David Gérard-Varet, Daniel Han-Kwan, & Frédéric Rousset
Quasineutral limit of the Euler-Poisson system for ions in a domain with boundaries II

L'accès aux articles de la revue « Journal de l'École polytechnique — Mathématiques »
(http://jep.cedram.org/), implique l'accord avec les conditions générales d'utilisation
(http://jep.cedram.org/legal/).

QUASINEUTRAL LIMIT OF
THE EULER-POISSON SYSTEM FOR
IONS IN A DOMAIN WITH BOUNDARIES II

by David Gérard-Varet, Daniel Han-Kwan & Frédéric Rousset

Abstract. — In this paper, we study the quasineutral limit of the isothermal Euler-Poisson equation for ions, in a domain with boundary. This is a follow-up to our previous work [5], devoted to no-penetration as well as subsonic outflow boundary conditions. We focus here on the case of supersonic outflow velocities. The structure of the boundary layers and the stabilization mechanism are different.

Résumé (Limite quasineutre du système d'Euler-Poisson pour les ions dans un domaine à bord II)

Dans cet article, nous étudions la limite quasineutre du système d'Euler-Poisson pour les ions dans un domaine à bord. Il s'agit de la suite de notre travail précédent [5], qui était consacré aux cas de conditions limites de type non-pénétration ou sortantes subsoniques. Nous nous focalisons ici sur le cas des vitesses sortantes supersoniques. La structure des couches limites ainsi que le mécanisme de stabilisation sont différents.

Contents

1. Introduction

This work is about the quasineutral limit of the isothermal Euler-Poisson equation, describing the dynamics of ions in a plasma. We focus on the role of the boundary of the plasma domain, and the associated boundary layer. It complements our previous work on the topic [5] (see also [4], [10]). We refer to the introduction of [5] for a substantial bibliography on the quasineutral limit and related issues.

In the Euler-Poisson model, ions are described by their density $n \geqslant 0$ and their velocity field $u \in \mathbb{R}^3$, while electrons are assumed to follow a *Maxwell-Boltzmann law*, i.e., their density is given by $e^{-\phi}$, where ϕ denotes the electric potential. We consider that the plasma is contained in the domain $\mathbb{R}_+^3 := \{x = (y,z) \in \mathbb{R}^2 \times \mathbb{R}^+\}$ (yet, general domains can actually be considered, see [5, Section 5.1]). The isothermal Euler-Poisson equation under study then reads, for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}_+^3$,

$$(1.1) \qquad \begin{cases} \partial_t n + \operatorname{div}(nu) = 0, \\ \partial_t u + u \cdot \nabla u + T^i \, \nabla \ln(n) = \nabla \phi, \\ \varepsilon^2 \Delta \phi + e^{-\phi} = n, \\ \phi|_{x_3=0} = \phi_b, \end{cases}$$

where $\varepsilon > 0$ is loosely speaking the ratio between the *Debye length* of the plasma and the typical length of observation. In all practical situations, it is a small parameter. The parameter $T^i > 0$ is the temperature of the ions. We consider in this work the case of supersonic outflow velocities. It means that we consider initial velocity fields $u(0) = (u_1(0), u_2(0), u_3(0))$ satisfying

$$u_3(0, y, 0) < -\sqrt{T^i}$$

in which case no boundary condition is needed for the Euler system, since there are only outgoing characteristics. On the other hand, we enforce a Dirichlet boundary condition on the electric potential.

We also fix some constant reference state: $n_{\mathrm{ref}} > 0$, $u_{\mathrm{ref}} = (0, 0, w_{\mathrm{ref}})$ with $w_{\mathrm{ref}} < 0$. We set $\phi_b = \phi_c + (\phi_{\mathrm{ref}})_{|x_3=0}$, with $\phi_{\mathrm{ref}} = -\ln(n_{\mathrm{ref}})$, and $\phi_c \in H^\infty(\mathbb{R}^2)$.

We are interested in the behaviour of the solutions of (1.1) as ε goes to 0, that is, in the so-called *quasineutral limit*. We refer once again to the introduction of [5] for an extensive discussion. The expected formal limit, obtained by taking directly $\varepsilon = 0$, is the isothermal Euler system:

$$(1.2) \qquad \begin{cases} \partial_t n + \operatorname{div}(nu) = 0, \\ \partial_t u + u \cdot \nabla u + (T^i + 1) \, \nabla \ln(n) = 0, \end{cases}$$

together with the neutrality relation $n = e^{-\phi}$. With regards to this last relation and the Dirichlet condition (1.1d) on ϕ, one would like to impose the boundary condition $n|_{x_3=0} = e^{-\phi_b}$. However, this condition is not *a priori* compatible with the hyperbolic system (1.2), therefore we expect the formation of a *boundary layer* in the vicinity of the boundary $\{x_3 = 0\}$.

In our previous work [5], we have considered subsonic outflow velocities. In this setting, a boundary condition on the normal velocity has to be added:

$$u \cdot n = \overline{u}.$$

In [5], we have notably focused on the non-penetration boundary condition (that is, $\overline{u} = 0$), and we have also considered the subsonic case $-\sqrt{T^i} < \overline{u} < 0$.

In this paper, we shall concentrate on the supersonic case

$$(1.3) \qquad\qquad u_3(0, y, 0) < -\sqrt{T^i + 1}.$$

We will comment briefly on the intermediate case $-\sqrt{T^i + 1} < u_3(0, y, 0) < -\sqrt{T^i}$, see Section 1.2.

The supersonic condition (1.3) is usually called *Bohm condition* (or *criterion*), while the boundary layer is usually referred to as the *sheath* in the physics literature. It plays an important role in the study of confined plasmas: see for instance the book of Lieberman and Lichtenberg [7, Chapter 6] and the review paper of Riemann [9].

1.1. **Main result.** — The main result proved in this paper is the following Theorem.

Theorem 1. — *Let (n^0, u^0) be a solution to (1.2) such that $(n^0 - n_{\mathrm{ref}}, u^0 - u_{\mathrm{ref}}) \in C^0([0, T], H^s(\mathbb{R}^3_+))$ with s large enough. We assume that*

$$\sup_{[0,T]} \sup_{y \in \mathbb{R}^2} \left(u_3^0(t, y, 0) + \sqrt{T^i + 1} \right) < 0$$

and that

$$(1.4) \qquad \sup_{[0,T]} \sup_{y \in \mathbb{R}^2} \left(|n^0(t, y, 0) - e^{-\phi_b}| + |u_{1,2}(t, y, 0)| \right) \leqslant \delta,$$

for some sufficiently small δ. Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, there exists a solution $(n^\varepsilon, u^\varepsilon)$ to (1.1) also defined on $[0, T]$ such that

$$\sup_{[0,T]} \left(\|n^\varepsilon - n^0\|_{L^2(\mathbb{R}^3_+)} + \|u^\varepsilon - u^0\|_{L^2(\mathbb{R}^3_+)} \right) \xrightarrow[\varepsilon \to 0]{} 0.$$

Furthermore, the rate of convergence is $O(\sqrt{\varepsilon})$.

Note that the supersonic condition forbids $y \mapsto u_3^0(t, y, 0)$ to vanish at infinity: more precisely, it behaves asymptotically in y like w_{ref}, which necessarily satisfies $w_{\mathrm{ref}} < -\sqrt{T^i + 1}$.

The smallness condition (1.4) is used in particular to ensure the stability of a boundary layer. The study of this boundary layer, which we build beforehand in Theorem 2, is crucial in the proof of Theorem 1, though it does not appear explicitly. We refer to Theorem 3 and Corollary 1 for more precise statements.

To prove Theorem 1 (and its refined version), we shall use a classical two-step argument as in our previous work [5]. The first step is a *consistency* step where we first build approximate solutions for (1.1) at a sufficiently high order, see Theorem 2. In the second step we combine linear estimates and a continuation argument to deduce the *stability* of those approximate solutions. The main differences compared to the analysis of [5] are as follows.

(1) Contrary to [5], the existence of the sequence of approximate solutions is not unconditional, and relies on the first part of the smallness assumption (1.4). Furthermore, at the main order, compared to the subsonic case treated in [5], there is a boundary layer for the third (i.e., the normal) component of the velocity.

(2) Because of this additional singular term, the linear estimates of [5] are not relevant. Instead, we shall consider weighted estimates, inspired by the classical work of Goodman [6]. The idea is to use the stabilizing effect of convection. In order to obtain relevant energy estimates, we shall also borrow some ideas from a recent paper of Nishibata, Ohnawa and Suzuki [8] on a related problem. Namely, this problem is the stability of a special solution of the unscaled Euler-Poisson system, that is, when $\varepsilon = 1$. Roughly, this special solution corresponds to the main order part of the boundary layer constructed in Theorem 2, but without any y- or ε-dependence. On that topic, one can also refer to the previous work of Suzuki [11], as well as the works of Ambroso, Méhats and Raviart [2], and Ambroso [1]).

Our analysis will combine three types of L^2 weighted estimates. In [5], only the "physical" unweighted energy estimate was needed. This was due to the fact that the physical energy, which is well adapted to symmetrize the singular term coming from the Poisson equation in the quasineutral limit, was compatible with the structure of the boundary layers. Here this is not the case and we have to use the stabilizing effect of convection via the weights and other energy functionals in order to control the new singular terms that arise.

1.2. About the case $-\sqrt{T^i + 1} < u_3(0, y, 0) < -\sqrt{T^i}$. — We shall explain in this very brief section how the intermediate case $-\sqrt{T^i + 1} < u_3(0, y, 0) < -\sqrt{T^i}$ can be handled. In this regime, no boundary condition can be imposed when $\varepsilon > 0$, but one boundary condition can (and actually must) be imposed when $\varepsilon = 0$, that is, for system (1.2). We can notably endow (1.2) with the boundary condition $n|_{x_3=0} = e^{-\phi_b}$. This additional boundary condition allows to satisfy the Dirichlet condition $\phi|_{x_3=0} = \phi_b$ in the limit $\varepsilon \to 0$, hence no boundary layer appears. Convergence of solutions of (1.1) to the solution of (1.2) is in this context straightforward to prove: one can build an accurate approximate solution of (1.1), whose first term is the solution of (1.2). As explained before, this approximate solution will not have any boundary layer part. Then, one can perform an energy estimate between the approximate and exact solutions of (1.1) (with the same initial data) to show convergence. In this way, we get

Proposition 1. — *Let (n^0, u^0) be a solution to (1.2) with the condition $n|_{x_3=0} = e^{-\phi_b}$, such that $(n^0 - n_{\mathrm{ref}}, u^0 - u_{\mathrm{ref}}) \in C^0([0, T], H^s(\mathbb{R}^3_+))$ with s large enough. We assume that*

$$\sup_{[0,T]} \sup_{y \in \mathbb{R}^2} \left(u^0(t, y, 0)_3 + \sqrt{T^i} \right) < 0, \quad \inf_{[0,T]} \inf_{y \in \mathbb{R}^2} \left(u^0(t, y, 0)_3 + \sqrt{T^i + 1} \right) > 0.$$

Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, there exists a solution $(n^\varepsilon, u^\varepsilon)$ to (1.1) also defined on $[0, T]$ such that

$$\sup_{[0,T]} \left(\|n^\varepsilon - n^0\|_{L^2(\mathbb{R}^3_+)} + \|u^\varepsilon - u^0\|_{L^2(\mathbb{R}^3_+)} \right) \xrightarrow[\varepsilon \to 0]{} 0,$$

$$\sup_{[0,T]} \left(\|n^\varepsilon - n^0\|_{L^\infty(\mathbb{R}^3_+)} + \|u^\varepsilon - u^0\|_{L^\infty(\mathbb{R}^3_+)} \right) \xrightarrow[\varepsilon \to 0]{} 0.$$

Furthermore, the rate of convergence is $O(\varepsilon)$.

Note that we get convergence in L^∞ due to the absence of boundary layers. In the setting of Theorem 1, boundary layers have to be included in order to describe the asymptotic behavior in L^∞ (see the refined version of the convergence in Corollary 1).

1.3. Overview and classification. — Summarizing the results obtained in [5] and in the present paper, we obtain the following classification of outflow boundary conditions for the study of the quasineutral limit of the Euler-Poisson equation.

Boundary condition for Euler-Poisson (1.1) ($\varepsilon > 0$)	Boundary condition for isothermal Euler (1.2) ($\varepsilon = 0$)	Main order boundary layer	Stability result
$u \cdot n = 0$ & $\phi = \phi_b$ (characteristic)	$u \cdot n = 0$	Density and Potential	[5, Theorem 2.1]
$u \cdot n = \overline{u}$, $\overline{u} < 0$ & $\phi = \phi_b$ $\|\overline{u}\| < \sqrt{T^i}$ (non-characteristic)	$u \cdot n = \overline{u}$	Density and Potential	[5, Section 5.2]
$\phi = \phi_b$ and initial datum s.t. $-\sqrt{T^i + 1} < u_3(0, y, 0) < -\sqrt{T^i}$ (non-characteristic)	$n = e^{-\phi_b}$	No boundary layer	Proposition 1
$\phi = \phi_b$ and initial datum s.t. $u_3(0, y, 0) < -\sqrt{T^i + 1}$ (non-characteristic)	None	Density, Potential and Velocity	Theorem 1 (under smallness assumption (1.4))

The rest of the paper is now entirely devoted to the proof of Theorem 1 (more precisely of the refined Theorem 3).

2. Derivation of the boundary layers

We construct in this section accurate approximate solutions of the Euler-Poisson system, of boundary layer type. They are expansions in powers of ε, of the form:

$$(2.1) \quad (n_a, u_a, \phi_a) = \sum_{i=0}^{K} \varepsilon^i \left(n^i(t, x), u^i(t, x), \phi^i(t, x) \right)$$
$$+ \sum_{i=0}^{K} \varepsilon^i \left(N^i(t, y, x_3/\varepsilon), U^i(t, y, x_3/\varepsilon), \Phi^i(t, y, x_3/\varepsilon) \right),$$

with K an arbitrarily large integer. These approximations split into two parts:

– a regular part, with coefficients (n^i, u^i, ϕ^i) depending on (t, x). This regular part models the macroscopic behaviour of the solutions.

– a singular part, with coefficients (N^i, U^i, Φ^i) depending on the regular variables t, y, but also on a rescaled variable $z = x_3/\varepsilon \in \mathbb{R}_+$. It models a boundary layer correction, of size ε near the boundary. Accordingly, we shall impose

$$(2.2) \qquad\qquad (N^i, U^i, \Phi^i) \longrightarrow 0, \quad \text{as } z \longrightarrow +\infty.$$

In order for the whole approximation ϕ_a to satisfy the proper Dirichlet condition, we shall further impose

(2.3) $\phi^0(t, y, 0) + \Phi^0(t, y, 0) = \phi_b, \quad \phi^i(t, y, 0) + \Phi^i(t, y, 0) = 0 \quad \text{for all } i \geqslant 1.$

Note that, far away from the boundary, the term (n^0, u^0, ϕ^0) will drive the dynamics of these approximate solutions, and will satisfy the quasineutral limit system:

(2.4) $$\begin{cases} \partial_t n + \operatorname{div}(nu) = 0, \\ \partial_t u + u \cdot \nabla u + (T^i + 1)\nabla \ln(n) = 0, \end{cases}$$

together with the relation $n^0 = e^{-\phi^0}$.

Our construction is summarized in the

THEOREM 2. — *There exists $\delta_0 > 0$ such that: for any $K \in \mathbb{N}^*$, for any (n_0^0, u_0^0) such that $\inf n_0^0 > 0$ satisfying the regularity assumption*

(2.5) $(n_0^0 - n_{\text{ref}}, u_0^0 - u_{\text{ref}}) \in H^\infty(\mathbb{R}_+^3),$

the Bohm condition

(2.6) $\sup u_{0,3}^0 < 0, \quad (\sup u_{0,3}^0))^2 > T^i + 1,$

and the smallness condition

(2.7) $\phi_c + \inf\limits_{y \in \mathbb{R}^2} \ln\left(\dfrac{n_0^0(y, 0)}{n_{\text{ref}}}\right) \geqslant -\delta_0, \quad \phi_c + \sup\limits_{y \in \mathbb{R}^2} \ln\left(\dfrac{n_0^0(y, 0)}{n_{\text{ref}}}\right) \leqslant \delta_0,$

one can find $T > 0$ and an expansion of type (2.1) with the following properties.

   (i) (n^0, u^0) *is a solution to (2.4) with initial data (n_0^0, u_0^0), satisfying*

$$(n^0 - n_{\text{ref}}, u^0 - u_{\text{ref}}) \in C^\infty([0, T], H^\infty(\mathbb{R}_+^3)).$$

Moreover, $\phi^0 = -\log n^0$.

   (ii) $\forall\, 1 \leqslant i \leqslant K, (n^i, u^i, \phi^i) \in C^\infty([0, T], H^\infty(\mathbb{R}_+^3)).$

   (iii) $\forall\, 0 \leqslant i \leqslant K, (N^i, U^i, \Phi^i) \in C^\infty([0, T], H_{y,z}^\infty(\mathbb{R}_+^3))$ *and decays together with its derivatives uniformly exponentially in z.*

   (iv) *Let us consider a solution $(n^\varepsilon, u^\varepsilon, \phi^\varepsilon)$ to (1.1) and define:*

$$n = n^\varepsilon - n_a, \quad u = u^\varepsilon - u_a, \quad \phi = \phi^\varepsilon - \phi_a.$$

Then (n, u, ϕ) satisfies the system of equations:

(2.8) $$\begin{cases} \partial_t n + (u_a + u) \cdot \nabla n + n \operatorname{div}(u + u_a) + \operatorname{div}(n_a u) = \varepsilon^K R_n, \\[2mm] \partial_t u + (u_a + u) \cdot \nabla u + u \cdot \nabla u_a + T^i \left(\dfrac{\nabla n}{n_a + n} - \dfrac{\nabla n_a}{n_a}\left(\dfrac{n}{n_a + n} \right) \right) \\[2mm] \hspace{6cm} = \nabla \phi + \varepsilon^K R_u, \\[2mm] \varepsilon^2 \Delta \phi = n - e^{-\phi_a}(e^{-\phi} - 1) + \varepsilon^{K+1} R_\phi, \end{cases}$$

where R_n, R_u, R_ϕ are remainders satisfying:

(2.9) $\sup\limits_{[0,T]} \|\nabla_x^\alpha R_{n,u,\phi}\|_{L^2(\mathbb{R}_+^3)} \leqslant C_\alpha \varepsilon^{-\alpha_3}, \quad \forall\, \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3, \quad |\alpha| \leqslant m,$

with $C_\alpha > 0$ independent of ε.

The whole section is devoted to the proof of this theorem. As already said in the introduction, the main difference with the boundary layer problem analyzed in [5] is that the third component of U^0 does not vanish: the velocity field has a boundary layer part of amplitude $O(1)$. This modifies all the boundary layer equations, compared to [5]. In particular, well-posedness is not unconditional anymore, which explains our condition (2.7). In fact, the construction of the first boundary layer term only requires a lower bound (first inequality in (2.7)), whereas the construction of the next terms requires an additional upper bound. Moreover, the constant δ_0 in this condition can be made explicit: see Section 2.2. Let us finally point out that the regularity assumption on the initial data (n_0^0, u_0^0) can be easily lowered to H^s, with s large enough.

2.1. The cascade of equations. — Plugging (2.1) in the Euler-Poisson system (1.1), we formally derive a whole set of equations. The well-posedness of such equations will be discussed in Section 2.2. For clarity of exposure, for any function $f = f(t, x)$, we denote by Γf the function $(t, y, z) \mapsto f(t, y, 0)$.

(a) Away from the boundary, we find as expected that (n^0, u^0) satisfies (2.4), plus the neutrality relation $\phi^0 = -\ln(n^0)$. The local existence and uniqueness of (n^0, u^0), starting from our supersonic data, will be stated rigorously in Section 2.2.

(b) In the boundary layer, with a Taylor expansion of the regular part of (2.1), the third line of (1.1) yields

$$(2.10) \qquad \partial_z^2 \Phi^0 = (\Gamma n^0 + N^0) - e^{-(\Gamma \phi^0 + \Phi^0)} = \Gamma n^0 \Big(\frac{\Gamma n^0 + N^0}{\Gamma n^0} - e^{-\Phi^0} \Big),$$

whereas the first line yields

$$\partial_z \big((\Gamma n^0 + N^0)(\Gamma u_3^0 + U_3^0) \big) = 0.$$

Taking into account (2.2), we obtain

$$(2.11) \qquad\qquad (\Gamma n^0 + N^0)(\Gamma u_3^0 + U_3^0) = \Gamma n^0 \, \Gamma u_3^0.$$

We then consider the vertical component of the momentum equation in (1.1). This gives

$$(\Gamma u_3^0 + U_3^0)\partial_z U_3^0 + T^i \partial_z \ln(\Gamma n^0 + N^0) = \partial_z \Phi^0.$$

We can integrate in z, and use (2.2) again to obtain

$$\frac{1}{2}(\Gamma u_3^0 + U_3^0)^2 + T^i \ln(\Gamma n^0 + N^0) = \Phi^0 + \frac{1}{2}(\Gamma u_3^0)^2 + T^i \ln(\Gamma n^0).$$

Inserting relation (2.11) in the last identity, we end up with

$$(2.12) \qquad \frac{(\Gamma n^0 \, \Gamma u_3^0)^2}{2(\Gamma n^0 + N^0)^2} + T^i \ln(\Gamma n^0 + N^0) = \Phi^0 + \frac{1}{2}(\Gamma u_3^0)^2 + +T^i \ln(\Gamma n^0).$$

This last equation can be written

$$(2.13) \qquad\qquad \Phi^0(t, y, z) = \mathscr{F}\Big(t, y, \frac{\Gamma n^0(t, y) + N^0(t, y, z)}{\Gamma n^0(t, y)}\Big)$$

where

(2.14) $$\mathscr{F}(t, y, N) = \frac{\Gamma u_3^0(t, y)^2}{2N^2} + T^i \ln(N) - \frac{\Gamma u_3^0(t, y)^2}{2}.$$

The derivative with respect to N is

$$\partial_N \mathscr{F}(t, y, N) = -\frac{\Gamma u_3^0(t, y)^2}{N^3} + \frac{T^i}{N}.$$

One has

$$\partial_N \mathscr{F}(t, y, N) < 0 \quad \text{over } (0, N_{\mathscr{F}}(t, y)), \quad N_{\mathscr{F}}(t, y) := \sqrt{\Gamma u_3^0(t, y)^2 / T^i}.$$

Hence, for all t, y, $N \mapsto \mathscr{F}(t, y, N)$ has a smooth inverse, and its inverse is decreasing from $(\Phi_{\mathscr{F}}(t, y), +\infty)$ to $(0, N_{\mathscr{F}}(t, y))$, where $\Phi_{\mathscr{F}}(t, y) := \mathscr{F}(t, y, N_{\mathscr{F}}(t, y) < 0$. *We make a slight abuse of notation, and denote $\mathscr{F}^{-1}(t, y, \Phi)$ such an inverse.*

Back to (2.10), we obtain

(2.15) $$\partial_z^2 \Phi^0(t, y, z) = \mathscr{X}(t, y, \Phi^0(t, y, z))$$

where

(2.16) $$\mathscr{X}(t, y, \Phi) := n^0(t, y, 0) \left(\mathscr{F}^{-1}(t, y, \Phi) - e^{-\Phi} \right).$$

This is an autonomous ODE in z, (t, y) playing the role of parameters. Of course, the r.h.s. is only defined as long as $\mathscr{F}^{-1}$ is. The ODE is completed by boundary conditions:

(2.17)
$$\Phi^0(t, y, 0) = \phi_{\text{ref}} + \phi_c(t, y) - \phi^0(t, y, 0) = \phi_c(t, y) + \ln\left(\frac{n^0(t, y, 0)}{n_{\text{ref}}} \right),$$
$$\lim_{z \to +\infty} \Phi^0(t, y, z) = 0.$$

We shall prove in Section 2.2 existence and uniqueness of a solution to this system, under a condition of type (2.7).

