Sandra Rozensztajn
Potentially semi-stable deformation rings for discrete series extended types

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POTENTIALLY SEMI-STABLE DEFORMATION RINGS
FOR DISCRETE SERIES EXTENDED TYPES

by Sandra Rozensztajn

Abstract. — We define deformation rings for potentially semi-stable deformations of fixed
discrete series extended type in dimension 2. In the case of representations of the Galois group
of $\mathbb{Q}_p$, we prove an analogue of the Breuil-Mézard conjecture for these rings. As an application,
we give some results on the existence of congruences modulo $p$ for newforms in $S_k(\Gamma_0(p))$.

Résumé (Anneaux de déformations potentiellement semi-stables pour les types étendus de la
série discrète)

Nous définissons des anneaux de déformations pour les déformations potentiellement semi-
stables ayant un type étendu de la série discrète fixé en dimension 2. Dans le cas des représenta-
tions du groupe de Galois de $\mathbb{Q}_p$, nous prouvons un analogue de la conjecture de Breuil-Mézard
pour ces anneaux. Nous donnons comme application de ceci des résultats sur l’existence de
congruences modulo $p$ pour les formes nouvelles dans $S_k(\Gamma_0(p))$.

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1. Introduction

Let $p > 2$ be a prime number, $K$ a finite extension of $\mathbb{Q}_p$ with absolute Galois
group $G_K$. Let $\overline{\rho}$ be a continuous representation of $G_K$ of dimension 2 with coef-
ficients in some finite field $F$ of characteristic $p$. Let $E$ be a finite extension of $\mathbb{Q}_p$
with residue field containing $F$. There exists an $\mathcal{O}_E$-algebra $R^\square(\overline{\rho})$ parametrizing the
framed deformations of $\overline{\rho}$ to $\mathcal{O}_E$-algebras. Kisin ([Kis08]) has shown that this ring has
quotients $R^{\mathbb{Q},\psi}(w, t, \mathfrak{p})$ that parametrize framed deformations $\rho$ that are potentially semi-stable of given determinant (encoded by $\psi$), fixed Hodge-Tate weights (encoded by the Hodge-Tate type $w$) and fixed inertial type $t$ (that is, the restriction to inertia of the Weil-Deligne representation $\text{WD}(\rho)$ associated to $\rho$ is isomorphic to a fixed smooth representation $t$). We are interested in a variant of this situation: instead of considering deformations with a fixed inertial type $t$, we consider deformations with a fixed extended type $t'$, that is, such that the restriction to the Weil group of $\text{WD}(\rho)$ is isomorphic to $t'$, in the case when $t'$ is a discrete series type (see the definition in Section 2.1). This problem was first considered in [BCDT01], in order to isolate some irreducible components of the deformation space parametrizing deformations in Section 2.1). This problem was first considered in [BCDT01], in order to isolate some irreducible components of the deformation space parametrizing deformations with fixed inertial type. For a discrete series inertial type $t$, we show that the ring $R^{\mathbb{Q},\psi}(w, t', \mathfrak{p})$ parametrizing deformations with fixed discrete series extended type $t'$ extending $t$ is the maximal reduced quotient of $R^{\mathbb{Q},\psi}(w, t, \mathfrak{p})$ supported in some set of irreducible components of $\text{Spec} R^{\mathbb{Q},\psi}(w, t, \mathfrak{p})$. More precisely, depending on $t$, adding the extra data of a $t'$ either does not give any additional information, or divides the set of irreducible components in two parts.

Some important information about the geometry of the rings $R^{\mathbb{Q},\psi}(w, t, \mathfrak{p})$ is given by the Breuil-Mézard conjecture ([BM02], proved for $K = \mathbb{Q}_p$ by Kisin [Kis09a] and Paškūnas [Paš15]) that relates the Hilbert-Samuel multiplicity of the special fiber of the ring to an automorphic multiplicity, computed in terms of smooth representations modulo $p$ of $\text{GL}_2(\mathcal{O}_K)$ attached to $w$ and $t$. Our main result is that when $K = \mathbb{Q}_p$ there is a similar formula for the Hilbert-Samuel multiplicity of the special fiber of $R^{\mathbb{Q},\psi}(w, t', \mathfrak{p})$ for a discrete series extended type $t'$. More precisely, Gee and Geraghty have shown in [GG15] that for discrete series inertial types $t$, the Breuil-Mézard conjecture can be reformulated using an automorphic multiplicity expressed in terms of representations not of $\text{GL}_2(\mathcal{O}_K)$, but of $\mathcal{O}_D^\vee$, where $\mathcal{O}_D$ is the ring of integers of the non-split quaternion algebra $D$ over $K$. The formula we give for the multiplicity of the special fiber of $R^{\mathbb{Q},\psi}(w, t', \mathfrak{p})$ is in terms of representations of a quotient $\mathcal{G}$ of $D^\times$ containing $\mathcal{O}_D^\vee$ as a subgroup of index 2. Using the local Langlands correspondence and the Jacquet-Langlands correspondence, we construct for each discrete series inertial type $t$ a smooth representation $\sigma_\mathcal{G}(t)$ of $\mathcal{G}$ with coefficients in $\overline{\mathbb{Q}}_p$ (or a pair of such representations, depending on the inertial type $t$). To a Hodge-Tate type $w$ we attach a representation $\sigma_w$ of $\mathcal{G}$ coming from an algebraic representation of $\text{GL}_2$ with highest weight given by $w$. The Hilbert-Samuel multiplicity is then given in terms of the multiplicity of the irreducible constituents of the reduction modulo $p$ of $\sigma_\mathcal{G}(t) \otimes \sigma_w$, seen as representations of a finite group $\Gamma$ through which all semi-simple representations modulo $p$ of $\mathcal{G}$ factor. For $K = \mathbb{Q}_p$ we have the following theorem (see Theorem 3.5.1 for a more precise statement).

**Theorem.** — Let $\mathfrak{p}$ be a continuous representation of $G_{\mathbb{Q}_p}$ of dimension 2 with coefficients in $\mathbb{F}_p$. There exists a positive linear form $\mu_\mathfrak{p}$ on the Grothendieck ring of representations of $\Gamma$ with values in $\mathbb{Z}$ satisfying the following property: for any discrete series inertial type $t$, Hodge-Tate type $w$, character $\psi$ lifting $\omega^{-1} \det \mathfrak{p}$, and extended...
type \( t' \) compatible with \((t, \psi)\), there exists a choice of representation \( \sigma_G(t) \) of \( G \) such that we have:

\[
e(\Omega(w, t', \rho)/\pi) = \mu_{\rho}(\sigma_G(t) \otimes \sigma_w).
\]

We deduce our result from the reformulation by [GG15] of the usual Breuil-Mézard conjecture, making use of modularity lifting theorems for modular forms on a quaternion algebra ramified at infinity and at primes dividing \( p \).

One consequence of this formula is Corollary 3.5.9: except when \( \rho \) has some very specific form, then when the addition of the data of the extended types divides the deformation ring in two parts, these parts have the same multiplicity. This is to be expected when \( \rho \) is irreducible, as in this case it can easily be seen that the deformation rings corresponding to the two extended types are in fact isomorphic (see Remark 2.3.5). But this is much more surprising when \( \rho \) is reducible, as in this case there does not seem to be a natural way to relate the deformation rings corresponding to the two extended types.

We give a concrete application of our result to the existence of congruences modulo \( p \) for some modular forms. When \( t \) is trivial, the ring \( \Omega(w, t', \rho) \) classifies semi-stable representations, and the extra data given by the extended type is the eigenvalues of the Frobenius of the associated filtered \((\phi, N)\)-module when the representation is not crystalline (there are only two possibilities for these eigenvalues if the determinant is fixed). If \( f \in S_k(\Gamma_0(p)) \) is a newform, this means that the extended type of \( \rho_{f,p}|_{G_{\mathbb{Q}_p}} \) gives the value of the coefficient \( a_p(f) = \pm p^{k/2-1} \). We give in Theorem 6.2.1 a criterion for the existence of a newform in \( S_k(\Gamma_0(p)) \) that is congruent to \( f \) modulo \( p \) but with the opposite value for \( a_p \).

1.1. Plan of the article. — We define the deformation rings \( \Omega(w, t', \rho) \) for discrete series extended types in Section 2. In Section 3 we introduce the groups and representations that play a role in the automorphic side for the formula for the Hilbert-Samuel multiplicity of the special fiber of the rings \( \Omega(w, t', \rho) \) and state our main theorems. We give in Section 4 some results about modular forms for quaternion algebras ramified at infinity and at primes dividing \( p \) that we need in Section 5, where we prove the theorems. Section 6 is devoted to the application to modular forms.

1.2. Notation. — We fix a prime number \( p > 2 \). We denote by \( K \) a finite extension of \( \mathbb{Q}_p \), and by \( q \) the cardinality of its residue field. Let \( G_K \) be the absolute Galois group of \( K \), \( I_K \) its inertia subgroup and \( W_K \) its Weil group. We denote by \( \varepsilon \) the \( p \)-adic cyclotomic character and \( \omega \) its reduction modulo \( p \). We normalize the Artin map of local class field theory \( \text{Art}_K : K^\times \to W_K^{ab} \) so that geometric Frobenius elements correspond to uniformizers. We denote by \( \text{unr}(a) \) the unramified character of \( W_K \) (or \( G_K \)) sending a geometric Frobenius to \( a \), and also the unramified character of \( K^\times \) sending a uniformizer to \( a \). We denote by \( \| \cdot \| \) the norm on \( W_K \), that is, the character \( \text{unr}(q^{-1}) \).
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2. Discrete series extended types and deformation rings

2.1. Extended types and Weil-Deligne representations. — An inertial type is a smooth representation \( t \) of \( I_K \) over \( \mathbb{Q}_p \) that extends to a representation of \( W_K \). We define an extended type to be a smooth representation of \( W_K \) over \( \mathbb{Q}_p \).

We recall the following well-known classification for inertial types and extended types in dimension 2 when \( p > 2 \) (see for example [Ima11, Lem.2.1] for a proof).

\[ \text{Lemma 2.1.1.} \quad \text{Let} \quad t \quad \text{be an extended type of degree 2. Then we are in exactly one of the following situations:} \]

- (scal) \( t|_{I_K} \) is scalar: there exist two smooth characters \( \chi, \chi' \) of \( W_K \) such that \( \chi|_{I_K} = \chi'|_{I_K} \) and \( t = \chi \oplus \chi' \).
- (char) There exist two smooth characters \( \chi_1, \chi_2 \) of \( W_K \) with distinct restrictions to \( I_K \) such that \( t = \chi_1 \oplus \chi_2 \).
- (red) Let \( K' \) be the unramified quadratic extension of \( K \). There exists a smooth character \( \chi \) of \( W_{K'} \) that does not extend to a character of \( W_K \) such that \( t = \text{ind}_{W_{K'}}^{W_K} \chi \). In this case \( t|_{I_K} \) is reducible and is a sum of characters that do not extend to \( W_K \).
- (irr) There exist a ramified quadratic extension \( L \) of \( K \) and a smooth character \( \chi \) of \( W_L \) that does not extend to a character of \( W_K \) such that \( t = \text{ind}_{W_L}^{W_K} \chi \). In this case \( t|_{I_K} \) is irreducible.

We call the inertial types corresponding to situation (scal), (red) or (irr) discrete series inertial types. We call the extended types corresponding to situation (scal) or (irr), or to situation (scal) with \( \chi' = \chi \otimes \| \cdot \|^{1/2} \) discrete series extended types.

The following Proposition is an immediate consequence of the classification.

\[ \text{Proposition 2.1.2.} \quad \text{Let} \quad t_1 \quad \text{and} \quad t_2 \quad \text{be two discrete series extended types with isomorphic restrictions to} \quad I_K. \quad \text{Then they differ by a twist by an unramified character.} \]

Let \( t \) be a discrete series extended type. If it is of the form (scal) or (irr) then \( t \) is not isomorphic to \( t \otimes \text{unr}(-1) \). If \( t \) is of the form (red) then \( t \) is isomorphic to \( t \otimes \text{unr}(-1) \).

Let \( t \) be a discrete series extended type. We call conjugate type of \( t \) the type \( t \otimes \text{unr}(-1) \). Two types with isomorphic restriction to \( I_K \) are conjugate if and only if they have the same determinant. When \( t \) is of the form (scal) or (irr), two conjugate extended types are distinct, but they are isomorphic when \( t \) is of the form (red). Let \( (r,N) \) be a Weil-Deligne representation of dimension 2, that is, a two-dimensional smooth representation \( r \) of the Weil group \( W_K \) and a nilpotent endomorphism \( N \) such that \( Nr(x) = \|x\|^{-1} r(x) N \) for any \( x \in W_K \). Let \( t \) be an inertial
type; we say that \((r, N)\) is of inertial type \(t\) if \(r|_{I_K}\) is isomorphic to \(t\). Let \(t'\) be an extended type; we say that \((r, N)\) is of extended type \(t'\) if \(r\) is isomorphic to \(t'\).

We say that \((r, N)\) is a discrete series Weil-Deligne representation if either \(r|_{I_K}\) is of the form \((\text{scal})\) and \(N \neq 0\) or \(r|_{I_K}\) is of the form \((\text{red})\) or \((\text{irr})\) (note that we can have \(N \neq 0\) only when \(r|_{I_K}\) is of the form \((\text{scal})\) and \(r\) is a twist of \(1 \oplus \| \cdot \|)\). With this definition, discrete series inertial (resp. extended) types are exactly the restriction to \(I_K\) (resp. \(W_K\)) of discrete series Weil-Deligne representations. See Section 3.1 for a justification of this terminology.

2.2. Potentially semi-stable representations and discrete series extended types

2.2.1. Filtered \((\phi, N)\)-modules with descent data. — Let \(F\) be a finite extension of \(\mathbb{Q}_p\), \(F_0\) the maximal unramified extension of \(\mathbb{Q}_p\) contained in \(F\). Let \(E\) be a finite extension of \(\mathbb{Q}_p\) (the coefficient field), that we suppose large enough.

A filtered \((\phi, N, F, E)\)-module is a free \(F_0 \otimes_{\mathbb{Q}_p} E\)-module \(D\) of finite rank, endowed with a \(F_0\)-semi-linear, \(E\)-linear endomorphism \(\phi\) and a \(F_0 \otimes E\) linear endomorphism \(N\) satisfying the commutation relation \(N\phi = p\phi N\), with \(N\) nilpotent, \(\phi\) an automorphism, and a decreasing filtration of \(F \otimes_{F_0} D\) by \(F \otimes_{\mathbb{Q}_p} E\)-submodules \(\text{Fil}^i(F \otimes_{F_0} D)\) such that \(\text{Fil}^i(F \otimes_{F_0} D) = F \otimes_{F_0} D\) when \(i\) is small enough and \(\text{Fil}^i(F \otimes_{F_0} D) = 0\) when \(i\) is large enough. We can define an admissibility condition for filtered \((\phi, N, F, E)\)-modules, we refer to [Fon94b] for the definition.

Let \(\rho : G_F \rightarrow \text{GL}(V)\) be a continuous representation, where \(V\) is a finite-dimensional \(E\)-vector space. If \(\rho\) is semi-stable, we can attach to it an admissible filtered \((\phi, N, F, E)\)-module by taking \(D_{\text{st}}(V) = (B_{\text{st}} \otimes_{\mathbb{Q}_p} V)^{G_F}\). The functor \(V \mapsto D_{\text{st}}(V)\) gives an equivalence of categories between the category of semi-stable representations of \(G_F\) and the category of admissible filtered \((\phi, N, F, E)\)-modules which preserves dimension, and the Hodge-Tate weights of \(\rho\) are the indices \(i\) with \(\text{Fil}^i(F \otimes_{F_0} D) \neq \text{Fil}^{i+1}(F \otimes_{F_0} D)\) (so that \(\varepsilon\) has its Hodge-Tate weights equal to 1).

Suppose now that we have \(\rho : G_K \rightarrow \text{GL}(V)\) a continuous representation such that \(\rho\) becomes semi-stable on a finite Galois extension \(F\) of \(K\). Then we can attach to it an admissible filtered \((\phi, N, F/K, E)\)-module, that is, an admissible filtered \((\phi, N, F, E)\)-module with descent data given by an action of \(\text{Gal}(F/K)\) which is \(F_0\)-semi-linear and \(E\)-linear and commutes with \(\phi\) and \(N\). The filtered \((\phi, N, F, E)\)-module is \(D_{\text{st}}^F(V)\), that is, \(D_{\text{st}}(V|_{G_F})\). This gives an equivalence of categories between the category of representations of \(G_K\) that become semi-stable over \(F\) and the category of admissible filtered \((\phi, N, F/K, E)\)-modules.

2.2.2. Weil-Deligne representation attached to a Galois representation

Let \(\rho : G_K \rightarrow \text{GL}(V)\) be a continuous representation of \(G_K\), where \(V\) is a finite-dimensional vector space over a finite extension \(E\) of \(\mathbb{Q}_p\). If \(\rho\) is potentially semi-stable, we attach to its filtered \((\phi, N, F/K, E)\)-module a Weil-Deligne representation \(WD(\rho)\) as in [Fon94a] (see also [CDT99, App.B] for more detailed explanations of the construction and its properties). It does not depend on the field \(F\) over which \(K\)
becomes semi-stable, and moreover it does not depend on the filtration but only on \( \phi \), \( N \) and the action of \( \text{Gal}(F/K) \).

