Arnaud Beauville

Some surfaces with maximal Picard number


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SOME SURFACES WITH MAXIMAL PICARD NUMBER

BY ARNAUD BEAUVILLE

Abstract. — For a smooth complex projective variety, the rank $\rho$ of the Néron-Severi group is bounded by the Hodge number $h^{1,1}$. Varieties with $\rho = h^{1,1}$ have interesting properties, but are rather sparse, particularly in dimension 2. We discuss in this note a number of examples, in particular those constructed from curves with special Jacobians.

Résumé (Quelques surfaces dont le nombre de Picard est maximal). — Le rang $\rho$ du groupe de Néron-Severi d’une variété projective lisse complexe est borné par le nombre de Hodge $h^{1,1}$. Les variétés satisfaisant à $\rho = h^{1,1}$ ont des propriétés intéressantes, mais sont assez rares, particulièrement en dimension 2. Dans cette note nous analysons un certain nombre d’exemples, notamment ceux construits à partir de courbes à jacobienne spéciale.

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1. Introduction

The Picard number of a smooth projective variety $X$ is the rank $\rho$ of the Néron-Severi group that is, the group of classes of divisors in $H^2(X, \mathbb{Z})$. It is bounded by the Hodge number $h^{1,1} := \dim H^1(X, \Omega^1_X)$. We are interested here in varieties with maximal Picard number $\rho = h^{1,1}$. As we will see in §2, there are many examples of such varieties in dimension $\geq 3$, so we will focus on the case of surfaces.
Apart from the well understood case of K3 and abelian surfaces, the quantity of known examples is remarkably small. In [Per82] Persson showed that some families of double coverings of rational surfaces contain surfaces with maximal Picard number (see Section 6.4 below); some scattered examples have appeared since then. We will review them in this note and examine in particular when the product of two curves has maximal Picard number – this provides some examples, unfortunately also quite sparse.

2. Generalities

Let \( X \) be a smooth projective variety over \( \mathbb{C} \). The Néron-Severi group \( \text{NS}(X) \) is the subgroup of algebraic classes in \( H^2(X, \mathbb{Z}) \); its rank \( \rho \) is the Picard number of \( X \). The natural map \( \text{NS}(X) \otimes \mathbb{C} \to H^2(X, \mathbb{C}) \) is injective and its image is contained in \( H^{1,1} \), hence \( \rho \leq h^{1,1} \).

**Proposition 1.** — The following conditions are equivalent:

(i) \( \rho = h^{1,1} \);

(ii) The map \( \text{NS}(X) \otimes \mathbb{C} \to H^{1,1} \) is bijective;

(iii) The subspace \( H^{1,1} \) of \( H^2(X, \mathbb{C}) \) is defined over \( \mathbb{Q} \).

(iv) The subspace \( H^2,0 \oplus H^0,2 \) of \( H^2(X, \mathbb{C}) \) is defined over \( \mathbb{Q} \).

**Proof.** — The equivalence of (iii) and (iv) follows from the fact that \( H^2,0 \oplus H^0,2 \) is the orthogonal of \( H^{1,1} \) for the scalar product on \( H^2(X, \mathbb{C}) \) associated to an ample class. The rest is clear. \( \square \)

When \( X \) satisfies these equivalent properties we will say for short that \( X \) is \( \rho \)-maximal (one finds the terms singular, exceptional or extremal in the literature).

**Remarks**

(1) A variety with \( H^{2,0} = 0 \) is \( \rho \)-maximal. We will implicitly exclude this trivial case in the discussion below.

(2) Let \( X, Y \) be two \( \rho \)-maximal varieties, with \( H^1(Y, \mathbb{C}) = 0 \). Then \( X \times Y \) is \( \rho \)-maximal. For instance \( X \times \mathbb{P}^n \) is \( \rho \)-maximal, and \( Y \times C \) is \( \rho \)-maximal for any curve \( C \).

(3) Let \( Y \) be a submanifold of \( X \); if \( X \) is \( \rho \)-maximal and the restriction map \( H^2(X, \mathbb{C}) \to H^2(Y, \mathbb{C}) \) is bijective, \( Y \) is \( \rho \)-maximal. By the Lefschetz theorem, the latter condition is realized if \( Y \) is a complete intersection of smooth ample divisors in \( X \), of dimension \( \geq 3 \). Together with Remark 2, this gives many examples of \( \rho \)-maximal varieties of dimension \( \geq 3 \); thus we will focus on finding \( \rho \)-maximal surfaces.

**Proposition 2.** — Let \( \pi : X \dashrightarrow Y \) be a rational map of smooth projective varieties.

(a) If \( \pi^* : H^{2,0}(Y) \to H^{2,0}(X) \) is injective (in particular if \( \pi \) is dominant), and \( X \) is \( \rho \)-maximal, so is \( Y \).

(b) If \( \pi^* : H^{2,0}(Y) \to H^{2,0}(X) \) is surjective and \( Y \) is \( \rho \)-maximal, so is \( X \).
Some surfaces with maximal Picard number

Note that since $\pi$ is defined on an open subset $U \subset X$ with codim$(X \setminus U) \geq 2$, the pull back map $\pi^* : H^2(Y, \mathbb{C}) \to H^2(U, \mathbb{C}) \cong H^2(X, \mathbb{C})$ is well defined.

**Proof.** — Hironaka’s theorem provides a diagram

$$
\begin{array}{c}
\hat{X} \\
\downarrow b \downarrow \hat{\pi} \\
X \xrightarrow{\pi} Y
\end{array}
$$

where $\hat{\pi}$ is a morphism, and $b$ is a composition of blowing-ups with smooth centers. Then $b^* : H^{2,0}(X) \to H^{2,0}(\hat{X})$ is bijective, and $\hat{X}$ is $\rho$-maximal if and only if $X$ is $\rho$-maximal; so replacing $\pi$ by $\hat{\pi}$ we may assume that $\pi$ is a morphism.

(a) Let $V := (\pi^*)^{-1}(\text{NS}(X) \otimes \mathbb{Q})$. We have

$$V \otimes \mathbb{C} = (\pi^*)^{-1}(\text{NS}(X) \otimes \mathbb{C}) = (\pi^*)^{-1}(H^{1,1}(X)) = H^{1,1}(Y)$$

(the last equality holds because $\pi^*$ is injective on $H^{2,0}(Y)$ and $H^{0,2}(Y)$), hence $Y$ is $\rho$-maximal.