REMARK 1. — From (2.17), we shall set

$$\sup_{t \in [0,T], \, y \in \mathbb{R}^2} |\Phi^0(t, y, 0)| = \sup_{t \in [0,T], \, y \in \mathbb{R}^2} |\phi_b + \ln n^0(t, y, 0)| =: \delta.$$

This parameter δ measures how far the quasineutral solution is from satisfying the Dirichlet condition. It will determine the size of the boundary layer.

Once Φ^0 has been determined, we obtain N^0 using the relation

$$\frac{n^0(t, y, 0) + N^0(t, y, z)}{n^0(t, y, 0)} = \mathscr{F}^{-1}(t, y, \Phi^0(t, y, z)).$$

In turn, (2.11) gives U_3^0.

Finally, we use the horizontal components of the momentum equation, that yield

$$(\Gamma u_3^0 + U_3^0) \partial_z U_y^0 = 0$$

which together with decay entails that $U_y^0 = 0$. This concludes the formal derivation of (n^0, u^0, ϕ^0), and (N^0, U^0, Φ^0).

(c) From there, one can derive (n^1, u^1, ϕ^1), and (N^1, U^1, Φ^1). More generally, knowing (n^k, u^k, ϕ^k), and (N^k, U^k, Φ^k), $k \leqslant i - 1$, one can derive equations on (n^i, u^i, ϕ^i), and then on (N^i, U^i, Φ^i). We shall stick to $i = 1$ for the sake of brevity, and refer to [5] for a full derivation in a close context.

Clearly,

$$
(2.18) \qquad
\begin{cases}
\partial_t n^1 + \mathrm{div}(n^0 u^1) + \mathrm{div}(n^1 u^0) = 0, \\[2mm]
\partial_t u^1 + u^1 \cdot \nabla u^0 + u^0 \cdot \nabla u^1 + T^i \Big(\dfrac{\nabla n^1}{n^0} - \dfrac{n^1 \nabla n^0}{(n^0)^2} \Big) = \nabla \phi^1, \\[3mm]
- e^{-\phi^0} \phi^1 = n^1.
\end{cases}
$$

The well-posedness of this system over $(0, T)$ (where T is the time up to which the supersonic boundary condition on u^0 is satisfied) will be reminded in the next paragraph.

As regards the boundary layer terms, we look as before at the Poisson equation of (1.1), which leads to

$$
(2.19) \qquad \partial_z^2 \Phi^1 = (z\Gamma \partial_3 n^0 + \Gamma n^1 + N^1) + e^{-(\Gamma \phi^0 + \Phi^0)}(z\Gamma \partial_3 \phi^0 + \Gamma \phi^1 + \Phi^1).
$$

Then, the mass equation yields (remember that $U_y^0 = 0$)

$$
\partial_z((\Gamma n^1 + N^1)(\Gamma u_3^0 + U_3^0)) + \partial_z \left((\Gamma n^0 + N^0)(\Gamma u_3^1 + U_3^1) \right) = F_1,
$$
$$
F_1 := -\partial_z(z\Gamma \partial_3 n^0 U_3^0 - N^0 z\Gamma \partial_3 u_3^0) - \mathrm{div}_y(N^0 \Gamma u_y^0) - \partial_t N^0.
$$

Integration from z to infinity yields

$$
(2.20) \quad (\Gamma n^1 + N^1)(\Gamma u_3^0 + U_3^0) + (\Gamma n^0 + N^0)(\Gamma u_3^1 + U_3^1)
$$
$$
= \int_{+\infty}^{z} F_1 + \Gamma n^1 \Gamma u_3^0 + \Gamma n^0 \Gamma u_3^1 =: F_2.
$$

Then, we write down ε^0 terms in the vertical component of the momentum equation:

$$
\partial_z((\Gamma u_3^0 + U_3^0)(\Gamma u_3^1 + U_3^1)) + T^i \partial_z \frac{\Gamma n^1 + N^1}{\Gamma n^0 + N^0} = \partial_z \Phi^1 + F_3,
$$
$$
F_3 := -\partial_z(z\Gamma \partial_3 u_3^0 U_3^0) - T^i \partial_z \Big(\frac{z \partial_3 n^0}{\Gamma n^0 + N^0} - \frac{z \partial_3 n^0}{\Gamma n^0} \Big) - \Gamma u_y^0 \cdot \nabla_y U_3^0 - \partial_t U_3^0.
$$

One can as before integrate with respect to z, to get

$$
(\Gamma u_3^0 + U_3^0)(\Gamma u_3^1 + U_3^1) + T^i \frac{\Gamma n^1 + N^1}{\Gamma n^0 + N^0} = \Phi^1 - \int_{+\infty}^{z} F_3 + \Gamma u_3^0 \Gamma u_3^1 + T^i \frac{\Gamma n^1}{\Gamma n^0} =: \Phi^1 + F_4.
$$

Using (2.11) and (2.20) to transform the first term at the l.h.s., we end up with

$$
\Big(- \frac{(\Gamma u_3^0 \Gamma n^0)^2}{(\Gamma n^0 + N^0)^3} + \frac{T^i}{\Gamma n^0 + N^0} \Big)(\Gamma n^1 + N^1) = \Phi^1 + F_5
$$

where

$$
F_5 := + \frac{\Gamma n^0 \Gamma u_3^0}{(\Gamma n^0 + N^0)^2} F_2 + F_4
$$

is known from the previous steps of the construction. This expression allows to express $\Gamma n^1 + N^1$ in terms of Φ^1, and to go back to (2.19) to have a closed equation on Φ^1. A tedious but straightforward calculation shows then that

$$(2.21) \qquad \partial_z^2 \Phi^1(t,y,z) = \partial_\Phi \mathscr{X}(t,y,\Phi^0(t,y,z))\Phi^1(t,y,z) + F_6(t,y,z)$$

with

$$F_6(t,y,z) := n^0(t,y,0)\left[\partial_N \mathscr{F}\left(t,y,\frac{n^0(t,y,0) + N^0(t,y,z)}{n^0(t,y,0)}\right)\right]^{-1} F_5(t,y,z)$$
$$+ n^0(t,y,0)(e^{-\Phi^0} - 1)z\partial_3\phi^0(t,y,0) + n^0(t,y,0)\phi^1(t,y,0).$$

This equation is completed with the boundary condition

$$(2.22) \qquad \Phi^1(t,y,0) = -\phi^1(t,y,0), \quad \lim_{z\to+\infty} \Phi^1(t,y,z) = 0.$$

The well-posedness of this equation will be discussed in the next section.

As soon as Φ^1 is determined, the expression of N^1 follows, and from (2.20), we get the expression of U_3^1. The horizontal part U_y^1 is obtained as for U_y^0 by considering the horizontal components of the momentum equation in (1.1). We leave all details to the reader. As mentioned before, the next order terms in the expansion satisfy similar problems, and we omit their construction for the sake of brevity.

2.2. Well-posedness. — We discuss here the well-posedness of the reduced models derived formally in the previous section.

(a) The supersonic condition (2.6) expresses that all characteristics of the Euler system (2.4) are outgoing, so that no boundary condition is necessary for solvability. More precisely, starting from the initial data satisfying (2.6)–(2.5), there exists a small time $T > 0$ and a smooth solution [3]

$$n^0 \in n_{\mathrm{ref}} + C^\infty([0,T], H^\infty(\mathbb{R}_+^3)), \quad u^0 \in u_{\mathrm{ref}} + C^\infty([0,T], H^\infty(\mathbb{R}_+^3)).$$

Moreover, one can assume that the supersonic condition is satisfied globally over $[0,T]$:

$$(2.23) \qquad \inf n^0 > 0, \quad \sup u_3^0 < 0, \quad (\sup u_3^0)^2 > T^i + 1.$$

Under this last condition, one can solve the equations on (n^i, u^i, ϕ^i), $i \geqslant 1$, over $[0,T]$. We remind that these equations are of the type

$$\begin{cases} \partial_t n^i + \mathrm{div}(n^0 u^i) + \mathrm{div}(n^i u^0) = f_n^i, \\[2mm] \partial_t u^i + u^i \cdot \nabla u^0 + u^0 \cdot \nabla u^i + T^i\left(\dfrac{\nabla n^i}{n^0} - \dfrac{n^i \nabla n^0}{(n^0)^2}\right) = \nabla\phi^i + f_u^i, \\[2mm] -e^{-\phi^0}\phi^i = n^i + f_\phi^i \end{cases}$$

(see (2.18) in the special case $i = 1$). These are linear hyperbolic systems with unknowns (n^i, u^i), and with outgoing characteristics over $[0,T]$ by (2.23). Starting (for instance) from zero initial data

$$n^i|_{t=0} = 0, \quad u^i|_{t=0} = 0,$$

these systems are shown inductively to possess unique regular solutions over $[0, T]$. More precisely, one can check inductively that all source terms $f_{n,u,\phi}^i$ belong to $C^\infty([0, T]; H^\infty(\mathbb{R}_+^3))$, so that

$$n^i \in C^\infty([0, T], H^\infty(\mathbb{R}_+^3)), \quad u^i \in C^\infty([0, T], H^\infty(\mathbb{R}_+^3)),$$

and the same holds for ϕ^i.

(b) We now have to address rigorously the construction of boundary layer systems. Note that the N^i's and U^i's are given in an extrinsic manner from the Φ^i's, so that we only need to focus on the latter. We begin by discussing the well-posedness of (2.15)–(2.16)–(2.17). We already noticed that t, y are just parameters, which makes it an ODE problem. This ODE problem has been addressed in [2], see also [11]. Omitting the dependence with respect to t, y, it reduces to the search of a trajectory $z \mapsto \Phi(z)$ of a second-order ODE

$$(2.24) \quad \Phi'' = \mathscr{X}(\Phi), \quad \mathscr{X}(\Phi) = \mathscr{F}^{-1}(\Phi) - e^{-\Phi}, \quad \mathscr{F}(N) = \frac{u_3^0(0)^2}{2N^2} + T^i \ln(N) - \frac{u_3^0(0)^2}{2}.$$

such that $\Phi(0) = \Phi_0$, $\lim_{z \to 0} \Phi = 0$ for some constant Φ_0. Note that we only allow solutions Φ with values in some interval $(\Phi_{\mathscr{F}}, +\infty)$, $\Phi_{\mathscr{F}} < 0$, in order for $\mathscr{F}^{-1}$ and $\mathscr{X}$ to be defined: see the discussion below (2.14).

This problem can be solved using the hamiltonian structure of the equation. Introducing the antiderivative $\mathscr{V} = \int_0^\Phi \mathscr{X}$, we get

$$(\Phi')^2 = 2\mathscr{V}(\Phi).$$

The variations of $\mathscr{V}$ can be determined using that $\mathscr{V}'(\Phi) = \mathscr{X}(\Phi)$ has the same sign as $\mathscr{Y}(N) = N - e^{-\mathscr{F}(N)}$, which in turn has the same sign as $\ln N + \mathscr{F}(N)$. We refer to [11] for more details. As a result, there exists $\delta_0 > 0$ such that: *for all* $\Phi_0 \in [-\delta_0, +\infty)$*, there is a unique (monotonic) solution of* (2.24) *connecting* Φ_0 *to* 0. Moreover, as $\mathscr{V}''(0) > 0$, $(0, 0)$ is a hyperbolic point for the 2×2 system associated to (2.24), which yields the exponential decay of Φ by the stable manifold theorem. More precisely, the rate of decay is given by

$$(2.25) \qquad\qquad \gamma := \sqrt{\mathscr{V}''(0)} = \sqrt{\frac{T^i + 1 - u_3^0(0)^2}{T^i - u_3^0(0)^2}}.$$

Back to the original problem (with dependence in t, y), this analysis provides a solution of (2.15)–(2.16)–(2.17) under the first inequality in (2.7). The regularity of Φ^0 with respect to (t, y) follows from standard arguments: using (for instance) the implicit function theorem, one can show that

$$\mathscr{X}(t, y, \phi) = \widetilde{\mathscr{X}}\left(n^0(t, y, 0), u_3^0(t, y), \Phi\right),$$

and then

$$\Phi^0(t, y, z) = \widetilde{\Phi^0}\left(n^0(t, y, 0), u_3^0(t, y), \phi_c(t, y), z\right)$$

for smooth functions

$$\widetilde{\mathscr{X}} = \widetilde{\mathscr{X}}(n, w, \Phi), \quad \widetilde{\Phi^0} = \widetilde{\Phi^0}(n, w, \phi, z).$$

Moreover, one can check that $\widetilde{\Phi^0}(n_{\text{ref}}, \cdot) = 0$. The smoothness in (t, y) and H^∞ integrability in y follow. For the sake of brevity, we leave the details to the reader.

Remark 2. — Note that the decay rate (2.25) is independent of the amplitude parameter defined in Remark 1. We thus get for Φ^0 an estimate under the form: there exists $C > 0$, $\gamma_0 > 0$ such that for every $\delta \in (0, \delta_0)$, we have the estimate

$$|\Phi^0| \leqslant C\delta e^{-\gamma_0 z}.$$

As regards the next order boundary layer terms, they all (formally) satisfy equations of the type

$$\partial_z^2 \Phi^i(t, y, z) = \partial_\Phi \mathscr{X}(t, y, \Phi^0(t, y, z))\Phi^i(t, y, z) + F^i$$

(see (2.21) for $i = 1$), where F^i depends on the lower order profiles. Their well-posedness (in the space of smooth functions, exponentially decaying in z) is shown inductively: freezing t, y, it reduces to show the well-posedness of a linear ODE of the type

$$\partial_z^2 \Phi = \mathscr{X}'(\Phi^0)\Phi + F,$$

for some smooth exponentially decaying F, with boundary conditions

$$\Phi|_{z=0} = \psi, \quad \Phi|_{z=\infty} = 0.$$

Note that, up to consider $\widetilde{\Phi}(z) = \Phi(z) - \psi\chi(z)$, with $\chi \in C_c^\infty(\mathbb{R}_+)$ satisfying $\chi(0) = 0$, we can always assume that $\psi = 0$.

As $\mathscr{X}'(\Phi) < 0$ for large Φ, the well-posedness of these systems is not clear under a mere lower bound on Φ^0. But as $\mathscr{X}'(0) > 0$, up to take a smaller δ_0 and impose the additional upper bound in (2.7), we ensure that $\mathscr{X}'(\phi^0) \geqslant \alpha > 0$ uniformly in z. The existence of a variational solution (say in $H_0^1(\mathbb{R}_+)$) follows then from Lax-Milgram's lemma. Smoothness in z is a consequence of standard elliptic regularity results, whereas the exponential decrease follows again from the stable manifold theorem.

The proof of Theorem 2 is thus complete.

3. Linear Stability

In this section, we establish L^2 type estimates for linear systems of the following form:

$$
(3.1) \quad
\begin{cases}
\partial_t \dot{n} + (u_a + u) \cdot \nabla \dot{n} + (n_a + n)\nabla \cdot \dot{u} = r_n, \\[2mm]
\partial_t \dot{u} + (u_a + u) \cdot \nabla \dot{u} + T^i \dfrac{\nabla \dot{n}}{n_a + n} = \nabla\dot{\phi} + r_u, \\[2mm]
\varepsilon^2 \Delta\dot{\phi} = \dot{n} + e^{-\phi_a}\,\dot{\phi}\,(1 + h(\phi)) + r_\phi,
\end{cases}
$$

where $r = (r_n, r_u, r_\phi)$ is a given source term, and where $h \in \{h_0, h_1\}$, where

$$h_0(\phi) := -\frac{e^{-\phi} - 1 + \phi}{\phi}, \quad h_1(\phi) := e^{-\phi} - 1,$$

cf. (4.6). We add to the system the boundary condition

$$\dot{\phi}|_{x_3=0} = 0. \tag{3.2}$$

To prove our main linear stability estimate, we shall use Goodman type weights in order to use the stabilizing effect of convection in the problem. From the construction of the approximate solution in the previous section, in particular, from Remarks 1 and 2, we know that the main boundary layer part of the approximate solution satisfies an estimate under the form

$$\sup_{(0,T)\times\mathbb{R}^2} |\partial_{x_3}(n_a, u_a, \phi_a)(\cdot, x_3)| + \varepsilon|\partial_{x_3}^2(n_a, u_a, \phi_a)(\cdot, x_3)| \leqslant \frac{\delta}{\varepsilon}\, e^{-\gamma_0\, x_3/\varepsilon}, \tag{3.3}$$

where δ will be assumed small whereas γ_0 is fixed. We recall that, as in the previous section, for any function $f = f(t, x)$, we denote by Γf the function $(t, y, z) \mapsto f(t, y, 0)$. We shall also assume that the trace on the boundary of the tangential velocity of the limit system is small, i.e.,

$$|\Gamma u_{1,2}^0| \leqslant \delta. \tag{3.4}$$

We introduce the *weight function*

$$\eta(x_3) := e^{(1-e^{-\mu x_3/\varepsilon})\delta/\mu^2},$$

where $\mu, \gamma > 0$ are some fixed parameters, to be specified later. Observe that η satisfies the following properties:

$$\begin{cases} \eta(0) = 1, \\ \eta'(x_3) = \dfrac{\delta}{\mu\varepsilon}\, e^{-\mu x_3/\varepsilon}\, \eta(x_3). \end{cases} \tag{3.5}$$

The main idea is that the parameter $\mu > 0$, $\mu \leqslant \gamma_0$ will be chosen small enough. We also assume that the parameter δ that measures the strength of the boundary layers is sufficiently small such that δ/μ^2 is also small. This yields in particular

$$\frac{1}{2} \leqslant \eta(x_3) \leqslant 2, \quad x_3\eta' \leqslant 4, \quad \forall\, x_3 \geqslant 0. \tag{3.6}$$

For example, we can choose $\mu = \delta^{1/4}$. The assumptions on μ further imply

$$\eta(x_3)\,\frac{C_a\delta}{\varepsilon}\, e^{-\gamma_0 x_3/\varepsilon} \leqslant C_a\,\mu\,\eta'(x_3), \quad \forall\, x_3 > 0, \tag{3.7}$$

This inequality will be crucial in many estimates of the paper. We also note for later purposes that

$$|\eta''| = \left|\frac{\delta}{\mu\varepsilon}\, e^{-\mu x_3/\varepsilon}\eta'(x_3) + \frac{\mu}{\varepsilon}\,\eta'(x_3)\right| \leqslant C\,\frac{\mu}{\varepsilon}\,\eta'(x_3). \tag{3.8}$$

We shall eventually assume that the forcing terms (r_n, r_u, r_ϕ) can be split into a small singular part and a regular part:

$$|(r_n, r_u, r_\phi/\varepsilon, \varepsilon\nabla r_n, \varepsilon\nabla r_u, \partial_t r_\phi)| \leqslant \mu\eta' R^s + R^r. \tag{3.9}$$

Note that since the derivative of η is of order $1/\varepsilon$, the first term in the r.h.s. of (3.9) is singular in ε. Concretely, the linearized system (3.1) will be obtained by taking ε-derivatives of (2.8). The remainders will contain commutators, and satisfy (3.9).

The crucial weighted L^2 estimate is given in the following proposition.

Proposition 2. — *Let (n_a, u_a, ϕ_a) be the approximate solution constructed in Theorem 2 and consider some smooth (n, u, ϕ) on $[0, T]$ such that for some $M > 0$*

$$n_a + n \geqslant 1/M, \quad e^{-\phi_a}(1 + h(\phi)) \geqslant 1/M,$$

(3.10)
$$\frac{\|(n, u, \phi)\|_{L^\infty}}{\varepsilon} + \|\nabla(n, u, \phi)\|_{L^\infty} + \|\partial_t(n, \phi)\|_{L^\infty} \leqslant M, \quad \forall t \in [0, T], \, x \in \mathbb{R}_+^3$$

and such that the Bohm condition is verified on the boundary

(3.11)
$$\Gamma(u_a + u)_3 < -\frac{1}{M}, \quad \left(\Gamma(u_a + u)_3\right)^2 \geqslant T^i + 1 + \frac{1}{M}.$$

Moreover, let us assume that the source term (r_n, r_u, r_ϕ) of (3.1) verifies the assumption (3.9). Then, there exists $\delta_0 > 0$, $\mu_0 > 0$ and $C(C_a, M)$ (C_a only depends on the approximate solution) such that for every $\varepsilon \in (0, \varepsilon_0]$, for every $\delta \in (0, \delta_0]$ and for every $\mu \in (0, \mu_0]$ with δ/μ^2 sufficiently small, we have for the solution $(\dot{n}, \dot{u}, \dot{\phi})$ of (3.1) the estimate

$$\sqrt{\mu}\Big(\big\|(\dot{n}, \dot{u}, \dot{\phi}, \varepsilon\nabla\dot{n}, \varepsilon\nabla\dot{u}, \varepsilon\nabla\dot{\phi})\big\|_{L_T^\infty L^2(\mathbb{R}_+^3)}^2 + \big\|\sqrt{\eta'}(\dot{n}, \dot{u}, \dot{\phi}, \varepsilon\nabla\dot{n}, \varepsilon\nabla\dot{u}, \varepsilon\nabla\dot{\phi})\big\|_{L_T^2 L^2(\mathbb{R}_+^3)}^2$$

$$+ \big\|\Gamma(\dot{n}, \dot{u}, \dot{\phi}, \varepsilon\nabla\dot{n}, \varepsilon\nabla\dot{u}, \varepsilon\partial_3\dot{\phi})\big\|_{L_T^2 L^2(\mathbb{R}^2)}^2\Big)$$

$$\leqslant C(C_a, M)\Big(\big\|(\dot{n}_0, \dot{u}_0, \dot{\phi}_0, \varepsilon\nabla\dot{n}_0, \varepsilon\nabla\dot{u}_0, \varepsilon\nabla\dot{\phi}_0)\big\|_{L^2(\mathbb{R}_+^3)}^2$$

$$+ \|(\dot{n}, \dot{u}, \varepsilon\nabla\dot{n}, \varepsilon\nabla\dot{u}, \dot{\phi}, \varepsilon\nabla\dot{\phi})\|_{L_T^2 L^2(\mathbb{R}_+^3)}^2 + \big\|R^r\big\|_{L_T^2 L^2(\mathbb{R}_+^3)}^2 + \mu\big\|\sqrt{\eta'}R^s\big\|_{L_T^2 L^2(\mathbb{R}_+^3)}^2\Big).$$

Here L_T^2 and L_T^∞ stand for $L^2([0, T])$ and $L^\infty([0, T])$. This whole section is dedicated to the proof of Proposition 2.

Proof of Proposition 2. — We first gather some useful bounds satisfied by the approximate solution (n_a, u_a, ϕ_a). In the proof, we shall denote by C_a and $C(C_a, M)$ numbers which depend only on the estimates of the approximate solution and on the number M defined in (3.10) and that may change from line to line. The important thing is that they are uniformly bounded for $\varepsilon \in (0, 1]$, $\delta \in (0, 1)$ and $T \in (0, T_0]$ where T_0 is the interval of time on which the approximate solution is defined. We first have, by construction (see Theorem 2):

(3.12)
$$\sup_{(0,T)\times\mathbb{R}_+^3} |\nabla_{x_1, x_2, t}^k(n_a, u_a, \phi_a)| \leqslant C_a, \quad C_a > 0,$$

with C_a independent of ε. In the other hand, we have for all $x_3 > 0$:

(3.13)
$$\sup_{(0,T)\times\mathbb{R}^2} |\varepsilon^l \partial_{x_3}^{1+l}\nabla_{x_1, x_2, t}^k(n_a, u_a, \phi_a)(\cdot, x_3)| \leqslant \frac{\delta}{\varepsilon}e^{-\gamma_0 x_3/\varepsilon} + C_a,$$

where we recall that $0 < \delta \ll 1$ can be considered as a small parameter.

