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Anti-holomorphic involutions of the moduli spaces of Higgs bundles

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ANTI-HOLOMORPHIC INVOLUTIONS OF THE MODULI SPACES OF HIGGS BUNDLES

BY INDRANIL BISWAS & OSCAR GARCÍA-PRADA

ABSTRACT. — We study anti-holomorphic involutions of the moduli space of G -Higgs bundles over a compact Riemann surface X , where G is a complex semisimple Lie group. These involutions are defined by fixing anti-holomorphic involutions on both X and G . We analyze the fixed point locus in the moduli space and their relation with representations of the orbifold fundamental group of X equipped with the anti-holomorphic involution. We also study the relation with branes. This generalizes work by Biswas–García-Prada–Hurtubise and Baraglia–Schaposnik.

RÉSUMÉ (Involutions anti-holomorphes des espaces de modules de fibrés de Higgs)

Nous étudions les involutions anti-holomorphes des espaces de modules de G -fibrés de Higgs sur une surface de Riemann compacte X , où G est un groupe de Lie semi-simple complexe. Ces involutions sont définies en fixant des involutions anti-holomorphes à la fois sur X et G . Nous en analysons le lieu des points fixes dans l'espace de modules et leur relation avec les représentations du groupe fondamental orbifold de X muni de l'involution anti-holomorphe. Nous étudions aussi la relation avec les « branes ». Ceci généralise les travaux de Biswas–García-Prada–Hurtubise et Baraglia–Schaposnik.

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1. INTRODUCTION

Let G be a complex semisimple affine algebraic group with Lie algebra \mathfrak{g} . Let X be a compact connected Riemann surface. A G -Higgs bundle over X is a pair (E, φ) , where E is a holomorphic principal G -bundle over X and φ is a holomorphic section of $E(\mathfrak{g}) \otimes K$ with $E(\mathfrak{g})$ being the vector bundle associated to E for the adjoint action of G on \mathfrak{g} and K being the canonical line bundle on X . We consider the moduli space of polystable G -Higgs bundles $\mathcal{M}(G)$. This has the structure of a hyper-Kähler manifold outside the singular locus.

Let $\alpha : X \rightarrow X$ and $\sigma : G \rightarrow G$ be anti-holomorphic involutions. We define the two involutions (see Section 4.1 for details)

$$(1.1) \quad \begin{aligned} \iota(\alpha, \sigma)^\pm : \mathcal{M}(G) &\longrightarrow \mathcal{M}(G) \\ (E, \varphi) &\longmapsto (\alpha^* \sigma(E), \pm \alpha^* \sigma(\varphi)). \end{aligned}$$

The goal of this paper is to describe the fixed points of these involutions. The fixed points are given by the image of moduli spaces of G -Higgs bundles satisfying a reality condition determined by α and σ , and an element $c \in Z_2^\sigma$, where Z is the center of G and Z_2^σ is the group of elements of order two in Z fixed by σ . For the involution $\iota(\alpha, \sigma)^+$, these are the moduli space of pseudo-real Higgs bundles considered in [5], to which we refer here as $(\alpha, \sigma, c, +)$ -pseudo-real G -Higgs bundles. For $\iota(\alpha, \sigma)^-$, the reality condition on the bundle E is the same as that for $\iota(\alpha, \sigma)^+$, but the different sign on φ gives a different reality condition on the moduli space of Higgs bundles, defining objects that we call $(\alpha, \sigma, c, -)$ -pseudo-real G -Higgs bundles. When the element $c \in Z_2^\sigma$ is trivial we call these objects real G -Higgs bundles.

The involution $\iota(\alpha, \sigma)^-$ is studied by Baraglia-Schaposnik [3] when σ is the anti-holomorphic involution τ corresponding to a compact real form of G (see also [19]). In [4], they consider the involutions $\iota(\alpha, \sigma)^+$ obtained as a result of composing $\iota(\alpha, \tau)^-$ with the holomorphic involution $\iota^-(\theta)$ of $\mathcal{M}(G)$ defined by $\iota^-(E, \varphi) = (\theta(E), -\theta(\varphi))$, where θ is the holomorphic involution of G given by $\theta = \sigma\tau$ (here one takes a compact conjugation τ commuting with σ). In fact, if we consider the involutions

$$(1.2) \quad \begin{aligned} \iota(\theta)^\pm : \mathcal{M}(G) &\longrightarrow \mathcal{M}(G) \\ (E, \varphi) &\longmapsto (\theta(E), \pm \theta(\varphi)), \end{aligned}$$

one has

$$\iota(\alpha, \sigma)^\pm = \iota^\mp(\theta) \circ \iota^-(\alpha, \tau).$$

The involutions (1.2) have been studied in [12, 13, 15].

In the language of branes [20], the fixed points of $\iota(\alpha, \sigma)^+$ are (A, A, B) -branes, while the fixed points of $\iota(\alpha, \sigma)^-$ are (A, B, A) -branes. What these mean is that the fixed points of $\iota(\alpha, \sigma)^-$ are complex Lagrangian submanifolds with respect to the complex structure J_2 defined on $\mathcal{M}(G)$ by the complex structure of G , while the fixed points of $\iota(\alpha, \sigma)^+$ are complex Lagrangian submanifolds with respect to the complex structure $J_3 = J_1 J_2$ obtained by combining J_2 with the natural complex structure J_1

defined on the moduli space of Higgs bundles for the Riemann surface X . The study of these branes is of great interest in connection with mirror symmetry and the Langlands correspondence in the theory of Higgs bundles (see [20, 18, 3, 2]).

We then identify these involutions in the moduli space of representations of the fundamental group of X in G , and describe the fixed points corresponding to the (α, σ, c, \pm) -pseudo-real G -Higgs bundles in terms of representations of the orbifold fundamental group of (X, α) in a group whose underlying set is $G \times \mathbb{Z}/2\mathbb{Z}$. The group structure on $G \times \mathbb{Z}/2\mathbb{Z}$ is constructed using the element $c \in Z_2^\sigma$ and an action of $\mathbb{Z}/2\mathbb{Z}$ on G which depends on the sign of the pseudo-reality condition; more precisely, this action is given by the conjugation σ in the “+” case, and the action of $\theta = \sigma\tau$ in the “−” case, where τ is a compact conjugation commuting with σ . When c is trivial we obtain the semi-direct products of G with $\mathbb{Z}/2\mathbb{Z}$ for the action σ .

The results of this paper have a straightforward generalization to the case in which G is reductive. In this situation the fundamental group of X is replaced by its universal central extension.

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2. G -HIGGS BUNDLES AND REPRESENTATIONS OF THE FUNDAMENTAL GROUP

2.1. MODULI SPACE OF G -HIGGS BUNDLES. — Let G be a complex semisimple affine algebraic group. Its Lie algebra will be denoted by \mathfrak{g} . Let X be an irreducible smooth projective curve defined over \mathbb{C} , equivalently, it is a compact connected Riemann surface. Let g_X be the genus of X ; throughout we assume that $g_X \geq 2$. The canonical line bundle of X will be denoted by K . For a principal G -bundle E , let $E(\mathfrak{g}) := E \times^G \mathfrak{g}$ be the adjoint vector bundle for E .

A G -Higgs bundle over X is a pair (E, φ) , where E is a holomorphic principal G -bundle over X and φ is a holomorphic section of $E(\mathfrak{g}) \otimes K$. Two G -Higgs bundles (E, φ) and (F, ψ) are isomorphic if there is a holomorphic isomorphism of principal G -bundles $f : E \rightarrow F$ such that the induced isomorphism

$$\mathrm{Ad}(f) \otimes \mathrm{Id}_K : E(\mathfrak{g}) \otimes K \longrightarrow F(\mathfrak{g}) \otimes K$$

sends φ to ψ .

There are notions of (semi)stability and polystability for G -Higgs bundles (see [8, 14, 7] for example). A G -Higgs bundle (E, φ) is said to be *stable* (respectively, *semistable*) if for every parabolic subgroup $P \subset G$, every holomorphic reduction $\sigma : E_P \rightarrow E$ of E to P such that

$$\varphi \in H^0(X, E_P(\mathfrak{p}) \otimes K) \subset H^0(X, E(\mathfrak{g}) \otimes K)$$

and every strictly antidominant character χ of P , we have that $\deg E_P(\chi) > 0$ (respectively, $\deg E_P(\chi) \geq 0$). A Higgs bundle (E, φ) is *polystable* if it is semistable and for every P , every reduction and every χ as above such that $\deg E_P(\chi) = 0$, there is a holomorphic reduction $E_L \subset E$ to a Levi subgroup $L \subset P$ such that $\varphi \in H^0(X, E_L(1) \otimes K)$.

Let $\mathcal{M}(G)$ denote the *moduli space of semistable G -Higgs bundles* of fixed topological type. This moduli space has the structure of a complex normal quasiprojective variety of dimension $\dim G(g_X - 1)$.

