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ON CLOSED SUBGROUPS OF THE GROUP OF HOMEOMORPHISMS OF A MANIFOLD

by Frédéric Le Roux

Abstract. — Let M be a triangulable compact manifold. We prove that, among closed subgroups of $\mathrm{Homeo}_0(M)$ (the identity component of the group of homeomorphisms of M), the subgroup consisting of volume preserving elements is maximal.

Résumé (Sur les sous-groupes du groupe des homéomorphismes d'une variété)

Soit M une variété triangulable compacte. Nous montrons que, parmi les sous-groupes de $\mathrm{Homeo}_0(M)$ (composante connexe de l'identité du groupe des homéomorphismes de M), le sous-groupe des homéomorphismes préservant le volume est maximal.

Contents

1. Introduction

The theory of groups acting on the circle is very rich (see in particular the monographs [Ghy01, Nav07]). The theory is far less developed in higher dimension, where it seems difficult to discover more than some isolated islands in a sea of chaos. In this note, we are interested in the closed subgroups of the group $\mathrm{Homeo}_0(M)$, the identity component of the group of homeomorphisms of some compact triangulable n-dimensional manifold M. We will show that, when $n \geqslant 2$, for any *good* (nonatomic and with total support) probability measure μ, the subgroup of elements that preserve μ is maximal among closed subgroups.

Mathematical subject classification (2010). — 57S05, 57M60, 37E30.
Keywords. — Transformation groups, homeomorphisms, maximal closed subgroups.

Let us recall some related results in the case when M is the circle. De La Harpe conjectured that $\mathrm{PSL}(2,\mathbb{R})$ is a maximal closed subgroup ([Bes04]). Ghys proposed a list of closed groups acting transitively, asking whether, up to conjugacy, the list was complete ([Ghy01]); the list consists of the whole group, $\mathrm{SO}(2)$, $\mathrm{PSL}(2,\mathbb{R})$, the group $\mathrm{Homeo}_{k,0}(\mathbb{S}^1)$ of elements that commute with some rotation of order k, and the group $\mathrm{PSL}_k(2,\mathbb{R})$ which is defined analogously. The first conjecture was solved by Giblin and Markovic in [GM06]. These authors also answered Ghys's question affirmatively, under the additional hypothesis that the group contains some non trivial arcwise connected component. Thinking of the two-sphere with these results in mind, one is naturally led to the following questions.

Question 1. — *Let G be a proper closed subgroup of* $\mathrm{Homeo}_0(\mathbb{S}^2)$ *acting transitively. Assume that G is not a (finite dimensional) Lie group. Is G conjugate to one of the two subgroups:* (1) *the centralizer of the antipodal map $x \mapsto -x$,* (2) *the subgroup of area-preserving elements?*

Note that the centralizer of the antipodal map is the group of lifts of homeomorphisms of the projective plane; it is the spherical analog of the groups $\mathrm{Homeo}_{k,0}(\mathbb{S}^1)$.

Question 2. — *Is* $\mathrm{PSL}(2,\mathbb{C})$ *maximal among closed subgroups of* $\mathrm{Homeo}_0(\mathbb{S}^2)$*?*

On the circle the group of measure-preserving elements coincides with $\mathrm{SO}(2)$. It is not maximal among closed subgroups since it is included in $\mathrm{PSL}(2,\mathbb{R})$. In contrast, we propose to prove that the closed subgroup of area-preserving homeomorphisms of the two-sphere is maximal. To put this into a general context, let M be a compact connected topological manifold whose dimension is greater than or equal to 2. For simplicity, we assume that M has no boundary. We also assume that M is triangulable, that is, it is homeomorphic to a simplicial complex. We do not assume that M is orientable. Let us equip M with a probability measure μ which is assumed to be *good*: this means that every finite set has measure zero, and every non-empty open set has positive measure. We consider the group $\mathrm{Homeo}_0(M)$ of homeomorphisms of M that are isotopic to the identity, and the subgroup $\mathrm{Homeo}_0(M,\mu)$ of elements that preserve the measure μ. We equip these transformation groups with the topology of uniform convergence, which turns them into topological groups. The subgroup $\mathrm{Homeo}_0(M,\mu)$ is easily seen to be closed in $\mathrm{Homeo}_0(M)$. Note that according to Fathi's theorem (first theorem in [Fat80]), $\mathrm{Homeo}_0(M,\mu)$ coincides with the identity component in the group of measure preserving homeomorphisms. The aim of the present note is to prove the following.

Theorem. — *The group* $\mathrm{Homeo}_0(M,\mu)$ *is maximal among closed subgroups of the group* $\mathrm{Homeo}_0(M)$*.*

Acknowledgements. — The author thanks the referees for their suggestions that has led to significant improvement in the exposition (and some cold sweats for the author...). After the first version of this paper has been written, Kwakkel and Tal

have announced an independent proof of this result in the case of the two-sphere (see [KT13]). Their paper also provides a partial description of the closed subgroups of the group of homeomorphisms of the two-sphere that contain SO(3).