We shall combine many energy estimates for the proof of Proposition 2. As already pointed out in the introduction, our computations share features with those led in [8, Section 2.1]. The starting point of all the energy estimates will be the following lemma:

Lemma 1. — *Under the assumptions of Proposition 2, we have the estimate*

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3_+}\eta\left((n_a+n)\frac{|\dot{u}|^2}{2}+\frac{|\dot{n}|^2}{n_a+n}\right)dx+\frac{1}{2}\int_{\mathbb{R}^3_+}\eta'\,Q^0(\dot{n},\dot{u})\,dx+\frac{1}{2}\int_{x_3=0}Q^0(\dot{n},\dot{u})-\mathscr{I}$$

$$\leqslant C(C_a,M)\Big(\|(\dot{n},\dot{u})\|_{L^2}^2+\|R^r\|_{L^2}^2+\mu\int_{\mathbb{R}^3_+}\eta'|R^s|^2\Big)$$

where the quadratic form Q^0 is associated to the symmetric matrix

$$\mathscr{Q}^0:=\begin{bmatrix}T^i\Gamma(\frac{|(u_a+u)_3|}{n^a+n}) & -T^i e^\top \\ -T^i e & \Gamma\big((n_a+n)|(u_a+u)_3|\big)\,\mathrm{Id}_3\end{bmatrix},\quad e^\top=(0,0,1)^\top$$

and

(3.14) $$\mathscr{I}:=\int_{\mathbb{R}^+_3}\eta\nabla\dot{\phi}\cdot(n_a+n)\dot{u}\,dx.$$

The main difficulty in proving Proposition 2 will be to handle the term $\mathscr{I}$, that involves the potential $\dot{\phi}$. We note that in the left hand side of the above estimate, the quadratic form Q^0 is positive thanks to the Bohm condition (3.11).

Proof of Lemma 1. — First, multiplying the velocity equation by $(n_a+n)\,\dot{u}\,\eta$, and performing standard manipulations, we obtain:

(3.15) $$\frac{d}{dt}\int_{\mathbb{R}^3_+}\eta\,(n_a+n)\frac{|\dot{u}|^2}{2}=\int_{\mathbb{R}^3_+}\eta\,(n_a+n)\dot{u}\cdot\partial_t\dot{u}+\int_{\mathbb{R}^3_+}\eta\,\partial_t(n_a+n)\frac{|\dot{u}|^2}{2}$$

$$=\mathscr{I}+I_1+I_2+\int_{\mathbb{R}^3_+}\eta\,(r_u\cdot((n_a+n)\dot{u}))+\int_{\mathbb{R}^3_+}\eta\,\partial_t(n_a+n)\frac{|\dot{u}|^2}{2},$$

where

$$I_1:=-T^i\int_{\mathbb{R}^3_+}\eta\,\frac{\nabla\dot{n}}{n_a+n}\cdot((n_a+n)\dot{u}),\quad I_2:=-\int_{\mathbb{R}^3_+}\eta\,[(u_a+u)\cdot\nabla\dot{u}]\cdot(n_a+n)\dot{u}.$$

We recall that $\mathscr{I}$ was defined in (3.14). The last two terms at the r.h.s. of (3.16) can be easily estimated through (3.12), (3.10), (3.6) and (3.9). We find

(3.16) $$\frac{d}{dt}\int_{\mathbb{R}^3_+}\eta\,(n_a+n)\frac{|\dot{u}|^2}{2}\leqslant\mathscr{I}+I_1+I_2$$

$$+C(C_a,M)\big(\|R^r(t)\|_{L^2(\mathbb{R}^3_+)}^2+\mu\big\|\sqrt{\eta'}R^s(t)\big\|_{L^2(\mathbb{R}^3_+)}^2+\|\dot{u}(t)\|_{L^2(\mathbb{R}^3_+)}+\mu\big\|\sqrt{\eta'}\dot{u}\big\|_{L^2(\mathbb{R}^3_+)}^2\big).$$

Integrating by parts, we have the identity

$$\frac{1}{T^i}I_1=-\int_{\mathbb{R}^3_+}\eta\,\nabla\dot{n}\cdot\dot{u}=\int_{\mathbb{R}^3_+}\eta\,\dot{n}\,\mathrm{div}\,\dot{u}+\int_{\mathbb{R}^3_+}\eta'\dot{n}\,\dot{u}_3+\int_{x_3=0}\dot{n}\,\dot{u}_3.$$

To write the boundary term, we have used that $\eta(0) = 1$, see (3.5). We can then use the evolution equation on $\dot{n}$ to express div $\dot{u}$ in terms of $\dot{n}$. We find

$$(3.17) \quad \frac{1}{T^i}I_1 = -\int_{\mathbb{R}^3_+} \frac{\eta}{n_a + n}\, (\partial_t + (u_a + u)\cdot\nabla)\,\frac{|\dot{n}|^2}{2} + \int_{\mathbb{R}^3_+} \frac{\eta}{n_a + n}\, r_n\,\dot{n}$$

$$+ \int_{\mathbb{R}^3_+} \eta'\dot{n}\,\dot{u}_3 + \int_{x_3=0} \dot{n}\,\dot{u}_3$$

$$= -\partial_t \int_{\mathbb{R}^3_+} \frac{\eta}{n_a + n}\,\frac{|\dot{n}|^2}{2} + \int_{\mathbb{R}^3_+} \eta'\,\frac{(u_a + u)_3}{n_a + n}\,\frac{|\dot{n}|^2}{2} + \int_{x_3=0} \frac{(u_a + u)_3}{n_a + n}\,\frac{|\dot{n}|^2}{2}$$

$$+ \int_{\mathbb{R}^3_+} \eta'\dot{n}\,\dot{u}_3 + \int_{x_3=0} \dot{n}\,\dot{u}_3 + J_1 + J_2$$

with

$$J_1 := \int_{\mathbb{R}^3_+} \text{div}\left(\frac{u_a + u}{n_a + n}\right)\frac{|\dot{n}|^2}{2}, \quad J_2 := \int_{\mathbb{R}^3_+} \partial_t\left(\frac{1}{n_a + n}\right)\frac{|\dot{n}|^2}{2} + \int_{\mathbb{R}^3_+} \frac{\eta}{n_a + n}\, r_n\,\dot{n}.$$

For the last term J_2, we use again (3.12), (3.10), (3.6) and (3.9) to get

$$(3.18) \quad J_2 \leqslant C(C_a, M)\left(\|R^r(t)\|^2_{L^2} + \|\dot{n}(t)\|^2_{L^2} + \mu\left\|\sqrt{\eta'}R^s(t)\right\|^2_{L^2} + \mu\left\|\sqrt{\eta'}\dot{n}(t)\right\|^2_{L^2}\right).$$

To bound J_1, we use (3.10) and (3.3) to state

$$J_1 \leqslant \int \eta\, C(C_a, M)\left(\frac{\delta}{\varepsilon}e^{-\gamma_0 x_3/\varepsilon} + 1\right)|\dot{n}|^2$$

$$(3.19) \qquad\qquad \leqslant C(C_a, M)\left(\mu\int_{\mathbb{R}^3_+} \eta'|\dot{n}|^2 + \|\dot{n}\|^2_{L^2}\right),$$

where we have used (3.7) to go from the first to the second line. We insert (3.18)–(3.19) in (3.17) to obtain

$$(3.20) \quad \frac{1}{T^i}I_1 \leqslant -\partial_t \int_{\mathbb{R}^3_+} \frac{\eta}{n_a + n}\,\frac{|\dot{n}|^2}{2} + \int_{\mathbb{R}^3_+} \eta'\,\frac{(u_a + u)_3}{n_a + n}\,\frac{|\dot{n}|^2}{2} + \int_{\mathbb{R}^3_+} \eta'\dot{n}\,\dot{u}_3$$

$$+ \int_{x_3=0} \frac{(u_a + u)_3}{n_a + n}\,\frac{|\dot{n}|^2}{2} + \int_{x_3=0} \dot{n}\,\dot{u}_3$$

$$+ C(C_a, M)\left(\mu\int_{\mathbb{R}^3_+} \eta'|\dot{n}|^2 + \|\dot{n}\|^2_{L^2} + \|R^r(t)\|^2_{L^2} + \mu\left\|\sqrt{\eta'}R^s(t)\right\|^2_{L^2}\right).$$

(2) *Treatment of I_2.* — We write

$$I_2 = -\int_{\mathbb{R}^3_+} \eta\,(n_a + n)(u_a + u)\cdot\nabla\frac{|\dot{u}|^2}{2}$$

$$= \int_{\mathbb{R}^3_+} \eta\,\text{div}((n_a + n)(u_a + u))\frac{|\dot{u}|^2}{2} + \int_{\mathbb{R}^3_+} \eta'\,(n_a + n)(u_a + u)_3\frac{|\dot{u}|^2}{2}$$

$$+ \int_{x_3=0} (n_a + n)(u_a + u)_3\frac{|\dot{u}|^2}{2}.$$

Relying once again on (3.3)–(3.12), and (3.7), we infer that:

$$(3.21) \quad I_2 \leqslant \int_{\mathbb{R}^3_+} \eta'\, (n_a + n)(u_a + u)_3 \frac{|\dot{u}|^2}{2} + \int_{x_3=0} (n_a + n)(u_a + u)_3 \frac{|\dot{u}|^2}{2}$$
$$+ C(C_a, M)\Big(\mu \int_{\mathbb{R}^3_+} \eta' |\dot{u}|^2 + \|\dot{u}\|^2_{L^2(\mathbb{R}^3_+)}\Big).$$

Conclusion. — Gathering (3.16), (3.20) and (3.21), we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^3_+} \eta\,(n_a + n) \frac{|\dot{u}|^2}{2} + \frac{d}{dt} \int_{\mathbb{R}^3_+} \eta \frac{T^i \dot{n}^2}{2(n_a + n)}$$
$$\leqslant \frac{1}{2} T^i \int_{\mathbb{R}^3_+} \eta' \frac{\dot{n}^2\,(u^a + u)_3}{n_a + n} + T^i \int_{\mathbb{R}^3_+} \eta' \dot{n}\, \dot{u}_3 + \int_{\mathbb{R}^3_+} \eta'\,(n_a + n)(u_a + u)_3 \frac{|\dot{u}|^2}{2}$$
$$+ T^i \int_{x_3=0} \frac{(u_a + u)_3}{n_a + n} \frac{|\dot{n}|^2}{2} + T^i \int_{x_3=0} \dot{n}\, \dot{u}_3 + \int_{x_3=0} (n_a + n)(u_a + u)_3 \frac{|\dot{u}|^2}{2}$$
$$+ C(C_a, M)\Big(\|(\dot{n}, \dot{u})\|^2_{L^2} + \|R^r\|^2_{L^2} + \mu \int_{\mathbb{R}^3_+} \eta'\,(\dot{n}^2 + |\dot{u}|^2 + |R^s|^2)\Big).$$

To conclude, it suffices to observe that the only significant contribution in the terms involving η' comes from the traces of the coefficients. Indeed, we have that

$$(3.22) \qquad u_a = u^0 + O(\varepsilon + \delta),\ n_a = n^0 + O(\varepsilon + \delta)$$

and that

$$(3.23) \qquad |(u^0, n^0) - \Gamma(u^0, n^0)| \leqslant C_a x_3, \quad |(u, n) - \Gamma(u, n)| \leqslant C(M) x_3$$

to obtain

$$(3.24) \quad \frac{(u^a + u)_3}{n_a + n} = \Gamma\Big(\frac{(u^0 + u)_3}{n^0 + n}\Big) + O(\delta + \varepsilon + x_3) = \Gamma\Big(\frac{(u_a + u)_3}{n_a + n}\Big) + O(\delta + \varepsilon + x_3),$$
$$(n_a + n)(u_a + u)_3 = \Gamma\big((n^0 + n)(u^0 + u)_3\big) + O(\varepsilon + \delta + x_3)$$
$$= \Gamma\big((n_a + n)(u_a + u)_3\big) + O(\varepsilon + \delta + x_3).$$

Since $\varepsilon \eta'$ and $x_3 \eta'$ are uniformly bounded (see (3.6)), we end up with

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3_+} \eta\Big((n_a + n)\frac{|\dot{u}|^2}{2} + \frac{|\dot{n}|^2}{n_a + n}\Big)\,dx + \frac{1}{2} \int_{\mathbb{R}^3_+} \eta'\, Q^0(\dot{n}, \dot{u})\,dx + \frac{1}{2} \int_{x_3=0} Q^0(\dot{n}, \dot{u}) - \mathscr{I}$$
$$\leqslant C(C_a, M)\Big(\|(\dot{n}, \dot{u})\|^2_{L^2} + \|R^r\|^2_{L^2} + \mu \int_{\mathbb{R}^3_+} \eta'|R^s|^2\Big) + (\mu + \delta) \int_{\mathbb{R}^3_+} \eta'\,(\dot{n}^2 + |\dot{u}|^2).$$

By the Bohm condition, the quadratic form Q_0 is positive definite: this allows to absorb the last term at the r.h.s. for μ and δ small enough. The estimate of Lemma 1 follows.

A. The estimate for the weighted physical energy. — We now use Lemma 1 in order to prove Proposition 2. The first estimate that we shall establish is the following

$$(3.25)\quad \|(\dot{n}, \dot{u}, \dot{\phi}, \varepsilon\nabla\dot{\phi})\|^2_{L^\infty_T L^2(\mathbb{R}^3_+)} + \big\|\sqrt{\eta'}(\dot{n}, \dot{u}, \dot{\phi}, \varepsilon\nabla\dot{\phi})\big\|^2_{L^2_T L^2(\mathbb{R}^3_+)}$$
$$+ \|\Gamma(\dot{n}, \dot{u}, \varepsilon\partial_{x_3}\dot{\phi})\|^2_{L^2_T L^2(\mathbb{R}^2)}$$
$$\leqslant C(C_a, M)\Big(\|(\dot{n}, \dot{u}, \dot{\phi}, \varepsilon\nabla\dot{\phi})(0)\|^2_{L^2(\mathbb{R}^3_+)} + \big\|\sqrt{\eta'}\,(\varepsilon\nabla\dot{n}, \varepsilon\,\mathrm{div}\,\dot{u})\big\|^2_{L^2_T L^2(\mathbb{R}^3_+)}$$
$$+ \|(\dot{n}, \dot{u}, \dot{\phi}, \varepsilon\nabla\dot{\phi})\|^2_{L^2_T L^2(\mathbb{R}^3_+)} + \|R^r\|^2_{L^2_T L^2(\mathbb{R}^3_+)} + \mu\big\|\sqrt{\eta'}R^s\big\|^2_{L^2_T L^2(\mathbb{R}^3_+)}\Big).$$

Note that the above estimate is not enough by itself because the singular term $\big\|\sqrt{\eta'}\,(\varepsilon\nabla\dot{n}, \varepsilon\,\mathrm{div}\,\dot{u})\big\|^2_{L^2_T L^2(\mathbb{R}^3_+)}$ in the right hand side is not controlled by the left hand side.

The first step in the proof of (3.25) consists in manipulating the integral $\mathscr{I}$ (see Lemma 1): it will give us some control on the potential $\dot{\phi}$. More precisely, we first integrate by parts to write:

$$\mathscr{I} = -\int_{\mathbb{R}^3_+} \eta\,\dot{\phi}\,\dot{u}\cdot\nabla(n_a + n) - \int_{\mathbb{R}^3_+} \eta\,\dot{\phi}\,(n_a + n)\,\mathrm{div}\,\dot{u} - \int_{\mathbb{R}^3_+} \eta'\,\dot{\phi}\,\big((n_a + n)\dot{u}_3\big).$$

There is no boundary term since $\dot{\phi}$ satisfies an homogeneous Dirichlet condition (see (3.2)). The first term at the r.h.s can be controlled thanks to (3.3)–(3.7) and (3.10). We get

$$(3.26)\quad \mathscr{I} \leqslant I_1 - \int_{\mathbb{R}^3_+} \eta'\,\dot{\phi}\,\big((n_a + n)\dot{u}_3\big) + C(C_a, M)\,\big(\mu\big\|\sqrt{\eta'}(\dot{u}, \dot{\phi})\big\|^2_{L^2} + \|(\dot{u}, \dot{\phi})\|^2_{L^2}\big)$$

with $I_1 := -\int_{\mathbb{R}^3_+} \eta\,\dot{\phi}\,(n_a + n)\,\mathrm{div}\,\dot{u}$. As above, we use the evolution equation on $\dot{n}$ to express $(n_a + n)\,\mathrm{div}\,\dot{u}$ in terms of $\dot{n}$:

$$I_1 := \int_{\mathbb{R}^3_+} \eta\dot{\phi}\partial_t\dot{n} + \int_{\mathbb{R}^3_+} \eta(u_a + u)\cdot\nabla\dot{n}\,\dot{\phi} - \int_{\mathbb{R}^3_+} \eta r_n\dot{\phi}.$$

We integrate by parts the second integral, and obtain

$$(3.27)\quad I_1 \leqslant J_1 + J_2 + J_3$$
$$+ C(C_a, M)\big(\|R_r\|^2_{L^2} + \mu\big\|\sqrt{\eta'}R^s\big\|^2_{L^2} + \mu\big\|\sqrt{\eta'}(\dot{\phi}, \dot{n})\big\|^2_{L^2} + \|(\dot{\phi}, \dot{n})\|^2_{L^2}\big)$$

with

$$J_1 := \int_{\mathbb{R}^3_+} \eta\dot{\phi}\partial_t\dot{n}, \quad J_2 := -\int_{\mathbb{R}^3_+} \eta(u_a + u)\cdot\nabla\dot{\phi}\,\dot{n} \quad J_3 := -\int_{\mathbb{R}^3_+} \eta'(u_a + u)_3\dot{\phi}\,\dot{n}.$$

Note that we have used implicitly the bound

$$-\int_{\mathbb{R}^3_+} \eta\,\mathrm{div}(u_a + u)\dot{\phi}\,\dot{n} \leqslant C(C_a, M)\,\big(\mu\big\|\sqrt{\eta'}(\dot{\phi}, \dot{n})\big\|^2_{L^2} + \|(\dot{\phi}, \dot{n})\|^2_{L^2}\big)$$

to handle the last term coming from the integration by parts.

(1) *Treatment of J_1.* — At first, by differentiating with respect to time the Poisson equation in (3.1), we can express $\partial_t \dot{n}$ in terms of $\dot{\phi}$, and substitute inside the expression for J_1:

$$J_1 = \int_{\mathbb{R}^3_+} \eta\, \dot{\phi}\, \left(\varepsilon^2 \partial_t \Delta\dot{\phi} - \partial_t(e^{-\phi_a}(1+h(\phi))\dot{\phi}) - \partial_t r_\phi \right)$$

$$(3.28) \qquad = -\varepsilon^2 \frac{d}{dt} \int_{\mathbb{R}^3_+} \frac{1}{2}\eta\, |\nabla\dot{\phi}|^2 - \frac{d}{dt}\int_{\mathbb{R}^3_+} \frac{1}{2}\eta\, \dot{\phi}^2 e^{-\phi_a}(1+h(\phi))$$

$$- \frac{1}{2}\int_{\mathbb{R}^3_+} \eta\, \partial_t(e^{-\phi_a}(1+h(\phi)))|\dot{\phi}|^2 - \int_{\mathbb{R}^3_+} \eta\, \dot{\phi}\, \partial_t r_\phi - \varepsilon^2 \int \eta'\dot{\phi}\, \partial_t \partial_{x_3}\dot{\phi}.$$

One has by using (3.12) and (3.10) the straightforward estimate

$$- \frac{1}{2}\int_{\mathbb{R}^3_+} \eta\, \partial_t(e^{-\phi_a}(1+h(\phi)))|\dot{\phi}|^2 \leqslant C(C_a, M)\|\dot{\phi}\|^2_{L^2(\mathbb{R}^3_+)}$$

and by using also (3.9)

$$- \int_{\mathbb{R}^3_+} \dot{\phi}\, \partial_t r_\phi \leqslant C(C_a, M)\big(\|R_r\|^2_{L^2} + \mu\big\|\sqrt{\eta'}R^s\big\|^2_{L^2} + \mu(\big\|\sqrt{\eta'}\dot{\phi}\big\|^2_{L^2} + \|\dot{\phi}\|^2_{L^2})\big).$$

As regards the last term, we claim

Lemma 2. — *The following inequality holds.*

$$(3.29) \quad \left|\int \varepsilon\eta'\dot{\phi}\,\varepsilon\partial_t\partial_{x_3}\dot{\phi}\right| \leqslant \big\|\sqrt{\eta'}\dot{\phi}\big\|_{L^2}\big\|\sqrt{\eta'}\big(\varepsilon\,\nabla\dot{n}, \varepsilon\,\mathrm{div}\,\dot{u}\big)\big\|^2_{L^2}$$

$$+ C(C_a, M)\big(\|R^r\|^2_{L^2} + \mu\big\|\sqrt{\eta'}R^s\big\|^2_{L^2} + \mu\big\|\sqrt{\eta'}\dot{\phi}\big\|^2_{L^2} + \|\dot{\phi}\|^2_{L^2}\big).$$

We postpone the proof of the lemma to the end of the section. Combining all the previous estimates, we deduce:

$$(3.30) \quad J_1 \leqslant -\varepsilon^2 \partial_t \int_{\mathbb{R}^3_+} \frac{1}{2}\eta\, |\nabla\dot{\phi}|^2 - \varepsilon\partial_t \int_{\mathbb{R}^3_+} \frac{1}{2}\eta\, \dot{\phi}^2 e^{-\phi_a}(1+h(\phi))$$

$$+ C(C_a, M)\Big(\|R^r\|^2_{L^2} + \mu\big\|\sqrt{\eta'}R^s\big\|^2_{L^2} + \mu\big\|\sqrt{\eta'}(\dot{n}, \dot{u}, \dot{\phi})\big\|^2_{L^2}$$

$$+ \big\|\sqrt{\eta'}\dot{\phi}\big\|_{L^2}\big\|\sqrt{\eta'}(\varepsilon\,\nabla\dot{n}, \varepsilon\,\mathrm{div}\,\dot{u})\big\|_{L^2} + \|(\dot{n}, \dot{\phi})\|^2_{L^2}\Big).$$

(2) *Treatment of J_2.* — We use the Poisson equation to express $\dot{n}$ in terms of $\dot{\phi}$. We get

$$(3.31) \quad J_2 = -\int_{\mathbb{R}^3_+} \eta\,\nabla\dot{\phi}\cdot(\varepsilon^2\Delta\dot{\phi}(u_a+u)) + \int_{\mathbb{R}^3_+} \eta\,\nabla\dot{\phi}\cdot(e^{-\phi_a}(1+h(\phi))\dot{\phi}(u_a+u))$$

$$+ \int_{\mathbb{R}^3_+} \eta\,\nabla\dot{\phi}\cdot(r_\phi(u_a+u)) := K_1 + K_2 + K_3.$$