2.2. G -HIGGS BUNDLES AND HITCHIN EQUATIONS. — As above, let G be a complex semisimple affine algebraic group. Let $H \subset G$ be a maximal compact subgroup. Let (E, φ) be a G -Higgs bundle over a compact Riemann surface X . By a slight abuse of notation, we shall denote the C^∞ -objects underlying E and φ by the same symbols. In particular, the Higgs field can be viewed as a $(1, 0)$ -form $\varphi \in \Omega^{1,0}(E(\mathfrak{g}))$ with values in $E(\mathfrak{g})$. Let

$$\tau: \Omega^{1,0}(E(\mathfrak{g})) \longrightarrow \Omega^{0,1}(E(\mathfrak{g}))$$

be the isomorphism induced by the compact conjugation of \mathfrak{g} (with respect to H) combined with the complex conjugation on complex 1-forms. Given a C^∞ reduction of structure group h of the principal G -bundle E to H , we denote by F_h the curvature of the unique connection compatible with h and the holomorphic structure on E ; see [1, pp. 191–192, Prop. 5] for the connection.

THEOREM 2.1. — *There is a reduction h of structure group of E from G to H that satisfies the Hitchin equation*

$$F_h - [\varphi, \tau(\varphi)] = 0$$

if and only if (E, φ) is polystable.

Theorem 2.1 was proved by Hitchin [17] for $G = \mathrm{SL}(2, \mathbb{C})$, and in [23, 24, 7] for the general case.

REMARK 2.2. — When G is reductive the equation in Theorem 2.1 is replaced by the equation

$$F_h - [\varphi, \tau(\varphi)] = c\omega,$$

where ω is a Kähler form on X and c is an element in the center of the Lie algebra of G , which is determined by the topology of E .

From the point of view of moduli spaces it is convenient to fix a C^∞ principal H -bundle \mathbf{E}_H and study the moduli space of solutions to *Hitchin's equations* for a pair (A, φ) consisting of a H -connection A on \mathbf{E}_H and a section $\varphi \in \Omega^{1,0}(X, \mathbf{E}_H(\mathfrak{g}))$:

$$(2.1) \quad \begin{aligned} F_A - [\varphi, \tau(\varphi)] &= 0 \\ \bar{\partial}_A \varphi &= 0. \end{aligned}$$

Here d_A is the covariant derivative associated to A , and $\bar{\partial}_A$ is the $(0, 1)$ part of d_A . The $(0, 1)$ part of d_A defines a holomorphic structure on \mathbf{E}_H . The gauge group \mathcal{H}

of \mathbf{E}_H acts on the space of solutions and the moduli space of solutions is

$$\mathcal{M}^{\text{gauge}}(G) := \{(A, \varphi) \text{ satisfying (2.1)}\} / \mathcal{H}.$$

Now, Theorem 2.1 can be reformulated as follows.

THEOREM 2.3. — *There is a homeomorphism*

$$\mathcal{M}(G) \cong \mathcal{M}^{\text{gauge}}(G).$$

To explain this correspondence we interpret the moduli space of G -Higgs bundles in terms of pairs $(\bar{\partial}_E, \varphi)$ consisting of a $\bar{\partial}$ -operator (holomorphic structure) $\bar{\partial}_E$ on the C^∞ principal G -bundle \mathbf{E}_G obtained from \mathbf{E}_H by the extension of structure group $H \hookrightarrow G$, and $\varphi \in \Omega^{1,0}(X, \mathbf{E}_G(\mathfrak{g}))$ satisfying $\bar{\partial}_E \varphi = 0$. Such pairs are in one-to-one correspondence with G -Higgs bundles (E, φ) , where E is the holomorphic G -bundle defined by the operator $\bar{\partial}_E$ on \mathbf{E}_G . The equation $\bar{\partial}_E \varphi = 0$ is equivalent to the condition that $\varphi \in H^0(X, E(\mathfrak{g}) \otimes K)$. The moduli space of polystable G -Higgs bundles $\mathcal{M}_d(G)$ can now be identified with the orbit space

$$\{(\bar{\partial}_E, \varphi) \mid \bar{\partial}_E \varphi = 0 \text{ and the } G\text{-Higgs bundle is polystable}\} / \mathcal{G},$$

where \mathcal{G} is the gauge group of \mathbf{E}_G , which is in fact the complexification of \mathcal{H} . Since there is a one-to-one correspondence between H -connections on \mathbf{E}_H and $\bar{\partial}$ -operators on \mathbf{E}_G , the correspondence given in Theorem 2.3 can be reformulated by saying that in the \mathcal{G} -orbit of a polystable G -Higgs bundle $(\bar{\partial}_{E_0}, \varphi_0)$ we can find another Higgs bundle $(\bar{\partial}_E, \varphi)$ whose corresponding pair (d_A, φ) satisfies the Hitchin equation $F_A - [\varphi, \tau(\varphi)] = 0$ with this pair (d_A, φ) being unique up to H -gauge transformations.

2.3. HIGGS BUNDLES AND REPRESENTATIONS. — Fix a base point $x_0 \in X$. By a *representation* of $\pi_1(X, x_0)$ in G we mean a homomorphism $\pi_1(X, x_0) \rightarrow G$. After fixing a representation of $\pi_1(X, x_0)$, the set of all such homomorphisms, $\text{Hom}(\pi_1(X, x_0), G)$, can be identified with the subset of G^{2g_X} consisting of $2g_X$ -tuples $(A_1, B_1, \dots, A_{g_X}, B_{g_X})$ satisfying the algebraic equation $\prod_{i=1}^{g_X} [A_i, B_i] = 1$. This shows that $\text{Hom}(\pi_1(X, x_0), G)$ is a complex algebraic variety.

The group G acts on $\text{Hom}(\pi_1(X, x_0), G)$ by conjugation:

$$(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1},$$

where $g \in G$, $\rho \in \text{Hom}(\pi_1(X, x_0), G)$ and $\gamma \in \pi_1(X, x_0)$. If we restrict the action to the subspace $\text{Hom}^+(\pi_1(X, x_0), G)$ consisting of reductive representations, the orbit space is Hausdorff. We recall that a *reductive representation* is one whose composition with the adjoint representation in \mathfrak{g} decomposes as a direct sum of irreducible representations. This is equivalent to the condition that the Zariski closure of the image of $\pi_1(X, x_0)$ in G is a reductive group. Define the *moduli space of representations* of $\pi_1(X, x_0)$ in G to be the orbit space

$$\mathcal{R}(G) = \text{Hom}^+(\pi_1(X, x_0), G) / G.$$

For another point $x' \in X$, the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x')$ are identified by an isomorphism unique up to an inner automorphism. Consequently, $\mathcal{R}(G)$ is independent of the choice of the base point x_0 .

One has the following (see e.g. [16], [25]).

THEOREM 2.4. — *The moduli space $\mathcal{R}(G)$ has the structure of a normal complex variety. Its smooth locus is equipped with a holomorphic symplectic form.*

Given a representation $\rho: \pi_1(X, x_0) \rightarrow G$, there is an associated flat principal G -bundle on X , defined as

$$E_\rho = \tilde{X} \times^\rho G,$$

where $\tilde{X} \rightarrow X$ is the universal cover associated to x_0 and $\pi_1(X, x_0)$ acts on G via ρ . This gives in fact an identification between the set of equivalence classes of representations $\text{Hom}(\pi_1(X), G)/G$ and the set of equivalence classes of flat principal G -bundles, which in turn is parametrized by the (nonabelian) cohomology set $H^1(X, G)$.

We have the following:

THEOREM 2.5. — *There is a homeomorphism $\mathcal{R}(G) \cong \mathcal{M}(G)$.*

The moduli spaces $\mathcal{M}(G)$ and $\mathcal{R}(G)$ are sometimes referred as the *Dolbeault* and *Betti* moduli spaces, respectively.

The proof of Theorem 2.5 is the combination of two existence theorems for gauge-theoretic equations. To explain this, let \mathbf{E}_G be, as above, a C^∞ principal G -bundle over X and \mathbf{E}_H a C^∞ reduction of structure group of it to H . Every G -connection D on \mathbf{E}_G decomposes uniquely as

$$D = d_A + \psi,$$

where d_A is an H -connection on \mathbf{E}_H and $\psi \in \Omega^1(X, \mathbf{E}_H(\sqrt{-1}\mathfrak{h}))$. Let F_A be the curvature of d_A . We consider the following set of equations for the pair (d_A, ψ) :

$$(2.2) \quad \begin{aligned} F_A + \frac{1}{2}[\psi, \psi] &= 0 \\ d_A\psi &= 0 \\ d_A^*\psi &= 0. \end{aligned}$$

These equations are invariant under the action of \mathcal{H} , the gauge group of \mathbf{E}_H . A theorem of Corlette [10], and Donaldson [11] for $G = \text{SL}(2, \mathbb{C})$, says the following.

