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TROPICAL AND NON-ARCHIMEDEAN LIMITS OF DEGENERATING FAMILIES OF VOLUME FORMS

by Sébastien Boucksom & Mattias Jonsson

Abstract. — We study the asymptotic behavior of volume forms on a degenerating family of compact complex manifolds. Under rather general conditions, we prove that the volume forms converge in a natural sense to a Lebesgue-type measure on a certain simplicial complex. In particular, this provides a measure-theoretic version of a conjecture by Kontsevich–Soibelman and Gross–Wilson, bearing on maximal degenerations of Calabi–Yau manifolds.

Résumé (Limites tropicales et non archimédiennes de familles de formes volumes qui dégénèrent)
Nous étudions le comportement asymptotique de formes volumes dans une famille de variétés complexes compactes qui dégénèrent. Sous des conditions assez générales, nous montrons que les formes volumes convergent en un sens naturel vers une mesure du type de Lebesgue sur un certain complexe simplicial. Ceci fournit en particulier une version en théorie de la mesure d'une conjecture de Kontsevich–Soibelman et Gross–Wilson portant sur les dégénérescences maximales de variétés de Calabi-Yau.

Contents

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Introduction

As is well-known, there is a natural bijection between (smooth, positive) volume forms on a complex manifold and smooth Hermitian metrics on its canonical bundle. Consequently, the data of a smooth family $(\nu_t)_{t\in\mathbb{D}^*}$ of volume forms on a holomorphic family $(X_t)_{t\in\mathbb{D}^*}$ of compact complex manifolds is equivalent to that of a proper holomorphic submersion $\pi\colon X\to\mathbb{D}^*$ together with a smooth metric ψ on the relative canonical bundle $K_{X/\mathbb{D}^*}$.

We say that the family (ν_t) has *analytic singularities* at $t=0$ if the following conditions hold:

(i) $\pi\colon X\to\mathbb{D}^*$ is meromorphic at $0\in\mathbb{D}$ in the sense that it extends to a proper, flat map $\pi\colon\mathscr{X}\to\mathbb{D}$, with $\mathscr{X}$ normal;

(ii) $\mathscr{X}$ can be chosen so that $K_{X/\mathbb{D}^*}$ extends to a $\mathbb{Q}$-line bundle $\mathscr{L}$ on $\mathscr{X}$, and ψ extends continuously to $\mathscr{L}$.

When (i) holds, we call $\mathscr{X}$ a *model* of X. Using resolution of singularities, we can always choose $\mathscr{X}$ as an *snc model*, that is, $\mathscr{X}$ is smooth and $\mathscr{X}_0=\sum_{i\in I}b_iE_i$ has simple normal crossing support. To $\mathscr{X}$ is then associated a dual complex $\Delta(\mathscr{X})$, with one vertex e_i for each E_i, and a face σ for each connected component Y of a non-empty intersection $E_J=\bigcap_{i\in J}E_i$ with $J\subset I$.

In the spirit of the Morgan-Shalen topological compactification of affine varieties [MS84], we introduce a natural "hybrid" space

$$\mathscr{X}^{\mathrm{hyb}}:=X\coprod\Delta(\mathscr{X})$$

associated to $\mathscr{X}$; it is equipped with a topology defined in terms of a tropicalization map $X\to\Delta(\mathscr{X})$, measuring the logarithmic rate of convergence of local coordinates compatible with $\mathscr{X}_0$.

Our first main result says that, after normalizing to unit mass, the volume forms ν_t admit a "tropical" limit inside $\mathscr{X}^{\mathrm{hyb}}$.

Theorem A. — *Let $(\nu_t)_{t\in\mathbb{D}^*}$ be a family of volume forms on a holomorphic family $X\to\mathbb{D}^*$ of compact complex manifolds, with analytic singularities at $t=0$. The asymptotic behavior of the total mass of ν_t is then given by*

$$\nu_t(X_t)\sim c|t|^{2\kappa_{\min}}(\log|t|^{-1})^d$$

with $c\in\mathbb{R}_+^*$, $\kappa_{\min}\in\mathbb{Q}$ and $d\in\mathbb{N}^*$, where $d\leqslant n:=\dim X_t$. Further, given any snc model $\mathscr{X}\to\mathbb{D}$ of $X\to\mathbb{D}^*$ such that $K_{X/\mathbb{D}^*}$ extends to a $\mathbb{Q}$-line bundle on $\mathscr{L}$ on $\mathscr{X}$ and ψ extends to a continuous metric on $\mathscr{L}$, the rescaled measures

$$\mu_t:=\frac{\nu_t}{|t|^{2\kappa_{\min}}(2\pi\log|t|^{-1})^d},$$

viewed as measures on $\mathscr{X}^{\mathrm{hyb}}$, converge weakly to a Lebesgue type measure μ_0 on a d-dimensional subcomplex $\Delta(\mathscr{L})$ of $\Delta(\mathscr{X})$.

The invariant $\kappa_{\min}$ and the subcomplex $\Delta(\mathscr{L})$ only depend on $\mathscr{L}$ (and not on the metric on $\mathscr{L}$). Consider the *logarithmic relative canonical bundle*

$$K^{\log}_{\mathscr{X}/\mathbb{D}} := K_{\mathscr{X}} + \mathscr{X}_{0,\mathrm{red}} - \pi^*(K_{\mathbb{D}} + [0]) = K_{\mathscr{X}/\mathbb{D}} + \mathscr{X}_{0,\mathrm{red}} - \mathscr{X}_0$$

and write $K^{\log}_{\mathscr{X}/\mathbb{D}} = \mathscr{L} + \sum_{i\in I} a_i E_i$ with $a_i \in \mathbb{Q}$. Setting $\kappa_i := a_i/b_i$, we then have $\kappa_{\min} = \min_{i\in I}\kappa_i$, and $\Delta(\mathscr{L})$ is the subcomplex of $\Delta(\mathscr{X})$ whose vertices e_i correspond to those $i \in I$ achieving the minimum.

On the other hand, the limit measure μ_0 does depend on ψ; it is given by

$$\mu_0 = \sum_{\sigma}\left(\int_{Y_\sigma} \mathrm{Res}_{Y_\sigma}(\psi)\right) b_\sigma^{-1}\lambda_\sigma.$$

Here, σ ranges over the d-dimensional faces of $\Delta(\mathscr{L})$, with corresponding strata $Y_\sigma \subset \mathscr{X}_0$, $\mathrm{Res}_{Y_\sigma}(\psi)$ is a naturally defined residual positive measure on Y_σ, λ_σ is the Lebesgue measure of σ normalized by its natural integral affine structure, and $b_\sigma \in \mathbb{Z}_{>0}$ is an arithmetic coefficient.

The study of the asymptotics of integrals is a very classical subject and has been pursued by many people; see for example the book [AGZV12]. The assertions in Theorem A are closely related to results by Chambert-Loir and Tschinkel (who also worked over general local fields and in an adelic setting). Specifically, the estimate for $\nu_t(X_t)$, suitably averaged over t, is essentially equivalent to [CLT10, Th. 1.2]. It also appears in [KS01, §3.1] and is exploited in [BHJ16].

The convergence result for the measures μ_t is also closely related to [CLT10, Cor. 4.8], where, however, the limit measure lives on $\mathscr{X}_0$ and not on $\Delta(\mathscr{X})$.[(1)] The main new feature of Theorem A is the precise and explicit convergence of the measure μ_t to a "tropical" limit μ_0, living on a simplicial complex.

The following examples illustrate Theorem A. First consider the subvariety

$$\mathscr{X} := \{(z_0^{n+1} + \cdots + z_n^{n+1}) + \varepsilon t z_0 \cdot \cdots \cdot z_n = 0\} \subset \mathbb{C} \times \mathbb{P}^n,$$

where $0 < \varepsilon \ll 1$. Write $X := \mathrm{pr}_1^{-1}(\mathbb{C}^*)$. The fiber X_t over $t \in \mathbb{D}^*$ is a Calabi-Yau manifold, and we can choose a nonvanishing holomorphic n-form η_t on X_t to define a smooth metric ψ on $K_{X/\mathbb{D}^*}$ that extends continuously to $\mathscr{L} = K_{\mathscr{X}/\mathbb{D}}$. In the terminology of Theorem A we have $\nu_t := 2^{-n} i^{n^2}\eta_t \wedge \overline{\eta}_t$. Here $\mathscr{X}_0$ is smooth, so $\Delta(\mathscr{X})$ is a single point. Thus $\nu_t(X_t) \sim c$ for some $c > 0$, and the limit measure μ_0 is a point mass.

Now consider instead

$$\mathscr{X} := \{t\varepsilon(z_0^{n+1} + \cdots + z_n^{n+1}) + z_0 \cdot \cdots \cdot z_n = 0\} \subset \mathbb{C} \times \mathbb{P}^n.$$

In this case, $\Delta(\mathscr{L}) = \Delta(\mathscr{X})$ is a union of $(n+1)$ simplices of dimension n, and topologically a sphere. We have $\nu_t(X_t) \sim c(\log|t|^{-1})^n$ and the limit measure is a weighted sum of Lebesgue measures on each simplex. In fact, it is clear by symmetry that the weights are equal; this also follows from Theorem C below.

(1) A. Chambert-Loir has pointed out that [CLT10, Cor. 4.8] is sufficiently precise, so that when applying it to toric blowups of $\mathscr{X}$ one can see the form of the limit measure μ_0 in Theorem A.

We also prove a logarithmic version of Theorem A, for a log smooth klt pair (X, B), and a metric ψ on $K_{(X,B)/\mathbb{D}^*}$, see Theorem 8.4.

The space $\mathscr{X}^{\mathrm{hyb}}$ and the measure μ_0 depend on the choice of snc model $\mathscr{X}$. We obtain a more canonical situation by considering all possible snc models simultaneously. Namely, the set of snc models of X is directed, and in §4 we define a locally compact (Hausdorff) topological space

$$X^{\mathrm{hyb}} := \varprojlim_{\mathscr{X}} \mathscr{X}^{\mathrm{hyb}},$$

fibering over $\mathbb{D}$, with central fiber $X_0^{\mathrm{hyb}} := \varprojlim \Delta(\mathscr{X})$. For any $\mathscr{X}$, the dual complex $\Delta(\mathscr{X})$ embeds in the central fiber X_0^{hyb} of X^{hyb}.

Corollary B. — *With assumptions and notation as in Theorem* A, *the measures μ_t, viewed as measures on X^{hyb}, converge weakly to a measure μ_0. Further, μ_0 is a Lebesgue type measure on a d-dimensional complex in X_0^{hyb}.*

Now consider the case when $X \to \mathbb{D}^*$ is projective. As we now explain, the central fiber of X^{hyb} is then a *non-Archimedean* space. Namely, X induces a smooth projective variety X_K over the non-Archimedean field K of complex formal Laurent series, to which we can associate a Berkovich analytification X_K^{an}. Similarly, any projective snc model $\mathscr{X} \to \mathbb{D}$ of X induces a projective model $\mathscr{X}_R$ over the valuation ring R of K. The dual complex $\Delta(\mathscr{X})$ then has a canonical realization as a compact $\mathbb{Z}$-PA subspace $\mathrm{Sk}(\mathscr{X}) \subset X_K^{\mathrm{an}}$, the *skeleton* of $\mathscr{X}$. In fact, it is well known (see e.g. [BFJ16]) that there is a homeomorphism $X_K^{\mathrm{an}} \xrightarrow{\sim} \varprojlim_{\mathscr{X}} \mathrm{Sk}(\mathscr{X})$, so we can identify the central fiber of the space X^{hyb} with the analytification X_K^{an}. In fact, as shown in Appendix A.6, using ideas from [Ber09], we can view the restriction of $X^{\mathrm{hyb}} \to \mathbb{D}$ to a closed subdisc $\overline{\mathbb{D}}_r$ as the analytification of the base change of X to a suitable Banach ring A_r.

Assuming $X \to \mathbb{D}^*$ is projective, we can describe the limit measure μ_0 and its support $\mathrm{Sk}(\mathscr{L}) \simeq \Delta(\mathscr{L})$ inside X_K^{an} in more detail. The skeleton $\mathrm{Sk}(\mathscr{L})$ is of purely non-Archimedean nature, and can be seen as a mild generalization of the *Kontsevich–Soibelman skeleton* introduced in [KS06] and studied in [MN15, NX16b, NX16a]. The *skeletal measure* μ_0, on the other hand, depends on both Archimedean and non-Archimedean data. Namely, it is supported on the skeleton $\mathrm{Sk}(\mathscr{L})$, but depends on the choice of metric on the restriction of the line bundle $\mathscr{L}$ to the central fiber $\mathscr{X}_0$ (viewed as a complex space) of any snc model $\mathscr{X}$.

We also study both the skeleton and the skeletal measure in the more general case when the model $\mathscr{X}$ is allowed to have mild (dlt) singularities.

One major motivation for studying the above general setting comes from degenerations of Calabi–Yau manifolds. Thus suppose $X \to \mathbb{D}^*$ is a projective holomorphic submersion, meromorphic at $0 \in \mathbb{D}$, such that $K_{X/\mathbb{D}^*} = \mathscr{O}_X$. Any trivializing section $\eta \in H^0(X, K_{X/\mathbb{D}^*})$ then defines a family $\eta_t := \eta|_{X_t}$ of trivializations of K_{X_t}, and hence a smooth family of volume forms $\nu_t := |\eta_t|^2$ with analytic singularities at $t = 0$. Indeed, for any snc model $\mathscr{X} \to \mathbb{D}$, η extends to a nowhere vanishing section of $\mathscr{L} := \mathscr{O}_{\mathscr{X}}$, and $\psi := \log|\eta|$ defines a smooth metric on $\mathscr{L}$.

The total mass $\nu_t(X_t) = \int_{X_t} |\eta_t|^2$ is then nothing but the L^2 (or Hodge) metric on the direct image of $K_{X/\mathbb{D}^*}$, whose asymptotic behavior at $t = 0$ is described in a very precise way by Schmid's nilpotent orbit theorem [Sch73, Th. 4.9] (compare for instance [GTZ16, Prop. 2.1]).

On the other hand, the skeleton $\mathrm{Sk}(\mathscr{L})$ described above coincides in the current context with the Kontsevich–Soibelman skeleton $\mathrm{Sk}(X)$ [KS06, MN15, NX16b]. Its dimension d, which features as the exponent of the log term in the asymptotics of the mass, measures how "bad" the degeneration is. Further, the family $X \to \mathbb{D}^*$ admits a relative minimal model $\mathscr{X}$, with certain mild (dlt) singularities [KNX15], and the essential skeleton can be identified with the dual complex of $\mathscr{X}$ [NX16b]. In particular, $d = 0$ if and only if X can filled in with a central fiber $\mathscr{X}_0$ which is a Calabi–Yau variety with klt singularities.

At the other end of the spectrum, $d = n = \dim X_t$ if and only if X is maximally degenerate, i.e., a "large complex structure limit". In that case, the essential skeleton $\mathrm{Sk}(X)$ is shown to be a pseudomanifold in [NX16b]. Building on this, we prove:

Theorem C. — *Let $X \to \mathbb{D}^*$ be a smooth projective family of Calabi–Yau varieties, meromorphic at $0 \in \mathbb{D}$. Assume that X is maximally degenerate and has semistable reduction. Then the skeletal measure μ_0 is a multiple of the integral affine Lebesgue measure on* $\mathrm{Sk}(X)$.

This theorem also holds in the purely non-Archimedean setting of Calabi–Yau varieties defined over the field of Laurent series. The semistable reduction condition means that X admits an snc model $\mathscr{X}$ with $\mathscr{X}_0$ reduced. This condition is always satisfied after a finite base change.

Theorem C describes measure-theoretic degenerations of Calabi–Yau varieties. Let us briefly discuss the case of *metric* degenerations. Consider a smooth projective family $X \to \mathbb{D}^*$ of Calabi–Yau varieties, meromorphic at $0 \in \mathbb{D}$, and suppose the family is polarized, that is, we are given a relative ample line bundle A on X. By Yau's theorem [Yau78], each fiber X_t carries a unique Ricci-flat Kähler metric ω_t in the cohomology class of A_t.

By [Wan03, Tos15, Tak15], the diameter D_t of (X_t, ω_t) remains bounded if and only if $d = 0$, that is, X admits a model $\mathscr{X}$ such that $\mathscr{X}_0$ has klt singularities. In this case, it is shown in [RZ11, RZ13], building in part on [DS14], that (X_t, ω_t) converges in the Gromov-Hausdorff sense to the Calabi–Yau variety $\mathscr{X}_0$, endowed with the metric completion of its singular Ricci-flat Kähler metric in the sense of [EGZ09].

The maximally degenerate case $d = n$ is the object of the *Kontsevich–Soibelman conjecture* [KS06][(2)], which states that $(X_t, D_t^{-2}\omega_t)$ (which has diameter one) converges in the Gromov-Hausdorff sense to the essential skeleton $\mathrm{Sk}(X)$ endowed with a piecewise smooth metric of Monge-Ampère type, i.e., locally given as the Hessian of a convex function satisfying a real Monge-Ampère equation. This conjecture

(2)Essentially the same conjecture was stated independently by Gross–Wilson [GW00] and Todorov.

has been verified for abelian varieties see e.g. [Oda14] but is largely open in general. The "mirror" situation, when one fixes the complex structure and degenerates the cohomology class of the Ricci-flat Kähler metric (along a line segment in the Kähler cone), is better understood [GW00, Tos09, Tos10, GTZ13, GTZ16, HT15, TWY14]. By performing a "hyper-Kähler rotation", this implies a version of the Kontsevich–Soibelman conjecture for special cases of Type III degenerations of K3 surfaces [GW00].

Theorems A and C indicate a possible approach to the Kontsevich–Soibelman conjecture. Indeed, recall that the metric ω_t for $t \in \mathbb{D}^*$ is constructed as the curvature form of a smooth metric ϕ_t on A_t, where ϕ_t in turn is obtained as a solution of the complex Monge-Ampère equation $\mathrm{MA}(\phi_t) = \mu_t$.

On the central fiber $X_0^{\mathrm{hyb}} = X_K^{\mathrm{an}}$ of X^{hyb}, it was shown in [BFJ15] that there exists a metric on the line bundle A_K^{an}, unique up to scaling, solving the non-Archimedean Monge-Ampère equation $\mathrm{MA}(\phi_0) = \mu_0$ (at least when X is defined over an algebraic curve). It is now tempting to approach the Kontsevich–Soibelman conjecture by studying the behavior of ϕ_t as $t \to 0$. However, this seems to be a delicate issue since there is no a priori reason why the weak continuity at $t = 0$ of $t \mapsto \mu_t$ would imply continuity of the solutions $t \mapsto \phi_t$.

Instead of Calabi-Yau manifolds, it would be interesting to study degenerating families $X \to \mathbb{D}^*$ of canonically polarized projective manifolds, where the metric on K_{X_t} would be the Kähler-Einstein metric or the Bergman metric, and prove versions of Theorems A and C in this context.

The paper is organized as follows. After recalling various facts in §1 we define in §2 the hybrid space $\mathscr{X}^{\mathrm{hyb}}$ associated to an SNC model $\mathscr{X}$. The proof of Theorem A is given in §3. In §4 we define the space X^{hyb} associated to a degeneration as an inverse limit of the spaces $\mathscr{X}^{\mathrm{hyb}}$, and prove Corollary B. Various notions of skeleta are defined and studied in §5, and in §6 we formalize the notion of a residually metrized model of the canonical bundle, and associate to such an object a positive measure on the relevant Berkovich space. Degenerations of Calabi–Yau varieties are studied in §7 where we prove Theorem C. In §8 we study various extensions, and in the appendix we recall the Berkovich analytification of a scheme over a Banach ring.

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1. Preliminaries

The goal of this section is to fix conventions and notation for metrics and measures, and to recall a few basic facts on integral affine structures. We also make a few calculations regarding tropicalizations that will be useful in the proof of Theorem A.

1.1. Metrics. — We use additive notation for line bundles and metrics over an analytic space X, both in the complex and non-Archimedean setting. This amounts to the following two rules:

(i) if for $i=1,2$, ϕ_i is a metric on a line bundle L_i and $a_i\in\mathbb{Z}$, then $a_1\phi_1+a_2\phi_2$ is a metric on $a_1L_1+a_2L_2$;

(ii) a metric on the trivial line bundle $\mathscr{O}_X$ is of the form $|\cdot|e^{-\phi}$ for a function ϕ on X, and we identify the metric with ϕ.

If s is a section of a line bundle L on X, then $\log|s|$ stands for the corresponding (possibly singular) metric on L in which s has length 1. For any metric ϕ on L, the above rules imply that $\log|s|-\phi$ is a function on X, and

$$|s|_\phi := |s|e^{-\phi} = \exp(\log|s|-\phi)$$

is the pointwise length of s in the metric ϕ.

A metric on a $\mathbb{Q}$-line bundle L is a collection $(\phi_m)_m$ of metrics on mL, for m sufficiently divisible, such that $\phi_{jm}=j\phi_m$.

The line bundle $\mathscr{O}_X(D)$ associated to any Cartier divisor D on X comes with a canonical singular metric ϕ_D, smooth outside D. This fact extends to $\mathbb{Q}$-divisors, by interpreting ϕ_D as a metric on a $\mathbb{Q}$-line bundle. In the complex case at least, the curvature current of ϕ_D, correctly normalized, coincides with the integration current on D.

1.2. Measures and forms. — Any finite-dimensional real vector space V comes equipped with a Lebesgue (or Haar) measure λ, uniquely defined up to a multiplicative constant. Any lattice $\Lambda\subset V$ allows us to normalize λ by $\lambda(V/\Lambda)=1$.

To any top-dimensional differential form ω on a C^∞ manifold X is associated a positive measure $|\omega|$ on X. For example, if $\Lambda\subset V$ is a lattice as above, $m_1,\dots,m_n$ is a basis of the dual lattice, then $|dm_1\wedge\dots\wedge dm_n|$ is a Lebesgue measure on V normalized by Λ.

If X is a complex manifold of dimension n, and Ω is a section of K_X, that is, a holomorphic n-form, we define $|\Omega|^2$ as the positive measure

$$|\Omega|^2 := \frac{i^{n^2}}{2^n}\,|\Omega\wedge\overline{\Omega}|.$$

The normalization is chosen so that the measure associated to the form $dz=dx+idy$ on $\mathbb{C}$ is Lebesgue measure $|dz|^2=|dx\wedge dy|$ on $\mathbb{C}\simeq\mathbb{R}^2$.

This construction induces a natural bijection between smooth metrics on the canonical bundle K_X and (smooth, positive) volume forms on X, which associates to a smooth metric ψ on K_X the volume form $e^{2\psi}$ locally defined by

$$e^{2\psi} := \frac{i^{n^2}|\Omega\wedge\overline{\Omega}|}{2^n|\Omega|^2_\psi} = \frac{|\Omega|^2}{|\Omega|^2e^{-2\psi}}$$

for any local section Ω of K_X. If ψ' is another metric on K_X, then

$$e^{2\psi'} = e^{2(\psi'-\psi)}e^{2\psi},$$

where $e^{2(\psi'-\psi)}$ is the usual exponential of the smooth function $2(\psi'-\psi)\in C^\infty(X)$. This can be used to make sense of $e^{2\psi}$ as a positive measure for any (possibly singular) metric ψ on K_X. Similarly, $e^{2\psi/m}$ is a volume form for every metric ψ on mK_X, $m\in\mathbb{Z}$.

Now assume (X,B) is a *pair* in the sense of the Minimal Model Program, i.e., X is a normal complex space and B is a (not necessarily effective) $\mathbb{Q}$-Weil divisor on X such that

$$K_{(X,B)} := K_X + B$$

is a $\mathbb{Q}$-line bundle. Denote by ϕ_B the canonical singular metric on $B|_{X_{\mathrm{reg}}}$, viewed as a $\mathbb{Q}$-line bundle. If ψ is smooth metric on the $\mathbb{Q}$-line bundle $K_{(X,B)}$, then $\psi-\phi_B$ is a smooth metric on $K_{X_{\mathrm{reg}}\smallsetminus B}$, and $e^{2(\psi-\phi_B)}$ is thus a volume form on $X_{\mathrm{reg}}\smallsetminus B$.[(3)]

A pair (X,B) is *subklt* if for some (or, equivalently, any) log resolution $\rho\colon X'\to X$ of (X,B), the unique $\mathbb{Q}$-divisor B' such that $\rho^*K_{(X,B)}=K_{(X',B')}$ and $\rho_*B'=B$ has coefficients <1. The pair (X,B) is *klt* if B is further effective.

Lemma 1.1. — *For any continuous metric ψ on $K_{(X,B)}$, (X,B) is subklt if and only if the measure $e^{2(\psi-\phi_B)}$ has locally finite mass near each point of X.*

Proof. — With the above notation it is immediate to check that

$$\rho^* e^{2(\psi-\phi_B)} = e^{2(\rho^*\psi-\phi_{B'})}.$$

We are thus reduced to a log smooth pair (X',B'), i.e., X' is smooth and B' has snc support, and the proof is then trivial. ☐

When (X,B) is subklt, we may thus view $e^{2(\psi-\phi_B)}$ as a finite positive (Radon) measure on X, putting no mass on Zariski closed subsets. Such measures are called *adapted* in [EGZ09, BBE+16].

1.3. Integral piecewise affine spaces. — The following discussion roughly follows [KKMSD73, p. 59] and [Ber04, §1].

If P is a rational polytope in $\mathbb{R}^n$, that is, the convex hull of a finite subset of $\mathbb{Q}^n$, denote by $M_P\subset C^0(P)$ the finitely generated free abelian group obtained by restricting to P affine functions with coefficients in $\mathbb{Z}$ (constant term included). Denote by 1_P the constant function on P with value 1, and set

$$\vec{M}_P := M_P/M_P\cap\mathbb{Q}1_P.$$

Denote also by $b_P\in\mathbb{N}$ the greatest integer such that $b_P^{-1}1_P\in M_P$.

The data of (P,M_P) modulo homeomorphism is called an *(abstract) $\mathbb{Z}$-polytope*. The functions in M_P are called *integral affine*, or $\mathbb{Z}$-affine.