2. Preliminaries: balls, triangulations and measure

In this part we first state the Annulus Theorem, and deduce that homeomorphisms on a manifold act transitively on locally flat balls. Then we state the Oxtoby-Ulam Theorem, and deduce that measure preserving homeomorphisms act transitively on good balls of some given measure. Finally we construct triangulations having good properties with respect to the measure.

In this section M denotes some compact connected topological manifold, possibly with boundary (as in the statement of the above theorem, M is not assumed to be orientable). In top of the two previously stated properties, good measures on a manifold with boundary are required to give measure zero to the boundary. We denote by $\mathrm{Homeo}_0(M, \partial M)$ the identity component of the group of homeomorphisms of M that are the identity on the boundary.

Balls. — A *ball* is any subset of M which is homeomorphic to a Euclidean closed ball in $\mathbb{R}^n$, where n is the dimension of M. Let us denote by $B_r(0)$ the Euclidean ball with radius r and center 0 in $\mathbb{R}^n$. A ball B will be called *locally flat* if it is the image of the Euclidean ball $B_1(0)$ under an embedding (continuous one-to-one map) $\gamma : B_2(0) \to M$. The Annulus Theorem deals with locally flat balls in the manifold $M = B_1(0)$. The usual statement is as follows (see [Kir69, Qui82]).

Theorem (Annulus Theorem, Kirby and Quinn). — *If B is a locally flat ball in the interior of the Euclidean ball $B_1(0)$, then $B_1(0) \smallsetminus \mathrm{Int}(B)$ is homeomorphic to the annulus $B_1(0) \smallsetminus \mathrm{Int}(B_{1/2}(0))$.*

We will need a stronger version, namely we would like the homeomorphism Φ between $B_1(0) \smallsetminus \mathrm{Int}(B)$ and $B_1(0) \smallsetminus \mathrm{Int}(B_{1/2}(0))$ to be the identity on the boundary of $B_1(0)$. This can be achieved as follows. We use the Stable Homeomorphism Theorem (see [Kir69]) which says that every orientation preserving homeomorphism of the $(n-1)$-sphere can be written as a composition of homeomorphisms whose fixed point set have non-empty interior. In particular, such a homeomorphism is isotopic to the identity. Up to composing Φ with a symmetry that preserves both $B_1(0)$ and $B_{1/2}(0)$, we can assume that the restriction of Φ to the boundary $\partial B_1(0)$ preserves the orientation. The Stable Homeomorphism Theorem provides an isotopy $(\Phi_t)_{t\in[0,1]}$ from $\Phi_{|\partial B_1(0)}$ to the identity in the space of homeomorphisms of the sphere $\partial B_1(0)$. We use this isotopy to extend Φ to the ball $B_2(0)$ by the formula

$$\Phi(sx) = s\Phi_{s-1}(x)$$

for $x \in \partial B_1(0)$ and $s \in [1, 2]$. This gives a homeomorphism between $B_2(0) \smallsetminus \mathrm{Int}(B)$ and $B_2(0) \smallsetminus \mathrm{Int}(B_{1/2}(0))$ which is the identity on $\partial B_2(0)$. Finally we consider a (radial) homeomorphism between $B_2(0)$ and $B_1(0)$ which is the identity outside some ball $B_r(0)$ containing B and $B_{1/2}(0)$, and we conjugate Φ by this homeomorphism to get the desired map.

Let σ denote that symmetry $(x_1,\dots,x_n)\to(x_1,\dots,-x_n)$ in $\mathbb{R}^n$. Note that σ reverses the orientation. For any embedding $\gamma : B_r(0)\to M$, we define $\overline{\gamma}=\gamma\sigma$.

Corollary 2.1. — *The group* $\mathrm{Homeo}_0(M,\partial M)$ *acts transitively on locally flat balls of M. More precisely, let $\gamma,\gamma' : B_2(0)\to M$ be two embeddings. Then there exists some $\Phi\in\mathrm{Homeo}_0(M)$ such that $\gamma'_{|B_1(0)}=\Phi\circ\gamma_{|B_1(0)}$ or $\gamma'_{|B_1(0)}=\Phi\circ\overline{\gamma}_{|B_1(0)}$.*

Note that, in case M happens to be orientable, we cannot do without $\overline{\gamma}$. Indeed, every $\Phi\in\mathrm{Homeo}_0(M,\partial M)$ preserves the orientation, and if $\gamma'\gamma^{-1}$ does not, there is no $\Phi\in\mathrm{Homeo}_0(M,\partial M)$ such that $\gamma'_{|B_1(0)}=\Phi\circ\gamma_{|B_1(0)}$.