THEOREM 2.6. — *There is a homeomorphism between*

$$\{\text{Reductive } G\text{-connections } D \mid F_D = 0\} / \mathcal{G}$$

and

$$\{(d_A, \psi) \text{ satisfying (2.2)}\} / \mathcal{H}.$$

The first two equations in (2.2) are equivalent to the flatness of $D = d_A + \psi$, and Theorem 2.6 simply says that in the \mathcal{G} -orbit of a reductive flat G -connection D_0 we can find a flat G -connection $D = \tilde{g}(D_0)$ such that if we write $D = d_A + \psi$, the additional condition $d_A^*\psi = 0$ is satisfied. This can be interpreted more geometrically in terms of the reduction $h = \tilde{g}(h_0)$ of \mathbf{E}_G to a principal H -bundle obtained by the

action of $\tilde{g} \in \mathcal{G}$ on h_0 . The equation $d_A^* \psi = 0$ is equivalent to the harmonicity of the $\pi_1(X)$ -equivariant map $\tilde{X} \rightarrow G/H$ corresponding to the new reduction of structure group h .

To complete the argument, leading to Theorem 2.5, we just need Theorem 2.1 and the following simple result.

PROPOSITION 2.7. — *The correspondence $(d_A, \varphi) \mapsto (d_A, \psi := \varphi - \tau(\varphi))$ defines a homeomorphism*

$$\{(d_A, \varphi) \text{ satisfying (2.1)}\} / \mathcal{H} \cong \{(d_A, \psi) \text{ satisfying (2.2)}\} / \mathcal{H}.$$

2.4. THE MODULI SPACE AS A HYPER-KÄHLER QUOTIENT. — We will see now that the moduli space $\mathcal{M}(G)$ has a hyper-Kähler structure. For this, recall first that a hyper-Kähler manifold is a differentiable manifold M equipped with a Riemannian metric g and complex structures J_i , $i = 1, 2, 3$ satisfying the quaternion relations $J_i^2 = -I$, $J_3 = J_1 J_2 = -J_2 J_1$, $J_2 = -J_1 J_3 = J_3 J_1$ and $J_1 = J_2 J_3 = -J_3 J_2$ such that if we define $\omega_i(\cdot, \cdot) = g(J_i \cdot, \cdot)$, then (g, J_i, ω_i) is a Kähler structure on M . Let Ω_i denote the holomorphic symplectic structure on $\mathcal{M}(G)$ with respect to the complex structure J_i . In fact, $\Omega_1 = \omega_2 + \sqrt{-1}\omega_3$, $\Omega_2 = \omega_3 + \sqrt{-1}\omega_1$ and $\Omega_3 = \omega_1 + \sqrt{-1}\omega_2$.

One way to understand the non-abelian Hodge theory correspondence mentioned above is through the analysis of the hyper-Kähler structure of the moduli spaces involved. We explain how these can be obtained as hyper-Kähler quotients. For this, let \mathbf{E}_G be a smooth principal G -bundle over X , and let \mathbf{E}_H be a fixed reduction of \mathbf{E}_G to the maximal compact subgroup H . The set \mathcal{A} of H -connections on \mathbf{E}_H is an affine space modelled on $\Omega^1(X, \mathbf{E}(\mathfrak{h}))$. Via the Chern correspondence, \mathcal{A} is in one-to-one correspondence with the set \mathcal{C} of holomorphic structures on \mathbf{E}_G [1, pp. 191–192, Prop. 5], which is an affine space modelled on $\Omega^{0,1}(X, \mathbf{E}_G(\mathfrak{g}))$. Let us denote $\Omega = \Omega^{1,0}(X, \mathbf{E}_G(\mathfrak{g}))$. We consider $\mathcal{X} = \mathcal{A} \times \Omega$. Via the identification $\mathcal{A} \cong \mathcal{C}$, we have for $\alpha \in \Omega^{0,1}(X, \mathbf{E}_G(\mathfrak{g}))$ and $\psi \in \Omega^{1,0}(X, \mathbf{E}_G(\mathfrak{g}))$ the following three complex structures on \mathcal{X} :

$$\begin{aligned} J_1(\alpha, \psi) &= (\sqrt{-1}\alpha, \sqrt{-1}\psi) \\ J_2(\alpha, \psi) &= (-\sqrt{-1}\tau(\psi), \sqrt{-1}\tau(\alpha)) \\ J_3(\alpha, \psi) &= (\tau(\psi), -\tau(\alpha)), \end{aligned}$$

where τ is the conjugation on \mathfrak{g} defining its compact form \mathfrak{h} (determined fiber-wise by the reduction to \mathbf{E}_H), combined with the complex conjugation on complex 1-forms.

One has also a Riemannian metric g defined on \mathcal{X} : for $\alpha \in \Omega^{0,1}(X, \mathbf{E}_G(\mathfrak{g}))$ and $\psi \in \Omega^{1,0}(X, \mathbf{E}_G(\mathfrak{g}))$,

$$g((\alpha, \psi), (\alpha, \psi)) = -2\sqrt{-1} \int_X B(\tau(\alpha), \alpha) + B(\psi, \tau(\psi)),$$

where B is the Killing form.

Clearly, J_i , $i = 1, 2, 3$, satisfy the quaternion relations, and define a hyper-Kähler structure on \mathcal{X} , with Kähler forms $\omega_i(\cdot, \cdot) = g(J_i \cdot, \cdot)$, $i = 1, 2, 3$. As shown in [17],

the action of the gauge group \mathcal{H} on \mathcal{X} preserves the hyper-Kähler structure and there are moment maps given by

$$\mu_1(A, \varphi) = F_A - [\varphi, \tau(\varphi)], \quad \mu_2(A, \varphi) = \operatorname{Re}(\bar{\partial}_E \varphi), \quad \mu_3(A, \varphi) = \operatorname{Im}(\bar{\partial}_E \varphi).$$

We have that $\boldsymbol{\mu}^{-1}(0)/\mathcal{H}$, where $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$ is the moduli space of solutions to the Hitchin equations (2.1). In particular, if we consider the irreducible solutions (equivalently, smooth) $\boldsymbol{\mu}_*^{-1}(0)$ we have that

$$\boldsymbol{\mu}_*^{-1}(0)/\mathcal{H}$$

is a hyper-Kähler manifold which, by Theorem 2.3, is homeomorphic to the subvariety of smooth points in moduli space $\mathcal{M}(G)$ of stable G -Higgs bundles with the topological class of \mathbf{E}_G .

Let us now see how the moduli of harmonic flat connections on \mathbf{E}_H can be realized as a hyper-Kähler quotient. Let \mathcal{D} be the set of G -connections on \mathbf{E}_G . This is an affine space modelled on $\Omega^1(X, \mathbf{E}_G(\mathfrak{g})) = \Omega^0(X, T^*X \otimes_{\mathbb{R}} \mathbf{E}_G(\mathfrak{g}))$. The space \mathcal{D} has a complex structure $I_1 = 1 \otimes \sqrt{-1}$, which comes from the complex structure of the bundle. Using the complex structure of X we have also the complex structure $I_2 = \sqrt{-1} \otimes \tau$. We can finally consider the complex structure $I_3 = I_1 I_2$.

The reduction to H of the G -bundle \mathbf{E}_G together with a Riemannian metric in the conformal class of X defines a flat Riemannian metric $g_{\mathcal{D}}$ on \mathcal{D} which is Kähler for the above three complex structures. Hence $(\mathcal{D}, g_{\mathcal{D}}, I_1, I_2, I_3)$ is also a hyper-Kähler manifold. As in the previous case, the action of the gauge group \mathcal{H} on \mathcal{D} preserves the hyper-Kähler structure and there are moment maps

$$\mu_1(D) = d_A^* \psi, \quad \mu_2(D) = \operatorname{Im}(F_D), \quad \mu_3(D) = \operatorname{Re}(F_D),$$

where $D = d_A + \psi$ is the decomposition of D defined by

$$\mathbf{E}_G(\mathfrak{g}) = \mathbf{E}_H(\mathfrak{h}) \oplus \mathbf{E}_H(\sqrt{-1}\mathfrak{h}).$$

Hence the moduli space of solutions to the harmonicity equations (2.2) is the hyper-Kähler quotient defined by

$$\boldsymbol{\mu}^{-1}(0)/\mathcal{H},$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$. The homeomorphism between the moduli spaces of solutions to the Hitchin and the harmonicity equations is induced from the affine map

$$\begin{aligned} \mathcal{A} \times \Omega &\longrightarrow \mathcal{D} \\ (d_A, \varphi) &\longmapsto d_A + \varphi - \tau(\varphi). \end{aligned}$$

One can see easily, for example, that this map sends $\mathcal{A} \times \Omega$ with complex structure J_2 to \mathcal{D} with complex structure I_1 (see [17]).

