) - N(q)| \geqslant \varepsilon$. The curvature leaf L_q passing through q is contained therefore in $M \smallsetminus C$. This implies that the compact curvature leaves L_{q_i} passing through q_i converge in the Hausdorff topology towards L_q. On the other hand

we have $L_{p_i} = L_{q_i}$ so that it also converges towards some L_p. This contradicts the fact that L_q is complete and thus it cannot contain points from C.

Eventually, if there does not exist some $\delta > 0$ such that $W_\varepsilon \supset V_\delta(C)$, we would find a sequence $p_n \in V_{1/n}(C) \smallsetminus W_\varepsilon$. If p is a limit point of the sequence p_n then $p \notin W_\varepsilon$, as W_ε is open. On the other hand $p \in C$, which is a contradiction. □

Now W_ε is a manifold containing C and whose frontier $\overline{W_\varepsilon} \smallsetminus W_\varepsilon \subset M \smallsetminus C$. Since W_ε is a saturated subset, its frontier consists of union of curvature leaves. From Lemma 4.2 these are round spheres, and in particular $\overline{W_\varepsilon}$ is a codimension zero submanifold of M.

Denote by $N_i(\varepsilon)$ the connected components of $\overline{W_\varepsilon}$. There are manifolds with boundary, which we call *standard neighborhoods of isotropic points*. The complementary $M \smallsetminus W_\varepsilon$ is a union of spherical cylinders, which will be called *necks*.

The first main result is the following chopping result:

Proposition 4.5. — *Assume that $\lambda(p) \geqslant c > 0$ for $p \in M \smallsetminus C$. Then there exist only finitely many components $N_i(\varepsilon)$ and also only finitely many necks.*

4.5. Preliminaries concerning the positive λ

Lemma 4.6. — *If $\lambda(p) > 0$, for all $p \in M \smallsetminus C$ and $H|_C > 0$, then there exists $\lambda_{\min}$ such that $\lambda(p) \geqslant \lambda_{\min} > 0$, for all $p \in M \smallsetminus C$.*

Proof. — As H is smooth on M and C is compact, there exists $\delta > 0$ and an open neighborhood V of C in M such that $H|_V > \delta$. On the other hand λ is smooth on $M \smallsetminus C$ and $M \smallsetminus V$ is compact hence $\inf_{p \in M \smallsetminus V} \lambda(p) \geqslant \delta' > 0$. We also have $\lambda(p) \geqslant H(p) \geqslant \delta$, if $p \in V$, so that we can take $\lambda_{\min} = \min(\delta, \delta')$. □

Lemma 4.7. — *There exists some constant $\lambda_{\max}(\varepsilon)$ such that whenever $p \notin W_\varepsilon$, then*

$$\lambda(p) \leqslant \lambda_{\max}(\varepsilon).$$

Proof. — We have

$$\lambda(p) = H(p) + \frac{\|\operatorname{grad}(H)_p\|}{4(H(p) - N(p))^2} \leqslant \sup_{p \in M} H(p) + \frac{1}{4\delta^2} \sup_{p \in M} \|\operatorname{grad} H|_p\|,$$

where $\delta = \delta(\varepsilon)$ is given in Lemma 4.4. □

Lemma 4.8. — *Assume that the compact n-manifold X can be isometrically immersed into $\mathbb{H}^{n+1}_\kappa$ so that its boundary ∂X goes onto the round sphere S_λ. Then the volume of X is greater than or equal to the volume of the standard n-ball B_λ with boundary S_λ lying in an n-plane $\mathbb{H}^n_\kappa$ embedded in $\mathbb{H}^{n+1}_\kappa$.*

Proof. — As the orthogonal projection $\mathbb{H}^{n+1}_\kappa \to \mathbb{H}^n_\kappa$ is distance-nonincreasing, the measure of X is greater than the measure of its projection. Now, the image of X is a homological n-cycle with boundary S_λ. Then, this n-cycle cannot miss a point of B_λ because the inclusion of S_λ into $\mathbb{H}^n_\kappa - \{\mathrm{pt}\}$ is a homology isomorphism. □

In particular, Lemma 4.8 shows that the n-dimensional Lebesgue measure of X is uniformly bounded from below by the volume of B_λ. Recall that a round sphere S_λ is a metric hypersphere in a totally geodesic $\mathbb{H}^n_\kappa \subset \mathbb{H}^{n+1}_\kappa$. Now a hypersphere of radius d in $\mathbb{H}^n_\kappa$, with $\kappa \neq 0$, has principal curvatures $\sqrt{\kappa}/\tanh(\sqrt{\kappa}d)$ and hence sectional curvature:

$$\lambda = \kappa\Big(\frac{1-\tanh^2(\sqrt{\kappa}d)}{\tanh^2(\sqrt{\kappa}d)}\Big).$$

Thus its volume of B_λ is a function $C_1(\lambda,\kappa,n) > 0$ decreasing as a function of the intrinsic curvature λ for all $\kappa \geqslant 0$, namely such that:

$$\lim_{\lambda\to\infty} C_1(\lambda,\kappa,n) = 0, \quad \lim_{\lambda\to 0} C_1(\lambda,\kappa,n) = \infty.$$

4.6. Compactness of $N_i(\varepsilon)$. — We first prove:

Proposition 4.9. — *Assume that $\lambda(p) \geqslant c > 0$ for $p \in M \smallsetminus C$. Then each $N_i(\varepsilon)$ is compact.*

Proof. — Such a component $N_i(\varepsilon)$ is compact, unless its boundary $\partial N_i(\varepsilon)$ consists of infinitely many $(n-1)$-spheres. All but finitely many such $(n-1)$-spheres should be separating, by Grushko's theorem since each non-separating $(n-1)$-sphere contributes a free factor $\mathbb{Z}$ to $\pi_1(M)$.