The evaluation map defines a canonical realization $P\hookrightarrow(M_P)^\vee_{\mathbb{R}}$ as a codimension one rational polytope, with tangent space T_P identified with $(\vec{M}_P)^\vee_{\mathbb{R}}$. Further, the lattice $T_{P,\mathbb{Z}}:=\mathrm{Hom}(\vec{M}_P,\mathbb{Z})\subset T_P$ yields a normalized Lebesgue measure λ_P on P.

The main example for us is as follows.

(3)Here and in what follows, we write $X\smallsetminus D$ for the complement of the support of a (not necessarily reduced) divisor D in a complex space X.

Lemma 1.2. — *Given $b_0, \dots, b_p \in \mathbb{N}^*$, view*
$$\sigma = \left\{ w \in \mathbb{R}_+^{p+1} \mid \textstyle\sum_{i=0}^p b_i w_i = 1 \right\}$$
as a $\mathbb{Z}$-simplex. Then $b_\sigma = \gcd(b_i)$, and
$$\mathrm{Vol}(\sigma) = \frac{b_\sigma}{p! \prod_i b_i}.$$

Proof. — Note that $T_{\sigma,\mathbb{Z}} = \{w \in \mathbb{Z}^{p+1} \mid \sum_i b_i w_i = 0\}$. The linear isomorphism $\phi \colon \mathbb{R}^{p+1} \to \mathbb{R}^{p+1}$ given by $\phi(w_j) = (b_j w_j)$ takes σ to the standard simplex
$$\sigma' = \left\{ w' \in \mathbb{R}_+^{p+1} \mid \textstyle\sum_i w'_j = 1 \right\},$$
and hence
$$[T_{\sigma',\mathbb{Z}} \colon \phi(T_{\sigma,\mathbb{Z}})] \, \mathrm{Vol}(\sigma) = \mathrm{Vol}(\sigma') = \frac{1}{p!}.$$
Write $T_{\sigma',\mathbb{Z}}$ as the kernel of $\chi \colon \mathbb{Z}^{p+1} \to \mathbb{Z}$ defined by $\chi(w') = \sum_i w'_i$. Then
$$\phi(T_{\sigma,\mathbb{Z}}) = \ker \chi \cap \phi(\mathbb{Z}^{p+1}), \quad \chi(\phi(\mathbb{Z}^{p+1})) = \gcd(b_i)\mathbb{Z},$$
and the exact sequence
$$0 \longrightarrow \frac{\ker \chi}{\ker \chi \cap \phi(\mathbb{Z}^{p+1})} \longrightarrow \frac{\mathbb{Z}^{p+1}}{\phi(\mathbb{Z}^{p+1})} \longrightarrow \frac{\mathbb{Z}}{\chi(\phi(\mathbb{Z}^{p+1}))} \longrightarrow 0$$
gives as desired
$$[T_{\sigma',\mathbb{Z}} \colon \phi(T_{\sigma,\mathbb{Z}})] = \frac{\prod_i b_i}{\gcd(b_i)}.$$
Finally, the first assertion is clear. □

Remark 1.3. — By setting $w_0 = b_0^{-1}(1 - \sum_{i=1}^p b_i w_i)$, we can identify σ with the simplex $\sum_1^p b_i w_i \leqslant 1$ in $\mathbb{R}_+^p$. The normalized Lebesgue measure on σ is then given by $\lambda_\sigma = b_\sigma^{-1} |dw_1 \wedge \cdots \wedge dw_p|$.

A *compact rational polyhedron* K in $\mathbb{R}^n$ is a finite union of rational polytopes P_i, which may then be arranged so that $P_i \cap P_j$ is either empty or a common face of P_i and P_j. We then say that (P_i) is a *subdivision* of K, and call the subdivision *simplicial* if each P_i is a simplex. A continuous function on K is *integral piecewise affine* ($\mathbb{Z}$-PA for short) if $f|_{P_i} \in M_{P_i}$ for some subdivision of K. These functions form a subgroup $\mathrm{PA}_\mathbb{Z}(K) \subset C^0(K)$, and the data of $(K, \mathrm{PA}_\mathbb{Z}(K))$ modulo homeomorphism is called a *compact $\mathbb{Z}$-PA space*.

The *normalized Lebesgue measure* of K is defined as
$$\lambda_K = \sum_{\dim P_i = \dim K} \mathbf{1}_{P_i} \lambda_{P_i}$$
for some (and hence any) subdivision into $\mathbb{Z}$-polytopes.

Note that a $\mathbb{Z}$-polytope P can be regarded as a $\mathbb{Z}$-PA space and that $M_P \subset \mathrm{PA}_\mathbb{Z}(P)$.

1.4. Tropicalizations and polar coordinates. — The material in this section is surely well known, but we include the details for lack of a suitable reference. The calculations here are used in the proof of Theorem 3.4 (which implies Theorem A).

Let $N \simeq \mathbb{Z}^{p+1}$ be a lattice, $M = \mathrm{Hom}(N, \mathbb{Z})$ the dual lattice, $\mathbb{C}[M]$ the semigroup ring and $T = \operatorname{Spec} \mathbb{C}[M] = N \otimes \mathbb{C}^*$ the algebraic torus. A basis for N induces a dual basis $(m_0, \dots, m_p)$ for M and elements $z_i \in \mathbb{C}[M]$, $0 \leqslant i \leqslant p$, such that $\mathbb{C}[M] = \mathbb{C}[z_0^{\pm 1}, \dots, z_p^{\pm 1}]$ and $T \simeq (\mathbb{C}^*)^{p+1}$.

Let $\Omega \in H^0(T, K_T)$ be the T-invariant global section given in coordinates by

$$\Omega = \frac{dz_0}{z_0} \wedge \cdots \wedge \frac{dz_p}{z_p}.$$

Note that Ω is independent of the choice of coordinates, up to a sign. Its associated measure

$$\rho := |\Omega|^2$$

is T-invariant, and hence a Haar measure on T.

We can write this measure in (logarithmic) polar coordinates via the canonical *tropicalization map* $L\colon T \to N_{\mathbb{R}}$, given in the basis above by

$$L = (-\log|z_0|, \dots, -\log|z_p|).$$

Note that L sits in the exact sequence $1 \to K \to T \to N_{\mathbb{R}}$ obtained by tensoring with N the exact sequence $1 \to S^1 \to \mathbb{C}^* \to \mathbb{R} \to 0$ induced by $z \mapsto -\log|z|$. In particular, $K = N \otimes S^1 \simeq (S^1)^{p+1}$ is a compact torus, and $L\colon T \to N_{\mathbb{R}}$ is a principal K-bundle.

On the one hand, let ω be the translation invariant real $(p+1)$-form on the tropical torus $N_{\mathbb{R}} \simeq \mathbb{R}^{p+1}$ given by

$$\omega = dm_0 \wedge \cdots \wedge dm_p.$$

This form is again independent of the choice of basis, up to a sign, and its associated measure $\lambda := |\omega|$ is the Lebesgue (or Haar) measure on $N_{\mathbb{R}}$ normalized by N.

On the other hand, since $L : T \to N_{\mathbb{R}}$ is a principal K-bundle, each fiber $K_w = L^{-1}(w)$ has a unique K-invariant probability measure ρ_w. Then ρ has a fiber decomposition

$$\rho = (2\pi)^{p+1} \lambda(dw) \otimes \rho_w,$$

i.e.,

$$\int_T f \, d\rho = (2\pi)^{p+1} \int_{N_{\mathbb{R}}} \left(\int_{K_w} f \, d\rho_w \right) \lambda(dw), \tag{1.1}$$

for any $f \in C_c^0(T)$. Concretely, we can use logarithmic polar coordinates on T:

$$z_j = \exp(-w_j + 2\pi i \theta_j)$$

for $0 \leqslant j \leqslant p$; then $\rho_w = |d\theta_0 \wedge \cdots \wedge d\theta_p|$, and

$$\rho = \left| \frac{dz_0}{z_0} \wedge \cdots \wedge \frac{dz_p}{z_p} \right|^2 = (2\pi)^{p+1} |dw_1 \wedge \cdots \wedge dw_n| \otimes \rho_w.$$

We will need the same analysis on certain subgroups of T. Fix an element $m \in M$ and let $\chi = \chi^m : T \to \mathbb{C}^*$ be the corresponding character. Let $b \in \mathbb{Z}_{>0}$ be the largest integer such that $b^{-1}m \in M$. In the bases above, we can write $m = \sum_{i=0}^p b_i m_i$ and $\chi = \prod_i z_i^{b_i}$, where $b_i \in \mathbb{Z}$; then $b = \gcd_i b_i$. On the other hand, we can pick a basis such that $m = bm_0$ and $\chi = z_0^b$. This is useful for computations.

For $t \in \mathbb{C}^*$, $T_t := \chi^{-1}(t)$ is a complex manifold with b connected components. Note that $T' := X_1$ is an algebraic subgroup of T and that T_t is a torsor for T' for any $t \in \mathbb{C}^*$. The T-invariant $(p+1)$-form Ω induces in a canonical way a T'-invariant p-form Ω_t on T_t, obtained as the restriction to T_t of any choice of holomorphic p-form Ω' on T such that $\frac{d\chi}{\chi} \wedge \Omega' = \Omega$. In general coordinates as above, we can pick

$$\Omega' = \frac{1}{\# J} \sum_{j \in J} \frac{(-1)^j}{b_j} \frac{dz_0}{z_0} \wedge \cdots \wedge \widehat{\frac{dz_j}{z_j}} \wedge \cdots \wedge \frac{dz_p}{z_p},$$

where $J = \{j \mid b_j \neq 0\}$. In special coordinates, so that $m = bm_0$ and $\chi = z_0^b$, we then have $\Omega' = \frac{1}{b} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_p}{z_p}$, and hence

$$T_t = \bigcup_{u^b = t} \{z_0 = u\} \quad \text{and} \quad \Omega_t = \frac{1}{b} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_p}{z_p}\bigg|_{T_t}.$$

Note that $\rho_1 := |\Omega_1|^2$ is Haar measure on T', whereas $\rho_t := |\Omega_t|^2$ is a T'-invariant measure on T_t. In the special case $p = 0$, T_t consists of b points, and ρ_t gives mass $1/b^2$ to each of them.

Next we study the analogous situation in the tropical torus $N_{\mathbb{R}}$. Viewing m as a linear form on $N_{\mathbb{R}}$, set $H_s := m^{-1}(s)$ for $s \in \mathbb{R}$. The lattice $N' = \operatorname{Ker} m \subset N$ defines an integral affine structure on H_s, and hence a normalized Lebesgue measure λ_s. Note that

$$|\omega'|_{H_s}| = \frac{1}{b} \lambda_s$$

for any choice of p-form ω' on $N_{\mathbb{R}}$ such that $dm \wedge \omega' = \omega$. In general coordinates, we pick

$$\omega' = \frac{1}{\# J} \sum_{j \in J} \frac{(-1)^j}{b_j} dm_0 \wedge \cdots \wedge \widehat{dm_j} \wedge \cdots \wedge dm_p,$$

where $J = \{j \mid b_j \neq 0\}$. In special coordinates, $\omega' = (1/b) dm_1 \wedge \cdots \wedge dm_p$.

Finally we describe ρ_t in polar coordinates. The tropicalization map $L : T \to N_{\mathbb{R}}$ induces a principal $T' \cap K$-bundle $T_t \to H_s$ with $s = -\log|t|$, and hence an invariant probability measure on $\rho_{t,w}$ on each fiber $K_{t,w} := T_t \cap K_w$. We claim that

$$\rho_t = \frac{(2\pi)^p}{b} \lambda_s(dw) \otimes \rho_{t,w},$$

i.e.,

$$\int_{T_t} f \, d\rho_t = \frac{(2\pi)^p}{b} \int_{H_s} \left(\int_{K_{t,w}} f \, \rho_{t,w} \right) \lambda_s(dw), \tag{1.2}$$

for any $f \in C_c^0(T_t)$, where $s = \log|t|^{-1}$.

The proof is essentially the same as that of (1.1). We work in special coordinates, so that $\chi = z_0^b$ and $m = bm_0$. Then $T_t = \{z_0^b = t\}$ has b connected components $T_t^{(\ell)}$, $1 \leqslant \ell \leqslant b$, and

$$\rho_t = |\Omega_t|^2 = \frac{1}{b^2}\left|\frac{dz_1}{z_1}\wedge\cdots\wedge\frac{dz_p}{z_p}\right|^2.$$

The restriction of the tropicalization map to $T_t^{(\ell)}$ amounts to the change of coordinates $z_j = u_j^{(\ell)}\exp(-w_j + 2\pi i\theta_j)$ for $1 \leqslant j \leqslant p$, where the $u_j^{(\ell)}$ are constants with $|u_j^{(\ell)}| = 1$. In these coordinates,

$$\rho_t|_{T_t^{(\ell)}} = \frac{(2\pi)^p}{b^2}\,|dm_1\wedge\cdots\wedge dm_n| \otimes |d\theta_1\wedge\cdots\wedge d\theta_p|.$$

Here $(1/b)|d\theta_1\wedge\cdots\wedge d\theta_p|$ induces the measure $\rho_{t,w}$ on $K_{t,w}$, whereas $|dm_1\wedge\cdots\wedge dm_s|$ is Lebesgue measure λ_s on H_s. Hence (1.2) follows.

2. The hybrid space associated to an snc model

In this section, we show how to perform a topological surgery in a complex manifold, replacing a simple normal crossing divisor with its dual complex. Our construction is similar to the one used by Morgan-Shalen in [MS84, §I.3], and can even be traced back to the pioneering work of Bergman [Ber71].

2.1. The dual complex. — Let D be an effective divisor with simple normal crossing (snc) support in a complex manifold $\mathscr{X}$. By definition, $D = \sum_{i\in I} b_i E_i$ with $b_i \in \mathbb{N}^*$ and $(E_i)_{i\in I}$ a finite family of smooth irreducible divisors such that

$$E_J := \bigcap_{i\in J} E_i$$

is either empty or smooth of codimension $|J|$ (with finitely many connected components) for each $\varnothing \neq J \subset I$. A connected component Y of a non-empty E_J is called a *stratum*. Together with $\mathscr{X} \smallsetminus D = E_\varnothing$, the locally closed submanifolds $\mathring{Y} := Y \smallsetminus \bigcup_{i\in I\smallsetminus J} E_i$ define a partition of $\mathscr{X}$.

The *dual complex* $\Delta(D)$ is the simplicial complex[(4)] defined as follows: to each stratum Y corresponds a simplex

$$\sigma_Y = \left\{w \in \mathbb{R}_+^J \mid \textstyle\sum_{i\in J} b_i w_i = 1\right\},$$

and σ_Y is a face of $\sigma_{Y'}$ if and only if $Y' \subset Y$. This description equips $\Delta(D)$ with an integral affine structure, by which we mean a compatible choice of integral affine structures on each simplex σ. This further induces a $\mathbb{Z}$-PA structure on $\Delta(D)$.

We write Y_σ for the stratum of a face σ. Each point $\xi \in D$ belongs to $\mathring{Y}_\xi$ for a unique stratum Y_ξ, obtained as the connected component of E_{J_ξ} containing ξ, with $J_\xi = \{i \in I \mid \xi \in E_i\}$. We denote by $\sigma_\xi := \sigma_{Y_\xi}$ the corresponding face of $\Delta(D)$.

(4) This is understood in the slightly generalized sense that the intersection of two faces is a *union* of common faces.

2.2. The hybrid topology. — Next we define a natural topology on the disjoint union

$$\mathscr{X}^{\mathrm{hyb}} := (\mathscr{X} \smallsetminus D) \coprod \Delta(D).$$

Consider a connected open set $\mathscr{U} \subset \mathscr{X}$ meeting D and local coordinates $z = (z_0, \ldots, z_n)$ on $\mathscr{U}$. We say that the pair $(\mathscr{U}, z)$ is *adapted* (to D) if the following conditions hold:

(i) if $E_0, \ldots, E_p$ are the irreducible components of D intersecting $\mathscr{U}$, then $\mathscr{U} \cap E_0 \cap \cdots \cap E_p$, if nonempty, equals $\mathscr{U} \cap \mathring{Y}$ for a component Y of $E_0 \cap \cdots \cap E_p$;

(ii) z_i is an equation of $E_i \cap \mathscr{U}$ with $|z_i| < 1$, $0 \leqslant i \leqslant p$.

We call $Y = Y_{\mathscr{U}}$ the stratum of $\mathscr{U}$, and denote by

$$\sigma_{\mathscr{U}} = \left\{ w \in \mathbb{R}^{p+1} \mid \textstyle\sum_{i=0}^{p} b_i w_i = 1 \right\}$$

the corresponding face of $\Delta(D)$. The function $f_{\mathscr{U},z} := \prod_{i=0}^{p} z_i^{b_i}$ is an equation of D in $\mathscr{U}$, with $|f_{\mathscr{U},z}| < 1$, and we get a continuous map $\mathrm{Log}_{\mathscr{U}} : \mathscr{U} \smallsetminus D \to \sigma_Y$ by setting

$$\mathrm{Log}_{\mathscr{U}} = \left(\frac{\log |z_i|}{\log |f_{\mathscr{U}}|} \right)_{0 \leqslant i \leqslant p}.$$

For any two adapted coordinate charts $(\mathscr{U}, z)$, $(\mathscr{U}', z')$, with the same stratum Y, we have $z_i' = u_i z_i$ with u_i nonvanishing on $\mathscr{U} \cap \mathscr{U}'$, for $i = 0, \ldots, p$ (after a possible reindexing); it follows that

$$\mathrm{Log}_{\mathscr{U}'} = \mathrm{Log}_{\mathscr{U}} + O\left(\frac{1}{\log |f_{\mathscr{U},z}|^{-1}} \right) \tag{2.1}$$

locally uniformly on $\mathscr{U} \cap \mathscr{U}'$. We next show how to globalize this construction.

Proposition 2.1. — *There exists an open neighborhood $\mathscr{V} \subset \mathscr{X}$ of D and a continuous map $\mathrm{Log}_{\mathscr{V}} : \mathscr{V} \smallsetminus D \to \Delta(D)$ such that for each adapted coordinate chart $(\mathscr{U}, z)$ with $\mathscr{U} \subset \mathscr{V}$ we have $\mathrm{Log}_{\mathscr{V}}(\mathscr{U} \smallsetminus D) \subset \sigma_{\mathscr{U}}$ and*

$$\mathrm{Log}_{\mathscr{V}} = \mathrm{Log}_{\mathscr{U}} + O\left(\frac{1}{\log |f_{\mathscr{U},z}|^{-1}} \right) \tag{2.2}$$

uniformly on compact subsets of $\mathscr{U}$.

This will be accomplished by means of a partition of unity, using the following elementary special case of [Cle77, Th. 5.7].

Lemma 2.2. — *There exists a family $((\mathscr{V}_\alpha, z_\alpha))_{\alpha \in A}$ of adapted coordinate charts, such that $(\mathscr{V}_\alpha)_\alpha$ forms a locally finite covering of D and such that the strata Y_α of the $\mathscr{V}_\alpha$ satisfy*

$$\bigcap_{\beta \in B} \mathscr{V}_\beta \neq \varnothing \Longrightarrow \bigcap_{\beta \in B} Y_\beta \neq \varnothing \tag{2.3}$$

for every finite $B \subset A$.

Proof of Proposition 2.1. — Pick an open cover $(\mathscr{V}_\alpha)_\alpha$ as in Lemma 2.2, and denote by $\mathrm{Log}_\alpha\colon \mathscr{V}_\alpha \smallsetminus D \to \sigma_\alpha$ the corresponding maps. Set $\mathscr{V} := \bigcup_\alpha \mathscr{V}_\alpha$, and pick a partition of unity (χ_α) subordinate to $(\mathscr{V}_\alpha)$. We claim that for each $\xi \in \mathscr{V}$ there exists an open neighborhood W of ξ and a face σ_W of $\Delta(D)$ such that

$$W \cap \operatorname{supp}\chi_\alpha \neq \varnothing \Longrightarrow \sigma_\alpha \subset \sigma_W$$

for any $\alpha \in A$. Indeed, using (2.3) it is easy to see that

$$W := \bigcap_{\alpha \mid \xi \in \mathscr{U}_\alpha} \mathscr{V}_\alpha \smallsetminus \bigcup_{\alpha \mid \xi \notin \operatorname{supp}\chi_\beta} \operatorname{supp}\chi_\beta$$

satisfies this property. By convexity of σ_W, it follows that $\mathrm{Log}_{\mathscr{V}} := \sum_\alpha \chi_\alpha \mathrm{Log}_{\mathscr{V}_\alpha}$ is well-defined on $W \smallsetminus D$, and hence yields a continuous map $\mathrm{Log}_{\mathscr{V}}\colon \mathscr{V} \smallsetminus D \to \Delta(D)$. The last property is a direct consequence of (2.1). □

We extend the previous map as

$$\mathrm{Log}_{\mathscr{V}}\colon \mathscr{V}^{\mathrm{hyb}} := (\mathscr{V} \smallsetminus D) \cup \Delta(D) \longrightarrow \Delta(D)$$

by setting $\mathrm{Log}_{\mathscr{V}} = \mathrm{id}$ on $\Delta(D)$.

Definition 2.3. — The *hybrid topology* on $\mathscr{X}^{\mathrm{hyb}} := (\mathscr{X} \smallsetminus D) \cup \Delta(D)$ is defined as the coarsest topology such that:

(i) $\mathscr{X} \smallsetminus D \hookrightarrow \mathscr{X}^{\mathrm{hyb}}$ is an open embedding;

(ii) For every open neighborhood $\mathscr{V}$ of D in $\mathscr{X}$, the set $(\mathscr{V} \smallsetminus D) \cup \Delta(\mathscr{X})$ is open in $\mathscr{X}^{\mathrm{hyb}}$;

(iii) $\mathrm{Log}_{\mathscr{V}}\colon \mathscr{V}^{\mathrm{hyb}} \to \Delta(D)$ is continuous.

Using (2.2), this definition is easily seen to be independent of the choice of map $\mathrm{Log}_{\mathscr{V}}$. If D is compact and $K \subset \mathscr{X}$ is a compact neighborhood of D, then one easily checks that the corresponding subset $K^{\mathrm{hyb}} = (K \smallsetminus D) \cup \Delta(D)$ is compact (Hausdorff). When $D = b_0 E_0$ has only one irreducible component, K^{hyb} is simply the Tychonoff one-point compactification of $K \smallsetminus D$.

Example 2.4. — Set $\mathscr{X} = \mathbb{D}^2$ and $D = E_0 + E_1$ the union of the coordinate axes, with coordinates (z_0, z_1). Then $\mathscr{U} = \mathscr{X}$ is itself an adapted coordinate chart. In these coordinates, $\mathrm{Log}_{\mathscr{U}}\colon \mathscr{U} \smallsetminus D \to \sigma_{\mathscr{U}}$ becomes the map $(\mathbb{D}^*)^2 \to [0,1]$ sending (z_0, z_1) to $\log|z_1| / \log|z_0 z_1|$. As a consequence, given $\zeta \in \mathbb{R}_+^*$ and $0 < \varepsilon \ll 1$, the closure in $\mathscr{X}^{\mathrm{hyb}}$ of the closed subset

$$F_\varepsilon := \{0 < |z_0|,\ |z_1| \leqslant \varepsilon,\ |z_0|^{\zeta+\varepsilon} \leqslant |z_1| \leqslant |z_0|^{\zeta-\varepsilon}\} \subset \mathbb{D}^2$$

is given by $\overline{F}_\varepsilon = F_\varepsilon \cup I_\varepsilon$, where

$$I_\varepsilon := \Big\{ w \in [0,1] \ \Big|\ \frac{\zeta - \varepsilon}{1 + \zeta - \varepsilon} \leqslant w \leqslant \frac{\zeta + \varepsilon}{1 + \zeta + \varepsilon} \Big\}.$$

Further, the sets $\overline{F}_\varepsilon$, for $0 < \varepsilon \ll 1$ form a basis of closed neighborhoods of the point $\zeta/(1+\zeta) \in [0,1]$ in $\mathscr{X}^{\mathrm{hyb}}$. See Figure 2.1.

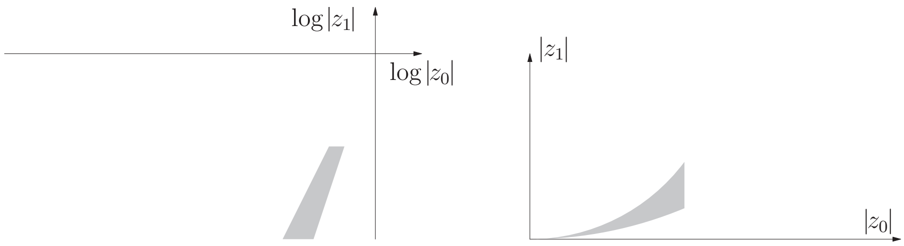

Figure 2.1. The figure shows the closed subset F_ε in Example 2.4.

3. Proof of Theorem A

In this section, we describe in more detail the objects involved in Theorem A, and then provide a proof. We work purely in the complex analytic category here.

3.1. Residual measures. — Let $\pi\colon \mathscr{X} \to \mathbb{D}$ be an snc degeneration, i.e., a proper, surjective holomorphic map from a connected complex manifold to the unit disc in $\mathbb{C}$, whose restriction to $X := \pi^{-1}(\mathbb{D}^*)$ is a submersion and such that $\mathscr{X}_0 := \pi^{-1}(0) = \sum_{i\in I} b_i E_i$ has snc support. Note that $X_t := \pi^{-1}(t)$ is non-singular for $t \in \mathbb{D}^*$. The *dual complex* $\Delta(\mathscr{X})$ is defined as that of $\mathscr{X}_0$; it is equipped with its natural $\mathbb{Z}$-PA structure. The *logarithmic canonical bundle* of $\mathscr{X}$ is

$$K_{\mathscr{X}}^{\log} := K_{\mathscr{X}} + \mathscr{X}_{0,\mathrm{red}}.$$

Setting $K_{\mathbb{D}}^{\log} := K_{\mathbb{D}} + [0]$, we define the *relative logarithmic canonical bundle* as

$$K_{\mathscr{X}/\mathbb{D}}^{\log} := K_{\mathscr{X}}^{\log} - \pi^* K_{\mathbb{D}}^{\log} = K_{\mathscr{X}/\mathbb{D}} + \mathscr{X}_{0,\mathrm{red}} - \mathscr{X}_0.$$

Now suppose we are given a $\mathbb{Q}$-line bundle $\mathscr{L}$ on $\mathscr{X}$ extending $K_{X/\mathbb{D}^*}$. We then have a unique decomposition

$$K_{\mathscr{X}/\mathbb{D}}^{\log} = \mathscr{L} + \sum_{i\in I} a_i E_i$$

with $a_i \in \mathbb{Q}$. Set $\kappa_i := a_i/b_i$ and $\kappa_{\min} := \min_i \kappa_i$.