Now, Theorems 2.3 and 2.6 can be regarded as existence theorems, establishing the non-emptiness of the hyper-Kähler quotient, obtained by focusing on different complex structures. For Theorem 2.3 one gives a special status to the complex structure J_1 . Combining the symplectic forms determined by J_2 and J_3 one has the J_1 -holomorphic symplectic form $\omega_c = \omega_2 + \sqrt{-1}\omega_3$ on $\mathcal{A} \times \Omega$. The gauge group $\mathcal{G} = \mathcal{H}^{\mathbb{C}}$ acts on $\mathcal{A} \times \Omega$ preserving ω_c . The symplectic quotient construction can also be extended

to the holomorphic situation (see e.g. [21]) to obtain the holomorphic symplectic quotient $\{(\bar{\partial}_E, \varphi) \mid \bar{\partial}_E \varphi = 0\} / \mathcal{G}$. What Theorem 2.3 says is that for a class $[(\bar{\partial}_E, \varphi)]$ in this quotient to have a representative (unique up to H -gauge) satisfying $\mu_1 = 0$ it is necessary and sufficient that the pair $(\bar{\partial}_E, \varphi)$ be polystable. This identifies the hyper-Kähler quotient to the set of equivalence classes of polystable G -Higgs bundles on \mathbf{E}_G . If one now takes J_2 on $\mathcal{A} \times \Omega$ or equivalently \mathcal{D} with I_1 and argues in a similar way, one gets Theorem 2.6 identifying the hyper-Kähler quotient to the set of equivalence classes of reductive flat connections on \mathbf{E}_G .

3. REAL G -HIGGS BUNDLES

3.1. INVOLUTIONS AND CONJUGATIONS OF COMPLEX LIE GROUPS. — Let G be a Lie group. We define

$$\text{Int}(G) := \{f \in \text{Aut}(G) \mid f(h) = ghg^{-1}, \text{ for every } h \in G\}.$$

We have that $\text{Int}(G) = \text{Ad}(G)$.

We define the group of outer automorphisms of G as

$$\text{Out}(G) := \text{Aut}(G) / \text{Int}(G).$$

We have a sequence

$$(3.1) \quad 1 \longrightarrow \text{Int}(G) \longrightarrow \text{Aut}(G) \longrightarrow \text{Out}(G) \longrightarrow 1.$$

It is well-known that if G is a connected complex reductive group then the extension (3.1) splits (see [22]).

Let G be a complex Lie group and let $G_{\mathbb{R}}$ be the underlying real Lie group. We will say that a real Lie subgroup $G_0 \subset G_{\mathbb{R}}$ is a *real form* of G if it is the fixed point set of a conjugation (anti-holomorphic involution) σ of G .

Now, let G be simple. A compact real form always exists. This follows from the fact that for a simple group there is a maximal compact subgroup $U \subset G$, such that $U^{\mathbb{C}} = G$. From this we can define a conjugation $\tau : G \rightarrow G$ such that $G^{\tau} = U$. Let $\text{Conj}(G)$ be the set of conjugations (i.e., anti-holomorphic involutions) of G . We can define the following equivalence relations in $\text{Conj}(G)$:

$$\sigma \sim \sigma' \text{ if there is } \alpha \in \text{Int}(G) \text{ such that } \sigma' = \alpha \sigma \alpha^{-1},$$

We can define a similar relation \sim in the set $\text{Aut}_2(G)$ of automorphisms of G of order 2.

REMARK 3.1. — The equivalence relation \sim for elements in $\text{Aut}_2(G)$ should not be confused with the inner equivalence, meaning the equivalence relation where two elements are equivalent if they map to the same element in $\text{Out}(G)$. It is easy to show that if $\theta \sim \theta'$ then they are inner equivalent.

Cartan [9] shows that there is a bijection

$$\text{Conj}(G) / \sim \longleftrightarrow \text{Aut}_2(G) / \sim .$$

More concretely, one has that given the compact conjugation τ , in each class $\text{Conj}(G)/\sim$ one can find a representative σ commuting with τ so that $\theta := \sigma\tau$ is an element of $\text{Aut}_2(G)$, and similarly if we start with a class in $\text{Aut}_2(G)/\sim$.

3.2. PSEUDO-REAL PRINCIPAL G -BUNDLES. — We use the notation of Section 3.1. Let G be a semisimple complex affine algebraic group. Let $\tau \in \text{Conj}(G)$ be a compact conjugation of G , and let $\sigma \in \text{Conj}(G)$ commuting with τ , and $\theta = \sigma\tau \in \text{Aut}_2(G)$.

Let $Z^\sigma \subset Z$ be the fixed point locus in the center $Z \subset G$. The subgroup of Z^σ generated by its elements of order two will be denoted by Z_2^σ .

Let X be a compact connected Riemann surface, of genus $g \geq 2$, equipped with an anti-holomorphic involution $\alpha : X \rightarrow X$.

DEFINITION 3.2. — Let E be a holomorphic principal G -bundle over X . Take any $c \in Z_2^\sigma$. We say that E is (α, σ, c) -pseudo-real if E is equipped with an anti-holomorphic map $\tilde{\alpha} : E \rightarrow E$ covering α such that

- $\tilde{\alpha}(eg) = \tilde{\alpha}(e)\sigma(g)$, for $e \in E$ and $g \in G$.
- $\tilde{\alpha}^2(e) = ec$.

If $c = 1$, we say that E is (α, σ) -real.

REMARK 3.3. — An alternative definition of pseudo-real bundle allows for c to be any element of Z . However we can modify $\tilde{\alpha}$ by the action of an element $a \in Z$ defining a covering map $\tilde{\alpha}' := \tilde{\alpha}.a$. By this, the element c gets modified by $c' = a\sigma(a)c$. In particular we can take a lying in Z^σ and the composition is modified by a^2 . Therefore if c lies in $(Z^\sigma)^2$, or more generally is of the form $\sigma(a)a$ we can normalize our pseudo-real structure to a real one. But since the natural homomorphism $Z_2^\sigma \rightarrow Z^\sigma/(Z^\sigma)^2$ is surjective we can always assume that c is of order 2, as we have done in our definition.

REMARK 3.4. — Sometimes to emphasize the pseudo-real structure we will write $(E, \varphi, \tilde{\alpha})$ for a G -Higgs bundle (E, φ) equipped with a pseudo-real structure $\tilde{\alpha}$.

Define the quotient

$$G_c := G/\langle c \rangle.$$

Note that $\langle c \rangle = \mathbb{Z}/2\mathbb{Z}$ if $c \neq 1$. Since c is fixed by σ , the involution σ induced an anti-holomorphic involution of G_c . This anti-holomorphic involution of G_c will be denoted by σ' . Let $(E_G, \tilde{\alpha})$ be a (α, σ, c) -pseudo-real principal G -bundle on X . Define $E_{G_c} := E_G/\langle c \rangle$. Note that E_{G_c} is the principal G_c -bundle obtained by extending the structure group of E_G using the quotient homomorphism $G \rightarrow G_c$. The above self-map $\tilde{\alpha}$ of E_G descends to a self-map

$$\tilde{\alpha}' : E_{G_c} \longrightarrow E_{G_c}.$$

Since $\tilde{\alpha}^2 = c$, we have $\tilde{\alpha}' \circ \tilde{\alpha}' = \text{Id}_{E_{G_c}}$. Therefore, $(E_{G_c}, \tilde{\alpha}')$ is a (α, σ') -real principal G_c -bundle.

The pair (X, α) defines a geometrically irreducible smooth projective curve defined over \mathbb{R} . This curve defined over \mathbb{R} will be denoted by X' . Assume that $c \neq 1$. Let G'

(respectively, G'_c) be the algebraic group, defined over \mathbb{R} , given by the pair (G, σ) (respectively, (G_c, σ')). Consider the short exact sequence of sheaves

$$1 \longrightarrow \langle c \rangle = \mathbb{Z}/2\mathbb{Z} \longrightarrow G' \longrightarrow G'_c \longrightarrow 1$$

on X' . Let

$$H_{\text{ét}}^1(X', G') \longrightarrow H_{\text{ét}}^1(X', G'_c) \xrightarrow{\beta} H_{\text{ét}}^2(X', \mathbb{Z}/2\mathbb{Z})$$

be the long exact sequence of étale cohomologies corresponding to the above short exact sequence of sheaves on the curve X' defined over \mathbb{R} . As noted above, a (α, σ, c) -pseudo-real principal G -bundle on X gives a (α, σ') -real principal G_c -bundle. Note that the isomorphism classes of principal G'_c -bundles on X' are parametrized by the elements of the cohomology $H_{\text{ét}}^1(X', G'_c)$. Indeed, this follows immediately from the fact that any principal G'_c -bundle on X' can be locally trivialized with respect to the étale topology. Therefore, a (α, σ, c) -pseudo-real principal G -bundle on X gives an element of $H_{\text{ét}}^1(X', G'_c)$.