(b) Let $W$ be the $\mathbb{Q}$-vector subspace of $H^2(Y, \mathbb{Q})$ such that

$$W \otimes \mathbb{C} = H^{2,0}(Y) \oplus H^{0,2}(Y).$$

Then $\pi^*W$ is a $\mathbb{Q}$-vector subspace of $H^2(X, \mathbb{Q})$, and

$$(\pi^*W) \otimes \mathbb{C} = \pi^*(W \otimes \mathbb{C}) = \pi^*H^{2,0}(Y) \oplus \pi^*H^{0,2}(Y) = H^{2,0}(X) \oplus H^{0,2}(X),$$

so $X$ is $\rho$-maximal. □

3. Abelian varieties

There is a nice characterization of $\rho$-maximal abelian varieties ([Kat75], [Lan75]):

**Proposition 3.** — Let $A$ be an abelian variety of dimension $g$. We have

$$\text{rk}_\mathbb{Z} \text{End}(A) \leq 2g^2.$$  

The following conditions are equivalent:

(i) $A$ is $\rho$-maximal;

(ii) $\text{rk}_\mathbb{Z} \text{End}(A) = 2g^2$;

(iii) $A$ is isogenous to $E^g$, where $E$ is an elliptic curve with complex multiplication.

(iv) $A$ is isomorphic to a product of mutually isogenous elliptic curves with complex multiplication.

(The equivalence of (i), (ii) and (iii) follows easily from Lemma 1 below; the only delicate point is (iii) $\Rightarrow$ (iv), which we will not use.)

Coming back to the surface case, suppose that our abelian variety $A$ contains a surface $S$ such that the restriction map $H^{2,0}(A) \to H^{2,0}(S)$ is surjective. Then $S$ is $\rho$-maximal if $A$ is $\rho$-maximal (Proposition 2(b)). Unfortunately this situation seems to be rather rare. We will discuss below (Proposition 6) the case of $\text{Sym}^2 C$ for a curve $C$. Another interesting example is the **Fano surface** $F_X$ parametrizing the lines...
contained in a smooth cubic threefold $X$, embedded in the intermediate Jacobian $JX$ [CG72]. There are some cases in which $JX$ is known to be $\rho$-maximal:

**Proposition 4**

(a) For $\lambda \in \mathbb{C}$, $\lambda^3 \neq 1$, let $X_\lambda$ (resp. $E_\lambda$) be the cubic in $\mathbb{P}^4$ (resp. $\mathbb{P}^2$) defined by

$$X_\lambda : X^3 + Y^3 + Z^3 - 3\lambda X Y Z = 0, \quad E_\lambda : X^3 + Y^3 + Z^3 - 3\lambda X Y Z = 0.$$ 

If $E_\lambda$ is isogenous to $E_0$, $JX_\lambda$ and $F_{X_\lambda}$ are $\rho$-maximal. The set of $\lambda \in \mathbb{C}$ for which this happens is countably infinite.

(b) Let $X \subset \mathbb{P}^4$ be the Klein cubic threefold $\sum_{i \in \mathbb{Z}/3} X_i^2 X_{i+1} = 0$. Then $JX$ and $F_X$ are $\rho$-maximal.

**Proof.** — Part (a) is due to Roulleau [Rou11], who proves that $JX_\lambda$ (for any $\lambda$) is isogenous to $E_\lambda^0 \times E_\lambda^2$. Since the family $(E_\lambda)_{\lambda \in \mathbb{C}}$ is not constant, there is a countably infinite set of $\lambda \in \mathbb{C}$ for which $E_\lambda$ is isogenous to $E_0$, hence $JX_\lambda$ and therefore $F_{X_\lambda}$ are $\rho$-maximal.

Part (b) follows from a result of Adler [Adl81], who proves that $JX$ is isogenous (actually isomorphic) to $E^5$, where $E$ is the elliptic curve whose endomorphism ring is the ring of integers of $\mathbb{Q}(\sqrt{-1})$ (see also [Rou09] for a precise description of the group $\text{NS}(X)$).

\[ \square \]

4. **Products of curves**

**Proposition 5.** — Let $C, C'$ be two smooth projective curves, of genus $g$ and $g'$ respectively. The following conditions are equivalent:

(i) The surface $C \times C'$ is $\rho$-maximal;

(ii) There exists an elliptic curve $E$ with complex multiplication such that $JC$ is isogenous to $E^g$ and $JC$ to $E^{g'}$.

**Proof.** — Let $p, p'$ be the projections from $C \times C'$ to $C$ and $C'$. We have

$$H^{1,1}(C \times C') = p^*H^2(C, \mathbb{C}) \oplus p'^*H^2(C', \mathbb{C}) \oplus (p^*H^{1,0}(C) \oplus p'^*H^{0,1}(C'))$$

$$\oplus (p^*H^{0,1}(C) \oplus p'^*H^{1,0}(C')),$$

hence $h^{1,1}(C \times C') = 2gg' + 2$. On the other hand we have

$$\text{NS}(C \times C') = p^*\text{NS}(C) \oplus p'^*\text{NS}(C') \oplus \text{Hom}(JC, JC'),$$

([LB92], Th. 11.5.1), hence $C \times C'$ is $\rho$-maximal if and only if $\text{rk Hom}(JC, JC') = 2gg'$. Thus the Proposition follows from the following (well-known) lemma:

**Lemma 1.** — Let $A$ and $B$ be two abelian varieties, of dimension $a$ and $b$ respectively. The $\mathbb{Z}$-module $\text{Hom}(A, B)$ has rank $\leq 2ab$; equality holds if and only if there exists an elliptic curve $E$ with complex multiplication such that $A$ is isogenous to $E^a$ and $B$ to $E^b$.
Proof: — There exist simple abelian varieties $A_1, \ldots, A_s$, with distinct isogeny classes, and nonnegative integers $p_1, \ldots, p_s, q_1, \ldots, q_s$ such that $A$ is isogenous to $A_{p_1}^{\times} \times \cdots \times A_{p_s}^{\times}$ and $B$ to $A_{q_1}^{\times} \times \cdots \times A_{q_s}^{\times}$. Then

$$\text{Hom}(A, B) \otimes \mathbb{Q} \cong M_{p_1, q_1}(K_1) \times \cdots \times M_{p_s, q_s}(K_s),$$

where $K_i$ is the (possibly skew) field $\text{End}(A_i) \otimes \mathbb{Q}$. Put $a_i := \dim A_i$. Since $K_i$ acts on $H^1(A_i, \mathbb{Q})$ we have $\dim_{\mathbb{Q}} K_i \leq b_1(A_i) = 2a_i$, hence

$$\text{rk} \, \text{Hom}(A, B) \leq \sum_i 2p_i q_i a_i \leq 2 \left( \sum_i p_i a_i \right) \left( \sum_i q_i a_i \right) = 2ab.$$