Definition 3.1. — We denote by $\Delta(\mathscr{L})$ the subcomplex of $\Delta(\mathscr{X})$ such that a face σ of $\Delta(\mathscr{X})$ is in $\Delta(\mathscr{L})$ if and only if each vertex of σ achieves $\min_i \kappa_i$.

In general, $\Delta(\mathscr{L})$ is neither connected nor pure dimensional. We say that a face of $\Delta(\mathscr{L})$ is *maximal* if it is not contained in a larger face of $\Delta(\mathscr{L})$.

Lemma 3.2. — *Let $Y \subset \mathscr{X}_0$ be a stratum corresponding to face σ of $\Delta(\mathscr{X})$, and denote by $J \subset I$ the set of irreducible components E_i cutting out Y. Then*

$$B_Y^{\mathscr{L}} := \sum_{i\notin J} (1 - (a_i - \kappa_{\min} b_i)) E_i|_Y$$

is a $\mathbb{Q}$-divisor on Y with snc support, and we have a canonical identification

$$\mathscr{L}|_Y = K_{(Y, B_Y^{\mathscr{L}})} := K_Y + B_Y^{\mathscr{L}}$$

as $\mathbb{Q}$-line bundles. If we further assume that σ is a maximal face of $\Delta(\mathscr{L})$, then $B_Y^{\mathscr{L}}$ has coefficients <1, so the pair $(Y, B_Y^{\mathscr{L}})$ is subklt.

Proof. — The first point is a simple consequence of the triviality of the normal bundle $\mathscr{O}_{\mathscr{X}_0}(\mathscr{X}_0)$ together with the adjunction formula

$$K_Y = (K_{\mathscr{X}} + \sum_{i\in J} E_i)|_Y,$$

canonically realized by Poincaré residues once an order on J has been chosen. When σ is a maximal face of $\Delta(\mathscr{L})$, each E_i meeting Y properly satisfies $\kappa_i > \kappa_{\min}$, which implies that $B_Y^{\mathscr{L}}$ has coefficients <1. $\square$

If ψ is a continuous metric on $\mathscr{L}$, $\psi|_Y$ may thus be viewed as a metric on $K_{(Y,B_Y^{\mathscr{L}})}$. When σ is a maximal face of $\Delta(\mathscr{L})$, the pair $(Y, B_Y^{\mathscr{L}})$ is subklt, and Lemma 1.1 applies. This leads to the following notion.

Definition 3.3. — Let Y be a stratum corresponding to a maximal face of $\Delta(\mathscr{L})$. The *residual measure* on Y of a continuous metric ψ on $\mathscr{L}$ is the (finite) positive measure on Y defined by

$$\operatorname{Res}_Y(\psi) := \exp\bigl(2(\psi|_Y - \phi_{B_Y^{\mathscr{L}}})\bigr).$$

This measure can be more explicitly described as follows. At each point $\xi \in Y$, pick local coordinates $(z_0, \dots, z_n)$ such that $z_0, \dots, z_p$ are local equations for the components $E_0, \dots, E_p$ of $\mathscr{X}_0$ that pass through ξ, indexed so that $J = \{0, \dots, d\}$, where $0 \leqslant d \leqslant p$, and such that $t = \prod_{j=0}^p z_j^{b_j}$ The logarithmic form

$$\Omega := \frac{dz_0}{z_0} \wedge \cdots \wedge \frac{dz_p}{z_p} \wedge dz_{p+1} \wedge \cdots \wedge dz_n$$

is a local trivialization of $K_{\mathscr{X}}^{\log}$, hence induces a local trivialization $\Omega^{\mathrm{rel}} = \Omega \otimes (dt/t)^{-1}$ of $K_{\mathscr{X}/\mathbb{D}}^{\log}$. We may then view $\tau := \prod_{i=0}^p z_i^{a_i} \Omega^{\mathrm{rel}}$ as a local $\mathbb{Q}$-generator of $\mathscr{L}$. Under the identification $\mathscr{L}|_Y = K_{(Y,B_Y^{\mathscr{L}})}$, we have

$$\tau|_Y = \prod_{i=d+1}^p z_i^{a_i - \kappa_{\min} b_i} \operatorname{Res}_Y(\Omega)$$

with

$$\operatorname{Res}_Y(\Omega) = \frac{dz_{d+1}}{z_{d+1}} \wedge \cdots \wedge \frac{dz_p}{z_p} \wedge dz_{p+1} \wedge \cdots \wedge dz_n \Big|_Y.$$

We infer

$$\operatorname{Res}_Y(\psi) = |\tau|_\psi^{-2} \prod_{i=d+1}^p |z_i|^{2(a_i - \kappa_{\min} b_i - 1)} \Big| \bigwedge_{i=d+1}^n dz_i \Big|^2. \tag{3.1}$$

3.2. Statement and first reductions. — It will be convenient to introduce the quantity

$$\lambda(t) := (\log |t|^{-1})^{-1},$$

for $t \in \mathbb{D}^*$. Note that $\lambda(t) \to 0$ as $t \to 0$.

Let $\mathscr{X}^{\mathrm{hyb}} := X \coprod \Delta(\mathscr{X})$ be the locally compact hybrid space constructed in §2. It comes with a proper map $\pi\colon \mathscr{X}^{\mathrm{hyb}} \to \Delta$ extending $\pi\colon X \to \mathbb{D}^*$ and such that $\Delta(\mathscr{X}) = \pi^{-1}(0)$. The next result implies Theorem A in the introduction.

Theorem 3.4. — *Let $\pi\colon \mathscr{X} \to \mathbb{D}$ be an snc degeneration, $\mathscr{L}$ a $\mathbb{Q}$-line bundle on $\mathscr{X}$ extending $K_{X/\mathbb{D}^*}$, and ψ a continuous metric on $\mathscr{L}$. Define $\kappa_{\min}$ as above, and set $d := \dim \Delta(\mathscr{L})$. Then, viewed as measures on $\mathscr{X}^{\mathrm{hyb}}$,*

$$\mu_t := \frac{\lambda(t)^d}{(2\pi)^d |t|^{2\kappa_{\min}}}\, e^{2\psi_t}$$

converges weakly to

$$\mu_0 := \sum_\sigma \left(\int_{Y_\sigma} \mathrm{Res}_{Y_\sigma}(\psi)\right) b_\sigma^{-1} \lambda_\sigma,$$

where σ ranges over the d-dimensional faces of $\Delta(\mathscr{L})$. Here λ_σ denotes normalized Lebesgue measure on σ and $b_\sigma = \gcd_{i\in J} b_i$, where $\mathscr{X}_0 = \sum_i b_i E_i$ and E_i, $i \in J$ are the divisors defining σ.

We start by making a few reductions. First, we may—and will—assume in what follows that $\kappa_{\min} = 0$. Indeed, t defines a nonvanishing section of $\mathscr{O}_{\mathscr{X}}(\mathscr{X}_0)$, and hence a smooth metric $\log |t|$, so we may replace $\mathscr{L}$ and ψ with $\mathscr{L} - \kappa_{\min}\mathscr{X}_0$ and $\psi - \kappa_{\min} \log |t|$, respectively, and end up with $\kappa_{\min} = 0$.

Since $\min_i a_i/b_i = \kappa_{\min} = 0$, we then have $a_i \geqslant 0$, with equality if and only if E_i corresponds to a vertex of $\Delta(\mathscr{L})$.

Next we reduce the assertion of Theorem 3.4 to a local problem. Let $Y \subset \mathscr{X}_0$ be the stratum of an arbitrary face σ of $\Delta(\mathscr{X})$, and denote by $E_0, \dots, E_p$ the components of $\mathscr{X}_0$ cutting out Y, ordered so that

$$\kappa_0 = \cdots = \kappa_q < \kappa_{q+1} \leqslant \cdots \leqslant \kappa_p.$$

We can then make the identification

$$\sigma = \left\{ w \in \mathbb{R}_+^{p+1} \;\middle|\; b \cdot w = 1 \right\}$$

with $b = (b_0, \dots, b_p) \in \mathbb{Z}_{>0}^{p+1}$. Set $b' = (b_0, \dots, b_q) \in \mathbb{Z}_{>0}^{q+1}$ and

$$\sigma' := \left\{ w' \in \mathbb{R}_+^{q+1} \;\middle|\; b' \cdot w' = 1 \right\}.$$

Then σ' is a face of σ under the embedding $\mathbb{R}_+^{q+1} \hookrightarrow \mathbb{R}_+^{p+1}$ given by $w' \to (w', 0)$. Let $Y' \supset Y$ be the corresponding stratum of $\mathscr{X}_0$.

Note that σ contains a face of $\Delta(\mathscr{L})$ if and only if $\kappa_0 = 0$; in that case, the face is unique, equal to σ' (which then implies $q \leqslant d$).

Pick $x \in \mathring{Y}$, and choose local coordinates $z = (z_0, \dots, z_n)$ at x such that z_i is a local equation of E_i for $0 \leqslant i \leqslant p$ and

$$t = \prod_{i=0}^{p} z_i^{b_i}.$$

We may assume that z is defined on a polydisc $\mathscr{U} \simeq \mathbb{D}(r)^{p+1} \times \mathbb{D}^{n-p}$ with $0 < r \ll 1$. Decompose

$$z = (z_0, \dots, z_n) \in \mathscr{U} \simeq \mathbb{D}(r)^{p+1} \times \mathbb{D}^{n-p}$$

as

$$z = (z', z'', y) \in \mathbb{D}(r)^{q+1} \times \mathbb{D}(r)^{p-q} \times \mathbb{D}^{n-p},$$

where we view y as a point of $\mathscr{U} \cap Y \simeq \mathbb{D}^{n-p}$, and (z'', y) as a point of $\mathscr{U} \cap Y' \simeq \mathbb{D}(r)^{p-q} \times \mathbb{D}^{n-p}$.

The coordinate chart $(\mathscr{U}, z)$ is adapted to $\mathscr{X}_0$ in the sense of §2.2, with

$$\mathrm{Log}_{\mathscr{U}} : \mathscr{U} \smallsetminus \mathscr{X}_0 \longrightarrow \sigma$$

given by

$$\mathrm{Log}_{\mathscr{U}} = \left(\frac{\log |z_i|}{\log |t|} \right)_{0 \leqslant i \leqslant p}.$$

We aim to establish the following result.

Lemma 3.5. — *Pick $\chi \in C_c^0(\mathscr{U})$. If $\kappa_0 = 0$ and $q = d$, then*

$$\lim_{t \to 0} (\mathrm{Log}_{\mathscr{U}})_*(\chi \mu_t) = \left(\int_{Y'} \chi \, \mathrm{Res}_{Y'}(\psi) \right) b_{\sigma'}^{-1} \lambda_{\sigma'}$$

in the weak topology of measures on σ, with σ' the unique d-dimensional face of $\Delta(\mathscr{L})$ contained in σ. Otherwise (i.e., if $\kappa_0 > 0$ or $q < d$) $(\mathrm{Log}_{\mathscr{U}})_*(\chi \mu_t) \to 0$.

Granted this result, let us show how to prove Theorem 3.4. For $0 < r \ll 1$, $\mathscr{V} := \pi^{-1}(\overline{\mathbb{D}}_r) \subset \mathscr{X}$ is an compact neighborhood of $\mathscr{X}_0$ with a map $\mathrm{Log}_{\mathscr{V}} : \mathscr{V}^{\mathrm{hyb}} \to \Delta(\mathscr{X})$ as in Proposition 2.1. We will use

Lemma 3.6. — *Let μ_t, $t \in \mathbb{D}_r$ be a family of probability measures on $\mathscr{X}^{\mathrm{hyb}}$ such that μ_t is supported on $\mathscr{X}_t$. Then $\lim_{t \to 0} \mu_t = \mu_0$ if and only if $\lim_{t \to 0} (\mathrm{Log}_{\mathscr{V}})_* \mu_t = \mu_0$. Here the limits are in the sense of weak convergence of measures on $\mathscr{X}^{\mathrm{hyb}}$ and $\Delta(\mathscr{X})$, respectively.*

By Lemma 3.6 we must show that

$$(\mathrm{Log}_{\mathscr{V}})_* \mu_t \longrightarrow \mu_0 = \sum_{\sigma'} \left(\int_{Y'} \mathrm{Res}_{Y'}(\psi) \right) b_{\sigma'}^{-1} \lambda_{\sigma'},$$

where σ' ranges over d-dimensional simplices in $\Delta(\mathscr{L})$. But this is easily seen to follow from Lemma 3.5, using a partition of unity argument as in the proof of Proposition 2.1.

Proof of Lemma 3.6. — The direct implication follows from the continuity of $\mathrm{Log}_{\mathscr{V}}$. For the reverse implication, assume that $\lim_{t\to 0}(\mathrm{Log}_{\mathscr{V}})_*\mu_t = \mu_0$ and consider the following three subsets of $C^0(\mathscr{V})$: A_1 is the set of functions of the form $\mathrm{Log}_{\mathscr{V}}^*\varphi$, where $\varphi \in C^0(\Delta(\mathscr{X}))$; A_2 is the set of functions of the form $\pi^* g$, where $g \in C^0(\mathbb{D}_r)$; and $A_3 = C^0_c(\mathscr{V} \smallsetminus \Delta(\mathscr{X}))$ together with the constant function 1. Then the real vector space $A \subset C^0(\mathscr{V})$ spanned by functions of the form $f_1 f_2 f_3$, with $f_i \in A_i$ is easily seen to be an $\mathbb{R}$-algebra that separates points and contains all constant functions. By the Stone-Weierstrass Theorem, A is dense in $C^0(\mathscr{V})$, so it suffices to prove that $\lim \int f\mu_t = \int f\mu_0$ for $f \in A$. By linearity, we may assume $f = f_1 f_2 f_3$ with $f_i \in A_i$. We may further assume $f_3 = 1$. Write $f_1 = \mathrm{Log}_{\mathscr{V}}^*\varphi$ and $f_2 = \pi^* g$. Then

$$\begin{aligned}\lim_{t\to 0}\int_{X_t} f\mu_t &= \lim_{t\to 0} g(t)\int_{X_t}\varphi\circ \mathrm{Log}_{\mathscr{V}}\,\mu_t\\ &= \lim_{t\to 0} g(t)\int_{\Delta(\mathscr{X})}\varphi\,(\mathrm{Log}_{\mathscr{V}})_*\mu_t = g(0)\int_{\Delta(\mathscr{X})}\varphi\mu_0 = \int f\mu_0,\end{aligned}$$

which completes the proof. ☐

3.3. Proof of Lemma 3.5. — As in §3.1, we introduce the logarithmic form

$$\Omega := \frac{dz_0}{z_0}\wedge\cdots\wedge\frac{dz_p}{z_p}\wedge dz_{p+1}\wedge\cdots\wedge dz_n,$$

and the corresponding local trivialization $\Omega^{\mathrm{rel}} = \Omega\otimes(dt/t)^{-1}$ of $K^{\log}_{\mathscr{X}/\mathbb{D}}$. The restriction Ω_t of Ω^{rel} to the fiber $U_t := \mathscr{X}_t\cap\mathscr{U}$ is a trivializing section of K_{U_t}, explicitly given by

$$\Omega_t = \frac{1}{p+1}\sum_{j=0}^{p}\frac{(-1)^j}{b_j}\,\frac{dz_0}{z_0}\wedge\cdots\wedge\widehat{\frac{dz_j}{z_j}}\wedge\cdots\wedge\frac{dz_p}{z_p}\wedge dz_{p+1}\wedge\cdots\wedge dz_n\Big|_{U_t}.$$

For $t\in\mathbb{D}^*$ close to 0, consider the map $\mathrm{Log}_t\colon U_t\to\sigma\times(Y\cap\mathscr{U})$ defined by

$$\mathrm{Log}_t = (\mathrm{Log}_{\mathscr{U}}, y) = \left(\frac{\log|z_0|}{\log|t|},\dots,\frac{\log|z_p|}{\log|t|}, z_{p+1},\dots,z_n\right).$$

Note the similarity to the situation considered in §1.4. More precisely, view $U := \mathscr{U}\cap X$ as embedded in $T\times\mathbb{C}^{n-p}$, where $T = (\mathbb{C}^*)^{p+1}$, and consider the character $\chi = \prod_{i=0}^p z_i^{b_i}$ on T. If $L\colon T\to\mathbb{R}^{p+1}$ is the tropicalization map, then

$$\mathrm{Log}_t = (\lambda(t)^{-1}L(z', z''), y).$$

Each fiber $\mathrm{Log}_t^{-1}(w,y)$ is a torsor for the (possibly disconnected) compact Lie group

$$K = \left\{\theta\in(\mathbb{R}/\mathbb{Z})^{p+1} \mid \textstyle\sum_i b_i\theta_i = 0\right\};$$

hence carries a unique K-invariant probability measure $\rho_{t,w,y}$.

The analysis in §1.4 now gives the following expression for the volume form $|\Omega_t|^2$ on U_t in logarithmic polar coordinates:

Lemma 3.7. — *For $h\in C^0_c(\mathscr{U})$ and $t\in\mathbb{D}^*$ close to 0, we have*

$$\int_{U_t} h|\Omega_t|^2=(2\pi)^p\lambda(t)^{-p}\int_{\sigma\times(Y\cap\mathscr{U})} b_\sigma^{-1}\lambda_\sigma(dw)\otimes|dy|^2\int_{\mathrm{Log}_t^{-1}(w,y)} h\,\rho_{t,w,y},\tag{3.2}$$

where $dy:=dz_{p+1}\wedge\cdots\wedge dz_n$.

As before, view $\tau:=\prod_{i=0}^p z_i^{a_i}\Omega^{\mathrm{rel}}$ as a local $\mathbb{Q}$-generator of $\mathscr{L}$, and set

$$g:=-\log|\tau|_\psi\in C^0(\mathscr{U}).$$

By definition, we have $\mu_t=(2\pi)^{-d}\lambda(t)^d|\Omega_t|^2/|\Omega_t|^2_{\psi_t}$, and hence

$$\begin{aligned}(2\pi)^{d-p}\lambda(t)^{p-d}|t|^{-2\kappa_0}&\int_{U_t} h\mu_t\\ &=\int_{\sigma\times(Y\cap\mathscr{U})} |t|^{2\sum_{i=q+1}^p b_iw_i(\kappa_i-\kappa_0)}b_\sigma^{-1}\lambda_\sigma(dw)\otimes|dy|^2\int he^{2g}\,\rho_{t,w,y}\\ &=\int_{\sigma\times(Y\cap\mathscr{U})} e^{-2\lambda(t)^{-1}\sum_{i=q+1}^p b_iw_i(\kappa_i-\kappa_0)}b_\sigma^{-1}\lambda_\sigma\otimes|dy|^2\int he^{2g}\,\rho_{t,w,y}\end{aligned}\tag{3.3}$$

for every $h\in C^0_c(\mathscr{U})$, thanks to Lemma 3.7.

We use the following change of variables. For $t\in\mathbb{D}^*$, consider the polytope

$$\sigma_t:=\Big\{(w',x'')\in\mathbb{R}_+^{q+1}\times\mathbb{R}_+^{p-q}\;\Big|\;b'\cdot w'=1,\ b''\cdot x''\leqslant\lambda(t)^{-1}\Big\}\subset\sigma'\times\mathbb{R}_+^{p-q}\subset\mathbb{R}_+^{p+1},$$

where $b'=(b_0,\dots,b_q)$ and $b''=(b_{q+1},\dots,b_p)$.

Lemma 3.8. — *The continuous map $Q_t\colon\sigma_t\to\sigma$ defined by*

$$Q_t(w',x'')=\big((1-\lambda(t)b''\cdot x'')\,w',\lambda(t)x''\big)$$

restricts to a homeomorphism between the interior of σ_t and the interior of σ. Further, its inverse maps the Lebesgue measure $b_\sigma^{-1}\lambda_\sigma$ on σ to the measure

$$(Q_t^{-1})_*b_\sigma^{-1}\lambda_\sigma=\big(1-\lambda(t)b''\cdot x''\big)^q\lambda(t)^{p-q}b_{\sigma'}^{-1}\lambda'_{\sigma'}\otimes|dx''|,$$

on σ_t, where $|dx''|$ is Lebesgue measure on $\mathbb{R}^{p-q}$ normalized by $\mathbb{Z}^{p-q}$.

Proof. — The first statement is elementary. To prove the second, we must make sure to handle the "multiplicities" b_σ and $b_{\sigma'}$ correctly. Parametrize the interior of σ by coordinates $(w_1,\dots,w_p)$ using $w_0=b_0^{-1}(1-\sum_1^p b_iw_i)$. By Remark 1.3 we have

$$b_\sigma\lambda_\sigma=|dw_1\wedge\cdots\wedge dw_p|.$$

Similarly, we parametrize the interiors of σ' and σ_t using coordinates $(w_1,\dots,w_q)$ and $(w_1,\dots,w_q,x''_{q+1},\dots,x''_p)$, respectively. Then

$$b_{\sigma'}\lambda_{\sigma'}=|dw_1\wedge\cdots\wedge dw_q|.$$

The required formula now follows from an elementary computation. □

Using the map Q_t and the fact that $\kappa_i-\kappa_0>0$ for $i>q$, it is easy to see that

$$\int_\sigma e^{-2\lambda(t)^{-1}\sum_{i=q+1}^p b_i w_i(\kappa_i-\kappa_0)}\lambda_\sigma(dw)=O(\lambda(t)^{p-q}).$$

By (3.3), it follows that

$$\mu_t(U_t)=O(\lambda(t)^{d-q}|t|^{2\kappa_0}), \tag{3.4}$$

and hence $\mu_t(U_t)\to 0$ unless $\kappa_0=0$ and $q=d$, which we henceforth assume. Given $\varphi\in C^0(\sigma)$, our goal is now to show

$$\int_{U_t}(\varphi\circ \mathrm{Log}_{\mathscr{U}})\chi\,\mu_t \longrightarrow \Big(\int_{\sigma'}\varphi b_{\sigma'}^{-1}\lambda_{\sigma'}\Big)\cdot\Big(\int_{Y'}\chi\,\mathrm{Res}_{Y'}(\psi)\Big). \tag{3.5}$$

Let us first express both sides of (3.5) in logarithmic polar coordinates. We start by the left-hand side. Set $f:=\chi e^{2g}\in C^0(\mathscr{U})$. By (3.3) and Lemma 3.8 we have

$$\begin{aligned}(3.6)\quad &(2\pi)^{d-p}\int_{U_t}(\varphi\circ\mathrm{Log}_{\mathscr{U}})\chi\,\mu_t\\ &=\lambda(t)^{d-p}\int_{\sigma\times(Y\cap\mathscr{U})}\varphi(w)e^{-2\lambda(t)^{-1}a''\cdot w''}b_\sigma^{-1}\lambda_\sigma(dw)\otimes|dy|^2\int f\,\rho_{t,w,y}\\ &=\int_{\sigma'\times\mathbb{R}_+^{p-d}\times(Y\cap\mathscr{U})}H_t(w',x'')b_{\sigma'}^{-1}\lambda_{\sigma'}(dw')\otimes|dx''|\otimes|dy|^2\int f\,\rho_{t,w',x'',y},\end{aligned}$$

where

$$H_t(w',x'')=\mathbf{1}_{\sigma_t}\varphi(Q_t(w',x''))e^{-2a''\cdot x''}(1-\lambda(t)b''\cdot x'')^d,$$

and $\rho_{t,w',x'',y}$ is the same measure as $\rho_{t,w,y}$ via the identification $Q_t(w',x'')=w$.

Note that $\lim_{t\to0}Q_t(w',x'')=(w',0)$, so

$$\lim_{t\to0}H_t(w',x'')=\mathbf{1}_{\sigma'\times\mathbb{R}_+^{p-d}}\varphi(w',0)e^{-2a''\cdot x''}.$$

Consider the tropicalization map

$$S\colon Y'\cap\mathscr{U}\longrightarrow\mathbb{R}_+^{p-d}\times(Y\cap\mathscr{U})$$

given by $S=(-\log|z_{d+1}|,\dots,-\log|z_p|,y)$. Each fiber $S^{-1}(x'',y)$ is a torsor for the compact torus $(\mathbb{R}/\mathbb{Z})^{p-d}$ and hence carries a unique invariant probability measure $\rho_{x'',y}$. As $t\to0$, the probability measure $\rho_{t,w',x'',y}$ converges weakly to $\rho_{x'',y}$ for any $w'\in\sigma'$.