We will give a necessary and sufficient condition for a given (α, σ') -real principal G_c -bundle on X to come from a (α, σ, c) -pseudo-real principal G -bundle.

Let $(E_{G_c}, \tilde{\alpha}')$ be a (α, σ') -real principal G_c -bundle on X . As explained above, $(E_{G_c}, \tilde{\alpha}')$ is equivalently a principal G'_c -bundle on X' . This principal G'_c -bundles on X' will be denoted by F_{G_c} . Consider the adjoint action of G on itself. Since c lies in the center of G , this action of G factors through the quotient group G_c . Let

$$E_{G_c}(G) := E_{G_c} \times^{G_c} G \longrightarrow X$$

be the fiber bundle associated to the principal G_c -bundle E_{G_c} for this action of G_c on G . Since the action of G_c on G preserves the group structure on G , each fiber of $E_{G_c}(G)$ is a group isomorphic to G . The action of G_c on G descends to an action of G_c on the quotient $G/\langle c \rangle = G_c$, and this descended action coincides with the adjoint action of G_c on itself. Therefore, the short exact sequence of groups

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow G \longrightarrow G_c \longrightarrow 1$$

produces a short exact sequence of fiber bundles with group structures

$$(3.2) \quad 1 \longrightarrow X \times (\mathbb{Z}/2\mathbb{Z}) \longrightarrow E_{G_c}(G) \longrightarrow \text{Ad}(E_{G_c}) \longrightarrow 1,$$

where $\text{Ad}(E_{G_c}) = E_{G_c} \times^{G_c} G_c$ is the adjoint bundle for E_{G_c} .

The involution $\tilde{\alpha}'$ of E_{G_c} and the involution σ of G together produce an anti-holomorphic involution of $E_{G_c}(G)$ covering α . Similarly, $\tilde{\alpha}'$ and σ' together produce an anti-holomorphic involution of $\text{Ad}(E_{G_c})$ covering α . Therefore, (3.2) produces a short exact sequence

$$(3.3) \quad 1 \longrightarrow X' \times (\mathbb{Z}/2\mathbb{Z}) \longrightarrow E_{G_c}(G)' \longrightarrow \text{Ad}(E_{G_c})' \longrightarrow 1$$

over the curve X' defined over \mathbb{R} . We note that $\text{Ad}(E_{G_c})'$ is the adjoint bundle for the principal G'_c -bundle F_{G_c} over X' defined by the pair $(E_{G_c}, \tilde{\alpha}')$.

The space of all isomorphism classes of principal G'_c -bundles on X' are parametrized by $H_{\text{ét}}^1(X', \text{Ad}(E_{G_c})')$. This identification is constructed as follows. First recall that $\text{Ad}(E_{G_c})'$ is the adjoint bundle for the principal G'_c -bundle F_{G_c} over X' . Given a principal G'_c -bundle on X' , by choosing étale local isomorphisms of it with F_{G_c} we

get an element of $H_{\text{ét}}^1(X', \text{Ad}(E_{G_c})')$. Conversely, given a 1-cocycle on X' with values in $\text{Ad}(E_{G_c})'$, by gluing back, using the cocycle, the restrictions of F_{G_c} to the open subsets for the cocycle, we get a principal G'_c -bundle on X' . Note that if F_{G_c} is the trivial principal G'_c -bundle, then $H_{\text{ét}}^1(X', \text{Ad}(E_{G_c})') = H_{\text{ét}}^1(X', G'_c)$.

The set $H_{\text{ét}}^1(X', \text{Ad}(E_{G_c})')$ has a distinguished base point t_0 . This point t_0 corresponds to the isomorphism class of the principal G'_c -bundle F_{G_c} .

Consider the short exact sequence of étale cohomologies

$$(3.4) \quad H_{\text{ét}}^1(X', E_{G_c}(G)') \xrightarrow{\gamma'} H_{\text{ét}}^1(X', \text{Ad}(E_{G_c})') \xrightarrow{\beta'} H_{\text{ét}}^2(X', \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

associated to (3.3). It can be shown that $(E_{G_c}, \tilde{\alpha}')$ is given by a (α, σ, c) -pseudo-real principal G -bundle if and only if the base point $t_0 \in H_{\text{ét}}^1(X', \text{Ad}(E_{G_c})')$ lies in the image of the map γ' in (3.4). Indeed, if $(E_G, \tilde{\alpha})$ is a (α, σ, c) -pseudo-real principal G -bundle on X that gives $(E_{G_c}, \tilde{\alpha}')$, then the adjoint bundle $\text{Ad}(E_G)$ equipped with the involution constructed using $\tilde{\alpha}$ and σ produces an element $t' \in H_{\text{ét}}^1(X', E_{G_c}(G)')$ such that $\gamma'(t') = t_0$. Conversely, any $t' \in H_{\text{ét}}^1(X', E_{G_c}(G)')$ produces a (α, σ, c) -pseudo-real principal G -bundle. If $\gamma'(t') = t_0$, then this (α, σ, c) -pseudo-real principal G -bundle gives the pair $(E_{G_c}, \tilde{\alpha}')$.

Therefore, we have the following.

PROPOSITION 3.5. — *A (α, σ') -real principal G_c -bundle $(E_{G_c}, \tilde{\alpha}')$ on X comes from a (α, σ, c) -pseudo-real principal G -bundle if and only if $\beta'(t_0) = 0$, where β' is the map in (3.4) and $t_0 \in H_{\text{ét}}^1(X', \text{Ad}(E_{G_c})')$ is the base point.*

The following proposition shows the relation between the reality conditions defined by conjugations of G that are inner equivalent. One has the following.

PROPOSITION 3.6. — *Let $\sigma, \sigma' \in \text{Conj}(G)$ such that $\sigma' = \text{Int}(g_0)\sigma$ for some $g_0 \in G$, i.e., $\sigma'(g) = g_0\sigma(g)g_0^{-1}$. Let E be a G -bundle over X . Then E is (α, σ, c) -pseudo-real if and only if it is (α, σ', c') -pseudo-real, where c and c' are related by g_0 and σ . In fact $c' = c$, if $\sigma(g_0) = g_0^{-1}$.*

Proof. — Let $(E, \tilde{\alpha})$ be a (α, σ, c) -pseudo-real principal G -bundle on X . Define

$$\tilde{\alpha}' : E \longrightarrow E, \quad e \longmapsto \tilde{\alpha}(e)g_0^{-1}.$$

Since $\tilde{\alpha}$ is anti-holomorphic and covers α , the map $\tilde{\alpha}'$ is also anti-holomorphic and covers α . For any $e \in E$ and $g \in G$, we have

$$\tilde{\alpha}'(eg) = \tilde{\alpha}(eg)g_0^{-1} = \tilde{\alpha}(e)\sigma(g)g_0^{-1} = \tilde{\alpha}(e)g_0^{-1}g_0\sigma(g)g_0^{-1} = \tilde{\alpha}'(e)\sigma'(g).$$

Also,

$$\tilde{\alpha}'(\tilde{\alpha}'(e)) = \tilde{\alpha}'(\tilde{\alpha}(e)g_0^{-1}) = \tilde{\alpha}(\tilde{\alpha}(e)g_0^{-1})g_0^{-1} = \tilde{\alpha}(\tilde{\alpha}(e))\sigma(g_0^{-1})g_0^{-1} = e c \sigma(g_0^{-1})g_0^{-1}.$$

Now, $\sigma'^2 = \text{Id}$ implies that $\sigma(g_0^{-1})g_0^{-1} \in Z$, and we can appeal to Remark 3.3 to claim that by modifying $\tilde{\alpha}'$ by an element of the center $c\sigma(g_0^{-1})g_0^{-1}$ is replaced by an element $c' \in Z_2^g$, and hence E has the structure of a (α, σ', c') -pseudo-real principal G -bundle on X . The last claim in the proposition is clear. \square

3.3. PSEUDO-REAL G -HIGGS BUNDLES. — Let $(E, \tilde{\alpha})$ be a (α, σ, c) -pseudo-real principal G -bundle on X as defined above. Let

$$\mathrm{ad}(E) := E \times^G \mathfrak{g} =: E(\mathfrak{g})$$

be the adjoint vector bundle for E . The self-map $\tilde{\alpha}$ of E produces an anti-holomorphic self-map

$$(3.5) \quad \tilde{\alpha}_0 : E(\mathfrak{g}) \longrightarrow E(\mathfrak{g})$$

such that $q \circ \tilde{\alpha}_0 = \alpha \circ q$, where q is the projection of $E(\mathfrak{g})$ to X . Since $c \in Z$, the adjoint action of c on \mathfrak{g} is trivial. This immediately implies that $\tilde{\alpha}_0$ is an involution. In other words, $(E(\mathfrak{g}), \tilde{\alpha}_0)$ is a real vector bundle (see [5]).