One has

$$K_2 = \frac{1}{2}\int_{\mathbb{R}^3_+} \eta\,\nabla|\dot\phi|^2\cdot(e^{-\phi_a}(1+h(\phi))(u_a+u))$$

$$= -\frac{1}{2}\int_{\mathbb{R}^3_+}\eta\,|\dot\phi|^2\,\mathrm{div}\,\big(e^{-\phi_a}(1+h(\phi))(u_a+u)\big)$$

$$\text{(3.32)} \hspace{4cm} -\frac{1}{2}\int_{\mathbb{R}^3_+}\eta'\,|\dot\phi|^2\,e^{-\phi_a}(1+h(\phi))(u_a+u)_3$$

$$\leqslant C(C_a,M)\big(\mu\big\|\sqrt{\eta'}\dot\phi\big\|_{L^2}^2+\|\dot\phi\|_{L^2}^2\big)$$
$$-\frac{1}{2}\int_{\mathbb{R}^3_+}\eta'\,|\dot\phi|^2\,e^{-\phi_a}(1+h(\phi))(u_a+u)_3.$$

Again, the bound on the first term at the right hand side is a consequence of (3.7). Thanks to (3.12) and (3.9) one has also

$$K_3 \leqslant C(C_a,M)\int_{\mathbb{R}^3_+}|r_\phi/\varepsilon|\,\|\varepsilon\nabla\dot\phi\|\,dx$$

$$\leqslant C(C_a,M)\big(\|R^r\|_{L^2(\mathbb{R}^3_+)}^2+\mu\big\|\sqrt{\eta'}R^s\big\|_{L^2}^2+\mu\big\|\sqrt{\eta'}\,\varepsilon\nabla\dot\phi\big\|_{L^2}^2+\|\varepsilon\nabla\dot\phi\|_{L^2}^2\big).$$

Finally, we compute

$$K_1 = \varepsilon^2\int_{\mathbb{R}^3_+}\eta\,((\nabla\dot\phi\cdot\nabla)(u_a+u))\cdot\nabla\dot\phi+\varepsilon^2\int_{\mathbb{R}^3_+}\eta\Big(\nabla\frac{|\nabla\dot\phi|^2}{2}\Big)\cdot(u_a+u)$$

$$\text{(3.33)} \hspace{2cm} +\varepsilon^2\int_{\mathbb{R}^3_+}\eta'\,\partial_3\dot\phi\,\nabla\dot\phi\cdot(u_a+u)+\varepsilon^2\int_{x_3=0}\partial_3\dot\phi\nabla\dot\phi\cdot(u_a+u)$$

$$=: L_1+L_2+L_3+\varepsilon^2\int_{x_3=0}|\partial_3\dot\phi|^2(u_a+u)_3.$$

We point out the simplification of the boundary term, due to the homogeneous boundary condition on $\dot\phi$. Once again, (3.7) leads to

$$|L_1|\leqslant C(C_a,M)\int_{\mathbb{R}^3_+}\eta\,\frac{\delta}{\varepsilon}\,e^{-\gamma_0 x_3/\varepsilon}\,\varepsilon^2|\nabla\dot\phi|^2+C(C_a,M)\|\varepsilon\nabla\dot\phi\|_{L^2}^2$$

$$\leqslant C(C_a,M)\Big(\mu\big\|\sqrt{\eta'}\,\varepsilon\nabla\dot\phi\big\|_{L^2}^2+\|\varepsilon\nabla\dot\phi\|_{L^2}^2\Big).$$

For L_2, by integration by parts, we can write:

$$L_2=-\varepsilon^2\int_{\mathbb{R}^3_+}\eta'\Big(\frac{|\nabla\dot\phi|^2}{2}\Big)(u_a+u)_3-\varepsilon^2\int_{\mathbb{R}^3_+}\eta\Big(\frac{|\nabla\dot\phi|^2}{2}\Big)\mathrm{div}(u_a+u)$$

$$-\varepsilon^2\int_{x_3=0}\Big(\frac{|\partial_{x_3}\dot\phi|^2}{2}\Big)(u_a+u)_3,$$

where we have used that $\nabla_{1,2}\dot\phi=0$ on the boundary. The first term has a "bad sign" but will be compensated by another contribution (coming from J_3). The second term satisfies:

$$\Big|\varepsilon^2\int_{\mathbb{R}^3_+}\eta\,\Big(\frac{|\nabla\dot\phi|^2}{2}\Big)\mathrm{div}(u_a+u)\Big|\leqslant C(C_a,M)\big(\mu\big\|\sqrt{\eta'}\,\varepsilon\nabla\dot\phi\big\|_{L^2}^2+\|\varepsilon\nabla\dot\phi\|_{L^2}^2\big).$$

Finally, the third (boundary) term of L_2 has also a bad sign, but will be compensated by the other boundary term of (3.33).

For L_3, we rely on the fact that

$$|(u^a + u)_{1,2} - (\Gamma u^0)_{1,2}| \leqslant C_a(\varepsilon + \delta + x_3) + \|u\|_{L^\infty}.$$

Together with our assumption (3.4) on the trace ($|(\Gamma u^0)_{1,2}| \leqslant \delta$), this implies

$$L_3 \leqslant \int_{\mathbb{R}^3_+} \eta' \, |\varepsilon \, \partial_3 \dot\phi|^2 \, (u^a(t,x) + u)_3 + \mu \int_{\mathbb{R}^3_+} \eta' \, |\varepsilon \nabla \dot\phi|^2 + \frac{\|u\|_{L^\infty}}{\varepsilon} \, \|\varepsilon \nabla \dot\phi\|_{L^2}^2.$$

Note that the last term in the right hand side of the above estimate is bounded by $C(M)\|\varepsilon \nabla \dot\phi\|_{L^2}^2$ thanks to (3.10).

Putting together the estimates on L_1, L_2 and L_3, we find

$$(3.34) \quad K_1 \leqslant -\varepsilon^2 \int_{\mathbb{R}^3_+} \eta' \, \frac{|\nabla \dot\phi|^2}{2}(u_a + u)_3 + \frac{\varepsilon^2}{2} \int_{x_3=0} |\partial_3 \dot\phi|^2 (u_a + u)_3$$

$$+ \int_{\mathbb{R}^3_+} \eta' \, |\varepsilon \, \partial_3 \dot\phi|^2 \, (u^a(t,x) + u)_3 + C(C_a, M)\big(\mu\big\|\sqrt{\eta'}\,\varepsilon \nabla \dot\phi\big\|_{L^2}^2 + \|\varepsilon \nabla \dot\phi\|_{L^2}^2\big).$$

Eventually, we derive an inequality on J_2:

$$(3.35) \quad J_2 \leqslant -\varepsilon^2 \int_{\mathbb{R}^3_+} \eta' \, \frac{|\nabla \dot\phi|^2}{2}(u_a + u)_3 + \frac{\varepsilon^2}{2} \int_{x_3=0} |\partial_3 \dot\phi|^2 (u_a + u)_3$$

$$- \frac{1}{2} \int_{\mathbb{R}^3_+} \eta' \, |\dot\phi|^2 \, e^{-\phi_a}(1 + h(\phi))(u_a + u)_3 + \int_{\mathbb{R}^3_+} \eta' \, |\varepsilon \, \partial_3 \dot\phi|^2 \, (u^a(t,x) + u)_3$$

$$+ C(C_a, M)\big(\|R^r\|_{L^2}^2 + \mu\big\|\sqrt{\eta'}\,R^s\big\|_{L^2}^2 + \|\varepsilon \nabla \dot\phi\|_{L^2}^2 + \mu\big\|\sqrt{\eta'}\,\varepsilon \nabla \dot\phi\big\|_{L^2}^2\big).$$

(3) *Treatment of J_3.* — First, we use the Poisson equation to replace $\dot{n}$:

$$J_3 = - \int_{\mathbb{R}^3_+} \eta' \, \dot\phi(u_a + u)_3 \left[\varepsilon^2 \Delta \dot\phi - e^{-\phi_a}(1 + h(\phi))\dot\phi - r_\phi\right] =: K_1 + K_2 + K_3.$$

Let us start with the estimate of K_1. With the help of an integration by parts, we get:

$$K_1 = \varepsilon^2 \int_{\mathbb{R}^3_+} \eta' \, (u_a + u)_3 |\nabla \dot\phi|^2 + \varepsilon^2 \int_{\mathbb{R}^3_+} \eta' \, \dot\phi \nabla (u_a + u)_3 \cdot \nabla \dot\phi + \varepsilon^2 \int_{\mathbb{R}^3_+} \eta'' \, \dot\phi(u_a + u)_3 \nabla \dot\phi.$$

The first term has a "good sign", and will be used to absorb the first term at the r.h.s. of (3.35). The second and third terms can be bounded using properties of the approximate solution and of the weight η (notably (3.8)). We end up with

$$(3.36) \quad K_1 \leqslant \varepsilon^2 \int_{\mathbb{R}^3_+} \eta' \, (u_a + u)_3 |\nabla \dot\phi|^2 + C(C_a, M)\big(\mu\big\|\sqrt{\eta'}(\dot\phi, \varepsilon \nabla \dot\phi)\big\|_{L^2}^2 + \|(\dot\phi, \varepsilon \nabla \dot\phi)\|_{L^2}^2\big).$$

K_2 is also a good term, that will absorb the third term at the r.h.s. of (3.35). For K_3, we use as usual (3.9) and write

$$K_3 \leqslant C(C_a, M)\big(\|R^r\|_{L^2}^2 + \mu\big\|\sqrt{\eta'}\,R^s\big\|_{L^2}^2 + \mu\big\|\sqrt{\eta'}\dot\phi\big\|_{L^2}^2 + \|\dot\phi\|_{L^2}^2\big).$$

Finally, these last bounds yield

$$(3.37) \quad J_3 \leqslant \varepsilon^2 \int_{\mathbb{R}^3_+} \eta' \, (u_a + u)_3 |\nabla \dot\phi|^2 + \int_{\mathbb{R}^3_+} \eta' \, (u_a + u)_3 e^{-\phi_a}(1 + h(\phi))|\dot\phi|^2$$
$$+ \, C(C_a, M)\big(\|R^r\|_{L^2}^2 + \mu\big\|\sqrt{\eta'}\, R^s\big\|_{L^2}^2 + \mu\big\|\sqrt{\eta'}(\dot\phi, \varepsilon\nabla\dot\phi)\big\|_{L^2}^2 + \|(\dot\phi, \varepsilon\nabla\dot\phi)\|_{L^2}^2\big).$$

Conclusion. — We can now collect the estimates (3.30), (3.35) and (3.37), and insert them into (3.27), followed by (3.26). We get

$$\mathscr{I} + \varepsilon^2 \frac{d}{dt} \int_{\mathbb{R}^3_+} \frac{1}{2}\eta \, |\nabla\dot\phi|^2 + \frac{d}{dt} \int_{\mathbb{R}^3_+} \frac{1}{2}\eta \, \dot\phi^2 e^{-\phi_a}(1 + h(\phi))$$
$$\leqslant -\int_{\mathbb{R}^3_+} \eta' \, \dot\phi \, \big((n_a + n)\dot u_3\big) + \frac{\varepsilon^2}{2} \int_{\mathbb{R}^3_+} \eta' \, |\nabla\dot\phi|^2 (u_a + u)_3 + \frac{\varepsilon^2}{2} \int_{x_3=0} |\partial_3\dot\phi|^2 (u_a + u)_3$$
$$+ \frac{1}{2} \int_{\mathbb{R}^3_+} \eta' \, (u_a + u)_3 \, e^{-\phi_a}(1 + h(\phi)) \, |\dot\phi|^2 + \int_{\mathbb{R}^3_+} \eta' \, |\varepsilon \, \partial_3\dot\phi|^2 \, (u^a(t, x) + u)_3 + \mathscr{R}_1,$$

where

$$\mathscr{R}_1 := C(C_a, M)\big(\|R^r\|_{L^2}^2 + \mu\big\|\sqrt{\eta'}R^s\big\|_{L^2}^2 + \mu\big\|\sqrt{\eta'}(\dot u, \dot n, \dot\phi, \varepsilon\nabla\dot\phi)\big\|_{L^2}^2$$
$$+ \, \|(\dot u, \dot n, \dot\phi, \varepsilon\nabla\dot\phi)\|_{L^2}^2 + \big\|\sqrt{\eta'}\dot\phi\big\|_{L^2} \big\|\sqrt{\eta'}(\varepsilon\,\nabla\dot n, \varepsilon\,\mathrm{div}\,\dot u)\big\|_{L^2}\big).$$

By combining this last estimate with the estimate of Lemma 1 and using the substitution (3.23) in the terms involving η', we obtain the estimate

$$(3.38) \quad \frac{d}{dt}\left[\int_{\mathbb{R}^3_+} \eta \, (n_a + n)\frac{|\dot u|^2}{2} + \int_{\mathbb{R}^3_+} \eta\frac{T^i \dot n^2}{2(n_a + n)} + \frac{1}{2}\int_{\mathbb{R}^3_+} \eta \, \varepsilon^2 |\nabla\dot\phi|^2 + \eta \, e^{\phi_a}(1 + h(\phi))\dot\phi^2 \right]$$
$$+ \frac{1}{2}\int_{\mathbb{R}^3_+} \eta' \, |\Gamma(u_a + u)_3||\varepsilon\,\nabla\dot\phi|^2 + \frac{1}{2}\int_{\mathbb{R}^3_+} \eta' \, |\Gamma(u_a + u)_3||\varepsilon\,\partial_3\dot\phi|^2$$
$$+ \frac{1}{2}\int_{\mathbb{R}^3_+} \eta' \, Q^A(\dot n, \dot u, \dot\phi) + \frac{1}{2}\int_{x_3=0} |(u_a + u)_3||\varepsilon\,\partial_3\dot\phi|^2 + \frac{1}{2}\int_{x_3=0} Q^0(\dot n, \dot u)$$
$$\leqslant \mathscr{R}_2,$$

where

$$\mathscr{R}_2 := \big\|\sqrt{\eta'}\dot\phi\big\|_{L^2}\big\|\sqrt{\eta'}(\varepsilon\,\nabla\dot n, \varepsilon\,\mathrm{div}\,\dot u)\big\|_{L^2} + C(C_a, M)\big(\|(\dot n, \dot u, \dot\phi, \varepsilon\,\nabla\dot\phi)\|_{L^2}^2$$
$$+ \, \mu\big\|\sqrt{\eta'}\,\varepsilon\nabla\dot\phi\big\|_{L^2}^2 + \|R^r\|_{L^2(\mathbb{R}^3_+)}^2 + \mu\big\|\sqrt{\eta'}R^s\big\|_{L^2(\mathbb{R}^3_+)}^2\big)$$

and Q^A is the positive quadratic form determined by the following symmetric matrix:

$$M^A = \Gamma\begin{bmatrix} T^i\frac{|(u_a+u)_3|}{n_a+n} & -T^i e^\top & 0 \\ -T^i e & (n_a + n)|(u_a + u)_3|\,\mathrm{Id}_3 & (n_a + n)e \\ 0 & (n_a + n)e^\top & e^{-\phi_a}(1 + h(\phi))|(u_a + u)_3| \end{bmatrix} - \mu C\,\mathrm{Id}_5\,.$$

Since μ can be chosen as small as we want, it suffices to prove that the leading above matrix is positive. According to Sylvester's criterion, we only have to check that the

leading principal minors, denoted by $(\Delta_i)_{1 \leqslant i \leqslant 5}$ are positive. We compute:

$$\Delta_1 = T^i \Gamma \Big(\frac{|(u_a + u)_3|}{n_a + n} \Big) > 0, \quad \Delta_2 = T^i \Gamma \Big(\frac{|(u_a + u)_3|^2}{n_a + n} \Big) > 0,$$

$$\Delta_3 = T^i \Gamma \Big(\frac{|(u_a + u)_3|^3}{n_a + n} \Big) > 0,$$

$$\Delta_4 = T^i \Gamma \Big((n_a + n)^2 |(u_a + u)_3|^2 \big(|(u_a + u)_3|^2 - T^i \big) \Big),$$

which is also positive as a straightforward consequence of Bohm condition (3.11). Finally the determinant is equal to:

$$\Delta_5 = T^i \Gamma \Big(|(u_a + u)_3|^3 (n_a + n)^2 \Big(\frac{e^{-\phi_a}(1 + h(\phi))}{n_a + n} |(u_a + u)_3|^2 - (T^i + 1) \Big) \Big),$$

which is positive as soon as $|(u_a + u)_3| > \sqrt{T^i + 1}$ as ensured by the Bohm condition (3.11). Indeed, we have that $|(u, n, \phi)| = O(\varepsilon)$ thanks to (3.10), and that since $n^0 = e^{\phi^0}$ that $n_a = e^{-\phi_a} + O(\delta + \varepsilon)$, therefore

$$\Gamma \Big(\frac{e^{-\phi_a}(1 + h(\phi))}{n_a + n} |(u_a + u)_3|^2 - (T^i + 1) \Big) = \Gamma \big(|(u_a + u)_3|^2 - (T^i + 1) \big) + O(\varepsilon + \delta) > 0,$$

for ε and δ sufficiently small. The quadratic form Q^0 is also positive as already observed.

To derive (3.25) from (3.38), it remains to note that, by positivity of Q^A, we can write the Young inequality

$$\big\| \sqrt{\eta'} \dot\phi \big\|_{L^2} \big\| \sqrt{\eta'} (\varepsilon \nabla \dot n, \varepsilon \operatorname{div} \dot u) \big\|_{L^2} \leqslant \widetilde\mu \big\| \sqrt{\eta'} \dot\phi \big\|_{L^2}^2 + C_{\widetilde\mu} \big\| \sqrt{\eta'} (\varepsilon \nabla \dot n, \varepsilon \operatorname{div} \dot u) \big\|_{L^2}^2,$$

and absorb the term $\widetilde\mu \| \sqrt{\eta'} \dot\phi \|_{L^2}^2$ in the left hand side by choosing $\widetilde\mu$ small enough. In the same spirit, the term $\mu \| \sqrt{\eta'} \, \varepsilon \nabla \dot\phi \|_{L^2}^2$ can be absorbed by the quadratic terms in $\varepsilon \nabla \dot\phi$ at the left hand side of (3.38) for μ small enough.

To conclude this section, we still have to provide the proof of Lemma 2.

Proof of Lemma 2. — To estimate the term $\varepsilon^2 \int \eta' \dot\phi \, \partial_t \partial_{x_3} \dot\phi$ in (3.28), we shall rely on an elliptic estimate on the Poisson equation. We first have the straightforward bound:

$$(3.39) \qquad \Big| \int \varepsilon \eta' \dot\phi \, \varepsilon \partial_t \partial_{x_3} \dot\phi \Big| \leqslant \big\| \sqrt{\eta'} \dot\phi \big\|_{L^2} \big\| \sqrt{\eta'} \, \varepsilon^2 \partial_t \partial_{x_3} \dot\phi \big\|_{L^2}.$$

Differentiating the Poisson with respect to time and taking the product with $\eta' \, \partial_t \dot\phi$, we obtain:

$$\int_{\mathbb{R}^3_+} \eta' \, e^{-\phi_a}(1 + h(\phi)) |\partial_t \dot\phi|^2 + \varepsilon^2 \int_{\mathbb{R}^3_+} \eta' \, |\nabla \partial_t \dot\phi|^2 + \varepsilon^2 \int_{\mathbb{R}^3_+} \eta'' \, \partial_t \partial_{x_3} \dot\phi \, \partial_t \dot\phi$$

$$= - \int_{\mathbb{R}^3_+} \eta' \, \partial_t \dot n \, \partial_t \dot\phi - \int_{\mathbb{R}^3_+} \eta' \, \partial_t (e^{-\phi_a}(1 + h(\phi))) \dot\phi \, \partial_t \dot\phi - \int_{\mathbb{R}^3_+} \eta' \partial_t r_\phi \partial_t \dot\phi.$$

We have chosen the parameters of the weight η so that δ/μ^2 is small enough. Hence,

$$\Big| \varepsilon^2 \int_{\mathbb{R}^3_+} \eta'' \, \partial_t \partial_{x_3} \dot\phi \, \partial_t \dot\phi \Big| \leqslant \frac{1}{2} \varepsilon^2 \int_{\mathbb{R}^3_+} \eta' \, |\nabla \partial_t \dot\phi|^2 + \frac{1}{4} \int_{\mathbb{R}^3_+} \eta' \, e^{-\phi_a}(1 + h(\phi)) |\partial_t \dot\phi|^2.$$

We also have by using (3.10) and the Young inequality that

$$\left| \int_{\mathbb{R}^3_+} -\eta' \, \partial_t \dot{n} \, \partial_t \dot{\phi} - \int_{\mathbb{R}^3_+} \eta' \, \partial_t (e^{-\phi_a}(1+h(\phi))) \dot{\phi} \, \partial_t \dot{\phi} \right|$$

$$\leqslant \frac{1}{8} \int_{\mathbb{R}^3_+} \eta' \, e^{-\phi_a}(1+h(\phi)) |\partial_t \dot{\phi}|^2 + C(C_a, M) \big(\big\| \sqrt{\eta'} \partial_t \dot{n} \big\|^2_{L^2} + \big\| \sqrt{\eta'} \dot{\phi} \big\|^2_{L^2} \big)$$

and by using (3.9) that

$$\int_{\mathbb{R}^3_+} \eta' \partial_t r_\phi \partial_t \dot{\phi} \leqslant C(C_a, M) \Big(\frac{1}{\varepsilon} \|R^r\|^2_{L^2} + \frac{\mu^2}{\varepsilon^2} \big\| \sqrt{\eta'} \, R^s \big\|^2_{L^2} \Big) + \frac{1}{8} \int_{\mathbb{R}^3_+} \eta' \, e^{-\phi_a}(1+h(\phi)) |\partial_t \dot{\phi}|^2.$$

We thus end up with the elliptic estimate:

$$(3.40) \quad \int_{\mathbb{R}^3_+} \eta' \, e^{-\phi_a}(1+h(\phi)) |\partial_t \dot{\phi}|^2 + \int_{\mathbb{R}^3_+} \eta' \, |\varepsilon \nabla \partial_t \dot{\phi}|^2$$

$$\leqslant C(C_a, M) \Big(\big\| \sqrt{\eta'} \partial_t \dot{n} \big\|^2_{L^2} + \big\| \sqrt{\eta'} \dot{\phi} \big\|^2_{L^2} \frac{1}{\varepsilon} \|R^r\|^2_{L^2} + \frac{\mu^2}{\varepsilon^2} \big\| \sqrt{\eta'} \, R^s \big\|^2_{L^2} \Big).$$

Going back to (3.39), this yields

$$\left| \int \varepsilon \eta' \dot{\phi} \, \varepsilon \partial_t \partial_{x_3} \dot{\phi} \right|$$

$$\leqslant C(C_a, M) \big\| \sqrt{\eta'} \dot{\phi} \big\|_{L^2} \big(\big\| \sqrt{\eta'} \varepsilon \partial_t \dot{n} \big\|_{L^2} + \varepsilon \big\| \sqrt{\eta'} \dot{\phi} \big\|_{L^2} + \sqrt{\varepsilon} \|R^r\|_{L^2} + \mu \big\| \sqrt{\eta'} \, R^s \big\|_{L^2} \big).$$

Now using the transport equation satisfied by $\dot{n}$, we can estimate $\big\| \sqrt{\eta'} \varepsilon \partial_t \dot{n} \big\|_{L^2}$. This yields:

$$\big\| \sqrt{\eta'} \varepsilon \partial_t \dot{n} \big\|_{L^2} \leqslant C_a \big\| \sqrt{\eta'} \, (\varepsilon \nabla \dot{n}, \varepsilon \operatorname{div} \dot{u}) \big\|_{L^2} + \sqrt{\varepsilon} \|R^r\|_{L^2} + \mu \big\| \sqrt{\eta'} R^s \big\|_{L^2}.$$

For the last term, we have used that $\varepsilon \eta' \leqslant \mu$ with the choice of the parameters. We have thus proven that

$$\left| \int \varepsilon \eta' \dot{\phi} \, \varepsilon \partial_t \partial_{x_3} \dot{\phi} \right| \leqslant \big\| \sqrt{\eta'} \dot{\phi} \big\|_{L^2} \big\| \sqrt{\eta'} (\varepsilon \, \nabla \dot{n}, \varepsilon \operatorname{div} \dot{u}) \big\|_{L^2}$$

$$+ C(C_a, M) \big(\|R^r\|^2_{L^2} + \mu \big\| \sqrt{\eta'} R^s \big\|^2_{L^2} + \mu \big\| \sqrt{\eta'} \dot{\phi} \big\|^2_{L^2} + \|\dot{\phi}\|^2_{L^2} \big). \quad \square$$

B. The second energy estimate. — The previous computation suggests to look for a control on $\varepsilon \, \nabla \dot{n}$ and $\varepsilon \operatorname{div} \dot{u}$, and this motivates the next energy estimate. We shall prove that

$$(3.41) \quad \|(\dot{n}, \dot{u}, \varepsilon \, \nabla \dot{n}, \varepsilon \operatorname{div} \dot{u})\|^2_{L^\infty_T L^2(\mathbb{R}^3_+)} + \big\| \sqrt{\eta'}(\dot{n}, \dot{u}, \varepsilon \, \nabla \dot{n}, \varepsilon \operatorname{div} \dot{u}) \big\|^2_{L^2_T L^2(\mathbb{R}^3_+)}$$

$$+ \|\Gamma(\dot{n}, \dot{u}, \varepsilon \partial_{x_3} \dot{n}, \varepsilon \operatorname{div} \dot{u})\|_{L^2_T L^2(\mathbb{R}^2)}$$

$$\leqslant C(C_a, M) \Big(\|(\dot{n}, \dot{u}, \varepsilon \nabla \dot{n}, \varepsilon \operatorname{div} \dot{u})(0)\|^2_{L^2(\mathbb{R}^3_+)} + \mu \big\| \sqrt{\eta'} \, \varepsilon \nabla \dot{u} \big\|^2_{L^2_T L^2(\mathbb{R}^3_+)}$$

$$+ \|(\dot{n}, \dot{u}, \varepsilon \nabla \dot{n}, \varepsilon \nabla \dot{u}, \dot{\phi}, \varepsilon \nabla \dot{\phi})\|^2_{L^2_T L^2(\mathbb{R}^3_+)} + \|R^r\|^2_{L^2_T L^2(\mathbb{R}^3_+)} + \mu \big\| \sqrt{\eta'} R^s \big\|^2_{L^2_T L^2(\mathbb{R}^3_+)} \Big).$$

Let us stress that this estimate alone does not allow to conclude: the right hand side involves the full $\varepsilon \nabla \dot{u}$, while the left hand side controls only $\varepsilon \operatorname{div} \dot{u}$. Later, we shall establish a third estimate, in order to handle $\varepsilon \nabla \dot{u}$ and close the argument.