Proof. — Let us first notice that given any two points $x,y\in M\smallsetminus\partial M$ we can find some ball in M whose interior contains x and y. Indeed, this is certainly true if the points are close enough, and we can deduce the general case by showing that the subgroup of homeomorphisms that fixes x acts transitively on $M\smallsetminus(\partial M\cup\{x\})$. Now consider a ball B_0 whose interior contains $\gamma(0)$ and $\gamma'(0)$. Let $B=\gamma(B_1(0))$, $B'=\gamma'(B_1(0))$. We first look for some $\Phi_0\in\mathrm{Homeo}_0(M,\partial M)$ such that $\Phi_0(B)=B'$. Let h be a (radial) homeomorphism of $B_2(0)$ that contracts $B_1(0)$ to a very small ball around 0. Then $\gamma h\gamma^{-1}$ is an element of $\mathrm{Homeo}_0(M,\partial M)$ which sends B to a small ball containing x, and thus included in B_0. Likewise we find a homeomorphism that sends B' inside B_0. By this construction we can assume that both B and B' are included in the interior of B_0. Then we apply the strong version of the Annulus theorem to find a homeomorphism Φ_0, supported in B_0 and sending B to B'. Note that, by the Alexander trick, Φ_0 is isotopic to the identity.

Now we prove the precise version of the corollary. Using the first part we can assume that $B=B'$. The maps $\gamma'_{|B_1(0)}\gamma^{-1}_{|B_1(0)}$ and $\gamma'_{|B_1(0)}\overline{\gamma}^{-1}_{|B_1(0)}$ are homeomorphisms of B, and one of those two maps preserves the orientation on B. Let Φ be equal to this map. The restriction of Φ to the boundary of B also preserves the orientation. Thus we can use the Stable Homeomorphism Theorem (as in the paragraph following the Annulus Theorem) to extend Φ to a ball B_0 whose interior contains B, leaving fixed every point on the boundary of B_0. We finally extend Φ to M by the identity outside B_0. □

The Oxtoby-Ulam theorem. — The key tool for dealing with good measures on manifolds is the Oxtoby-Ulam Theorem ([OU41], Corollary 1 in Part II, see also [Fat80], Theorem 3.1).

Theorem (Oxtoby-Ulam). — *Let μ,μ' be two good measures on M with the same (finite) total mass. Then there exists some homeomorphism $\Phi\in\mathrm{Homeo}_0(M,\partial M)$ such that $\Phi_*\mu=\mu'$.*

The paper [GP75] gives an easy proof for the particular case when M is the unit cube $[0,1]^n$ (the fact that Φ is the identity on the boundary is not included in the statement there but follows immediately from the proof). The general case is deduced from this particular case by using a theorem of M. Brown, which provides a map p from the cube onto M whose restriction to the interior of the cube is an embedding (see [Bro62] or [Fat80], Proposition 2.3 and its complement). The fact that Φ is isotopic to the identity is not part of the statement in [OU41] nor [Fat80] but follows from the proof, since Φ is obtained as $p\widetilde{\Phi}p^{-1}$ for some homeomorphism $\widetilde{\Phi}$ of the cube, and an isotopy in the cube from the identity to $\widetilde{\Phi}$ (Alexander trick) induces an isotopy in M from the identity to Φ.

From now on we fix some good probability measure μ on M. Let us denote by $\mathrm{Homeo}_0(M, \partial M, \mu)$ the subgroup of $\mathrm{Homeo}_0(M, \partial M)$ of homeomorphisms that preserves μ. A *good ball* in M is a locally flat ball whose boundary has measure zero. Note that, due to countable additivity, if $\gamma : B_1(0) \to M$ is any topological embedding, then for almost every $r \in (0, 1)$ the ball $\gamma(B_r(0))$ is good. As a consequence of Oxtoby-Ulam theorem we get a measure-preserving version of Corollary 2.1. (This Corollary is just a variation on well-known properties, see for example Proposition 3.6 in [Fat80].)

Corollary 2.2. — *For every $c > 0$, the group $\mathrm{Homeo}_0(M, \partial M, \mu)$ acts transitively on good balls of M with measure c.*

More precisely, let $\gamma, \gamma' : B_2(0) \to M$ be two embeddings. Let $B = \gamma(B_1(0))$, $B' = \gamma'(B_1(0))$, assume that $\mu(\partial B) = \mu(\partial B') = 0$ and that $\gamma'\gamma^{-1}$ preserves the measure μ in the sense that $(\gamma'\gamma^{-1})_*\mu_{|B} = \mu_{|B'}$. Then there exists some $\Phi \in \mathrm{Homeo}_0(M, \partial M, \mu)$ such that $\gamma'_{|B_1(0)} = \Phi \circ \gamma_{|B_1(0)}$ or $\gamma'_{|B_1(0)} = \Phi \circ \overline{\gamma}_{|B_1(0)}$.

Proof. — Given two good balls with the same measure, the Oxtoby-Ulam theorem provides embeddings γ, γ' as in the second part of the statement. Thus the first part will follow from the second.