By dominated convergence it follows that

$$\begin{aligned}(3.7)\quad &\lim_{t\to0}(2\pi)^{d-p}\int_{U_t}(\varphi\circ\mathrm{Log}_{\mathscr{U}})\chi\,d\mu_t\\ &=\int_{\sigma'\times\mathbb{R}_+^{p-d}\times(Y\cap\mathscr{U})}\varphi(w',0)e^{-2a''\cdot x''}b_{\sigma'}^{-1}\lambda_{\sigma'}(dw')\otimes|dx''|\otimes|dy|^2\int f\,\rho_{x'',y}\\ &=\Big(\int_{\sigma'}\varphi b_{\sigma'}^{-1}\lambda_{\sigma'}\Big)\Big(\int_{\mathbb{R}_+^{p-d}}e^{-2a''\cdot x''}|dx''|\int_{Y\cap\mathscr{U}}|dy|^2\int f\,\rho_{x'',y}\Big).\end{aligned}$$

It only remains to compare the second factor of (3.7) to the second factor in (3.5). To this end, we again use logarithmic polar coordinates. We have

$$\chi\,\mathrm{Res}_{Y'}(\psi) = f \prod_{i=d+1}^{p} |z_i|^{2a_i-2}|dz''|^2 \otimes |dy|^2. \tag{3.8}$$

For $d < j \leqslant p$, set $z_j = e^{-x_j+2\pi i\theta_j}$ with $x'' \in \mathbb{R}_+^{p-d}$ and $\theta'' \in (\mathbb{R}/\mathbb{Z})^{p-d}$. Then

$$\int_{Y'\cap\mathscr{U}} \chi\,\mathrm{Res}_{Y'}(\psi) = (2\pi)^{p-d}\int_{\mathbb{R}_+^{p-d}} e^{-2a''\cdot x''}|dx''| \int_{Y\cap\mathscr{U}} |dy|^2 \int f\,\rho_{x'',y}, \tag{3.9}$$

which completes the proof of (3.5), and hence of Theorem 3.4.

4. The limit hybrid model

Let $\pi\colon X\to\mathbb{D}^*$ be a proper submersion, with X a connected complex manifold. Assume that π is meromorphic over $0\in\mathbb{D}$ in the sense that it admits a *model* $\pi\colon\mathscr{X}\to\mathbb{D}$, that is, $\mathscr{X}$ is a normal complex space, π is a flat proper map, and we are given an isomorphism $X\simeq\pi^{-1}(\mathbb{D}^*)$ over $\mathbb{D}^*$. We say that $\mathscr{X}$ is an *snc model* (of X) if $\mathscr{X}$ is smooth and the Cartier divisor $\mathscr{X}_0 := \pi^{-1}(0)$ has simple normal crossing support. Such models always exist by Hironaka's theorem.

To any snc model $\mathscr{X}$ we can associate as in §2 a hybrid space $\mathscr{X}^{\mathrm{hyb}}$, that of course depends on $\mathscr{X}$. In this section we define a canonical hybrid space X^{hyb}, obtained as the inverse limit of the $\mathscr{X}^{\mathrm{hyb}}$, that does not have this defect. We then prove Theorem B from the introduction.

In the projective case, we show that the both the central fiber X_0^{hyb} and the closed subset $X^{\mathrm{hyb}}_{\overline{\mathbb{D}}_r}$ can be viewed as analytifications in the sense of Berkovich.

4.1. Snc models and simple blowups.

— Given any two models $\mathscr{X}$, $\mathscr{X}'$ of X, there is a canonical bimeromorphic map $\mathscr{X}'\dashrightarrow\mathscr{X}$, and we say that $\mathscr{X}'$ *dominates* $\mathscr{X}$ if this map is a morphism. Any two models $\mathscr{X}$, $\mathscr{X}'$ are dominated by a third, for instance the normalization of the graph of $\mathscr{X}\dashrightarrow\mathscr{X}'$. By Hironaka's theorem, any model is dominated by an snc model. Thus the set of models forms a directed set, in which snc models are cofinal.

Suppose $\mathscr{X}$ is an snc model and that $\mathscr{X}'$ is another model that dominates $\mathscr{X}$ via $\rho\colon\mathscr{X}'\to\mathscr{X}$. As in [KS06, Def. 22] we say that ρ is a *simple blowup* if it is a blowup along a smooth, connected complex subspace W of $\mathscr{X}_0$ meeting transversely (or not at all) every irreducible component of $\mathscr{X}_0$ that does not contain it. In this case, $\mathscr{X}'$ is also an snc model.

Lemma 4.1. — *Suppose $\mathscr{X}$ and $\mathscr{X}'$ are snc models, and that $\mathscr{X}'$ dominates $\mathscr{X}$ via $\rho\colon\mathscr{X}'\to\mathscr{X}$. Then there exists a third snc model $\mathscr{X}''$ dominating $\mathscr{X}'$, such that the induced map $\mathscr{X}''\to\mathscr{X}$ is a composition of simple blowups.*

We are grateful to Bernard Teissier for help with the following argument.

Proof. — By Hironaka's version of the Chow theorem (in turn a consequence of the flattening theorem), see [Hir75, Cor. 2], there exists a complex manifold $\mathscr{X}''$ and a projective bimeromorphic morphism $\mathscr{X}'' \to \mathscr{X}$ such that $\mathscr{X}''$ dominates $\mathscr{X}'$. Since $\mathscr{X}' \to \mathscr{X}$ is an isomorphism above X, the construction in [Hir75] further guarantees that $\mathscr{X}'' \to \mathscr{X}$ is an isomorphism above X. Indeed, the proof proceeds by blowing up well-chosen smooth centers contained in the non-flat locus of $\mathscr{X}' \to \mathscr{X}$, see Déf. 4.4.3 (2) in loc. cit.

We may therefore assume that $\mathscr{X}' \to \mathscr{X}$ itself is projective, and more precisely the blowup of an ideal I cosupported on $\mathscr{X}_0$. By the principalization theorem for ideals, there exists a projective bimeromorphic morphism $\mathscr{X}'' \to \mathscr{X}$ that is a composition of simple blowups, such that the pullback of I to $\mathscr{X}''$ is a principal ideal, see [Kol07, Th. 3.45] or [Wło09, Th. 2.0.3]. In particular, $\mathscr{X}''$ dominates $\mathscr{X}'$. □

4.2. Induced maps between dual complexes. — Suppose $\mathscr{X}'$ and $\mathscr{X}$ are snc models with $\mathscr{X}'$ dominating $\mathscr{X}$ via $\rho\colon \mathscr{X}' \to \mathscr{X}$. There is then an integral affine map

$$r_{\mathscr{X}\mathscr{X}'}\colon \Delta(\mathscr{X}') \longrightarrow \Delta(\mathscr{X}),$$

defined as follows. Consider any simplex σ' of $\Delta(\mathscr{X}')$ and let Y' be the corresponding stratum. There exists a unique minimal stratum Y of $\mathscr{X}_0$ such that $\rho(Y') \subset Y$. Let $\sigma = \sigma_Y$ be the corresponding simplex. Let E_i, $0 \leqslant i \leqslant p$ (resp. E'_j, $0 \leqslant j \leqslant p'$) be the irreducible components of $\mathscr{X}_0$ cutting out Y (resp. Y'). Then

$$\rho^* E_i = \sum_{j=0}^{p'} a_{ij} E'_j,$$

for $0 \leqslant i \leqslant p$, where $a_{ij} \in \mathbb{Z}_{>0}$.

We can realize the simplex σ (resp. σ') as the subset $\{\sum_{i=0}^{p} b_i w_i = 1\} \subset \mathbb{R}_+^{p+1}$ (resp. $\{\sum_{j=0}^{p'} b'_j w'_j = 1\} \subset \mathbb{R}_+^{p'+1}$), where b_i (resp. b'_j) is the multiplicity of E_i in $\mathscr{X}_0$ (resp. of E'_j in $\mathscr{X}'_0$). The restriction of $r_{\mathscr{X}\mathscr{X}'}$ to σ' is then given by

$$w_i = \sum_{j=0}^{p'} a_{ij} w'_j. \tag{4.1}$$

for $0 \leqslant i \leqslant p$. It is clear that $r_{\mathscr{X}\mathscr{X}'}$ defines a continuous, integral affine map from $\Delta(\mathscr{X}')$ to $\Delta(\mathscr{X})$. Further, if $\mathscr{X}$, $\mathscr{X}'$ and $\mathscr{X}''$ are snc models with $\mathscr{X}''$ dominating $\mathscr{X}'$, and $\mathscr{X}'$ dominating $\mathscr{X}$, then $r_{\mathscr{X}\mathscr{X}'} \circ r_{\mathscr{X}'\mathscr{X}''} = r_{\mathscr{X}\mathscr{X}''}$.

In general, it may happen that $\rho(Y')$ is a strict subvariety of Y, and the linear map defining $r_{\mathscr{X}\mathscr{X}'}|_{\sigma'}$ could fail to be injective or surjective.

Definition 4.2. — With notation as above, we say that σ' is *active* for $r_{\mathscr{X}\mathscr{X}'}$ if the restriction $\rho|_{Y'}\colon Y' \to Y$ is a bimeromorphic morphism and the $\mathbb{Q}$-linear map defining $r_{\mathscr{X}\mathscr{X}'}|_{\sigma'}$ is an isomorphism. In this case, σ' and σ have the same dimension, and $r_{\mathscr{X}\mathscr{X}'}$ maps σ' homeomorphically onto a $\mathbb{Z}$-subsimplex of σ of the same dimension.

Denote by $A_{\mathscr{X}\mathscr{X}'}$ the union of all simplices in $\Delta(\mathscr{X}')$ that are active for $r_{\mathscr{X}\mathscr{X}'}$. Our goal in this subsection is to prove the following result.

Proposition 4.3. — *Let $\mathscr{X}$ and $\mathscr{X}'$ be snc models, with $\mathscr{X}'$ dominating $\mathscr{X}$. Then $r_{\mathscr{X}\mathscr{X}'}$ maps $A_{\mathscr{X}\mathscr{X}'}$ homeomorphically onto $\Delta(\mathscr{X})$.*

Corollary 4.4. — *The images under $r_{\mathscr{X}\mathscr{X}'}$ of the active simplices in $\Delta(\mathscr{X}')$ form a simplicial $\mathbb{Z}$-subdivision of $\Delta(\mathscr{X})$. As a consequence, there exists a unique, $\mathbb{Z}$-PA map $i_{\mathscr{X}'\mathscr{X}}\colon \Delta(\mathscr{X}) \to \Delta(\mathscr{X}')$ such that $i_{\mathscr{X}'\mathscr{X}}(\Delta(\mathscr{X})) = A_{\mathscr{X}\mathscr{X}'}$ and $r_{\mathscr{X}\mathscr{X}'} \circ i_{\mathscr{X}'\mathscr{X}} = \mathrm{id}$.*

When π, π' and ρ are projective, one can prove Proposition 4.3 using the algebraic tool of valuations. Here we follow an ad hoc approach, based on Lemma 4.1.

Lemma 4.5. — *Suppose $\mathscr{X}$, $\mathscr{X}'$ and $\mathscr{X}''$ are snc models, with $\mathscr{X}'$ dominating $\mathscr{X}$ and $\mathscr{X}''$ dominating $\mathscr{X}'$. Let σ'' be a simplex of $\Delta(\mathscr{X}'')$, and let σ' be the smallest simplex of $\Delta(\mathscr{X}')$ containing $r_{\mathscr{X}'\mathscr{X}''}(\sigma'')$. Then σ'' is active for $r_{\mathscr{X}\mathscr{X}''}$ iff σ'' is active for $r_{\mathscr{X}'\mathscr{X}''}$ and σ' is active for $r_{\mathscr{X}\mathscr{X}'}$. As a consequence,*

$$A_{\mathscr{X}\mathscr{X}''} = A_{\mathscr{X}'\mathscr{X}''} \cap r^{-1}_{\mathscr{X}'\mathscr{X}''}(A_{\mathscr{X}\mathscr{X}'}).$$

Proof. — To ease notation, set $r' := r_{\mathscr{X}'\mathscr{X}''}$ and $r := r_{\mathscr{X}\mathscr{X}'}$. Let σ be the smallest simplex of $\Delta(\mathscr{X})$ containing $r(\sigma')$. Write Y, Y' and Y'' for the strata of $\mathscr{X}_0$, $\mathscr{X}_0'$ and $\mathscr{X}_0''$ corresponding to σ, σ' and σ'', respectively. The restrictions $r'|_{\sigma''}\colon \sigma'' \to \sigma'$ and $r_{\sigma'}\colon \sigma' \to \sigma$ are given by $\mathbb{Q}$-linear maps, and we have induced morphisms $Y'' \to Y'$ and $Y' \to Y$.

First suppose that σ'' is active for r' and σ' is active for r. Then $r'|_{\sigma''}$ and $r|_{\sigma'}$ are given by $\mathbb{Q}$-linear isomorphisms; hence so is the composition $r_{\mathscr{X}\mathscr{X}''}|_{\sigma''}$. Similarly, the maps $Y'' \to Y'$ and $Y' \to Y$ are bimeromorphic morphisms; hence so is the composition $Y'' \to Y$. It follows that σ'' is active for $r_{\mathscr{X}\mathscr{X}''}$.

Conversely, suppose σ'' is active for $r_{\mathscr{X}\mathscr{X}''}$. Since the map $Y'' \to Y$ is a bimeromorphic morphism, the map $Y'' \to Y'$ (resp. $Y' \to Y$) must be injective (resp. surjective). In particular, $\dim Y'' \leqslant \dim Y'$ and $\dim Y \leqslant \dim Y'$. Similarly, since the $\mathbb{Q}$-linear map defining $r_{\mathscr{X}\mathscr{X}''}|_{\sigma''} = r|_{\sigma'} \circ r'|_{\sigma''}$ is an isomorphism, the $\mathbb{Q}$-linear map defining $r'|_{\sigma''}$ (resp. $r|_{\sigma'}$) must be injective (resp. surjective). In particular, $\dim \sigma'' \leqslant \dim \sigma'$ and $\dim \sigma \leqslant \dim \sigma'$. Now

$$\dim Y'' + \dim \sigma'' = \dim Y' + \dim \sigma' = \dim Y + \dim \sigma = n - 1,$$

so we infer that $\dim Y'' = \dim Y' = \dim Y$ and $\dim \sigma = \dim \sigma' = \dim \sigma''$. This further implies that the maps $Y'' \to Y'$ and $Y' \to Y$ are bimeromorphic morphisms, and that the $\mathbb{Q}$-linear maps defining $r'|_{\sigma''}$ and $r|_{\sigma'}$ are isomorphisms. Hence σ'' and σ' are active for r' and r, respectively. $\square$

Lemma 4.6. — *Suppose $\mathscr{X}$, $\mathscr{X}'$ and $\mathscr{X}''$ are snc models, with $\mathscr{X}'$ dominating $\mathscr{X}$ and $\mathscr{X}''$ dominating $\mathscr{X}'$.*

(a) *If $r_{\mathscr{X}\mathscr{X}''}\colon A_{\mathscr{X}\mathscr{X}''} \to \Delta(\mathscr{X})$ is surjective, then so is $r_{\mathscr{X}\mathscr{X}'}\colon A_{\mathscr{X}\mathscr{X}'} \to \Delta(\mathscr{X})$.*

(b) *If $r_{\mathscr{X}\mathscr{X}''}\colon A_{\mathscr{X}\mathscr{X}''} \to \Delta(\mathscr{X})$ is injective and $r_{\mathscr{X}'\mathscr{X}''}\colon A_{\mathscr{X}'\mathscr{X}''} \to \Delta(\mathscr{X}')$ is surjective, then $r_{\mathscr{X}\mathscr{X}'}\colon A_{\mathscr{X}\mathscr{X}'} \to \Delta(\mathscr{X})$ is injective.*

(c) *If $r_{\mathscr{X}\mathscr{X}'}\colon A_{\mathscr{X}\mathscr{X}'} \to \Delta(\mathscr{X})$ and $r_{\mathscr{X}'\mathscr{X}''}\colon A_{\mathscr{X}'\mathscr{X}''} \to \Delta(\mathscr{X}')$ are both surjective, then so is $r_{\mathscr{X}\mathscr{X}''}\colon A_{\mathscr{X}\mathscr{X}''} \to \Delta(\mathscr{X})$.*

(d) *If $r_{\mathscr{X}\mathscr{X}'}\colon A_{\mathscr{X}\mathscr{X}'} \to \Delta(\mathscr{X})$ and $r_{\mathscr{X}'\mathscr{X}''}\colon A_{\mathscr{X}'\mathscr{X}''} \to \Delta(\mathscr{X}')$ are both injective, then so is $r_{\mathscr{X}\mathscr{X}''}\colon A_{\mathscr{X}\mathscr{X}''} \to \Delta(\mathscr{X})$.*

Proof. — This is formal consequence of the relations $r_{\mathscr{X}\mathscr{X}''} = r_{\mathscr{X}\mathscr{X}'} \circ r_{\mathscr{X}'\mathscr{X}''}$ and $A_{\mathscr{X}\mathscr{X}''} = A_{\mathscr{X}'\mathscr{X}''} \cap r_{\mathscr{X}'\mathscr{X}''}^{-1}(A_{\mathscr{X}\mathscr{X}'})$. For example, let us prove (a). Pick any point $w \in \Delta(\mathscr{X})$. The assumption implies that we can find $w'' \in A_{\mathscr{X}\mathscr{X}''}$ with $r_{\mathscr{X}\mathscr{X}''}(w'') = w$. Then $w' := r_{\mathscr{X}'\mathscr{X}''}(w'') \in A_{\mathscr{X}\mathscr{X}'}$ and $r_{\mathscr{X}\mathscr{X}'}(w') = w$. Thus (a) holds. The proofs of (b)–(d) are similar and left to the reader. $\square$

Lemma 4.7. — *The assertions of Proposition* 4.3 *hold when ρ is a simple blowup.*

Proof. — This is well known (see e.g. [KS06, p. 381]) but we supply a proof for the convenience of the reader. To simplify notation, we set $r := r_{\mathscr{X}\mathscr{X}'}$, $A := A_{\mathscr{X}\mathscr{X}'}$, $\Delta := \Delta(\mathscr{X})$ and $\Delta' := \Delta(\mathscr{X}')$.

Let W be the center of the blowup ρ, and Z the smallest stratum of $\mathscr{X}_0$ containing W. Let E_i, $i \in I$ be the irreducible components of $\mathscr{X}_0$, $J \subset I$ the subset such that Z is an component of E_J, and σ_Z the simplex defined by Z. Let E'_i, $i \in I$ be the strict transform of E_i to $\mathscr{X}'$. Finally, let E' be the exceptional divisor of ρ. It corresponds to a vertex $v' = v'_{E'}$ of Δ'.

First assume $W \subsetneq Z$. In this case, Δ' is obtained from Δ by "raising a tent over the simplex σ_Z". Let us be more precise. Consider a simplex σ of Δ, corresponding to a stratum Y of $\mathscr{X}_0$. By the definition of a simple blowup, W meets every irreducible component of $\mathscr{X}_0$ transversely (if at all). It follows that Y cannot be contained in W, so ρ is a biholomorphism above a general point of Y. Thus the strict transform Y' of Y defines a stratum of $\mathscr{X}'_0$ as well as a simplex σ' of Δ', whose vertices correspond to the strict transforms of the vertices of σ. In this case, r maps σ' onto σ, and $\rho\colon Y' \to Y$ is a bimeromorphic morphism, so σ' is active for r.

This proves that $r\colon A \to \Delta$ is surjective. To prove injectivity, consider a stratum Y' of $\mathscr{X}'_0$, with corresponding simplex σ' of Δ'. If Y' is not contained in E', then ρ is a biholomorphism at the general point of Y', $Y := \rho(Y')$ is a stratum of $\mathscr{X}_0$ of the same dimension as Y', and Y' is the strict transform of Y. Thus we are in the situation above. On the other hand, if Y' is contained in E', then there exist irreducible components E_i, $i \in J$ of $\mathscr{X}_0$, having strict transforms E'_i, $i \in J$, such that σ' has v' and v'_i, $i \in J$ as vertices. Since W is not a stratum of $\mathscr{X}_0$, the smallest stratum Y containing $\rho(Y')$ is cut out by E_i, $i \in J$. It follows that r maps the simplex σ' onto the lower-dimensional simplex σ, so σ' is not active for r. Hence $r\colon A \to \Delta$ is injective.

Now assume $W = Z$ is stratum of $\mathscr{X}_0$, defining a simplex σ with vertices v_i, $i \in J$. In this case, Δ' is obtained from Δ by a barycentric subdivision of the simplex σ_Z. Again, let us be more precise. The same argument as above shows that if Y is a stratum of $\mathscr{X}_0$ that is not contained in W, and Y' is the strict transform, then the simplex $\sigma'_{Y'}$ is active for r and $r(\sigma'_{Y'}) = \sigma_Y$. Further, $\sigma'_{Y'}$ is the unique simplex in $\mathscr{X}'_0$ that is active for r and whose image under r meets the interior of σ_Y.

It remains to consider strata of $\mathscr{X}_0$ contained in Z. This becomes a toroidal calculation. Let Y be such a stratum, cut out by E_i, $i \in K$, where $J \subset K$. Then $\rho^{-1}(Y)$ consists of $|J|$ strata Y'_i, $i \in J$, each cut out by E' and E'_j, $j \in K \smallsetminus \{i\}$. The restriction $\rho|_{Y'_i}\colon Y'_i \to Y$ is a bimeromorphic morphism, and the corresponding simplex σ'_i

is active for r and maps homeomorphically onto a simplex contained in σ_Y. Further, these simplices $r(\sigma'_i)$ have disjoint interiors and cover σ_Y. Finally, if Y' is a stratum of $\mathscr{X}'_0$ contained in $E=\rho^{-1}(Z)$, then $Y=\rho(Y')$ is a stratum contained in Z, hence $Y'=Y'_i$ is one of the strata above. This completes the proof. $\square$

Proof of Proposition 4.3. — Since $r_{\mathscr{X}\mathscr{X}'}$ is continuous, $A_{\mathscr{X}\mathscr{X}'}$ is compact, and $\Delta(\mathscr{X})$ is Hausdorff, it suffices to prove that $r_{\mathscr{X}\mathscr{X}'}\colon A_{\mathscr{X}\mathscr{X}'}\to\Delta(\mathscr{X})$ is bijective.

Using Lemma 4.6 (c)–(d) and Lemma 4.7, one proves by induction on the number of blowups that $r_{\mathscr{X}\mathscr{X}'}\colon A_{\mathscr{X}\mathscr{X}'}\to\Delta(\mathscr{X})$ is bijective when $\mathscr{X}'\to\mathscr{X}$ is a composition of simple blowups.

Now consider the general case. Using Lemma 4.1 we find an snc model $\mathscr{X}''$ dominating both $\mathscr{X}$ and $\mathscr{X}'$ and such that the morphism $\mathscr{X}''\to\mathscr{X}$ is a composition of simple blowups. Thus $r_{\mathscr{X}\mathscr{X}''}\colon A_{\mathscr{X}\mathscr{X}''}\to\Delta(\mathscr{X})$ is bijective. By Lemma 4.6 (a), it follows that $r_{\mathscr{X}\mathscr{X}'}\colon A_{\mathscr{X}\mathscr{X}''}\to\Delta(\mathscr{X})$ is surjective. Since $\mathscr{X}$ and $\mathscr{X}'$ were arbitrary snc models with $\mathscr{X}'$ dominating $\mathscr{X}$, it follows that $r_{\mathscr{X}'\mathscr{X}''}\colon A_{\mathscr{X}'\mathscr{X}''}\to\Delta(\mathscr{X})$ is also surjective. It now follows from Lemma 4.6 (b) that $r_{\mathscr{X}\mathscr{X}'}\colon A_{\mathscr{X}\mathscr{X}''}\to\Delta(\mathscr{X})$ is injective, which completes the proof. $\square$

4.3. Induced maps between hybrid spaces. — To any snc model $\mathscr{X}$ of X we associated in §2 a hybrid space $\mathscr{X}^{\mathrm{hyb}}=X\coprod\Delta(\mathscr{X})$. Let us briefly recall the topology on $\mathscr{X}^{\mathrm{hyb}}$ in the present context. Extend $\pi\colon X\to\mathbb{D}^*$ to a map

$$\pi\colon\mathscr{X}^{\mathrm{hyb}}\longrightarrow\mathbb{D}$$

by declaring $\pi=0$ on $\Delta(\mathscr{X})$. For $0<r\leqslant 1$, define $\mathscr{X}_{\mathbb{D}_r}:=\pi^{-1}(\mathbb{D}_r)$. The construction in §2 yields, for $0<r\ll 1$, a tropicalization map

$$\log_{\mathscr{X}}\colon\mathscr{X}_{\mathbb{D}_r}\longrightarrow\Delta(\mathscr{X})$$

uniquely defined up to an additive error term of size $O((\log|t|)^{-1})$. The topology on $\mathscr{X}^{\mathrm{hyb}}$ is the coarsest one such that $\log_{\mathscr{X}}$ is continuous, π is continuous, and the inclusion $X\subset\mathscr{X}^{\mathrm{hyb}}$ is an open embedding.

Now suppose $\mathscr{X}'$ and $\mathscr{X}$ are snc models, with $\mathscr{X}'$ dominating $\mathscr{X}$ via $\rho\colon\mathscr{X}'\to\mathscr{X}$. Define the map $\rho^{\mathrm{hyb}}\colon\mathscr{X}'^{\mathrm{hyb}}\to\mathscr{X}^{\mathrm{hyb}}$ to be the identity on $X\subset\mathscr{X}'$ and equal to the map $r_{\mathscr{X}\mathscr{X}'}$ on $\Delta(\mathscr{X}')$ defined in §4.2.

Proposition 4.8. — *The map ρ^{hyb} is continuous and surjective. Further, we have*

$$\operatorname{Log}_{\mathscr{X}}\circ\rho^{\mathrm{hyb}}=r_{\mathscr{X}\mathscr{X}'}\circ\operatorname{Log}_{\mathscr{X}'}+O((\log|t|)^{-1})\tag{4.2}$$

on $X_{\mathbb{D}_r^*}$ for $0<r\ll 1$.

Proof. — Surjectivity follows from Proposition 4.3, and continuity from (4.2) after unwinding the definitions. It remains to establish (4.2). Consider any point $\xi'\in\mathscr{X}_0$ and set $\xi=\pi(\xi')$. We can find adapted coordinate charts $(\mathscr{U}',z')$ at ξ' on $\mathscr{X}'$ and $(\mathscr{U},z)$ at ξ on $\mathscr{X}$ such that $\rho(\mathscr{U}')\subset\mathscr{U}$ and such that the following holds: $t=\prod_{i=0}^p z_i^{b_i}$ in $\mathscr{U}$, $t=\prod_{j=0}^{p'}(z'_j)^{b'_j}$ in $\mathscr{U}'$ and $\rho^*z_i=\prod_j(z'_j)^{a_{ij}}$. Since the map $r_{\mathscr{X}\mathscr{X}'}$ is given by (4.1), the result now follows from Proposition 2.1. $\square$

4.4. The limit hybrid space. — Proposition 4.8 allows us to introduce

Definition 4.9. — The *hybrid space associated to* X is the topological space

$$X^{\mathrm{hyb}} := \varprojlim_{\mathscr{X}} \mathscr{X}^{\mathrm{hyb}},$$

where $\mathscr{X}$ runs over all snc models of X.