We say that \( \rho \) is of inertial type \( t \) if \( \text{WD}(\rho) \) is of type \( t \), and of extended type \( t' \) if \( \text{WD}(\rho) \) is. Note that \( \text{WD}(\rho) \) is of scalar inertial type if and only if \( \rho \) is semi-stable up to twist, and in this case \( N \neq 0 \) if and only if \( \rho \) is semi-stable but not crystalline up to twist.

2.3. **Deformation rings.** — In this section we fix a discrete series inertial type \( t \). Note that the notions of this Section will be interesting only when \( t \) is of the form (scal) or (irr) as we explain later.

Let \( \overline{\rho} \) be a continuous representation of \( G_K \) of dimension 2 over a finite field \( \mathbb{F} \) of characteristic \( p \). Let \( E \) be a finite extension of \( \mathbb{Q}_p \) with residue field containing \( F \). We denote by \( R^{\square, \psi}(\overline{\rho}) \) the universal framed deformation \( \mathcal{O}_E \)-algebra of \( \overline{\rho} \).

2.3.1. **Deformation rings of fixed inertial type**

Let \( w = (m_r, m_\tau) \in (\mathbb{Z}_{\geq 0} \times \mathbb{Z})^{\text{Hom}(K, \overline{\mathbb{F}}_p)} \) be a Hodge-Tate type, \( t \) be an inertial type, and \( \psi \) a character of \( G_K \). We are interested in lifts \( \rho \) of \( \overline{\rho} \) that are potentially semi-stable, with determinant \( \psi \mathbb{e} \), Hodge-Tate weights \((m_r, m_\tau + n_\tau + 1)_\tau \) (we then say that \( \rho \) has Hodge-Tate type \( w \)), and inertial type \( t \).

In [Kis08], Kisin shows that, after possibly enlarging \( E \), there exists a quotient \( R^{\square, \psi}(w, t, \overline{\rho}) \) of \( R^{\square}(\overline{\rho}) \) that has the following properties.

**Theorem 2.3.1**

1. \( R^{\square, \psi}(w, t, \overline{\rho}) \) is \( p \)-torsion free, \( R^{\square, \psi}(w, t, \overline{\rho})[1/p] \) is reduced and equidimensional.

2. For any finite extension \( E'/E \), a map \( x : R^{\square}(\overline{\rho}) \to E' \) factors through \( R^{\square, \psi}(w, t, \overline{\rho}) \) if and only if the representation \( \rho_x \) is of determinant \( \psi \mathbb{e} \), potentially semi-stable of Hodge-Tate type \( w \), and of inertial type \( t \).

**Remark 2.3.2.** — The ring \( R^{\square, \psi}(w, t, \overline{\rho}) \) can be non-zero only if \( w, \psi \) and \( t \) satisfy the equality: \( \psi|_{I_K} = (\det t) \prod \mathbb{e}^{n_\tau + 2m_\tau} \), where \( \mathbb{e}_\tau \) is the Lubin-Tate character corresponding to \( \tau \), so that \( \mathbb{e} = \prod \mathbb{e}_\tau \). In this case we say that \( \psi \) is compatible with \( t \) and \( w \). Note that by [EG14, Lem. 4.3.1] the isomorphism class of \( R^{\square, \psi}(w, t, \overline{\rho}) \) does not depend on \( \psi \) as long as it is compatible with \( t \) and \( w \).

2.3.2. **Irreducible components and extended types.** — Suppose that \( t \) is a discrete series inertial type, and let \( t' \) be an extended type such that \( t'|_{I_K} \) is isomorphic to \( t \). We define a subset of the set of irreducible components of \( \text{Spec} R^{\square, \psi}(w, t, \overline{\rho}) \) by saying that an irreducible component is of type \( t' \) if:

1. when \( t \) is of the form (red) or (irr), the irreducible component has an \( E' \)-point \( x \) with \( \rho_x \) of extended type \( t' \) for some finite extension \( E'/E \);  
2. when \( t \) is of the form (scal), the irreducible component has an \( E' \)-point \( x \) with \( \rho_x \) of extended type \( t' \) that is not potentially crystalline.
There can exist a component of type $t'$ only if $\det t' = \text{WD}(\psi \varepsilon)$, hence there are at most two such extended types and then they are conjugate if $t$ is of the form (scal) or (irr), and at most one such extended type if $t$ is of the form (red). We say that $t'$ is compatible with $(t, \psi)$ if $t'|_{I_K}$ is isomorphic to $t$ and $\det t' = \text{WD}(\psi \varepsilon)$. If $t$ is of the form (red) and $t'$ is compatible to $(t, \psi)$ then all irreducible components of $\text{Spec} \, R^\wedge(w, t, \pi)$ are of type $t'$.

If a component is of type $t'$, then for all closed points $x$ over a finite extension $E'/E$, the representation $\rho_x$ is of type $t'$. In particular a component is of at most one extended type. This follows from the next proposition.

**Proposition 2.3.3.** — Let $\mathcal{O}$ be an affinoid algebra which is a domain. Let $\rho : G_K \to \text{GL}_2(\mathcal{O})$ be a continuous $\mathcal{O}$-linear representation such that for any closed point $x \in \text{Max}(\mathcal{O})$, the representation $\rho_x$ is potentially semi-stable, with the same discrete series inertial type $t$, Hodge-Tate weights and determinant for all $x$. If $t$ is scalar and at least one representation in the family is not potentially crystalline, or if $t$ is not scalar, then the extended type is constant in the family.

**Proof.** — Let $F$ be a finite extension of $K$ such that $\rho_x|_{G_F}$ is semi-stable for all $x$. Such an $F$ exists and is determined by $t$. Using the results of [BC08, §6.3] we can then construct a free $F_0 \otimes \mathcal{O}$-module $D_{st}(\rho)$ with a Frobenius $\phi$, a monodromy operator $N$ and an action of $\text{Gal}(F/K)$ that are $\mathcal{O}$-linear and such that for all $x$, $\mathcal{O}/\mathfrak{m}_x \otimes \mathcal{O} D_{st}(\rho)$ is isomorphic to $D^\text{st}_x(\rho_x)$. We can apply the method of the construction of the Weil-Deligne representation as given in [CDT99, App. B] to $D_{st}(\rho)$. This gives a continuous representation $r : W_K \to \text{GL}_2(\mathcal{O})$ and $N \in M_2(\mathcal{O})$ such that for all $x$, $(r_x, N_x)$ is the Weil-Deligne representation attached to $\rho_x$. By assumption all representations $r_x$ have the same restriction to inertia and the same determinant. If $t$ is of the form (red), this implies that the isomorphism class of $r_x$ is constant and hence the extended type is constant in the family.

Suppose now that $t$ is of the form (irr), that is, $t = (\text{ind}^{I_K}_{I_L} \chi)|_{I_K}$ for some ramified quadratic extension $L$ of $K$ and some character $\chi$ of $W_L$ that does not extend to $W_K$. Then $t|_{I_L} = \chi \otimes \chi'$ for characters $\chi, \chi'$ that do not extend to $I_K$. Fix also an element $\alpha \in I_K \setminus I_L$. We can choose a basis $(e_1, e_2)$ of $\mathcal{O}^2$ (after possibly replacing $\text{Max}(\mathcal{O})$ by some admissible covering) such that for all $x$, we have $r_x(\beta)e_1 = \chi(\beta)e_1$ and $r_x(\beta)e_2 = \chi'(\beta)e_2$ for all $\beta \in I_L$, and $r_x(\alpha)e_1 = e_2$. Let Frob be any Frobenius element of $W_K$. Then the matrix in this basis of $r_x(\text{Frob}^2)$ and of $r_x(\beta)$ for any $\beta \in I_K$ is constant, and the matrix of $r_x(\text{Frob})$ can take only two possible values that determine the isomorphism class of $r_x$. As the matrix varies continuously with $x$ and $\mathcal{O}$ is integral, it is constant, hence the isomorphism class of $r_x$ is constant.

Suppose now that $t$ is scalar. Let Frob be any Frobenius element of $W_K$. Then the isomorphism class of $r_x$ is determined by the characteristic polynomial of $r_x(\text{Frob})$. Let $U$ be the Zariski open subset of $\text{Max}(\mathcal{O})$ defined by the condition $N \not= 0$. Then on $U$ the eigenvalues of $r_x(\text{Frob})$ are of the form $\alpha_x$ and $q\alpha_x$ for some $\alpha_x$. As the determinant of $r_x$ is constant, $\alpha_x^2$ is constant, hence the characteristic polynomial
of \( r_x(\text{Frob}) \) can take only two possible values on \( U \). As \( U \) is Zariski dense in Max(\( \mathcal{A} \)) (because \( \mathcal{A} \) is a domain), it can only take two possible values on Max(\( \mathcal{A} \)), and in fact only one by continuity. Hence the isomorphism class of \( r_x \) is constant. □

2.3.3. Deformation rings of fixed discrete series extended type. — We define a quotient \( R^{\psi}(w, t', \overline{\mathcal{P}}) \) of \( R^{\psi}(w, t, \overline{\mathcal{P}}) \) by taking the maximal reduced quotient supported on the set of irreducible components of \( \text{Spec} R^{\psi}(w, t, \overline{\mathcal{P}}) \) that are of type \( t' \). We also define, following [GG15, §5], a ring \( R^{\psi}(w, t^{\text{ds}}, \overline{\mathcal{P}}) \) corresponding to all the irreducible components of some extended type \( t' \).

If \( t \) is of the form (red), there is exactly one extended type \( t' \) that is compatible with \( (t, \psi) \), and we have

\[
R^{\psi}(w, t', \overline{\mathcal{P}}) = R^{\psi}(w, t, \overline{\mathcal{P}}) = R^{\psi}(w, t^{\text{ds}}, \overline{\mathcal{P}}).
\]

If \( t \) is of the form (irr), \( R^{\psi}(w, t, \overline{\mathcal{P}}) = R^{\psi}(w, t^{\text{ds}}, \overline{\mathcal{P}}) \), but there are two extended types that are compatible with \( (t, \psi) \) so \( R^{\psi}(w, t', \overline{\mathcal{P}}) \) can be different from \( R^{\psi}(w, t^{\text{ds}}, \overline{\mathcal{P}}) \).

If \( t \) is scalar, \( R^{\psi}(w, t^{\text{ds}}, \overline{\mathcal{P}}) \) is a quotient of \( R^{\psi}(w, t, \overline{\mathcal{P}}) \) supported only on the components containing points corresponding to representations that are not potentially crystalline and is generally different from \( R^{\psi}(w, t, \overline{\mathcal{P}}) \) (see also [GG15, Lem. 5.5] and the remarks preceding it). If \( t' \) is an extended type compatible with \( (t, \psi) \), then \( R^{\psi}(w, t', \overline{\mathcal{P}}) \) is a quotient of \( R^{\psi}(w, t^{\text{ds}}, \overline{\mathcal{P}}) \), but it can be different from it as there are two possibilities for \( t' \).

Then we have the following properties.

**Proposition 2.3.4**

1. \( R^{\psi}(w, t', \overline{\mathcal{P}}) \) is \( p \)-torsion free, \( R^{\psi}(w, t', \overline{\mathcal{P}})[1/p] \) is reduced and equidimensional.

2. For all finite extensions \( E'/E \), if a map \( x : R^{\psi}(\overline{\mathcal{P}}) \to E' \) factors through \( R^{\psi}(w, t', \overline{\mathcal{P}}) \) then the representation \( \rho_x \) is of determinant \( \psi \), potentially semi-stable of Hodge-Tate type \( w \) and of extended type \( t' \).

3. For all finite extensions \( E'/E \), a map \( x : R^{\psi}(\overline{\mathcal{P}}) \to E' \) such that the representation \( \rho_x \) is of determinant \( \psi \), potentially semi-stable of Hodge-Tate type \( w \) and of extended type \( t' \) factors through \( R^{\psi}(w, t', \overline{\mathcal{P}}) \).

**Proof.** — Properties (1) and (2) follow from the analogous properties for the inertial type \( t \), and Proposition 2.3.3.

In the case not of the form (scal), property (3) follows from the fact that any \( \rho_x \) of inertial type \( t \) is of some discrete series extended type \( t' \).

Suppose now that \( t \) is scalar, we can suppose that \( t \) is trivial. Let \( x \) be as in (3): the map \( x \) factors through \( R^{\psi}(w, t, \overline{\mathcal{P}}) \) by Theorem 2.3.1 and it defines a representation \( \rho : G_K \to \text{GL}_2(E_{\mathcal{E}'}) \) lifting \( \overline{\mathcal{P}} \) for some finite extension \( E' \) of \( E \). If \( \rho \) is not crystalline, then by definition \( x \) factors through \( R^{\psi}(w, t', \overline{\mathcal{P}}) \). Suppose now that \( \rho \) is crystalline. Then WD(\( \rho \)) is of the form \((r, 0) \) with \( r \) isomorphic to \( t' \). As \( t' \) is a discrete series extended type with trivial restriction to inertia, this means that \( r = \psi \otimes (1 + \| \cdot \|) \) for
some unramified character $\psi$ of $W_K$. In particular, we see that there exists a nonzero map $\text{WD}(\rho) \to \text{WD}(\rho) \otimes \| \cdot \|$. By [All14, Th. D], this means that the point $x$ defining the representation $\rho$ is a non-smooth point on $\text{Spec} \, R^{\Box, \psi}(w, t, \mathcal{P})[1/p]$. We know from the proof of [Kis09a, Lem. A.3] that the union of the crystalline irreducible components of $\text{Spec} \, R^{\Box, \psi}(w, t, \mathcal{P})[1/p]$ is smooth. So this means that $x$ is a point of $\text{Spec} \, R^{\Box, \psi}(w, t, \mathcal{P})[1/p]$ which is also on an irreducible component that contains non-crystalline points, and so is a point on $\text{Spec} \, R^{\Box, \psi}(w, t', \mathcal{P})[1/p]$, and $x$ factors through $R^{\Box, \psi}(w, t', \mathcal{P})$. □

Remark 2.3.5. — Suppose that $\mathcal{P}$ is irreducible. Then as $\mathcal{P}$ is isomorphic to $\mathcal{P} \otimes \text{unr}(-1)$, the map $\rho \mapsto \rho \otimes \text{unr}(-1)$ induces an involution of $R^{\Box, \psi}(w, t^\text{ds}, \mathcal{P})$ that exchanges $R^{\Box, \psi}(w, t', \mathcal{P})$ and $R^{\Box, \psi}(w, t'', \mathcal{P})$, where $t'$ and $t''$ are the conjugate extended types compatible with $(t, \psi)$. In particular $R^{\Box, \psi}(w, t', \mathcal{P})$ and $R^{\Box, \psi}(w, t'', \mathcal{P})$ are isomorphic.

Remark 2.3.6. — For an inertial type of the form (char), the extended type is not constant on irreducible components. In fact it follows from [GM09, §3.2] that, in the case $K = \mathbb{Q}_p$, for an inertial type of this form there exists only a finite number of isomorphism classes of potentially semi-stable representations of given regular Hodge-Tate weights and extended type.

3. Multiplicities

Let $D$ be the non-split quaternion algebra over $K$. In this section we consider all smooth representations as having coefficients in $\mathbb{Q}_p$, unless otherwise specified.

3.1. Local Langlands and Jacquet-Langlands. — We denote by $\text{JL}$ the local Jacquet-Langlands correspondence, that attaches to every irreducible smooth admissible representation of $D^\times$ a discrete series smooth representation of $\text{GL}_2(K)$ (that is, supercuspidal or a twist of the Steinberg representation).

We denote by $\text{rec}_p$ the local Langlands correspondence that attaches to each irreducible smooth admissible representation of $\text{GL}_2(K)$ a Weil-Deligne representation of degree 2 with the normalization of [HT01, Introd.].

We set $\text{LL}_D(\pi) = (\text{rec}_p \circ \text{JL}(\pi)) \otimes \| \cdot \|^1/2$, so that the image of $\text{LL}_D$ is exactly the discrete series Weil-Deligne representations $(r, N)$ (see Sections 3.5 and 4.5.1 for a justification of the normalization). We give some properties of $\text{LL}_D$: let $\psi$ a character of $K^\times$, and denote by $N_D$ the reduced norm $D \to K$. For $x \in \overline{\mathbb{Q}}_p^\times$, we denote by $\text{unr}_D(x)$ the character of $D^\times$ given by $\text{unr}(x) \circ N_D$, and more generally for $\psi$ a character of $K^\times$ we denote by $\psi_D$ the character $\psi \circ N_D$ of $D^\times$. Then $\text{LL}_D(\psi_D) = (\psi \circ \psi^p \| \cdot \|, N)$ with $N \neq 0$.

Let $\varpi_D$ be a uniformizer of $D$. If $a \geq 1$, we set $U^a_D = 1 + \varpi_D^a \mathcal{O}_D$. It is an open compact subgroup of $D^\times$, which does not depend on the choice of $\varpi_D$. It follows from the explicit description of smooth representations of $D^\times$ (as can be found for example in [BH06, Chap. 13]) that any irreducible smooth representation of $D^\times$ that is not a character has one of the following forms:
(1) \( \pi_D = \text{ind}_{L^* U_D^0} \psi \) for some character \( \psi \) and some ramified quadratic extension \( L \) of \( K \).

(2) \( \pi_D = \text{ind}_{L^* U_D^0} \rho \) for some irreducible representation \( \rho \) of \( L^* U_D^0 \) of dimension 1 or \( q \) and \( L \) the unramified quadratic extension of \( K \).