Let us prove the second part. Given γ, γ' as in the statement, Corollary 2.1 provides some $\Phi_1 \in \mathrm{Homeo}_0(M, \partial M)$ with all the wanted properties except that it does not preserve the measure. Then the Oxtoby-Ulam theorem yields a homeomorphism Φ_2 supported in $M \smallsetminus B'$ and sending the measure $(\Phi_{1*}\mu)_{|M \smallsetminus B'}$ to the measure $\mu_{|M \smallsetminus B'}$. The final map Φ is obtained as $\Phi_2\Phi_1$. □

Triangulations. — From now on we assume that M has empty boundary, and that it is homeomorphic to a simplicial complex. Let $\mathscr{T}$ be any triangulation of M. We would like the $(n-1)$-skeleton of $\mathscr{T}$ to have measure zero. We will get this additional property as follows. Each n-dimensional simplex s of $\mathscr{T}$ is homeomorphic to the standard n-dimensional simplex in $\mathbb{R}^{n+1}$; let μ_s be a probability measure on s which is the homeomorphic image of the n-dimensional Lebesgue measure on the standard simplex. The measure

$$\mu' = \frac{1}{N} \sum \mu_s$$

(where N denotes the number of n-dimensional simplices of $\mathscr{T}$) is a good probability measure on M for which the $(n-1)$-dimensional simplices have measure zero. We apply the Oxtoby-Ulam theorem to get a homeomorphism h of M sending μ' to μ. Then we consider the image triangulation $\mathscr{T}_0 = h_*(\mathscr{T})$, whose $(n-1)$-skeleton has measure zero. In addition to this, all the n-simplices of $\mathscr{T}_0$ have the same mass.

Note that in the barycentric subdivision of any simplex, all n-dimensional simplices have the same Lebesgue measure (one way to see this is to consider the group of affine maps that permute the vertices of the simplex; these maps preserve the volume and the barycentric subdivision; furthermore this group acts transitively on the set of n-dimensional simplices of the barycentric subdivision). Thus using successive barycentric subdivisions we get the following.

Proposition 2.3. — *There exists a sequence $(\mathscr{T}_p)_{p\geqslant 0}$ of nested triangulations such that*

(1) *for every p, the $(n-1)$-skeleton of $\mathscr{T}_p$ has measure zero,*
(2) *for every p, all the n-simplices of $\mathscr{T}_p$ have the same measure,*
(3) *the sequences (m_p) and (d_p) tend to zero, where m_p denotes the common measure of the simplices of $\mathscr{T}_p$, and d_p denotes the supremum of the diameters of the simplices of $\mathscr{T}_p$ (for some metric which is compatible with the topology on M).*

Here is a useful consequence. Let O be any open subset of M. We define inductively $\mathscr{O}_p$ as the set of all the n-dimensional open simplices of $\mathscr{T}_p$ that are included in O but not in some $s \in \mathscr{O}_{p-1}$. The elements of $\mathscr{O} := \bigcup \mathscr{O}_p$ are pairwise disjoint and their closures cover O. Since the $(n-1)$-skeleton of our triangulations have no mass, we have the equality

$$\mu(O) = \sum_{U\in\mathscr{O}} \mu(U). \tag{1}$$

We call a (closed) simplex of some $\mathscr{T}_p$ *good* if it is a good ball in M. We notice that for every $p > 0$, all the n-dimensional simplices that are disjoint from the $(n-1)$-skeleton of $\mathscr{T}_0$ are good[(1)]. Thus Equality (1) still holds if, in the definition of the $\mathscr{O}_p$'s, we replace the simplices by the simplices whose closure is good. This yields the following.

Corollary 2.4. — *If two probability measures μ, μ' give the same mass to all the good n-simplices of $\mathscr{T}_p$ for every p, then they are equal.*

3. Localization

Now we begin the proof of the theorem. Let M be a compact connected triangulable manifold without boundary. We consider some element $f \in \mathrm{Homeo}_0(M)$ that does not preserve the measure μ, and we denote by G_f the subgroup of $\mathrm{Homeo}_0(M)$ generated by

$$\{f\} \cup \mathrm{Homeo}_0(M, \mu).$$

[(1)] Note that there can be simplices in $\mathscr{T}_0$ that fail to be good balls if $\mathscr{T}_0$ is a triangulation but not a PL-triangulation.

Our aim is to show that the group G_f is dense in $\mathrm{Homeo}_0(M)$. In the first lemma we look for an element of the group G_f that does not preserve the measure and has small support.