Here X^{hyb} is equipped with the inverse limit topology. The maps $\pi\colon \mathscr{X}^{\mathrm{hyb}} \to \mathbb{D}$ define a continuous and proper map

$$\pi\colon X^{\mathrm{hyb}} \longrightarrow \mathbb{D}.$$

We can identify X with the open subset $\pi^{-1}(\mathbb{D}^*)$. Similarly, the compact subset $X_0^{\mathrm{hyb}} := \pi^{-1}(0)$ can be identified with $\varprojlim_{\mathscr{X}} \Delta(\mathscr{X})$. For every snc model $\mathscr{X}$ we have, by the definition of the inverse limit, a continuous proper map $r_{\mathscr{X}}\colon X^{\mathrm{hyb}} \to \mathscr{X}^{\mathrm{hyb}}$. We also have an embedding $i_{\mathscr{X}}\colon \Delta(\mathscr{X}) \to X^{\mathrm{hyb}}$ of $\Delta(\mathscr{X})$ onto a closed subset of X_0^{hyb}. It satisfies $r_{\mathscr{X}} \circ i_{\mathscr{X}} = \mathrm{id}$ on $\Delta(\mathscr{X})$.

Remark 4.10. — It is not clear how to define a map $\mathrm{Log}\colon X^{\mathrm{hyb}} \to \varprojlim_{\mathscr{X}} \Delta(\mathscr{X})$, since each tropicalization map $\mathrm{Log}_{\mathscr{X}}$ is only defined on $X_{\mathbb{D}^*(r)}$, where $r = r_{\mathscr{X}}$ depends on $\mathscr{X}$. See §4.6 for a substitute in the projective case.

4.5. Convergence of measures. — For any locally compact Hausdorff space Z, let $\mathscr{M}(Z)$ denote the space of signed Radon measures on Z. By definition we have $X^{\mathrm{hyb}} = \varprojlim_{\mathscr{X}} \mathscr{X}^{\mathrm{hyb}}$, and this induces a homeomorphism

$$\mathscr{M}(X^{\mathrm{hyb}}) \xrightarrow{\sim} \varprojlim_{\mathscr{X}} \mathscr{M}(\mathscr{X}^{\mathrm{hyb}}).$$

Theorem 3.4 now implies the following result, which is equivalent to Corollary B in the introduction.

Corollary 4.11. — *Let $\pi\colon X \to \mathbb{D}^*$ be a proper submersion that is meromorphic at $0 \in \mathbb{D}$, and let ψ be a continuous metric on $K_{X/\mathbb{D}^*}$ with analytic singularities. Then there exists a positive measure μ_0 on X_0^{hyb} such that if $\mu_t := \lambda(t)^d e^{2\psi_t}/|t|^{2\kappa_{\min}}(2\pi)^d$, then $\lim_{t\to 0} \mu_t = \mu_0$ in the sense of weak convergence of measures on X^{hyb}. Further, there exists a snc model $\mathscr{X} \to \mathbb{D}$ and a $\mathbb{Q}$-line bundle $\mathscr{L}$ on $\mathscr{X}$ extending $K_{X/\mathbb{D}^*}$ such that ψ extends to a smooth metric on $\mathscr{L}$, and*

$$\mu_0 := \sum_\sigma \left(\int_{Y_\sigma} \mathrm{Res}_{Y_\sigma}(\psi) \right) b_\sigma^{-1} \lambda_\sigma,$$

where σ ranges over the d-dimensional faces of $\Delta(\mathscr{L})$. Here λ_σ denotes normalized Lebesgue measure on σ and $b_\sigma = \gcd_{i\in J} b_i$, where $\mathscr{X}_0 = \sum_i b_i E_i$ and E_i, $i \in J$ are the divisors defining σ.

4.6. The projective case. — Now consider the case when $X \to \mathbb{D}^*$ is projective.[5] As we now explain, we can then view X^{hyb} and its central fiber as analytic spaces.

The projectivity assumption means that X can be viewed as a smooth subspace $\mathbb{P}^N \times \mathbb{D}^*$, defined by homogeneous polynomials with coefficients that are holomorphic functions on $\mathbb{D}^*$ and meromorphic at $0 \in \mathbb{D}$.

We can view these coefficients as complex formal Laurent series, that is, elements of the field $K := \mathbb{C}((t))$. Given $r \in (0,1)$, this field admits a natural non-Archimedean absolute value that is trivial on $\mathbb{C}^*$ and normalized by $|t| = r$. In other words, we have $|\sum_j a_j t^j| = r^{\min\{j \mid a_j \neq 0\}}$.

Further, the equations defining X now define a smooth projective variety X_K over the field K. To this variety we can associate a non-Archimedean space X_K^{an}, namely the Berkovich analytification of X_K with respect to non-Archimedean norm on K. This is a connected and locally connected compact (Hausdorff) space.

We claim that X_0^{hyb} is homeomorphic on X_K^{an}. To see this, we note that, for the same reasons as above, every projective snc model $\mathscr{X} \to \mathbb{D}$ of X defines a projective snc model $\mathscr{X}_R$ of X_K over the valuation ring $R = \mathbb{C}[\![t]\!]$ of K. Further, the dual complex $\Delta(\mathscr{X})$ of $\mathscr{X}$ can be identified with the dual complex $\Delta(\mathscr{X}_R)$ of $\mathscr{X}_R$. Now, there exists a canonical retraction map $r_{\mathscr{X}} \colon X_K \to \Delta(\mathscr{X}_K)$, and we have

$$X_K^{\mathrm{an}} \xrightarrow{\sim} \varprojlim_{\mathscr{X} \text{ projective snc}} \Delta(\mathscr{X}). \tag{4.3}$$

This was announced in [KS06, Th. 10, p. 383]; see e.g. [BFJ16, Cor. 3.2] for details. On the other hand, Lemma 4.1 implies that in $X_0^{\mathrm{hyb}} = \varprojlim_{\mathscr{X}} \Delta(\mathscr{X})$, we may take the limit over projective snc models. This implies that $X_0^{\mathrm{hyb}} \simeq X_K^{\mathrm{an}}$.

Next we analyze the space X^{hyb} itself, using the appendix. Fix $0 < r < 1$ and consider the Banach ring

$$A_r := \left\{ f = \textstyle\sum_{\alpha \in \mathbb{Z}} c_\alpha t^\alpha \in \mathbb{C}((t)) \;\middle|\; \|f\|_{\mathrm{hyb}} := \sum_{\alpha \in \mathbb{Z}} \|c_\alpha\|_{\mathrm{hyb}} r^\alpha < +\infty \right\},$$

where $\|\cdot\|_{\mathrm{hyb}}$ is the maximum of the usual norm and the trivial norm on $\mathbb{C}$. The Berkovich spectrum $\mathscr{M}(A_r)$ of A_r is homeomorphic to $\overline{\mathbb{D}}_r$.

Every function that is holomorphic on $\mathbb{D}^*$ and meromorphic at $0 \in \mathbb{D}$ defines an element of A_r. Hence we can define the base change $X_{A_r} \subset \mathbb{P}^N_{A_r}$ using the same homogeneous equations as above. Then X_{A_r} is a scheme of finite type over A_r, so its analytification $X_{A_r}^{\mathrm{An}}$ is a compact Hausdorff space with a continuous map π_r onto $(\operatorname{Spec} A_r)^{\mathrm{An}} = \mathscr{M}(A_r) \simeq \overline{\mathbb{D}}_r$. (In Appendix A.6, this analytification is denoted by X^{hyb}, but here we use $X_{A_r}^{\mathrm{An}}$ for clarity.) We have a homeomorphism

$$\tau^* \colon \pi_r^{-1}(\overline{\mathbb{D}}_r^*) \xrightarrow{\sim} X_{\overline{\mathbb{D}}_r^*} \xrightarrow{\sim} X^{\mathrm{hyb}}_{\overline{\mathbb{D}}_r^*} \tag{4.4}$$

and another homeomorphism

$$\tau_0 \colon \pi^{-1}(0) \xrightarrow{\sim} X_0^{\mathrm{hyb}} \xrightarrow{\sim} X_K^{\mathrm{an}}. \tag{4.5}$$

We glue these together to a map $\tau \colon X_{A_r}^{\mathrm{An}} \to X^{\mathrm{hyb}}_{\overline{\mathbb{D}}_r}$.

[5] In the projective case, the existence of the spaces $\mathscr{X}^{\mathrm{hyb}}$ and X^{hyb} was observed by Kontsevich and Soibelman, see [KS06, p. 383].

Proposition 4.12. — *The map $\tau\colon X^{\mathrm{An}}_{A_r}\to X^{\mathrm{hyb}}_{\mathbb{D}_r}$ is homeomorphism.*

Proof. — It follows from (4.4) and (4.5) that τ is a bijection. Since $X^{\mathrm{An}}_{A_r}$ is compact and $X^{\mathrm{hyb}}_{\mathbb{D}_r}$ is Hausdorff, it only remains to prove that τ is continuous. It suffices to show that the corresponding map $\tau_{\mathscr{X}}: X^{\mathrm{An}}_{A_r}\to\mathscr{X}^{\mathrm{hyb}}_{\mathbb{D}_r}$ is continuous for a given snc model $\mathscr{X}$. For this, in turn, it suffices to show that $\mathrm{Log}_{\mathscr{X}}\circ\tau_{\mathscr{X}}$ is continuous near the central fiber.

Consider a coordinate chart $(\mathscr{U},z)$ adapted to $\mathscr{X}_0$ in the sense of §2.2. Let $E_0,\dots,E_p$ be the irreducible components of $\mathscr{X}_0$ intersecting $\mathscr{U}$. Let $\widehat{\mathscr{U}}\subset X^{\mathrm{An}}_{A_r}$ be the set of seminorms satisfying $|z_i|<1$ for $0\leqslant i\leqslant p$. Then we have

$$\begin{aligned}\mathrm{Log}_{\mathscr{X}}\circ\tau_{\mathscr{X}}&=\left(\frac{\log|z_i|_\infty}{\log|t|_\infty}\right)_{0\leqslant i\leqslant p}+O((\log|t|_\infty)^{-1})\\&=(\log|z_i|^{-1})_{0\leqslant i\leqslant p}+O((\log|t|_\infty)^{-1})\end{aligned}$$

on $\widehat{\mathscr{U}}\smallsetminus\pi^{-1}(0)$. Now the function $(\log|z_i|^{-1})_i$ is continuous on $\widehat{\mathscr{U}}$ with values in the simplex $\sigma=\mathbb{R}^{p+1}_+\cap\{\sum_0^p b_iw_i=1\}\subset\Delta(\mathscr{X})$. This completes the proof, since we can cover a neighborhood of the central fiber in $X^{\mathrm{An}}_{A_r}$ with sets of the type $\widehat{\mathscr{U}}$. $\square$

5. Berkovich spaces and skeleta

Our goal in this section and the next is to study the limit measure μ_0 appearing in Corollary B in more detail. This measure lives on a Berkovich space and its support has an integral piecewise affine structure.

In this section we undertake a fairly general study of metrics on the canonical bundle of a projective variety defined over a discretely valued field of residue characteristic zero. To such a metric is associated a skeleton, a subset of the underlying Berkovich space. In the setting of Corollary B, the skeleton will be the support of the measure μ_0.

The material here has overlap with [MN15, NX16b] and also draws on [Tem16], but we present some details for the convenience of the reader.

Until further notice, X denotes a smooth, proper, geometrically connected variety over the field $K:=k(\!(t)\!)$ of formal Laurent series with coefficients in an algebraically closed field k of characteristic 0. We set $n:=\dim X$ and denote by X^{an} the Berkovich analytification of X with respect the non-Archimedean absolute value $|\cdot|=r^{\mathrm{ord}_0}$ on K, for some fixed $r\in(0,1)$.

While X^{an} comes equipped with a structure sheaf, we shall merely consider it as a topological space. Since X is proper, X^{an} is compact. There is a natural continuous surjective map $X^{\mathrm{an}}\to X$ such that the preimage of a (scheme) point $\xi\in X$ is identified with the set of real-valued valuations[(6)] v on the residue field of ξ satisfying $v|_{k^*}\equiv 0$ and $v(t)=1$. In particular, the preimage X^{val} of the generic point of X consists of real-valued valuations of the function field $F(X)$.

[(6)]Here we use additive terminology; the multiplicative norm associated to v is r^v.

5.1. Models. — Set $S := \operatorname{Spec} k[\![t]\!]$. Following the convention of [MN15], we define a *model* of X to be a normal separated scheme $\mathscr{X}$, flat and of finite type (but possibly non-proper) over S, together with an identification of the generic fiber of the structure morphism $\pi\colon \mathscr{X} \to S$ with X.

For any two models $\mathscr{X}$, $\mathscr{X}'$, the identifications of the generic fibers with X induces a unique birational map $\mathscr{X}' \dashrightarrow \mathscr{X}$. We say that $\mathscr{X}'$ *dominates* $\mathscr{X}$ if this map is a proper morphism. Any two models can be dominated by a third.

For any model $\mathscr{X}$ and every irreducible component E of $\mathscr{X}_0$, we set $b_E := \operatorname{ord}_E(t)$, and view the divisorial valuation

$$v_E := b_E^{-1} \operatorname{ord}_E$$

as an element of $X^{\mathrm{val}} \subset X^{\mathrm{an}}$. The set of such points is a dense subset $X^{\div} \subset X^{\mathrm{an}}$.

We usually denote by $\mathscr{X}_0 = \sum_{i \in I} b_i E_i$ the irreducible decomposition of the central fiber, and write $E_J := \bigcap_{i \in J} E_i$ for $J \subset I$. We say that $\mathscr{X}$ is *snc* if ($\mathscr{X}$ is regular and) $\mathscr{X}_{0,\mathrm{red}}$ has simple normal crossing support. Since k has characteristic 0, this means that each non-empty E_J is smooth over k, of codimension $|J|$ in $\mathscr{X}$.

More generally, a model $\mathscr{X}$ is *toroidal* if $\mathscr{X} \smallsetminus \mathscr{X}_0 \subset \mathscr{X}$ is a strict toroidal embedding in the sense of [KKMSD73], i.e., is formally isomorphic, at each closed point of $\mathscr{X}_0$, to the inclusion of $\mathbb{G}_{m,k}^{n+1}$ in a toric k-variety, and such that each E_i is normal (which then implies that each non-empty E_J is normal).

Every model $\mathscr{X}$ contains a largest snc Zariski open subset $\mathscr{X}_{\mathrm{snc}} \subset \mathscr{X}$. By Hironaka's theorem, $\mathscr{X}$ is dominated by an snc model $\mathscr{X}'$ such that the induced birational morphism $\mu\colon \mathscr{X}' \to \mathscr{X}$ is projective, and an isomorphism over $\mathscr{X}_{\mathrm{snc}}$.

If $\mathscr{X}$ is a model of X, the set $\mathscr{X}^{\mathrm{an}} \subset X^{\mathrm{an}}$ of semivaluations that admit a center (or reduction) on $\mathscr{X}_0$ is a closed subset; it can be viewed as the generic fiber of a suitable formal scheme [MN15, 2.2.2]. By the valuative criterion of properness, we have $\mathscr{X}^{\mathrm{an}} = \mathscr{X}'^{\mathrm{an}}$ for each proper morphism of models $\mathscr{X}' \to \mathscr{X}$, and $\mathscr{X}^{\mathrm{an}} = X^{\mathrm{an}}$ if $\mathscr{X}$ is proper (over S, that is). The reduction map $c_{\mathscr{X}}\colon \mathscr{X}^{\mathrm{an}} \to \mathscr{X}_0$, taking a semivaluation to its center, is anticontinuous.(7)

The set $\mathscr{X}^{\div} := \mathscr{X}^{\mathrm{an}} \cap X^{\div}$ consists of all divisorial valuations v on $F(\mathscr{X}) = F(X)$ that are centered on $\mathscr{X}_0$, trivial on k and such that $v(t) = 1$.

5.2. Model metrics. — If L is a line bundle on X, a *model* $\mathscr{L}$ of L is a $\mathbb{Q}$-line bundle $\mathscr{L}$ on a *proper* model $\mathscr{X}$, together with an identification $\mathscr{L}|_X = L$. It defines a *model metric* $\phi_{\mathscr{L}}$ on the Berkovich analytification L^{an} of L. If $\mathscr{L}'$ is another model of L, determined on a proper model $\mathscr{X}'$ of X, then $\phi_{\mathscr{L}} = \phi_{\mathscr{L}'}$ if and only if the pull-backs of $\mathscr{L}$ and $\mathscr{L}'$ to some higher model $\mathscr{X}''$ coincide.

A model of $\mathscr{O}_X$ is given by a $\mathbb{Q}$-Cartier divisor D supported on the central fiber of a proper model $\mathscr{X}$; the corresponding model metric will then be identified with the *model function* $\phi_D\colon X^{\mathrm{an}} \to \mathbb{R}$ defined by $\phi_D(v) = v(D)$. It satisfies

$$\inf_{X^{\mathrm{an}}} \phi_D = \min_E \phi_D(v_E), \tag{5.1}$$

where E runs over the irreducible components of $\mathscr{X}_0$.

(7) Anticontinuity means that the inverse image of an open set is closed.

5.3. Log canonical divisors. — If $\mathscr{X}$ is a regular model, $\pi\colon \mathscr{X} \to S$ is a locally complete intersection morphism, so the dualizing sheaf $\omega_{\mathscr{X}/S}$ is a well-defined line bundle (see [MN15, §4.1] for a more detailed discussion). For an arbitrary (normal) model, we may thus introduce the *relative canonical divisor (class)* $K_{\mathscr{X}/S}$ as the Weil divisor class on $\mathscr{X}$ such that $\mathscr{O}_{\mathscr{X}_{\mathrm{reg}}}(K_{\mathscr{X}/S}) = \omega_{\mathscr{X}_{\mathrm{reg}}/S}$. We then define:

(i) the *canonical divisor* $K_{\mathscr{X}} := K_{\mathscr{X}/S} + \pi^* K_S$;

(ii) the *log canonical divisor* $K^{\log}_{\mathscr{X}} := K_{\mathscr{X}} + \mathscr{X}_{0,\mathrm{red}}$;

(iii) the *relative log canonical divisor*

$$K^{\log}_{\mathscr{X}/S} := K^{\log}_{\mathscr{X}} - \pi^* K^{\log}_S = K_{\mathscr{X}/S} + \mathscr{X}_{0,\mathrm{red}} - \mathscr{X}_0.$$

Note that $K^{\log}_{\mathscr{X}}$ is $\mathbb{Q}$-Cartier if and only if $K^{\log}_{\mathscr{X}/S}$ is $\mathbb{Q}$-Cartier.

Example 5.1. — Assume that $\mathscr{X}$ is snc, and write as above $\mathscr{X}_0 = \sum_{i\in I} b_i E_i$. Pick a closed point $\xi \in \mathscr{X}_0$, and denote by $J = \{0,\dots,p\} \subset I$ the set of components of $\mathscr{X}_0$ passing through ξ. We may choose a regular system of parameters $z_0,\dots,z_n \in \mathscr{O}_{\mathscr{X},\xi}$ such that z_i is a local equation of E_i for $0 \leqslant i \leqslant p$, i.e., $t = u z_0^{b_0} \dots z_p^{b_p}$ for some unit $u \in \mathscr{O}^*_{\mathscr{X},\xi}$. The logarithmic form

$$\Omega := \frac{dz_0}{z_0} \wedge \cdots \wedge \frac{dz_p}{z_p} \wedge dz_{p+1} \wedge \cdots \wedge dz_n$$

is then a local generator of $K^{\log}_{\mathscr{X}}$, and induces a local generator

$$\Omega^{\mathrm{rel}} := \Omega \otimes (dt/t)^{-1}$$

of $K^{\log}_{\mathscr{X}/S}$.

Remark 5.2. — When $\mathscr{X}$ is snc, $\mathscr{O}_{\mathscr{X}}(K^{\log}_{\mathscr{X}/S})$ coincides with the relative logarithmic dualizing sheaf $\omega_{\mathscr{X}^+/S^+}$ of [NX16b, (3.2.2)]. When $\mathscr{X}$ is regular, $\mathscr{O}_{\mathscr{X}}(K_{\mathscr{X}})$ is described in [dFEM11, App. A] as the determinant of the locally free sheaf $\Omega'_{\mathscr{X}/k} \subset \Omega^1_{\mathscr{X}/k}$ of *special differentials*, corresponding to derivations D of $\mathscr{O}_{\mathscr{X}}$ such that $D(f) = f'(t)dt$ for $f \in k[\![t]\!]$.

5.4. Log discrepancies. — We refer to [dFKX12], [NX16b, §2.2] and [KNX15] for more details and references on what follows.

Let $\mathscr{X}$ be a model with $K^{\log}_{\mathscr{X}}$ $\mathbb{Q}$-Cartier, and recall that $\mathscr{X}^{\div}$ denotes the set of divisorial valuations v on $\mathscr{X}$ such that $v(t) = 1$. We define the *log discrepancy* $A_{\mathscr{X}}(v)$ as the log discrepancy of v with respect to the pair $(\mathscr{X}, \mathscr{X}_{0,\mathrm{red}})$, in the usual sense of the Minimal Model Program.

The log discrepancy function $A_{\mathscr{X}}\colon \mathscr{X}^{\div} \to \mathbb{Q}$ is characterized by the following property: if $\mathscr{X}'$ is a model over $\mathscr{X}$ with proper birational morphism $\rho\colon \mathscr{X}' \to \mathscr{X}$, then

$$K^{\log}_{\mathscr{X}'} = \rho^* K^{\log}_{\mathscr{X}} + \sum_E b_E A_{\mathscr{X}}(v_E) E, \tag{5.2}$$

with E running over the irreducible components of $\mathscr{X}'_0$.

We say that a model $\mathscr{X}$ is *log canonical* (*lc* for short) and *divisorially log terminal* (*dlt*) if the pair $(\mathscr{X}, \mathscr{X}_{0,\mathrm{red}})$ has the corresponding property, in the sense of the Minimal Model Program.

Since the generic fiber X is smooth, a model $\mathscr{X}$ is thus lc if and only if $K^{\log}_{\mathscr{X}}$ is $\mathbb{Q}$-Cartier, with log discrepancy function $A_{\mathscr{X}} : \mathscr{X}^{\div} \to \mathbb{Q}$ taking non-negative values. If $\mathscr{X}$ is lc, then the center $c_{\mathscr{X}}(v) \in \mathscr{X}_0$ of a valuation $v \in \mathscr{X}^{\div}$ with $A_{\mathscr{X}}(v) = 0$ is called an *lc center* of $\mathscr{X}$, and an lc model $\mathscr{X}$ is dlt if and only if $\mathscr{X}_{\mathrm{snc}}$ contains all lc centers. The irreducible components of each non-empty E_J are then normal, with generic point contained in $\mathscr{X}_{\mathrm{snc}}$ [Kol13, 4.16].

Example 5.3. — Assume that $\dim X = 1$, and let $\mathscr{X}$ be a dlt model. Each irreducible component E_i is then a smooth curve. At a point $\xi \in E_i \cap E_j$, $i \neq j$, $\mathscr{X}$ is snc. At a closed point $\xi \in \mathring{E}_i$, $\mathscr{X}$ is either regular, or has a cyclic quotient singularity.

Example 5.4. — If $\mathscr{X}$ is toroidal, then $\mathscr{X}$ is lc, and $\mathscr{X}$ is dlt if and only if it is snc. Following [dFKX12, KNX15], we could say that an lc model $\mathscr{X}$ is *qdlt* (for *quotient of dlt*) if its lc centers are contained in a toroidal open subset $\mathscr{U} \subset \mathscr{X}$.

Example 5.5. — If $\mathscr{X}$ is any model such that $\mathscr{X}_0$ has klt singularities (and hence is reduced), then $\mathscr{X}$ is dlt, by inversion of adjunction.

5.5. The skeleton of a dlt model. — The *dual complex* $\Delta(\mathscr{X})$ of an snc model $\mathscr{X}$ is defined as the dual complex of the snc divisor $\mathscr{X}_0 = \sum_{i\in I} b_i E_i$, as in §2.1. It is equipped with a natural integral affine structure, in which the face σ corresponding to a component Y of a non-empty E_J is identified with the simplex

$$\sigma = \left\{ w \in \mathbb{R}^J_+ \mid \textstyle\sum_{i\in J} b_i w_i = 1 \right\},$$

in such a way that $M_\sigma = \mathbb{Z}^J$.

As explained in [BFJ16, §3] and [MN15, §3], there is a natural embedding

$$\mathrm{emb}_{\mathscr{X}} : \Delta(\mathscr{X}) \longrightarrow \mathscr{X}^{\mathrm{an}}$$

that takes a point $w \in \sigma$ to the corresponding monomial valuation. In particular, the vertex corresponding to E_i is sent to the divisorial valuation $v_{E_i} = b_i^{-1} \operatorname{ord}_{E_i}$. The value group of a valuation $v = \mathrm{emb}_{\mathscr{X}}(w)$, $w \in \sigma$, is given by

$$v(F(X)^*) = M_\sigma(w) := \{f(w) \mid f \in M_\sigma\}.$$

Further, if $w \in \mathring{\sigma}$, then Y_σ is the closure of the center of $\mathrm{emb}_{\mathscr{X}}(w)$.

The resulting subspace $\mathrm{Sk}(\mathscr{X}) := \mathrm{emb}_{\mathscr{X}}(\Delta_X) \subset \mathscr{X}^{\mathrm{an}} \subset X^{\mathrm{an}}$ is called the *skeleton* of $\mathscr{X}$. It is naturally a $\mathbb{Z}$-PA space, the $\mathbb{Z}$-PA functions on $\mathrm{Sk}(\mathscr{X})$ being precisely the restrictions of model functions ϕ_D determined by a Cartier divisor D on some proper modification $\mathscr{X}' \to \mathscr{X}$.