**Proposition 3.1.1.** — Let \( t \) be a Weil-Deligne representation of dimension 2 that is of the form (red) or (irr) of Lemma 2.1.1. Then the following conditions are equivalent:

(1) the type of \( t \) is of the form (red).

(2) \( \text{LL}_{D}^{-1}(r) \simeq \text{LL}_{D}^{-1}(r) \otimes \text{unr}_{D}(-1) \).

(3) \( \text{LL}_{D}^{-1}(r) = \text{ind}_{L^* U_D^0} \psi \) for \( L \) the unramified quadratic extension of \( K \), some \( a \) and some representation \( \rho \) of \( L^* U_D^0 \).

(4) the restriction of \( \text{LL}_{D}^{-1}(r) \) to \( O_D^x \) is the sum of two irreducible representations that differ by conjugation by \( \varpi_D \).

And the following conditions are equivalent:

(1) the type of \( t \) is of the form (irr).

(2) \( \text{LL}_{D}^{-1}(r) \not\simeq \text{LL}_{D}^{-1}(r) \otimes \text{unr}_{D}(-1) \).

(3) \( \text{LL}_{D}^{-1}(r) = \text{ind}_{L^* U_D^0} \psi \) for some ramified quadratic extension \( L \) of \( K \), some \( a \) and some character \( \psi \) of \( L^* U_D^0 \).

(4) the restriction of \( \text{LL}_{D}^{-1}(r) \) to \( O_D^x \) is irreducible.

**Proof.** — Note first that \( \text{LL}_{D} \) is compatible with twists by characters, as this is the case for \( \text{rec}_{p} \) and \( \text{JL} \).

(1) \( \Leftrightarrow \) (2) comes from Proposition 2.1.2.

(1) \( \Leftrightarrow \) (3) comes from the explicit descriptions of the local Langlands and Jacquet-Langlands correspondence (see [BH06]).

(3) \( \Leftrightarrow \) (4) is [GG15, Prop. 3.8] (see also [Gér78, §§5 & 6]). \( \square \)

### 3.2. Representations attached to a discrete series inertial type

**3.2.1. Representations of \( D^x \) and \( O_D^x \).** — Let \( t \) be some discrete series inertial type. Let \( (r, N) \) be some discrete series Weil-Deligne representation with \( t = r|_{I_K} \). Let \( \pi_t = \text{LL}_{D}^{-1}(r, N) \), which depends on the choice of \( (r, N) \) only up to unramified twist. If \( (r, N) \) is of the form (scal) then \( \pi_t \) is a character of \( D^x \), so the restriction of \( \pi_t \) to \( O_D^x \) is irreducible. As we have seen in Proposition 3.1.1, if \( (r, N) \) is of the form (irr) then \( \pi_t \) is still irreducible after restriction to \( O_D^x \), and if \( (r, N) \) is of the form (red) then the restriction of \( \pi_t \) to \( O_D^x \) is the sum of two irreducible constituents that differ by conjugation by \( \varpi_D \). Let \( \sigma_D(t) \) be one of the irreducible constituents of the restriction of \( \pi_t \) to \( O_D^x \); it depends only on \( t \) and not on the choice of \( (r, N) \) (this is the same as the representation \( \sigma_D(t) \) of \([GG15, \S3.1]\)). We can recover \( \pi_t \) from \( \sigma_D(t) \) up to unramified twist. Hence we have the following property.

**Proposition 3.2.1.** — Let \( \pi_D \) be a smooth irreducible representation of \( D^x \). Then \( \text{Hom}_{O_D^x}(\sigma_D(t), \pi_D) \neq 0 \) if and only if \( \text{LL}_{D}(\pi_D)|_{I_K} \simeq t \).
Remark 3.2.2. — As was already noted in [GG15], contrary to the case of $\text{GL}_2$, we see that the type for $D^\times$ is not unique, at least for representations of the form (red). On the other hand, as there are only one or two irreducible constituents for the restriction to $\mathcal{O}_D^\times$ of a smooth irreducible representation of $D^\times$, it is much easier to find a type.

3.2.2. The group $\mathcal{G}_{\varpi_K}$. — Let $\varpi_K$ be a uniformizer of $K$ and $\varpi_D$ a uniformizer of $D$ with $\varpi_D^2 = \varpi_K$. Let $\mathcal{G}_{\varpi_K} = D^\times / \varpi_D^\times$. Then $\mathcal{G}_{\varpi_K}$ is isomorphic to the semi-direct product $\mathcal{O}_D^\times \rtimes \varpi_D \{1, \iota\}$, where the action of $\iota$ on $\mathcal{O}_D^\times$ is by conjugation by $\varpi_D$. As a group, $\mathcal{G}_{\varpi_K}$ depends on $\varpi_K$, but not on the choice of $\varpi_D$ such that $\varpi_D^2 = \varpi_K$. Let $\xi = \unr_D(-1)$, that is, the character of $\mathcal{G}_{\varpi_K}$ that is trivial on $\mathcal{O}_D^\times$ and sends $\iota$ to $-1$.

Let $\mathfrak{t}$ be a discrete series inertial type. Using the representation $\pi_\mathfrak{t}$ of Section 3.2.1, we can attach to $\mathfrak{t}$ a smooth representation $\sigma_{\mathfrak{t}}(\varpi)$ of $\mathcal{G}_{\varpi_K}$, or equivalently a smooth representation of $D^\times$ that is trivial on $\varpi_K$: there is an unramified twist of $\pi_\mathfrak{t}$ that is trivial on $\varpi_K$, as $\pi_\mathfrak{t}(\varpi_K)$ is scalar.

The relation between $\sigma_{\mathfrak{t}}(\varpi)$ and $\sigma_D(\varpi)$ is then given by Proposition 3.1.1. If $\varpi$ is of the form (scal) or (irr) then the representation $\sigma_{\mathfrak{t}}(\varpi)$ is defined only up to twist by $\xi$, and the restriction of $\sigma_{\mathfrak{t}}(\varpi)$ to $\mathcal{O}_D^\times$ is isomorphic to $\sigma_D(\varpi)$, which is irreducible. If $\varpi$ is of the form (red), then there is only one possibility for $\sigma_{\mathfrak{t}}(\varpi)$ and the restriction of $\sigma_{\mathfrak{t}}(\varpi)$ to $\mathcal{O}_D^\times$ is isomorphic to the direct sum of $\sigma_D(\varpi)$ and the representation $\sigma_D(\varpi)^{\varpi_D}$ obtained from $\sigma_D(\varpi)$ by conjugation by a uniformizer.

Remark 3.2.3. — The representation $\sigma_{\mathfrak{t}}(\varpi)$ of $\mathcal{G}_{\varpi_K}$ is the analogue in our situation of the representation $\sigma_{\mathfrak{t}}$ of $\hat{U}_0(\ell)$ in [BCDT01, §1.2].

3.2.3. Realizations of $\mathcal{G}_{\varpi_K}$. — Let $K'$ be the unramified quadratic extension of $K$. By fixing an embedding of $K'$ into $D$ and a basis of $D$ as a $K'$-vector space, we can define an embedding $D^\times \to \text{GL}_2(K')$, hence, after choosing $K' \to \overline{\mathbb{Q}}_p$, an embedding $u : D^\times \to \text{GL}_2(\overline{\mathbb{Q}}_p)$. All such embeddings are conjugate in $\text{GL}_2(\overline{\mathbb{Q}}_p)$ by the Skolem-Noether theorem.

Fix now a uniformizer $\varpi_K$ of $K$, a square root $\varpi_D$ of $\varpi_K$ in $D$ and a square root $\sqrt{\varpi_K}$ of $\varpi_K$ in $\overline{\mathbb{Q}}_p$. With these choices we can define an embedding $\tilde{u} : \mathcal{G}_{\varpi_K} \to \text{GL}_2(\overline{\mathbb{Q}}_p)$ by setting $\tilde{u}|_{\mathcal{O}_D^\times} = u|_{\mathcal{O}_D^\times}$ and $\tilde{u}(\iota) = \sqrt{\varpi_K}^{-1} u(\varpi_D)$.

Note that for each choice of $\varpi_K$ and $\sqrt{\varpi_K}$, all the possible $\tilde{u}$ corresponding to the various choices of $u$ and $\varpi_D$ are conjugate in $\text{GL}_2(\overline{\mathbb{Q}}_p)$. Moreover, for varying choices of $\varpi_K$ all the $\tilde{u}|_{\mathcal{O}_D^\times}$ are conjugate.

3.3. Representations of $\Gamma_K$

3.3.1. The group $\Gamma_K$. — Let $k$ be the residue field of $K$ and $\ell$ its quadratic extension, so that $\mathcal{O}_D / \varpi_D \simeq \ell$. We define the group $\Gamma_K = \ell^\times \rtimes \{1, \iota\}$, where $\iota$ acts on $\ell^\times$ by the non-trivial $k$-automorphism of $\ell$.

The quotient $\mathcal{G}_{\varpi_K} / (1 + \varpi_D \mathcal{O}_D)$ is naturally isomorphic to the group $\Gamma_K$, and the map $\mathcal{G}_{\varpi_K} \to \Gamma_K$ extends the natural morphism $\mathcal{O}_D^\times \to \ell^\times$. As $1 + \varpi_D \mathcal{O}_D$ is a proper $p$-group, any semi-simple representation of $\mathcal{G}_{\varpi_K}$ in characteristic $p$ factors through $\Gamma_K$. 

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3.3.2. Irreducible representations of $\Gamma_K$ in characteristic $p$. — Let $\mathbb{F}$ be an algebraic closure of $k$. Fix $\ell \to \mathbb{F}$, and let $q$ be the cardinality of $k$.

For an element $a$ in $\mathbb{Z}/(q^2 - 1)\mathbb{Z}$, we denote by $\chi_a$ the character of $\ell^\times$ sending $x$ to $x^a$.

For an element $a$ in $\mathbb{Z}/(q - 1)\mathbb{Z}$, we denote by $\delta_a$ the character of $\Gamma_K$ such that $\delta_a$ coincides with $\chi_{a(q+1)} = \chi_a|_{\ell^\times} \circ N_{\ell/k}$ on $\ell^\times$ and $\delta_a(\ell) = 1$.

Let $\xi$ be the character of $\Gamma_K$ that is trivial on $\ell^\times$ and $\xi(\ell) = -1$.

Let $r_a = \text{ind}^\Gamma_{\ell^\times} \chi_a$ for $a \in \mathbb{Z}/(q^2 - 1)\mathbb{Z}$. Then $r_a$ is irreducible if and only if $a$ is not divisible by $q + 1$. If $a = (q + 1)b$ then $r_a$ is isomorphic to $\delta_b \oplus \xi \delta_b$.

**Proposition 3.3.1.** — The irreducible representations of $\Gamma_K$ with coefficients in $\mathbb{F}$ are exactly the following:

- The characters $\delta_a$ for $a \in \mathbb{Z}/(q - 1)\mathbb{Z}$.
- The characters $\xi \delta_a$ for $a \in \mathbb{Z}/(q - 1)\mathbb{Z}$.
- The representations $r_a$ for $a \in \mathbb{Z}/(q^2 - 1)\mathbb{Z}$ not divisible by $q + 1$.

Moreover, these representations are all distinct, except for the relation $r_a = r_{qa}$. Finally, $\xi r_a$ is isomorphic to $r_a$, and $r_a|_{\ell^\times} = \chi_a \oplus \chi_{qa}$.

The irreducible representations of $\Gamma_K$ are the analogue in our situation of the Serre weights.

3.3.3. Reduction modulo $p$ of representations of $G_{\mathbb{F}K}$ attached to discrete series types

Let $\sigma_{\mathbb{F}}(\mathfrak{t})$ be a representation of $G_{\mathbb{F}K}$ attached to a discrete series inertial type $\mathfrak{t}$ as in Section 3.2.2. As $G_{\mathbb{F}K}$ is compact, we can find an invariant lattice in $\sigma_{\mathbb{F}}(\mathfrak{t})$, and consider the semi-simplification of the reduction modulo $p$ of this representation. We denote by $\sigma_{\mathbb{F}}(\mathfrak{t})$ the representation of $\Gamma_K$ that we obtain (it is semi-simple, independent of the choice of the invariant lattice and its restriction to $\ell^\times$ is independent of any choice).

**Proposition 3.3.2.** — Each irreducible representation of $\Gamma_K$ over $\mathbb{F}$ has a lift in characteristic 0 that is of the form $\sigma_{\mathbb{F}}(\mathfrak{t})$ for some discrete series inertial type $\mathfrak{t}$.

**Proof.** — Let $\delta$ be an irreducible representation of $\Gamma_K$ of dimension 1. It is of the form $\chi \circ N_{\ell/k}$ or $\xi \chi \circ N_{\ell/k}$ for some character $\chi$ of $K^\times$. We define a scalar inertial type $\mathfrak{t}_3$ by $\mathfrak{t}_3 = (\bar{\chi} \oplus \bar{\chi})|_{I_K}$, where $\bar{\chi}$ denotes the image by local class field theory of the Teichmüller lift of the character $\chi \circ (K^\times \to k^\times)$. Then we can choose $\sigma_{\mathbb{F}}(\mathfrak{t}_3)$ so that $\sigma_{\mathbb{F}}(\mathfrak{t}_3)$ is isomorphic to $\delta$ (note that $\sigma_{\mathbb{F}}(\mathfrak{t}_3)$ depends on $\delta$ and not only on $\mathfrak{t}_3$).

Let $r$ be an irreducible representation of $\Gamma_K$ of dimension 2. There exists $a \in \mathbb{Z}/(q^2 - 1)\mathbb{Z}$ not divisible by $q + 1$ such that $r|_{\ell^\times} = \chi_a \oplus \chi_{qa}$. Let $K^\ell$ be the unramified quadratic extension of $K$, and $\bar{\chi}_a : W_K^\ell \to \overline{\mathbb{Q}}_p^\times$ the tame character given by the Teichmüller lift of $\chi_a \circ (K^{\ell \times} \to \ell^{\times})$. We define an inertial type $\mathfrak{t}_r$ of the form (red) by $\mathfrak{t}_r = (\text{ind}_{W_K^\ell}^{\overline{\mathbb{Q}}_p} \bar{\chi}_a)|_{I_K}$. Then $\sigma_{\mathbb{F}}(\mathfrak{t}_r)$ is isomorphic to $r$, as follows from the explicit constructions in [BH06, Chap. 13].
We denote by \( \mathcal{R}(\Gamma_K) \) the Grothendieck ring of representations of \( \Gamma_K \) with coefficients in \( \mathbb{F} \). We denote by \( [\sigma] \) the image in \( \mathcal{R}(\Gamma_K) \) of a representation \( \sigma \) of \( \Gamma_K \).

We now compute the reduction of some representations of \( \mathcal{G}_{\wp_K} \) attached to discrete series types. Let \( L \) be the ramified quadratic extension of \( K \) generated by the square root of \( \wp_K \), and fix an embedding of \( L \) into \( D \). Any semi-simple representation of \( L^\times U_D^2 \) in characteristic \( p \) is trivial on \( U_D^1 \cap L^\times U_D^2 \) as this is a pro-\( p \)-group (recall that \( U_D^2 \) was defined in Section 3.1). Any representation of \( L^\times U_D^2 \) that is trivial on the subgroup generated by \( \wp_K \) factors through the image of \( L^\times U_D^2 \) inside \( \mathcal{G}_{\wp_K} \). Hence any semi-simple representation of \( L^\times U_D^2 \) in characteristic \( p \) that is trivial on \( \wp_K \) factors through the subgroup \( L^\times U_D^2 / \langle U_D^1, \wp_K^\mathcal{R} \rangle \) of \( \Gamma_K \). Let us call this subgroup \( \Delta \), it is equal to \( k^\times \times \{1, \iota\} \subset \ell^\times \times \{1, \iota\} \).

**Proposition 3.3.3.** — Let \( L \) be as above and let \( \theta = \text{ind}_{L^\times U_D^2}^{L^\times U_D^2} \psi \) for some smooth character \( \psi \) of \( L^\times U_D^2 \) with \( \psi(\wp_K) = 1 \). Then \( \theta \) factors through \( \mathcal{G}_{\wp_K} \). Denote by \( \mathcal{G} \) the semi-simple representation of \( \Gamma_K \) which is the reduction modulo \( p \) of \( \theta \). Let \( n \in \mathbb{Z} / (q - 1) \mathbb{Z} \) and \( \alpha \in \{0, 1\} \) be such that \( \overline{\psi} = \xi^\alpha \chi_n |\Delta \) as a representation of \( \Delta \). We denote by \( I(\overline{\psi}) \) the set of irreducible representations of \( \Gamma_K \) with central character equal to \( \overline{\psi}|_{k^\times} \). Then we have in \( \mathcal{R}(\Gamma_K) \):

1. if \( \psi(-1) = -1 \), that is, \( n \) is odd, then \( I(\overline{\psi}) \) consists of \( (q + 1)/2 \) representations of dimension 2, and
   \[
   [\mathcal{G}] = q^{a-1} \left( \sum_{r \in I(\overline{\psi})} |r| \right),
   \]

2. if \( \psi(1) = 1 \), that is, \( n \) is even, then \( I(\overline{\psi}) \) consists of \( (q - 1)/2 \) representations of dimension 2 and 4 representations of dimension 1, and:
   \[
   [\mathcal{G}] = q^{a-1} \left( \sum_{r \in I(\overline{\psi})} |r| \right) + \frac{q^{a-1} + 1}{2} [\xi^\alpha](\delta_{n/2}) + \left[\delta_{(n+q-1)/2}\right] + \frac{q^{a-1} - 1}{2} [\xi^{\alpha+1}](\delta_{n/2}) + \left[\delta_{(n+q-1)/2}\right]).
   \]

**Proof.** — We proceed as in [BD14, §4]. We have \( [L^\times U_D^1 : L^\times U_D^2] = q^{a-1} \) (note that the essential conductor of \( \theta \) is \( 2a + 1 \)). The reduction modulo \( p \) of \( \text{ind}_{L^\times U_D^2}^{L^\times U_D^2} \psi \) is the sum of \( (q^{a-1} + 1)/2 \) copies of \( \overline{\psi} \) and of \( (q^{a-1} - 1)/2 \) copies of \( \xi \overline{\psi} \). Let \( \mu \) be a smooth character of \( L^\times U_D^1 \) in characteristic \( p \) with \( \mu(\wp_K) = 1 \), then \( \text{ind}_{L^\times U_D^2}^{\Gamma_K} \mu \) factors through \( \Gamma_K \), and the representation of \( \Gamma_K \) that we obtain is \( \text{ind}_{\Gamma_K}^{\Gamma_K} \mu \), which can be computed via Brauer characters.