Now, let M_j be the compact connected submanifolds of M bounded by the separating $(n-1)$-spheres in $\partial N_i(\varepsilon)$. By van Kampen's theorem and Grushko's theorem all but finitely many M_j should be simply connected. Since M_j is simply connected, the inclusion into M lifts to an embedding of M_j into $\widetilde{M}$, and thus M_j is isometrically immersed into $\mathbb{H}^{n+1}_\kappa$ such that its boundary goes onto some S_λ.

Now, each ∂M_j is a curvature leaf isometrically embedded into $\mathbb{H}^{n+1}_\kappa$ as a round $(n-1)$-sphere S_λ of curvature λ. From lemma 4.8 the n-dimensional measure of M_j is uniformly bounded from below by $C_1(\lambda,\kappa,n) > 0$.

Since $\sup_{p\in\partial W}\lambda \leqslant \lambda_{\max}(\varepsilon) < \infty$ by Lemma 4.7 the n-dimensional measure of M_j is uniformly bounded from below by $C_1(\lambda_{\max}(\varepsilon),\kappa,n) > 0$. Since M has finite volume there are only finitely many pairwise disjoint domains M_j and hence only finitely many components of $\partial N_i(\varepsilon)$. □

Remark 4.10. — The proof above shows that the number of separating boundary components which separate simply connected domains is uniformly bounded by $\mathrm{vol}(M)/C_1(\sup_{p\in M\smallsetminus W_\varepsilon}\lambda,\kappa,n)$.

4.7. Proof of Proposition 4.5. — As above, all but finitely many $N_i(\varepsilon)$ are simply connected, by van Kampen's and Grushko's theorems.

Lemma 4.11. — *If $N_i(\varepsilon)$ is simply connected then it is diffeomorphic to a sphere with finitely many disjoint open balls removed.*

Proof. — Since $N_i(\varepsilon) \subset M$ is compact and simply connected it can be isometrically immersed into $\mathbb{H}^{n+1}_\kappa$. Moreover, the boundary $\partial N_i(\varepsilon)$ consists of finitely many $(n-1)$-spheres. Since $N_i(\varepsilon)$ is compact and any ball in $\mathbb{H}^{n+1}_\kappa$ is conformally equivalent to a ball in the Euclidean space $\mathbb{R}^{n+1}$ it follows that $N_i(\varepsilon)$ can be conformally immersed into $\mathbb{R}^{n+1}$. There is then a standard procedure of gluing balls along the boundary within the realm of conformally flat manifolds (see [46, §2]). We obtain then a conformally flat closed manifold $\widehat{N_i(\varepsilon)}$ which is simply connected. Kuiper's theorem ([35]) implies that $\widehat{N_i(\varepsilon)}$ is diffeomorphic to a sphere, and hence the lemma. □

Lemma 4.12. — *There are only finitely many components $N_i(\varepsilon)$ other than cylinders.*

Proof. — We already saw above that there are finitely many components $N_i(\varepsilon)$ with one boundary component or not simply connected. It remains to show that we cannot have infinitely many holed spheres $N_i(\varepsilon)$ with at least three holes. Assume the contrary. Then all but finitely many of them should be connected among themselves using cylinders. But a trivalent graph has at least that many free generators of its fundamental group as vertices. This implies that $\pi_1(M)$ has a free factor of arbitrarily large number of generators, contradicting the Grushko theorem. □

The only possibility left is to have infinitely many manifolds $N_i(\varepsilon)$ which are diffeomorphic to cylinders $S^{n-1} \times [0,1]$, to be called *tubes*. Only finitely many of these tubes can be non-separating. We say that two separating tubes $N_i(\varepsilon)$ and $N_j(\varepsilon)$ are equivalent if the component of $M \setminus (N_i(\varepsilon) \cup N_j(\varepsilon))$ joining the boundary components of the tubes is a cylinder, i.e., the tubes are isotopic in M.

A separating tube is *inessential* if one of the two components of its complement is a ball and *essential* otherwise. There are also only finitely many equivalence classes of separating essential tubes $N_i(\varepsilon)$.

In an infinite family of essential pairwise equivalent $N_i(\varepsilon)$ we have necks C_i connecting them in a chain. Their union is then an open spherical cylinder. By Lemma 4.2 its closure is a closed spherical cylinder or a ball. Moreover, every $N_i(\varepsilon)$ contains a round sphere $X_i \subset N_i(\varepsilon)$ obtained as a limit curvature leaf. Then, the limit X_∞ of X_i exists and is a round sphere possibly degenerate, by the arguments from the proof of Lemma 4.3 and [8]. On the other hand $X_\infty \subset C$ as $H(p) = N(p)$ for all $p \in X_i$. Thus $X_\infty \subset W_\varepsilon$ and, according to Lemma 4.4, sufficiently closed X_i are also contained in W_ε. This contradicts the fact that there is a neck C_i separating X_i from X_{i+1} which is disjoint from W_ε.

If we have an infinite family of inessential tubes $N_i(\varepsilon)$ accumulating to a point q of C, then all but finitely many will be contained within the neighborhood $N_j(\varepsilon)$ containing q, leading again to a contradiction. This proves Proposition 4.5.

4.8. The geometry of $N_i(\varepsilon)$ in the case $H|_{M \setminus C} > 0$. — We give here a simple argument which permits to conclude the proof when $H > 0$.

Lemma 4.13. — *Assume that $H > 0$ and $n \geqslant 3$. Then $N_i(\varepsilon)$ is conformally equivalent to a spherical space form with boundary.*

Proof. — We have $|N(p) - H(p)| \leqslant \varepsilon$, for any $p \in N_i(\varepsilon)$. As H was supposed to be positive on M there exists some K such that $\min(H(p), N(p)) \geqslant K > 0$ for any $p \in N_i(\varepsilon)$.