The real structure of the canonical line bundle K of X given by α and the above real structure $\tilde{\alpha}_0$ of $E(\mathfrak{g})$ combine to define a real structure on the vector bundle $E(\mathfrak{g}) \otimes K$. For notational convenience, this real structure on $E(\mathfrak{g}) \otimes K$ will also be denoted by $\tilde{\alpha}$. So

$$\tilde{\alpha} : E(\mathfrak{g}) \otimes K \longrightarrow E(\mathfrak{g}) \otimes K$$

is an anti-holomorphic involution over α .

DEFINITION 3.7. — Let (E, φ) be a G -Higgs bundle. We say that (E, φ) is $(\alpha, \sigma, c, +)$ -pseudo-real (respectively, $(\alpha, \sigma, c, -)$ -pseudo-real) if E is (α, σ, c) -pseudo-real, and φ satisfies

$$\tilde{\alpha}(\varphi) = \varphi \quad (\text{respectively, } \tilde{\alpha}(\varphi) = -\varphi).$$

The concept of $(\alpha, \sigma, c, +)$ -pseudo-real Higgs bundle was introduced in [5], where notions of (semi)stability and polystability for these objects were defined. These notions are identical for the $(\alpha, \sigma, c, -)$ -pseudo-real case. For the benefit of the reader we recall the basic definitions and facts (see [5] for details).

Let $\mathrm{Ad}(E) := E \times^G G$ be the group-scheme over X associated to E for the adjoint action of G on it self. The bundle $\mathrm{Ad}(E)$ is equipped with an anti-holomorphic involution

$$(3.6) \quad \tilde{\alpha} : \mathrm{Ad}(E) \longrightarrow \mathrm{Ad}(E)$$

(abusing notation again) covering α . Note that $\tilde{\alpha}^2 = \mathrm{Id}_{\mathrm{Ad}(E)}$ since the adjoint action of Z^σ on G is trivial.

A *parabolic subgroup scheme* of $\mathrm{Ad}(E)$ is a Zariski closed analytically locally trivial subgroup scheme $\underline{P} \subset \mathrm{Ad}(E)$ such that $\mathrm{Ad}(E)/\underline{P}$ is compact. For such a parabolic subgroup scheme \underline{P} let $\underline{\mathfrak{p}} \subset \mathrm{ad}(E)$ be the corresponding bundle of Lie algebras.

A (α, σ, c, \pm) -pseudo-real G -Higgs bundle $(E, \varphi, \tilde{\alpha})$ is *semistable* (respectively *stable*) if for every proper parabolic subgroup scheme $\underline{P} \subset \mathrm{Ad}(E)$ such that $\tilde{\alpha}(\underline{P}) \subset \underline{P}$, where $\tilde{\alpha}$ is given by (3.6), and $\varphi \in H^0(X, \underline{\mathfrak{p}} \otimes K)$,

$$\mathrm{deg}(\underline{\mathfrak{p}}) \leq 0 \quad (\text{respectively, } \mathrm{deg}(\underline{\mathfrak{p}}) < 0),$$

where $\underline{\mathfrak{p}}$ is the vector bundle associated to \underline{P} defined above.

One has the following (see [5]).

PROPOSITION 3.8. — *Let $(E, \varphi, \tilde{\alpha})$ be a (α, σ, c, \pm) -pseudo-real G -Higgs bundle.*

- (1) *If (E, φ) is semistable (respectively stable), in the sense of Section 2.1, then $(E, \varphi, \tilde{\alpha})$ is semistable (respectively stable).*
- (2) *If $(E, \varphi, \tilde{\alpha})$ is semistable then (E, φ) is semistable.*
- (3) *If $(E, \varphi, \tilde{\alpha})$ is stable then (E, φ) is polystable (in the sense given in Section 2.1).*

To define polystability for a pseudo-real G -Higgs bundle let $\underline{\mathfrak{p}} \subset \text{ad}(E)$ be a parabolic subalgebra bundle such that $\tilde{\alpha}_0(\underline{\mathfrak{p}}) = \underline{\mathfrak{p}}$, where $\tilde{\alpha}_0$ is the involution defined in (3.5).

Let $R_u(\underline{\mathfrak{p}}) \subset \underline{\mathfrak{p}}$ be the holomorphic subbundle over X whose fiber over a point $x \in X$ is the nilpotent radical of the parabolic subalgebra $\underline{\mathfrak{p}}_x$. Therefore, the quotient $\underline{\mathfrak{p}}/R_u(\underline{\mathfrak{p}})$ is a bundle of reductive Lie algebras. Note that $\tilde{\alpha}_0(R_u(\underline{\mathfrak{p}})) = R_u(\underline{\mathfrak{p}})$. A *Levi subalgebra bundle* of $\underline{\mathfrak{p}}$ is a holomorphic subbundle

$$\ell(\underline{\mathfrak{p}}) \subset \underline{\mathfrak{p}}$$

such that for each $x \in X$, the fiber $\ell(\underline{\mathfrak{p}})_x$ is a Lie subalgebra of $\underline{\mathfrak{p}}_x$, and the composition

$$\ell(\underline{\mathfrak{p}}) \hookrightarrow \underline{\mathfrak{p}} \twoheadrightarrow \underline{\mathfrak{p}}/R_u(\underline{\mathfrak{p}})$$

is an isomorphism, where $\underline{\mathfrak{p}} \twoheadrightarrow \underline{\mathfrak{p}}/R_u(\underline{\mathfrak{p}})$ is the quotient map.

A semistable (α, σ, c, \pm) -pseudo-real G -Higgs bundle $(E, \varphi, \tilde{\alpha})$ is *polystable* if either is stable, or there is a proper parabolic subalgebra bundle $\underline{\mathfrak{p}} \subsetneq \text{ad}(E)$, and a Levi subalgebra bundle $\ell(\underline{\mathfrak{p}}) \subset \underline{\mathfrak{p}}$, such that

$$\tilde{\alpha}_0(\underline{\mathfrak{p}}) = \underline{\mathfrak{p}}, \quad \tilde{\alpha}_0(\ell(\underline{\mathfrak{p}})) = \ell(\underline{\mathfrak{p}}), \quad \varphi \in H^0(X, \ell(\underline{\mathfrak{p}}) \otimes K),$$

and for every parabolic subalgebra bundle $\underline{\mathfrak{q}} \subset \ell(\underline{\mathfrak{p}})$ with $\tilde{\alpha}_0(\underline{\mathfrak{q}}) = \underline{\mathfrak{q}}$ we have

$$\deg(\underline{\mathfrak{q}}) < 0.$$

We have the following (see [5]).

PROPOSITION 3.9. — *A (α, σ, c, \pm) -pseudo-real G -Higgs bundle $(E, \varphi, \tilde{\alpha})$ is polystable if and only if (E, φ) is polystable.*

We can thus define the moduli space $\mathcal{M}(G, \alpha, \sigma, c, \pm)$ of isomorphism classes of polystable (α, σ, c, \pm) -pseudo-real G -Higgs bundles, and, as a consequence of Proposition 3.9, define maps

$$(3.7) \quad \mathcal{M}(G, \alpha, \sigma, c, \pm) \longrightarrow \mathcal{M}(G)$$

that forget the pseudo-real structure.

4. INVOLUTIONS OF MODULI SPACES

4.1. INVOLUTIONS OF HIGGS BUNDLE MODULI SPACES. — As before, let $\alpha : X \rightarrow X$ and $\sigma : G \rightarrow G$ be anti-holomorphic involutions. For a holomorphic principal G -bundle E on X , let $\sigma(E)$ be the C^∞ principal G -bundle on X obtained by extending the structure group of E using the homomorphism σ . So the total space of $\sigma(E)$ is identified with that of E , but the action of $g \in G$ on $e \in E$ coincides with the action of $\sigma(g)$ on $e \in \sigma(E)$. Consequently, the pullback $\alpha^*\sigma(E)$ has a holomorphic structure given by the holomorphic structure of E . Let

$$\tilde{\sigma} : E(\mathfrak{g}) \longrightarrow E(\mathfrak{g})$$

be the conjugate linear isomorphism that sends the equivalence class of any $(e, v) \in E \times \mathfrak{g}$ to the equivalence class of $(e, d\sigma(v))$, where $d\sigma$ is the automorphism of \mathfrak{g} corresponding to σ . Let φ be a Higgs field on E . Let $\sigma(\varphi)$ be the C^∞ section of $E(\mathfrak{g}) \otimes K$ defined by $\tilde{\sigma}$ and the C^∞ isomorphism $K \rightarrow \overline{K}$ defined by $df \mapsto d\overline{f}$, where f is any locally defined holomorphic function on X .