We further have a natural retraction $r_{\mathscr{X}} : \mathscr{X}^{\mathrm{an}} \to \mathrm{Sk}(\mathscr{X})$, mapping a valuation v centered on $\mathscr{X}_0$ to the monomial valuation $r_{\mathscr{X}}(v)$ taking the same values on the E_i's.

These retractions induce a homeomorphism

$$X^{\mathrm{an}} \xrightarrow{\sim} \varprojlim_{\mathscr{X}} \mathrm{Sk}(\mathscr{X}),$$

where $\mathscr{X}$ runs over all proper (or projective) snc models, compare (4.3).

If $\mathscr{X}' \to \mathscr{X}$ is a proper morphism of snc models, then, by [MN15, 3.1.7],

$$\mathrm{Sk}(\mathscr{X}) \subset \mathrm{Sk}(\mathscr{X}') \subset \mathscr{X}'^{\mathrm{an}} = \mathscr{X}^{\mathrm{an}},$$

the first inclusion being $\mathbb{Z}$-PA. Further,

$$\bigcup_{\mathscr{X}\ \mathrm{snc}} \mathrm{Sk}(\mathscr{X}) \subset X^{\mathrm{an}}$$

coincides with the set of (quasi)monomial, or Abhyankar, valuations.

For a dlt model $\mathscr{X}$, the dual complex $\Delta(\mathscr{X})$ and skeleton $\mathrm{Sk}(\mathscr{X}) \subset X^{\mathrm{an}}$ are simply defined as those of $\mathscr{X}_{\mathrm{snc}}$, cf. [NX16b]. The retraction $r_{\mathscr{X}}\colon \mathscr{X}^{\mathrm{an}} \to \mathrm{Sk}(\mathscr{X})$ can be defined as above when $\mathscr{X}$ is $\mathbb{Q}$-factorial, but its existence is otherwise unclear (at least to us!).

By [KKMSD73], any toroidal model $\mathscr{X}$ has a dual complex $\Delta(\mathscr{X})$ endowed with a natural integral affine structure. This dual complex is canonically realized as a subspace $\mathrm{Sk}(\mathscr{X}) \subset X^{\mathrm{an}}$, for instance by setting $\mathrm{Sk}(\mathscr{X}) := \mathrm{Sk}(\mathscr{X}')$ for any toroidal modification $\mathscr{X}' \to \mathscr{X}$ with $\mathscr{X}'$ snc. Thus $\mathrm{Sk}(\mathscr{X})$ is equipped with a $\mathbb{Z}$-PA structure.

5.6. From log discrepancies to Temkin's metric. — As noted in [FJ04, BFJ08, JM12] in increasing order of generality, log discrepancy functions extend in a natural way to Berkovich spaces. More precisely, let $\mathscr{X}$ be any model of X such that $K^{\log}_{\mathscr{X}}$ is $\mathbb{Q}$-Cartier, with log discrepancy function $A_{\mathscr{X}}\colon \mathscr{X}^{\div} \to \mathbb{Q}$. For each snc model $\mathscr{X}'$ properly dominating $\mathscr{X}$, a simple computation going back (at least) to [Kol97, Lem. 3.11] shows the following:

(i) the restriction of $A_{\mathscr{X}}$ to $\mathrm{Sk}(\mathscr{X}')$ is $\mathbb{Z}$-affine on each face of $\Delta(\mathscr{X}')$;

(ii) we have $A_{\mathscr{X}} \geqslant A_{\mathscr{X}} \circ r_{\mathscr{X}'}$, the inequality being strict outside $\mathrm{Sk}(\mathscr{X}')$.

We may thus extend $A_{\mathscr{X}}$ to an lsc function $A_{\mathscr{X}}\colon \mathscr{X}^{\mathrm{an}} \to [0,+\infty]$ by setting

$$A_{\mathscr{X}}(v) := \sup_{\mathscr{X}'} A_{\mathscr{X}}(r_{\mathscr{X}'}(v)) \tag{5.3}$$

for any $v \in \mathscr{X}^{\mathrm{an}}$. When $\mathscr{X}$ is dlt, the log discrepancy function $A_{\mathscr{X}}$ determines the skeleton as follows.

Proposition 5.6. — *If $\mathscr{X}$ is dlt, then* $\mathrm{Sk}(\mathscr{X}) = \{v \in \mathscr{X}^{\mathrm{an}} \mid A_{\mathscr{X}}(v) = 0\}$.

Lemma 5.7. — *Assume that $\mathscr{X}$ is lc, and pick $v \in \mathscr{X}^{\mathrm{an}}$ with $A_{\mathscr{X}}(v) = 0$. Then $c_{\mathscr{X}}(v)$ is an lc center of $\mathscr{X}$.*

Proof. — We claim that, for every sufficiently high snc model $\mathscr{X}'$ proper over $\mathscr{X}$, $v' := r_{\mathscr{X}'}(v)$ and v have the same center on $\mathscr{X}$. Indeed, the center of v on $\mathscr{X}'$ is a specialization of that of $r_{\mathscr{X}'}(v)$, and hence $c_{\mathscr{X}}(v) \in \overline{c_{\mathscr{X}}(r_{\mathscr{X}'}(v))}$. On the other hand, we have $\lim_{\mathscr{X}'} r_{\mathscr{X}'}(v) = v$. Since $c_{\mathscr{X}}\colon \mathscr{X}^{\mathrm{an}} \to \mathscr{X}_0$ is anticontinuous, $c_{\mathscr{X}}^{-1}(\overline{\{c_{\mathscr{X}}(v)\}})$ is open, and hence contains $v' := r_{\mathscr{X}'}(v)$ for some snc model $\mathscr{X}'$ proper over $\mathscr{X}$. As a result, $c_{\mathscr{X}}(v')$ is a specialization of $c_{\mathscr{X}}(v)$, and the claim follows.

By (5.3), we have $A_{\mathscr{X}}(v') = 0$, and it is thus enough to prove the result for $v' \in \mathrm{Sk}(\mathscr{X}')$. If σ is the unique face of $\Delta(\mathscr{X}')$ containing v' in its interior, then $A_{\mathscr{X}} \equiv 0$ on σ, since $A_{\mathscr{X}}$ is non-negative and affine on σ. For any divisorial point w in the relative interior of σ, we thus have $A_{\mathscr{X}}(w) = 0$ and $c_{\mathscr{X}'}(v') = c_{\mathscr{X}'}(w)$, which shows that $c_{\mathscr{X}}(v') = c_{\mathscr{X}}(w)$ is an lc center. □

Proof of Proposition 5.6. — When $\mathscr{X}$ is snc, the result is a direct consequence of (i) and (ii) above. When $\mathscr{X}$ is dlt, we have by definition

$$\mathrm{Sk}(\mathscr{X}) = \mathrm{Sk}(\mathscr{X}_{\mathrm{snc}}) \subset \mathscr{X}_{\mathrm{snc}}^{\mathrm{an}} \subset \mathscr{X}^{\mathrm{an}},$$

and $A_{\mathscr{X}} = A_{\mathscr{X}_{\mathrm{snc}}}$ on $\mathscr{X}_{\mathrm{snc}}^{\mathrm{an}}$. It is thus enough to show that any $v \in \mathscr{X}^{\mathrm{an}}$ with $A_{\mathscr{X}}(v) = 0$ belongs to $\mathscr{X}_{\mathrm{snc}}^{\mathrm{an}}$, i.e., satisfies $c_{\mathscr{X}}(v) \in \mathscr{X}_{\mathrm{snc}}$. But $c_{\mathscr{X}}(v)$ is an lc center by Lemma 5.7, and hence $c_{\mathscr{X}}(v) \in \mathscr{X}_{\mathrm{snc}}$ by definition of dlt singularities. □

Let $\mathscr{X}$ be a proper model with $K^{\log}_{\mathscr{X}/S}$ $\mathbb{Q}$-Cartier. Viewed as a $\mathbb{Q}$-line bundle, the latter is then a model of K_X, and hence defines a model metric $\phi_{K^{\log}_{\mathscr{X}/S}}$ on K_X^{an}. Further, (5.2) shows that the lsc metric

$$A_X := \phi_{K^{\log}_{\mathscr{X}/S}} + A_{\mathscr{X}} \tag{5.4}$$

on K_X^{an} is independent of $\mathscr{X}$. This is a special case of Temkin's canonical metrization of the canonical bundle [Tem16].(8) The *weight function* of [MN15] associated to a pluricanonical form $\omega \in H^0(X, mK_X)$ is the function $A_X - \frac{1}{m}\log|\omega|$ on X^{an}.

5.7. The skeleton of a metric on K_X. — The purpose of this section is to introduce and study a slight generalization of the *Kontsevich–Soibelman skeleton* introduced in [KS06] and further analyzed in [MN15, NX16b].

Definition 5.8. — If ψ is a continuous (or usc) metric on K_X^{an}, set $\kappa := A_X - \psi$ and $\kappa_{\min} := \inf_{X^{\mathrm{an}}} \kappa$. The *skeleton* of ψ is the compact set

$$\mathrm{Sk}(\psi) = \{x \in X^{\mathrm{an}} \mid \kappa(x) = \kappa_{\min}\}.$$

Note that κ is an lsc function $X^{\mathrm{an}} \to (-\infty, +\infty]$, and hence achieves its infimum.

Definition 5.9. — Let $\mathscr{L}$ be a model of K_X determined on a proper dlt model $\mathscr{X}$. We denote by $\Delta(\mathscr{L})$ the subcomplex of $\Delta(\mathscr{X})$ such that a face σ of $\Delta(\mathscr{X})$ is in $\Delta(\mathscr{L})$ if and only if each vertex of σ achieves $\min_i \kappa(v_i)$ with $\kappa = A_X - \phi_{\mathscr{L}}$.

Concretely, the values $\kappa(v_i)$ are computed as follows: we have

$$K^{\log}_{\mathscr{X}/S} = \mathscr{L} + \sum_{i \in I} a_i E_i$$

(8) That we obtain Temkin's metric follows from [Tem16, Th. 8.1.2]. Note that Temkin uses multiplicative terminology.

with $a_i \in \mathbb{Q}$, and $\kappa(v_{E_i}) = a_i/b_i$. Note that each face of $\Delta(\mathscr{X})$ contains at most one maximal face of $\Delta(\mathscr{L})$.

Proposition 5.10. — *Assume that ψ is a model metric on K_X^{an}, determined by a model $\mathscr{L}$ of K_X on a proper dlt model $\mathscr{X}$ of X. Then $\mathrm{Sk}(\psi) \subset \mathrm{Sk}(\mathscr{X})$, and $\kappa = A_X - \psi$ is affine on each face of $\Delta(\mathscr{X})$. In particular,*

$$\kappa_{\min} = \min_i \kappa(v_i), \tag{5.5}$$

where v_i runs over the vertices in $\Delta(\mathscr{X})$, and $\mathrm{Sk}(\psi)$ is the subset of $\mathrm{Sk}(\mathscr{X}) \subset X^{\mathrm{an}}$ corresponding to the subcomplex $\Delta(\mathscr{L})$ of $\Delta(\mathscr{X})$.

Proof. — Since the relative log canonical divisor $K^{\log}_{\mathscr{X}/S}$ and $\mathscr{L}$ are both models of K_X, $D := K^{\log}_{\mathscr{X}/S} - \mathscr{L}$ is a $\mathbb{Q}$-Cartier divisor supported on $\mathscr{X}_0$. The corresponding model function ϕ_D satisfies $\kappa = A_{\mathscr{X}} + \phi_D$, which shows that $\kappa|_{\mathrm{Sk}(\mathscr{X})} = \phi_D|_{\mathrm{Sk}(\mathscr{X})}$ is affine on each face of $\Delta(\mathscr{X})$. Now pick $v \in \mathrm{Sk}(\psi)$. By (5.1), we get

$$\kappa(v) = A_{\mathscr{X}}(v) + \phi_D(v) \geqslant \inf \phi_D = \min_i \phi_D(v_i) = \min_i (A_{\mathscr{X}} + \phi_D)(v_i) \geqslant \inf_{X^{\mathrm{an}}} \kappa.$$

It follows that $A_{\mathscr{X}}(v) = 0$, and hence $v \in \mathrm{Sk}(\mathscr{X})$, by Proposition 5.6. □

5.8. Residual boundaries. — The following construction plays a crucial role for the understanding of the limit measure appearing in Corollary B.

Consider a model metric $\mathscr{L}$ of K_X defined on a proper dlt model $\mathscr{X}$. Following §3.1 we explain how to associate a subklt pair $(Y, B_Y^{\mathscr{L}})$ to each stratum Y of $\mathscr{X}_0$ corresponding to a maximal simplex in $\Delta(\mathscr{L})$.

Let us first recall a few facts about adjunction. When $\mathscr{X}$ is an snc model, each stratum Y comes with a boundary $B_Y := \sum_{i \notin J_Y} E_i \cap Y$. Here (Y, B_Y) is log smooth, and

$$K^{\log}_{\mathscr{X}/S}\big|_Y = K_{(Y,B_Y)} := K_Y + B_Y, \tag{5.6}$$

the identification being provided by Poincaré residues. When $\mathscr{X}$ is merely dlt, each stratum Y is normal, and comes with a canonically defined effective $\mathbb{Q}$-divisor B_Y such that (Y, B_Y) is dlt and still satisfies (5.6) (cf. [Kol13, 4.19]). We have

$$B_Y = \sum_{i \notin J_Y} E_i \cap Y + B'_Y,$$

where B'_Y is an effective $\mathbb{Q}$-divisor supported in the complement of $\mathscr{X}_{\mathrm{snc}}$.

Example 5.11. — For each i, $E_i \cap (\mathscr{X} \smallsetminus \mathscr{X}_{\mathrm{snc}})$ contains finitely many prime divisors F_{ik} of E_i. At the generic point of F_{ik}, $\mathscr{X}$ has cyclic quotient singularities, and

$$B_{E_i} = \sum_{j \neq i} E_j \cap E_i + \sum_k \Big(1 - \frac{1}{m_{ik}}\Big) F_{ik}$$

with m_{ik} the order of the corresponding cyclic groups, cf. [Kol13, 3.36.3].

Now let ψ be a model metric on K_X^{an}, determined by a model $\mathscr{L}$ of K_X on a proper dlt model $\mathscr{X}$ of X. Introduce as before the function $\kappa := A_X - \psi$ on X^{an}, and note that the $\mathbb{Q}$-Cartier divisor

$$D := K^{\log}_{\mathscr{X}/S} - \mathscr{L} - \kappa_{\min}\mathscr{X}_0 = \sum_i (\kappa(v_{E_i}) - \kappa_{\min}) b_i E_i$$

is effective.

Lemma 5.12. — *If Y is a stratum of $\mathscr{X}_0$ corresponding to a face σ of $\Delta(\mathscr{L})$, then $Y \not\subset \operatorname{supp} D$. It follows that the $\mathbb{Q}$-Cartier divisor*

$$B_Y^{\mathscr{L}} := B_Y - D|_Y$$

is well-defined, and we have a canonical identification $\mathscr{L}|_Y = K_{(Y, B_Y^{\mathscr{L}})}$ as $\mathbb{Q}$-line bundles. Further, if σ is a maximal face of $\Delta(\mathscr{L})$, then the pair $(Y, B_Y^{\mathscr{L}})$ is subklt.

We emphasize that $B_Y^{\mathscr{L}}$ is not effective in general.

Proof. — The first two points are clear. When σ is a maximal face, each E_i meeting Y satisfies $\kappa(v_{E_i}) > \kappa_{\min}$. As a result, $D|_Y$ contains each lc center $E_i \cap Y$ of (Y, B_Y), which yields the last assertion. $\square$

5.9. Skeleta and base change. — Now we study how skeleta of snc models and of metrics behave under base change.

For $m \in \mathbb{Z}_{>0}$ consider the Galois extension $K' := k(\!(t^{1/m})\!)$ of $K = k(\!(t)\!)$, with Galois group $G = \mathbb{Z}/m\mathbb{Z}$, and set $X' = X_{K'}$. Then G acts on X'^{an} and the canonical map $p\colon X'^{\mathrm{an}} \to X^{\mathrm{an}}$ induces a homeomorphism

$$X'^{\mathrm{an}}/G \xrightarrow{\sim} X^{\mathrm{an}}.$$

If $\mathscr{X}$ is a model of X, then its normalized base change yields a model $\mathscr{X}'$ of X' with a finite morphism $\rho\colon \mathscr{X}' \to \mathscr{X}$. If D is a $\mathbb{Q}$-divisor on $\mathscr{X}$ defining a model function ϕ_D on X^{an}, then

$$\phi_{\rho^* D} = m p^* \phi_D. \tag{5.7}$$

When $\mathscr{X}$ is an snc model, $\mathscr{X}'$ is toroidal, by [KKMSD73, pp. 98–102]. The following rather detailed description will be useful later on.

Lemma 5.13. — *We have $p^{-1}(\mathrm{Sk}(\mathscr{X})) = \mathrm{Sk}(\mathscr{X}')$. Further, for each face σ of $\Delta(\mathscr{X})$, there exist positive integers e_σ, f_σ and g_σ satisfying*

$$e_\sigma = \frac{m}{\gcd(m, b_\sigma)} \quad \textit{and} \quad f_\sigma g_\sigma = \gcd(m, b_\sigma)$$

and such that the following properties hold: $p^{-1}(\sigma)$ is a union of g_σ faces σ'_α of $\Delta(\mathscr{X}')$, and these are permuted by G. For each α:

(a) *p induces a $\mathbb{Q}$-affine isomorphism $\sigma'_\alpha \xrightarrow{\sim} \sigma$;*
(b) *p induces a generically finite map $Y_{\sigma'_\alpha} \to Y_\sigma$, of degree f_σ;*
(c) *$mp^* M_\sigma \subset M_{\sigma'_\alpha}$, and $[M_{\sigma'_\alpha} : mp^* M_\sigma] = e_\sigma$.*

Furthermore, we have:

(i) $M_{\sigma'_\alpha} = p^*(mM_\sigma + \mathbb{Z}1_\sigma)$;
(ii) $\mathrm{Vol}(\sigma'_\alpha) = m^{\dim\sigma}\,\mathrm{Vol}(\sigma)$;
(iii) $b_{\sigma'_\alpha} = b_\sigma/\gcd(m, b_\sigma)$.

Proof. — The proof uses the toroidal theory of [KKMSD73] together with elementary ramification theory of valuations [ZS75].

Let σ be the face of $\Delta(\mathscr{X})$ corresponding to an irreducible component Y of $E_0 \cap \cdots \cap E_p$. Set $b_i = \mathrm{ord}_{E_i}(t)$. With the identification

$$\sigma = \{w \in \mathbb{R}_+^{p+1} \mid \textstyle\sum_i b_i w_i = 1\},$$

the integral affine structure M_σ is given by the lattice $\mathbb{Z}^{p+1}$. Note that $b_\sigma = \gcd_i b_i$.

Given a closed point $\xi \in \mathring{Y}$, we can find local coordinates $z_0, \ldots, z_n$ in the formal completion $\widehat{\mathscr{O}}_{\mathscr{X},\xi} \simeq k[\![z_0, \ldots, z_n]\!]$ such that $t = \prod_{i=0}^p z_i^{b_i}$. A toric computation (cf. [KKMSD73, pp. 98–102]) shows that ξ has $\gcd(m, b_\sigma)$ preimages ξ'_α in $\mathscr{X}'_0$, with $\mathscr{X}'$ formally isomorphic, at each ξ'_α, to the product of $\mathbb{A}_k^{n-p}$ with the affine toric k-variety corresponding to the cone $\mathbb{R}_+^{p+1} \subset \mathbb{R}^{p+1}$ with lattice

$$M' := \mathbb{Z}^{p+1} + \mathbb{Z}\,(b_0/m, \ldots, b_p/m)\,.$$

It follows that $p^{-1}(\sigma)$ is the union of the corresponding faces σ'_α of $\Delta(\mathscr{X}')$, each isomorphic to

$$\sigma' = \bigl\{w' \in \mathbb{R}_+^{p+1} \bigm| \textstyle\sum_i b_i w'_i = m\bigr\},$$

with integral affine structure induced by M'. Now p restricts to a homeomorphism $\sigma'_\alpha \xrightarrow{\sim} \sigma$ given by $w = w'/m$. Thus $M_{\sigma'_\alpha} = mp^*M_\sigma + \mathbb{Z}1_{\sigma'_\alpha}$. This implies (i), and (ii)–(iii) easily follow.

Now note that

$$\begin{aligned}[][M'_{\sigma'_\alpha} : mp^*M_\sigma] &= [mp^*M_\sigma + \mathbb{Z}1_{\sigma'_\alpha} : mp^*M_\sigma] \\ &= [\mathbb{Z}^{p+1} + \mathbb{Z}(b_0/m, \ldots, b_p/m) : \mathbb{Z}^{p+1}] = \frac{m}{\gcd(m, b_\sigma)} =: e_\sigma.\end{aligned}$$

It remains to analyze the degree f_σ of the restriction $Y_{\sigma'_\alpha} \to Y_\sigma$. For this we use ramification theory.

The function field $F(X') = F(X)(t^{1/m})$ is a Galois extension of $F(X)$ of degree m, with Galois group G. For any valuation $v' \in X'^{\,\mathrm{val}}$, we have $v'|_{F(X)} = mp(v')$.

Let $v \in X^{\mathrm{an}}$ be a valuation corresponding to a point $w \in \sigma$. Assume w is "general" in the sense that $\dim_{\mathbb{Q}} \sum_{i=0}^p \mathbb{Q}w_i = p$. The point w has g_σ preimages w'_α under p, one in each σ'_α, and the valuations $v'_\alpha := m^{-1}w'_\alpha$ are all the extensions of v to $F(X')$. Let us compute the residue degree and ramification index of these extensions.

The residue fields of v and v'_α are exactly the function fields of Y and Y'_α, respectively, so the residue degree of the extension v'_α of v is equal to f_σ.

The value group $\Gamma_v = v(F(X))$ of v is given by $\Gamma_v = \sum_{i=0}^p \mathbb{Z}w_i$. Similarly, the value group of v'_α is given by $\Gamma_{v'_\alpha} = \frac{1}{m}\mathbb{Z} + \frac{1}{m}\sum_{i=0}^p \mathbb{Z}w'_i = \frac{1}{m}\mathbb{Z} + \sum_{i=0}^p \mathbb{Z}w_i$. It follows that the ramification index of the extension v'_α of v is given by

$$[\Gamma_{v'_\alpha}:\Gamma_v]=\big[\tfrac{1}{m}\mathbb{Z}+\textstyle\sum_{i=0}^p\mathbb{Z}w_i:\sum_{i=0}^p\mathbb{Z}w_i\big]=\gcd(\mathbb{Z}\cap m\textstyle\sum_{i=0}^p\mathbb{Z}w_i)=\dfrac{m}{\gcd(m,b_\sigma)}=e_\sigma.$$

By [ZS75, p. 77] we now have $e_\sigma f_\sigma g_\sigma=m$, which completes the proof. $\square$

Next we study skeleta of metrics. Generalizing [NX16b, Lem. 4.1.9], we prove:

Lemma 5.14. — *Let ψ be a continuous metric on K_X^{an}, ψ' the metric on $K_{X'}^{\mathrm{an}}\simeq p^*K_X^{\mathrm{an}}$ corresponding to $p^*\psi$, and set $\kappa':=A_{X'}-\psi'$. Then $\kappa'=mp^*\kappa$. As a consequence,* $\mathrm{Sk}(\psi')=p^{-1}\,\mathrm{Sk}(\psi)$ *and* $\kappa'_{\min}=m\kappa_{\min}$.

Proof. — By [Gub98, Th. 7.12] (see also [BFJ16, Cor. 2.3]), model metrics are dense in the set of continuous metrics on K_X^{an}. Hence we may assume ψ is a model metric. Using (5.3), it is enough to show that $\kappa'(v')=m\kappa(p(v'))$ for a divisorial valuation $v'\in X'^{\div}$. Let $\mathscr{X}$ be an snc model with $p(v')\in\mathrm{Sk}(\mathscr{X})$, and such that $\psi=\phi_{\mathscr{L}}$ for a model $\mathscr{L}$ of K_X on $\mathscr{X}$. Since the normalized base change $\mathscr{X}'$ of $\mathscr{X}$ is toroidal, we can choose a toroidal modification $\mathscr{X}''\to\mathscr{X}'$ with $\mathscr{X}''$ snc. The induced morphism $\rho\colon\mathscr{X}''\to\mathscr{X}$ is toroidal; hence it satisfies the log ramification formula

$$mK^{\log}_{\mathscr{X}''/S'}=\rho^*K^{\log}_{\mathscr{X}/S}.$$

By (5.7), we infer $\phi_{K^{\log}_{\mathscr{X}''/S'}}-\psi'=p^*(\phi_{K^{\log}_{\mathscr{X}/S}}-\psi)$, which gives the desired result since $v'\in\mathrm{Sk}(\mathscr{X}'')$, $p(v')\in\mathrm{Sk}(\mathscr{X})$ imply $A_{\mathscr{X}''}(v')=A_{\mathscr{X}}(p(v'))=0$. $\square$

6. Skeletal measures

From now on, we assume that $k=\mathbb{C}$, and that X is a smooth, projective, geometrically connected variety over the non-Archimedean field $K=\mathbb{C}(\!(t)\!)$. Our goal is to construct measures of the types appearing in Theorem A and Corollary B.

6.1. Residually metrized models.

— As explained above, to any model $\mathscr{L}$ of a line bundle L on X, defined on a proper dlt model $\mathscr{X}$ of X, we can associate a skeleton $\mathrm{Sk}(\mathscr{L})\subset\mathrm{Sk}(\mathscr{X})\subset X^{\mathrm{an}}$. To produce a measure on $\mathrm{Sk}(\mathscr{L})$ we need additional data.