The last inequality is strict unless $s = a_1 = 1$, in which case the first one is strict unless $\dim_{\mathbb{Q}} K_1 = 2$. The lemma, and therefore the Proposition, follow. \(\square\)

The most interesting case occurs when $C = C'$. Then:

**Proposition 6.** — Let $C$ be a smooth projective curve. The following conditions are equivalent:

(i) The Jacobian $JC$ is $\rho$-maximal;

(ii) The surface $C \times C$ is $\rho$-maximal;

(iii) The symmetric square $\text{Sym}^2 C$ is $\rho$-maximal.

Proof: — The equivalence of (i) and (ii) follows from Proposition 5. The Abel-Jacobi map $\text{Sym}^2 C \to JC$ induces an isomorphism

$$H^{2,0}(JC) \cong \wedge^2 H^0(C, K_C) \xrightarrow{\sim} H^{2,0}(\text{Sym}^2 C),$$

thus (i) and (iii) are equivalent by Proposition 2. \(\square\)

When the equivalent conditions of Proposition 6 hold, we will say that $C$ has **maximal correspondences** (the group $\text{End}(JC)$ is often called the group of divisorial correspondences of $C$).

By Proposition 3 the Jacobian $JC$ is then isomorphic to a product of isogenous elliptic curves with complex multiplication. Though we know very few examples of such curves, we will give below some examples with $g = 4$ or 10.

For $g = 2$ or 3, there is a countably infinite set of curves with maximal correspondences ([HN65], [Hof91]). The point is that any indecomposable principally polarized abelian variety of dimension 2 or 3 is a Jacobian; thus it suffices to construct an indecomposable principal polarization on $E^g$, where $E$ is an elliptic curve with complex multiplication, and this is easily translated into a problem about hermitian forms of rank $g$ on certain rings of quadratic integers.

This approach works only for $g = 2$ or 3; moreover it does not give an explicit description of the curves. Another method is by using automorphism groups, with the help of the following easy lemma:
**Lemma 2.** — Let $G$ be a finite group of automorphisms of $C$, and let $H^0(C,K_C) = \bigoplus_{i \in I} V_i$ be a decomposition of the $G$-module $H^0(C,K_C)$ into irreducible representations. Assume that there exists an elliptic curve $E$ and for each $i \in I$, a nontrivial map $\pi_i : C \to E$ such that $\pi_i^* H^0(E,K_E) \subset V_i$. Then $JC$ is isogenous to $E^g$.

In particular if $H^0(C,K_C)$ is an irreducible $G$-module and $C$ admits a map onto an elliptic curve $E$, then $JC$ is isogenous to $E^g$.

**Proof.** — Let $\eta$ be a generator of $H^0(E,K_E)$. Let $i \in I$; the forms $g^* \pi_i^* \eta$ for $g \in G$ generate $V_i$, hence there exists a subset $A_i$ of $G$ such that the forms $g^* \pi_i^* \eta$ for $g \in A_i$ form a basis of $V_i$.

Put $\Pi_i = (g \circ \pi_i)_{g \in A_i} : C \to E^{A_i}$, and $\Pi = (\Pi_i)_{i \in I} : C \to E^g$. By construction $\Pi^* : H^0(E^g,\Omega_{E^g}) \to H^0(C,K_C)$ is an isomorphism. Therefore the map $JC \to E^g$ deduced from $\Pi$ is an isogeny. \hfill \Box

In the examples which follow, and in the rest of the paper, we put $\omega := e^{2\pi i/3}$.

**Example 1.** — We consider the family $(C_t)$ of genus 2 curves given by $y^2 = x^6 + tx^3 + 1$, for $t \in \mathbb{C} \setminus \{\pm 2\}$. It admits the automorphisms

$$\tau : (x, y) \mapsto \left(\frac{1}{x}, \frac{y}{x^3}\right) \quad \text{and} \quad \psi : (x, y) \mapsto (\omega x, y).$$

The forms $dx/y$ and $xdx/y$ are eigenvectors for $\psi$ and are exchanged (up to sign) by $\tau$; it follows that the action of the group generated by $\psi$ and $\tau$ on $H^0(C_t,K_{C_t})$ is irreducible.

Let $E_t$ be the elliptic curve defined by $v^2 = (u + 2)(u^3 - 3u + t)$; the curve $C_t$ maps onto $E_t$ by

$$(x, y) \mapsto \left(x + \frac{1}{x}, \frac{y(x + 1)}{x^2}\right).$$

By Lemma 2 $JC_t$ is isogenous to $E_t^g$. Since the $j$-invariant of $E_t$ is a non-constant function of $t$, there is a countably infinite set of $t \in \mathbb{C}$ for which $E_t$ has complex multiplication, hence $C_t$ has maximal correspondences.

**Example 2.** — Let $C$ be the genus 2 curve $y^2 = x(x^4 - 1)$; its automorphism group is a central extension of $\mathfrak{S}_4$ by the hyperelliptic involution $\sigma$ ([LB92], 11.7); its action on $H^0(C,K_C)$ is irreducible.

Let $E$ be the elliptic curve $E : v^2 = u(u+1)(u-2\alpha)$, with $\alpha = 1 - \sqrt{2}$. The curve $C$ maps to $E$ by

$$(x, y) \mapsto \left(\frac{x^2 + 1}{x - 1}, \frac{y(x - \alpha)}{(x - 1)^2}\right).$$

The $j$-invariant of $E$ is 8000, so $E$ is the elliptic curve $\mathbb{C}/\mathbb{Z}[\sqrt{2}]$ ([Sil94], Prop. 2.3.1).