To prove (3.41), we shall again rely on Lemma 1, but $\mathscr{I}$ will be handled in a different way. By integration by parts:

$$\mathscr{I} = \int_{\mathbb{R}^3_+} \eta \nabla \dot{\phi}\,(n_a + n) \cdot \dot{u}$$

$$= -\int_{\mathbb{R}^3_+} \eta \dot{\phi}\,(n_a + n)\,\mathrm{div}\,\dot{u} - \int_{\mathbb{R}^3_+} \eta' \dot{\phi}\,(n_a + n)\dot{u}_3 - \int_{\mathbb{R}^3_+} \eta \dot{\phi}\,\nabla(n_a + n) \cdot \dot{u}$$

$$(3.42) \qquad \leqslant I_1 + I_2 + C(C_a, M)\big(\mu\big\|\sqrt{\eta'}(\dot{u}, \dot{\phi})\big\|^2_{L^2} + \|(\dot{u}, \dot{\phi})\|^2_{L^2}\big)$$

with

$$I_1 := -\int_{\mathbb{R}^3_+} \eta \dot{\phi}\,(n_a + n)\,\mathrm{div}\,\dot{u}, \quad I_2 := -\int_{\mathbb{R}^3_+} \eta' \dot{\phi}\,(n_a + n)\dot{u}_3.$$

The bilinear singular term I_2 will be "compensated" in the end (thanks to the Bohm condition). Therefore we first focus on I_1.

(1) *Treatment of I_1*. — A first idea is to use the Poisson equation satisfied by $\dot{\phi}$:

$$\dot{\phi} = \frac{1}{1 + h(\phi)} e^{\phi_a} \Big[\varepsilon^2 \Delta \dot{\phi} - \dot{n} - r_\phi\Big],$$

to express $\dot{\phi}$ in terms of $\varepsilon^2 \Delta \dot{\phi}$ and $\dot{n}$. We get:

$$I_1 = -\int_{\mathbb{R}^3_+} \eta(n_a + n)\frac{1}{1 + h(\phi)} e^{\phi_a}\,\mathrm{div}\,\dot{u}\,\Big[\varepsilon^2 \Delta \dot{\phi} - \dot{n} - r_\phi\Big]$$

$$\leqslant J_1 + J_2 + C \int_{\mathbb{R}^3_+} |\varepsilon\,\mathrm{div}\,\dot{u}|\,|r_\phi/\varepsilon|$$

where

$$J_1 := -\int_{\mathbb{R}^3_+} \eta(n_a + n)\frac{1}{1 + h(\phi)} e^{\phi_a}\,\mathrm{div}\,\dot{u}\,(\varepsilon^2 \Delta \dot{\phi}),$$

$$J_2 := \int_{\mathbb{R}^3_+} \eta(n_a + n)\frac{1}{1 + h(\phi)} e^{\phi_a}\,\mathrm{div}\,\dot{u}\,\dot{n}.$$

By (3.9),

$$(3.43) \quad I_1 \leqslant J_1 + J_2$$

$$+ C(C_a, M)\big(\|R^r\|^2_{L^2} + \mu\big\|\sqrt{\eta'}R^s\big\|^2_{L^2} + \mu\big\|\sqrt{\eta'}\varepsilon\,\mathrm{div}\,\dot{u}\big\|^2_{L^2} + \|\varepsilon\,\mathrm{div}\,\dot{u}\|^2_{L^2}\big).$$

We shall now study the terms J_1, J_2.

(a) *Study of J_1*. — To evaluate J_1, the idea is to take the divergence in the equation satisfied by $\dot{u}$, in order to express $\Delta \dot{\phi}$ in terms of $\dot{u}$ and $\dot{n}$. This reads:

$$(\partial_t + (u_a + u) \cdot \nabla)(\varepsilon\,\mathrm{div}\,\dot{u}) + \sum_{i=1}^3 \varepsilon \partial_i (u_a + u) \cdot \nabla \dot{u}_i + \varepsilon T^i\,\mathrm{div}\Big(\frac{\nabla \dot{n}}{n_a + n}\Big) = \varepsilon \Delta \dot{\phi} + \varepsilon\,\mathrm{div}\,r_u.$$

Hence, $J_1 = \sum_{i=1}^{4} K_i$, with

$$K_1 := -\int_{\mathbb{R}^3_+} \eta \frac{n_a + n}{1 + h(\phi)} e^{\phi_a} (\partial_t + (u_a + u) \cdot \nabla) \frac{|\varepsilon \operatorname{div} \dot{u}|^2}{2},$$

$$K_2 := -\int_{\mathbb{R}^3_+} \eta \frac{n_a + n}{1 + h(\phi)} e^{\phi_a} (\varepsilon \partial_i u_a \cdot \nabla) \dot{u}_i \,(\varepsilon \operatorname{div} \dot{u}),$$

$$K_3 := -T^i \int_{\mathbb{R}^3_+} \eta \frac{n_a + n}{1 + h(\phi)} e^{\phi_a} \varepsilon \operatorname{div} \left(\frac{\nabla \dot{n}}{n_a + n} \right) (\varepsilon \operatorname{div} \dot{u}),$$

$$K_4 := \int_{\mathbb{R}^3_+} \eta \frac{n_a + n}{1 + h(\phi)} e^{\phi_a} (\varepsilon \operatorname{div} r_u)\,(\varepsilon \operatorname{div} \dot{u}).$$

The terms K_2 and K_4 can be bounded through standard arguments. With (3.7) and (3.9) in mind, we obtain

(3.44) $K_2 + K_4 \leqslant C(C_a, M) \left(\|R^r\|_{L^2}^2 + \mu \|\sqrt{\eta'} R^s\|_{L^2}^2 + \mu \|\sqrt{\eta'} \varepsilon \nabla \dot{u}\|_{L^2}^2 + \|\varepsilon \nabla \dot{u}\|_{L^2}^2 \right).$

Note the presence of $\varepsilon \nabla \dot{u}$ at the r.h.s., due to K_2. We now evaluate the contribution of the convection term K_1. Standard manipulations yield

$$K_1 = -\frac{d}{dt} \int_{\mathbb{R}^3_+} \eta \frac{n_a + n}{1 + h(\phi)} e^{\phi_a} \frac{(\varepsilon \operatorname{div} \dot{u})^2}{2} + \int_{\mathbb{R}^3_+} \eta \frac{d}{dt}\left[\frac{n_a + n}{1 + h(\phi)} e^{\phi_a} \right] \frac{(\varepsilon \operatorname{div} \dot{u})^2}{2}$$

$$+ \int_{\mathbb{R}^3_+} \eta \operatorname{div} \left(\frac{n_a + n}{1 + h(\phi)} e^{\phi_a} (u_a + u) \right) \frac{(\varepsilon \operatorname{div} \dot{u})^2}{2}$$

$$+ \int_{\mathbb{R}^3_+} \eta' \frac{n_a + n}{1 + h(\phi)} e^{\phi_a} (u_a + u)_3 \frac{(\varepsilon \operatorname{div} \dot{u})^2}{2}$$

$$+ \int_{x_3 = 0} \left[\frac{n_a + n}{1 + h(\phi)} e^{\phi_a} (u_a + u)_3 \frac{(\varepsilon \operatorname{div} \dot{u})^2}{2} \right].$$

The second term can be bounded by a crude L^2 estimate, the third one can be bounded thanks to (3.3)–(3.7): we get

(3.45) $K_1 \leqslant -\dfrac{d}{dt} \displaystyle\int_{\mathbb{R}^3_+} \eta \dfrac{n_a + n}{1 + h(\phi)} e^{\phi_a} \dfrac{(\varepsilon \operatorname{div} \dot{u})^2}{2}$

$+ \displaystyle\int_{\mathbb{R}^3_+} \eta' \dfrac{n_a + n}{1 + h(\phi)} e^{\phi_a} (u_a + u)_3 \dfrac{(\varepsilon \operatorname{div} \dot{u})^2}{2} + \displaystyle\int_{x_3 = 0} \left[\dfrac{n_a + n}{1 + h(\phi)} e^{\phi_a} (u_a + u)_3 \dfrac{(\varepsilon \operatorname{div} \dot{u})^2}{2} \right]$

$+ C(C_a, M)\left(\mu \|\sqrt{\eta'} \varepsilon \operatorname{div} \dot{u}\|_{L^2}^2 + \|\varepsilon \operatorname{div} \dot{u}\|_{L^2}^2 \right).$

The second term is non-positive, and will be one of the crucial terms to control all singular terms (thanks to the Bohm condition). Note finally that the boundary contribution is non-positive: this will also help later.

Now we turn to the integral that involves the pressure term, for which we perform an integration by parts

$$K_3 = L_1 + L_2 + L_3 + L_4,$$

where

$$L_1 := T^i \int_{\mathbb{R}^3_+} \eta\Big(\frac{1}{n_a+n}\frac{1}{1+h(\phi)}e^{\phi_a}\Big)\nabla[(n_a+n)(\varepsilon\operatorname{div}\dot{u})]\cdot\varepsilon\nabla\dot{n},$$

$$L_2 := T^i \int_{\mathbb{R}^3_+} \eta\,\nabla\Big(\frac{1}{n_a+n}\frac{1}{1+h(\phi)}e^{\phi_a}\Big)(n_a+n)(\varepsilon\operatorname{div}\dot{u})\cdot\varepsilon\nabla\dot{n},$$

$$L_3 := T^i \int_{\mathbb{R}^3_+} \eta'\Big(\frac{1}{1+h(\phi)}e^{\phi_a}\Big)(\varepsilon\operatorname{div}\dot{u})\,\varepsilon\partial_3\dot{n},$$

$$L_4 := T^i \int_{x_3=0} \Big(\frac{1}{1+h(\phi)}e^{\phi_a}\Big)(\varepsilon\operatorname{div}\dot{u})\,\varepsilon\partial_3\dot{n}.$$

We use the equation satisfied by $\dot{n}$ in order to rewrite the $\nabla[...]$ part in L_1. To this end, we take the $\varepsilon\nabla$ operator on the transport equation for n to get:

$$(\partial_t + (u_a+u)\cdot\nabla)(\varepsilon\nabla\dot{n}) + \nabla\left[(n_a+n)\,\varepsilon\operatorname{div}\dot{u}\right] + \varepsilon\sum_{i=1}^{3}(\partial_i(u_a+u)\cdot\nabla\dot{n})e_i = \varepsilon\nabla r_n,$$

where $e_1 = (1,0,0)^\top, e_2 = (0,1,0)^\top, e_3 = (0,0,1)^\top$. As before the convection term on $(\varepsilon\nabla\dot{n})$ is very useful. With the usual manipulations, we obtain:

$$L_1 \leqslant -\frac{d}{dt}T^i \int_{\mathbb{R}^3_+} \eta\Big(\frac{1}{n_a+n}\frac{1}{1+h(\phi)}e^{\phi_a}\Big)\frac{|\varepsilon\nabla\dot{n}|^2}{2}$$

$$+ T^i \int_{\mathbb{R}^3_+} \eta'\Big(\frac{1}{n_a+n}\frac{1}{1+h(\phi)}e^{\phi_a}\Big)(u_a+u)_3\frac{|\varepsilon\nabla\dot{n}|^2}{2}$$

$$+ T^i \int_{x_3=0} \Big(\frac{1}{n_a+n}\frac{1}{1+h(\phi)}e^{\phi_a}\Big)(u_a+u)_3\frac{|\varepsilon\nabla\dot{n}|^2}{2}$$

$$+ C(C_a, M)\big(\|R^r\|^2_{L^2} + \mu\big\|\sqrt{\eta'}R^s\big\|^2_{L^2} + \mu\big\|\sqrt{\eta'}\varepsilon\nabla\dot{n}\big\|^2_{L^2} + \|\varepsilon\nabla\dot{n}\|^2_{L^2}\big).$$

By now standard manipulations, we also have

$$L_2 \leqslant C(C_a, M)\big(\mu\big\|\sqrt{\eta'}(\varepsilon\nabla\dot{n}, \varepsilon\operatorname{div}\dot{u})\big\|^2_{L^2} + \|(\varepsilon\nabla\dot{n}, \varepsilon\operatorname{div}\dot{u})\|^2_{L^2}\big).$$

From the estimates on the L_i's, one can deduce an inequality on K_3. Combining this inequality with (3.44) and (3.45), we end up with

$$(3.46)\quad J_1 \leqslant -\frac{d}{dt}\int_{\mathbb{R}^3_+} \eta\frac{n_a+n}{1+h(\phi)}e^{\phi_a}\frac{(\varepsilon\operatorname{div}\dot{u})^2}{2}$$

$$-\frac{d}{dt}T^i \int_{\mathbb{R}^3_+} \eta\Big(\frac{1}{n_a+n}\frac{1}{1+h(\phi)}e^{\phi_a}\Big)\frac{|\varepsilon\nabla\dot{n}|^2}{2} + N_1 + N_2 + N_3,$$

with

$$N_1 := \int_{\mathbb{R}^3_+} \eta'\frac{n_a+n}{1+h(\phi)}e^{\phi_a}(u_a+u)_3\frac{(\varepsilon\operatorname{div}\dot{u})^2}{2}$$

$$+ \int_{x_3=0} \Big[\frac{n_a+n}{1+h(\phi)}e^{\phi_a}(u_a+u)_3\frac{(\varepsilon\operatorname{div}\dot{u})^2}{2}\Big],$$

$$N_2 := T^i \int_{\mathbb{R}^3_+} \eta' \Big(\frac{1}{n_a + n} \frac{1}{1 + h(\phi)} e^{\phi_a} \Big) (u_a + u)_3 \frac{|\varepsilon \nabla \dot{n}|^2}{2}$$

$$+ T^i \int_{\mathbb{R}^3_+} \eta' \Big(\frac{1}{1 + h(\phi)} e^{\phi_a} \Big) (\varepsilon \operatorname{div} \dot{u}) \, \varepsilon \partial_3 \dot{n}$$

$$+ T^i \int_{x_3 = 0} \Big(\frac{1}{n_a + n} \frac{1}{1 + h(\phi)} e^{\phi_a} \Big) (u_a + u)_3 \frac{|\varepsilon \nabla \dot{n}|^2}{2}$$

$$+ T^i \int_{x_3 = 0} \Big(\frac{1}{1 + h(\phi)} e^{\phi_a} \Big) (\varepsilon \operatorname{div} \dot{u}) \, \varepsilon \partial_3 \dot{n},$$

and

$$N_3 := C(C_a, M) \big(\|R^r\|_{L^2}^2 + \mu \big\| \sqrt{\eta'} R^s \big\|_{L^2}^2 + \mu \big\| \sqrt{\eta'} (\varepsilon \nabla \dot{n}, \varepsilon \nabla \dot{u}) \big\|_{L^2}^2 + \|(\varepsilon \nabla \dot{n}, \varepsilon \nabla \dot{u})\|_{L^2}^2 \big).$$

(b) *Study of J_2.* — Let us now work on J_2, which has to be treated very carefully. Indeed, a rough L^2 estimate only shows that this is a singular term in $1/\varepsilon$, which does not seem small. Instead, we use the transport equation satisfied by $\dot{n}$ to express $(n_a + n) \operatorname{div} \dot{u}$ in terms of other quantities. With similar manipulations as before, we obtain:

$$(3.47) \quad J_2 \leqslant - \frac{d}{dt} \int_{\mathbb{R}^3_+} \eta \frac{1}{1 + h(\phi)} e^{\phi_a} \frac{|\dot{n}|^2}{2}$$

$$+ C(C_a, M) \big(\|R^r\|_{L^2}^2 + \mu \big\| \sqrt{\eta'} R^s \big\|_{L^2}^2 + \mu \big\| \sqrt{\eta'} (\dot{n}, \dot{u}) \big\|_{L^2}^2 + \|(\dot{n}, \dot{u})\|_{L^2}^2 \big)$$

$$+ \frac{1}{2} \int_{\mathbb{R}^3_+} \eta' \frac{1}{1 + h(\phi)} e^{\phi_a} (u_a + u)_3 |\dot{n}|^2 + \frac{1}{2} \int_{x_3 = 0} \frac{1}{1 + h(\phi)} e^{\phi_a} (u_a + u)_3 |\dot{n}|^2.$$

Observe here that the last two terms are non-positive. Together with (3.46) and (3.47), the bound (3.43) leads to

$$(3.48) \quad I_1 \leqslant - \frac{d}{dt} \int_{\mathbb{R}^3_+} \eta \frac{n_a + n}{1 + h(\phi)} e^{\phi_a} \frac{(\varepsilon \operatorname{div} \dot{u})^2}{2}$$

$$- \frac{d}{dt} T^i \int_{\mathbb{R}^3_+} \eta \Big(\frac{1}{n_a + n} \frac{1}{1 + h(\phi)} e^{\phi_a} \Big) \frac{|\varepsilon \nabla \dot{n}|^2}{2}$$

$$- \frac{d}{dt} \int_{\mathbb{R}^3_+} \eta \frac{1}{1 + h(\phi)} e^{\phi_a} \frac{|\dot{n}|^2}{2} + N_4 + N_5 + N_6,$$

with

$$N_4 := \frac{1}{2} \int_{\mathbb{R}^3_+} \eta' \frac{1}{1 + h(\phi)} e^{\phi_a} (u_a + u)_3 |\dot{n}|^2 + \frac{1}{2} \int_{x_3 = 0} \frac{1}{1 + h(\phi)} e^{\phi_a} (u_a + u)_3 |\dot{n}|^2$$

$$+ \int_{\mathbb{R}^3_+} \eta' \frac{n_a + n}{1 + h(\phi)} e^{\phi_a} (u_a + u)_3 \frac{(\varepsilon \operatorname{div} \dot{u})^2}{2}$$

$$+ \int_{x_3 = 0} \Big[\frac{n_a + n}{1 + h(\phi)} e^{\phi_a} (u_a + u)_3 \frac{(\varepsilon \operatorname{div} \dot{u})^2}{2} \Big],$$

$$N_5 := T^i \int_{\mathbb{R}^3_+} \eta' \Big(\frac{1}{n_a + n} \frac{1}{1 + h(\phi)} e^{\phi_a} \Big) (u_a + u)_3 \frac{|\varepsilon \nabla \dot{n}|^2}{2}$$

$$+ T^i \int_{\mathbb{R}^3_+} \eta' \Big(\frac{1}{1 + h(\phi)} e^{\phi_a} \Big) (\varepsilon \operatorname{div} \dot{u}) \, \varepsilon \partial_3 \dot{n}$$

$$+ T^i \int_{x_3 = 0} \Big(\frac{1}{n_a + n} \frac{1}{1 + h(\phi)} e^{\phi_a} \Big) (u_a + u)_3 \frac{|\varepsilon \nabla \dot{n}|^2}{2}$$

$$+ T^i \int_{x_3 = 0} \Big(\frac{1}{1 + h(\phi)} e^{\phi_a} \Big) (\varepsilon \operatorname{div} \dot{u}) \, \varepsilon \partial_3 \dot{n},$$

and

$$N_6 := C(C_a, M) \Big(\|R^r\|^2_{L^2} + \mu \big\| \sqrt{\eta'} R^s \big\|^2_{L^2}$$

$$+ \mu \big\| \sqrt{\eta'} (\dot{n}, \dot{u}, \varepsilon \nabla \dot{n}, \varepsilon \nabla \dot{u}) \big\|^2_{L^2} + \|(\dot{n}, \dot{u}, \varepsilon \nabla \dot{n}, \varepsilon \nabla \dot{u})\|^2_{L^2} \Big).$$

(2) *Treatment of I_2.* — By using (3.23), we have the straightforward bound:

$$I_2 \leqslant \frac{1}{2} \int_{\mathbb{R}^3_+} \eta' \, (n_a + n) |(u_a + u)_3| |\dot{\phi}|^2 + \frac{1}{2} \int_{\mathbb{R}^3_+} \eta' \frac{n_a + n}{|(u_a + u)_3|} |\dot{u}_3|^2$$

$$\leqslant \frac{1}{2} \int_{\mathbb{R}^3_+} \eta' \, \Gamma n^0 |\Gamma(u^0)_3| |\dot{\phi}|^2 + \frac{1}{2} \int_{\mathbb{R}^3_+} \eta' \frac{n_a + n}{|(u_a + u)_3|} |\dot{u}_3|^2 + \mu \int_{\mathbb{R}^3_+} \eta' |\dot{\phi}|^2 \, dx$$

$$+ C(C_a, M) \big(1 + \frac{\|(u, n)\|_{L^\infty}}{\varepsilon} \big) \|\dot{\phi}\|^2_{L^2}.$$

Note that the last term is actually not singular, thanks to (3.10).