Consider a connected component V_i of $\pi^{-1}(N_i(\varepsilon))$, where $\pi : \widetilde{M} \to M$ is the universal covering projection. Since the sectional curvature of V_i is bounded from below by $K > 0$, the Bonnet-Myers theorem implies that V_i has bounded diameter and hence it is compact. Each boundary component of V_i has constant positive curvature and hence is diffeomorphic to a sphere. This implies that $\pi_1(V_i)$ is a free factor of $\pi_1(\widetilde{M}) = 1$ and so V_i must be simply connected. Then the conformal gluing of balls described in [46, §2] along the boundary components of V_i produces a closed simply connected conformally flat manifold $\widehat{V}_i$ which can be immersed into $\mathbb{R}^{n+1}$. Therefore by Kuiper's theorem ([35]) $\widehat{V}_i$ is conformally equivalent to a sphere. We derive that V_i is conformally equivalent to a holed sphere, since it is obtained from a sphere by deleting out the interiors of several disjoint round balls.

Now, notice that any closed loop in $N_i(\varepsilon)$ can be lifted to a path in V_i, so that the action of the subgroup $\pi_1(N_i(\varepsilon))$ of $\pi_1(M)$ as a deck transformation subgroup acting on $\widetilde{M}$ keeps V_i globally invariant. Thus V_i is endowed with a free action by $\pi_1(N_i(\varepsilon))$. Since V_i is compact $\pi_1(N_i(\varepsilon))$ must be finite.

The action of $\pi_1(N_i(\varepsilon))$ on the holed sphere V_i is by conformal diffeomorphisms. Then Liouville's theorem (see [38, Th. 3.1]) implies that each one of these diffeomorphisms is the restriction of some global conformal diffeomorphism of the round sphere S^n, as $n \geqslant 3$. The group generated by the global conformal diffeomorphisms of S^n is still isomorphic to $\pi_1(N_i(\varepsilon))$, as any conformal diffeomorphism which is identity on an open subset of S^n is the identity. The spherical metric induces then a spherical space form structure on $N_i(\varepsilon)$. Moreover, $N_i(\varepsilon)$ is obtained from the quotient of S^n by the finite group $\pi_1(N_i(\varepsilon))$ by removing several open balls. □

Remark 4.14. — Observe that the immersability property and the strict positive curvature enables us to find an elementary proof disposing of the differentiable sphere theorem under a pointwise pinched curvature assumption.

Remark 4.15. — Note that the $\pi_1(N_i(\varepsilon))$-action on S^n might possibly not be free. Its point stabilizers might be finite, in which case the quotient $S^n/\pi_1(N_i(\varepsilon))$ is a so-called orbifold. The proof actually shows that M is conformally equivalent to a connected sum of spherical space forms and classical Schottky manifolds (see [46]).

Remark 4.16. — The proof of Theorem 1.6 shows that closed manifolds having positive horizontal curvature for some 1-QC *analytic* metric should be conformally equivalent to either the n-sphere, $S^1 \times S^{n-1}$ or $S^1 \times_{-1} S^{n-1}$. Here $S^1 \times_{-1} S^{n-1}$ denotes the non-orientable S^{n-1}-bundle over S^1 with structure group $\mathrm{O}(n)$, i.e., obtained when the monodromy map is a reflection in a hyperplane.

4.9. Proof of Theorem 1.8. — The more general situation when $H|_C > 0$ is treated along the same lines as before. A small neighborhood of isotropic points has boundary

consisting of spheres. We wish to glue along each boundary component a ball by identifying the corresponding boundary spheres by means of some (gluing) diffeomorphism. This operation will be called *capping off* boundary spheres. The homeomorphism type of the closed manifold obtained by capping off its boundary components is well-defined, namely independent on the gluing diffeomorphisms. We should stress that, in contrast, the diffeomorphism type of the resulting manifold is determined by the classes of the gluing diffeomorphisms in the group of sphere diffeomorphisms up to isotopy, which can be non-trivial – for instance when $n = 7$. However, if the boundary spheres are endowed with Riemannian metrics of constant curvature metrics and the gluing diffeomorphism is an isometry, then the capped off manifold is well-defined up to diffeomorphism, since orientation preserving isometries are isotopic.

Our aim is to realize a *geometric capping off* using spherical caps of constant curvature as balls and isometries as gluing diffeomorphisms. Both the manifold and the spherical caps will be isometrically immersed in $\mathbb{H}^{n+1}_\kappa$. The resulting manifold inherits a $\mathscr{C}^1$ Riemannian metric structure, which is $\mathscr{C}^\infty$ outside the joint spheres. A slight perturbation of the metric will produce a $\mathscr{C}^\infty$-metric.

Our key result below shows that small neighborhoods of isotropic points are standard.

Proposition 4.17. — *If ε is small enough then those $N_i(\varepsilon)$ which have positive sectional curvature are diffeomorphic to spherical space forms with several disjoint open balls removed.*

We postpone the rather involved proof until the next Section 4.10.

End of proof of Theorem 1.8. — By hypothesis $\lambda > 0$ on $M \smallsetminus C$ and $H|_C > 0$. From Lemma 4.6, λ has a positive lower bound on $M \smallsetminus C$ and by Proposition 4.5 there are only finitely many $N_i(\varepsilon)$ and finitely many necks, which are spherical cylinders. By choosing ε small enough we can assume that all $N_i(\varepsilon)$ have positive sectional curvature, because $N|_C = H|_C > 0$ and $N_i(\varepsilon)$ are saturated by Lemma 4.4. Proposition 4.17 above shows that $N_i(\varepsilon)$ are diffeomorphic to spherical space forms with several disjoint open balls removed. Thus M can be obtained by gluing together the holed spherical space forms and spherical cylinders along their boundary spheres, which correspond to attaching 1-handles. □

4.10. Proof of Proposition 4.17. — The saturation Lemma 4.4 shows that given $\nu > 0$ there exists $\varepsilon(\nu) > 0$ small enough such that $|N(p) - H(p)| \leqslant \nu$, for any $p \in N_i(\varepsilon)$, meaning that $N_i(\varepsilon)$ has pointwise pinched curvature.

Consider the manifold $\widehat{N_i(\varepsilon)}$ obtained topologically from $N_i(\varepsilon)$ by capping off boundary spheres by balls. We will show that $\widehat{N_i(\varepsilon)}$ also admits a pointwise pinched Riemannian metric.