**Example 3** (The $\mathfrak{S}_4$-invariant quartic curves). — Consider the standard representation of $\mathfrak{S}_4$ on $\mathbb{C}^3$. It is convenient to view $\mathfrak{S}_4$ as the semi-direct product $(\mathbb{Z}/2)^2 \rtimes \mathfrak{S}_3$,
with \( \mathfrak{S}_3 \) (resp. \((\mathbb{Z}/2)^2\)) acting on \( \mathbb{C}^3 \) by permutation (resp. change of sign) of the basis vectors. The quartic forms invariant under this representation form the pencil

\[(C_t)_{t \in \mathbb{P}^1}: x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2) = 0.\]

According to [DK93], this pencil was known to Ciani. It contains the Fermat quartic \((t = 0)\) and the Klein quartic \((t = \frac{3}{4}(1 \pm i \sqrt{7}))\).

Let us take \( t \notin \{2, -1, -2, \infty\} \); then \( C_t \) is smooth. The action of \( \mathfrak{S}_4 \) on \( H^0(C_t, K) \), given by the standard representation, is irreducible. Moreover the involution \( x \mapsto -x \) has 4 fixed points, hence the quotient curve \( E_t \) has genus 1. It is given by the degree 4 equation

\[ u^2 + tu(y^2 + z^2) + y^4 + z^4 + ty^2z^2 = 0 \]

in the weighted projective space \( \mathbb{P}(2,1,1) \). Thus \( E_t \) is a double covering of \( \mathbb{P}^1 \) branched along the zeroes of the polynomial \((t + 2)(y^4 + z^4) + 2ty^2z^2\). The cross-ratio of these zeroes is \(- (t + 1)\), so \( E_t \) is the elliptic curve \( y^2 = x(x - 1)(x + t + 1) \). By Lemma 2 \( JC_t \) is isogenous to \( E_t^3 \). For a countably infinite set of \( t \) the curve \( E_t \) has complex multiplication, thus \( C_t \) has maximal correspondences. For \( t = 0 \) we recover the well known fact that the Jacobian of the Fermat quartic curve is isogenous to \((\mathbb{C}/\mathbb{Z}[i])^3\).

**Example 4.** — Consider the genus 3 hyperelliptic curve \( H: y^2 = x(x^6 + 1) \). The space \( H^0(H, K_H) \) is spanned by \( dx/y, xdx/y, x^2dx/y \). This is a basis of eigenvectors for the automorphism \( \tau : (x, y) \mapsto (\omega x, \omega^2 y) \). On the other hand the involution \( \sigma : (x, y) \mapsto (1/x, -y/x^4) \) exchanges \( dx/y \) and \( x^2dx/y \), hence the summands of the decomposition

\[ H^0(H, K_H) = \left\langle \frac{dx}{y}, x^3 \frac{dx}{y} \right\rangle \oplus \left\langle x \frac{dx}{y} \right\rangle \]

are irreducible under the group \( \mathfrak{S}_3 \) generated by \( \sigma \) and \( \tau \).

Let \( E_i \) be the elliptic curve \( u^2 = u^3 + u \), with endomorphism ring \( \mathbb{Z}[i] \). Consider the maps \( f \) and \( g \) from \( H \) to \( E_i \) given by

\[ f(x, y) = (x^2, xy) \quad g(x, y) = \left( \lambda^2 \left(x + \frac{1}{x}\right), \frac{\lambda^3 y}{x^2} \right) \quad \text{with} \quad \lambda^{-4} = -3. \]

We have

\[ f^* \frac{du}{v} = \frac{2x - 1}{y} \quad \text{and} \quad g^* \frac{du}{v} = \lambda^{-1} (x^2 - 1) \frac{dx}{y}. \]

Thus we can apply Lemma 2, and we find that \( JH \) is isogenous to \( E_i^3 \).

Thus \( JH \) is isogenous to the Jacobian of the Fermat quartic \( F_4 \) (Example 3). In particular we see that the surface \( H \times F_4 \) is \( \rho \)-maximal.

We now arrive to our main example in higher genus. Recall that we put \( \omega = e^{2\pi i/3} \).

**Proposition 7.** — The Fermat sextic curve \( C_6: X^6 + Y^6 + Z^6 = 0 \) has maximal correspondences. Its Jacobian \( JC_6 \) is isogenous to \( E_{10}^{10} \), where \( E_{10} \) is the elliptic curve \( \mathbb{C}/\mathbb{Z}[\omega] \).
The first part can be deduced from the general recipe given by Shioda to compute the Picard number of $C_d \times C_d$ for any $d$ [Shi81]. Let us give an elementary proof. Let $G := T \rtimes \mathfrak{S}_3$, where $\mathfrak{S}_3$ acts on $\mathbb{C}^3$ by permutation of the coordinates and $T$ is the group of diagonal matrices $t$ with $t^6 = 1$.

Let $\Omega = \frac{X \, dY - Y \, dX}{Z^5} = \frac{Y \, dZ - Z \, dY}{X^5} = \frac{Z \, dX - X \, dZ}{Y^5} \in H^0(C, K_C(-3))$.

A basis of eigenvectors for the action of $T$ on $H^0(C_6, K)$ is given by the forms $X^a Y^b Z^c \Omega$, with $a + b + c = 3$; using the action of $\mathfrak{S}_3$ we get a decomposition into irreducible components:

\[ H^0(C_6, K) = V_{3,0,0} \oplus V_{2,1,0} \oplus V_{1,1,1}, \]

where $V_{a,b,c}$ is spanned by the forms $X^a Y^b Z^c \Omega$ with $\{a, b, c\} = \{\alpha, \beta, \gamma\}$.

Let us use affine coordinates $x = X/Z$, $y = Y/Z$ on $C_6$. We consider the following maps from $C_6$ onto $E_0$: $v^2 = u^3 - 1$:

\[ f(x, y) = (-x^2, y^3), \quad g(x, y) = \left(2^{-2/3} x^{-2} y^4, \frac{1}{2}(x^3 - x^{-3})\right); \]

and, using for $E_0$ the equation $\xi^3 + \eta^3 + 1 = 0$, $h(x, y) = (x^2, y^2)$.

We have

\[ f^* \frac{du}{v} = -\frac{2x \, dx}{y^5} = -2XY^2 \Omega \in V_{2,1,0}, \]
\[ g^* \frac{du}{v} = -2^{4/3}Y^3 \Omega \in V_{3,0,0}, \]
\[ h^* \frac{d\xi}{\eta^2} = 2XYZ \Omega \in V_{1,1,1}, \]

so the Proposition follows from Lemma 2.