Lemma 3.1. — *For every positive ε there exists a good ball B of measure less than ε and an element $g \in G_f$ which is supported in B and does not preserve the measure μ.*

Proof. — By hypothesis the probability measures μ and $f_*\mu$ are not equal. According to Corollary 2.4, there exists some $p > 0$ and some simplex of the triangulation $\mathscr{T}_p$ whose closure B_1 is a good ball, and such that $\mu(B_1) \neq \mu(f^{-1}(B_1))$. To fix ideas let us assume that

$$\mu(f^{-1}(B_1)) > \mu(B_1).$$

This implies the same inequality for at least one of the simplices of $\mathscr{T}_{p+1}$ that are included in B_1 ; thus, by induction, we see that we can choose p to be arbitrarily large. Note that we have $\mu(f^{-1}(M \smallsetminus B_1)) < \mu(M \smallsetminus B_1)$. Thus the same reasoning, applied to $M \smallsetminus B_1$, provides a (closed) simplex B_2 of some $\mathscr{T}_{p'}$, disjoint from B_1, such that

$$\mu(f^{-1}(B_2)) < \mu(B_2).$$

Again, by induction, we can assume that $p' = p$ and this is an arbitrarily large integer. In particular B_1 and B_2 are good balls with the same mass. Furthermore, by compactness we can find sequences of such B_1, B_2 that converges respectively to some points x_1, x_2 in M. We choose a ball B' whose interior contains x_1, x_2 and whose $f_*\mu$-measure is less than ε, and balls B_1, B_2 included in the interior of B'.

Since B_1 and B_2 have the same measure, by Corollary 2.2 there exists $\phi \in \mathrm{Homeo}_0(M, \mu)$ supported in B' and sending B_1 onto B_2. Now we consider the element

$$g = f^{-1}\phi f$$

of the group G_f. It has support in the ball $B := f^{-1}(B')$, whose measure is less than ε. It sends the ball $f^{-1}(B_1)$ to the ball $f^{-1}(B_2)$, and we have

$$\mu(f^{-1}(B_1)) > \mu(B_1) = \mu(B_2) > \mu(f^{-1}(B_2)),$$

so that g does not preserve the measure μ, as required by the lemma. $\square$

The next lemma will allow us to find some element of the group G_f that transfers some little mass from one half of some ball to the other half; the key point is that we will be able to choose the value of the transferred mass in some open interval around 0. We need some notations. We subdivide the Euclidean unit ball $B_1(0)$ of $\mathbb{R}^n$ into the half-balls $B_1^- = B_1(0) \cap \{x_1 \leqslant 0\}$ and $B_1^+ = B_1(0) \cap \{x_1 \geqslant 0\}$, where x_1 denotes the first coordinate in $\mathbb{R}^n$. Let Σ be the $(n-1)$-dimensional ball $B_1^- \cap B_1^+$ that separates the half-balls. We consider a given ball B (which is not necessarily a good ball) and some homeomorphism g supported in B. For every topological embedding $\gamma : B_1(0) \to M$

we let $\gamma^{\pm} = \gamma(B_1^{\pm})$; we say that γ is *thin* if $\gamma(\Sigma)$ has measure zero. We now consider the set $\mathscr{I}$ of all the numbers of the type

$$\mu(g(\gamma^+)) - \mu(\gamma^+),$$

where $\gamma(B_1(0)) = B$ and γ is thin.

Lemma 3.2. — *If g does not preserve the measure μ then $\mathscr{I}$ contains an interval $[a^-, a^+]$ with $a^- < 0 < a^+$.*

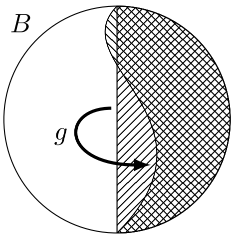

Figure 3.1. $\mu(g(\gamma^+)) < \mu(\gamma^+)$

Proof. — First we want to prove that there exists some $\gamma : B_1(0) \to B$ which is thin and such that $\mu(g(\gamma^+)) \neq \mu(\gamma^+)$. Since g does not preserve the measure μ, we can find some good ball b in the interior of B such that $\mu(b) \neq \mu(g(b))$. To fix ideas we assume that $\mu(b) < \mu(g(b))$. Using the Oxtoby-Ulam Theorem and Corollary 2.2 we can identify (only for the duration of this paragraph) B with a Euclidean ball in $\mathbb{R}^n$, b with another Euclidean ball inside B, and μ with the restriction of the Lebesgue measure on $\mathbb{R}^n$. All our balls are centered at the origin. Let b' be a Euclidean ball slightly greater than b, and T be a thin tube in $B \smallsetminus b'$ connecting the boundary of B and that of b'. There exists a homeomorphism $\gamma : B_1(0) \to B$ such that $\gamma^+ = T \cup b'$. The construction can be done so that the (Lebesgue) measure of γ^+ is arbitrarily close to that of b and the measure of $g(\gamma^+)$ is arbitrarily close to that of $g(b)$, and then we have $\mu(\gamma^+)) < \mu(g(\gamma^+))$, as wanted.