Let S be a boundary component of $N_i(\varepsilon)$ and F be a collar of S in $N_i(\varepsilon)$ which is saturated by curvature leaves. In particular, F is diffeomorphic to $S^{n-1} \times [0, 1]$. As F is simply connected, the inclusion $F \subset M$ could be lifted to $\widetilde{M}$ and thus F is

embedded into $\mathbb{H}^{n+1}_\kappa$. We say that F has *width* w if the distance between the two boundary components is w.

Before we proceed, recall that the *envelope* of a one-parameter family of hypersurfaces in $\mathbb{H}^{n+1}_\kappa$ is a hypersurface tangent to each member of the family (see [19] for the case of surfaces). Alternatively, it is the boundary of the region filled by the family of hypersurfaces. If the family is given by equations $F_t(x) = 0$, depending smoothly on the parameter t, then its envelope is the set of points x for which there exists a such that $F_a(x) = \partial F_t/\partial t|_{t=a}(x) = 0$. The envelope of a one-parameter family of hyperspheres is usually called a *canal* hypersurface in the literature.

We will first need the following lemma, whose proof is postponed to Section 4.11 below:

Lemma 4.18

(1) *Assume $H(S) > 0$. If the width w of the collar F is small enough then we can cap off smoothly F along the boundary component S by attaching a hyperspherical cap $\mathscr{H}^+(S)$ using an isometry between its boundary $\partial\mathscr{H}^+(S)$ and S and of radius*

$$d = \frac{1}{\sqrt{\kappa}} \operatorname{arctanh} \sqrt{\kappa/(H(S)+\kappa)} = \frac{1}{2\sqrt{\kappa}} \log\left(\frac{1+\sqrt{\kappa/(H(S)+\kappa)}}{1-\sqrt{\kappa/(H(S)+\kappa)}}\right),$$

so that $F \cup_S \mathscr{H}^+(S)$ embeds isometrically in $\mathbb{H}^{n+1}_\kappa$.

(2) *If $H(S) \leqslant 0$ then there is no hyperspherical cap tangent to F along S.*

We suppose from now on that $H(S) > 0$. Let now $\widehat{N_i(\varepsilon)}$ denote the manifold obtained from $N_i(\varepsilon)$ by adding to each boundary component S the corresponding spherical cap $\mathscr{H}^+(S)$. Its diffeomorphism type is uniquely determined, independently on the choice of the orientation preserving isometries used for gluing.

Note that both F and $\mathscr{H}^+(S)$ are embedded in $\mathbb{H}^{n+1}_\kappa$, because they are simply connected. By choosing a collar F of small enough width we can assume that their union $F \cup \mathscr{H}^+(S)$ remains embedded in $\mathbb{H}^{n+1}_\kappa$. We actually only need this union to be immersed in $\mathbb{H}^{n+1}_\kappa$. Then, the Riemannian metrics on F and $\mathscr{H}^+(S)$ are restrictions of the metric induced on their union by the embedding $F \cup_S \mathscr{H}^+(S) \hookrightarrow \mathbb{H}^{n+1}_\kappa$.

We define a tensor on $\widehat{N_i(\varepsilon)}$ to be the 1-QC Riemannian metric on points of $N_i(\varepsilon)$ and the Riemannian metric on $F \cup_S \mathscr{H}^+(S)$ which is induced by its embedding in $\mathbb{H}^{n+1}_\kappa$. Since these two metrics coincide on F this tensor is a Riemannian metric on $\widehat{N_i(\varepsilon)}$.

The regularity of the metric depends on the regularity of the embedding

$$F \cup_S \mathscr{H}^+(S) \hookrightarrow \mathbb{H}^{n+1}_\kappa.$$

Thus the metric is $\mathscr{C}^\infty$ everywhere except possibly at the joint spheres $\partial N_i(\varepsilon)$ where the metric might be only of class $\mathscr{C}^1$, because $F \cup_S \mathscr{H}^+(S)$ is only a $\mathscr{C}^1$-submanifold of $\mathbb{H}^{n+1}_\kappa$, in general. However, an arbitrarily small perturbation of the embedding (or the metric) by an infinitely flat function along a small collar of the boundary sphere S will make it smooth. Recall that an infinitely flat function is a smooth real function which vanishes on $(-\infty, 0]$ and equals 1 on $[\delta, \infty)$.

Further, we need to introduce another metric invariant of a Riemannian manifold. Recall from [44] that the *pointwise anisotropy* of the metric is the quantity:

$$I(p) = \sup_{\sigma} \Big| K_p(\sigma) - \frac{R(p)}{n(n-1)} \Big|,$$

where $K_p(\sigma)$ is the curvature of the plane σ at p and $R(p)$ is the scalar curvature at p. Furthermore, the *integral anisotropy* of the manifold N is the number

$$\int_N I(p)\, d\,\mathrm{vol}\,.$$

Lemma 4.19. — *Assume $H(S) > 0$. For $n \geqslant 3$ there exist two constants C_1, C_2 such that for any $\delta > 0$ and for any for small enough ε we have:*

(1) $\mathrm{vol}(\widehat{N_i(\varepsilon)}) \geqslant C_1 > 0$*;*
(2) $\mathrm{diam}^2(\widehat{N_i(\varepsilon)}) \sup_{p \in \widehat{N_i(\varepsilon)}, \sigma} |K_p(\sigma)| < C_2$*;*
(3) *and the integral isotropy of $\widehat{N_i(\varepsilon)}$ is smaller than δ.*

The proof of Lemma 4.19 is deferred to Section 4.12. The last ingredient needed is the following result of Nikolaev (see [44]):

Proposition 4.20. — *Given any constants C_1, C_2 there exists some $\delta(C_1, C_2, n) > 0$ such that any closed n-manifold N satisfying*

$$\mathrm{vol}(N) \geqslant C_1 > 0, \qquad \mathrm{diam}^2(N) \sup_{p \in N, \sigma} |K_p(\sigma)| < C_2,$$

and whose integral isotropy is smaller than δ, is diffeomorphic to a space form.