We now claim that $\dot{\phi}$ satisfies the following bound:

$$(3.49) \quad \frac{1}{2} \int_{\mathbb{R}^3_+} \eta' \, \Gamma n^0 |\Gamma(u^0)_3| |\dot{\phi}|^2 \leqslant \frac{1}{2} \int_{\mathbb{R}^3_+} \eta' \frac{|(u_a + u)_3|}{n_a + n} |\dot{n}|^2$$

$$+ C(C_a, M) \big(\|R^r\|^2_{L^2} + \mu \big\| \sqrt{\eta'} R^s \big\|^2_{L^2} + \mu \big\| \sqrt{\eta'} (\dot{n}, \dot{\phi}) \big\|^2_{L^2} + \|(\dot{n}, \dot{\phi})\|^2_{L^2} \big).$$

The term $\frac{1}{2} \int_{\mathbb{R}^3_+} \eta' \frac{|(u_a + u)_3|}{n_a + n} |\dot{n}|^2$ has the bad sign but will be compensated by other terms later.

To prove estimate (3.49), we write the Poisson equation satisfied by $\dot{\phi}$:

$$-\varepsilon^2 \Delta \dot{\phi} + e^{-\phi_a} (1 + h(\phi)) \dot{\phi} = -\dot{n} - r_\phi.$$

We multiply by $m \dot{\phi}$, with the weight $m := \eta' |(\Gamma u^0)_3|$. By a standard energy estimate:

$$\big\| \sqrt{m} \, \varepsilon \nabla \dot{\phi} \big\|^2_{L^2} + \int_{\mathbb{R}^3} m e^{-\phi_a} (1 + h(\phi)) |\dot{\phi}|^2 \, dx \leqslant \int_{\mathbb{R}^3} m (|\dot{n}| + |r_\phi|) |\dot{\phi}| \, dx + \varepsilon^2 \int_{\mathbb{R}^3} |\nabla^2 m| |\dot{\phi}|^2.$$

We use the Young inequality for the first term, resulting in

$$\int_{\mathbb{R}^3} m |\dot{n}| \, |\dot{\phi}| \leqslant \frac{1}{2} \int_{\mathbb{R}^3} \eta' \Big| \frac{\Gamma(u^0)_3}{\Gamma n^0} \Big| |\dot{n}|^2 + \frac{1}{2} \int_{\mathbb{R}^3} \eta' (\Gamma n^0 |\Gamma(u^0)_3|) |\dot{\phi}|^2.$$

Next, we observe that thanks to (3.9), we have

$$\int_{\mathbb{R}^3_+} m|r_\phi|\,|\dot{\phi}| \leqslant C(C_a, M) \int_{\mathbb{R}^3_+} \frac{|r_\phi|}{\varepsilon}|\dot{\phi}|$$
$$\leqslant C(C_a, M)\big(\|R^r\|^2_{L^2} + \mu\big\|\sqrt{\eta'}R^s\big\|^2_{L^2} + \mu\big\|\sqrt{\eta'}\dot{\phi}\big\|^2_{L^2} + \|\dot{\phi}\|^2_{L^2}\big).$$

By the choice of the weight function, we have

$$\varepsilon^2\,|\nabla^2 m| \lesssim \big(\mu^2 + \delta + \varepsilon^2\big)\,m \lesssim \mu^2 m,$$

which implies

$$\varepsilon^2 \int_{\mathbb{R}^3} |\nabla^2 m||\dot{\phi}|^2 \lesssim \mu^2 \big\|\sqrt{m}\dot{\phi}\big\|^2_{L^2} \leqslant C_a\mu^2\big\|\sqrt{\eta'}\dot{\phi}\big\|^2_{L^2}.$$

Finally, since $n^0 = e^{-\phi_0}$, relations (3.22), (3.23) give

$$\int_{\mathbb{R}^3} m e^{-\phi_a}(1 + h(\phi))|\dot{\phi}|^2$$
$$\geqslant \int_{\mathbb{R}^3} \eta'(\Gamma n^0|\Gamma(u^0)_3| - C_a(\delta + \varepsilon))|\dot{\phi}|^2 - C_a \int_{\mathbb{R}^3} (1 + \frac{\|\phi\|_{L^\infty}}{\varepsilon})|\dot{\phi}|^2$$
$$\geqslant \int_{\mathbb{R}^3} \eta'(\Gamma n^0|\Gamma(u^0)_3||\dot{\phi}|^2 - C(C_a, M)\big(\mu\big\|\sqrt{\eta'}\dot{\phi}\big\|^2_{L^2} + \|\dot{\phi}\|^2_{L^2}\big).$$

We thus get that

$$(3.50) \quad \frac{1}{2}\int_{\mathbb{R}^3} \eta'(\Gamma n^0|\Gamma(u^0)_3|)|\dot{\phi}|^2 \leqslant \frac{1}{2}\int_{\mathbb{R}^3} \eta'\Big|\frac{\Gamma(u^0)_3}{\Gamma n^0}\Big|\,|\dot{n}|^2$$
$$+ C(C_a, M)\big(\|R^r\|^2_{L^2} + \mu\big\|\sqrt{\eta'}R^s\big\|^2_{L^2} + \mu\big\|\sqrt{\eta'}\dot{\phi}\big\|^2_{L^2} + \|\dot{\phi}\|^2_{L^2}\big).$$

Applying (3.24) to the first term at the r.h.s., we obtain the desired inequality (3.49). It follows that

$$(3.51) \quad I_2 \leqslant \frac{1}{2}\int_{\mathbb{R}^3_+} \eta'\frac{|(u_a + u)_3|}{n_a + n}|\dot{n}|^2 + \frac{1}{2}\int_{\mathbb{R}^3_+} \eta'\frac{n_a + n}{|(u_a + u)_3|}|\dot{u}_3|^2$$
$$+ C(C_a, M)\big(\|R^r\|^2_{L^2} + \mu\big\|\sqrt{\eta'}R^s\big\|^2_{L^2} + \mu\big\|\sqrt{\eta'}\dot{\phi}\big\|^2_{L^2} + \|\dot{\phi}\|^2_{L^2}\big).$$

(3) *Conclusion of estimate B.* — We can then inject (3.48) and (3.51) into (3.42). To ease reading, we set

$$\mathscr{I} := \int_{\mathbb{R}^3_+} \eta\Big(\frac{n_a + n}{1 + h(\phi)}e^{\phi_a}\frac{(\varepsilon\,\mathrm{div}\,\dot{u})^2}{2} + \Big(\frac{T^i}{n_a + n}\frac{1}{1 + h(\phi)}e^{\phi_a}\Big)\frac{|\varepsilon\nabla\dot{n}|^2}{2} + \frac{1}{1 + h(\phi)}e^{\phi_a}\frac{|\dot{n}|^2}{2}\Big).$$

We get

$$\mathscr{I} + \frac{d}{dt}\mathscr{J} \leqslant \int_{\mathbb{R}^3_+} \eta' \frac{n_a + n}{1 + h(\phi)} e^{\phi_a}(u_a + u)_3 \frac{(\varepsilon \operatorname{div} \dot{u})^2}{2}$$
$$+ T^i \int_{\mathbb{R}^3_+} \eta' \Big(\frac{1}{1 + h(\phi)} e^{\phi_a}\Big)(\varepsilon \operatorname{div} \dot{u}) \, \varepsilon \partial_3 \dot{n}$$
$$+ \frac{1}{2} \int_{\mathbb{R}^3_+} \eta' \Big(\frac{e^{\phi_a}}{1 + h(\phi)}(u_a + u)_3 \, |\dot{n}|^2 + \frac{|(u_a + u)_3|}{n_a + n} \, |\dot{n}|^2 + \frac{n_a + n}{|(u_a + u)_3|} |\dot{u}_3|^2 \Big)$$
$$+ T^i \int_{\mathbb{R}^3_+} \eta' \Big(\frac{1}{n_a + n} \frac{1}{1 + h(\phi)} e^{\phi_a}\Big)(u_a + u)_3 \frac{|\varepsilon \nabla \dot{n}|^2}{2} + B.T. + \mathscr{G} + \mathscr{R}_3.$$

Here $B.T.$ denotes the boundary contributions,

$$B.T. := \int_{x_3=0} \Big(\frac{1}{2} e^{\phi_a}(n_a + n)(u_a + u)_3(\varepsilon \operatorname{div} \dot{u})^2 + \frac{T^i}{2} \frac{e^{\phi_a}}{(n_a + n)(1 + h(\phi))}(u_a + u)_3 |\varepsilon \nabla \dot{n}|^2$$
$$+ T^i \frac{e^{\phi_a}}{(1 + h(\phi))} \varepsilon \operatorname{div} \dot{u} \, \varepsilon \partial_3 \dot{n} + \frac{1}{2} \frac{e^{\phi_a}}{1 + h(\phi)}(u_a + u)_3 |\dot{n}|^2 \Big),$$

$\mathscr{G}$ stands for a good term, that is,

$$(3.52) \quad \mathscr{G} \leqslant C(C_a, M) \Big(\|(\dot{n}, \dot{u}, \varepsilon \nabla \dot{n}, \varepsilon \nabla \dot{u}, \dot{\phi}, \varepsilon \nabla \dot{\phi})\|^2_{L^2(\mathbb{R}^3)}$$
$$+ \|R^r\|^2_{L^2(\mathbb{R}^3)} + \mu \|\sqrt{\eta'} R^s\|^2_{L^2(\mathbb{R}^3)} \Big),$$

and $\mathscr{R}_3$ is for

$$\mathscr{R}_3 := \mu \|\sqrt{\eta'} \, \varepsilon \nabla \dot{u}\|^2_{L^2} + \mu \|\sqrt{\eta'} \, \dot{\phi}\|^2_{L^2} + C(C_a, M)\mu \|\sqrt{\eta'}(\dot{n}, \dot{u}, \varepsilon \nabla \dot{n}, \varepsilon \operatorname{div} \dot{u})\|^2_{L^2(\mathbb{R}^3_+)}.$$

Let us stress that some singular terms vanish at leading order: indeed, the relation $e^{\phi_0} = 1/n_0$ implies

$$\frac{1}{2} \int_{\mathbb{R}^3_+} \eta' \Big(\frac{e^{\phi_a}}{1 + h(\phi)}(u_a + u)_3 \, |\dot{n}|^2 + \frac{|(u_a + u)_3|}{n_a + n} \, |\dot{n}|^2 \leqslant C(C_a, M)\big(\mu \|\sqrt{\eta'}\dot{n}\|^2_{L^2} + \|\dot{n}\|^2_{L^2}\big).$$

By combining the previous estimate and Lemma 1, we conclude that we have
$$(3.53)$$
$$\frac{d}{dt}\Big[\int_{\mathbb{R}^3_+} \eta \, (n_a + n)\frac{|\dot{u}|^2}{2} + \int_{\mathbb{R}^3_+} \eta \frac{T^i|\dot{n}|^2}{2(n_a + n)} + \int_{\mathbb{R}^3_+} \eta \frac{1}{1 + h(\phi)} e^{\phi_a}(n_a + n)\frac{(\varepsilon \operatorname{div} \dot{u})^2}{2}$$
$$+ T^i \int_{\mathbb{R}^3_+} \eta \Big(\frac{1}{n_a + n}\frac{1}{1 + h(\phi)} e^{\phi_a}\Big)\frac{|\varepsilon \nabla \dot{n}|^2}{2} + \int_{\mathbb{R}^3_+} \eta \frac{1}{1 + h(\phi)} e^{\phi_a} \frac{|\dot{n}|^2}{2}\Big]$$
$$+ \frac{1}{2} \int_{\mathbb{R}^3_+} \eta' \, Q_1^B(\dot{n}, \dot{u}) + \frac{1}{2} \int_{\mathbb{R}^3_+} \eta' \, Q_2^B(\varepsilon \operatorname{div} \dot{u}, \varepsilon \nabla \dot{n}) + \frac{1}{2} \int_{x_3=0} Q^0(\dot{n}, \dot{u})$$
$$+ \frac{1}{2} \int_{x_3=0} Q_2^B(\varepsilon \operatorname{div} \dot{u}, \varepsilon \nabla \dot{n}) \leqslant \mu \|\sqrt{\eta'} \, |\varepsilon \nabla \dot{u}\|^2_{L^2} + \mu \|\sqrt{\eta'} \, \dot{\phi}\|^2_{L^2} + \mathscr{G}.$$

with Q_1^B, Q_2^B are quadratic forms determined by the two following symmetric matrices:

$$M_1^B = \Gamma \begin{bmatrix} T^i \frac{|(u_a+u)_3|}{n^a+n} & -T^i e^\top \\ -T^i e & (n_a+n)\big(|(u_a+u)_3|\operatorname{Id}_3 - \frac{1}{|(u_a+u)_3|}e\,e^\top\big) \end{bmatrix} - C\mu\operatorname{Id}_4,$$

$$M_2^B = \Gamma \begin{bmatrix} \frac{e^{\phi_a}}{1+h(\phi)}(n_a+n)|(u_a+u)|_3 & -\frac{T^i e^{\phi_a}}{1+h(\phi)}e^\top \\ -\frac{T^i e^{\phi_a}}{1+h(\phi)}e & T^i \frac{e^{\phi_a}|(u_a+u)|_3}{(1+h(\phi))(n_a+n)}\operatorname{Id}_3 \end{bmatrix} - C\mu\operatorname{Id}_4 .$$

We recall that $\mu > 0$ is a parameter that can be taken arbitrarily small. Observe that M_1^B is positive for μ sufficiently small, as soon as the Bohm condition (3.11) is satisfied. We also get that M_2^B is positive thanks to the Bohm condition.

To obtain (3.41), we still have to get rid of the term $\mu\|\sqrt{\eta'}\,\dot\phi\|_{L^2}^2$ at the r.h.s. of (3.53). But as the quadratic form Q_1^B controls $\|\sqrt{\eta'}\dot n\|_{L^2}^2$, it also controls $\|\sqrt{\eta'}\dot\phi\|_{L^2}^2$ thanks to (3.50), up to add some good term and some $\mu\|\sqrt{\eta'}\dot n\|_{L^2}^2$ term at the r.h.s. By taking μ small enough, this singular term is controlled by the l.h.s., which gives (3.41).

This estimate by itself allows us almost to conclude; the only problematic term comes from the singular contribution $\int_{\mathbb{R}_+^3} \eta'\,|\varepsilon\nabla\dot u|^2$ in the right hand side which involves a full derivative of $\dot u$, and that will be treated below.

C. The third energy estimate. — The previous computations suggest that we also need an estimate of a full ε-derivative of $\dot u$. Note that we can not work with the curl of $\dot u$ (together with the previous estimate on the divergence), because we would not be able to control the trace of $\dot u$ on the boundary $\{x_3 = 0\}$.

We shall prove that

$$(3.54) \quad \|(\dot n, \dot u, \varepsilon\nabla\dot n, \varepsilon\nabla\dot u)\|_{L_T^\infty L^2(\mathbb{R}_+^3)}^2 + \big\|\sqrt{\eta'}(\dot n, \dot u, \varepsilon\nabla\dot n, \varepsilon\nabla\dot u)\big\|_{L_T^2 L^2(\mathbb{R}_+^3)}^2$$

$$+ \|\Gamma(\dot n, \dot u, \varepsilon\nabla\dot n, \varepsilon\nabla\dot u)\|_{L_T^2 L^2(\mathbb{R}^2)}^2 \leqslant C(C_a, M)\Big(\|(\dot n, \dot u, \varepsilon\nabla\dot n, \varepsilon\nabla\dot u)(0)\|_{L^2(\mathbb{R}_+^3)}^2$$

$$+ \big\|\sqrt{\eta'}(\dot n, \dot u, \dot\phi, \varepsilon\nabla\dot n, \varepsilon\operatorname{div}\dot u, \varepsilon\nabla\dot\phi)\big\|_{L_T^2 L^2(\mathbb{R}_+^3)}^2 + \|\Gamma(\dot n, \dot u, \varepsilon\partial_{x_3}\dot\phi, \varepsilon\nabla\dot n, \varepsilon\operatorname{div}\dot u)\|_{L_T^2 L^2(\mathbb{R}^2)}^2$$

$$+ \|(\dot n, \dot u, \varepsilon\nabla\dot n, \varepsilon\nabla\dot u, \dot\phi, \varepsilon\nabla\dot\phi)\|_{L_T^2 L^2(\mathbb{R}^3)}^2 + \|R^r\|_{L_T^2 L^2(\mathbb{R}^3)}^2 + \mu\big\|\sqrt{\eta'}R^s\big\|_{L_T^2 L^2(\mathbb{R}_+^3)}^2\Big).$$

We shall use again the estimate of Lemma 1, but through a different approach of the integral $\mathscr{I}$. The idea will be to combine the estimate of Lemma 1 with an estimate on the ε-derivatives of the system to cancel the singular term through the Poisson equation.

Let us first first apply ε-derivatives on the equation on the velocity field $\dot u$: for $i = 1, 2, 3$, this yields

$$(3.55) \quad \partial_t(\varepsilon\,\partial_i\dot u) + (u_a+u)\cdot\nabla(\varepsilon\,\partial_i\dot u) + \varepsilon\,\partial_i(u_a+u)\cdot\nabla\dot u + T^i\frac{\nabla(\varepsilon\,\partial_i\dot n)}{n_a+n}$$

$$+ T^i\varepsilon\,\partial_i\Big(\frac{1}{n_a+n}\Big)\nabla\dot n = \varepsilon\,\partial_i\nabla\dot\phi + \varepsilon\,\partial_i r_u.$$

We then take the scalar product with $\eta\,\varepsilon\,\partial_i \dot{u}$ and make the sum in i. We find

$$(3.56) \quad \sum_{i=1}^{3}\Big(\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3_+}\eta|\varepsilon\,\partial_i\dot{u}|^2 - \frac{1}{2}\int_{\mathbb{R}^3_+}\eta'\,(u_a+u)_3\,|\varepsilon\,\partial_i\dot{u}|^2 - \frac{1}{2}\int_{x_3=0}(u_a+u)_3\,|\varepsilon\,\partial_i\dot{u}|^2\Big)$$
$$= I_1 + I_2 + \mathscr{G} + \mathscr{S}$$

where

$$I_1 := -\sum_{i=1}^{3}\Big(\int_{\mathbb{R}^3_+}\eta\,T^i\frac{\nabla(\varepsilon\,\partial_i\dot{n})}{n_a+n}\cdot\varepsilon\,\partial_i\dot{u}\Big), \quad I_2 := \sum_{i=1}^{3}\int_{\mathbb{R}^3_+}\eta\,\varepsilon\,\partial_i\nabla\dot{\phi}\cdot\varepsilon\,\partial_i\dot{u}.$$

The term $\mathscr{S}$ above will refer from now and on to any admissible singular term, that is, a quantity which is singular but bounded as

$$(3.57) \qquad |\mathscr{S}| \leqslant \mu\big\|\sqrt{\eta'}\varepsilon\nabla\dot{u}\big\|_{L^2}^2 + C\big\|\sqrt{\eta'}\big(\dot{n},\dot{u},\dot{\phi},\varepsilon\nabla\dot{n},\varepsilon\operatorname{div}\dot{u},\varepsilon\nabla\dot{\phi}\big)\big\|_{L^2}^2.$$

The first term in $\mathscr{S}$ will be absorbed by positive terms in the left hand side for μ small enough, while the second term will be absorbed by the left hand sides of Estimates A and B. The term $\mathscr{G}$ above will refer from now and on to a good term, that is, satisfying (3.52).

(1) *Treatment of I_1.* — We use the identity:

$$(3.58) \quad I_1 = \int_{\mathbb{R}^3_+}\eta\,T^i\frac{(\varepsilon\,\partial_i\dot{n})}{n_a+n}\operatorname{div}(\varepsilon\,\partial_i\dot{u}) + \int_{\mathbb{R}^3_+}\eta\,T^i(\varepsilon\,\partial_i\dot{n})\nabla\Big(\frac{1}{n_a+n}\Big)\cdot\varepsilon\,\partial_i\dot{u}$$
$$+ \int_{\mathbb{R}^3_+}\eta'\,T^i\frac{(\varepsilon\,\partial_i\dot{n})}{n_a+n}\,\varepsilon\,\partial_i\dot{u}_3 + \int_{x_3=0}T^i\frac{(\varepsilon\,\partial_i\dot{n})}{n_a+n}\,\varepsilon\,\partial_i\dot{u}_3.$$

We shall now use the transport equation satisfied by $\varepsilon\,\partial_i\dot{n}$, which reads:

$$(\partial_t + (u_a+u)\cdot\nabla)(\varepsilon\,\partial_i\dot{n}) + \varepsilon\,\partial_i(n_a+n)\operatorname{div}\dot{u}$$
$$+ (n_a+n)\operatorname{div}(\varepsilon\,\partial_i\dot{u}) + \varepsilon\,\partial_i(u_a+u)\cdot\nabla\dot{n} = \varepsilon\nabla r_n.$$

Therefore, we have, relying on the standard manipulations:

$$(3.59) \quad \int_{\mathbb{R}^3_+}\eta\,T^i\frac{(\varepsilon\,\partial_i\dot{n})}{n_a+n}\operatorname{div}(\varepsilon\,\partial_i\dot{u}) = -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3_+}\eta\frac{T^i}{(n_a+n)^2}\,|\varepsilon\partial_i\dot{n}|^2$$
$$+ \frac{1}{2}\int_{\mathbb{R}^3_+}\eta'\,\frac{T^i(u_a+u)_3}{(n_a+n)^2}\,|\varepsilon\partial_i\dot{n}|^2 + \frac{1}{2}\int_{x_3=0}\frac{T^i(u_a+u)_3}{(n_a+n)^2}\,|\varepsilon\partial_i\dot{n}|^2 + \mathscr{G} + \mathscr{S}.$$

We end up with

$$(3.60) \quad I_1 = -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3_+}\eta\frac{T^i}{(n_a+n)^2}|\varepsilon\partial_i\dot{n}|^2$$
$$+ \frac{1}{2}\int_{\mathbb{R}^3_+}\eta'\,\frac{T^i(u_a+u)_3}{(n_a+n)^2}|\varepsilon\partial_i\dot{n}|^2 + \frac{1}{2}\int_{x_3=0}\frac{T^i(u_a+u)_3}{(n_a+n)^2}|\varepsilon\partial_i\dot{n}|^2$$
$$+ \int_{\mathbb{R}^3_+}\eta'\,T^i\frac{(\varepsilon\,\partial_i\dot{n})}{n_a+n}\varepsilon\,\partial_i\dot{u}_3 + \int_{x_3=0}T^i\frac{(\varepsilon\,\partial_i\dot{n})}{n_a+n}\varepsilon\,\partial_i\dot{u}_3 + \mathscr{G} + \mathscr{S}.$$

(2) *Treatment of I_2*. — After some integration by parts, we can rewrite I_2 as

$$I_2 = -\sum_{i=1}^{3} \left(\int_{\mathbb{R}^3_+} \eta \, \varepsilon \, \partial_i \dot\phi \, \varepsilon \operatorname{div} \partial_i \dot u + \int_{\mathbb{R}^3_+} \eta' \, \varepsilon \, \partial_i \dot\phi \, \varepsilon \, \partial_i \dot u_3 + \int_{x_3=0} \varepsilon \, \partial_i \dot\phi \, \varepsilon \, \partial_i \dot u_3 \right).$$