We have involutions

$$(4.1) \quad \begin{aligned} \iota(\alpha, \sigma)^\pm : \mathcal{M}(G) &\longrightarrow \mathcal{M}(G) \\ (E, \varphi) &\longmapsto (\alpha^*\sigma(E), \pm\alpha^*\sigma(\varphi)). \end{aligned}$$

PROPOSITION 4.1. — *The image of the map*

$$\mathcal{M}(G, \alpha, \sigma, c, +) \longrightarrow \mathcal{M}(G)$$

in (3.7) is contained in the fixed point locus of the involution $\iota(\alpha, \sigma)^+$. Moreover, the fixed point locus of $\iota(\alpha, \sigma)^+$ in the smooth locus $\mathcal{M}(G)^{\text{sm}} \subset \mathcal{M}(G)$ is the intersection of $\mathcal{M}(G)^{\text{sm}}$ with the union of the images of $\mathcal{M}(G, \alpha, \sigma, c, +)$ as c runs over Z_2^σ , where Z_2^σ as before is the subgroup of Z^σ generated by the order two points.

Similarly, the fixed point locus of $\iota(\alpha, \sigma)^-$ in $\mathcal{M}(G)^{\text{sm}}$ is the intersection of $\mathcal{M}(G)^{\text{sm}}$ with the union of the images of $\mathcal{M}(G, \alpha, \sigma, c, -)$ as c runs over Z_2^σ .

Proof. — From the definition of $\iota(\alpha, \sigma)^+$ (respectively, $\iota(\alpha, \sigma)^-$) it follows immediately that $\mathcal{M}(G, \alpha, \sigma, c, +)$ (respectively, $\mathcal{M}(G, \alpha, \sigma, c, -)$) is contained in the fixed point locus of $\iota(\alpha, \sigma)^+$ (respectively, $\iota(\alpha, \sigma)^-$).

A G -Higgs bundle (E, φ) lies in $\mathcal{M}(G)^{\text{sm}}$ if (E, φ) is stable and the automorphism group of (E, φ) coincides with the center Z of G , i.e., if the Higgs bundle is simple as defined in [14, 13, 6] (we recall that such bundles are called regularly stable). Suppose that $(E, \varphi) \in \mathcal{M}(G)^{\text{sm}}$ is fixed under the involution $\iota(\alpha, \sigma)^+$ (respectively $\iota(\alpha, \sigma)^-$). This means that there exists an isomorphism

$$f : E \longrightarrow \alpha^*\sigma(E)$$

such that $\alpha^*\sigma(f) \circ f \in \text{Aut}(E, \varphi)$, but since $\text{Aut}(E, \varphi) = Z$, we have that $\alpha^*\sigma(f) \circ f = c \in Z$. We can interpret f as a map $f' : E \rightarrow \sigma(E)$ such that $\sigma(f') \circ f' = c \in Z$. Identifying $\sigma(E)$ with E with multiplication on the right defined by $e \cdot g = e\sigma(g)$, where $g \in G$ and $e \in E$, we are indeed defining a $(\alpha, \sigma, c, +)$ (respectively $(\alpha, \sigma, c, -)$) pseudo-real structure on (E, φ) , since we can always assume that $c \in Z_2^\sigma$, as explained

in Remark 3.3. In other words, (E, φ) lies in the image of $\mathcal{M}(G, \alpha, \sigma, c, +)$ (respectively $\mathcal{M}(G, \alpha, \sigma, c, -)$). \square

REMARK 4.2. — In Definition 3.2 we could have defined a pseudo-real structure replacing $\tilde{\alpha}$ by an anti-holomorphic map $\tilde{\alpha}' : E \rightarrow \sigma(E)$ of G -bundles covering α . Although $\sigma(E)$ is no longer a holomorphic bundle, its total space is a complex manifold because it is identified with the total space of E , and hence the anti-holomorphicity condition makes sense. The condition $\tilde{\alpha}(eg) = \tilde{\alpha}(e)\sigma(g)$ in Definition 3.2 is now automatic since $\tilde{\alpha}'$ is a G -bundle map.

PROPOSITION 4.3. — Let σ and σ' be inner equivalent elements in $\text{Conj}(G)$, i.e., they define the same element in $\text{Out}_2(G)$. Then

$$\iota(\alpha, \sigma)^+ = \iota(\alpha, \sigma')^+ \text{ (respectively, } \iota(\alpha, \sigma)^- = \iota(\alpha, \sigma')^-).$$

Proof. — If we replace σ by $\sigma' := g_0\sigma g_0^{-1}$, where $g_0 \in G$, then the corresponding anti-holomorphic involution of the moduli space is replaced by its composition with the holomorphic automorphism of the moduli space corresponding to the automorphism of G defined by $g \mapsto g_0 g g_0^{-1}$. But this automorphism of G produces the identity map of the moduli space. Therefore, the anti-holomorphic involution of the moduli space is unchanged if σ is replaced by σ' . \square

REMARK 4.4. — Consider the identification between the (α, σ, c) -pseudo-real principal G -bundles and the (α, σ', c') -pseudo-real principal G -bundles on X given by Proposition 3.6 when σ and σ' are inner equivalent. Note that a Higgs field on a (α, σ', c') -pseudo-real principal G -bundle produces a Higgs field on the corresponding (α, σ, c) -pseudo-real principal G -bundle, and vice versa. We thus have that by Proposition 3.6 $\mathcal{M}(G, \alpha, \sigma, c, +)$ is isomorphic to $\mathcal{M}(G, \alpha, \sigma', c', +)$ (respectively $\mathcal{M}(G, \alpha, \sigma, c, -)$ is isomorphic to $\mathcal{M}(G, \alpha, \sigma', c', -)$) giving the same image under the corresponding maps to $\mathcal{M}(G)$.

4.2. CORRESPONDENCE WITH REPRESENTATIONS FOR $\iota(\alpha, \sigma)^+$. — We have the orbifold fundamental group of (X, α) that we will denote $\Gamma(X, \alpha)$ (see [5] for example). This fits into an exact sequence

$$(4.2) \quad 1 \longrightarrow \pi_1(X, x_0) \longrightarrow \Gamma(X, x_0) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

Let $\text{Map}'(\Gamma(X, x_0), G \times (\mathbb{Z}/2\mathbb{Z}))$ be the space of all maps

$$\delta : \Gamma(X, x_0) \longrightarrow G \times (\mathbb{Z}/2\mathbb{Z})$$

such that the following diagram is commutative:

$$(4.3) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(X, x_0) & \longrightarrow & \Gamma(X, x_0) & \xrightarrow{\eta} & \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \\ & & \downarrow & & \downarrow \delta & & \parallel \\ 1 & \longrightarrow & G & \longrightarrow & G \times (\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \longrightarrow 1. \end{array}$$

Take an element $c \in Z_2^\sigma$ in the subgroup generated by the elements of Z^σ order two. Using it, we will define another group structure on $G \times (\mathbb{Z}/2\mathbb{Z})$. The group operation is given by

$$(g_1, e_1) \cdot (g_2, e_2) = (g_1(\sigma)^{e_1}(g_2)c^{e_1e_2}, e_1 + e_2).$$

Note that when $c = 1$ we obtain a semidirect product.

Let $\text{Hom}_c(\Gamma(X, x_0), G \times (\mathbb{Z}/2\mathbb{Z}))$ be the space of all maps

$$\delta \in \text{Map}'(\Gamma(X, x_0), G \times (\mathbb{Z}/2\mathbb{Z}))$$

such that δ is a homomorphism with respect to this group structure.

Two elements $\delta, \delta' \in \text{Hom}_c(\Gamma(X, x_0), G \times (\mathbb{Z}/2\mathbb{Z}))$ are called *equivalent* if there is an element $g \in G$ such that $\delta'(z) = g^{-1}\delta(z)g$ for all $z \in \pi_1(X, \alpha)$.

THEOREM 4.5. — *The moduli space $\mathcal{M}(G, \alpha, \sigma, c, +)$ is identified with the space of equivalence classes of reductive elements of $\text{Hom}_c(\Gamma(X, x_0), G \times (\mathbb{Z}/2\mathbb{Z}))$.*

Proof. — This follows from Proposition 5.6 of [5]. □

THEOREM 4.6. — *Consider the involution $\iota(\alpha, \sigma)^+$ of $\mathcal{M}(G)$. It is anti-holomorphic with respect to the almost complex structures J_1 and J_2 , and it is holomorphic with respect to J_3 .*

Proof. — The almost complex structure J_1 is the almost complex structure of the Dolbeault moduli space (the moduli space of Higgs bundles). Therefore, $\iota(\alpha, \sigma)^+$ is anti-holomorphic with respect to J_1 .

The almost complex structure J_2 is the almost complex structure of the Betti moduli space (the representation space $\mathcal{R}(G)$). Note that the almost complex structure of the Betti moduli space coincides with that of the de Rham moduli space.