End of proof of Proposition 4.17. — Recall that for every $\nu > 0$ there exists $\varepsilon(\nu) > 0$ such that $|N(p) - H(p)| \leqslant \nu$, for any $p \in N_i(\varepsilon)$. Let $\nu > 0$ be such that $\nu \cdot \mathrm{vol}(M) < \delta(C_1, C_2)$. We choose $\varepsilon < \varepsilon(\nu)$. Then, the Riemannian manifold $\widehat{N_i(\varepsilon)}$ satisfies all conditions of Proposition 4.20. Thus $\widehat{N_i(\varepsilon)}$ is diffeomorphic to a space form and our claim follows.

4.11. Proof of Lemma 4.18. — The image of the collar F in $\mathbb{H}^{n+1}_\kappa$ by the immersion is a canal hypersurface. This was already proved in [46, 22] for the case where $\kappa = 0$.

Assume now that $\kappa > 0$. We want to show that each curvature leaf S the hypersurface F is tangent to a fixed round hypersphere $\mathscr{H}(S)$. Let ψ be the unitary normal vector field on the image of F in $\mathbb{H}^{n+1}_\kappa$ and $\overline{\nabla}$ be the Levi-Civita connection on $\mathbb{H}^{n+1}_\kappa$. The Weingarten (or shape) operator L_ψ is defined as $L_\psi X = (\overline{\nabla}_X \psi)^T$, where the superscript ^{T} means the tangent part. Moreover, $g(L_\psi X, Y) = h(X, Y)$, where X, Y are tangent to M and h is the second fundamental form of M. It follows that $L_\psi X = -\sqrt{H + \kappa} \cdot X$ for any vector field X tangent to the curvature leaf S.

Lemma 4.21. — *The image of a curvature leaf S with $H(S) > 0$ by the immersion into $\mathbb{H}^{n+1}_\kappa$ is a round sphere. Furthermore, either the focal locus of the image of F is empty or else there is a point $q \in \mathbb{H}^{n+1}_\kappa$ such that a hypersphere $\mathscr{H}(S)$ with center q contains S.*

Proof. — This seems to be widely known (see [22] for $\kappa = 0$). The first part follows from the case $\kappa = 0$, as balls in $\mathbb{H}^{n+1}_\kappa$ are conformally equivalent to balls in the Euclidean space and curvatures leaves are sent into rounds spheres.

An alternate proof goes as follows. According to [11] the focal locus of F is diffeomorphic to an interval, because the focal points have multiplicity $n-1$. An endpoint of the focal locus of F is equidistant to the points of S, since the derivative of the focal map ([11, §1.e, Th.. 2.1 & proof of Th. 3.1]) is trivial. This shows that S is contained in a hypersphere $\mathscr{H}(S)$.

Still another proof of this statement could be found in [31] in a slightly different wording. □

In order to compute the radius of the hypersphere $\mathscr{H}(S)$, we recall that for a unit speed geodesic γ parametrized by $[0, l]$ issued from a point p and normal to the submanifold W of $\mathbb{H}^{n+1}_\kappa$ the Jacobi field $Y(t)$ along γ is a W-Jacobi field if $Y(0)$ is tangent to W at p, $Y(t)$ is orthogonal to $\dot\gamma(t)$ for all t, and $\nabla_{\dot\gamma(0)}Y(0) + L_{\dot\gamma(0)}Y(0)$ is orthogonal to W, where $L_{\dot\gamma(0)}$ is the Weingarten (or shape) operator of W. The point q on γ is called a focal point of W along γ if there exists a non-trivial W-Jacobi field along γ vanishing at q.

If $X_1, X_2, \ldots, X_n$ is a set of parallel vector fields along γ which form along with $\dot\gamma$ a basis of the ambient tangent space to $\mathbb{H}^{n+1}_\kappa$, then a Jacobi field on $\mathbb{H}^{n+1}_\kappa$ along γ has the form (see [13]):

$$Y(t) = \sum_{i=1}^{n} (a_i \sinh \sqrt{\kappa} t + b_i \cosh \sqrt{\kappa} t)\, X_i(t),$$

where $a_i, b_i \in \mathbb{R}$.

An immediate computation yields the fact that focal points of S along geodesics pointing in the direction of ψ exist if and only if $H(S) > 0$, in which case they are all at the same distance:

$$d = \frac{1}{\sqrt{\kappa}} \operatorname{arctanh} \sqrt{\kappa/(H+\kappa)}$$

Moreover $\mathscr{H}(S)$ and F are tangent along S since they have the same normal vector field ψ at the points of S.

Eventually, S bounds two balls in $\mathscr{H}(S)$ and one of them is a spherical cap $\mathscr{H}^+(S)$ with the property $F \cup_S \mathscr{H}^+(S)$ is $\mathscr{C}^1$. This proves Lemma 4.18.

Remark 4.22. — We have results of similar nature when $H(S) \leqslant 0$. The focal locus of F is empty as the focal map is only defined when $|\kappa/(H(S)+\kappa)| < 1$. Further, the normal geodesics in the direction $\pm\psi$ diverge. Nevertheless there exists a totally umbilical embedded n-plane of constant negative intrinsic curvature $H(S)$ which plays this time the role of the round sphere $\mathscr{H}(S)$. However, now the cap $\mathscr{H}^+(S)$ is unbounded. These are called *equidistant planes* (or hypersurfaces) and were first considered in [12]. In fact they are the locus of points at a given distance r from a hyperbolic hyperplane of $\mathbb{H}^{n+1}_\kappa$, or a horosphere respectively. For instance, when $H(S) = 0$, $\mathscr{H}(S)$ is a horosphere centered at the point at infinity where all normal

geodesics to S abut. Further, when $H(S) < 0$ we have an equidistant plane for the distance $r = \operatorname{arccosh}\big(1/\sqrt{|H(S)|}\big)$.