Then we have

$$-\sum_{i=1}^{3} \int_{\mathbb{R}^3_+} \eta \, \varepsilon \, \partial_i \dot\phi \, \varepsilon \operatorname{div} \partial_i \dot u$$
$$= \int_{\mathbb{R}^3_+} \eta \, \varepsilon^2 \, \Delta \dot\phi \operatorname{div} \dot u + \int_{\mathbb{R}^3_+} \eta' \, \varepsilon \, \partial_3 \dot\phi \, \varepsilon \operatorname{div} \dot u + \int_{x_3=0} \varepsilon \, \partial_3 \dot\phi \, \varepsilon \operatorname{div} \dot u$$
$$= L + \mathscr{S} + \mathscr{B},$$

where we set

$$(3.61) \qquad\qquad L := \int_{\mathbb{R}^3_+} \eta \, \varepsilon^2 \, \Delta \dot\phi \operatorname{div} \dot u.$$

Similarly to $\mathscr{B}$ stands for an admissible boundary term which is bounded by boundary terms in the left hand sides of Estimates A and B, that is to say

$$|\mathscr{B}| \leqslant C \|\Gamma(\dot n, \dot u, \varepsilon\partial_{x_3}\dot\phi, \varepsilon\nabla\dot n, \varepsilon \operatorname{div} \dot u)\|^2_{L^2(\mathbb{R}^2)}.$$

Hence,

$$(3.62) \qquad I_2 = -\sum_{i=1}^{3} \left(\int_{\mathbb{R}^3_+} \eta' \, \varepsilon \, \partial_i \dot\phi \, \varepsilon \, \partial_i \dot u_3 + \int_{x_3=0} \varepsilon \, \partial_i \dot\phi \, \varepsilon \, \partial_i \dot u_3 \right) + L + \mathscr{S} + \mathscr{B}.$$

Putting together (3.60), (3.62) and (3.56), we deduce

$$(3.63) \quad \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3_+} \eta |\varepsilon\nabla\dot u|^2 + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3_+} \eta \frac{T^i}{(n_a+n)^2}|\varepsilon\nabla\dot n|^2 \leqslant \frac{1}{2}\int_{\mathbb{R}^3_+} \eta' \, (u_a+u)_3 \, |\varepsilon\nabla\dot u|^2$$
$$+ \frac{1}{2}\int_{x_3=0} (u_a+u)_3 \, |\varepsilon\nabla\dot u|^2\Big) + \frac{1}{2}\int_{\mathbb{R}^3_+} \eta' \, \frac{T^i(u_a+u)_3}{(n_a+n)^2}|\varepsilon\nabla\dot n|^2$$
$$+ \frac{1}{2}\int_{x_3=0} \frac{T^i(u_a+u)_3}{(n_a+n)^2}|\varepsilon\nabla\dot n|^2 + \int_{\mathbb{R}^3_+} \eta' \, T^i \frac{(\varepsilon\nabla\dot n)}{n_a+n} \cdot \varepsilon\nabla\dot u_3 + \int_{x_3=0} T^i \frac{(\varepsilon\nabla\dot n)}{n_a+n} \cdot \varepsilon\nabla\dot u_3$$
$$- \left(\int_{\mathbb{R}^3_+} \eta' \, \varepsilon\nabla\dot\phi \cdot \varepsilon\nabla\dot u_3 + \int_{x_3=0} \varepsilon\nabla\dot\phi \cdot \varepsilon\nabla\dot u_3 \right) + L + \mathscr{G} + \mathscr{S} + \mathscr{B}.$$

(3) *Treatment of $\mathscr{I}$*. — We recall that the $\mathscr{I}$ term defined in Lemma 1 can be written as

$$\mathscr{I} = -\int_{\mathbb{R}^3_+} \eta \, \dot\phi \operatorname{div} \big((n_a+n)\dot u\big) - \int_{\mathbb{R}^3_+} \eta' \, \dot\phi \, \big((n_a+n)\dot u_3\big)$$
$$= J + \mathscr{G} + \mathscr{S}.$$

with $J := -\int_{\mathbb{R}^3_+} \eta \, \dot\phi \, (n_a+n) \operatorname{div} \dot u$. We shall study $\mathscr{I}$ (and J) in a different way than that followed for the first two energy estimates. The idea is to combine J with I_2

through the term L (see (3.61)–(3.21)). Therefore, we decompose J as follows:

$$J = \int_{\mathbb{R}_+^3} \eta \operatorname{div} \dot{u} \big(\varepsilon^2 \, \Delta \dot{\phi} - \dot{\phi} \, (n_a + n) \big) - L = J_1 + J_2 - L,$$

with

$$J_1 := \int_{\mathbb{R}_+^3} \eta \operatorname{div} \dot{u} \big(\varepsilon^2 \, \Delta \dot{\phi} - e^{-\phi_a} \dot{\phi}(1 + h(\phi)) \big),$$

$$J_2 := \int_{\mathbb{R}_+^3} \eta \operatorname{div} \dot{u} \, \dot{\phi} \big(e^{-\phi_a}(1 + h(\phi)) - (n_a + n) \big).$$

Since by construction of the approximation solution, we have $e^{-\phi^0} = n^0$, J_2 can be bounded as follows:

$$J_2 \leqslant \mu \int_{\mathbb{R}_+^3} \eta' \left((\varepsilon \operatorname{div} \dot{u})^2 + \dot{\phi}^2 \right) + \mathscr{G}.$$

Considering J_1, by the Poisson equation, we have:

$$J_1 = \int_{\mathbb{R}_+^3} \eta \operatorname{div} \dot{u} \, \dot{n} + \int_{\mathbb{R}_+^3} \eta \operatorname{div} \dot{u} \, r_\phi.$$

One can rely on the transport equation satisfied by $\dot{n}$ to replace $\operatorname{div} \dot{u}$, which yields:

$$\int_{\mathbb{R}_+^3} \eta \operatorname{div} \dot{u} \, \dot{n} = \int_{\mathbb{R}_+^3} \eta \, \frac{\dot{n}}{n_a + n} \left(-\partial_t \dot{n} - (u_a + u) \cdot \nabla \dot{n} + r_n \right),$$

which we shall recast, after the usual manipulations, as

$$\int_{\mathbb{R}_+^3} \eta \operatorname{div} \dot{u} \, \dot{n} = -\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}_+^3} \eta \, \frac{\dot{n}^2}{n_a + n} + \frac{1}{2} \int_{\mathbb{R}_+^3} \eta' \, \frac{(u_a + u)_3}{n_a + n} \dot{n}^2 + \frac{1}{2} \int_{x_3 = 0} (u_a + u)_3 \, \dot{n}^2$$

$$+ \mu \int_{\mathbb{R}_+^3} \eta' \, (\dot{n}^2 + |\dot{u}|^2) + C_a \int_{\mathbb{R}_+^3} \eta \, \dot{n} r_n.$$

We obtain

$$(3.64) \quad \mathscr{I} \leqslant -L + -\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}_+^3} \eta \, \frac{\dot{n}^2}{n_a + n} + \frac{1}{2} \int_{\mathbb{R}_+^3} \eta' \, \frac{(u_a + u)_3}{n_a + n} \dot{n}^2$$

$$+ \frac{1}{2} \int_{x_3 = 0} (u_a + u)_3 \, \dot{n}^2 + \mathscr{G} + \mathscr{S}.$$

Consequently, by combining (3.63) and (3.64), we obtain

$$(3.65) \quad \mathscr{I} + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}_+^3} \eta \Big(|\varepsilon \, \nabla \dot{u}|^2 + \frac{T^i}{(n_a + n)^2} |\varepsilon \nabla \dot{n}|^2 + \frac{\dot{n}^2}{n_a + n} \Big)$$

$$\leqslant \frac{1}{2} \int_{\mathbb{R}_+^3} \eta' \, (u_a + u)_3 \, |\varepsilon \nabla \dot{u}|^2 + \frac{1}{2} \int_{\mathbb{R}_+^3} \eta' T^i \, \frac{(u_a + u)_3}{(n_a + n)^2} |\varepsilon \nabla \dot{n}|^2$$

$$+ \frac{1}{2} \int_{\mathbb{R}_+^3} \eta' \, \frac{(u_a + u)_3}{n_a + n} \dot{n}^2 - \int_{\mathbb{R}_+^3} \eta' \, \varepsilon \, \nabla \dot{\phi} \cdot \varepsilon \, \nabla \dot{u}_3 + \int_{\mathbb{R}_+^3} \eta' \, T^i \frac{(\varepsilon \, \nabla \dot{n})}{n_a + n} \cdot \varepsilon \, \nabla \dot{u}_3$$

$$+ B.T. + \mathscr{G} + \mathscr{S} + \mathscr{B}.$$

Here $B.T.$ denotes the important boundary contributions:

$$B.T. = \int_{x_3=0} \left(-\varepsilon \partial_3 \dot{\phi} \, \varepsilon \partial_3 \dot{u} + \frac{T^i}{n_a + n} \varepsilon \nabla \dot{n} \cdot \varepsilon \nabla \dot{u}_3 + \frac{1}{2} T^i \frac{(u_a+u)_3}{(n_a+n)^2} |\varepsilon \nabla \dot{n}|^2 + \frac{1}{2}(u_a+u)_3 |\dot{n}|^2 \right).$$

(3) *Conclusion of estimate C.* — By combining, (3.65) and Lemma 1, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^3_+} \eta \left((n_a+n) \frac{|\dot{u}|^2}{2} + \frac{T^i |\dot{n}|^2}{2(n_a+n)} + \frac{1}{2} |\varepsilon \partial \dot{u}|^2 + \frac{1}{2} \frac{T^i}{(n_a+n)^2} |\varepsilon \nabla \dot{n}|^2 + \frac{\dot{n}^2}{n_a+n} \right)$$

$$+ \frac{1}{2} \int_{\mathbb{R}^3_+} \eta' \, Q^0(\dot{n}, \dot{u}) + \frac{1}{2} \int_{\mathbb{R}^3_+} \eta' \left(Q^C(\varepsilon \nabla \dot{n}, \varepsilon \nabla \dot{u}_3) + |(u_a+u)_3| \, |\varepsilon \nabla (u_1, u_2)|^2 \right)$$

$$+ \frac{1}{2} \int_{x_3=0} Q^0(\dot{n}, \dot{u}_3) + \frac{1}{2} \int_{x_3=0} Q^C(\varepsilon \nabla \dot{n}, \varepsilon \nabla \dot{u}_3)$$

$$\leqslant - \int_{\mathbb{R}^3_+} \eta' \, \varepsilon \nabla \dot{\phi} \cdot \varepsilon \nabla \dot{u}_3 - \int_{x_3=0} \varepsilon \partial_3 \dot{\phi} \, \varepsilon \partial_3 \dot{u}_3 + \mathscr{G} + \mathscr{S} + \mathscr{B},$$

where Q^C is the quadratic form of the symmetric matrix:

$$M_C = \Gamma \begin{bmatrix} T^i \frac{|u_{a,3}|}{n_a^2} \operatorname{Id}_3 & -\frac{T^i}{n_a} \operatorname{Id}_3 \\ -\frac{T^i}{n_a} \operatorname{Id}_3 & |u_{a,3}| \operatorname{Id}_3 \end{bmatrix} - \mu \operatorname{Id}_6,$$

which is positive thanks to the Bohm condition (3.11).

   To conclude, we can handle the two first terms in the right hand side of the above estimates by using the Young inequality. Indeed, we write

$$\left| \int_{\mathbb{R}^3_+} \eta' \, \varepsilon \nabla \dot{\phi} \cdot \varepsilon \nabla \dot{u}_3 \right| \leqslant \frac{\widetilde{\mu}}{2} \left\| \sqrt{\eta'} \varepsilon \nabla \dot{u}_3 \right\|_{L^2}^2 + \frac{1}{2\widetilde{\mu}} \left\| \sqrt{\eta'} \varepsilon \nabla \dot{\phi} \right\|_{L^2}^2$$

and for $\widetilde{\mu}$ sufficiently small, the term $\| \sqrt{\eta'} \varepsilon \nabla \dot{u}_3 \|_{L^2}^2$ can be absorbed in the left hand side since Q^C is positive while the other term can be incorporated in $\mathscr{S}$. We proceed in the same way for the boundary term by writing

$$\left| \int_{x_3=0} \varepsilon \partial_3 \dot{\phi} \, \varepsilon \partial_3 \dot{u}_3 \right| \leqslant \frac{\widetilde{\mu}}{2} \| \varepsilon \partial_3 \dot{u} \|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2\widetilde{\mu}} \| \varepsilon \partial_3 \dot{\phi} \|_{L^2(\mathbb{R}^2)}$$

and we absorb the first term in the left hand side by the positivity of Q^C while the other term can be incorporated in $\mathscr{B}$.

   To obtain (3.54), it suffices to observe that the term $\mu \| \sqrt{\eta'} \varepsilon \nabla \dot{u} \|_{L^2}^2$ can be absorbed in the left hand side for μ sufficiently small and to integrate in time.

D. End of the proof of Proposition 2. — We can first combine estimates (3.25) and (3.41) in the following way. We consider $(3.41) + \epsilon(3.25)$ with ϵ fixed sufficiently small (independently of the other involved parameters μ and ε) so that the singular term $\| \sqrt{\eta'} (\varepsilon \nabla \dot{n}, \varepsilon \operatorname{div} \dot{u}) \|_{L^2_T L^2(\mathbb{R}^3_+)}^2$ in the right hand side of (3.25) can be absorbed by the

left hand side of (3.41). This yields

(3.66)
$$\|(\dot{n},\dot{u},\dot{\phi},\varepsilon\nabla\dot{n},\varepsilon\operatorname{div}\dot{u},\varepsilon\nabla\dot{\phi})\|^2_{L^\infty_T L^2(\mathbb{R}^3_+)} + \left\|\sqrt{\eta'}(\dot{n},\dot{u},\dot{\phi},\varepsilon\nabla\dot{n},\varepsilon\operatorname{div}\dot{u},\varepsilon\nabla\dot{\phi})\right\|^2_{L^2_T L^2(\mathbb{R}^3_+)}$$
$$+ \|\Gamma(\dot{n},\dot{u},\varepsilon\nabla\dot{n},\varepsilon\operatorname{div}\dot{u},\varepsilon\partial_{x_3}\dot{\phi})\|^2_{L^2_T L^2(\mathbb{R}^2)}$$
$$\leqslant C(C_a,M)\Big(\|(\dot{n},\dot{u},\dot{\phi},\varepsilon\nabla\dot{n},\varepsilon\nabla\dot{u},\varepsilon\nabla\dot{\phi})(0)\|^2_{L^2(\mathbb{R}^3_+)} + \mu\left\|\sqrt{\eta'}\varepsilon\nabla u\right\|^2_{L^2_T L^2(\mathbb{R}^3_+)}$$
$$+ \|(\dot{n},\dot{u},\varepsilon\nabla\dot{n},\varepsilon\nabla\dot{u},\dot{\phi},\varepsilon\nabla\dot{\phi})\|^2_{L^2_T L^2(\mathbb{R}^3_+)} + \|R^r\|^2_{L^2_T L^2(\mathbb{R}^3_+)} + \mu\left\|\sqrt{\eta'}R^s\right\|^2_{L^2_T L^2(\mathbb{R}^3_+)}\Big).$$

To handle the singular term $\mu\left\|\sqrt{\eta'}\varepsilon\nabla u\right\|^2_{L^2_T L^2(\mathbb{R}^3_+)}$ in the right hand side of (3.66), we consider $(3.66) + \sqrt{\mu}(3.54)$. To ease reading, we set

$$\mathscr{P}_1 := \|(\dot{n},\dot{u},\dot{\phi},\varepsilon\nabla\dot{n},\varepsilon\operatorname{div}\dot{u},\varepsilon\nabla\dot{\phi})\|^2_{L^\infty_T L^2(\mathbb{R}^3_+)} + \sqrt{\mu}\|\varepsilon\nabla\dot{u}\|^2_{L^\infty_T L^2(\mathbb{R}^3_+)}$$
$$+ \left\|\sqrt{\eta'}(\dot{n},\dot{u},\dot{\phi},\varepsilon\nabla\dot{n},\varepsilon\operatorname{div}\dot{u},\varepsilon\nabla\dot{\phi})\right\|^2_{L^2_T L^2(\mathbb{R}^3_+)} + \sqrt{\mu}\left\|\sqrt{\eta'}\varepsilon\nabla\dot{u}\right\|^2_{L^2_T L^2(\mathbb{R}^3_+)}$$
$$+ \|\Gamma(\dot{n},\dot{u},\varepsilon\nabla\dot{n},\varepsilon\operatorname{div}\dot{u},\varepsilon\partial_{x_3}\dot{\phi})\|^2_{L^2_T L^2(\mathbb{R}^2)} + \sqrt{\mu}\|\Gamma\varepsilon\nabla\dot{u}\|^2_{L^2_T L^2(\mathbb{R}^2)},$$

and

$$\mathscr{P}_2 := C(C_a,M)\Big(\|(\dot{n},\dot{u},\dot{\phi},\varepsilon\nabla\dot{n},\varepsilon\operatorname{div}\dot{u},\varepsilon\nabla\dot{\phi})(0)\|^2_{L^2(\mathbb{R}^3_+)}$$
$$+ \mu\left\|\sqrt{\eta'}\varepsilon\nabla u\right\|^2_{L^2_T L^2(\mathbb{R}^3_+)} + \sqrt{\mu}\left\|\sqrt{\eta'}(\dot{n},\dot{u},\dot{\phi},\varepsilon\nabla\dot{n},\varepsilon\operatorname{div}\dot{u},\varepsilon\nabla\dot{\phi})(t)\right\|^2_{L^2_T L^2(\mathbb{R}^3_+)}$$
$$+ \sqrt{\mu}\|\Gamma(\dot{n},\dot{u},\varepsilon\nabla\dot{n},\varepsilon\operatorname{div}\dot{u},\varepsilon\partial_{x_3}\dot{\phi})\|^2_{L^2_T L^2(\mathbb{R}^2)}$$
$$+ \|(\dot{n},\dot{u},\varepsilon\nabla\dot{n},\varepsilon\nabla\dot{u},\dot{\phi},\varepsilon\nabla\dot{\phi})\|^2_{L^2_T L^2(\mathbb{R}^3_+)} + \|R^r\|^2_{L^2_T L^2(\mathbb{R}^3_+)} + \mu\left\|\sqrt{\eta'}R^s\right\|^2_{L^2_T L^2(\mathbb{R}^3_+)}\Big).$$

We get

(3.67)
$$\mathscr{P}_1 \leqslant \mathscr{P}_2.$$

To conclude, we observe that by taking μ sufficiently small, the singular terms

$$\mu\left\|\sqrt{\eta'}\varepsilon\nabla u\right\|^2_{L^2_T L^2(\mathbb{R}^3_+)} + \sqrt{\mu}\left\|\sqrt{\eta'}(\dot{n},\dot{u},\dot{\phi},\varepsilon\nabla\dot{n},\varepsilon\operatorname{div}\dot{u},\varepsilon\nabla\dot{\phi})(t)\right\|^2_{L^2_T L^2(\mathbb{R}^3_+)}$$
$$+ \sqrt{\mu}\|\Gamma(\dot{n},\dot{u},\varepsilon\nabla\dot{n},\varepsilon\operatorname{div}\dot{u},\varepsilon\partial_{x_3}\dot{\phi})\|^2_{L^2_T L^2(\mathbb{R}^2)}$$

can be absorbed in the right hand side.

We end the proof by integrating in time, by applying a Gronwall estimate and by using (3.10). □

4. Nonlinear stability

In this section, we shall prove our main Theorem 1. As already mentioned, we shall actually prove a more precise version, see Theorem 3.

Let us write the solution $(n^\varepsilon, u^\varepsilon, \phi^\varepsilon)$ of (1.1) under the form

$$n^\varepsilon = n_a + n, \quad u^\varepsilon = u_a + u, \quad \phi^\varepsilon = \phi_a + \phi,$$

where n_a, u_a, ϕ_a are shorthands for $n_{\mathrm{app}}^\varepsilon, u_{\mathrm{app}}^\varepsilon, \phi_{\mathrm{app}}^\varepsilon$ (defined in (2.1)). Then, we get for (n, u, ϕ) the system

$$(4.1) \quad \begin{cases} \partial_t n + (u_a + u) \cdot \nabla n + n \operatorname{div}(u + u_a) + \operatorname{div}(n_a u) = \varepsilon^K R_n, \\[2mm] \partial_t u + (u_a + u) \cdot \nabla u + u \cdot \nabla u_a + T^i \Big(\dfrac{\nabla n}{n_a + n} - \dfrac{\nabla n_a}{n_a} \Big(\dfrac{n}{n_a + n} \Big) \Big) \\[2mm] \hspace{6cm} = \nabla \phi + \varepsilon^K R_u, \\[2mm] \varepsilon^2 \Delta \phi = n - e^{-\phi_a}(e^{-\phi} - 1) + \varepsilon^{K+1} R_\phi, \end{cases}$$

together with the boundary condition

$$(4.2) \qquad\qquad\qquad\qquad \phi|_{x_3=0} = 0$$

and the initial condition

$$(4.3) \qquad\qquad u|_{t=0} = \varepsilon^{K+1} u_0, \quad n|_{t=0} = \varepsilon^{K+1} n_0.$$

Observe here that in (4.1), $\varepsilon^K R_n$, $\varepsilon^K R_u$ and $\varepsilon^{K+1} R_\phi$ are remainders that appear because (n_a, u_a, ϕ_a) is not an exact solution of (1.1).

The main result of this section is

THEOREM 3. — *Let $m \geqslant 3$, $(n_0, u_0) \in H^m(\mathbb{R}_+^3)$. Let $K \in \mathbb{N}^*$, $K \geqslant m$ and (n_a, u_a, ϕ_a) an approximate solution at order K, given by Theorem 2, which is defined on $[0, T_0]$. There exists $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0]$ and for every $\varepsilon \in (0, \varepsilon_0]$, there is $C > 0$ independent of ε such that the solution of (4.1)–(4.2)–(4.3) is defined on $[0, T_0]$ and satisfies the estimate*

$$\varepsilon^{|\alpha|} \|\partial^\alpha(n, u, \phi, \varepsilon \nabla \phi)\|_{L^2(\mathbb{R}_+^3)} \leqslant C \varepsilon^K, \quad \forall t \in [0, T_0], \quad \forall \alpha \in \mathbb{N}^3, |\alpha| \leqslant m.$$

We thus obtain:

COROLLARY 1. — *Under the assumptions of Theorem 3, we have*

$$(4.4) \qquad \big\| \big(n^\varepsilon - n_a, u^\varepsilon - u_a, \phi^\varepsilon - \phi_a\big) \big\|_{H^m(\mathbb{R}_+^3)} \leqslant C \varepsilon^{K-m}, \quad \forall t \in [0, T_0].$$

In particular, we get the L^2 and L^∞ convergences as $\varepsilon \to 0$:

$$\sup_{[0,T_0]} \big(\big\| n^\varepsilon - n^0 - N^0\big(\cdot, \cdot, \tfrac{\cdot}{\varepsilon}\big) \big\|_{L^\infty(\mathbb{R}_+^3)} + \|n^\varepsilon - n^0\|_{L^2(\mathbb{R}_+^3)} \big) \longrightarrow 0,$$

$$(4.5) \qquad \sup_{[0,T_0]} \big(\big\| u^\varepsilon - u^0 - U^0\big(\cdot, \cdot, \tfrac{\cdot}{\varepsilon}\big) \big\|_{L^\infty(\mathbb{R}_+^3)} + \|u^\varepsilon - u^0\|_{L^2(\mathbb{R}_+^3)} \big) \longrightarrow 0,$$

$$\sup_{[0,T_0]} \big(\big\| \phi^\varepsilon - \phi^0 - \Phi^0\big(\cdot, \cdot, \tfrac{\cdot}{\varepsilon}\big) \big\|_{L^\infty(\mathbb{R}_+^3)} + \|\phi^\varepsilon - \phi^0\|_{L^2(\mathbb{R}_+^3)} \big) \longrightarrow 0.$$

Then Theorem 1 is a straightforward consequence of this corollary.