As before, fix a base point $x_0 \in X$. The involution α of X produces an isomorphism

$$\alpha' : \pi_1(X, x_0) \longrightarrow \pi_1(X, \alpha(x_0)).$$

This in turn gives a biholomorphism

$$\alpha'' : \text{Hom}^+(\pi_1(X, x_0), G)/G \longrightarrow \text{Hom}^+(\pi_1(X, \alpha(x_0)), G)/G.$$

As noted before, $\mathcal{R}(G) = \text{Hom}^+(\pi_1(X, x_0), G)/G$ is independent of the choice of the base point. So α'' is a biholomorphism

$$(4.4) \quad \alpha'' : \mathcal{R}(G) \longrightarrow \mathcal{R}(G).$$

Since α is an involution, it follows that α'' is also an involution.

Let

$$b : \mathcal{R}(G) = \text{Hom}^+(\pi_1(X, x_0), G)/G \longrightarrow \mathcal{R}(G)$$

be the anti-holomorphic involution defined by $\rho \mapsto \sigma \circ \rho$. In other words, b sends a homomorphism $\rho : \pi_1(X) \rightarrow G$ to the composition

$$\pi_1(X, x_0) \xrightarrow{\rho} G \xrightarrow{\sigma} G.$$

Clearly b commutes with the above involution α'' in (4.4). Therefore, $b \circ \alpha''$ is also an involution. Note that $b \circ \alpha''$ is anti-holomorphic because α'' is holomorphic and b is anti-holomorphic.

The above involution $b \circ \alpha''$ of $\mathcal{R}(G)$ coincides with the involution $\iota(\alpha, \sigma)^+$ of $\mathcal{M}(G)$ under the correspondence $\mathcal{M}(G) \cong \mathcal{R}(G)$. Therefore, $\iota(\alpha, \sigma)^+$ is anti-holomorphic with respect to J_2 .

We recall that $J_3 = J_1 J_2$. Since $\iota(\alpha, \sigma)^+$ is anti-holomorphic with respect to both J_1 and J_2 , from the above identity it follows immediately that $\iota(\alpha, \sigma)^+$ is holomorphic with respect to J_3 . \square

Since $\mathcal{R}(G)$ is hyper-Kähler, the holomorphic symplectic form Ω_2 on it is flat with respect to the Kähler structure ω_2 corresponding to J_2 . Similarly, the holomorphic symplectic form Ω_1 with respect to J_1 is flat with respect to the Kähler structure ω_1 corresponding to J_1 . In particular, $\mathcal{R}(G)$ and $(\mathcal{M}(G), J_1, \omega_1, \Omega_1)$ are Calabi-Yau.

THEOREM 4.7. — *The moduli space $\mathcal{M}(G, \alpha, \sigma, c, +)$ is a special Lagrangian subspace of $\mathcal{R}(G)$. Similarly, it is special Lagrangian with respect to $(\mathcal{M}(G), J_1, \omega_1, \Omega_1)$. Also, it is complex Lagrangian with respect to (J_3, Ω_3) .*

Proof. — Since the involution $\iota(\alpha, \sigma)^+$ is holomorphic with respect to J_3 , it follows that $\mathcal{M}(G, \alpha, \sigma, c, +)$ is a holomorphic subspace with respect to J_3 . Recall that $\Omega_3 = \omega_1 + \sqrt{-1}\omega_2$. The involution $\iota(\alpha, \sigma)^+$ is anti-holomorphic with respect to J_1 and J_2 and it is an isometry. Hence $\iota(\alpha, \sigma)^+$ takes ω_1 and ω_2 to $-\omega_1$ and $-\omega_2$ respectively. Hence $\iota(\alpha, \sigma)^+$ takes Ω_3 to $-\Omega_3$. This immediately implies that $\mathcal{M}(G, \alpha, \sigma, c, +)$ is Lagrangian with respect to Ω_3 .

Since $\mathcal{M}(G, \alpha, \sigma, c, +)$ is the fixed point locus of an isometric anti-holomorphic involution of the Calabi-Yau space $\mathcal{R}(G)$, it follows that $\mathcal{M}(G, \alpha, \sigma, c, +)$ is a special Lagrangian subspace of $\mathcal{R}(G)$. For a similar reason, $\mathcal{M}(G, \alpha, \sigma, c, +)$ is a special Lagrangian subspace of $(\mathcal{M}(G), J_1, \omega_1)$. \square

4.3. CORRESPONDENCE WITH REPRESENTATIONS FOR $\iota(\alpha, \sigma)^-$. — Next we consider the involution $\iota(\alpha, \sigma)^-$.

Consider the holomorphic involution $\theta = \sigma\tau$ of G as defined earlier in Section 3.1. Using $c \in Z_2^\sigma$, we will define yet another group structure on $G \times (\mathbb{Z}/2\mathbb{Z})$. The group operation is given by

$$(g_1, e_1) \cdot (g_2, e_2) = (g_1(\theta)^{e_1}(g_2)c^{e_1 e_2}, e_1 + e_2).$$

Let $\text{Hom}_c^-(\pi_1(X, \alpha), G \times (\mathbb{Z}/2\mathbb{Z}))$ be the space of all maps

$$\delta \in \text{Map}'(\pi_1(X, \alpha), G \times (\mathbb{Z}/2\mathbb{Z}))$$

such that δ is a homomorphism with respect to this new group structure.

Two elements $\delta', \delta'' \in \text{Hom}_c^-(\pi_1(X, \alpha), G \times (\mathbb{Z}/2\mathbb{Z}))$ are called *equivalent* if there is an element $g \in G$ such that $\delta'(z) = g^{-1}\delta''(z)g$ for all $z \in \pi_1(X, \alpha)$.

THEOREM 4.8. — *The moduli space $\mathcal{M}(G, \alpha, \sigma, c, -)$ is identified with the space of equivalence classes of reductive elements of $\text{Hom}_c^-(\pi_1(X, \alpha), G \times (\mathbb{Z}/2\mathbb{Z}))$.*

Proof. — This follows from Proposition 5.6 of [5]. \square

THEOREM 4.9. — *Consider the involution $\iota(\alpha, \sigma)^-$ of $\mathcal{M}(G)$. It is anti-holomorphic with respect to the almost complex structures J_1 and J_3 , and it is holomorphic with respect to J_2 .*

Proof. — The involution $\iota(\alpha, \sigma)^-$ is clearly anti-holomorphic with respect to J_1 because J_1 coincides with the complex structure of the Dolbeault moduli space.

Let

$$\tilde{b} : \mathcal{R}(G) = \text{Hom}^+(\pi_1(X, x_0), G)/G \longrightarrow \mathcal{R}(G)$$

be the holomorphic involution defined by $\rho \mapsto \theta \circ \rho$. In other words, \tilde{b} sends a homomorphism $\rho : \pi_1(X) \rightarrow G$ to the composition

$$\pi_1(X) \xrightarrow{\rho} G \xrightarrow{\theta} G.$$

Clearly \tilde{b} commutes with the above involution α'' in (4.4). Therefore, $\tilde{b} \circ \alpha''$ is also an involution. The composition $\tilde{b} \circ \alpha''$ is holomorphic because both α'' and \tilde{b} are holomorphic.

The above involution $\tilde{b} \circ \alpha''$ of $\mathcal{R}(G)$ coincides with $\iota(\alpha, \sigma)^-$, and the complex structure of the Betti moduli space $\mathcal{R}(G)$ is given by J_2 . Therefore, $\iota(\alpha, \sigma)^-$ is holomorphic with respect to J_2 .

Since $J_3 = J_1 J_2$, and $\iota(\alpha, \sigma)^-$ is anti-holomorphic with respect to J_1 and holomorphic with respect to J_2 , we conclude that $\iota(\alpha, \sigma)^-$ is anti-holomorphic with respect to J_3 . \square

Consider the complex structure J_3 and the corresponding holomorphic symplectic form Ω_3 . Since $\mathcal{R}(G)$ is hyper-Kähler, Ω_3 is flat with respect to the Kähler structure for J_3 . Now we have following analog of Theorem 4.7.

THEOREM 4.10. — *The moduli space $\mathcal{M}(G, \alpha, \sigma, c, -)$ is a special Lagrangian subspace of $(\mathcal{M}(G), J_1, \omega_1, \Omega_1)$. Similarly, it is special Lagrangian with respect to $(\mathcal{M}(G), J_3, \omega_3, \Omega_3)$. Also, it is a complex Lagrangian subspace with respect to $(\mathcal{R}(G), J_2, \Omega_2)$.*

COROLLARY 4.11. — *The fixed point locus of the involution $\iota(\alpha, \sigma)^-$ is a complex subspace of $\mathcal{M}(G)$ with the complex structure induced by J_2 , i.e., the natural complex structure of the moduli space of representations $\mathcal{R}(G)$.*

REMARK 4.12. — Corollary 4.11 is obtained by Baraglia–Schaposnik [3] in the case when σ is the compact conjugation τ .

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