By Proposition 2 every quotient of $C_6$ has again maximal correspondences. There are four such quotient which have genus 4:

- The quotient by an involution $\alpha \in T$, which we may take to be $\alpha : (X, Y, Z) \mapsto (X, Y, -Z)$. The canonical model of $C_6/\alpha$ is the image of $C_6$ by the map

\[ (X, Y, Z) \mapsto (X^2, XY, Y^2, Z^2); \]

its equations in $\mathbb{P}^3$ are $xz - y^2 = x^3 + z^3 + t^3 = 0$. Projecting onto the conic $xz - y^2 = 0$ realizes $C_6/\alpha$ as the cyclic triple covering $u^3 = u^6 + 1$ of $\mathbb{P}^1$.

- The quotient by an involution $\beta \in \mathfrak{S}_3$, say $\beta : (X, Y, Z) \mapsto (Y, X, Z)$. The canonical model of $C_6/\beta$ is the image of $C_6$ by the map

\[ (X, Y, Z) \mapsto ((X + Y)^2, Z(X + Y), Z^2, XY); \]

its equations are $xz - y^2 = x(x - 3t)^2 + z^3 - 2t^3 = 0$.

Since the quadric containing their canonical model is singular, the two genus 4 curves $C_6/\alpha$ and $C_6/\beta$ have a unique $g^3_1$. The associated triple covering $C_6/\alpha \to \mathbb{P}^1$ is cyclic, while the corresponding covering $C_6/\beta \to \mathbb{P}^1$ is not. Therefore the two curves are not isomorphic.
• The quotient by an element of order 3 of $T$ acting freely, say $\gamma : (X, Y, Z) \mapsto (X, \omega Y, \omega^2 Z)$. The canonical model of $C_6/\gamma$ is the image of $C_6$ by the map

$$(X, Y, Z) \mapsto (X^3, Y^3, Z^3, XYZ);$$

its equations are $x^2 + y^2 + z^2 = t^3 - xyz = 0$. Projecting onto the conic $x^2 + y^2 + z^2 = 0$ realizes $C_6/\gamma$ as the cyclic triple covering $v^3 = u(u^4 - 1)$ of $\mathbb{P}^3$; thus $C_6/\gamma$ is not isomorphic to $C_6/\alpha$ or $C_6/\beta$.

• The quotient by an element of order 3 of $S_3$ acting freely, say $\delta : (X, Y, Z) \mapsto (Y, Z, X)$. The canonical model of $C_6/\delta$ is the image of $C_6$ by the map

$$(X, Y, Z) \mapsto (X^3 + Y^3 + Z^3, XYZ, X^2Y + Y^2Z + Z^2X, XY^2 + YZ^2 + ZX^2).$$

It is contained in the smooth quadric $(x+y)^2 + 5y^2 - 2zt = 0$, so $C_6/\delta$ is not isomorphic to any of the 3 previous curves.

Thus we have found four non-isomorphic curves of genus 4 with Jacobian isogenous to $E^4_4$. The product of any two of these curves is a $\rho$-maximal surface.

Corollary 1. — The Fermat sextic surface $S_6 : X^6 + Y^6 + Z^6 + T^6 = 0$ is $\rho$-maximal.

Proof. — This follows from Propositions 7, 2 and Shioda’s trick: there exists a rational dominant map $\pi : C_6 \times C_6 \dashrightarrow S_6$, given by

$$\pi((X,Y,Z),(X',Y',Z')) = (XZ',YZ',ixX',iyY').$$

□

Remark 4. — Since the Fermat plane quartic has maximal correspondences (Example 2), the same argument gives the classical fact that the Fermat quartic surface is $\rho$-maximal. It follows from the explicit formula for $\rho(S_d)$ given in [Aok83] that $S_d$ is $\rho$-maximal (for $d \geq 4$) only for $d = 4$ and 6.

Again every quotient of the Fermat sextic is $\rho$-maximal. For instance, the quotient of $S_6$ by the automorphism $(X, Y, Z, T) \mapsto (X, Y, Z, \omega T)$ is the double covering of $\mathbb{P}^2$ branched along $C_6$: it is a $\rho$-maximal K3 surface. The quotient of $S_6$ by the involution $(X, Y, Z, T) \mapsto (X, Y, -Z, -T)$ is given in $\mathbb{P}^5$ by the equations

$$y^2 - xz = v^2 - uw = x^3 + z^3 + u^3 + w^3 = 0;$$

it is a complete intersection of degrees $(2, 2, 3)$, with 12 ordinary nodes. Other quotients have $p_g$ equal to 2, 3, 4 or 6.

5. Quotients of self-products of curves

The method of the previous section may sometimes allow to prove that certain quotients of a product $C \times C$ have maximal Picard number. Since we have very few examples we will refrain from giving a general statement and contend ourselves with one significant example.

Let $C$ be the curve in $\mathbb{P}^4$ defined by

$$u^2 = xy, \quad v^2 = x^2 - y^2, \quad w^2 = x^2 + y^2.$$
It is isomorphic to the modular curve $X(8)$ [FSM13]. Let $\Gamma \subset \text{PGL}(5, \mathbb{C})$ be the subgroup of diagonal elements changing an even number of signs of $u, v, w$; $\Gamma$ is isomorphic to $(\mathbb{Z}/2)^2$ and acts freely on $C$.

**Proposition 8**

(a) $JC$ is isogenous to $E_i^3 \times E_{\sqrt{-2}}^2$, where $E_0 = \mathbb{C}/\mathbb{Z}[\alpha]$ for $\alpha = i$ or $\sqrt{-2}$.

(b) The surface $(C \times C)/\Gamma$ is $\rho$-maximal.

**Proof**

(a) The form $\Omega := (xdy - ydx)/uvw$ generates $H^0(C, K_C(-1))$, and is $\Gamma$-invariant; thus multiplication by $\Omega$ induces a $\Gamma$-equivariant isomorphism

$$H^0(\mathbb{P}^4, O_{\mathbb{P}^4}(1)) \xrightarrow{\sim} H^0(C, K_C).$$

Let $V$ and $L$ be the subspaces of $H^0(C, K_C)$ corresponding to $\langle u, v, w \rangle$ and $\langle x, y \rangle$. The projection $\langle u, v, w, x, y \rangle \mapsto \langle u, v, w \rangle$ maps $C$ onto the quartic curve $F: 4u^4 + v^4 - w^4 = 0$; the induced map $f: C \to F$ identifies $F$ with the quotient of $C$ by the involution $(u, v, w, x, y) \mapsto (u, v, w, -x, -y)$, and we have $f^* H^0(F, K_F) = V$.