Definition 6.1. — Let L be a line bundle on X. A *residually metrized model* of L is a pair $\mathscr{L}^\#=(\mathscr{L},\psi_0)$ where $\mathscr{L}$ is a model of L, determined on a proper dlt model $\mathscr{X}$ of X, and ψ_0 is a continuous Hermitian metric on $\mathscr{L}_0:=\mathscr{L}|_{\mathscr{X}_0}$, viewed as a holomorphic line bundle over the complex space $\mathscr{X}_0$. A *residually metrized model metric* $\psi^\#$ on L is an equivalence class of such pairs, modulo pull-back to a higher model.

Example 6.2. — If L is trivial, then any choice of trivialization $s\in H^0(X,L)$ defines a residually metrized model metric $\psi^\#$ on L, determined on any model $\mathscr{X}$ by $\mathscr{L}=\mathscr{O}_{\mathscr{X}}$ and ψ_0 the trivial metric on $\mathscr{O}_{\mathscr{X}_0}$.

6.2. Residual measures. — Let $\mathscr{L}^\# = (\mathscr{L}, \psi_0)$ be a residually metrized model of K_X, determined on a proper dlt model $\mathscr{X}$. If Y is a stratum corresponding to a top-dimensional face of $\Delta(\mathscr{L})$, Lemma 5.12 shows that the restriction of ψ_0 to $\mathscr{L}|_Y$ induces a Hermitian metric ψ_Y on $K_{(Y,B_Y^{\mathscr{L}})} := K_Y + B_Y^{\mathscr{L}}$, with $(Y, B_Y^{\mathscr{L}})$ subklt. By Lemma 1.1, we may thus introduce:

Definition 6.3. — Let Y be a stratum corresponding to a top-dimensional face of $\Delta(\mathscr{L})$. The *residual measure* of $\mathscr{L}^\#$ on Y is the (finite) positive measure

$$\operatorname{Res}_Y(\mathscr{L}^\#) := \exp\big(2(\psi_Y - \phi_{B_Y^{\mathscr{L}}})\big).$$

This definition is of course compatible with one in §3.1, and can be more explicitly described as follows. Let ξ be a (closed) point of $Y \cap \mathscr{X}_{\mathrm{snc}}$, index the irreducible components $E_0, \ldots, E_p$ passing through ξ so that Y is a component of $\bigcap_{0\leqslant i\leqslant d} E_i$ with $d = \dim \Delta(\mathscr{L}) \leqslant p$. In the notation of Example 5.1, the Poincaré residue

$$\operatorname{Res}_Y(\Omega) = \left(\frac{dz_{d+1}}{z_{d+1}} \wedge \cdots \wedge \frac{dz_p}{z_p} \wedge dz_{p+1} \wedge \cdots \wedge dz_n\right)\Big|_Y$$

is a generator of $K_{(Y,B_Y)} = K^{\log}_{\mathscr{X}/S}\big|_Y$. Setting $a_i := \kappa(v_{E_i}) b_i \in \mathbb{Q}$, we have

$$K^{\log}_{\mathscr{X}/S} = \mathscr{L} + \sum_i a_i E_i,$$

and we may thus view

$$\tau := t^{\kappa_{\min}} \prod_{i=0}^{p} z_i^{a_i - \kappa_{\min} b_i} \Omega^{\mathrm{rel}} = t^{\kappa_{\min}} \prod_{i=d+1}^{p} z_i^{a_i - \kappa_{\min} b_i} \Omega^{\mathrm{rel}}$$

as a local $\mathbb{Q}$-generator of $\mathscr{L}$. Further, $B_Y^{\mathscr{L}} = \sum_{i=d+1}^{p} (1 - (a_i - \kappa_{\min} b_i)) E_i|_Y$, and $\tau|_Y$ corresponds to

$$\prod_{i=d+1}^{p} z_i^{a_i - \kappa_{\min} b_i} \operatorname{Res}_Y(\Omega)$$

under the identification $\mathscr{L}|_Y = K_{(Y,B_Y^{\mathscr{L}})}$. We arrive at

$$\begin{aligned}
\operatorname{Res}_Y(\mathscr{L}^\#) &= \frac{\prod_{i=d+1}^{p} |z_i|^{2(a_i - \kappa_{\min} b_i)}}{\left| t^{\kappa_{\min}} \prod_{i=d+1}^{p} z_i^{a_i - \kappa_{\min} b_i} \Omega^{\mathrm{rel}} \right|_{\psi_0}^2} \, |\operatorname{Res}_Y(\Omega)|^2 \\
&= \frac{\prod_{i=d+1}^{p} |z_i|^{2(a_i - \kappa_{\min} b_i - 1)}}{\left| t^{\kappa_{\min}} \prod_{i=d+1}^{p} z_i^{a_i - \kappa_{\min} b_i} \Omega^{\mathrm{rel}} \right|_{\psi_0}^2} \, |dz_{d+1} \wedge \cdots \wedge dz_n|^2. \qquad (6.1)
\end{aligned}$$

6.3. Measures on dual complexes. — We now define measures associated to residually metrized model metrics.

Definition 6.4. — Let $\mathscr{L}^\#$ be a residually metrized model of K_X, determined on a proper dlt model $\mathscr{X}$ of X. To $\mathscr{L}^\#$ we associate a positive measure $\mu_{\mathscr{L}^\#}$ on

$\Delta(\mathscr{L}) \subset \Delta(\mathscr{X})$ defined by

$$\mu_{\mathscr{L}^\#} = \sum_\sigma \left(\int_{Y_\sigma} \operatorname{Res}_{Y_\sigma}(\mathscr{L}^\#) \right) b_\sigma^{-1} \lambda_\sigma,$$

where σ runs over the top-dimensional faces of $\Delta(\mathscr{L})$.

By Lemma 1.2, we have

$$\mu_{\mathscr{L}^\#}(\sigma) = \frac{\int_{Y_\sigma} \operatorname{Res}_{Y_\sigma}(\mathscr{L}^\#)}{d! \prod_{i\in J} b_i}$$

for each face σ corresponding to a component of some E_J.

6.4. Skeletal measures on Berkovich spaces. — Now consider a residually metrized model metric $\psi^\#$ on K_X. Pick any representative $\mathscr{L}^\# = (\mathscr{L}, \psi_0)$ for $\psi^\#$, where $\mathscr{L}$ is a model of K_X determined on a proper dlt model $\mathscr{X}$ of X, and where ψ_0 is a continuous metric on $\mathscr{L}_0 := \mathscr{L}|_{\mathscr{X}_0}$.

Definition 6.5. — The *skeletal measure* $\mu_{\psi^\#}$ is the image of the measure $\mu_{\mathscr{L}^\#}$ under the embedding $\Delta(\mathscr{L}) \hookrightarrow X^{\mathrm{an}}$. We view it as a positive measure on X^{an}, supported on the skeleton $\operatorname{Sk}(\psi^\#) := \operatorname{Sk}(\phi_{\mathscr{L}})$.

This definition makes sense, in view of the following result.

Lemma 6.6. — *The skeletal measure $\mu_{\mathscr{L}^\#}$ is independent of the choice of representative $\mathscr{L}^\#$ for $\psi^\#$.*

Proof. — Let $\mathscr{X}$, $\mathscr{X}'$ be proper dlt models of X, with $\mathscr{X}'$ dominating $\mathscr{X}$ via a proper birational morphism $\rho\colon \mathscr{X}' \to \mathscr{X}$. Let $\mathscr{L}^\# = (\mathscr{L}, \psi_0)$ be a residually metrized model of K_X consisting of a model $\mathscr{L}$ of K_X determined on $\mathscr{X}$ and a continuous metric ψ_0 on $\mathscr{L}_0$. Set $\mathscr{L}' = \rho^*\mathscr{L}$, $\psi_0' = \rho^*\psi_0$ and $\mathscr{L}'^\# = (\mathscr{L}', \psi_0')$. We must prove that $\mu_{\mathscr{L}'^\#} = \mu_{\mathscr{L}^\#}$.

Let σ' be a top-dimensional face of $\Delta(\mathscr{L}')$, Y' the associated stratum of $\mathscr{X}_0'$, Y the minimal stratum of $\mathscr{X}_0$ containing $\rho(Y')$ and $\sigma = \sigma_Y$ the associated simplex of $\Delta(\mathscr{X})$. Then σ and σ' have the same dimension, and if we (somewhat abusively) identify σ and σ' with their images in $\operatorname{Sk}(\phi_{\mathscr{L}}) \subset X^{\mathrm{an}}$, then σ' is a rational subsimplex of σ. It suffices to prove that $\mu_{\mathscr{L}'^\#}(\sigma') = \mu_{\mathscr{L}^\#}(\sigma')$.

Now ρ restricts to a birational morphism of $Y' \to Y$, so since $\lambda_\sigma|_{\sigma'} = \lambda_{\sigma'}$ and $b_\sigma = b_{\sigma'}$, it suffices to prove that $\operatorname{Res}_{Y'}(\mathscr{L}'^\#) = \rho^* \operatorname{Res}_Y(\mathscr{L}^\#)$. But this is formal. Indeed, we have $(\rho|_Y)_*(B_{Y'}^{\mathscr{L}'}) = B_Y^{\mathscr{L}}$ and we can identify $(\rho|_Y)^* K_{(Y, B_Y^{\mathscr{L}})}$ with $K_{(Y', B_{Y'}^{\mathscr{L}'})}$ in such a way that the restriction of ψ_0' to $\mathscr{L}'|_{Y'} = K_{(Y', B_{Y'}^{\mathscr{L}'})}$ coincides with the pullback under $\rho|_{Y'}$ of the restriction of ψ_0 to $\mathscr{L}|_Y = K_{(Y, B_Y^{\mathscr{L}})}$. $\square$

6.5. Behavior under base change. — Fix $m \in \mathbb{Z}_{>0}$. As before, denote by X' the base change of X to $K' = \mathbb{C}((t^{1/m}))$, with induced map $p\colon X'^{\mathrm{an}} \to X^{\mathrm{an}}$.

Theorem 6.7. — *Let $\psi^{\#}$ be a residually metrized model metric on K_X, and let $\psi'^{\#}$ be its pull-back to X'. Then*

$$p_*\mu_{\psi'^{\#}} = m^d \mu_{\psi^{\#}}$$

with $d = \dim \mathrm{Sk}(\psi^{\#})$.

Proof. — Pick a representative $\mathscr{L}^{\#} = (\mathscr{L}, \psi_0)$ of $\psi^{\#}$ such that $\mathscr{L}$ is defined on a proper snc model $\mathscr{X}$. Let $\mathscr{X}'$ be the normalized base change by $t = t'^m$.

Let σ be a d-dimensional face of $\Delta(\mathscr{L})$. By Lemma 5.13, $p^{-1}(\sigma)$ is the union of g_σ distinct isomorphic faces σ'_α of $\Delta(\mathscr{X})$ such that

$$b_{\sigma'_\alpha} = b_\sigma / \gcd(m, b_\sigma) \tag{6.2}$$

$$\mathrm{Vol}(\sigma'_\alpha) = m^d \,\mathrm{Vol}(\sigma). \tag{6.3}$$

Further, the induced map $Y'_{\sigma'_\alpha} \to Y$ is generically finite, of degree f_σ independent of α, and we have $f_\sigma g_\sigma = \gcd(m, b_\sigma)$. Pick a toroidal modification $\mathscr{X}'' \to \mathscr{X}'$ with $\mathscr{X}''$ snc, denote by $\rho\colon \mathscr{X}'' \to \mathscr{X}$ the composition, and set $\mathscr{L}'' := \rho^*\mathscr{L}$.

Each face σ'_α above is subdivided into simplices $\sigma''_{\alpha\beta}$ of $\Delta(\mathscr{L}'')$ of dimension d, each corresponding to a stratum $Y''_{\alpha\beta}$ of $\mathscr{X}''_0$, and $\rho|_{Y''_{\alpha\beta}}\colon Y''_{\alpha\beta} \to Y$ is generically finite, of degree f_σ. Further, (6.2) implies that

$$b_{\sigma''_{\alpha\beta}} = b_{\sigma'_\alpha} = b_\sigma / \gcd(m, b_\sigma) \quad \text{for all } \alpha, \beta. \tag{6.4}$$

We shall need the following result:

Lemma 6.8. — *With notation as above, we have, for all α, β:*

$$\mathrm{Res}_{Y''_{\alpha\beta}}(\mathscr{L}''^{\#}) = \gcd(m, b_\sigma)^{-2} \rho^* \,\mathrm{Res}_Y(\mathscr{L}^{\#}).$$

Grant this result for the moment. Lemma 6.8 implies

$$\int_{Y''_{\alpha\beta}} \mathrm{Res}_{Y''_{\alpha\beta}}(\mathscr{L}''^{\#}) = f_\sigma \gcd(m, b_\sigma)^{-2} \int_Y \mathrm{Res}_Y(\mathscr{L}^{\#}),$$

and hence

$$\begin{aligned}(p_*\mu')(\sigma) = \sum_{\alpha,\beta} \mu'(\sigma''_{\alpha\beta}) &= \sum_{\alpha,\beta} \left(\int_{Y''_{\alpha\beta}} \mathrm{Res}_{Y''_{\alpha\beta}}(\mathscr{L}''^{\#}) \right) b^{-1}_{\sigma''_{\alpha\beta}} \mathrm{Vol}(\sigma''_{\alpha\beta}) \\ &= f_\sigma \gcd(m, b_\sigma)^{-2} \left(\int_Y \mathrm{Res}_Y(\mathscr{L}^{\#}) \right) b_\sigma^{-1} \gcd(m, b_\sigma) \sum_\alpha \mathrm{Vol}(\sigma'_\alpha) \\ &= m^d \left(\int_Y \mathrm{Res}_Y(\mathscr{L}^{\#}) \right) b_\sigma^{-1} \mathrm{Vol}(\sigma) = m^d \mu(\sigma),\end{aligned}$$

thanks to (6.3) and (6.4). □

Proof of Lemma 6.8. — Pick a closed point $\xi''\in\mathring{Y}''$ and set $\xi=\rho(\xi'')\in\mathring{Y}$. We use the notation at the end of §6.2 with $p=d$. Namely, pick local coordinates $(z_i)_{0\leqslant i\leqslant n}$ at ξ and $(z''_j)_{0\leqslant j\leqslant n}$ at ξ'' such that $E_i=\{z_i=0\}$ for $0\leqslant i\leqslant d$ and $E''_j=\{z''_j=0\}$ for $0\leqslant j\leqslant d$. We have $\rho^*z_i=u_i\prod_{j=0}^d(z''_j)^{c_{ij}}$ for $0\leqslant i\leqslant d$, where $c_{ij}\in\mathbb{Z}_{\geqslant 0}$ and $u_i\in\mathscr{O}_{\mathscr{X}'',\xi''}$ is a unit. Further, by Lemma 5.13, the matrix (c_{ij}) has determinant $\pm e_\sigma$, where $e_\sigma=m/\gcd(m,b_\sigma)$.

Set

$$\Omega_1:=\frac{dz_0}{z_0}\wedge\cdots\wedge\frac{dz_d}{z_d}\quad\text{and}\quad\Omega_2:=dz_{d+1}\wedge\cdots\wedge dz_n,$$

and define Ω''_1, Ω''_2 similarly. Then $\Omega:=\Omega_1\wedge\Omega_2$ and $\Omega'':=\Omega''_1\wedge\Omega''_2$ are local $\mathbb{Q}$-generators of $K^{\log}_{\mathscr{X}}$ and $K^{\log}_{\mathscr{X}''}$ at ξ and ξ'', respectively. Further,

$$\operatorname{Res}_Y(\Omega)=\Omega_2|_Y\quad\text{and}\quad\operatorname{Res}_{Y''}(\Omega'')=\Omega''_2|_{Y''}.$$

Now

$$\rho^*\Omega_1=\pm e_\sigma\Omega''_1+\frac{1}{z''_0\ldots z''_d}\,\widetilde{\Omega}''_1,$$

where $\widetilde{\Omega}''_1$ is a regular $(d+1)$-form vanishing at ξ'', and

$$\rho^*\Omega_2=q\Omega''_2+\widetilde{\Omega}''_2,$$

where $q\in\mathscr{O}_{\mathscr{X},\xi''}$ and $\widetilde{\Omega}''_2$ is a regular $(n-d)$-form at ξ'' satisfying $\Omega''_1\wedge\widetilde{\Omega}''_2=0$. On the one hand, this leads to

$$(\rho|_{Y''})^*\operatorname{Res}_Y(\Omega)=(\rho|_{Y''})^*(\Omega_2|_Y)=q\Omega''_2|_{Y''}=q\operatorname{Res}_{Y''}(\Omega'').$$

On the other hand, we also get

$$\rho^*\Omega=\pm qe_\sigma(1+h)\Omega'',$$

with q as above and $h\in\mathscr{O}_{\mathscr{X}'',\xi''}$ vanishing along Y''.

Define Ω^{rel} and Ω''^{rel} by $\frac{dt}{t}\otimes\Omega^{\mathrm{rel}}=\Omega$ and $\frac{dt'}{t'}\otimes\Omega''^{\mathrm{rel}}=\Omega''$, respectively. Then

$$m\frac{dt'}{t'}\otimes\rho^*\Omega^{\mathrm{rel}}=\rho^*\Big(\frac{dt}{t}\Big)\otimes\rho^*\Omega^{\mathrm{rel}}=\rho^*\Omega=\pm qe_\sigma(1+h)\Omega''=\pm qe_\sigma(1+h)\frac{dt'}{t'}\otimes\Omega''^{\mathrm{rel}},$$

so that

$$\rho^*\Omega^{\mathrm{rel}}=\pm\frac{e_\sigma}{m}\,q(1+h)\Omega''^{\mathrm{rel}}.$$

As a consequence,

$$\rho^*|t^{\kappa_{\min}}\Omega^{\mathrm{rel}}|_{\psi_0}=\frac{e_\sigma}{m}\,|q|\,|(1+h)|\,|(t')^{\kappa'_{\min}}\Omega''^{\mathrm{rel}}|_{\psi'_0}.$$

Since h vanishes along Y'', this finally leads to

$$\begin{aligned}(\rho|_{Y''})^*\operatorname{Res}_Y(\mathscr{L}^\#)&=\frac{(\rho|_{Y''})^*|\operatorname{Res}_Y(\Omega)|^2}{(\rho|_{Y''})^*|t^{\kappa_{\min}}\Omega^{\mathrm{rel}}|^2_{\psi_0}}=\frac{|(\rho|_{Y''})^*(\Omega_2|_Y)|^2}{(e_\sigma^2/m^2)|q|^2|(t')^{\kappa'_{\min}}\Omega''^{\mathrm{rel}}|^2_{\psi'_0}}\\&=\Big(\frac{m}{e_\sigma}\Big)^2\frac{|\operatorname{Res}_{Y''}(\Omega'')|^2}{|(t')^{\kappa'_{\min}}\Omega''^{\mathrm{rel}}|^2_{\psi'_0}}=\Big(\frac{m}{e_\sigma}\Big)^2\operatorname{Res}_{Y''}(\mathscr{L}''^\#),\end{aligned}$$

which completes the proof since $e_\sigma=m/\gcd(b_\sigma,m)$. □

7. The Calabi–Yau case

As in §6, we assume that X is a smooth, projective, geometrically connected variety over $\mathbb{C}(\!(t)\!)$. Now we further assume that K_X is trivial. Pick a trivializing section $\eta \in H^0(X, K_X)$, and denote by $\log|\eta|$ the associated model metric on K_X^{an}, determined on any model $\mathscr{X}$ by $\mathscr{L} = \mathscr{O}_{\mathscr{X}}$, with η providing the identification $\mathscr{L}|_X \simeq K_X$. Denote also by $\log|\eta|^{\#}$ the residually metrized model metric induced by the trivial Hermitian metric $\psi_0 = 0$ on $\mathscr{O}_{\mathscr{X}_0}$.

The function $\kappa := A_X - \log|\eta| = -\log|\eta|_{A_X}$ coincides with the weight function of [MN15, NX16b]. By definition, the *Kontsevich–Soibelman skeleton* of X is $\mathrm{Sk}(X) := \mathrm{Sk}(\log|\eta|^{\#})$. It is indeed independent of the choice of η, since any other trivializing section of K_X is of the form $\eta' = f\eta$ with $f \in \mathbb{C}(\!(t)\!)^*$, and hence $\kappa' = \kappa + \mathrm{ord}_0(f)$.

7.1. Topology of the skeleton. — By [NX16b, Th. 4.2.4], the $\mathbb{Z}$-PA-space $\mathrm{Sk}(X)$ is connected, of pure dimension d, and is a deformation retract of X^{an}. Further, $\mathrm{Sk}(X)$ is a *pseudomanifold with boundary*, i.e., for some (or, equivalently, any) triangulation Δ of $\mathrm{Sk}(X)$, we have:

(a) Non-branching property: every $(d-1)$-simplex of Δ is contained in at most two d-simplices

(b) Strong connectedness: every pair of n-simplices σ, σ' is joined by a chain of n-simplices $\sigma = \sigma_1, \dots, \sigma_N = \sigma'$ with σ_i and σ_{i+1} sharing a common $(n-1)$-face.

In the maximally degenerate case $d = n$, if X has semistable reduction, then $\mathrm{Sk}(X)$ is even a *pseudomanifold*, i.e., (a) is replaced by

(a') every $(n-1)$-simplex of Δ is contained in exactly two n-simplices.

See also [KX16] for even more precise results on the structure of $\mathrm{Sk}(X)$. For example, $\mathrm{Sk}(X)$ is homeomorphic to a sphere if $n \leqslant 3$ (still in the maximally degenerate case and X having semistable reduction).

7.2. The skeletal measure. — Consider the skeletal measure $\mu_{\log|\eta|^{\#}}$ on $\mathrm{Sk}(X)$. Choose an snc model $\mathscr{X}$, and write as usual $\mathscr{X}_0 = \sum_{i\in I} b_i E_i$. The form η defines an identification $K^{\log}_{\mathscr{X}/S} = \sum_{i\in I} a_i E_i$, and Proposition 5.10 yields

$$\kappa_{\min} = \min_i \frac{a_i}{b_i}. \tag{7.1}$$

If $\kappa_{\min} \in \mathbb{Z}$, then

$$\omega := \frac{dt}{t^{\kappa_{\min}+1}} \wedge \eta$$

is a logarithmic form on $\mathscr{X}$. For each face σ of $\Delta(\mathscr{L})$, ordering the set $J \subset I$ of components cutting out the stratum $Y = Y_\sigma$ yields a well-defined Poincaré residue $\mathrm{Res}_Y(\omega)$. By Lemma 5.12, $\mathrm{Res}_Y(\omega)$ is a rational section of K_Y, with divisor

$$-B_Y^{\mathscr{L}} = \sum_{i\notin J} (a_i - \kappa_{\min} b_i - 1) E_i|_Y.$$

When σ is a maximal face, $\mathrm{Res}_Y(\omega)$ is thus a holomorphic form on Y; using the formulas in §6.2, it is easy to see that the residual measure on Y is given by

$$\mathrm{Res}_Y(\log|\eta|^{\#}) = |\operatorname{Res}_Y(\omega)|^2.$$

The following result corresponds to Theorem C in the introduction.

Theorem 7.1. — *Assume that X is maximally degenerate, i.e., $\dim \mathrm{Sk}(X) = n$, and that X has semistable reduction, Then the skeletal measure $\mu_{\log|\eta|^{\#}}$ is a multiple of the integral Lebesgue measure of $\mathrm{Sk}(X)$.*

Proof. — Let $\mathscr{X}$ be a semistable model, i.e., $\mathscr{X}$ is snc with $\mathscr{X}_0$ reduced. By (7.1), we have $\kappa_{\min} \in \mathbb{Z}$. Since some non-empty E_J might have several components, the dual complex $\Delta(\mathscr{X})$ is possibly not a triangulation of $\mathrm{Sk}(X)$. However, the barycentric subdivision Δ' of $\Delta(\mathscr{X})$ is a triangulation; the corresponding toroidal modification $\mathscr{X}'$ is snc, with $\mathscr{X}'_0$ is possibly non-reduced, but $b_\sigma = 1$ for each n-simplex σ of Δ'. Applying the above discussion to $\mathscr{X}'$, we infer

$$\mu_{\log|\eta|^{\#}} = \sum_\sigma |\operatorname{Res}_{y_\sigma}(\omega)|^2 \lambda_\sigma,$$

with σ ranging over the n-dimensional faces of Δ', with corresponding strata $y_\sigma \in \mathscr{X}'_0$ reduced to single points. It will thus be enough to show that $|\operatorname{Res}_{y_\sigma}(\omega)|$ is independent of σ.

By the strong connectedness property, any two n-simplices σ, σ' of Δ' can be joined by a chain of n-simplices $\sigma = \sigma_1, \dots, \sigma_N = \sigma'$ with σ_i and σ_{i+1} sharing a common $(n-1)$-face τ_i. Denoting by $y_i = y_{\sigma_i}$ and $Y_i = Y_{\tau_i}$ the corresponding strata in $\mathscr{X}'$, we thus have $y_i, y_{i+1} \in Y_i$. Further, the Poincaré residue $\mathrm{Res}_{Y_i}(\omega)$ has poles precisely at y_i, y_{i+1}, since any other pole would correspond to an n-simplex of Δ' containing τ_i, contradicting the non-branching property. Since $\mathrm{Res}_{y_i} \mathrm{Res}_Y(\omega) = \mathrm{Res}_{y_i}(\omega)$, the residue theorem applied to the Riemann surface Y_i yields $\mathrm{Res}_{y_i}(\omega) + \mathrm{Res}_{y_{i+1}}(\omega) = 0$, and hence $|\operatorname{Res}_{y_1}(\omega)| = \cdots = |\operatorname{Res}_{y_N}(\omega)|$. □

Remark 7.2. — Theorem 7.1 fails in general when X does not have semistable reduction. Indeed, the semistable reduction theorem [KKMSD73] shows that the base change $p\colon X' \to X$ to $\mathbb{C}(\!(t^{1/m})\!)$ has semistable reduction for some m divisible enough. By Lemma 5.14, $\dim \mathrm{Sk}(X') = n$, and $\mu_{\log|\eta'|^{\#}}$ is thus a multiple of the integral Lebesgue measure λ' of $\mathrm{Sk}(X')$, by Theorem 7.1. By Theorem 6.7, $\mu_{\log|\eta|^{\#}} = m^{-n} p_* \lambda'$. However, $p_*\lambda'$ is not proportional to the integral Lebesgue measure λ of $\mathrm{Sk}(X)$ in general. Indeed, for each n-simplex σ of $\Delta(\mathscr{L})$, Lemma 5.13 shows that $(p_*\lambda')_\sigma = m^n b_\sigma \lambda_\sigma$, and b_σ is in general not independent of σ.