4.12. Proof of Lemma 4.19. — The argument used in the proof of Lemma 4.9 shows that the measure of the spherical cap $\mathscr{H}^+(S)$ is at least that of the hyperbolic ball B_λ lying in a n-plane and having the same boundary sphere S, and thus uniformly bounded from below by the positive constant $C_1(\lambda_{\max}(\varepsilon), \kappa, n)$. Therefore, the volume of $\widehat{N_i(\varepsilon)}$ is uniformly bounded from below by $C_1(\lambda_{\max}(\varepsilon), \kappa, n) > 0$.

Each boundary component of $N_i(\varepsilon)$ is either *separating* M in two disjoint components or *non-separating*. The separating spheres are of two types: either one separated component is simply connected, or else none of the components are simply connected. The boundary component is accordingly called trivial and respectively non-trivial.

Each non-separating or separating non-trivial boundary component induces a non-trivial splitting of the fundamental group $\pi_1(M)$ as a free amalgamated product. It follows that their total number is bounded from above by the maximal number $r(\pi_1(M))$ of factors in a free splitting of the fundamental group $\pi_1(M)$. Further, according to Remark 4.10, the number of trivial separating boundary components is uniformly bounded by $\operatorname{vol}(M)/C_1(\lambda_{\max}(\varepsilon), \kappa, n)$. Moreover, each spherical cap $\mathscr{H}^+(S)$ lies on a hypersphere of radius d, where d is given in Lemma 4.18. It follows that the diameter of $\widehat{N_i(\varepsilon)}$ is uniformly bounded from above by

$$C_2 = C_2(\operatorname{vol}(M), \lambda, H, \kappa, n) = r(\pi_1 M) \cdot \operatorname{diam}(M) + 2\pi d \cdot \frac{\operatorname{vol}(M)}{C_1(\lambda_{\max}(\varepsilon), \kappa, n)},$$

while the volume $\widehat{N_i(\varepsilon)}$ is at least $C_1 = C_1(\lambda_{\max}(\varepsilon), \kappa, n)$. Observe that d is uniformly bounded in terms of $|\inf_{p\in M} H|$.

Now, hyperspheres $\mathscr{H}(S)$ of radius d in $\mathbb{H}^{n+1}_\kappa$ have constant sectional curvature. As they are totally umbilical, their principal curvatures are all equal to $\sqrt{\kappa}/\tanh(\sqrt{\kappa}d)$. It follows that the sectional curvature of the hypersphere $\mathscr{H}(S)$ is equal to $H(S)$. By choosing the smoothing collar F of S of small enough width we obtain:

$$\sup_{p\in\widehat{N_i(\varepsilon)},\sigma} |K_p(\sigma)| \leqslant \max\Big(\sup_{p\in M} |H|, \sup_{p\in M} |N|\Big)$$

On the other hand the pointwise anisotropy $I(p)$ is non-zero only on the subset $N_i(\varepsilon)$, as spherical caps have constant curvature. Thus the integral anisotropy of $\widehat{N_i(\varepsilon)}$ is bounded by above by $\nu \int_M |H| d\operatorname{vol}$. The claim of Lemma 4.19 follows.

Remark 4.23. — We don't know whether pieces $N_i(\varepsilon)$ for small enough ε can be assigned a well-defined sign according to the curvature at their isotropic points. This would permit to use Gromov's pointwise pinching result in negative curvature (see [25]) under bounded volume assumptions. This would be so if locally the dimension of C were at least 3, by F. Schur ([50]).

Remark 4.24. — Isotropic points where the sectional curvature is negative can only be approached by leaves S with $H(S) < 0$ and the method considered above cannot

provide an isometric capping off for the boundary components. For instance, if some negatively curved Gromov-Thurston manifold (see [26]) admits a spine which can be realized by points of constant sectional curvature, then we could provide examples of 1-QC manifolds which are not obtained by gluing together space forms. Recall that a spine for a closed manifold M is a polyhedron $P \subset M$ with the property that $M \setminus D^n$ collapses onto P, where $D^n \subset M$ denotes an n-ball. A typical spine could be geometrically constructed using the cut locus of nice metrics.

5. More about the equivariant immersions of 1-QC manifolds

Throughout this section M is a closed orientable n-manifold endowed with a 1-QC Riemannian metric, which is assumed conformally flat when $n = 3$.

Proposition 5.1. — *Suppose that $n = 3$ and κ is such that $H + \kappa > 0$ on M.*

(1) *If $N + \kappa \neq 0$ then there exists an unique isometric immersion $f : \widetilde{M} \to \mathbb{H}^{n+1}_\kappa$, up to an isometry of $\mathbb{H}^{n+1}_\kappa$.*

(2) *Assume that $N + \kappa = 0$. Then the set of such equivariant isometric immersions of a component E of $M \setminus C$ is in bijection with the set of paths in the space of complex structures over the surface L corresponding to a curvature leaf.*

Proof. — If $\{X_1, X_2, X_3 = \xi\}$ is a local orthonormal basis and h_{ij} are components of the second fundamental form of the immersion then Gauss equations read:

$$h_{11}h_{22} - h_{12}^2 = \mu, h_{11}h_{33} - h_{13}^2 = \nu, h_{22}h_{33} - h_{23}^2 = \nu$$
$$h_{23}h_{11} = h_{12}h_{13}, h_{13}h_{11} = h_{12}h_{23}, h_{12}h_{33} = h_{13}h_{23},$$

where $\mu = H + \kappa$, $\nu = N + \kappa$. The homogeneous equations above imply that either $h_{11} = h_{22}$ or else $\nu(h_{11} + h_{22}) = \mu h_{33}$. The first alternative leads to the immersion arising from the form h in Proposition 2.14, as happened for $n = 4$. The second one coupled with the non-homogeneous equations leads to $\nu = 0$. In this case we obtain the solution consisting of h_{ij} with

$$h_{33} = h_{13} = h_{23} = 0,$$

while the other components are satisfying:

$$h_{11}h_{22} - h_{12}^2 = \mu.$$

It follows that the restriction of h to each lift $\pi^{-1}(L)$ of a curvature leaf $L \subset E$ (of constant curvature H_0) in $\widetilde{M}$ is the second fundamental form of an immersion $f_{\lambda_0} : \pi^{-1}(L) \to \mathbb{H}^n_\kappa$. This immersion f_{λ_0} is a slice of the immersion $f : \widetilde{M} \to \mathbb{H}^{n+1}_\kappa$, with respect to the foliation of $\mathbb{H}^{n+1}_\kappa$ by hyperbolic hyperplanes $\mathbb{H}^n_\kappa$.