The quotient curve $H := C/\Gamma$ is the genus 2 curve $z^2 = t(t^4 - 1)$ [Bea13]. The pull-back of $H^0(H, K_H)$ is the subspace invariant under $\Gamma$, that is $L$. Thus $JC$ is isogenous to $JF \times JH$. From examples 1 and 2 of §4 we conclude that $JC$ is isogenous to $E_i^3 \times E_{\sqrt{-2}}^2$.

(b) We have $\Gamma$-equivariant isomorphisms

$$H^{1,1}(C \times C) = H^2(C, \mathbb{C}) \oplus H^2(C, \mathbb{C}) \oplus (H^{1,0} \boxtimes H^{0,1}) \oplus (H^{0,1} \boxtimes H^{1,0})$$

$$= \mathbb{C}^2 \oplus \text{End}(H^0(C, K_C))^\oplus 2$$

(where $\Gamma$ acts trivially on $\mathbb{C}^2$), hence

$$H^{1,1}((C \times C)/\Gamma) = \mathbb{C}^2 \oplus \text{End}_\Gamma(H^0(C, K_C))^\oplus 2.$$
6. Other examples

6.1. Elliptic modular surfaces. — Let $\Gamma$ be a finite index subgroup of $\text{SL}_2(\mathbb{Z})$ such that $-I \notin \Gamma$. The group $\text{SL}_2(\mathbb{Z})$ acts on the Poincaré upper half-plane $\mathbb{H}$; let $\Delta_\Gamma$ be the compactification of the Riemann surface $\mathbb{H}/\Gamma$. The universal elliptic curve over $\mathbb{H}$ descends to $\mathbb{H}/\Gamma$, and extends to a smooth projective surface $B_\Gamma$ over $\Delta_\Gamma$, the \textit{elliptic modular surface} attached to $\Gamma$. In [Shi69] Shioda proves that $B_\Gamma$ is $\rho$-maximal.\(^{(1)}\)

Now take $\Gamma = \Gamma(5)$, the kernel of the reduction map $\text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/5)$. In [Liv81] Livné constructed a $\mathbb{Z}/5$-covering $X \to B_{\Gamma(5)}$, branched along the sum of the 25 5-torsion sections of $B_{\Gamma(5)}$. The surface $X$ satisfies $c_1^2 = 3c_2 (= 225)$, hence it is a ball quotient and therefore rigid. By analyzing the action of $\mathbb{Z}/5$ on $H^{1,1}(X)$ Livné shows that $H^{1,1}(X)$ is not defined over $\mathbb{Q}$, hence $X$ \textit{is not} $\rho$-maximal. This seems to be the only known example of a surface which cannot be deformed to a $\rho$-maximal surface.

6.2. Surfaces with $p_g = K^2 = 1$. — The minimal surfaces with $p_g = K^2 = 1$ have been studied by Catanese [Cat79] and Todorov [Tod80]. Their canonical model is a complete intersection of type $(6,6)$ in the weighted projective space $\mathbb{P}(1,2,2,3,3)$. The moduli space $\mathcal{M}$ is smooth of dimension 18.

**Proposition 9.** — \textit{The $\rho$-maximal surfaces are dense in $\mathcal{M}$.}

\textbf{Proof.} — We can replace $\mathcal{M}$ by the Zariski open subset $\mathcal{M}_a$ parametrizing surfaces with ample canonical bundle. Let $S \in \mathcal{M}_a$, and let $f : S \to (B, o)$ be a local versal deformation of $S$, so that $S \cong S_0$. Let $L$ be the lattice $H^2(S, \mathbb{Z})$, and $k \in L$ the class of $K_S$. We may assume that $B$ is simply connected and fix an isomorphism of local systems $R^2f_*\mathbb{Z} \cong L_B$, compatible with the cup-product and mapping the canonical class $[K_S/B]$ onto $k$. This induces for each $b \in B$ an isometry $\varphi_b : H^2(S_b, \mathbb{C}) \cong L_C$, which maps $H^{2,0}(S_b)$ onto a line in $L_C$; the corresponding point $\varphi(b)$ of $\mathbb{P}(L_C)$ is the period of $S_b$. It belongs to the complex manifold

$$\Omega := \{ [x] \in \mathbb{P}(L_C) \mid x^2 = 0, \ x \cdot k = 0, \ x \cdot \bar{x} > 0 \}.$$ 

Associating to $x \in \Omega$ the real 2-plane $P_x := \langle \text{Re}(x), \text{Im}(x) \rangle \subseteq L_R$ defines an isomorphism of $\Omega$ onto the Grassmannian of positive oriented 2-planes in $L_R$.

The key point is that the image of the \textit{period map} $\varphi : B \to \Omega$ is open [Cat79]. Thus we can find $b$ arbitrarily close to $o$ such that the 2-plane $P_b$ is defined over $\mathbb{Q}$, hence $H^{2,0}(S_b) \oplus H^{0,2}(S_b) = P_b \oplus \mathbb{R} \mathbb{C}$ is defined over $\mathbb{Q}$.

\textbf{Remark 5.} — The proof applies to all surfaces with $p_g = 1$ for which the image of the period map is open (for instance to K3 surfaces); unfortunately this seems to be a rather exceptional situation.

\(^{(1)}\) I am indebted to I. Dolgachev and B. Totaro for pointing out this reference.
6.3. Todorov surfaces. — In [Tod81] Todorov constructed a series of regular surfaces with \( p_g = 1, 2 \leq K^2 \leq 8 \), which provide counter-examples to the Torelli theorem. The construction is as follows: let \( K \subset \mathbb{P}^3 \) be a Kummer surface. We choose \( k \) double points of \( K \) in general position (this can be done with \( 0 \leq k \leq 6 \)), and a general quadric \( Q \subset \mathbb{P}^3 \) passing through these \( k \) points. The Todorov surface \( S \) is the double covering of \( K \) branched along \( K \cap Q \) and the remaining \( 16 - k \) double points. It is a minimal surface of general type with \( p_g = 1, K^2 = 8 - k, q = 0 \). If moreover we choose \( K \rho \)-maximal (that is, \( K = E^2/\{\pm 1\} \), where \( E \) is an elliptic curve with complex multiplication), then \( S \) is \( \rho \)-maximal by Proposition 2(b).