8. Extensions

In this section we extend the main results in various directions.

8.1. A singular version of Theorem A. — Let $\pi\colon \mathscr{X} \to \mathbb{D}$ be a projective, flat holomorphic map of a normal complex space onto the disc, with $X := \pi^{-1}(\mathbb{D}^*)$ smooth over $\mathbb{D}^*$. Since π is projective, it defines a smooth projective variety $X_{\mathbb{C}(\!(t)\!)}$ over $\mathbb{C}(\!(t)\!)$, as well as a model $\mathscr{X}_{\mathbb{C}[\![t]\!]}$.

Let $\mathscr{L}$ be a $\mathbb{Q}$-line bundle on $\mathscr{X}$ extending $K_{X/\mathbb{D}^*}$, and ψ a continuous Hermitian metric on $\mathscr{L}$. This data induces a continuous Hermitian metric ψ_t on K_{X_t} for $t \in \mathbb{D}^*$, as well as a residually metrized model $\mathscr{L}^\#$ of $K_{X_{\mathbb{C}(\!(t)\!)}}$, the model given by $\mathscr{L}_{\mathbb{C}[\![t]\!]}$ and the metric by the restriction of ψ to $\mathscr{L}_0 = \mathscr{L}|_{\mathscr{X}_0}$. Thus we obtain a skeletal measure $\mu_{\mathscr{L}^\#}$ on $X^{\mathrm{an}}_{\mathbb{C}(\!(t)\!)}$.

Denote by $\mathscr{L}'$ (resp. ψ') the pull-back of $\mathscr{L}$ (resp. ψ) to a log resolution $\mathscr{X}' \to \mathscr{X}$. By invariance of skeletal measures under pull-back, we have $\mu_{\mathscr{L}'^\#} = \mu_{\mathscr{L}^\#}$, and Theorem 3.4 therefore implies:

Theorem 8.1. — *The rescaled measures*

$$\mu_t := \frac{e^{2\psi_t}}{|t|^{2\kappa_{\min}}(2\pi \log|t|^{-1})^d},$$

viewed as measures on $\mathscr{X}'^{\mathrm{hyb}}$, converge weakly to $\mu_{\mathscr{L}^\#}$.

Corollary 8.2. — *If $\mathscr{X}$ (i.e., the pair $(\mathscr{X}, \mathscr{X}_{0,\mathrm{red}})$) is dlt, then*

$$\lim_{t\to 0} \frac{\int_{\mathscr{X}_t} e^{2\psi_t}}{|t|^{2\kappa_{\min}}(2\pi \log|t|^{-1})^d} = \sum_\sigma \left(\int_{Y_\sigma} \mathrm{Res}_{Y_\sigma}(\mathscr{L}^\#)\right) b_\sigma^{-1} \operatorname{Vol}(\sigma),$$

where σ runs over the d-dimensional faces of $\Delta(\mathscr{L})$.

When $d = 0$, this implies the following slight generalization of [Li15, Lem. 1].

Corollary 8.3. — *Assume that $\mathscr{X}_0$ has klt singularities (and hence $\mathscr{X}$ is dlt by inversion of adjunction). Let ψ be a continuous metric on $K_{\mathscr{X}/\mathbb{D}}$. Then $t \mapsto \int_{\mathscr{X}_t} e^{2\psi_t}$ is continuous at $t = 0$.*

8.2. Corollary B for pairs. — Suppose (X, B) is a projective subklt pair over $\mathbb{D}^*$ that is meromorphic at $0 \in \mathbb{D}$.

By Bertini's theorem (see [Kol97, 4.8] and also below), the pair (X_t, B_t) is subklt for all $t \in \mathbb{D}^*$ outside a discrete subset Z. Let ψ be a continuous metric on $K_{(X,B)/\mathbb{D}^*}$. As explained in §1.2, ψ induces a finite positive measure $e^{2(\psi_t - \phi_{B_t})}$ on X_t for $t \in \mathbb{D}^* \smallsetminus Z$.

Assume that ψ has analytic singularities in the sense that there exists a flat projective map $\mathscr{X} \to \mathbb{D}$ extending $X \to \mathbb{D}^*$, with $\mathscr{X}$ normal, and a $\mathbb{Q}$-line bundle $\mathscr{L}$ on $\mathscr{X}$ extending $K_{(X,B)/\mathbb{D}^*}$ such that ψ extends continuously to $\mathscr{L}$.

Our assumptions imply that X is defined over the Banach ring A_r described in the appendix for $0 < r \ll 1$. Let X^{hyb} be the analytification of the base change X_{A_r}. Recall that X^{hyb} naturally fibers over $\overline{\mathbb{D}}_r$, with $X^{\mathrm{hyb}}_{\overline{\mathbb{D}}^*_r} \simeq X_{\overline{\mathbb{D}}^*_r}$ and $X^{\mathrm{hyb}}_0 \simeq X^{\mathrm{an}}_{\mathbb{C}(\!(t)\!)}$.

Theorem 8.4. — *The pair (X_t, B_t) is subklt for $0 < |t| \ll 1$. Further, there exist $\kappa_{\min} \in \mathbb{Q}$ and $d \in \mathbb{N}^*$ such that the rescaled measures*

$$\mu_t := \frac{e^{2\psi_t}}{|t|^{2\kappa_{\min}}(2\pi \log|t|^{-1})^d},$$

viewed as measures on X^{hyb}, converge weakly, as $t \to 0$, to a finite positive measure μ_0 on $X_0^{\mathrm{hyb}} = X^{\mathrm{an}}_{\mathbb{C}((t))}$.

A special case of Theorem 8.4 is the log Calabi–Yau setting, when the $\mathbb{Q}$-line bundle $K_{(X,B)/\mathbb{D}^*}$ is trivial. In general, we are not able to give a very precise description of the limit measure μ_0, but the proof will show that μ_0 is a skeletal measure when the pair (X, B) is log smooth.

Proof of Theorem 8.4. — Let us first treat the case when (X, B) is log smooth. In this case we need not assume that $X \to \mathbb{D}^*$ is projective. It follows from the normal crossings condition that (X_t, B_t) is subklt for $0 < |t| \ll 1$. After reparametrizing we may assume this is true for all $t \in \mathbb{D}^*$, that is, $Z = \varnothing$. Set

$$\nu_t = e^{2(\psi_t - \phi_{B_t})}.$$

This is a positive measure on X_t, smooth outside the support of B_t. Pick an snc model $(\mathscr{X}, \mathscr{B})$ of (X, B), where $\mathscr{B}$ is the closure of B in $\mathscr{X}$, such that ψ extends to a continuous metric on a $\mathbb{Q}$-line bundle $\mathscr{L}$ on $\mathscr{X}$ extending $K_{(X,B)/\mathbb{D}^*}$.

We can then prove a version of Theorem A inside the hybrid space $\mathscr{X}^{\mathrm{hyb}}$. By letting $\mathscr{X}$ vary, we obtain Theorem 8.4 as a consequence, just as Corollary B follows from Theorem A.

The proof is very similar to the proof of Theorem A, so we will only indicate the modifications needed. Let us write

$$K^{\log}_{(\mathscr{X},\mathscr{B})/\mathbb{D}} = \mathscr{L} + \sum_{i \in I} a_i E_i$$

with $a_i \in \mathbb{Q}$. Set $\kappa_i := a_i/b_i$ and $\kappa_{\min} := \min_i \kappa_i$. Here $\mathscr{X}_0 = \sum_i b_i E_i$ as before.

Define $\Delta(\mathscr{L})$ as the subcomplex of $\Delta(\mathscr{X})$ spanned by the vertices such that $\kappa_i = \kappa_{\min}$. This will be the support of the measure μ_0. For every stratum Y corresponding to a maximal face of $\Delta(\mathscr{L})$, define a subklt pair $(Y, B_Y^{\mathscr{L}})$ using

$$B_Y^{\mathscr{L}} := \mathscr{B}|_Y + \sum_{i \notin J} (1 - (a_i - \kappa_{\min} b_i)) E_i|_Y.$$

The residual measure $\mathrm{Res}_Y(\psi)$ is given by

$$\mathrm{Res}_Y(\psi) := \exp\bigl(2(\psi|_Y - \phi_{B_Y^{\mathscr{L}}})\bigr).$$

Finally set

$$\mu_0 := \sum_\sigma \left(\int_{Y_\sigma} \mathrm{Res}_{Y_\sigma}(\psi) \right) b_\sigma^{-1} \lambda_\sigma,$$

where σ ranges over the d-dimensional faces of $\Delta(\mathscr{L})$, with $d = \dim \Delta(\mathscr{L})$.

We then prove a version of Theorem 3.4. Namely, if

$$\mu_t := \frac{\lambda(t)^d}{(2\pi)^d|t|^{2\kappa_{\min}}}\, e^{2(\psi_t-\phi_{B_t})}.$$

then we show that μ_t converges to μ_0 in $\mathscr{X}^{\mathrm{hyb}}$ as $t\to 0$. This is done via a local convergence result as in Lemma 3.5. Namely, given a point $\xi\in\mathscr{X}_0$, we choose local coordinates $(z_0,\dots,z_n)$ at ξ as in §3.2, but further require that these coordinates also cut out the irreducible components of $\mathscr{B}$ containing ξ. More precisely, there exist m with $p\leqslant m\leqslant n$ such that these irreducible components are given by $B_i=\{z_i=0\}$ for $p<i\leqslant m$. Also set $c_i:=\mathrm{ord}_{B_i}(\mathscr{B})<1$.

A local $\mathbb{Q}$-generator for $\mathscr{L}$ at ξ is then given by

$$\tau=\prod_{i=0}^{p} z_i^{a_i}\prod_{i=p+1}^{m} z_i^{-c_i}\;\Omega^{\mathrm{rel}},$$

with Ω^{rel} as before. For a stratum Y corresponding to a d-dimensional simplex in $\Delta(\mathscr{L})$, the residual measure is given by

$$\mathrm{Res}_Y(\psi)=|\tau|_\psi^{-2}\prod_{i=d+1}^{p}|z_i|^{2(a_i-\kappa_{\min}b_i-1)}\prod_{i=p+1}^{m}|z_i|^{-2c_i}\Big|\bigwedge_{i=d+1}^{n}dz_i\Big|^2. \tag{8.1}$$

The measure μ_t can be written near ξ as

$$\mu_t=\frac{\lambda(t)^d}{(2\pi)^d}\,\frac{\prod_{i=p+1}^m|z_i|^{-2c_i}|\Omega_t|^2}{\big|\prod_{i=p+1}^m z_i^{-c_i}\Omega_t\big|_{\psi_t}^2}.$$

The proof now proceeds exactly as in §3.3 except that we need to insert a factor $\prod_{i=p+1}^m|z_i|^{-2c_i}$ in the last two lines of (3.3) and (3.6), the second line and the second factor of the last line of (3.7), and the right-hand sides of (3.8) and (3.9). This completes the proof in the log smooth case.

Now we consider the general case, assuming $X\to\mathbb{D}^*$ is projective. Pick a log resolution $q\colon(X',B')\to(X,B)$. Since (X'_t,B'_t) is subklt for $0<|t|\ll 1$, the same is true for (X_t,B_t). We have an induced continuous map $q^{\mathrm{hyb}}\colon(X')^{\mathrm{hyb}}\to X^{\mathrm{hyb}}$. By what precedes, there exist $\kappa\in\mathbb{Q}$ and $d\in\mathbb{N}$ such that the measure

$$\mu'_t:=\frac{e^{2(\psi'_t-\phi_{B'_t})}}{|t|^{2\kappa_{\min}}(2\pi\log|t|^{-1})^d}$$

on $X'_t=X^{\mathrm{hyb}}$ converges to a nonzero positive measure μ'_0 on $(X')^{\mathrm{hyb}}$. By continuity, it follows that $\mu_t=q_*^{\mathrm{hyb}}\mu'_t$ converges to the nonzero positive measure $\mu_0=q_*^{\mathrm{hyb}}\mu'_0$ on X^{hyb}. This completes the proof. □

8.3. Degenerations of Ricci-flat Kähler manifolds. — Let M be a Ricci-flat Kähler manifold, i.e., a compact Kähler manifold with trivial first Chern class $c_1(M)\in H^2(M,\mathbb{C})$. Then M carries a canonical probability measure μ, given by $\mu=e^{2\psi}/\int_M e^{2\psi}$ where ψ is a Hermitian metric on K_M with curvature 0 (and hence unique up to a constant).

By the Calabi-Yau theorem, each Kähler $(1,1)$-class on M further contains a unique Ricci-flat Kähler metric ω, characterized by

$$\frac{\omega^n}{\int_M \omega^n} = \mu.$$

Recall also that K_M is torsion, i.e., $rK_M \simeq \mathscr{O}_M$ for some positive integer r. Indeed, this is a consequence of the Beauville-Bogomolov theorem [Bea83, Bog74], which implies that M admits a finite étale cover $p : M' \to M$ with $K_{M'} = p^*K_M$ trivial. A trivializing section η of rK_M defines a metric $\psi = (1/r)\log|\eta|$ on K_M as above, and hence $\mu = |\eta|^{2/r} / \int |\eta|^{2/r}$.

As a consequence of Theorem A, we shall prove:

Theorem 8.5. — *Let $\pi\colon X \to \mathbb{D}^*$ be a holomorphic family of Calabi-Yau Kähler manifolds X_t, meromorphic at $t = 0$, and let μ_t be the corresponding family of canonical probability measures. For any snc model $\mathscr{X}$, μ_t converges in $\mathscr{X}^{\mathrm{hyb}}$ to a skeletal measure μ_0 supported in $\Delta(\mathscr{X})$.*

Proof. — As recalled above, K_{X_t} is torsion for each fixed t. Equivalently,

$$h^0(X_t, rK_{X_t}) = 1 \quad \text{for some positive integer } r.$$

Since $t \mapsto h^0(X_t, rK_{X_t})$ is upper semicontinuous in the Zariski topology, it follows that rK_{X_t} is trivial for a fixed r independent of t. Given any snc model $\pi\colon \mathscr{X} \to \mathbb{D}$, $\pi_*\mathscr{O}(rK_{\mathscr{X}/\mathbb{D}})$ is torsion free of rank one, and hence a line bundle. The choice of a trivializing section yields a holomorphic section η of $K_{\mathscr{X}/\mathbb{D}}$, inducing a holomorphic family η_t of trivializing sections of rK_{X_t} for $t \neq 0$. As a consequence, the family of volume forms $\nu_t := |\eta_t|^{2/r}$ has analytic singularities at $t = 0$, and the result is thus a consequence of Theorem A, since $\mu_t = \nu_t/\nu_t(X_t)$. $\square$

Appendix. Berkovich spaces over Banach rings

In this appendix we review the construction of the analytification of a scheme of finite type defined over a Banach ring. The main reference for this is [Ber09]; see also [Poi10, Poi13, Jon16]. For suitable choices of Banach rings, this leads to spaces that contain both Archimedean and non-Archimedean data.

A.1. Berkovich spectra. — Let A be a Banach ring, that is, a commutative ring that is complete with respect to a submultiplicative norm $\|\cdot\|$. The Berkovich spectrum $\mathscr{M}(A)$ is the set of all bounded multiplicative seminorms on A. In other words, a point $x \in \mathscr{M}(A)$ corresponds to a function $|\cdot|_x\colon A \to \mathbb{R}_{\geqslant 0}$ such that $|\cdot|_x \leqslant \|\cdot\|$, $|1|_x = 1$, $|f+g|_x \leqslant |f|_x + |g|_x$ and $|fg|_x = |f|_x|g|_x$ for $f, g \in A$. The spectrum is a nonempty, compact Hausdorff space with respect to the topology of pointwise convergence.

For $x \in \mathscr{M}(A)$, denote by $\mathfrak{p}_x$ the kernel of $|\cdot|_x$. This is a prime ideal of A, and $|\cdot|_x$ defines a multiplicative norm on $A/\mathfrak{p}_x$. The completion of the fraction field of $A/\mathfrak{p}_x$ with respect to this norm is a valued field $\mathscr{H}(x)$. We write $f(x)$ for the image of $f \in A$ in $\mathscr{H}(x)$; then $|f(x)| = |f|_x$. The assignment $x \mapsto \mathfrak{p}_x$ yields a map $\mathscr{M}(A) \to \operatorname{Spec}(A)$ that is continuous for the Zariski topology $\operatorname{Spec}(A)$.

Example A.1. — If k is a valued field (i.e., a field with a multiplicative norm), then $\mathscr{M}(k)$ is a singleton.

Example A.2. — When A is a complex Banach algebra, the Gelfand-Mazur Theorem implies that the Berkovich spectrum agrees with the maximal ideal spectrum.

A.2. Analytification of a scheme. — To any scheme X of finite type over a Banach ring A, Berkovich associates an *analytification*[(9)] X^{An}, a locally compact topological space with a continuous morphism $X^{\mathrm{An}} \to \mathscr{M}(A)$, defined as follows.

When $X = \operatorname{Spec} B$ is affine, with B a finitely generated A-algebra, X^{An} is defined as the set of multiplicative seminorms $|\cdot|_x$ on B whose restriction to A is bounded by the given norm on A, i.e., belongs to $\mathscr{M}(A)$. The topology on X^{An} is the weakest one for which $x \mapsto |f|_x = |f(x)|$ is continuous for every $f \in B$.

In the general case, the analytification X^{An} is defined by gluing together the analytifications of an affine open cover, and yields a covariant functor $X \mapsto X^{\mathrm{An}}$. If $X \hookrightarrow Y$ is an open (resp. closed) embedding, then so is $X^{\mathrm{An}} \hookrightarrow Y^{\mathrm{An}}$. If $X \to Y$ is surjective, then so is $X^{\mathrm{An}} \to Y^{\mathrm{An}}$.

The topological space X^{An} is Hausdorff (resp. compact) if X is separated (resp. projective). The assignment $x \mapsto \mathfrak{p}_x$ above globalizes to a continuous map

$$\pi\colon X^{\mathrm{An}} \longrightarrow X,$$

where X is equipped with the Zariski topology.

When A is a valued field, it is more common to write X^{an} instead of X^{An} [Ber90].

Example A.3. — For $A = \mathbb{C}$, the Gelfand-Mazur theorem shows that X^{An} coincides with the usual analytification of X, i.e., the set $X(\mathbb{C})$ of complex points of X endowed with the euclidean topology.

A.3. The hybrid norm on $\mathbb{C}$. — Denote by $\mathbb{C}_{\mathrm{hyb}}$ the Banach field $(\mathbb{C}, \|\cdot\|_{\mathrm{hyb}})$, where the *hybrid norm* is defined as

$$\|\cdot\|_{\mathrm{hyb}} := \max\{|\cdot|_0, |\cdot|_\infty\},$$

with $|\cdot|_0$ the trivial absolute value and $|\cdot|_\infty$ the usual absolute value.

The elements of the Berkovich spectrum $\mathscr{M}(\mathbb{C}_{\mathrm{hyb}})$ are of the form $|\cdot|_\infty^\rho$ for $\rho \in [0,1]$, interpreted as the trivial absolute value $|\cdot|_0$ for $\rho = 0$. This yields a homeomorphism $\mathscr{M}(\mathbb{C}_{\mathrm{hyb}}) \simeq [0,1]$.

A.4. Hybrid geometry over $\mathbb{C}$. — If X is a scheme of finite type over $\mathbb{C}$, we denote by $X^{\mathrm{hol}} = X(\mathbb{C})$ its analytification with respect to the usual absolute value $|\cdot|_\infty$, by X_0^{an} its analytification with respect to the trivial absolute value, and by X^{hyb} its analytification with respect to the hybrid norm $\|\cdot\|_{\mathrm{hyb}}$.

From the structure morphism $X \to \operatorname{Spec} \mathbb{C}$ we obtain a continuous map $\lambda\colon X^{\mathrm{hyb}} \to \mathscr{M}(\mathbb{C}_{\mathrm{hyb}}) \simeq [0,1]$. The fiber $\lambda^{-1}(\rho)$ is equal to the analytification of X with respect

(9) We use the term analytification even though we shall only consider X^{An} as a topological space. In particular, X^{An} only depends on the reduced scheme structure of X.

to the multiplicative norm $|\cdot|_\infty^\rho$ on $\mathbb{C}$. In particular, we have canonical identifications $\lambda^{-1}(1) \simeq X^{\mathrm{hol}}$ and $\lambda^{-1}(0) \simeq X_0^{\mathrm{An}}$. For $0 < \rho \leqslant 1$, the fiber $\lambda^{-1}(\rho)$ is also homeomorphic to X^{hol}. In fact, we have a a homeomorphism

$$\lambda^{-1}\left((0,1]\right) \simeq (0,1] \times X^{\mathrm{hol}},$$

see [Ber09, Lem. 2.1].

A.5. The hybrid circle. — Now consider the *hybrid circle* of radius $r \in (0,1)$, that is, $C_{\mathrm{hyb}}(r) := \{|t| = r\} \subset \mathbb{A}^{1,\mathrm{hyb}} = (\operatorname{Spec}\mathbb{C}[t])^{\mathrm{hyb}}$. By [Poi10, Prop. 2.1.1], this is compact and realized as the Berkovich spectrum of the Banach ring

$$A_r := \big\{f = \textstyle\sum_{\alpha\in\mathbb{Z}} c_\alpha t^\alpha \in \mathbb{C}(\!(t)\!) \mid \|f\|_{\mathrm{hyb}} := \sum_{\alpha\in\mathbb{Z}} \|c_\alpha\|_{\mathrm{hyb}} r^\alpha < +\infty\big\}.$$

Since $\|c_\alpha\|_{\mathrm{hyb}} \geqslant |c_\alpha|_\infty$, every $f \in A_r$ defines a continuous function f^{hol} on the punctured closed disc $\overline{\mathbb{D}}_r^*$ that is holomorphic on $\mathbb{D}_r^*$ and meromorphic at 0.

Proposition A.4. — *There is a homeomorphism $\overline{\mathbb{D}}_r \xrightarrow{\sim} \mathscr{M}(A_r) \simeq C_{\mathrm{hyb}}(r)$, that maps $z \in \overline{\mathbb{D}}_r \subset \mathbb{C}$ to the seminorm on A_r defined by*

$$|f| = \begin{cases} r^{\mathrm{ord}_0(f)} & \text{if } z = 0,\\ r^{\log|f^{\mathrm{hol}}(z)|_\infty/\log|z|_\infty} & \text{otherwise,}\end{cases} \tag{A.1}$$

and via which the map $\lambda\colon C_{\mathrm{hyb}}(r) \to [0,1]$ is given by $\lambda(z) = \log r/\log|z|_\infty$.

Proof. — The map $\tau\colon \overline{\mathbb{D}}_r \to \mathscr{M}(A_r)$ given by (A.1) is clearly well defined. It is also continuous on $\overline{\mathbb{D}}_r^*$. To prove continuity at 0, we note that for each $f \in A_r$, we can write $f^{\mathrm{hol}} = z^{\mathrm{ord}_0(f)}u$, where u is a continuous function on $\overline{\mathbb{D}}_r$ that is holomorphic on $\mathbb{D}_r$ with $u(0) \neq 0$. As a consequence, we get $\lim_{z\to 0}\log|f^{\mathrm{hol}}(z)|_\infty/\log|z|_\infty = \mathrm{ord}_0(f)$.

Now, for each $\rho \in (0,1]$, $\lambda^{-1}(\rho) \subset C_{\mathrm{hyb}}(r)$ can be identified with the circle of radius r with respect to the absolute value $|\cdot|_\infty^\rho$, while $\lambda^{-1}(0)$ is the non-Archimedean absolute value $r^{-\mathrm{ord}_0}$ on $\mathbb{C}(\!(t)\!)$. This proves that the map τ above is bijective, and hence a homeomorphism by compactness. $\square$

Remark A.5. — When $r < s$, the identity gives a bounded map from A_s to A_r, and $\varinjlim_{r\to 0} A_r$ is the fraction field of $\mathscr{O}_{\mathbb{C},0}$, i.e., the ring of meromorphic germs at the origin of $\mathbb{C}$.

A.6. Geometry over the hybrid circle. — Let now X be a scheme of finite type over A_r. We will associate to X three kinds of analytic spaces.

First, since X is obtained by gluing together finitely many affine schemes cut out by polynomials with coefficients holomorphic on $\mathbb{D}_r^* \subset \mathbb{C}$ and meromorphic at 0, we can associate to X in a functorial way a complex analytic space X^{hol} over $\mathbb{D}_r^*$, which we call its *holomorphic analytification*.

Second, since A_r is contained in $\mathbb{C}(\!(t)\!)$, we may also consider the base change $X_{\mathbb{C}(\!(t)\!)}$ and its *non-Archimedean analytification* $X_{\mathbb{C}(\!(t)\!)}^{\mathrm{an}}$ with respect to the non-Archimedean absolute value r^{ord_0} on $\mathbb{C}(\!(t)\!)$.

Third, we denote by X^{hyb} the analytification of X as a scheme of finite type over the Banach ring A_r, and call it the *hybrid analytification* of X. In view of Proposition A.4, it comes with a continuous structure map

$$\pi\colon X^{\mathrm{hyb}} \longrightarrow \overline{\mathbb{D}}_r \simeq \mathscr{M}(A_r),$$

Recall further that X^{hyb} is locally compact, Hausdorff if X is separated, and compact if X is proper over A_r. The discussion above implies:

Lemma A.6. — *We have canonical homeomorphisms*

$$\pi^{-1}(0) \simeq X^{\mathrm{an}}_{\mathbb{C}((t))} \quad \textit{and} \quad \pi^{-1}(\mathbb{D}_r^*) \simeq X^{\mathrm{hol}} \tag{A.2}$$

compatible with the projection to $\mathbb{D}_r$.

In §4 we give a topological description of X^{hyb}.

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