It follows from Proposition 3.1.1 that Proposition 3.3.3 gives \( \overline{\sigma}(t) \) when \( t \) is of type (irr) under some compatibility condition between \( t \) and \( \wp_K \). As we will see in Section 3.5 this compatibility condition is harmless. When \( t \) is scalar, \( \overline{\sigma}(t) \) is easy to compute as \( \sigma(\mu) \) is of dimension 1. The value of \( \overline{\sigma}(t) \) when \( t \) is of the form (red) could be immediately obtained from [BD14, Prop. 4.6]: as \( \overline{\sigma}(t) = \xi \overline{\sigma}(t) \), it

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is entirely determined by its restriction to \( \ell^\times \). We do not give details as they are not really needed. Indeed, our goal in computing \( \overline{\sigma_\mathcal{G}}(\mathfrak{t}) \) is to allow us to compute the multiplicity of some deformation rings as we shall see in Theorems 3.5.1 and 3.5.2, but for \( \mathfrak{t} \) of the form (red) this multiplicity can be computed by the formula coming from the Breuil-Mézard conjecture for \( \text{GL}_2 \). Note that complete results and computations can be found in [Tok15].

### 3.4. A reformulation of a result of Gee and Geraghty.

Let \( \Gamma = \Gamma_K \). We fix a uniformizer \( \varpi_K \) of \( K \). We denote by \( w_0 \) the Hodge-Tate type \((0,0)_{\tau \in \text{Hom}(K, \mathbb{Z}_p)}. \)

In the case \( K = \mathbb{Q}_p \), let \( w = (n,m) \) be a Hodge-Tate type. We set \( |w| = n + 2m \).

We define a representation \( \sigma_w = \text{Sym}^n \otimes \text{det}^m \) of \( \text{GL}_2(\mathbb{Q}_p) \), hence of \( \mathcal{G} = \mathcal{G}_{\sigma_{\mathfrak{q}_p}} \) via a realization of \( \mathcal{G} \) as in Section 3.2.3. In particular \( \sigma_{w_0} \) is the trivial representation of \( \mathcal{G} \). The isomorphism class of the restriction of \( \sigma_w \) to \( \mathcal{O}_D^\times \) does not depend on the particular choice of a realization, as they are all conjugate in restriction to \( \mathcal{O}_D^\times \). We can see the reduction modulo \( p \) of \( \sigma_w \) as a representation of \( \Gamma \) by restriction. We denote its image in \( \mathcal{H}(\Gamma) \) by \( \overline{\sigma_w} \), it does not depend on any choices made (including \( \varpi_K \)): indeed it is the restriction to \( \Gamma \) of the representation \( \text{Sym}^n \otimes \text{det}^m \) of \( \text{GL}_2(\mathbb{F}) \) via any embedding of \( \Gamma \) into \( \text{GL}_2(\mathbb{F}) \), and all such embeddings are conjugate.

We denote by \( \pi \) a uniformizer of the field \( E \) of Section 2.3. For any noetherian local ring \( A \), we denote by \( e(A) \) the Hilbert-Samuel multiplicity \( e(A,A) \) (see [Mat89] for the definition of the Hilbert-Samuel multiplicity, and also [Kis09a, §1.3] for properties relevant to our situation).

Let \( \ell = \mathbb{F}_p^\times \), and let \( \mathcal{H}(\ell^\times) \) be the Grothendieck ring of representations of \( \ell^\times \) with coefficients in \( \mathbb{F}_p^\times \).

For \( K = \mathbb{Q}_p \), we recall the following result ([GG15, Cor. 5.7]), which is the consequence of the main result of [GG15] and the usual formulation of the Breuil-Mézard conjecture proved in [Kis09a], [Pas15] and [HT15]. Here \( \sigma_D(\mathfrak{t}) \) is, as in Section 3.2.1, a choice of irreducible sub-representation of the restriction of \( \pi_\mathfrak{t} \) to \( \mathcal{O}_D^\times \).

**Theorem 3.4.1.** Let \( \mathfrak{p} : G_{\mathbb{Q}_p} \to \text{GL}_2(\mathbb{F}) \), and suppose that \( p \geq 5 \) if \( \mathfrak{p} \) is a twist of an extension of the trivial representation by the cyclotomic character. There exists a positive linear functional \( i_{D,\mathfrak{p}} : \mathcal{H}(\ell^\times) \to \mathbb{Z} \) such that for each discrete series inertial type \( \mathfrak{t} \), and each choice of \( \sigma_D(\mathfrak{t}) \), we have

\[
e(e(R^{\ell,\psi}(w,\mathfrak{t}^\text{ds},\mathfrak{p})/\pi)) = i_{D,\mathfrak{p}}([\overline{\sigma_D}(\mathfrak{t}) \otimes \overline{\sigma_\mathcal{G}|_{\ell^\times}}]).
\]

We return to the case of a general \( K \). We have the following well-known result (see for example [GS11, Lem. 3.5]).

**Proposition 3.4.2.** Let \( \overline{\mathfrak{p}} \) be a continuous representation of \( G_K \) of dimension 2 with coefficients in \( \mathbb{F} \). Suppose that \( \overline{\mathfrak{p}} \) has a potentially semi-stable lift with scalar type \( \mathfrak{t} = \psi \oplus \psi \) and Hodge-Tate weights \((0,1)_{\tau \in \text{Hom}(K, \mathbb{Q}_p)} \) which is not potentially crystalline. Then \( \overline{\mathfrak{p}} \) is an unramified twist of \((\varpi_1^\psi) \otimes \overline{\psi}).\)
We deduce from this that when \( \overline{\rho} \) is not a twist of an extension of the trivial character by the cyclotomic character, \( R^{\square, \psi}(w_0, t^{\text{ds}}, \overline{\rho}) = R^{\square, \psi}_G(w_0, t^{\text{ds}}, \overline{\rho}) \) for any discrete series type \( t \), where the second ring parametrizes only representations that are potentially crystalline. Hence we can deduce from the main result of [GG15] and [GK14, Th. A] the following.

**Theorem 3.4.3.** — Let \( \overline{\rho} : G_K \to \text{GL}_2(\mathbb{F}) \) a continuous representation that is not a twist of an extension of the trivial representation by the cyclotomic character. There exists a positive linear functional \( i_{D, \overline{\rho}} : \mathcal{H}(\ell^\times) \to \mathbb{Z} \) such that for each discrete series inertial type \( t \), and each choice of \( \sigma_D(t) \), we have

\[
e_i(R^{\square, \psi}(w_0, t^{\text{ds}}, \overline{\rho})/\pi) = i_{D, \overline{\rho}}(\overline{\sigma}(t)|_{\ell^\times}).
\]

Let \( d_t = 1 \) if \( t \) has the form (scal) or (irr), and \( d_t = 2 \) if \( t \) has the form (red), so that \( d_t \) is the number of irreducible components of \( \sigma_D(t)|_{\ell^\times} \). Then we can give a reformulation of Theorems 3.4.1 and 3.4.3 in terms of representations of \( \Gamma \).

**Theorem 3.4.4.** — Let \( \overline{\rho} : G_{A_K} \to \text{GL}_2(\mathbb{F}) \), and suppose that \( p \geq 5 \) if \( \overline{\rho} \) is a twist of an extension of the trivial representation by the cyclotomic character. There exists a positive linear functional \( i_{\overline{\rho}} : \mathcal{H}(\Gamma) \to \mathbb{Z} \) such that for each discrete series inertial type \( t \) and each choice of \( \sigma_D(t) \) we have

\[
d_t e_i(R^{\square, \psi}(w_0, t^{\text{ds}}, \overline{\rho})/\pi) = i_{\overline{\rho}}(\overline{\sigma}(t) \otimes \overline{\psi}_{|_{\ell^\times}}).
\]

**Theorem 3.4.5.** — Let \( \overline{\rho} : G_K \to \text{GL}_2(\mathbb{F}) \) a continuous representation that is not a twist of an extension of the trivial representation by the cyclotomic character. There exists a positive linear functional \( i_{\overline{\rho}} : \mathcal{H}(\Gamma) \to \mathbb{Z} \) such that for each discrete series inertial type \( t \) and each choice of \( \sigma_D(t) \) we have

\[
d_t e_i(R^{\square, \psi}(w_0, t^{\text{ds}}, \overline{\rho})/\pi) = i_{\overline{\rho}}(\overline{\sigma}(t)|_{\ell^\times}).
\]

**Proof of Theorems 3.4.4 and 3.4.5.** — It follows from Theorems 3.4.1 (resp. 3.4.3) and the definition of \( \sigma_D(t) \) that we have the equality

\[
d_t e_i(R^{\square, \psi}(w, t^{\text{ds}}, \overline{\rho})/\pi) = i_{D, \overline{\rho}}(\overline{\sigma}(t) \otimes \overline{\psi}_{|_{\ell^\times}}).
\]

Set \( i_{\overline{\rho}}(\gamma) = i_{D, \overline{\rho}}(\gamma|_{\ell^\times}) \) for all irreducible representations \( \gamma \) of \( \Gamma \) to get the result. \( \square \)

In particular, we observe that

\[
i_{\overline{\rho}}(\gamma) = i_{\overline{\rho}}(\xi_{\gamma}) = (\dim \gamma)e_i(R^{\square, \psi}(w_0, t^{\text{ds}}, \overline{\rho})/\pi)
\]

for any irreducible representation \( \gamma \) of \( \Gamma \), where \( t_{\gamma} \) is the inertial type defined in the proof of Proposition 3.3.2.

We denote by \( W_{\Gamma}(\overline{\rho}) \) the set of \( \gamma \) such that \( i_{\overline{\rho}}(\gamma) \neq 0 \). This is the translation in the setting of representations of \( \Gamma \) of the (predicted) quaternionic Serre weights of [GS11]. Note in particular that, as in [GS11], the set \( W_{\Gamma}(\overline{\rho}) \) is determined by the existence of certain lifts of \( \overline{\rho} \) that have all their Hodge-Tate weights equal to \((0, 1)\), which makes the situation with quaternion algebras simpler than the situation of
Serre weights for GL$_2$, since, for GL$_2$, one cannot in general lift a Serre weight as a type in characteristic 0.

3.5. Multiplicity formulas. — We now state our main theorems for the multiplicity of the special fiber of the discrete series extended type deformation rings, which we prove in Section 5.

**Theorem 3.5.1.** — Let $\rho$ be a continuous representation of $G_{\mathbb{Q}_p}$ of dimension 2 with coefficients in $\mathbb{F}$. Suppose that $p \geq 5$ if $\rho$ is a twist of an extension of the trivial character by the cyclotomic character. Let $\psi = \omega^{-1} \det \rho$, $\varpi_{\mathbb{Q}_p}$ a uniformizer of $\mathbb{Q}_p$ and $\alpha$ a square root of $\overline{\psi}(\varpi_{\mathbb{Q}_p})^{-1}$.

There exists a positive linear form $\mu_{\rho}$ on $R(\Gamma)$ with values in $\mathbb{Z}$ satisfying the following property: for any discrete series inertial type $t$, Hodge-Tate type $w$, character $\psi$ lifting $\overline{\psi}$ compatible with $t$ and $w$, and $t^+$ a discrete series extended type compatible with $(t, \psi)$ we have:

$$e(R^{\square, \psi}(w, t^+, \rho)/\pi) = \mu_{\rho}(\varpi_{\mathbb{Q}_p}(t) \otimes \varpi_{w})$$

for the choice of representation $\sigma_{\mathfrak{g}}(t)$ of $\mathfrak{g}_{\mathbb{Q}_p}$ such that $t^+ = LL_D(\sigma_{\mathfrak{g}}(t)) \otimes \text{unr}(a \varpi_{\mathbb{Q}_p})^{-1}$ for some $a \in \mathbb{Z}_p$ lifting $\alpha$.

**Theorem 3.5.2.** — Let $\rho$ be a continuous representation of $G_K$ of dimension 2 with coefficients in $\mathbb{F}$ that is not a twist of an extension of the trivial character by the cyclotomic character. Let $\psi = \omega^{-1} \det \rho$, $\varpi_K$ a uniformizer of $K$ and $\alpha$ a square root of $\overline{\psi}(\varpi_K)^{-1}$.

There exists a positive linear form $\mu_{\rho}$ on $R(\Gamma)$ with values in $\mathbb{Z}$ satisfying the following property: for any discrete series inertial type $t$, character $\psi$ lifting $\overline{\psi}$ compatible with $t$ and $w_0$, and $t^+$ a discrete series extended type compatible with $(t, \psi)$ we have:

$$e(R^{\square, \psi}(w_0, t^+, \rho)/\pi) = \mu_{\rho}(\varpi_{\mathbb{Q}_p}(t))$$

for the choice of representation $\sigma_{\mathfrak{g}}(t)$ of $\mathfrak{g}_{\mathbb{Q}_p}$ such that $t^+ = LL_D(\sigma_{\mathfrak{g}}(t)) \otimes \text{unr}(a \varpi_{\mathbb{Q}_p})^{-1}$ for some $a \in \mathbb{Z}_p$ lifting $\alpha$.

**Remark 3.5.3.** — It follows from the definition of the compatibility of $t^+$ with $(t, \psi, w)$ that there exists indeed a choice of $\sigma_{\mathfrak{g}}(t)$ satisfying the condition. If $t^-$ is the extended type conjugate to $t^+$, then the choices of $\sigma_{\mathfrak{g}}(t)$ for $t^+$ and $t^-$ differ by multiplication by $\xi$.

In the case when $t$ is of the form (red), recall that there is only one extended type $t^+$ compatible with $(t, \psi)$, and

$$R^{\square, \psi}(w, t^{\text{red}}, \rho) = R^{\square, \psi}(w, t^+, \rho).$$

There is no choice to be made for $\sigma_{\mathfrak{g}}(t)$ as it is isomorphic to $\xi \sigma_{\mathfrak{g}}(t)$.

We have the following proposition, which is a consequence of [Kis09a, Prop. 1.3.9].
Proposition 3.5.4. — Let \( t^+, t^- \) be the two distinct conjugate extended types compatible with \( (t, \psi) \) with \( t \) of the form (scal) or (irr). Then
\[
e(R^{\square, \psi}(w, t^+, \overline{\rho})/\pi) + e(R^{\square, \psi}(w, t^-, \overline{\rho})/\pi) = e(R^{\square, \psi}(w, t^{\text{disp}}, \overline{\rho})/\pi).
\]
We have the following corollary (\( t_\tau \) and \( t_\delta \) are the inertial types defined in the proof of Proposition 3.3.2).

Corollary 3.5.5. — We have
\[
\mu_\varphi([r]) = e(R^{\square, \psi}(w_0, t^{\text{disp}}_+, \overline{\rho})/\pi)
\]
for any irreducible representation \( r \) of \( \Gamma \) of dimension 2, and
\[
\mu_\varphi([\delta] + [\xi]) = e(R^{\square, \psi}(w_0, t^{\text{disp}}_+, \overline{\rho})/\pi)
\]
for any irreducible representation \( \delta \) of \( \Gamma \) of dimension 1. In particular, for any irreducible representation \( \gamma \) of \( \Gamma \), we have
\[
\mu_\varphi([\gamma]) + \mu_\varphi([\xi \gamma]) = i_\varphi([\gamma]).
\]

Proof. — Let \( r \) be an irreducible representation of \( \Gamma \) of dimension 2. Then \( r = \pi_\varphi(t_\tau) \) for some inertial type \( t_\tau \) of the form (red) by Proposition 3.3.2. Then as remarked in Section 2.3.3, if \( t^+_\tau \) is the extended type compatible with \( (t_\tau, \psi) \), then
\[
R^{\square, \psi}(w_0, t^+_\tau, \overline{\rho}) = R^{\square, \psi}(w_0, t^{\text{disp}}_+, \overline{\rho}),
\]
hence the formula in this case. Let \( \delta \) be an irreducible representation of \( \Gamma \) of dimension 1. Then
\[
e(R^{\square, \psi}(w_0, t^{\text{disp}}_+, \overline{\rho})/\pi) = e(R^{\square, \psi}(w_0, t^+_\tau, \overline{\rho})/\pi) + e(R^{\square, \psi}(w_0, t^-_\delta, \overline{\rho})/\pi),
\]
where \( t^+_\delta \) and \( t^-_\delta \) are the two conjugate extended types compatible with \( (t_\delta, \psi) \). So we deduce the formula from Remark 3.5.3. The formula with \( i_\varphi \) then follows from Theorems 3.4.4 and 3.4.5.