We can find a continuous family $(R_t)_{t\in[0,1]}$ of rotations of $B_1(0)$ such that R_0 is the identity and R_1 is a rotation that exchanges B_1^- and B_1^+. Setting $\gamma_t := \gamma \circ R_t$, we have $\gamma_1^+ = \gamma_0^- = \gamma^-$. Note that it may happen that $\gamma_t(\Sigma)$ has positive measure for some t. To remedy this we consider $\gamma' = \phi \circ \gamma$, where $\phi : B \to B$ is a homeomorphism that fixes $\gamma(\Sigma)$, such that the image under γ' of the Lebesgue measure on $B_1(0)$ is equivalent to the restriction of μ to the ball B, in the sense that both measures share the same measure zero sets; such a ϕ is provided by the Oxtoby-Ulam theorem. This ensures that $\gamma'_t := \gamma' \circ R_t$ is thin for every t. Note that ${\gamma'_0}^{\pm} = \gamma_0^{\pm}$ and ${\gamma'_1}^{\pm} = \gamma_1^{\pm}$. We have

$$\begin{aligned}\mu(g(\gamma_1'^{+})) - \mu(\gamma_1'^{+}) &= \mu(g(\gamma_0'^{-})) - \mu(\gamma_0'^{-}) \\ &= (\mu(B) - \mu(g(\gamma_0'^{+}))) - (\mu(B) - \mu(\gamma_0'^{+})) \\ &= -(\mu(g(\gamma_0'^{+})) - \mu(\gamma_0'^{+})) \neq 0.\end{aligned}$$

Thus the set $\mathscr{I}$ contains the interval

$$\{\mu(g(\gamma_t'^{+})) - \mu(\gamma_t'^{+}), t \in [0,1]\}$$

which contains both a positive and a negative number, as required by the lemma. □

A repeated use of the previous lemma will enable us to transfer some macroscopic amount of mass, getting the following corollary.

Corollary 3.3. — *Let $\gamma_0 : B_1(0) \to M$ be a topological embedding in M with $\mu(\gamma_0(\Sigma)) = 0$, let $B_0 = \gamma_0(B_1(0))$, and let $c > 0$ be strictly less than the measure of γ_0^+. Then there exists some element $h \in G_f$, supported in B_0, such that*

$$\mu(h(\gamma_0^+)) = \mu(\gamma_0^+) - c.$$

In the situation of the corollary we will say that h *transfers a mass c from γ_0^+ to γ_0^-*.

Proof. — Lemma 3.1 provides some element $g \in G_f$ that does not preserve the measure μ, and which is supported on a good ball B whose measure is less than the minimum of $\mu(\gamma_0^+) - c$ and $\mu(\gamma_0^-)$. Then Lemma 3.2 provides some homeomorphism $\gamma : B_1(0) \to B$ which is thin and such that g transfers some mass a from γ^+ to γ^-:

$$\mu(g(\gamma^+)) = \mu(\gamma^+) - a.$$

Since such a number a can be chosen freely in an open interval containing 0, we can assume that $a = c/N$ for some positive integer N.

Let $\gamma' : B_1(0) \to B_0$ be an embedding such that (see Figure 3.2)

(1) $B' := \gamma'(B_1(0))$ is a good ball,

(2) $\gamma'^+ \subset \gamma_0^+$, $\gamma'^- \subset \gamma_0^-$,

(3) $\mu(\gamma'^+) = \mu(\gamma^+)$, $\mu(\gamma'^-) = \mu(\gamma^-)$.

Note that "there is enough room", that is, the third condition is compatible with the second one: indeed the measures of γ^+ and γ^- are less than the measure of B which is less than the measures of γ_0^+ and γ_0^-.

The map $\gamma'\gamma^{-1}$ does not necessarily preserve the measure μ, as is required for the use of Corollary 2.2. But this problem can be solved by post-composing γ' with a first map supported in γ'^+, and with a second map supported in γ'^-, both maps being provided by the Oxtoby-Ulam Theorem, thanks to the equality of total masses given by third item above. Also note that since B, B' are good balls, we can extend γ, γ' to $B_2(0)$. Thus Corollary 2.2 applies and gives a homeomorphism $\Phi_1 \in \mathrm{Homeo}_0(M, \mu)$ such that $\gamma'_{|B_1(0)} = \Phi_1 \circ \gamma_{|B_1(0)}$ (up to changing γ to $\overline{\gamma} = \gamma\sigma$; note that the symmetry σ preserves the Euclidean half-balls B_1^-, B_1^+).

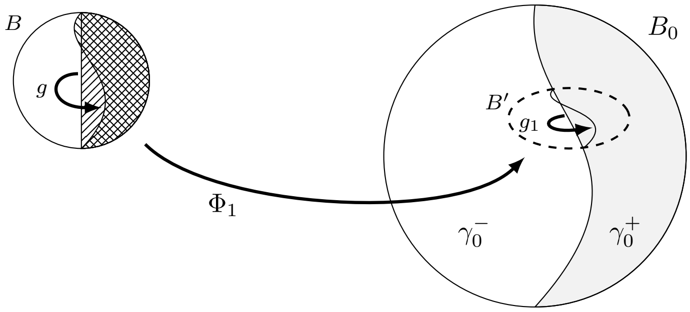

Figure 3.2. g_1 transfers some positive mass from γ_0^+ to γ_0^-

Now the conjugate $g_1 = \Phi_1 g \Phi_1^{-1}$ transfers a mass a from γ_0^+ to γ_0^-:

$$\mu(g_1(\gamma_0^+)) = \mu(\gamma_0^+) - a,$$
$$\mu(g_1(\gamma_0^-)) = \mu(\gamma_0^-) + a.$$