In [41, Prop. 2.5] one identified the 2-tensors satisfying the Gauss and Codazzi equations for a constant curvature metric on a surface with the complex structures on the surface. Therefore, the complex structures of the slices of E give us a path of complex structures on the slice surface. Recall that the sectional curvature of the slice surface is the not necessarily constant function λ on E, so that we need to rescale it in every slice in order to fit exactly into the framework of [41, Prop. 2.5]. □

Observe that the immersion $f : \widetilde{M} \to \mathbb{H}^{n+1}_\kappa$ is covered by an equivariant immersion into the (unit) tangent bundle $\widehat{f} : \widetilde{M} \to T\mathbb{H}^{n+1}_\kappa$, defined by

$$\widehat{f}(p) = (f(p), n_{f(p)}) \in T\mathbb{H}^{n+1}_\kappa,$$

where $n_{f(p)}$ denotes the unit (positive) normal vector at $f(\widetilde{M})$ at the point $f(p)$ associated to the local sheet defined around p.

Let us further consider the hyperbolic Gauss map $G : T\mathbb{H}^{n+1}_\kappa \to S^n$ which sends the pair (p, v), where $p \in \mathbb{H}^{n+1}_\kappa, v \in T_p\mathbb{H}^{n+1}_\kappa$, to the point at infinity to which the geodesic issued from p in direction v will reach the boundary. Here, S^n is identified with the boundary at infinity $\partial\mathbb{H}^{n+1}_\kappa$ of the hyperbolic space.

Proposition 5.2. — *Suppose that $H + \kappa > 0$, $N + \kappa > 0$ and $n \geqslant 3$. The composition $D_\infty = G \circ \widehat{f} : \widetilde{M} \to S^n$ is an immersion. Moreover the flat conformal structure defined by the 1-QC metric on M is the one given by the developing map F and the holonomy homomorphism ρ.*

Proof. — If $H+\kappa > 0$ and $N+\kappa > 0$, then $f(\widetilde{M})$ is infinitesimally convex, namely the eigenvalues of its second fundamental form are positive. According to a well-known result of Bishop (see [3]) $f(\widetilde{M})$ is locally convex. This implies that D_∞ is an immersion. Moreover, by the arguments above the immersion F is equivariant with respect to the homomorphism $\rho : \pi_1(M) \to \mathrm{SO}(n+1, 1)$, this time interpreting $\mathrm{SO}(n+1, 1)$ as the group of Möbius (conformal) transformations of the sphere. It follows that the conformally flat structure on M is actually given by the developing map D_∞ and the holonomy homomorphism ρ. □

Proposition 5.3. — *To any closed n-manifold M, $n \geqslant 3$, endowed with a 1-QC Riemannian metric we can associate a hyperbolic metric on $M \times [0, \infty)$ which restricts to (a homothetic of) the initial 1-QC metric on the boundary.*

Proof. — Propositions 2.14 and 2.16 for $n \geqslant 4$ and Proposition 5.1 for $n = 3$ show that for large enough κ there exists a unique equivariant immersion $f : \widetilde{M} \to \mathbb{H}^{n+1}_\kappa$. We can construct under the conditions of Proposition 5.2 an equivariant immersion $\phi : \widetilde{M} \times [0, \infty) \to \mathbb{H}^{n+1}_\kappa$ by setting:

$$\phi(p, t) = \exp(t n_{f(p)}).$$

This is well-defined for all $t \in [0, \infty)$, and it is an immersion, since $f(\widetilde{M})$ is locally convex. In particular, we have a constant curvature metric $\phi^* g_{\mathbb{H}^{n+1}_\kappa}$ induced by this immersion as pull-back of the usual Riemannian metric $g_{\mathbb{H}^{n+1}_\kappa}$ on $\mathbb{H}^{n+1}_\kappa$. Since ϕ is equivariant the metric descends to $M \times [0, \infty)$ and it becomes of constant curvature -1 after rescaling. In order to apply Proposition 5.2 we have to choose κ large enough. However, a different κ will lead to the same hyperbolic metric. □

Proposition 5.4. — *The hyperbolic metric $\phi^* g_{\mathbb{H}^{n+1}_\kappa}$ on $M \times [0, \infty)$ is complete if the map $\phi : \widetilde{M} \times [0, \infty) \to \mathbb{H}^{n+1}_\kappa$ is proper. In particular, this is true if $2N < H \leqslant 0$.*

Proof. — Observe that $M \times [0,\infty)$ has a complete product metric constructed from the 1-QC Riemannian metric on the factor M and the standard metric on $[0,\infty)$. The pull-back metric $\phi^* g_{\mathbb{H}^{n+1}_\kappa}$ on $M \times [0,\infty)$ is complete if and only if every countable bounded subset in this metric is also bounded in the product metric.

Suppose that the pull-back metric is not complete. Thus, there exists a countable set $A \subset M \times [0,\infty)$ which is unbounded in the product metric – i.e., its projection on $[0,\infty)$ is unbounded – while A is bounded in the hyperbolic metric. Then, a lift $\widetilde{A}$ of A to $\widetilde{M} \times [0,\infty)$ is also bounded with respect to the lift of the hyperbolic metric. Since ϕ is an isometric immersion, it contracts distances and hence $\phi(\widetilde{A})$ is contained in a compact metric ball B in $\mathbb{H}^{n+1}_\kappa$. Therefore, $\phi^{-1}(B)$ is not compact in $\widetilde{M} \times [0,\infty)$ endowed with its usual topology, because it contains the unbounded subset $\widetilde{A}$ and hence ϕ is not proper. This settles the first claim.