It follows from this corollary that \( \mu_\varphi([\gamma]) = 0 \) if \( \gamma \not\in W_\Gamma(\overline{\rho}) \). We begin the definition of \( \mu_\varphi \) by setting \( \mu_\varphi([\gamma]) = 0 \) for any irreducible \( \gamma \) not in \( W_\Gamma(\overline{\rho}) \). With this definition, the equalities of Theorem 3.5.1 and 3.5.2 hold for all \( t, \psi, w \) (with \( w = w_0 \) if \( K \neq \mathbb{Q}_p \)) such that \( R^{\square, \psi}(w, t^{\text{disp}}, \overline{\rho}) = 0 \).

From Proposition 3.4.2 we deduce the next proposition.

Proposition 3.5.6. — If \( \overline{\rho} \) is a representation such that \( W_\Gamma(\overline{\rho}) \) contains a representation \( \delta \) of dimension 1 then \( \overline{\rho} \) is a twist of an extension of the trivial character by the cyclotomic character and there is at most one possible value for \( \delta \) for which \( \mu_\varphi(\delta) \neq 0 \).

Remark 3.5.7. — When \( K = \mathbb{Q}_p \), \( \overline{\rho} \) is a twist of an extension of the trivial character by the cyclotomic character if and only if \( W_\Gamma(\overline{\rho}) \) contains a representation \( \delta \) of dimension 1, and then \( i_\varphi(\delta) = 1 \). This follows from the explicit computations of deformation rings that can be found in [BM02, §5.2].

Proposition 3.5.8. — If \( \overline{\rho} \) is not a twist of an extension of the trivial character by the cyclotomic character then for any representation \( \gamma \) of \( \Gamma \) we have \( \mu_\varphi([\gamma]) = \mu_\varphi([\xi \gamma]) \).
Proof. — It suffices to prove this for representations \( \gamma \) that are irreducible. If \( \dim \gamma = 2 \) then \( \xi \gamma = \gamma \) so the statement holds. If \( \dim \gamma = 1 \) then by Proposition 3.5.6 both sides of the equality are zero.

Corollary 3.5.9. — Let \( \overline{\rho} \) be a continuous representation of \( G_K \) of dimension 2 with coefficients in \( \mathbb{F} \) which is not a twist of an extension of the trivial character by the cyclotomic character. Then for any discrete series inertial type \( t \), any Hodge-Tate type \( w \) if \( K = \mathbb{Q}_p \), or for \( w = w_0 \) if \( K \neq \mathbb{Q}_p \), we have

\[
e(\overline{\rho}^\psi)(w, t^+, \overline{\rho})/\pi) = e(\overline{\rho}^\psi)(w, t^-, \overline{\rho})/\pi) = \frac{d_t}{2} e(\overline{\rho}^\psi)(w, t_{\text{ds}}, \overline{\rho})/\pi).
\]

Proof. — The first equality comes from Proposition 3.5.8 and Remark 3.5.3. The last equality follows from Proposition 3.5.4.

Remark 3.5.10. — If \( \overline{\rho} \) is irreducible Corollary 3.5.9 holds even without Theorems 3.5.1 and 3.5.2, because of Remark 2.3.5.

Corollary 3.5.11. — Let \( K = \mathbb{Q}_p \). Suppose that there exists a representation \( \delta \) of dimension 1 of \( \Gamma \) with \( i_\rho(\delta) \neq 0 \) (that is, \( \overline{\rho} \) is a twist of an extension of the trivial character by the cyclotomic character, and then \( i_\rho(\delta) = 1 \)). Then for any discrete series inertial type \( t \), Hodge-Tate type \( w \), character \( \psi \) and pair of conjugate extended types \( (t^+, t^-) \) compatible with \( (t, \psi) \), we have

\[
e(\overline{\rho}^\psi)(w, t^+, \overline{\rho})/\pi) = e(\overline{\rho}^\psi)(w, t^-, \overline{\rho})/\pi),
\]

or \( \ne(\overline{\rho}^\psi)(w, t^+, \overline{\rho})/\pi) - e(\overline{\rho}^\psi)(w, t^-, \overline{\rho})/\pi) = 1.\)

The former takes place in particular when \( \overline{\rho}^\psi(w, t_{\text{ds}}, \overline{\rho}) = 0 \), or \( t \) is of the form (red), or \( t \) is of the form (irr) with \( \pi_t \) of the form \( \text{ind}_{L^\times U_K^0} \psi \) for some ramified quadratic extension \( L \) of \( K \), some \( a \) and some character \( \psi \) of \( L^\times U_K^0 \) with \( \psi(-1) = -1 \) (see Proposition 3.1.1 for the notations).

We see examples where we have \( \ne(\overline{\rho}^\psi)(w, t^+, \overline{\rho})/\pi) - e(\overline{\rho}^\psi)(w, t^-, \overline{\rho})/\pi) = 1 \) in Section 6.

Proof. — Note that we can choose \( \varpi_{\mathbb{Q}_p} \), as we wish to compute the multiplicities. Let \( \sigma_\varphi(t) \) be a choice of representation attached to \( t \) as in Section 3.2.2. We need to compute

\[
[\sigma_\varphi(t) \otimes \varpi_w : \delta] - [\xi \sigma_\varphi(t) \otimes \varpi_w : \delta].
\]

We do this using the results of Proposition 3.3.3 and the remarks that follow for \( \sigma_\varphi(t) \), and the Lemma below for \( \varpi_w \).

Lemma 3.5.12. — In \( \mathfrak{H}(\Gamma_{\mathbb{Q}_p}) \) we have that \( [\det^m] = [\xi^m \delta_n] \) for all \( m \) and

\[
[\text{Sym}^{2n} \mathbb{P}^2] = [\delta_n] + \sum_{i=1}^n [r_{n(p+1)+i(p-1)}] \quad \text{and} \quad [\text{Sym}^{2n+1} \mathbb{P}^2] = \sum_{i=0}^n [r_{n(p+1)+i(p-1)}]
\]

for all \( n \geq 0 \). Moreover, \( r_{n(p+1)+i(p-1)} \) is irreducible for

\[
0 < i < (p + 1)/2 \quad \text{and} \quad (p + 1)/2 < i < p + 1,
\]

J.E.P. — M., 2015, tome 3
Our proof of Theorems 3.5.1 and 3.5.2 is by deducing them from the usual version of the Breuil-Mézard conjecture, in the cases where it is already known. We can hope that this method generalizes to the cases of the Breuil-Mézard conjecture that are not yet known, which leads us to the following.

**Conjecture 3.5.13.** Let \( \mathfrak{p} \) be a continuous representation of \( G_K \) of dimension 2 with coefficients in \( \mathbb{F} \). There exists a positive linear form \( \mu_\mathfrak{p} \) on \( \mathcal{S}(\Gamma) \) with values in \( \mathbb{Z} \) satisfying the following property: for any discrete series inertial type \( \pi \), Hodge-Tate type \( w \), character \( \psi \) lifting \( \omega^{-1} \det \mathfrak{p} \) compatible with \( \pi \) and \( w \), and extended type \( \tau \) compatible with \( (\pi, \psi) \), there exists a choice of representation \( \sigma_\mathfrak{g}(\pi) \) of \( \mathfrak{g} \) such that we have:

\[
e(\mathcal{R}^\square_\psi(w, \pi^+) - (\pi), \pi^+)\) = \mu_\mathfrak{p}[\mathcal{S}(\pi) \otimes \mathcal{S}(w)].
\]

When \( \mathfrak{p} \) is not a twist of an extension of the trivial character by the cyclotomic character, we have

\[
e(\mathcal{R}^\square_\psi(w, \pi^-) - (\pi), \pi^-)\) = \mu_\mathfrak{p}[\mathcal{S}(\pi) \otimes \mathcal{S}(w)],
\]

where \( \pi^- \) is the extended type that is conjugate to \( \pi^+ \).

### 4. Quaternionic modular forms

#### 4.1. Global setting

Let \( F \) be a totally real number field such that for all places \( v \mid p \), \( F_v \) is isomorphic to \( K \). We denote by \( \Sigma_p \) the set of places above \( p \), and we assume that the number of infinite places of \( F \) has the same parity as the cardinality of \( \Sigma_p \). Let \( B \) be the quaternion algebra with center \( F \) that is ramified exactly at the infinite places of \( F \) and at \( \Sigma_p \), which exists thanks to the parity condition.

For all \( v \in \Sigma_p \), we fix an isomorphism between \( B_v \) and the quaternion algebra \( D \) of Section 3. For any finite place \( v \) of \( F \) that is not in \( \Sigma_p \), fix an isomorphism between \( B_v \) and \( M_2(F_v) \) so that \( \mathcal{O}_B \) corresponds to \( \text{GL}_2(\mathcal{O}_{F_v}) \). We fix \( v_0 \in \Sigma_p \) and denote \( \Sigma_p \setminus \{v_0\} \) by \( \Sigma'_p \).

Let \( \mathfrak{w}_K \) be a uniformizer of \( K \). We denote by \( \mathcal{G} \) the group \( \mathcal{G}_{\mathfrak{w}_K} \) of Section 3.2.2. We fix a uniformizer \( \mathfrak{w}_D \) of \( D \) with \( \mathfrak{w}_D^2 = \mathfrak{w}_K \).

#### 4.2. Modular forms

We recall the theory of quaternionic modular forms (see for example [Tay06, §1], and also [Kha01, §4.1] and [GS11, §2] for the situation with a quaternion algebra ramified at \( p \)).

Denote by \( \mathcal{A}_F \subset \mathcal{A}_K \) the ring of finite adeles of \( F \). Let \( U = \prod_v U_v \) be a compact open subgroup of \( (B \otimes F \mathcal{A}_F) \) such that for all finite places \( v \), \( U_v \subset \mathcal{O}_{B_v}^\times \), and for all \( v \in \Sigma_p \), \( U_v = \mathcal{O}_{B_v}^\times \).

Let \( A \) be a topological \( \mathbb{Z}_p \)-algebra. For all \( v \mid p \), let \( (\sigma_v, V_v) \) be a representation of \( U_v \) on a finite free \( A \)-module. We define a representation \( \sigma \) of \( U \) on \( V = \otimes_{v \mid p} V_v \) by letting \( U_v \) act by \( \sigma_v \) for \( v \mid p \) and letting \( U_v \) act trivially for \( v \nmid p \). Let \( \eta \) be a
continuous character $\left(\mathbb{A}_F^I\right)^{\times}/F^\times \to A^\times$ such that for all $v$, the restriction of $\sigma$ and of $\eta$ to $U_v \cap \mathcal{O}_F^\times$ coincide (such a character does not necessarily exist).

Let $S_{\sigma,\eta}(U,A)$ be the set of continuous functions $f : B^\times \setminus (B \otimes_F \mathbb{A}_F^I)^{\times} \to V$ such that:

- for all $g \in (B \otimes_F \mathbb{A}_F^I)^{\times}$ and $u \in U$, $f(gu) = \sigma(u)^{-1}f(g)$,
- for all $g \in (B \otimes_F \mathbb{A}_F^I)^{\times}$ and $z \in (\mathbb{A}_F^I)^{\times}$, $f(gz) = \eta(z)^{-1}f(g)$.

We can extend the action of $U$ on $(\sigma,V)$ to an action of $U(\mathbb{A}_F^I)^{\times}$: we let $(\mathbb{A}_F^I)^{\times}$ act via $\eta$. We say that $U$ is small enough (see for example [Kis09a, §2.1.1]) if

$$\text{for all } t \in (B \otimes_F \mathbb{A}_F^I)^{\times}, \quad (U(\mathbb{A}_F^I)^{\times} \cap t^{-1}D^\times t)/F^\times = 1.$$  

In this case, the functor $(\sigma,V) \mapsto S_{\sigma,\eta}(U,A)$ is exact in $(\sigma,V)$. In the following we will always assume that $U$ is small enough.

Let now $(\hat{\sigma}_{v_0},V_{v_0})$ be a representation of $\mathcal{G}$ with coefficients in $A$, and for $v \in \Sigma_p^\prime$, let $(\sigma_v,V_v)$ be a representation of $U_v \cong \mathcal{O}_D^\times$ as before. Let $\tilde{\sigma}$ be the representation $\tilde{\sigma}_{v_0} \otimes (\otimes_{v \in \Sigma_p} \sigma_v)$ of $\mathcal{G} \times (\prod_{v \in \Sigma_p} U_v)$ on $\otimes_{v \in \Sigma_p} V_v$. Let $\sigma_{v_0}$ be the restriction of $\tilde{\sigma}_{v_0}$ to $U_{v_0} = \mathcal{O}_D^\times$. We define as before $\sigma$ a representation of $U$ on $\otimes_{v \in \Sigma_p} V_v$, and we suppose that the character $\eta$ exists. We define a space of modular forms $S_{\sigma,\eta}(U,A)$ by setting $S_{\sigma,\eta}(U,A) = S_{\sigma,\eta}(U,A)$. We will endow the space $S_{\sigma,\eta}(U,A)$ with an additional structure (a Hecke operator at $v_0$) in Section 4.4.1.

### 4.3. Hecke Algebra

The group $(B \otimes_F \mathbb{A}_F^I)^{\times}$ acts on the set of functions on $(B \otimes_F \mathbb{A}_F^I)^{\times}$ by $g \cdot f(z) = f(zg)$.

Let $S$ be a finite set of places of $F$ containing all places above $p$ and all $v$ such that $U_v$ is not $\mathcal{O}_{\mathcal{D}}^\times$, and $S' \subset S$ the set of places $w$ such that $U_w$ is not $\mathcal{O}_{\mathcal{B}}^\times$. Let $T_S = \mathbb{Z}[T_v,S_v,U_{\varpi_w},v \in S,v \in S']$ be a polynomial ring. We define an action of $T_S$ on $S_{\sigma,\eta}(U,A)$ by:

- $T_v$ is the action of $U_v \left( \begin{array}{cc} \varpi_v & 0 \\ 0 & 1 \end{array} \right) U_v$,
- $S_v$ is the action of $U_v \left( \begin{array}{cc} 0 & \varpi_v \\ 0 & 0 \end{array} \right) U_v$,
- $U_{\varpi_w}$ is the action of $U_w \left( \begin{array}{cc} \varpi_w & 0 \\ 0 & \varpi_w \end{array} \right) U_w$,

where $\varpi_v$ is a uniformizer of $F_v$. The actions of these operators commute, and the definition of the Hecke operators $T_v$ and $S_v$ does not depend on the choice of $\varpi_v$.

Let $T_{\sigma,\eta}(U,A)$ be the $A$-algebra generated by the image of $T_S$ in the ring of endomorphisms of $S_{\sigma,\eta}(U,A)$.

### 4.4. Hecke Operators at Places above $p$

#### 4.4.1. Hecke operators

We fix a representation $\tilde{\sigma}_{v_0}$ of $\mathcal{G}$, and representations $\sigma_v$ of $U_v$ for $v \in \Sigma_p^\prime$ as in Section 4.2. Consider the space of modular forms $S_{\sigma,\eta}(U,A)$ of Section 4.2.

We define an operator $W_{v_0}$ acting on $S_{\sigma,\eta}(U,A)$ by $W_{v_0}(f)(g) = \tilde{\sigma}_{v_0}(v)f(g\varpi_D,v_0)$, where $\varpi_D,v_0$ is the element of $(B \otimes_F \mathbb{A}_F^I)^{\times}$ that is equal to $\varpi_D$ at $v_0$ and 1 everywhere else. One easily checks that $W_{v_0}f$ is indeed an element of $S_{\sigma,\eta}(U,A)$ if $f$ is. Note
that $W_{v_0}^2$ is multiplication by $\eta(\varpi_{K,v_0})^{-1}$, where $\varpi_{K,v_0} = \varpi_{D, v_0}^2$. It is clear from the
definition that $W_{v_0}$ commutes with the action of the Hecke algebra $T_{\sigma, \eta}(U)$.

Suppose that $A$ contains a square root $\alpha$ of $\eta(\varpi_{K,v_0})^{-1}$. Then we get a decomposition

$$S_{\sigma, \eta}(U) = S_{\sigma, \eta}(U, A)^\perp \oplus S_{\sigma, \eta}(U, A)^-,\,$$

where $S_{\sigma, \eta}(U, A)^\perp$ denotes the subspace of $S_{\sigma, \eta}(U, A)$, where $W_{v_0}$ acts by $\pm \alpha$. If we
replace $\sigma_{v_0}$ by $\xi \sigma_{v_0}$ without changing $\alpha$, the space of modular forms $S_{\sigma, \eta}(U, A)$ is
unchanged, $W_{v_0}$ is replaced by $-W_{v_0}$ and the $+$ and $-$ subspaces are exchanged.

4.4.2. The case of type (red). — Consider the following special case: let $(\sigma_{v_0}, V_{v_0})$ be a
representation of $U_{v_0} \simeq \mathcal{O}_{K_v}^N$ over $A$, and $\sigma'_{v_0}$ the representation on $V_{v_0}$ defined by
$\sigma'_{v_0}(g) = \sigma_{v_0}(\varpi_D g^{-1})$. We can define a representation $(\tilde{\sigma}_{v_0}, \tilde{V}_{v_0})$ of $G$ by
$\tilde{V}_{v_0} = V_{v_0} \otimes V_{v_0}, U_{v_0}$ acts by $(\sigma_{v_0}, \sigma'_{v_0})$ and $i$ acts by $\tilde{\sigma}_{v_0}(x,y) = (y,x)$. Fix representations
at places $v \in \Sigma_p$, a character $\eta$ and representations $\sigma$ and $\tilde{\sigma}$ as in Section 4.2.