If $N = 1$ the proof is complete. If $N > 1$ we consider $\gamma_1 = g_1 \circ \gamma_0$. Note that both $\mu(\gamma_1^+)$ and $\mu(\gamma_1^-)$ are bigger than $\mu(B)$: there is "still enough room". We repeat the process with γ_1 instead of γ_0. That is, we choose $\Phi_2 \in \mathrm{Homeo}_0(M, \mu)$ sending $\gamma^\pm$ inside $\gamma_1^\pm$ and define $g_2 = \Phi_2 g \Phi_2^{-1}$. This is an element of G_f that transfers a mass a from γ_1^+ to γ_1^-. Then $g_2 g_1$ transfers a mass $2a$ from γ_0^+ to γ_0^-:

$$\begin{aligned}\mu(g_2 g_1(\gamma_0^+)) &= \mu(g_2(\gamma_1^+)) \\ &= \mu(\gamma_1^+) - a \\ &= \mu(g_1(\gamma_0^+)) - a \\ &= \mu(\gamma_0^+) - 2a.\end{aligned}$$

Since we have chosen the measure of B to be less than $\mu(\gamma_0^+) - c$, there is "enough room" to repeat the process N times. We get the final homeomorphism h as a composition of the N homeomorphisms $g_N, \ldots, g_1$. □

4. Proof of the theorem

We consider as before some element $f \in \mathrm{Homeo}_0(M) \smallsetminus \mathrm{Homeo}_0(M, \mu)$. Let g be some other element in $\mathrm{Homeo}_0(M)$. In order to prove the theorem we want to approximate g with some element in the group G_f generated by f and $\mathrm{Homeo}_0(M, \mu)$. We fix a triangulation $\mathscr{T}_0$ for which the $(n-1)$-skeleton has measure zero. The first step of the proof consists in finding an element $g' \in G_f$ satisfying the following property: *for every n-simplex s of $\mathscr{T}_0$, the measure of $g'(s)$ coincides with the measure of $g^{-1}(s)$* (see Figure 4.1). To achieve this, the (very natural) idea is to use Corollary 3.3 to progressively transfer some mass from the simplices s whose mass is larger than the mass of their image under g^{-1}, to those for which the opposite holds.

Here are some details. Given a triangulation $\mathscr{T}$ for which the $(n-1)$-skeleton has measure zero, we choose two n-dimensional simplices s, s' of $\mathscr{T}$, and some positive constant c less than $\mu(s)$; let us explain how to transfer a mass c from s to s'. First assume that s and s' are adjacent. Then we can choose an embedding $\gamma : B_1(0) \to s \cup s'$ with $\gamma(\Sigma) \subset s \cap s'$, $\gamma^+ \subset s$ and $\gamma^- \subset s'$, and we apply Corollary 3.3. Thus we get an element $h \in G_f$, supported in $s \cup s'$, such that $\mu(h(s)) = \mu(s) - c$, and consequently $\mu(h(s')) = \mu(s') + c$. Now consider the general case, when s and s' are not adjacent. Since M is connected, there exists a sequence $s_0 = s, \dots, s_\ell = s'$ of simplices of $\mathscr{T}$ in which two successive elements are adjacent. As described before we can transfer a mass c from s_0 to s_1, then from s_1 to s_2, and so on. Thus by successive adjacent transfers of mass we get some element in $h \in G_f$ that transfers a mass c from s to s'. Note that the masses of all the other elements do not change, that is, $\mu(h(\sigma)) = \mu(\sigma)$ for every simplex σ of $\mathscr{T}$ different from s and s'.

Now we go back to our triangulation $\mathscr{T}_0$, and we construct g' the following way. If each simplex s has the same measure as its inverse image $g^{-1}(s)$ then there is nothing to do. In the opposite case there exists some simplex s of $\mathscr{T}_0$ such that $\mu(s) > \mu(g^{-1}(s))$. We also select some other simplex s' such that $\mu(s') \neq \mu(g^{-1}(s'))$, and we use the previously described construction of a homeomorphism $g_1 \in G_f$ that transfers the mass $\mu(s) - \mu(g^{-1}(s))$ from the simplex s to the simplex s'. After doing so the number of simplices $g_1(s) \in g_{1*}\mathscr{T}_0$ whose mass differs from the mass of $g^{-1}(s)$ has decreased by at least one compared to $\mathscr{T}_0$. We proceed inductively until we get an element $g' \in G_f$ such that $\mu(g'(s)) = \mu(g^{-1}(s))$ for every simplex s in $\mathscr{T}_0$, as wanted for this first step.