Further, Alexander proved in [1, Prop. 2] that an isometric hypersurface immersion of a complete, connected Riemannian manifold into a simply connected complete manifold of sectional curvature at most $-\kappa$ having the absolute value of the eigenvalues of its second fundamental form bounded by $\sqrt{\kappa}$ must be an embedding. Recall that the eigenvalues of $\mathrm{II}_{f(\widetilde{M})}$ are $\sqrt{H+\kappa}$ and $(N+\kappa)/\sqrt{H+\kappa}$. Then the immersion f satisfies the hypothesis of Alexander's theorem above if $2N < H \leqslant 0$. In this case f is an embedding and $f(\widetilde{M})$ is convex. It follows that ϕ is a proper embedding. □

Remark 5.5. — If the holonomy group $\rho(\pi_1(M)) \subset \mathrm{SO}(n+1,1)$ is not discrete, then the immersion f is not proper and hence ϕ is not proper either.

Remark 5.6. — A construction due to Thurston in the case $n = 2$ (see [33]), further extended to all dimensions by Apanasov ([2]), Kulkarni and Pinkall ([39]), provides a canonical $\mathscr{C}^{1,1}$ Riemannian metric on M and a canonical totally geodesic stratification, using the convex hull construction on the limit set of $\rho(\pi_1(M))$. They defined a continuous ρ-equivariant map $D_0 : \widetilde{M} \to \mathbb{H}^{n+1}_\kappa$ with locally convex image (under the assumption that $S^n \smallsetminus D_\infty(\widetilde{M})$ contains at least two points). Specifically, we set $D_0(p) = q$, for $p \in \widetilde{M}$, where q is the unique closest point to $D_\infty(p)$ which belongs to the intersection of all half-spaces in $\mathbb{H}^{n+1}_\kappa$ determined by the round spherical balls $S^n \smallsetminus B$, where $D_\infty(p) \subset B \subset D_\infty(\widetilde{M})$. There is then an equivariant stratification of $\widetilde{M}$, such that D_0 maps each stratum of $\widetilde{M}$ isometrically onto some pleat in $\mathbb{H}^{n+1}_\kappa$, namely some k-dimensional convex hull of a subset in S^n, for $1 \leqslant k \leqslant n+1$.

We believe that 1-QC manifolds have only strata of dimensions n and $n-1$, namely they are channel Möbius manifolds in the terminology of [39]. This will provide an action of $\pi_1(M)$ on a $\mathbb{R}$-tree.

We can extend D_0 and D_∞ to an equivariant $\mathscr{C}^1$-immersion $D : \widetilde{M} \times (0,\infty) \to \mathbb{H}^{n+1}_\kappa$ by sending (p,t) into the point sitting at hyperbolic distance t from $D_0(p)$ on the geodesic ray joining $D_0(p)$ to $D_\infty(p)$. Each slice $D(*,t)$, $t > 0$, is a locally convex $\mathscr{C}^1$-immersion. Then the pull-back $D^* g_{\mathbb{H}^{n+1}_\kappa}$ also gives a hyperbolic (after rescaling) metric on $M \times (0,\infty)$. The relationship between the two hyperbolic structures induced by D and ϕ respectively has still to be clarified.

Remark 5.7. — When $n = 2$ the situation is different, and it was described by Labourie in [41]. The space $\mathscr{I}(k_0)$ of codimension-one isometric equivariant isometries of a surface of a surface S of genus at least 2 into the hyperbolic space $\mathbb{H}^3_1$, such that the pull-back metric is of constant curvature $k_0 \in (-1, 0)$ – up to left composition by isometries of $\mathbb{H}^3_1$ and right composition by lifts of diffeomorphisms isotopic to identity – is not one point as for $n \geqslant 4$. In fact, there is a natural map of $\mathscr{I}(k_0)$ into the Teichmüller space $\mathscr{T}(S)$ which associates to an immersion the conformal class ι_0 of the induced constant curvature metric. The fiber $\mathscr{I}(k_0, \iota_0)$ of this map can also be mapped homeomorphically onto the Teichmüller space $\mathscr{T}(S)$ by sending the class of an immersion to the conformal class of its second (or third) fundamental form or, equivalently, of its associated complex structure. Then the map from $\mathscr{I}(k_0)$ to the space of $\mathbb{C}P^1$ (i.e., conformally flat) structures on S which associates to an immersion f the $\mathbb{C}P^1$-structure given by F is also a homeomorphism. This gives a one parameter family of parametrizations of the space of $\mathbb{C}P^1$ structures by $\mathscr{T}(S) \times \mathscr{T}(S)$.

Thus, for each $\mathbb{C}P^1$-structure on S and $k \in (-1, 0)$ there exists some metric of constant curvature k on S and an equivariant isometric embedding $f_k : S \to \mathbb{H}^3_1$, whose associated $\mathbb{C}P^1$ structure is the one with which we started. Further, the immersion ϕ_k associated to f_k provides a hyperbolic structure $\mathscr{M}(k)$ on $S \times [0, \infty)$, called a geometrically finite end (see [41]). In [41, 40] one proved that for each constant $k \in [k_0, 0)$ there exists a unique incompressible embedded surface $S_k \subset S \times [0, \infty)$ of constant curvature k homeomorphic to S and the family S_k foliates the geometrically finite end $\mathscr{M}(k_0)$. This proves that actually $\mathscr{M}(k_1) \subset \mathscr{M}(k_2)$ if $k_1 > k_2$. When k converges to -1 then $\mathscr{M}(k)$ converges to a geometrically finite end. The $\mathbb{C}P^1$-structure associated to this geometrically finite end is also the one from the beginning. Moreover, the finite boundary of $\mathscr{M}(0) = \bigcup_k \mathscr{M}(k)$ is isometric to a surface of constant curvature -1 but its embedding into $\mathbb{H}^3_1$ is a pleated surface along a geodesic lamination, according to Thurston (see [41] for more details). Higher dimensional versions of Labourie's results concerning moduli of hyperbolic flat conformal were obtained by Smith in [51].

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