From now on, our goal is to prove Theorem 3. First, for each $\varepsilon > 0$, one can solve the Poisson equation in (4.1), and express ϕ in terms of n; more precisely, by elliptic regularity, $\nabla \phi$ can be seen as a semi-linear term in n and the known local existence results for the compressible Euler equation with strictly dissipative boundary conditions ([3]) can be applied to the system (4.1). Let us assume that $(n_0, u_0) \in H^m(\mathbb{R}_+^3)$ for $m \geqslant 3$, then there exists $T^\varepsilon > 0$ and a unique solution

of (4.1) defined on $[0, T^\varepsilon]$ such that $u \in \mathscr{C}([0, T^\varepsilon), H^m)$ and that there exists $M > 0$ independent of ε such that the assumptions (3.10) and (3.11) of Proposition 2 are satisfied with

$$(4.6) \qquad h_0(\phi) := -\frac{e^{-\phi} - 1 + \phi}{\phi}, \quad h_1(\phi) := e^{-\phi} - 1.$$

Thanks to the well-posedness in H^m for $m \geqslant 3$ of the system (4.1), we can define

$$T^\varepsilon = \sup \left\{ T \in [0, T_0], \ \forall\, t \in [0, T], \ \|(n, u, \phi)\|_{H^m_\varepsilon(\mathbb{R}^3_+)} \leqslant \varepsilon^r \right\}$$

where r is chosen such that

$$(4.7) \qquad\qquad\qquad\qquad 5/2 < r < K$$

and the H^m_ε norm is defined by

$$\|f\|_{H^m_\varepsilon(\mathbb{R}^3_+)} = \sum_{|\alpha| \leqslant m} \varepsilon^{|\alpha|} \|\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} f\|_{L^2(\mathbb{R}^3_+)}.$$

The difficulty is thus to prove that the solution actually exists on an interval of time independent of ε. We shall get this result by proving uniform energy estimates combined with the previous local existence result when the initial data and the source term are sufficiently small (i.e., when the approximate solution (n_a, u_a) is sufficiently accurate).

For $m \geqslant 3$, we shall prove by using a priori estimates that under the assumptions of Theorem 3, we have for every $T \in [0, T^\varepsilon]$ the control

$$(4.8) \qquad \|(n, u, \phi)(T)\|_{H^m_\varepsilon(\mathbb{R}^3_+)} \leqslant C(C_a, M) \varepsilon^K e^{T C(C_a, M)/\sqrt{\mu}}$$

with $C(C_a, M)$ independent of ε, μ and T for $\varepsilon \in (0, 1]$, $\mu \in (0, 1]$ and $T \in [0, T_0]$.

To prove this estimate, we can apply the operator $\mathscr{Z}^\alpha := (\varepsilon \partial)^\alpha$ to (4.1) for $|\alpha| \leqslant m - 1$. We obtain for $\mathscr{Z}^\alpha(n, u, \phi)$ the system

$$(4.9) \quad \begin{cases} \partial_t \mathscr{Z}^\alpha n + (u_a + u) \cdot \nabla \mathscr{Z}^\alpha n + (n_a + n) \operatorname{div} \mathscr{Z}^\alpha u = \mathscr{C}_n + \varepsilon^K \mathscr{Z}^\alpha R_n, \\[2mm] \partial_t \mathscr{Z}^\alpha u + (u_a + u) \cdot \nabla \mathscr{Z}^\alpha u + T^i \dfrac{\nabla \mathscr{Z}^\alpha n}{n_a + n} = \nabla \mathscr{Z}^\alpha \phi + \mathscr{C}_u + \varepsilon^K \mathscr{Z}^\alpha R_u, \\[2mm] \varepsilon^2 \Delta \mathscr{Z}^\alpha \phi = \mathscr{Z}^\alpha n + e^{-\phi_a} \mathscr{Z}^\alpha \phi (1 + h) + \mathscr{C}_\phi + \varepsilon^{K+1} \mathscr{Z}^\alpha R_\phi. \end{cases}$$

One has $h = h_0$ for $|\alpha| = 0$ and $h = h_1$ for $|\alpha| \geqslant 1$. The functions $\mathscr{C}_n, \mathscr{C}_u$ and $\mathscr{C}_\phi$ are remainders mostly due to commutators; in $\mathscr{C}_n$, we include the term $-\mathscr{Z}^\alpha[u \cdot \nabla n_a] - \mathscr{Z}^\alpha[n \operatorname{div} u_a]$ as well, while in $\mathscr{C}_u$ we also include $-\mathscr{Z}^\alpha[u \cdot \nabla u_a] + T^i \mathscr{Z}^\alpha\left[\frac{\nabla n_a}{n_a}\left(\frac{n}{n_a + n}\right)\right]$. One can observe that this corresponds to the "abstract" system that we have studied in Proposition 2.

The assumptions (3.10), (3.11) of Proposition 2 are matched on $[0, T^\varepsilon]$ for ε sufficiently small since they are verified by the approximate solution. The only estimate that does not follow directly from the definition of T^ε is the estimate of $\|\partial_t(n, \phi)\|_{L^\infty}$. For $\partial_t n$, we immediately get by using the first line of (4.1) that

$$\|\partial_t n\|_{L^\infty} \leqslant C_a(\varepsilon^{r-1} + \varepsilon^K) \leqslant M,$$

for M sufficiently small. In a similar way, by taking the time derivative in the elliptic equation for ϕ, we have

$$(4.10) \qquad -\varepsilon^2 \Delta \partial_t \phi + e^{-(\phi_a + \phi)} \partial_t \phi = -\partial_t n + \partial_t(e^{-\phi_a})(e^\phi - 1) + \varepsilon^{K+1} \partial_t R_\phi.$$

By using the maximum principle for this elliptic equation, we thus get that

$$\|\partial_t \phi\|_{L^\infty} \leqslant C(C_a, M)\big(\|\partial_t n\|_{L^\infty} + \|\phi\|_{L^\infty} + \varepsilon^{K+1}\big) \leqslant C(C_a, M)\varepsilon^{r-1}.$$

To use the result of Proposition 2, we also need to check that the remainders $\mathscr{C}_n$, $\mathscr{C}_u$ can be split as in the assumption (3.9).

To describe the structure of these remainders, we can start with the study of $\mathscr{C}_n$. We can distinguish between a linear part which corresponds to

$$\mathscr{C}_n^l = -[\mathscr{L}^\alpha, u_a] \cdot \nabla n - \mathscr{L}^\alpha(n \operatorname{div} u_a) - [\mathscr{L}^\alpha, n_a]\operatorname{div} u - \mathscr{L}^\alpha(u \cdot \nabla n_a)$$

and a nonlinear part which is

$$\mathscr{C}_n^{nl} = -[\mathscr{L}^\alpha, u] \cdot \nabla n - [\mathscr{L}^\alpha, n]\operatorname{div} u.$$

By using standard tame estimates in Sobolev spaces, the nonlinear part can be seen as a regular part in the decomposition (3.9). Indeed, we easily get that for some $C > 0$ independent of ε, we have

$$\|\mathscr{C}_n^{nl}\|_{H_\varepsilon^1} \leqslant C\big(\|\nabla u\|_{L^\infty} + \|\nabla n\|_{L^\infty}\big)\|(n, u)\|_{H_\varepsilon^{|\alpha|+1}}$$

and hence by using the Sobolev embedding, we obtain that on $[0, T^\varepsilon]$,

$$\|\mathscr{C}_n^{nl}\|_{H_\varepsilon^1} \leqslant C\varepsilon^{r-5/2}\|(n, u)\|_{H_\varepsilon^m}.$$

To estimate the linear part $\mathscr{C}_n^l$, we can use the Leibniz formula and the estimates (3.12), (3.13) on the approximate solution. By using the ε weighted derivatives, the terms that lead to singular terms are the ones of the first type

$$\varepsilon^k \partial^\beta n_a \cdot \nabla \partial^\gamma u$$

or the ones of the second type

$$\varepsilon^k \partial_{x_3} \partial^\beta n_a \cdot \partial^\gamma u$$

with $|\beta| + |\gamma| = k$. Indeed, for the terms of the second type, we obtain a singular term as soon as β involves only x_3 derivatives and when they all hit the boundary layer term in n_a, in this case, we can use (3.7) to estimate them by

$$(4.11) \qquad C_a\big(\mu \, \eta' |\langle \varepsilon \partial \rangle^m (n, u)| + |\langle \varepsilon \partial \rangle^m (n, u)|\big)$$

with the notation

$$|\langle \varepsilon \partial \rangle^m (n, u)| = \sum_{|\beta| \leqslant m} |(\varepsilon \partial)^\beta (n, u)|.$$

We thus see the first term in (4.11) as a singular part of the source term and the second one as a regular part in the decomposition (3.9).

For the terms of the first type, in order to estimate them with u in a H_ε^m space, we need to write them under the form

$$\varepsilon^{|\beta|-1} \partial^\beta n_a \cdot (\varepsilon \nabla)(\varepsilon \partial)^\gamma u$$

and hence we obtain again a singular term as soon as ∂^β is made only of x_3 derivatives and that they all hit the boundary layer terms in n_a. By using (3.7), we can estimate this contribution again by

$$C_a\big(\mu\,\eta'|\langle\varepsilon\partial\rangle^m(n,u)| + |\langle\varepsilon\partial\rangle^m(n,u)|\big)$$

We thus obtain an estimate

$$|\mathscr{C}^l| + |\varepsilon\nabla\mathscr{C}^l| \leqslant C_a\big(\mu\,\eta'|\langle\varepsilon\partial\rangle^m(n,u)| + |\langle\varepsilon\partial\rangle^m(n,u)|\big).$$

We can proceed in the same way to handle $\mathscr{C}_u$. For example, for the commutator

$$\mathscr{C}^p := \Big[\mathscr{Z}^\alpha, \frac{1}{n_a+n}\Big]\nabla n,$$

we can expand

$$(\mathscr{C}^p, \varepsilon\nabla\mathscr{C}^p) = \sum_{|\beta|+|\gamma|\leqslant m,\,\beta\neq 0} \star_{\beta,\gamma}\big(\mathscr{Z}^\beta n_a \nabla\mathscr{Z}^\gamma n + \mathscr{Z}^\beta n\nabla\mathscr{Z}^\gamma n\big),$$

where $\star_{\beta,\gamma}$ stand for terms which are uniformly bounded in L^∞ on $[0,T^\varepsilon]$. By using the same arguments as before, we thus get that

$$|\mathscr{C}^p| + |\varepsilon\nabla\mathscr{C}^p| \leqslant C(C_a,M)\big(\mu\,\eta'|\langle\varepsilon\partial\rangle^m(n,u)| + |\langle\varepsilon\partial\rangle^m(n,u)|\big) + C(C_a,M)|\widetilde{\mathscr{C}^p}|$$

where $|\widetilde{\mathscr{C}^p}|$ is bounded in L^2 by

$$C(C_a,M)\varepsilon^{r-5/2}\|n\|_{H^m_\varepsilon}$$

by using the usual product estimates in Sobolev spaces.

For $\varepsilon^{-1}\mathscr{C}_\phi$, we can use again the bounds on the approximate solution (3.12), (3.13) and standard tame estimates in Sobolev spaces. Indeed, we have that $\varepsilon^{-1}\mathscr{C}_\phi$ can be expanded as a sum of terms either of the form

$$\varepsilon^{-1}(\mathscr{Z}^\alpha e^{-\phi_a})(e^{-\phi}-1)$$

which is bounded by $C(C_a,M)\big(\mu\,\eta'|\phi| + |\phi|\big)$ or of the form

$$\varepsilon^{-1}\mathscr{Z}^\beta(e^{-\phi_a})\,\mathscr{Z}^\gamma\phi\,e^{-\phi}$$

with $|\beta|+|\gamma| \leqslant m$ which can also be bounded by $C(C_a,M)\big(\mu\,\eta'|\phi| + |\phi|\big)$ or

$$\varepsilon^{-1}\mathscr{Z}^\beta(e^{-\phi_a})\,\mathscr{Z}^{\gamma_1}\phi\cdots\mathscr{Z}^{\gamma_r}\phi\,e^{-\phi}$$

with $r \geqslant 2$, $|\beta|+|\gamma_1|+\cdots+|\gamma_r| \leqslant m$ and $\beta \neq 0$. Since this term is at least quadratic in ϕ, we can use tame estimates to bound it in L^2 by

$$\varepsilon^{-1}C(C_a,M)\|\phi\|_{L^\infty}\|\phi\|_{H^{m-1}_\varepsilon} \leqslant C(C_a,M)\varepsilon^{r-5/2}\|\phi\|_{H^{m-1}_\varepsilon}.$$

We thus get that we can split $\varepsilon^{-1}\mathscr{C}_\phi$ according to (3.9):

$$|\varepsilon^{-1}\mathscr{C}_\phi| \leqslant C(C_a,M)\big(\mu\eta'|\langle\varepsilon\partial\rangle^{m-1}\phi| + |\widetilde{\mathscr{C}_\phi}|\big)$$

with

$$\|\widetilde{\mathscr{C}_\phi}\|_{L^2} \leqslant C(C_a,M)\|\phi\|_{H^{m-1}_\varepsilon}.$$

By using the above expansions of $\mathscr{C}_\phi$, we also easily obtain that

$$|\partial_t\mathscr{C}_\phi| \leqslant C(C_a,M)\big(\varepsilon\eta'|\langle\varepsilon\partial\rangle^{m-1}\partial_t\phi| + |\mathscr{C}^1_\phi|$$

where $\mathscr{C}_\phi^1$ can be estimated as

$$\|\mathscr{C}_\phi^1\|_{L^2} \leqslant C(C_a, M)\varepsilon\|\partial_t\phi\|_{H_\varepsilon^{m-1}}.$$

In summary, we have proven that the commutator terms can be split under the form

$$\left|\left(\mathscr{C}_n, \mathscr{C}_u, \varepsilon\nabla\mathscr{C}_n, \varepsilon\nabla\mathscr{C}_u, \mathscr{C}_\phi/\varepsilon, \partial_t\mathscr{C}_\phi\right)\right| \leqslant C(C_a, M)\left(\mu\eta'\mathscr{C}^s + \mathscr{C}^r\right)$$

with

$$\left\|\sqrt{\eta'}\mathscr{C}^s\right\|_{L^2} \leqslant C(C_a, M)\left(\left\|\sqrt{\eta'}(n, u, \phi)\right\|_{H_\varepsilon^m} + \varepsilon\left\|\sqrt{\eta'}\partial_t\phi\right\|_{H_\varepsilon^{m-1}}\right),$$
$$\left\|\mathscr{C}^r\right\|_{L^2} \leqslant C(C_a, M)\left(\left\|(n, u, \phi)\right\|_{H_\varepsilon^m} + \varepsilon\left\|\partial_t\phi\right\|_{H_\varepsilon^{m-1}}\right).$$

Consequently, by using Proposition 2, we obtain for every $T \in [0, T^\varepsilon]$

$$(4.12) \quad \sqrt{\mu}\left(\|(n, u, \phi)\|_{L_T^\infty H_\varepsilon^m}^2 + \left\|\sqrt{\eta'}(n, u, \phi)\right\|_{L_T^2 H_\varepsilon^m}^2\right)$$
$$\leqslant C(C_a, M)\Big(\varepsilon^{2K} + \mu\left\|\sqrt{\eta'}(n, u, \phi)\right\|_{L_T^2 H_\varepsilon^m}^2$$
$$+ \mu\left\|\sqrt{\eta'}\,\varepsilon\partial_t\phi\right\|_{L_T^2 H_\varepsilon^{m-1}}^2 + \|(n, u, \phi)\|_{L_T^2 H_\varepsilon^m}^2 + \|\varepsilon\partial_t\phi\|_{L_T^2 H_\varepsilon^{m-1}}^2\Big).$$

To conclude, it remains to estimate the terms involving $\partial_t\phi$ in the right hand side of the above estimate. By using the elliptic equation (4.10) and standard energy estimates, we get that

$$\|\varepsilon\nabla\partial_t\phi(t)\|_{H_\varepsilon^{m-1}} + \|\partial_t\phi(t)\|_{H_\varepsilon^{m-1}} \leqslant C(C_a, M)\left(\|\partial_t n(t)\|_{H_\varepsilon^{m-1}} + \|\phi(t)\|_{H_\varepsilon^{m-1}} + \varepsilon^{K+1}\right)$$

for every $t \in [0, T^\varepsilon]$. Therefore, we obtain in particular that

$$\varepsilon\|\partial_t\phi(t)\|_{H_\varepsilon^{m-1}} \leqslant C(C_a, M)\left(\varepsilon\|\partial_t n(t)\|_{H_\varepsilon^{m-1}} + \|\phi(t)\|_{H_\varepsilon^{m-1}} + \varepsilon^{K+1}\right).$$

We can then use the first line of (4.1) to express $\varepsilon\partial_t n$ in terms of space derivatives, this yields that on $[0, T^\varepsilon]$, we have

$$(4.13) \qquad \varepsilon\|\partial_t\phi(t)\|_{H_\varepsilon^{m-1}} \leqslant C(C_a, M)\left(\|(n, u)(t)\|_{H_\varepsilon^m} + \|\phi(t)\|_{H_\varepsilon^{m-1}} + \varepsilon^{K+1}\right).$$

Finally, we can estimate $\left\|\sqrt{\eta'}\,\varepsilon\partial_t\phi\right\|_{L_T^2 H_\varepsilon^{m-1}}$. This follows again from estimates on the elliptic equation (4.10). We just use the weight η' in the estimates as in the derivation of (3.40). This yields for $t \in [0, T^\varepsilon]$,

$$\left\|\sqrt{\eta'}\,\partial_t\phi(t)\right\|_{H_\varepsilon^{m-1}} \leqslant C(C_a, M)\left(\left\|\sqrt{\eta'}\partial_t n(t)\right\|_{H_\varepsilon^{m-1}} + \left\|\sqrt{\eta'}\,\phi(t)\right\|_{H_\varepsilon^{m-1}} + \varepsilon^K\right).$$

Consequently, by using again the first line of (4.1), to express $\partial_t n$, we obtain

$$(4.14) \quad \varepsilon\left\|\sqrt{\eta'}\,\partial_t\phi(t)\right\|_{H_\varepsilon^{m-1}}$$
$$\leqslant C(C_a, M)\left(\left\|\sqrt{\eta'}(n, u, \phi)(t)\right\|_{H_\varepsilon^m} + \|(n, u, \phi)(t)\|_{H_\varepsilon^m} + \varepsilon^{K+1}\right).$$

From (4.12), (4.13), (4.14), we have thus proven that for every $T \in [0, T^\varepsilon]$, we have

$$\sqrt{\mu}\left(\|(n, u, \phi)\|_{L_T^\infty H_\varepsilon^m}^2 + \left\|\sqrt{\eta'}(n, u, \phi)\right\|_{L_T^2 H_\varepsilon^m}^2\right)$$
$$\leqslant C(C_a, M)\left(\varepsilon^{2K} + \mu\left\|\sqrt{\eta'}(n, u, \phi)\right\|_{L_T^2 H_\varepsilon^m}^2 + \|(n, u, \phi)\|_{L_T^2 H_\varepsilon^m}^2\right).$$

For μ sufficiently small, the singular term $\mu\|\sqrt{\eta'}(n,u,\phi)\|^2_{L^2_T H^m_\varepsilon}$ can be absorbed in the left hand side, this yields

$$\sqrt{\mu}\|(n,u,\phi)\|^2_{L^\infty_T H^m_\varepsilon} \leqslant C(C_a,M)\big(\varepsilon^{2K} + \|(n,u,\phi)\|^2_{L^2_T H^m_\varepsilon}\big)$$

and hence from the Gronwall inequality, we obtain

$$\|(n,u,\phi)(T)\|^2_{H^m_\varepsilon} \leqslant C(C_a,M)\varepsilon^{2K}e^{TC(C_a,M)/\sqrt{\mu}}, \quad \forall\, T \in [0,T^\varepsilon].$$

We have thus proven (4.8).

The parameter μ can now be considered as fixed. By standard continuation arguments, we next obtain that for ε sufficiently small $T^\varepsilon \geqslant T_0$ and that on $[0,T_0]$, we have

$$\sup_{T\in[0,T_0]} \|(n,u,\phi)(T)\|^2_{H^m_\varepsilon} \leqslant C(C_a,M)\varepsilon^{2K}e^{T_0 C(C_a,M)/\sqrt{\mu}}.$$

This ends the proof of Theorem 3.

References

[1] A. Ambroso – "Stability for solutions of a stationary Euler-Poisson problem", *Math. Models Methods Appl. Sci.* **16** (2006), no. 11, p. 1817–1837.

[2] A. Ambroso, F. Méhats & P. A. Raviart – "On singular perturbation problems for the nonlinear Poisson equation", *Asymptot. Anal.* **25** (2001), no. 1, p. 39–91.

[3] S. Benzoni-Gavage & D. Serre – *Multidimensional hyperbolic partial differential equations*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, Oxford, 2007, First-order systems and applications.

[4] S. Cordier & E. Grenier – "Quasineutral limit of an Euler-Poisson system arising from plasma physics", *Comm. Partial Differential Equations* **25** (2000), no. 5-6, p. 1099–1113.

[5] D. Gérard-Varet, D. Han-Kwan & F. Rousset – "Quasineutral limit of the Euler-Poisson system for ions in a domain with boundaries", *Indiana Univ. Math. J.* **62** (2013), p. 359–402.

[6] J. Goodman – "Nonlinear asymptotic stability of viscous shock profiles for conservation laws", *Arch. Rational Mech. Anal.* **95** (1986), no. 4, p. 325–344.

[7] M. Lieberman & A. Lichtenberg – *Principles of plasma discharges and materials processing*, Cambridge University Press, 1994.

[8] S. Nishibata, M. Ohnawa & M. Suzuki – "Asymptotic stability of boundary layers to the Euler-Poisson equations arising in plasma physics", *SIAM J. Math. Anal.* **44** (2012), no. 2, p. 761–790.

[9] K.-U. Riemann – "The Bohm criterion and sheath formation", *J. Phys. D: Applied Physics* **24** (1991), no. 4, p. 493.

[10] M. Slemrod & N. Sternberg – "Quasi-neutral limit for Euler-Poisson system", *J. Nonlinear Sci.* **11** (2001), no. 3, p. 193–209.

[11] M. Suzuki – "Asymptotic stability of stationnary solutions to the Euler-Poisson equations arising in plasma physics", *Kinet. and Relat. Mod.* **4** (2011), no. 2, p. 569–588.

David Gérard-Varet, Institut de Mathématiques de Jussieu (UMR 7586), Université Paris-Diderot
Bâtiment Sophie Germain, 75205 Paris Cedex 13, France
E-mail : `gerard-varet@math.jussieu.fr`
Url : `http://webusers.imj-prg.fr/~david.gerard-varet/`

Daniel Han-Kwan, CNRS & Centre de Mathématiques Laurent Schwartz (UMR 7640), École
polytechnique
91128 Palaiseau Cedex, France
E-mail : `daniel.han-kwan@math.polytechnique.fr`
Url : `http://www.math.polytechnique.fr/~daniel.han-kwan/`

Frédéric Rousset, Laboratoire de Mathématiques d'Orsay (UMR 8628), Université Paris-Sud et
Institut Universitaire de France
Bâtiment 425, 91405 Orsay Cedex, France
E-mail : `frederic.rousset@math.u-psud.fr`
Url : `http://www.math.u-psud.fr/anm_edp/donnees/rousset.html`