Note that by varying the quadric \( Q \) we get a continuous, non-constant family of \( \rho \)-maximal surfaces.

6.4. Double covers. — In [Per82] Persson constructs \( \rho \)-maximal double covers of certain rational surfaces by allowing the branch curve to acquire some simple singularities (see also [BE87]). He applies this method to find \( \rho \)-maximal surfaces in the following families:

- Horikawa surfaces, that is, surfaces on the “Noether line” \( K^2 = 2p_g - 4 \), for \( p_g \not\equiv -1 \) (mod. 6);
- Regular elliptic surfaces;
- Double coverings of \( \mathbb{P}^2 \).

In the latter case the double plane admits (many) rational singularities; it is unknown whether there exists a \( \rho \)-maximal surface \( S \) which is a double covering of \( \mathbb{P}^2 \) branched along a smooth curve of even degree \( \geq 8 \).

6.5. Hypersurfaces and complete intersections. — Probably the most natural families to look at are smooth surfaces in \( \mathbb{P}^3 \), or more generally complete intersections. Here we may ask for a smooth surface \( S \), or for the minimal resolution of a surface with rational double points (or even any surface deformation equivalent to a complete intersection of given type). Here are the examples that we know of:

- The quintic surface \( x^3yz + y^3zt + z^3tx + t^3xy = 0 \) has four \( A_9 \) singularities; its minimal resolution is \( \rho \)-maximal [Sch11]. It is not yet known whether there exists a smooth \( \rho \)-maximal quintic surface.
- The Fermat sextic is \( \rho \)-maximal (§4, Corollary 1).
- The complete intersection \( y^2 - xz = v^2 - uw = x^3 + z^3 + u^3 + w^3 = 0 \) of type \((2, 2, 3)\) in \( \mathbb{P}^5 \) has 12 nodes; its minimal desingularization is \( \rho \)-maximal (end of §4).
- The surface of cuboids is a complete intersection of type \((2, 2, 2, 2)\) in \( \mathbb{P}^6 \) with 48 nodes; its minimal desingularization is \( \rho \)-maximal (§5, Corollary 2).

7. The complex torus associated to a \( \rho \)-maximal variety

For a \( \rho \)-maximal variety \( X \), let \( T_X \) be the \( \mathbb{Z} \)-module \( H^2(X, \mathbb{Z})/\text{NS}(X) \). We have a decomposition

\[
T_X \otimes \mathbb{C} = H^{2,0} \oplus H^{0,2}
\]
defining a weight 1 Hodge structure on $T_X$, hence a complex torus $T := H^{0,2}/p_2(T_X)$, where $p_2 : T_X \otimes \mathbb{C} \to H^{2,0}$ is the second projection. Via the isomorphism $H^{0,2} = H^2(X, O_X)$, $T_X$ is identified with the cokernel of the natural map $H^2(X, \mathbb{Z}) \to H^2(X, O_X)$.

The exponential exact sequence gives rise to an exact sequence

$$0 \to \text{NS}(X) \to H^2(X, \mathbb{Z}) \to H^2(X, O_X) \to H^3(X, \mathbb{Z}) \to 0,$$

hence to a short exact sequence

$$0 \to T_X \to H^2(X, O_X') \xrightarrow{\partial} H^3(X, \mathbb{Z}),$$

so that $T_X$ appears as the “continuous part” of the group $H^2(X, O_X')$.

**Example 5.** Consider the elliptic modular surface $B_\Gamma$ of Section 6.1. The space $H^0(B_\Gamma, K_{B_\Gamma})$ can be identified with the space of cusp forms of weight 3 for $\Gamma$; then the torus $T_{B_\Gamma}$ is the complex torus associated to this space by Shimura (see [Shi69]).

**Example 6.** Let $X = C \times C'$, with $JC$ isogenous to $E^\tau$ and $JC'$ to $E'^\tau$ (Proposition 5). The torus $T_X$ is the cokernel of the map

$$i \otimes i' : H^1(C, \mathbb{Z}) \otimes H^1(C', \mathbb{Z}) \to H^1(C, O_C) \otimes H^1(C', O_{C'}),$$

where $i$ and $i'$ are the embeddings

$$H^1(C, \mathbb{Z}) \hookrightarrow H^1(C, O_C) \quad \text{and} \quad H^1(C', \mathbb{Z}) \hookrightarrow H^1(C', O_{C'}).$$

We want to compute $T_X$ up to isogeny, so we may replace the left hand side by a finite index sublattice. Thus, writing $E = C/\Gamma$, we may identify $i$ with the diagonal embedding $\Gamma^g \hookrightarrow \mathbb{C}^g$, and similarly for $i'$; therefore $i \otimes i'$ is the diagonal embedding of $(\Gamma \otimes \Gamma)^{g\gamma}$ in $\mathbb{C}^{g\gamma}$. Put $\Gamma = \mathbb{Z} + Z\tau$; the image $\Gamma'$ of $\Gamma \otimes \Gamma$ in $C$ is spanned by $1, \tau, \tau^2$; since $E$ has complex multiplication, $\tau$ is a quadratic number, hence $\Gamma$ has finite index in $\Gamma'$. Finally we obtain that $T_X$ is isogenous to $E^{g\gamma}$. For the surface $X = (C \times C)/\Gamma$ studied in §5 an analogous argument shows that $T_X$ is isogenous to $E^{g\gamma}$. This is still an abelian variety of type CM, in the sense that $\text{End}(A) \otimes \mathbb{Q}$ contains an étale $\mathbb{Q}$-algebra of maximal dimension $2 \dim(A)$. There seems to be no reason why this should hold in general. However it is true in the special case $h^{2,0} = 1$ (e.g. for holomorphic symplectic manifolds):