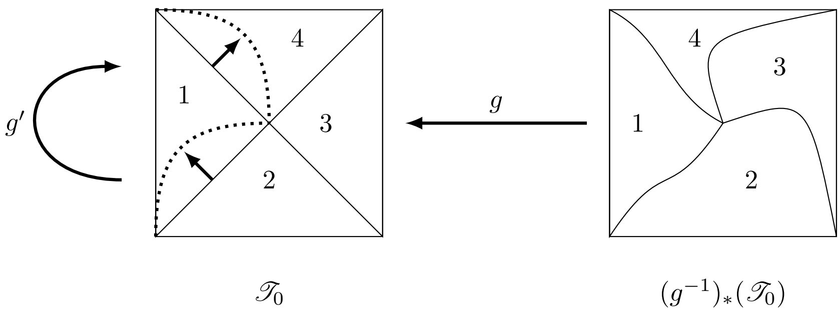

Figure 4.1. First step: the triangle $g^{-1}T_2$ is too big, and $g^{-1}T_4$ is too small. We transfer mass from T_4 to T_2 (via T_1, for instance) to get a homeomorphism g' such that the images of a given triangle under g' and g^{-1} have the same mass

For the second and last step we consider the triangulations $(g^{-1})_*(\mathscr{T}_0)$ and $g'_*(\mathscr{T}_0)$ (see Figure 4.2). The homeomorphism $g'g$ sends the first one to the second one, and each simplex $g^{-1}(s) \in (g^{-1})_*(\mathscr{T}_0)$ has the same measure as its image $g'(s) \in g'_*(\mathscr{T}_0)$. We apply Oxtoby-Ulam theorem independently on each $g'(s)$ to get a homeomorphism

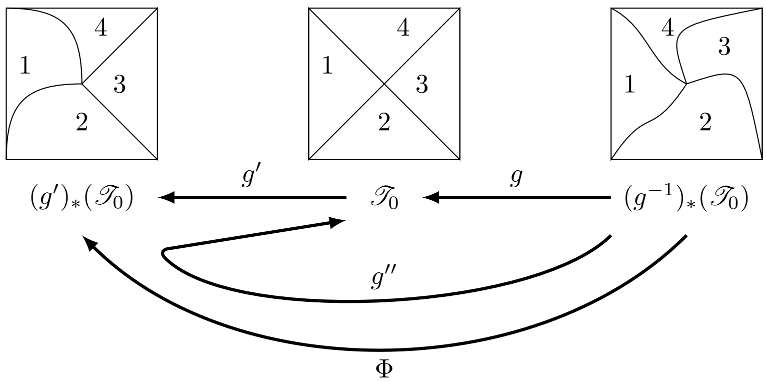

Figure 4.2. Second step: every triangle of $(g^{-1})_*(\mathscr{T}_0)$ has the same image under g and g''

$\Phi_s : g'(s) \to g'(s)$, which is the identity on $\partial g'(s)$, and which sends the measure $(g'g)_*(\mu_{|g^{-1}(s)})$ to the measure $\mu_{|g'(s)}$. The homeomorphism

$$\Phi := \Big(\prod_s \Phi_s\Big) g'g$$

preserves the measure μ. Furthermore by Alexander's trick each Φ_s is isotopic to the identity, thus Φ is isotopic to the identity, and belongs to the group $\mathrm{Homeo}_0(M,\mu)$.

Now the homeomorphism $g'' = g'^{-1}\Phi$ belongs to the group G_f and for each simplex s of the triangulation $\mathscr{T}_0$ we have $g''^{-1}(s) = g^{-1}(s)$. We can have chosen the triangulation $\mathscr{T}_0$ so that each simplex has diameter less than some given ε. Every point x in M belongs to some n-dimensional closed simplex $g^{-1}(s)$ of the triangulation $(g^{-1})_*\mathscr{T}_0$, and since both $g(x)$ and $g''(x)$ belong to s they are at a distance less than ε apart. In other words the uniform distance from g to g'' is less than ε. This proves that g belongs to the closure of G_f, and completes the proof of the theorem. $\square$

5. Concluding remarks

We have proved that the group of volume preserving homeomorphisms is maximal among closed subgroups of $\mathrm{Homeo}_0(M)$. In contrast, it is *not* maximal among *all* subgroups of $\mathrm{Homeo}_0(M)$. Indeed, consider for example the homeomorphisms h for which the image of the volume is equivalent to the volume, or (equivalently) such that for every measurable set E, $h(E)$ has measure zero if and only if E has measure zero. The set of such homeomorphisms constitutes an intermediate subgroup.

Finally, the group $\mathrm{Homeo}_0(\mathbb{S}^2, \mathrm{area})$ is far from being the only maximal closed subgroup. For example, one can prove that the stabilizer of a point is a maximal closed subgroup, and so is the group of homeomorphisms preserving the equator, or some given Cantor set.

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