Let $\alpha$ be a square root of $\eta(\varpi_{K,v_0})^{-1}$. We have two embeddings

$$i_+, i_- : S_{\sigma, \eta}(U, A) \longrightarrow S_{\sigma, \eta}(U, A),$$

given by $i_+(f)(g) = (f(g), \pm \alpha^{-1} \tilde{\sigma}_{v_0}(i)f(g \varpi_{D, v_0}))$. The image of $i_{\pm}$ is $S_{\sigma, \eta}(U, A)^\pm$
and $i_+ + i_-$ is an isomorphism from $S_{\sigma, \eta}(U, A)^2$ to $S_{\sigma, \eta}(U, A)$.

We will make use of this remark in the following situation: $\tilde{\sigma}_{v_0}$ is of the form $\sigma(\mathfrak{t}) \otimes \sigma_{\text{alg}}$ for $\sigma_{\text{alg}}$ the restriction to $\mathcal{G}$ of some algebraic representation of $\text{GL}_2$
(by an embedding as in Section 3.2.3), $\mathfrak{t}$ is an inertial type of the form (red) and $\sigma(\mathfrak{t})$ is the $\mathcal{G}$-representation attached to $\mathfrak{t}$ in Section 3.2.2.

4.5. Galois representations attached to quaternionic modular forms

4.5.1. General results. — Suppose now that $A$ is a $p$-adic field $E$ containing the
unramified quadratic extension $K'$ of $K$ and a square root $\sqrt{\varpi_K}$ of $\varpi_K$. Then there is
an embedding $\mathfrak{u}$ of $\mathcal{G}$ into $\text{GL}_2(E)$ as in Section 3.2.3. Suppose that for all $v \mid p$, the
representation $\sigma_v$ of Section 4.2 is of the form $\sigma_{v,\text{alg}} \otimes \sigma_{v,\text{sm}}$, where $\sigma_{v,\text{sm}}$ is a smooth
representation of $U_v$, and $\sigma_{v,\text{alg}}$ is the restriction to $U_v$ of an algebraic representation of $\text{GL}_2$
via $\mathfrak{u}|_{\mathcal{O}_K^\times}$. We always assume that either $K = \mathbb{Q}_p$ or $\sigma_{v,\text{alg}}$ is trivial for all $v$.
If $K = \mathbb{Q}_p$, $\sigma_{v,\text{alg}}$ is the restriction of a representation of the form $\text{Sym}^n \otimes \det^m$.
and $k = n_v + 2m_v + 1$ does not depend on $v$.

We recall the construction and properties of Galois representations associated to
eigenforms in $S_{\sigma, \eta}(U, E)$. See for example [Kis09b, §3.1.14] for the link between these
spaces of modular forms and the classical spaces of automorphic representations, from
which we deduce the properties of the Galois representations attached to them. Choose
embeddings $i_p, i_\infty$ of $E$ into $\overline{\mathbb{Q}}_p$ and $\mathbb{C}$ respectively.

Let $\sigma_{p,\text{alg}} = \otimes_v \sigma_{v,\text{alg}}$ and $\sigma_{p,\text{sm}} = \otimes_v \sigma_{v,\text{sm}}$. Let $\sigma_{\mathbb{C},\text{alg}} = \sigma_{\text{alg}} \otimes_E \mathbb{C}$, $\sigma_{\mathbb{C},\text{sm}} =
\sigma_{\text{sm}} \otimes_E \mathbb{C}$ and $\sigma_{\mathcal{C}} = \sigma_{\text{alg}} \otimes \sigma_{\text{sm}}$, acting on the space $W_{\mathcal{C}}$. Then $\sigma_{\mathcal{C},\text{alg}}$ can be viewed as
a representation of $B_{\mathcal{C}}^\times = (B \otimes \mathbb{R})^\times$, and $\sigma_{\mathcal{C},\text{sm}}$ is a smooth representation of $U_p = \otimes_v U_v$. Let $U'_p$ be a compact open subgroup of $U_p$ contained in $\ker \sigma_{p,\text{sm}}$, and $U'$

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the compact subgroup of \((B \otimes_F \mathbb{A}_F)^\times\) which is the same as \(U\) but with \(U_p\) replaced by \(U_p^\prime\).

Let \(C^\infty(B^\times \backslash (B \otimes_F \mathbb{A}_F)^\times / U')\) the space of smooth functions with values in \(\mathbb{C}\). It is endowed with a right action of \(B^\infty\).

We denote by \((\sigma^\vee, W^\vee)\) the dual of a representation \((\sigma, W)\). Let

\[
\phi : S_{\sigma, \eta}(U, E) \rightarrow \text{Hom}_{B^\infty}(W^\vee_{C}, C^\infty(B^\times \backslash (B \otimes_F \mathbb{A}_F)^\times / U'))
\]

be the map defined by

\[
\phi(f) = w \mapsto (x \mapsto w(\sigma_{C, \text{alg}}(x_\infty)^{-1}\sigma_{p, \text{alg}}(f(x_\infty))) ),
\]

where \(x = (x_\infty, x_\mathrm{sm}) \in (B \otimes_F \mathbb{A}_F)^\times \times B^\infty\). We denote by

\[
\phi_w(f) \in C^\infty(B^\times \backslash (B \otimes_F \mathbb{A}_F)^\times / U')
\]

the element \(\phi(f)(w)\) for \(w \in W^\vee_{C}\).

Let \(\pi = \otimes_v \pi_v\) be the irreducible automorphic representation of \(B^\times\) generated by some non-zero \(\phi_w(f)\) for \(w \in W^\vee_{C}\) and \(f \in S_{\sigma, \eta}(U, E)\) that is an eigenform for \(T_g\). Then \(\pi_\mathrm{sm}\) is isomorphic to \(W^\vee_{\text{alg}, C}\), and has central character \(\eta(z) = N_{F'/F}(z_\infty)^{1-k}N_{F/F}(z_p)^{k-1}\eta(z_p)^{-1}\).

Let \(\rho_f : G_F \rightarrow \text{GL}_2(\mathbb{Q}_p)\) be the Galois representation attached to \(\pi\), so that for all \(v\) not in \(S\), the characteristic polynomial \(\rho_f(\text{Frob}_v)\) is \(X^2 - t_v X + N(v)s_v\), where \(\text{Frob}_v\) is an arithmetic Frobenius at \(v\), and \(t_v\) and \(s_v\) are the eigenvalues of the Hecke operators \(T_v\) and \(S_v\) acting on \(f\). Then \(\rho_f\) has determinant \(\varepsilon\eta\) and for all \(v \mid p\), \(\rho_f|_{G_{F_p}}\) is potentially semi-stable with Hodge-Tate weights \((m_v, m_v + n_v + 1)\) if \(K = \mathbb{Q}_p\) and \((0,1)\)'s \(\in \text{Hom}(K, \pi_p)\) otherwise, and \(\text{WD}(\rho_f|_{G_{F_p}})_{F-ss}\) is isomorphic to \(\text{LL}_{D}(\pi^\vee)\).

4.5.2. Structure at \(p\). – Let \(v\) be in \(\Sigma_\mathbb{P}\). Let \(\varphi : \sigma^\vee_v \rightarrow \pi\) be given by \(w \mapsto \phi_w(f)\), and \(\varphi_p : \sigma^\vee_v \rightarrow \pi_p\) be the projection to \(\pi_p\). It is a non-zero \(U_v\)-equivariant morphism (where \(U_v\) acts on \(\sigma^\vee_v\) via its action by \(\sigma^\vee_v\)), hence \(\pi_p|_{U_v}\) contains some irreducible constituent of \(\sigma^\vee_v|_{U_v}\). In particular if \(\sigma|_{U_v}\) is a copy of representations \(\sigma\) of \(\tau_v\) attached to some discrete series inertial type \(\tau_v\) as in Section 3.2.1, \(\rho_f|_{G_{F_p}}\) is of type \(\tau_v\) by Proposition 3.2.1.

Fix \(v_0\) in \(\Sigma_\mathbb{P}\) and suppose that \(\sigma|_{U_v}\) is in fact a representation of \(\mathcal{G}\). By the embedding \(\tilde{u}\), the representation \(\sigma_{v_0, \text{alg}}\) also extends to a representation of \(\mathcal{G}\). Suppose that \(W_{v_0} f = \alpha f\), where \(W_{v_0}\) is the Hecke operator defined in Section 4.4.1. Let \(\mu\) be the central character of \(\sigma_{v_0, \text{alg}}\). We extend the representation \(\sigma^\vee_{v_0}\) of \(\mathcal{G}\) to a representation of \(B^\times\) by \(\sigma^\vee_{v_0} \otimes \text{unr}(\alpha(\sqrt{\mathbb{A}_K}))\) (here \(\sigma|_{U_v}\) is seen as a representation of \(D^\times\) via the canonical map \(D^\times \rightarrow \mathcal{G}\)). Then \(\varphi_{v_0}\) is equivariant for the action of the group \(D^\times\) so that \(\pi_{v_0}\) is isomorphic to \(\sigma^\vee_{v_0} \otimes \text{unr}(\alpha(\sqrt{\mathbb{A}_K}))\) if \(\sigma|_{U_v}\) is irreducible. This gives the following.

**Lemma 4.5.1.** – If \(\sigma|_{U_v} = \sigma_\mathcal{G}(\tau)\) for some discrete series inertial type \(\tau\), \(\sigma_{v_0, \text{alg}}\) has central character \(\mu\) and if \(W_{v_0} f = \alpha f\) then \(\rho_f|_{G_{v_0}}\) is of extended type \(\text{LL}_{D}(\sigma_\mathcal{G}(\tau)) \otimes \text{unr}(\alpha(\sqrt{\mathbb{A}_K}))^{-1}\).
5. Proof of the main theorems

5.1. Notation. — In this section we fix $K$ and a continuous representation $\rho : G_K \to \text{GL}_2(F)$.

When $K = \mathbb{Q}_p$, we assume that $p \geq 5$ when $\rho$ is a twist of an extension of the trivial character by the cyclotomic character (we need this condition to apply the results of [Paš15] and [HT15]).

When $K \neq \mathbb{Q}_p$, we assume that $\rho$ is not a twist of an extension of the trivial character by the cyclotomic character, and whenever a Hodge-Tate type $w$ appears we always mean $w = w_0$.

We fix a uniformizer $\varpi_K$ of $K$.

Let $\overline{\psi}$ be the character $\omega^{-1} \det \rho$ of $G_K$, which we see also as a character $K^\times$ via local class field theory. We fix $\alpha \in \mathbb{F}_p$ such that $\alpha^2 = \overline{\psi}(\varpi_K)^{-1}$.

For any irreducible representation $\gamma$ of $\Gamma = \Gamma_K$, we fix an inertial type $t_\gamma$ and a representation $\sigma_\gamma(t_\gamma)$ as in the proof of Proposition 3.3.2, a lift $\psi_\gamma$ of $\overline{\psi}$ that is compatible with $t_\gamma$ and $w_0$, and an extended type $t^+_\gamma$ such that $t^+_\gamma$ is compatible with $(t_\gamma, \psi_\gamma)$ and $t^+_\gamma = \text{LL}_D(\sigma_\gamma(t_\gamma)) \otimes \text{unr}(\alpha_\gamma)^{-1}$ for an $\alpha_\gamma$ lifting $\alpha$.

5.2. Definition of $\mu_\rho$. — We are now able to define the linear form $\mu_\rho$; we define it to be the linear form on $\mathcal{F}(\Gamma)$ such that $\mu_\rho([\gamma]) = c(R^\psi, \psi(t_\gamma, \rho)/\pi)$ for any irreducible representation $\gamma$ of $\Gamma$. It is clear that $\mu_\rho(\gamma) = 0$ if $i_\rho(\gamma) = 0$.

We must now prove that $\mu_\rho$ satisfies the properties claimed in Theorems 3.5.1 and 3.5.2.

Let $t$ be a discrete series inertial type, $w$ a Hodge-Tate type (with $w = w_0$ if $K \neq \mathbb{Q}_p$), $\psi$ a lift of $\overline{\psi}$ that is compatible with $t$ and $w$, and $t^+$ an extended type compatible with $(t, \psi)$.

If $R^\psi(w, t^+, \rho) = 0$ then we have $\mu_\rho([\overline{\psi}(w) \otimes \overline{\psi}_w]) = 0$ by the results of Section 3.4. So we need only prove the equalities of Theorems 3.5.1 and 3.5.2 when $R^\psi_2(w, t^+, \rho) \neq 0$. This is the object of the rest of this section.

5.3. Global realization in characteristic $p$. — We start by realizing $\rho$ in some global Galois representation.

Proposition 5.3.1. — There exist a totally real field $F$ and a continuous irreducible representation $\tau : G_F \to \text{GL}_2(\mathbb{F}_p)$, such that:

1. the number of places of $F$ above $p$ has the same parity as the number of infinite places of $F$,
2. for all $v | p$, $F_v$ is isomorphic to $K$,
3. for all $v | p$, $\tau|_{G_{F_v}} \simeq \rho$,
4. $\tau$ is unramified outside $p$,
5. $\tau$ is totally odd,
6. $\tau$ is modular,
7. the restriction of $\tau$ to $G_{F(\zeta_p)}$ is absolutely irreducible, and if $p = 5$, $\tau$ does not have projective image isomorphic to $\text{PGL}_2(\mathbb{F}_5)$.
Proof: — All conditions except the first follow from [GK14, Cor. A.3]. We can ensure that the first condition is satisfied by taking an \( F \) such that the number of places of \( F \) above \( p \) and the number of infinite places of \( F \) are both even. Indeed, note that the proof of [GK14, Cor. A.3] starts (in [GK14, Prop. A.1]) by considering an auxiliary number field \( E \) which is totally real and such that for all \( v \mid p \), \( E_v \) is isomorphic to \( K \), and the \( F \) we get is a finite extension of \( E \). If we impose to \( E \) the additional condition that \( 2[K : \mathbb{Q}_p] \) divides \([E : \mathbb{Q}]\), then the parity condition will be satisfied.

\[
\]

5.4. Global realizations in characteristic 0

5.4.1. Global data. — From now on we fix a field \( F \) and a representation \( \pi \) satisfying the conditions of Proposition 5.3.1.

Let \( \Sigma_p \) be the set of places of \( F \) above \( p \). We fix a \( v_0 \in \Sigma_p \) and denote \( \Sigma_p \setminus \{v_0\} \) by \( \Sigma_p \) as before.

Let \( B \) be the quaternion algebra with center \( F \) that is ramified exactly at the infinite places of \( F \) and at all places in \( \Sigma_p \). Such a \( B \) exists thanks to condition (1) of Proposition 5.3.1. Let \( D \) be the non-split quaternion algebra over \( K \). Let \( \mathcal{O}_B \) be a maximal order in \( B \). For all \( v \) not dividing \( p \), we fix an isomorphism \( \mathcal{O}_B^\times \cong \text{GL}_2(\mathcal{O}_{F_v}) \), and for all \( v \in \Sigma_p \), we fix an isomorphism \( \mathcal{O}_B^\times \cong \mathcal{O}_D^\times \).

Let \( \varpi_D \) be a uniformizer of \( D = B_{v_0} \) such that \( \varpi_D^2 = \varpi_K \), where \( \varpi_K \) is our fixed uniformizer of \( K \). We set \( \mathcal{G} = \mathcal{G}_{\varpi_K} \).

We choose an auxiliary place \( v_1 \nmid p \) such that \( Nv_1 \neq 1 \pmod{p} \), the ratio of the eigenvalues of \( \pi(\text{Frob}_{v_1}) \) is not \( Nv_1^{1/4} \), and the characteristic of \( v_1 \) is large enough so that for any quadratic extension \( F' \) of \( F \) and any \( \zeta \) a root of unity in \( F' \), \( v_1 \nmid \zeta + \zeta^{-1} - 2 \). The existence of such a place \( v_1 \) follows from [DDT97, Lem. 4.11] and [Kis09a, Lem. 2.2.1].

We let \( U \) be the compact open subgroup of \( (B \otimes_F H_F^1)^\times \) such that \( U_v = \mathcal{O}_D^\times \) for \( v \in \Sigma_p \), \( U_v = \text{GL}_2(\mathcal{O}_{F_v}) \) for \( v \notin \Sigma_p \) and \( v \neq v_1 \), and finally \( U_{v_1} \) is the set of elements of \( \text{GL}_2(\mathcal{O}_{F_{v_1}}) \) that are upper triangular unipotent modulo \( v_1 \). The last condition we imposed on \( v_1 \) ensures that \( U \) is small enough in the sense of Section 4.2 (see [Kis09a, §2.1.1]).

5.4.2. Modular lift. — We want know to show that the representation \( \pi \) can be lifted to an appropriate modular Galois representation.

Lemma 5.4.1. — For all \( v \in \Sigma_p \), let \( \tau_v \) be an inertial type such that \( \gamma_v = \sigma_D(\tau_v) \) is an irreducible representation of \( \ell^\times \), and \( \psi_v \) be a character of \( G_{F_v} \). Suppose that the ring

\[
R_p = \oplus_{v \in \Sigma_p} R_{\ell^\times, \psi_v}(w_0, t_v^{da}, \mathcal{O})
\]

is not zero. Let \( \sigma = \otimes_v \sigma_D(\tau_v) \). Then there exists \( \eta \) satisfying the compatibility conditions with \( \sigma \) of Section 4.2, and which restricts to \( \psi_v \) on \( F_v^\times \) for all \( v \in \Sigma_p \), and the space of modular forms \( S_{\sigma, \eta}(U, \sigma) \) contains an eigenform \( f \) whose associated Galois representation has its reduction modulo \( p \) isomorphic to \( \pi \).
Proof: — For the existence and construction of $\eta$, see [GK14, §5.4.1].