**Proposition 10.** If $h^{2,0}(X) = 1$, the torus $T_X$ is an elliptic curve with complex multiplication.

**Proof.** Let $T_X'$ be the pull back of $H^{2,0} + H^{0,2}$ in $H^2(X, \mathbb{Z})$; then $p_2(T_X')$ is a sublattice of finite index in $p_2(T_X)$. Choosing an ample class $h \in H^2(X, \mathbb{Z})$ defines a quadratic form on $H^2(X, \mathbb{Z})$ which is positive definite on $T_X$. Replacing again $T_X'$ by a finite index sublattice we may assume that it admits an orthogonal basis $(e, f)$ with $e^2 = a$, $f^2 = b$. Then $H^{2,0}$ and $H^{0,2}$ are the two isotropic lines of $T_X' \otimes \mathbb{C}$; they are spanned by the vectors $\omega = e + \tau f$ and $\overline{\omega} = e - \tau f$, with $\tau^2 = -a/b$. We have $e = \frac{1}{2}(\omega + \overline{\omega})$ and $f = \frac{1}{2\tau}(\omega - \overline{\omega})$; therefore multiplication by $\frac{1}{2\tau}$ induces an
isomorphism of \( \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau) \) onto \( H^{0,2}/p_{2}(T'_{X}) \), hence \( T_{X} \) is isogenous to \( \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau) \) and 
\[
\text{End}(T_{X}) \otimes \mathbb{Q} = \mathbb{Q}(\tau) = \mathbb{Q}\left(\sqrt{-\text{disc}(T'_{X})}\right).
\]

8. Higher codimension cycles

A natural generalization of the question considered here is to look for varieties \( X \) for which the group \( H^{2p}(X, \mathbb{Z})_{\text{alg}} \) of algebraic classes in \( H^{2p}(X, \mathbb{Z}) \) has maximal rank \( h^{p,p} \). Very few nontrivial cases seem to be known. The following is essentially due to Shioda:

**Proposition 11.** — Let \( F_{d}^{n} \) be the Fermat hypersurface of degree \( d \) and even dimension \( n = 2\nu \). For \( d = 3,4 \), the group \( H^{n}(F_{d}^{n}, \mathbb{Z})_{\text{alg}} \) has maximal rank \( h^{\nu,\nu} \).

**Proof.** — According to [Shi79] we have
\[
\text{rk} H^{n}(F_{d}^{n}, \mathbb{Z})_{\text{alg}} = 1 + \frac{n!}{(n+1)!} \quad \text{and} \quad \text{rk} H^{n}(F_{d}^{n}, \mathbb{Z})_{\text{alg}} = \sum_{k=0}^{\nu+1} \frac{(n+2)!}{(k!)^{2}(n+2-2k)!}.
\]

On the other hand, let \( R_{d}^{n} := \mathbb{C}[X_{0}, \ldots, X_{n+1}]/(X_{0}^{d} - 1, \ldots, X_{n+1}^{d} - 1) \) be the Jacobian ring of \( F_{d}^{n} \); Griffiths theory [Gri69] provides an isomorphism of the primitive cohomology \( H^{\nu,\nu}(F_{d}^{n})_{\text{alg}} \) with the component of degree \((\nu+1)(d-2)\) of \( R_{d}^{n} \). Since this ring is the tensor product of \((n+2)\) copies of \( \mathbb{C}[T]/(T^{d-1}) \), its Poincaré series \( \sum_{k} \dim(R_{d}^{n})_{k} T^{k} \)
\[
is (1 + T + \cdots + T^{d-2})^{n+2}.
\]
Then an elementary computation gives the result.

In the particular case of cubic fourfolds we have more examples:

**Proposition 12.** — Let \( F \) be a cubic form in \( 3 \) variables, such that the curve \( F(x, y, z) = 0 \) in \( \mathbb{P}^{2} \) is an elliptic curve with complex multiplication; let \( X \) be the cubic fourfold defined by \( F(x, y, z) + F(u, v, w) = 0 \) in \( \mathbb{P}^{5} \). The group \( H^{3}(X, \mathbb{Z})_{\text{alg}} \) has maximal rank \( h^{3,3}(X) \).

**Proof.** — Let \( u \) be the automorphism of \( X \) defined by
\[
u(x, y, z; u, v, w) = (x, y, z; \omega u, \omega v, \omega w).
\]

We observe that \( u \) acts trivially on the (one-dimensional) space \( H^{3,1}(X) \). Indeed Griffiths theory [Gri69] provides a canonical isomorphism
\[
\text{Res} : H^{0}(\mathbb{P}^{5}, K_{\mathbb{P}^{5}}(2X)) \rightarrow H^{3,1}(X);
\]
the space \( H^{0}(\mathbb{P}^{5}, K_{\mathbb{P}^{5}}(2X)) \) is generated by the meromorphic form \( \Omega/G^{2} \), with
\[
\Omega = xdy \wedge dz \wedge du \wedge dv \wedge dw - ydx \wedge dz \wedge du \wedge dv \wedge dw + \cdots,
\]
\[
G = F(x, y, z) + F(u, v, w).
\]
The automorphism \( u \) acts trivially on this form, and therefore on \( H^{3,1}(X) \).

Let \( F \) be the variety of lines contained in \( X \). We recall from [BD85] that \( F \) is a holomorphic symplectic fourfold, and that there is a natural isomorphism of Hodge structures \( \alpha : H^{3}(X, \mathbb{Z}) \rightarrow H^{2}(F, \mathbb{Z}) \). Therefore the automorphism \( u_{F} \) of \( F \) induced by \( u \) is symplectic. Let us describe its fixed locus.
The fixed locus of $u$ in $X$ is the union of the plane cubics $E$ given by $x = y = z = 0$ and $E'$ given by $u = v = w = 0$. A line in $X$ preserved by $u$ must have (at least) two fixed points, hence must meet both $E$ and $E'$; conversely, any line joining a point of $E$ to a point of $E'$ is contained in $X$, and preserved by $u$. This identifies the fixed locus $A$ of $u_F$ to the abelian surface $E \times E'$. Since $u_F$ is symplectic $A$ is a symplectic submanifold, that is, the restriction map $H^{2,0}(F) \to H^{2,0}(A)$ is an isomorphism. By our hypothesis $A$ is $\rho$-maximal, so $F$ is $\rho$-maximal by Proposition 2. Since $\alpha$ maps $H^4(X, \mathbb{Z})_{alg}$ onto $\text{NS}(F)$ this implies the Proposition. \hfill $\square$

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Arnaud Beauville, Laboratoire J.-A. Dieudonné, UMR 7351 du CNRS, Université de Nice
Parc Valrose, F-06108 Nice cedex 2, France
E-mail : arnaud.beauville@unice.fr
Url : http://math1.unice.fr/~beauvill/