We now prove the existence of the eigenform $f$.

Suppose first that none of the $t_v$ is scalar. Then each $R^{\square, \psi_s}(w_0, t_v^{ds}, \rho)$ parametrizes only potentially crystalline representations. Corollary 3.1.7 of [Gee11] applied to our situation gives that for each irreducible component of $\text{Spec} \, R_p[1/p]$, there exists a lift $r$ of $\mathfrak{g}$ that is modular, unramified outside $p$, with determinant $\eta$, potentially crystalline with Hodge-Tate weights $(0, 1)$ at each $v \in \Sigma_p$ and for each $F_v \rightarrow \overline{\mathbb{Q}}_p$ and defining a point on the given irreducible component. The representation $r$ comes from some automorphic form on a quaternion algebra over $F$. Thanks to the local conditions on $r$, we can suppose that $r$ comes from a modular form $f$ on $B$, and that $f \in S_{\sigma, \eta}(U, \rho)$. This proves the claim in this case, and in particular whenever $\overline{\rho}$ is not isomorphic to a twist of $(\begin{smallmatrix} \eta & 0 \\ 0 & 1 \end{smallmatrix})$ by Proposition 3.5.6.

When $\overline{\rho}$ is isomorphic to a twist of $(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ (in the case $K = \mathbb{Q}_p$), Corollary 3.1.7 of [Gee11] is not enough: for dim $\gamma_v = 1$, we need points on $\text{Spec} \, R^{\square, \psi_s}(w_0, t_v^{ds}, \overline{\rho})[1/p]$ corresponding to potentially semi-stable representations that are not potentially crystalline. We make use of [BD14, Th.3.2.2] instead: it allows us to choose $r$ such that each $r|_{G_{F_v}}$ is not potentially crystalline when dim $\gamma_v = 1$. Note that in this situation, each $R^{\square, \psi_s}(w_0, t_v^{ds}, \overline{\rho})$ is irreducible.

Let $t$ be a discrete series inertial type and $w$ be a Hodge-Tate type, and let $\psi$ be the character defined in Section 5.1.

**Proposition 5.4.2.** — Suppose that $R^{\square, \psi}(w, t^{ds}, \rho) \neq 0$. There exists a character $\eta$ of $(K_F^0)_{\infty/\infty}$ which restricts to $\psi$ on $F_v^\times$ for all $v \in \Sigma_p$ such that, for $\sigma = \bigotimes_{v \in \Sigma_p} (\sigma_D(t) \otimes \sigma_w)$, the space of modular forms $S_{\sigma, \eta}(U, \rho)$ contains an eigenform $f$ whose associated Galois representation has its reduction modulo $p$ isomorphic to $\mathfrak{g}$.

**Proof:** — As before, the existence of $\eta$ satisfying the compatibility conditions with $\sigma$ of Section 4.2 and whose restriction to $F_v^\times$ coincides $\psi$ for all $v \in \Sigma_p$, comes from [GK14, 5.4.1].

If $R^{\square, \psi}(w, t^{ds}, \rho) \neq 0$, by Theorems 3.4.1 and 3.4.3 there exist an irreducible representation $\gamma$ of $\prod_{v \in \Sigma_p} \ell^\times$, $\gamma = \bigotimes_{v \in \Sigma_p} \gamma_v$ appearing as an irreducible constituent of $\bigotimes_{v \in \Sigma_p} (\tau_{F_v}(t) \otimes \tau_{w})$ and characters $\psi_v$ that are equal to a finite order character times the cyclotomic character such that $\bigotimes_{v \in \Sigma_p} R^{\square, \psi_s}(w_0, t_v^{ds}, \rho) \neq 0$, where $t_v$ is the inertial type attached to the representation $\gamma_v$ as in the proof of Proposition 3.3.2.

We can apply Lemma 5.4.1 to the family of types $(t_v)$, as by construction $\sigma_D(t_v) = \gamma_v$ is an irreducible representation of $\ell^\times$. Then the result follows from [GS11, Lem. 2.1].

In particular in the conditions of Proposition 5.4.2, $r_f|_{G_{F_v}}$ has determinant $\epsilon \psi$, inertial type $t$ and Hodge-Tate type $w$, where $r_f$ is the Galois representation attached to $f$ as in Section 4.5.1.
5.5. Patching. — Let \((t_v, w_v)_{v \in \Sigma_p}\) be a family of discrete series inertial types and Hodge-Tate types, with \(t_{v_0} = t\) and \(w_{v_0} = w\). Let also \((\psi_v)_{v \in \Sigma_p}\) be a family of characters of \(G_K = G_{F_v}\).

We suppose in this paragraph that there exists a character \(\eta\) of \((\mathbb{A}_F^\times / F^\times)\) that restricts to \(\psi_v\) on \(F_v^\times\) for all \(v \in \Sigma_p\) and such that for \(\sigma = \otimes_{v \in \Sigma_p} (\sigma_D(t_v) \otimes \sigma_{w_v})\), the space of modular forms \(S_{\sigma,\eta}(U, \mathcal{O})\) contains an eigenform \(f\) whose associated Galois representation has its reduction modulo \(p\) isomorphic to \(\tau\).

Consider now
\[
\tilde{\sigma} = (\sigma_\mathcal{O}(t)) \otimes \sigma_w \otimes (\otimes_{v \in \Sigma_p} (\sigma_D(t_v) \otimes \sigma_{w_v})).
\]
It is a representation of \(\mathcal{O} \times \prod_{v \in \Sigma_p} U_v\), where \(U_v = \mathcal{O}_v^\times\). The space of modular forms \(S_{\tilde{\sigma},\eta}(U, \mathcal{O})\) is either the same as \(S_{\sigma,\eta}(U, \mathcal{O})\) as an \(\mathcal{O}\)-module and a \(T_S\)-algebra (if \(t\) is of the form (scq) or (irr), as in this case \(\sigma_\mathcal{O}(t)\) acts on the same space as \(\sigma_D(t)\)) or is two copies of \(S_{\sigma,\eta}(U, \mathcal{O})\) (if \(t\) is of the form (red), as in this case \(\sigma_\mathcal{O}(t)\) acts on a space which is two copies of \(\sigma_D(t)\), see Section 4.4.2). In any case, it contains a copy of the form \(f\) given by Proposition 5.4.2. Let \(m\) be a maximal ideal of \(T_S\) containing \(p\) such that \(f\) is in \(S_{\sigma,\eta}(U, \mathcal{O})_m\).

Let \(T_{\tilde{\sigma},\eta}(U, \mathcal{O})_m\) be the image of \(T_{S,m}\) in the endomorphism ring of \(S_{\tilde{\sigma},\eta}(U, \mathcal{O})_m\). Any eigenform in \(S_{\tilde{\sigma},\eta}(U, \mathcal{O})_m\) has an associated Galois representation with residual representation isomorphic to \(\tau\), which is absolutely irreducible. By the main result of [Tay89] and the Jacquet-Langlands correspondence (see [Tay06, Lem. 1.3]) we deduce that we have a Galois representation
\[
\tilde{\tau}_m : G_F \longrightarrow \text{GL}_2(T_{\tilde{\sigma},\eta}(U, \mathcal{O})_m)
\]
coming from all the eigenforms in \(S_{\tilde{\sigma},\eta}(U, \mathcal{O})_m\). In particular, \(\tilde{\tau}_m\) is unramified outside \(p\).

Let \(\mathcal{R}\) be the universal deformation ring for deformations of \(\tau\) that are unramified outside \(\Sigma_p\) and \(\mathcal{R}^\square\) the framed analogue. Then we have a map \(\mathcal{R} \rightarrow T_{\tilde{\sigma},\eta}(U, \mathcal{O})_m\) coming from \(\tilde{\tau}_m\), so \(\mathcal{R}\) acts on \(S_{\tilde{\sigma},\eta}(U, \mathcal{O})_m\) via the Hecke operators.

Let \(R^\square(\overline{\eta})\) the ring classifying lifts of \(\overline{\eta}\), and \(\mathcal{R}^\square_\mathcal{O} = \otimes_{v \in \Sigma_p} R^\square(\overline{\eta}_{v})\). Let \(R_0\) be
\[
\otimes_{v \in \Sigma_p} R^\square(\overline{\eta}_{v}, w_{v,0}, \epsilon_{v}),
\]
seen as a quotient of \(\mathcal{R}^\square_\mathcal{O}\), and \(\mathcal{R}^\prime = \mathcal{R}^\square \otimes_{\mathcal{O}} R_0\). The ring \(\mathcal{R}^\prime\) is the universal ring for lifts of \(\tau\) that are unramified outside \(p\) and potentially semi-stable of inertial type \(t_v\), of some discrete series extended type, of Hodge-Tate type \(w_v\) and determinant \(\epsilon_{v}\) at each place \(v \in \Sigma_p\). Let \(M_0 = \mathcal{R}^\square \otimes_{\mathcal{O}} S_{\tilde{\sigma},\eta}(U, \mathcal{O})_m\). Then the action of \(\mathcal{R}^\square\) on \(M_0\) factors through \(\mathcal{R}^\prime\) by the results recalled in Section 4.5.2.

We decompose the reduction \(\overline{\tau}\) of \(\tilde{\tau}\) modulo \(p\) as a direct sum \(\overline{\tau} = \otimes_{v} \gamma_{v} \otimes \gamma_{v}^\times\), with \(\gamma = \gamma_v \otimes (\otimes_{v \in \Sigma_p} \gamma_v)\), of representations of \(\Gamma \times \prod_{v \in \Sigma_p} \mathcal{U}_v\). This gives a decomposition
\[
\sigma = \otimes_{v} \gamma_v \otimes \gamma_v^\times\text{ as a representation of } \mathcal{O} \times \prod_{v \in \Sigma_p} \mathcal{U}_v.
\]

Using the techniques of [Kis09a, §2.2] we construct the following objects:

1. a ring \(R_\infty\) which is a power series ring on \(R_0\) (this is \(R_\infty\) of [Kis09a]),
2. a ring \(S_\infty\) which is a power series ring on \(\mathcal{O}\) (this is \(R_\infty[[\Delta_\infty]]\) of [Kis09a]),
(3) an \((S_\infty, R_\infty)\)-module \(M_\infty\) that is free as an \(S_\infty\)-module, and such that \(M_0\) is a quotient of \(M_\infty\).

(4) a \((S_\infty, R_\infty)\)-linear operator \(W_{v_0}\) on \(M_\infty\) compatible with the Hecke operator \(W_{v_0}\) on \(M_0\) defined in Section 4.4.1, with \(W_{v_0}^2 = \psi_{v_0}(\varpi_K)^{-1}\),

(5) a decomposition \(M_\infty \otimes \overline{F} = \oplus \gamma \overline{M}_{\infty, \gamma}\) as \((S_\infty, R_\infty)\)-module, such that each \(\overline{M}_{\infty, \gamma}\) is a finite free \(S_\infty \otimes \overline{F}\)-module, and such that moreover \(\overline{M}_{\infty, \gamma}\) does not depend on \((t_{v_0}, w_{v_0})\) or the \((t_v, w_v)\), in a sense that is explained below.

The only part which is not a copy of the arguments of [Kis09a] is (4). The module \(M_\infty\) is built by patching modules \(M_n = \mathcal{A} \otimes \mathbb{F} S_{\check{r}, n}(U_n, \vartheta_m)\), for some choice of compact open subgroup \(U_n\) which is maximal at \(v\) for all \(v \mid p\), and for some choice of maximal ideal \(m_n\) (with \(U_0 = U\) and \(m_0 = m\)). In particular, for each \(n\) there is a Hecke operator \(W_{v_0}\) on \(M_n\) as in Section 4.4.1 with \(W_v^2 = \eta(\varpi_{K,v_0})^{-1} = \psi_{v_0}(\varpi_K)^{-1}\) which is compatible with the surjective map \(M_n \to M_0\). Moreover, the action of \(W_{v_0}\) commutes with the right action of the subgroup \(B \otimes \mathbb{F} k_v^\times\) of elements that are trivial at \(v_0\), hence \(W_{v_0}\) is \(R_\infty\)- and \(S_\infty\)-linear. Once a square root of \(\eta(\varpi_{K,v_0})^{-1}\) is fixed, then for each \(n\) we have a decomposition \(M_n = M_n^+ \oplus M_n^-\) in sub-\((S_\infty, R_\infty)\) modules according to the eigenvalues of \(W_{v_0}\), and this decomposition is compatible to the decomposition \(M_0^+ \oplus M_0^-\). We apply the patching argument not only to \(M_n\), but to the decomposition \(M_n^+ \oplus M_n^-\), which gives a decomposition \(M_\infty = M_\infty^+ \oplus M_\infty^-\) and an operator \(W_{v_0}\) on \(M_\infty\) with the required properties. Note that \(M_\infty^+\) and \(M_\infty^-\) are also free as \(S_\infty\)-modules.

Let \(\gamma\) be an irreducible smooth representation of \(\mathcal{G} \times \prod_{v \in \Sigma_v} U_v\) in characteristic \(p\), and \(\check{\gamma}\) a smooth lift of \(\gamma\) (as in Proposition 3.3.2 for the part which is a representation of \(\mathcal{G}\) and by Teichmüller lift for the parts which are representations of \(U_v = \mathcal{O}_p^\times\)). By Lemma 5.4.1, there exists a character \(\eta_\gamma\) of \((k_p^\times) / F^\times\) lifting \(\eta\) and characters \((\psi_\gamma,v)_{v \in \Sigma_v}\), satisfying the conditions at the beginning of this Paragraph, so we can make the constructions with the space of modular forms \(S_{\check{r}, \eta_\gamma}(U, \vartheta)\). We denote by \(M_{\infty, \gamma}\) the patched module we obtain (although it depends not only on \(\gamma\) but also on other data). Then (5) means that \(\overline{M}_{\infty, \gamma}\) is isomorphic to the reduction modulo \(p\) of \(M_{\infty, \gamma}\). We also have a Hecke operator \(W_{v_0}\) on \(M_{\infty, \gamma}\) and a decomposition \(M_{\infty, \gamma} = M_{\infty, \gamma}^+ \oplus M_{\infty, \gamma}^-\) that reduces to \(\overline{M}_{\infty, \gamma} = \overline{M}_{\infty, \gamma}^+ \oplus \overline{M}_{\infty, \gamma}^-\).

5.6. Equality of multiplicities. — Recall that \(d_\mathfrak{k} = 2\) if \(\mathfrak{k}\) is of the form (red) and \(d_\mathfrak{k} = 1\) otherwise. As in [Kis09a, Lem. (2.2.11)], \(M_\infty\) has rank 0 or \(d_\mathfrak{k}\) at each generic point of \(R_\infty\) and

\[
e(M_\infty / \pi M_\infty, R_\infty / \pi R_\infty) \leq d_\mathfrak{k} e(R_\infty / \pi R_\infty),
\]

with equality if and only if the support of \(M_\infty\) is all of \(\text{Spec } R_\infty\) (we already know that it is a union of irreducible components of \(\text{Spec } R_\infty\)).

Our main ingredient is the following Proposition, which we prove using the results of Section 3.4 and the methods of [Kis09a, §(2.2)].

Proposition 5.6.1. — We have \(e(M_\infty / \pi M_\infty, R_\infty / \pi R_\infty) = d_\mathfrak{k} e(R_\infty / \pi R_\infty)\).
Proof: — Suppose first that \( w_v = w_0 \) for all \( v \in \Sigma_p \). By reasoning as in the proof of Proposition 5.4.2, we see that the support of the module \( M_0 \) meets each irreducible component of Spec \( R_\infty[1/p] \). The irreducible components of Spec \( R_\infty[1/p] \) are connected components, hence we can apply the criterion of [GK14, Lem. 4.3.8]: the equality of Proposition 5.6.1 holds if and only if the support of \( M_0 \) meets every irreducible component of Spec \( R_\infty[1/p] \).

Let us now return to the case without conditions on the \( w_v \) (in the case \( K = \mathbb{Q}_p \)). It follows from the decomposition \( M_\infty \otimes \mathbb{F} = \oplus_{\gamma} \overline{M}_{\infty, \gamma} \) that
\[
e(M_\infty/\pi, R_\infty/\pi) = \sum_\gamma n_\gamma \epsilon(\overline{M}_{\infty, \gamma}, R_\infty/\pi).
\]
Moreover,
\[
\epsilon(R_\infty/\pi R_\infty) = \prod_{v \in \Sigma_p} \epsilon(R^{\otimes \psi_v}(w_v, t_{\psi_v}^{ds}, \mathfrak{p})/\pi).
\]
As \( K = \mathbb{Q}_p \), it follows from the theorems of Section 3.4 that we have
\[
d_\epsilon \epsilon(R_\infty/\pi R_\infty) = \sum_\gamma n_\gamma (\dim_{\gamma(v_0)} \prod_{v \in \Sigma_p} \epsilon(R^{\otimes \psi_v}(w_v, t_{\psi_v}^{ds}, \mathfrak{p})/\pi).
\]
We denote by \( R_\infty, \gamma \) the ring that is the analogue of \( R_\infty \) but with \((w_v, t_{\psi_v})\) instead of \((w_v, t_v)\) for all \( v \in \Sigma_p \), and the characters \((\psi_{\gamma(v)}, v)\) defined at the end of Section 5.5 instead of \((\psi_v)\). Let also \( R_p, \gamma \) be the analogue of \( R_p \).

Then the image of \( R_\infty \) and \( R_\infty, \gamma \) in the endomorphisms of \( \overline{M}_{\infty, \gamma} \) is the same, as follows from (4) of the properties of patching, hence
\[
e(\overline{M}_{\infty, \gamma}, R_\infty/\pi) = \epsilon(\overline{M}_{\infty, \gamma}, R_\infty, \gamma/\pi).
\]
Moreover,
\[
\epsilon(R_\infty, \gamma/\pi) = \prod_{v \in \Sigma_p} \epsilon(R^{\otimes \psi_{\gamma(v)}}(w_0, t_{\psi_v}^{ds}, \mathfrak{p})/\pi),
\]
and we have \( \epsilon(\overline{M}_{\infty, \gamma}, R_\infty, \gamma/\pi) = (\dim_{\gamma(v_0)}) \epsilon(R_\infty, \gamma/\pi) \) by applying the part of Proposition 5.6.1 that we have already proved to \( \overline{M}_{\infty, \gamma} \) and \( R_\infty, \gamma \), which concludes the proof of Proposition 5.6.1 in the general case. \( \square \)

5.7. Action of the Hecke operator at \( p \). — We apply the results of the preceding paragraphs in the situation coming from Proposition 5.4.2, so in particular \( w_v = w \) and \( t_v = t \) and \( \psi_v = \psi \) for all \( v \in \Sigma_p \). Recall that we have chosen a square root \( \alpha \) of \( \eta(w_K, v_0) = \eta(w_K)^{-1} \).

Let \((t^+, t^-) = (t_{\psi_0}^+, t_{\psi_0}^-)\) be a pair of conjugate extended types compatible to \((t, \psi)\). We set
\[
R^+ = R^{\otimes \psi_i}(w_0, t_{\psi_0}^+, \mathfrak{p}) \otimes R^{\otimes \psi_i}(w_0, t_{\psi_0}^{ds}, \mathfrak{p}) R_\infty,
\]
\[
R^- = R^{\otimes \psi_i}(w_0, t_{\psi_0}^-, \mathfrak{p}) \otimes R^{\otimes \psi_i}(w_0, t_{\psi_0}^{ds}, \mathfrak{p}) R_\infty
\]
(so that \( R_\infty = R^{+}_\infty = R^{+}_\infty \) when \( t \) is of the form (red), and these rings differ only when \( t \) is of the form (scal) or (irr)).
We make a choice for $\sigma_g(t)$ so that

$$t^+ = LL_D(\sigma_g(t)) \otimes \text{unr}(a\sqrt{\omega K})^{-1}$$

for a lift $a$ of $\alpha$ with $a^2 = \psi(\omega K)^{-1}$ (recall that $|w| = n + 2m$ if $K = \mathbb{Q}_p$ and $w = (n, m)$, and set $|v_0| = 0$ for any $K$).

We can consider the Hecke operator $W_{v_0}$ acting on all the spaces of modular forms that we have defined. This gives a decomposition $M_\infty = M_\infty^+ \oplus M_\infty^-$ as in Section 5.5, where $M_\infty^+$ is the submodule on which $W_{v_0}$ acts by a lift of $\alpha$, and decompositions $M_{n} = M_{n}^+ \oplus M_{n}^-$ for the modules $M_{n}$.

The action of $R_{\infty}$ on $M_\infty^+$ factors through $R_{\infty}^+$, and similarly for $R_{\infty}^-$. Indeed this is true for each $M_{n} = M_{n}^+ \oplus M_{n}^-$ by Lemma 4.5.1.

We can do the same thing for each irreducible representation $\gamma$ of $\Gamma$; recall that for each $\gamma$ we made a choice in Section 5.1 of an inertial type $t_{\gamma}$ and a representation $\sigma\gamma(t_{\gamma})$ of $H$ such that $\overline{\sigma}\gamma(t_{\gamma})$ is isomorphic to $\gamma$ and an extended type

$$t^+_\gamma = LL_D(\sigma\gamma(t_{\gamma})) \otimes \text{unr}(a_{\gamma})^{-1}$$

for an $a_{\gamma}$ lifting $\alpha$ with $W_{v_0}^2 = a_{\gamma}^2$ on $M_{\infty, \gamma}$. Let

$$t^-_{\gamma} = LL_D(\xi\sigma\gamma(t_{\gamma})) \otimes \text{unr}(a_{\gamma})^{-1}
= LL_D(\sigma\gamma(t_{\xi\gamma})) \otimes \text{unr}(a_{\gamma})^{-1}
= LL_D(\sigma\gamma(t_{\gamma})) \otimes \text{unr}(-a_{\gamma})^{-1}.$$

For $\gamma$ of dimension 2 we set $R_{\infty, \gamma}^+ = R_{\infty, \gamma}^- = R_{\infty, \gamma}$ and for $\gamma$ of dimension 1 we set

$$R_{\infty, \gamma}^+ = R_{\eta, \gamma}^{\psi}(w_0, t_{\gamma}^+, \overline{\rho}) \otimes R_{\eta, \gamma}^{\psi}(w_0, t_{\gamma}^+, \rho) R_{\infty, \gamma},
R_{\infty, \gamma}^- = R_{\eta, \gamma}^{\psi}(w_0, t_{\gamma}^-, \overline{\rho}) \otimes R_{\eta, \gamma}^{\psi}(w_0, t_{\gamma}^-, \rho) R_{\infty, \gamma}.$$

Then for all $\gamma$ the action of $R_{\infty, \gamma}$ on $\overline{M}_{\infty, \gamma}$ factors through $R_{\infty, \gamma}^\pm$ as before. Note that $t_{\gamma}^- = t_{\xi\gamma}^-$ and $R_{\infty, \gamma}^- = R_{\infty, \xi\gamma}^+$. Note that the decompositions $M_\infty = M_\infty^+ \oplus M_\infty^-$ and $\overline{M}_{\infty, \gamma} = \overline{M}_{\infty, \gamma}^+ \oplus \overline{M}_{\infty, \gamma}^-$ for all $\gamma$ are compatible, that is,

$$\overline{M}_{\infty}^\pm \otimes F = \oplus \gamma(\overline{M}_{\infty, \gamma}^\pm)^{n_\gamma}.$$

In particular, we have $e(M_{\infty}^\pm/\pi, R_{\infty}^\pm/\pi) = e(M_{\infty}^\pm/\pi, R_{\infty}^-/\pi)$, hence

$$e(M_{\infty}^\pm/\pi, R_{\infty}^\pm/\pi) + e(M_{\infty}^\pm/\pi, R_{\infty}^-/\pi) = e(M_{\infty}^\pm/\pi, R_{\infty}^\pm/\pi).$$

We also have that $e(M_{\infty}^\pm/\pi, R_{\infty}^\pm/\pi) \leq e(R_{\infty}^\pm/\pi)$ by the same argument as in [Kis09a, Lem. (2.2.11)]. Moreover,

$$d_{\gamma} e(R_{\infty}^+/\pi) = e(R_{\infty}^+/\pi) + e(R_{\infty}^-/\pi)$$

(see Proposition 3.5.4). Hence we deduce from Proposition 5.6.1 that

$$e(M_{\infty}^+/\pi, R_{\infty}^+/\pi) = e(R_{\infty}^+/\pi) \text{ and } e(M_{\infty, \gamma}^+/\pi, R_{\infty, \gamma}^+/\pi) = e(R_{\infty, \gamma}^+/\pi) $$

for all $\gamma$. 

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Finally we get that
\[ e(R^{\Box, \psi}(w, t^+, \overline{p})/\pi) = \sum_{\gamma} e(R^{\Box, \psi}(w_0, t^+_0, \overline{p})/\pi) e(R^{\Box, \psi}(w, t^+, \overline{p})/\pi), \]
as \( e(R_{\infty}/\pi) \neq 0 \). As the right-hand side is \( \mu_\overline{p}([\overline{\sigma}_\overline{\psi}(t) \otimes \overline{\sigma}_w]) \) by the definition of \( \mu_\overline{p} \) in Section 5.2, we get that
\[ e(R^{\Box, \psi}(w, t^+, \overline{p})/\pi) = \mu_\overline{p}([\overline{\sigma}_\overline{\psi}(t) \otimes \overline{\sigma}_w]), \]
which finishes the proof of Theorems 3.5.1 and 3.5.2.

6. Application

6.1. Computation of weights. — In this section we suppose \( p \geq 5 \).

Let \( \overline{p} \) be a continuous representation \( G_{Q_p} \rightarrow \text{GL}_2(F) \) such that \( \overline{p}|_{I_p} = (\begin{smallmatrix} \omega & 1 \\ 0 & 1 \end{smallmatrix}) \otimes \omega^n \).

We compute \( \mu_\overline{p} \) in this case for the choice \( \overline{\omega}|_{Q_p} = \overline{p} \).

For a representation of the form
\[ \overline{p} = \left( \begin{array}{cc} \omega & * \\ 0 & 1 \end{array} \right) \otimes \omega^n \otimes \text{unr}(x) \]
then \( \overline{\omega}(p) = x^2 \). In order to apply Theorem 3.5.1 we need to make a choice of a square root of \( \overline{\omega}(p)^{-1} \), and we take this square root to be \( \alpha = x^{-1} \).

Lemma 6.1.1. — Let \( \overline{p}|_{I_p} = (\begin{smallmatrix} \omega & 1 \\ 0 & 1 \end{smallmatrix}) \otimes \omega^n \) for some \( n \) with * “très ramifié” (and non-zero), then \( \mu_\overline{p}(\xi^n \delta_n) = 1 \) and all other \( \mu_\overline{p}(\gamma) \) are 0.

Let \( \overline{p}|_{I_p} = (\begin{smallmatrix} \omega & 1 \\ 0 & -1 \end{smallmatrix}) \otimes \omega^n \) for some \( n \) with * “peu ramifié” but non-zero, then \( \mu_\overline{p}(\xi^n \delta_n) = 1 \) and \( \mu_\overline{p}(r_n(p+1)+p-1) = 2 \) and all other \( \mu_\overline{p}(\gamma) \) are 0.

Let \( \overline{p}|_{I_p} = (\begin{smallmatrix} \omega & 1 \\ 0 & 0 \end{smallmatrix}) \otimes \omega^n \) for some \( n \), then \( \mu_\overline{p}(\xi^n \delta_n) = 1 \) and \( \mu_\overline{p}(r_n(p+1)+p-1) = 4 \) and all other \( \mu_\overline{p}(\gamma) \) are 0.

Proof. — We can compute \( e(R^{\Box, \psi}(w, \text{triv}^{ds}, \overline{p})/\pi) \) for any Hodge-Tate type \( w \) using the formula coming from the Breuil-Mézard conjecture for \( \text{GL}_2 \) and the list of the Serre weights with their multiplicities given in [BM02, §2.1.2]. Then we compare this to the formula for this multiplicity given by Theorem 3.5.1, using also the formula given by Lemma 3.5.12.

We compute first \( e(R^{\Box, \psi}(w, \text{triv}^{ds}, \overline{p})/\pi) \) for Hodge-Tate types of the form \( w = (0, m) \). We get that \( \mu_\overline{p}(\xi^n \delta_m) = 0 \) for \( \alpha = 0, 1 \) if \( m \) is not equal to \( n \) modulo \( p-1 \), and \( \mu_\overline{p}(\delta_n) + \mu_\overline{p}(\xi^n \delta_n) = 1 \). Using the computations in [BM02, §5.2.1] we see that in fact \( \mu_\overline{p}(\xi^n \delta_n) = 1 \).

By computing \( e(R^{\Box, \psi}(w, \text{triv}^{ds}, \overline{p})/\pi) \) for Hodge-Tate types \( w \) of the form \( (a, b) \) for \( a > 0 \) we can find the value of \( \mu_\overline{p}(r) \) for representations \( r \) of dimension 2.

6.2. An application to congruences modulo \( p \) in \( \mathcal{S}_k(\Gamma_0(p)) \). — Let \( f \) be a newform in \( \mathcal{S}_k(\Gamma_0(p)) \). Then \( a_p(f) = \pm p^{k/2-1} \). We denote by \( \rho_f \) the \( p \)-adic Galois representation associated to \( f \), \( r_f \) its reduction modulo \( p \), and \( r_{f,p} \) its restriction to a decomposition group \( G_{Q_p} \) at \( p \).

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Theorem 6.2.1. — Let \( k > 2 \) be an even integer, \( f \) a newform in \( S_k(\Gamma_0(p)) \).

1. Suppose that \( r_{f,p} \) is of the form
   \[
   \begin{pmatrix}
     \omega & * \\
     0 & 1
   \end{pmatrix} \otimes \omega^{k/2-1} \otimes \text{unr}(x)
   \]
   for some \( x \) and \( * \) “très ramifié” (and non-zero) and \( k \leq 2p + 2 \). Then
   \[
   x^{-1} = (-1)^{k/2-1}(a_p(f)/p^{k/2-1}).
   \]
   In particular, there does not exist a newform \( g \) in \( S_k(\Gamma_0(p)) \) congruent to \( f \) modulo \( p \) such that \( a_p(g) = -a_p(f) \).

2. Suppose that either \( r_{f,p}|_{\Gamma_0} \) is not of the form \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \omega^{k/2-1} \) with \( * \) “très ramifié” or that \( k > 2p + 2 \). If \( r_{f|G_0(p)} \) is absolutely irreducible then there exists a newform \( g \in S_k(\Gamma_0(p)) \) congruent to \( f \) modulo \( p \) such that \( a_p(g) = -a_p(f) \).

Proof. — Let \( u_p(f) = a_p(f)p^{1-k/2} \). By the results of [Sai09], \( \rho_f \) is a semi-stable, non-crystalline representation with extended type
\[
\begin{pmatrix}
(\omega) \\
0 & 1
\end{pmatrix} \otimes \omega^{k/2-1} \otimes \text{unr}(x),
\]
with \( * \) “très ramifié”. By the existence of \( f, R^{\square,\psi_k}(w_k, \tau_f, r_{f,p}) \) is non-zero. With the normalization \( \varpi_{Q_p} = \sqrt{p} \) as before, there is a choice of \( i \in \mathbb{Z}/2\mathbb{Z} \) with \( \sigma_{g^i}(\text{triv}) = \xi^i \) such that
\[
\tau_f = (1 \oplus \| \cdot \|) \otimes \text{unr}((-1)^i) \otimes \text{unr}(y^{-1}p^{1-k/2})
\]
for some \( y \) lifting \( x^{-1} \), and then
\[
e_{R^{\square,\psi_k}(w_k, \tau_f, r_{f,p})/\pi} = \mu_{\tau_{f,p}}([\xi^i \sigma_{w_k}]).
\]
As \( k < 2p + 2 \), by Lemma 3.5.12 and Lemma 6.1.1 this can be non-zero only if \( i = k/2 - 1 \), that is, \( y = (-1)^{k/2-1}u_p(f) \), which gives the result (note that we could apply the same method for \( f \in S_k(\Gamma_0(Np)) \) new at \( p \) for any \( N \) such that \( p \nmid N \).

Suppose now that either \( r_{f,p}|_{\Gamma_0} \) is not of the form \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \omega^{k/2-1} \) with \( * \) “très ramifié” or that \( k > 2p + 2 \). By the existence of \( f, R^{\square,\psi_k}(w_k, \tau_f, r_{f,p}) \) is non-zero and then Corollary 3.5.9 or the computations of Lemma 6.1.1 show that both \( R^{\square,\psi_k}(w_k, \tau_f, r_{f,p}) \) and \( R^{\square,\psi_k}(w_k, \tau'_f, r'_{f,p}) \) are non-zero when \( k > 2 \), where \( \tau'_f \) is the extended type conjugate to \( \tau_f \).

Suppose now that moreover \( r_{f|G_0(p)} \) is absolutely irreducible. Let \( B \) be the quaternion algebra over \( \mathbb{Q} \) that is ramified exactly at \( p \) and \( \infty \). There exists a modular form \( f' \) on \( B \) such that the automorphic representations attached to \( f \) and \( f' \) correspond to each other via Jacquet-Langlands, and more precisely we can take for \( f' \) an eigenform in \( S_{\sigma,\eta}(U, \theta) \) for \( \sigma = \sigma_{alg} = \text{Sym}^{k-2} \theta^2 \) and some character \( \eta \) that restricts to \( \psi_k \) at \( p \), and \( U \) as in Section 5.3. Then we are in the situation of Section 5.5, from
which we retain the notations. In particular Proposition 5.6.1 holds, hence the module $M_0[1/p]$ meets each irreducible component of Spec $R_p[1/p]$. As Spec $R_p[1/p]$ has irreducible components of both possible extended types, there exists an eigenform $g'$ in $S_{\pi, \sigma(U, \theta)}$ such that the Galois representation attached to $g'$ has an extended type at $p$ which is conjugate to that of $r_f$. Let $g \in S_\lambda(\Gamma_0(p))$ be an eigenform such that the automorphic representations attached to $g$ and $g'$ correspond via Jacquet-Langlands, then $g$ is the form we were looking for.

The first part of Theorem 6.2.1 can be seen as a generalization of Conjecture 4 of [CS04] which was proved in [AB07] (see also [BP11]).

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